A STUDY OF SOME PROBLEMS ASSOCIATED WITH INTUITIONISTIC FUZZY IDEALS OF Γ-RINGS

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DECLARATION

I, hereby declared that the presented work in the thesis entitled "A Study Of Some Problems Associated With Intuitionistic Fuzzy Ideals Of Γ-Rings" in fulfilment of degree of **Doctor of Philosophy (Ph.D.)** is outcome of research work carried out by me under the supervision of Dr. Nitin Bhardwaj, working as Professor, in the Mathematics Department of Lovely Professional University, Punjab, India and Co-Supervision of Dr. Poonam Kumar Sharma, working as Associate Professor, in the Mathematics Department of D.A.V. College Jalandhar. In keeping with the general practice of reporting scientific observations, due acknowledgments have been made whenever work described here has been based on the findings of another investigator. This work has not been submitted in part or full to any other University or Institute for the award of any degree.

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CERTIFICATE

This is to certify that the work reported in the Ph. D. thesis entitled "A Study Of Some Problems Associated With Intuitionistic Fuzzy Ideals Of Γ-Rings" submitted in fulfillment of the requirement for the reward of degree of **Doctor of Philosophy (Ph.D.)** in the Mathematics Department (School of Chemical Engineering and Physical Sciences), is a research work carried out by Hem Lata, 42000059, is bonafide record of her original work carried out under my supervision and that no part of thesis has been submitted for any other degree, diploma or equivalent course.

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ABSTRACT

Intuitionistic Fuzzy Set (IFS) is a complex function $f = (f_1, f_2)$ whose domain is the universal set X and the range set is $[0,1] \times [0,1]$, with the condition that $f_1(x) + f_2(x) \leq 1$, for all $x \in X$, where the coordinate functions $f_1, f_2 : X \rightarrow [0,1]$ are called membership function and non-membership function respectively. The study of these functions was first proposed and dealt with by K.T. Atanassov [4] in 1983 and as a result, a new theory on sets has come into existence which is known as the IFS (Intuitionistic fuzzy set) theory. This theory has captivated the attention of many researchers all over the world who have contributed particularly to its development and application. In recent years, the rapid growth of IFS theory and its applications have been witnessed worldwide and extensive research has been done to study the comparison of the theory of IFS with other theories of uncertainties and vagueness. Some authors replaced an algebraic structure with the universal set and studied the notion of intuitionistic fuzzy algebraic structures.

Nobusawa [39] coined the concept of Γ-Ring. Barnes [8] weakened slightly the conditions in the definition of the Γ-Ring in the sense of Nobusawa. Since then, a lot of studies has been undertaken by researchers to inquire about the different properties of this Γ-Ring. By choosing Γ suitably a part of the ring may be seen as a Γ-Ring. Numerous results which are based on ring theory have been put forth in Γ-Ring.

The work of intuitionistic fuzzify of ideals of Γ-Ring was first determined by Kim et al. in [34] and further many relevant results and intuitionistic fuzzification of ideals of Γ-Ring can be seen in the work of Palaniappan et al. in [42,43,44,45,46]. The thesis aims to intuitionistic fuzzify some other concepts such as Characteristic Ideal, Primary Ideal, Irreducible ideal, 2-Absorbing Ideal, $2-$Absorbing Primary Ideal, Prime radical, Primary decomposition of an ideal in the Γ-Ring. Furthermore, we also investigate the topological aspects of the set of all IFPIs (intuitionistic fuzzy prime ideals) of Γ-Ring. An attempt has been made to unify the concepts of the intuitionistic fuzzy prime ideal (2-absorbing ideal) and (IFPrI) intuitionistic fuzzy primary ideal ($2-$absorbing primary ideal) into (IFf-PrI) intuitionistic fuzzy f-primary ideal (2-f-absorbing primary ideals) and studied their properties, where f is a function from set of all IFIs (Intuitionistic Fuzzy Ideals) of Γ-

Rings to itself satisfying certain properties. Also, the concept of extension of an ideal with respect to an arbitrary point of the Γ-Ring has been explored and many properties of it has been also studied.

In Chapter 3, the concept of IFCI (Intuitionistic Fuzzy Characteristic Ideal) in Γ-Rings is examined. An illustrative example is provided to demonstrate an IFI that does not qualify as an IFCI. The relationship between IFCI and its level cut sets is explored, alongside investigations into the correspondence between the set of all automorphisms of a Γ-Ring and the corresponding automorphisms of its operator rings. Furthermore, it is demonstrated that a one-to-one map exists between IFCIs(H) (the set of all intuitionistic fuzzy characteristic ideals of a Γ-ring) and IFCIs(OR) (the set of all intuitionistic fuzzy characteristic ideals of an operator-ring). These structures prove valuable in developing concepts such as IFPI (Intuitionistic Fuzzy Prime Ideal), IFPrIs (Intuitionistic Fuzzy Primary Ideals), and IFSPI (Intuitionistic Fuzzy Semi-Prime Ideal) in a Γ-Ring framework.

In Chapter 4, the foundational concepts of IFPrI and IFPR (Intuitionistic Fuzzy Prime Radical) in Γ-Ring H are thoroughly examined. It is proven that IFPrI of a Γ-Ring constitutes a two-valued IFS, with the base set defined as a primary ideal (The base set of IFS Q is defined as the set$\{h \in H: \mu_Q(h) = 1, \nu_Q(h) = 0\}$). The concept of IFPR in Γ-Ring H is introduced, demonstrating that the IFPR of an IFPrI yields an IFPI. Furthermore, the homeomorphic behaviour of IFPrI as well as IFPR in Γ-Ring is investigated. The study of these notions lays the foundation for a crucial property in Γ-ring theory: the decomposition of ideals into primary ideals in the intuitionistic fuzzy environment for Γ-Ring.

In Chapter 5, introduces and explores the concept of irreducibility of an IFI in a Γ-Ring. It is proven that every IFI in a Noetherian Γ-Ring can be expressed as an intersection of a finite number of IFIrIs (Intuitionistic Fuzzy Irreducible Ideals). Additionally, the IF version of the Lasker-Noether theorem for a commutative Noetherian Γ-Ring is established, demonstrating that every IFI G in such a ring can be decomposed into a finite intersection of IFPrIs. This decomposition is referred to as an IF primary decomposition.

The independence of the set of all IF-associated prime ideals of G in the case of minimal intuitionistic fuzzy primary decomposition is also shown. The chapter sets a new horizon in the study of IF primary decomposition, paving the way for further research in other algebraic structures.

In Chapter 6, a topology is defined on $\mathcal{X} = IFSpec(H)$, which represents the collection of all IFPIs of a commutative Γ-Ring H with unity, referred to as the Zariski topology. The compactness of the subspace $\mathcal{Y}$ of $\mathcal{X}$ is established using bases for the Zariski topology. It is demonstrated that the space $\mathcal{X}$ is always T_0 but not T_2, though it becomes a T_2 space when H is a Boolean Γ-Ring. It has been also shown that subspace $\mathcal{Y}$ is T_1 if and only if every singleton element of $\mathcal{Y}$ is IF maximal ideal of H. Further for a homomorphism f from a Γ-Ring H_1 onto a Γ-Ring H_2, it is shown that $\mathcal{X}' = IFSpec(H_2)$ is homeomorphic to the subset $\mathcal{X}^* = \{G \in \mathcal{X}: G$ is f- invariant $\}$ consisting of f-invariant elements of $\mathcal{X} = IFSpec(H_1)$. Also, the space $\mathcal{X}$ is irreducible if and only if the intersection of all the elements of $\mathcal{X}$ is also an element of $\mathcal{X}$. However the space $\mathcal{X}$ is connected iff 0_H and e are the only idempotent elements in H.

In Chapter 7, the concept of IFf-PrIs (2-absorbing f-primary ideals) is introduced, which unifies the notions of IFPIs (2-absorbing ideals) and IFPrIs (2-APrIs) in a Γ-Ring. This study sets the foundation for the exploration of the decomposition property for IFf-PrI (2-absorbing f-primary ideal).

In Chapter 8, the notion of extensions of IFI with respect to an element in the Γ-Ring is investigated, and characterizations of IFPI and IFSPI are developed, providing valuable insights into the properties of these structures.

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LIST OF ABBREVIATIONS AND SYMBOLS USED

IFS	Intuitionistic fuzzy set
$IFS(X)$	Set of all intuitionistic fuzzy sets of X
IFP(H)	Intuitionistic fuzzy point of Γ-ring H
$\mu_G(x)$	Degree of membership of the element x in the IFS G
$\nu_G(x)$	Degree of non-membership of the element x in the IFS G
Aut(H) or Aut(OR)	Set of all automorphism of Γ-ring H or Set of all automorphism of operator ring of Γ-ring H
$Hom_{H_1}^{H_2}$	Homomorphism of a Γ-Ring H_1 into a Γ-Ring H_2
Γ-$Hom_{H_1}^{H_2}$	Γ-Homomorphism $from$ H_1 to H_2 where H_1 and H_2 are $\Gamma - rings$
$IFI(H)$ or $IFI(L)$ or $IFI(R)$	Set of intuitionistic fuzzy ideals of Γ-ring H or Set of intuitionistic fuzzy ideals of left operator ring L or Set of intuitionistic fuzzy ideals of right operator ring R
$FI(H)$ or $I(H)$	Set of fuzzy ideals of Γ-ring H or Set of ideals of Γ-ring H
$FLI(H)$ or $FRI(H)$	Set of fuzzy left ideals of Γ-ring H or Set of fuzzy right ideals of Γ-ring H
$FI(OR)$ or $I(L)$ or $I(R)$	Set of fuzzy ideals of operator ring of Γ-ring H or Set of ideals of left operator ring of Γ-ring H or Set of ideals of right operator ring of Γ-ring H
$IFCI$ or FCI or CI	Intuitionistic fuzzy characteristic ideal or fuzzy characteristic ideal or characteristic ideal
$IFCI(H)$ or $CI(H)$ or $CI(L)$	Set of all intuitionistic fuzzy characteristic ideals of Γ-ring H or Set of all characteristic ideals of Γ-ring H or Set of all characteristic ideals of left operator ring of Γ-ring H
$IFCI(OR)$	Set of all intuitionistic fuzzy characteristic ideals of operator-ring
$IFCF$	Intuitionistic fuzzy characteristic function
$IFPI$ or $IFSPI$ or FPI or $FSPI$ PI or SPI	Intuitionistic fuzzy prime ideal or Intuitionistic fuzzy semi prime ideal or fuzzy prime ideal or fuzzy semi prime ideal or prime ideal or semi prime ideal
$IFPR$ or $IFRI$	Intuitionistic fuzzy prime radical or intuitionistic fuzzy radical ideal
$IFPrI$ or $FPrI$ or PrI	Intuitionistic fuzzy primary ideal or fuzzy primary ideal or primary ideal
$IFMI$ or MI	Intuitionistic fuzzy Maximal ideal or Maximal ideal
$IF2 - AI$ or $2 - AI$	Intuitionistic fuzzy 2-absorbing ideal or 2-absorbing ideal

$\boldsymbol{IF2-API}$ or $\boldsymbol{2-API}$	Intuitionistic fuzzy 2-absorbing prime ideal or 2-absorbing prime ideal
$\boldsymbol{IF2-Af-PI}$ or $\boldsymbol{2-Af-PI}$	Intuitionistic fuzzy 2-absorbing f-prime ideal or 2-absorbing f-prime ideal
$\boldsymbol{IF2-APrI}$ or $\boldsymbol{IF2-ASPrI}$ or $\boldsymbol{2-APrI}$	Intuitionistic fuzzy 2-absorbing primary ideal or Intuitionistic fuzzy 2-absorbing semi primary ideal or 2-absorbing primary ideal
$\boldsymbol{IF2-Af-PrI}$ or $\boldsymbol{IF2-Ag-PrI}$ or $\boldsymbol{2-Af-PrI}$	Intuitionistic fuzzy 2-absorbing f-primary ideal or Intuitionistic fuzzy 2-absorbing g-primary ideal or 2-absorbing f-primary ideal
$\boldsymbol{IFP}$ **−PrI or** $\boldsymbol{IFf}$ **−PrI or** $\boldsymbol{IFg}$ **−PrI**	Intuitionistic fuzzy P-primary ideal or Intuitionistic fuzzy f-primary ideal or Intuitionistic fuzzy g-primary ideal
$\boldsymbol{IFP-2-APrI}$ or $\boldsymbol{2-A\delta-PrI}$	Intuitionistic fuzzy P-2-absorbing primary ideal or 2-absorbing δ-primary ideal
$\boldsymbol{IFIrI}$ or $\boldsymbol{IrI}$	Intuitionistic fuzzy irreducible ideal or irreducible ideal
$\boldsymbol{G_{(\eta,\theta)}}$	(η,θ)-cut set of the IFS G
$\boldsymbol{G^*}$	Support of the intuitionistic fuzzy set G
$\boldsymbol{G_*}$	$G_{(\eta,\theta)}$, where $\eta=\mu_G(0)$ and $\theta=\nu_G(0)$
$\langle\boldsymbol{G}\rangle$	Intuitionistic fuzzy ideal generated by G
$\sqrt{\boldsymbol{G}}$	Intuitionistic fuzzy radical of G
$\boldsymbol{f(G)}$	Image of the IFS G under the map f
$\boldsymbol{f^{-1}(G)}$	Inverse image of the IFS G under the map f
$\boldsymbol{Kerf}$	Kernal of the map f
$\boldsymbol{Img(G)}$	Set of values of the IFS G
$\boldsymbol{Sup(\Lambda)}$	Supremum of the index set Λ
$\boldsymbol{Inf(\Lambda)}$	Infimum of the index set Λ
$\boldsymbol{\chi_Y}$	Intuitionistic fuzzy characteristic function on a subset Y of X
$\boldsymbol{h_{(\eta,\theta)}}$	Intuitionistic fuzzy point (IFP) of X with support h
$\boldsymbol{Spec(H)}$	Set of all prime ideals of Γ-ring H
$\boldsymbol{IFSpec(H)}$	Set of all IFPIs of the Γ-ring H
$\boldsymbol{IFSpec(OR)}$	Set of all IFPIs of the operator ring of Γ-ring H
$(₲_1:₲_2)$	IF residual quotient of $₲_1$ by $₲_2$
$₲_1\boldsymbol{\Gamma}₲_2$	Γ-product of IFSs $₲_1$ and $₲_2$
$\mathbb{N}$	Set of natural numbers
$\mathbb{Z}$	Set of integers
$\mathbb{R}$	Set of real numbers
$\mathbf{Z_n}$	Additive (multiplicative) group of integers modulo n
$<\boldsymbol{G},\boldsymbol{h}>$	Extension of an IFS G with respect to h

Chapter 1

Introduction

In this chapter, the history and chronological development of Γ-Ring, fuzzification of some ring theoretic structures in Γ-Ring has been given briefly, and also some results on IFI in Γ-Ring obtained so far. A subsequent chapter-wise summary of the research carried out in the thesis is discussed.

1.1 History and Development

1.1.1 Γ-Rings

Among generalizations of rings, the concept of Γ-Ring holds a unique position. Algebraic structure of all rectangular matrices of the same type over a division ring under addition have a crucial role in classical ring theory. Although a binary multiplication on this set is possible but it lacks suitable interpretations. To address this, M.R. Hestenes [25], in 1962, introduced a ternary multiplication on the set of all m×n matrices over the division ring D, defined as $abc = ab^t.c$, where b^t denotes the transpose of matrix b. This ternary multiplication involves the usual multiplication of three matrices, as further developed by N. Nobusawa [39], in 1964 and the algebraic structure defined was more generalized than a ring. Additionally, Γ was endowed with a ternary multiplication that meets the same conditions as explained by Hesten

The conditions described by Nobusawa in the definition of Γ-Ring was slightly weakened by W. E. Barnes [8], in 1966. After that, J. Luh [38], in the year 1969 and S. Kyuno [36], in 1978, deliberated the structure of Γ-Rings and discovered various generalized results parallel to ring theory. Z. K. Warsi [68] in 1978, explored the decomposition of primary ideals on Γ-Ring. In 1982, S. Kyuno [37] gave complete notes on the Jacobson radical of Γ-Rings. In 2009, A.C. Paul and M.S. Uddin [47] further extended the work of S.Kyuno for Jacobson radical of Γ-Rings and in 2011, A.C. Paul and M.S. Uddin [48] also developed the decomposition in Neotherian Γ-Rings using sub Γ_H-modules. In 2015, R. Paul [49] deliberated various types of ideals of Γ-Rings and the corresponding ORs. In 2016, M. Y. Elkettani and A. Kasem [19] introduced the notion of δ-primary Γ-ideals of Γ-Rings and studied the properties of these classes of Γ-ideals. In 2018, A. H. Rezaei and B. Davvaz [54] have constructed Γ-algebra and Γ-Lie admissible algebras.

1.1.2 Fuzzification of some of the concepts analyzed in Γ-Rings

The notion of FIs in Γ-Rings was introduced by Y. B. Jun and C. Y. Lee [30] in 1992 and they also studied preliminary properties of FIs. Further, the concept of fuzzy characteristic Γ-ideals and FPI of a Γ-Ring was introduced by S.M Hong and Y.B Jun [26,27] in 1994 and 1995, they elucidated numerous characterizations for an FI to be an FPI. Ozturk et al. [41] in the year 2002, gave a result for a Γ-Ring to be Artinian by characterizing Noetherian Γ-Rings with a use of fuzzy ideals. In the year 2005, T.K. Dutta and T. Chanda [15] defined some compositions of FIs of a Γ-Ring and studied the structures of FI(H). They established an analogous between FI(H) and the FI(OR) of the Γ-Ring. Also, they characterized Γ-field and Noetherian Γ-Ring.

In year 2007, different depictions for an FI to be an FPI which was obtained by Jun was given by T.K. Dutta and T. Chanda [16] and also they proved a few more new depictions of an FPI. M. Dumitru [17] in 2009, has given a direct way to study some kinds of radicals in Γ-Rings. One can study the same radicals in the associated rings to a Γ-Ring, namely the ring of left and right operators over the Γ-Ring. Interestingly, there exists a correspondence between the ideals of these operator rings and the ideals of the Γ-Ring. In 2010, B.A. Erosy [21] defined FSPIs of a Γ-Ring via operator rings and obtained a few more characterizations of FSPIs. In the year 2017, Serkan et al. [57] introduced the concept of F2-APr gamma ideals in Γ-Rings which is an abstraction of the idea of FPI and FPrI in Γ-Rings. Also in year 2017, Yesilkurt et al. [67] introduced the notion of a fuzzy weakly & partial weakly prime ideals and fuzzy semiprime Γ-ideals of a Γ-Ring and obtained their characterization. In 2018, the concept of extensions of fuzzy ideal w.r.t. an element in the Γ-semiring was introduced by B. Venkateshwarlu, M.M.K. Rao, and Y.A. Narayana in [70]. In 2019, A. K. Agrawal, P. K. Mishra, Sandhya Verma, and Roopali Saxena [1] studied some theorems on FPI of Γ-Ring and found a characterization of FPrI of a Γ-Ring. In 2019, Goswami et al. [24] in year 2019 studied the Fuzzy Structure Space of Semirings and Γ-Semirings and examined many separation axioms of this space.

1.1.3 Intuitionistic fuzzification of some of the concepts analyzed in Γ-Rings

In 1986, K.T. Atanassov, have defined the concept of IFSs as a generalization of Fuzzy sets, an example was given to support the definition and its generalization. In 2001, K.H. Kim, Y.B. Jun, and M.A. Ozturk [34] coined the concept of IFIs of Γ-Ring and have seen various properties of them. In 2008, K.H. Kim, and J.G. Lee [35] studied the notion of intuitionistic (T, S)-normed FI of Γ-Ring. Palaniappan et al. [43], in 2010, had given a suitable characterization of IFI of a Γ-rings and many related results were proved. Palaniappan et al. [46] in 2011, introduced the concept of IFPI (IFSPI) in Γ-Ring. They also established a relation between the $IFSpec$(H) and $IFSpec$(OR). A characterization of IF Artinian and noetherian Γ-Rings has been established. In 2017, D. Ezhilmaran and A. Dhandapani [22] studied IF bi-ideals in Γ-near rings. In 2018, S. Yavuza, D. Onara, B.A. Ersoya, G. Yesilot [69] introduced the concept of IF2-APrIs of commutative rings. In 2020, Y.A. Bhargavi, [9] introduced the concepts of translational invariant vague set and ideals generated by it in a Γ-semiring.

The main objectives of the thesis are

1. To enrich the knowledge of intuitionistic fuzzy set on algebraic structures of Γ-Rings.
2. To extend the concepts of ring theory to intuitionistic fuzzy ring theory associated with Γ-Rings.
3. To define new concepts in Γ-rings in the intuitionistic fuzzy environment.

4. To study the topological aspect of the set of all intuitionistic fuzzy prime ideals associated with Γ-Rings.

5. To unifying some ideals in the intuitionistic fuzzy environment associated with Γ-Rings.

1.2 Chapter Wise Summary

During the voyage of research, the compilation of work done is a major part. In this thesis the work has been tried to compile as follows:

In Chapter 1, a brief history and the subsequent advancement in the concept of Γ-Ring is furnished. The details of work done on the intuitionistic fuzzification of some algebraic structures in Γ-Rings have been given. Also, the research work carried out in the thesis is presented concisely.

In Chapter 2, some basic definitions, results, and properties of Γ-rings, ideals in Γ-rings, and IFI in Γ-rings which are mandatory for the research work are accentuated.

In Chapter 3, the concept of IFCI of a Γ-Ring which was an analog of a characteristic ideal in the ordinary ring theory has been defined, and various new results has been derived. The correlation between the Aut(H) and the corresponding Aut(OR) have been innovated. Then a one-to-one correlation between IFCI(H) and that of its operator ring has been constituted. This is used to obtain a similar bijection for characteristic ideals.

In Chapter 4, The notion of IFPR of an IFI in Γ-Rings has been introduced. The IFPrI of Γ-Rings have also been characterized. The homomorphic behavior of IFPrI and IFPR of Γ-Rings have also been analyzed. The study of these notions laid down the foundation of the most important property in ring theory: the decomposition of ideals in terms of primary ideals in the IF environment for Γ-Ring.

In Chapter 5, the IF version of the Lasker-Noether theorem for a commutative Γ-Ring has been established. It has been proved that in a commutative Noetherian Γ-Ring, every IFI G, can be broken down as an intersection of a finite number of IFIrIs (PrIs). This decomposition is called an IF primary decomposition. Further, in the case of a minimal IF primary decomposition of G, it has been proved that the set of all IF-associated PI of G, is independent of the particular decomposition. Some other fundamental results of this concept have also been discussed.

In Chapter 6, The IF structure space of a Γ-Ring set up by the class of IFPIs of Γ-Ring called the IF prime spectrum of Γ-Ring has also been investigated and deliberated. Apart from studying the basic properties of this structure space, some important properties like separation axioms, compactness, irreducibility, and connectedness in this structure space have also been explored.

In chapter 7, the notion of expansion of IFIs of a commutative Γ-Ring has been introduced and using this concept, the notion of IFf-PrIs (2-Af-PrIs) has been developed which unifies the concept of IFPIs (2-AIs) and IFPrIs ($2-$APrIs) of a Γ-Ring. Several important results about IFPIs (2-AIs) and IFPrIs (2-APrIs) have been extended into this general framework.

In chapter 8, extension of IFI w.r.t. to a point of Γ-Ring was investigated and characterization of IFPIs and IFSPIs has been innovated.

1.3 Applications of Intuitionistic fuzzy logic in Γ-ring

Intuitionistic fuzzy logic and Gamma-ring theory are sophisticated mathematical frameworks employed across domains such as computer science, decision-making, and logic. Integrating intuitionistic fuzzy logic with Gamma-ring theory creates innovative possibilities for tackling intricate problems characterized by uncertainty, imprecision, and complex mathematical structures. Some potential applications include:

1. Decision-Making in Uncertain Environments

Intuitionistic fuzzy logic is well-suited for decision-making problems involving uncertainty, as it incorporates both membership and non-membership functions. Meanwhile, gamma-ring theory offers a structural framework to mathematically operate on these sets, providing powerful tools for making optimized decisions in uncertain environments.

2. Multi-Criteria Optimization Problems

In challenges such as resource allocation, product design, or financial portfolio optimization, decision-makers often face competing criteria that are not precisely defined. Intuitionistic fuzzy logic enables the management of degrees of truth, uncertainty, and hesitation in these scenarios. Gamma-rings provide a mathematical framework to model the algebraic relationships among these criteria, facilitating the development of effective optimization strategies.

3. Fuzzy Relational Databases and Information Retrieval

Intuitionistic fuzzy logic improves relational databases' capacity to manage vague or imprecise data. Simultaneously, Gamma-rings can define operations such as union, intersection, and complement within this fuzzy relational model, enabling queries and information retrieval under uncertainty while ensuring algebraic consistency.

4. Fault Diagnosis in Complex Systems

Intuitionistic fuzzy logic is effective for assessing the degree of fault in components of complex systems, such as power grids, manufacturing plants, or transportation networks. Gamma-ring theory aids in modeling the relationships between different components and diagnostic tests, enabling the development of more robust diagnostic algorithms.

5. Image Processing and Pattern Recognition

Intuitionistic fuzzy logic facilitates the segmentation and classification of images with uncertain pixel data, while Gamma-ring theory offers algebraic tools to manage operations on such image data structures. It can model processes like image transformations, blurring, or noise reduction, ensuring consistent algebraic operations within intuitionistic fuzzy sets. This combination can enhance pattern recognition accuracy in applications such as medical imaging and automated inspection systems.

6. Knowledge Representation and Reasoning

Intuitionistic fuzzy logic is valuable for representing knowledge in expert systems where certainty levels are not absolute. Gamma-rings can structure and integrate diverse sources of fuzzy knowledge, ensuring logical consistency and supporting more efficient inference and decision-making processes.

7. Control Systems and Automation

In industrial control systems, sensor data may be imprecise due to noise or environmental influences. Intuitionistic fuzzy logic aids in handling these uncertainties during decision-making. Gamma-ring theory models the algebraic relationships between control actions and environmental factors, enabling optimal control strategies while accounting for system imprecision.

Conclusion:

The integration of intuitionistic fuzzy logic and gamma-ring theory provides robust mathematical tools for addressing uncertainty and imprecision across diverse applications. Intuitionistic fuzzy sets enable the handling of vague or incomplete information, while gamma-ring theory facilitates the organization and processing of intricate relationships. This combined approach can enhance solutions in areas such as decision-making, optimization, database management, fault diagnosis, and beyond.

Chapter 2

Literature Review

This chapter is divided into two sections. In the first section, an introduction to Γ-Ring theory has been provided and crucial definitions and results pertinent to Γ-Rings, which are imperative for subsequent chapters has been articulated. In the second section, fundamental definitions and concepts related to IFS theory, as introduced by K.T. Atanassov—an abstraction of the theory of fuzzy sets has been provided. Outline of elementary operations on IFSs has been provided and instances where the notion of IFS has been applied to various algebraic concepts has been explored. This exploration naturally leads to the introduction of IF subrings and ideals within the context of Γ-Ring.

2.1 Introduction To *Γ*-Ring Theory And Some Important Results

This section contains some definitions and results on Γ-Ring which are mainly taken from [8,13,17,36,37,39,49,68].

Definition 2.1.1 [8,39] "(Γ-Ring) If $(H,+)$ and $(\Gamma,+)$ are additive Abelian groups, then H is called a Γ-Ring if there exists mapping $H\times\Gamma\times H\rightarrow H$ [image of (h_1,α,h_2) is denoted by $h_1\alpha h_2$, where $h_1,h_2\in H$, and $\alpha\in\Gamma$ satisfying the following conditions:

1. $h_1\alpha h_2\in H$.
2. $(h_1+h_2)\alpha h_3=h_1\alpha h_3+h_2\alpha h_3$, $h_1(\alpha+\beta)h_2=h_1\alpha h_2+h_1\beta h_2$, $h_1\alpha(h_2+h_3)=h_1\alpha h_2+h_1\alpha h_3$
3. $(h_1\alpha h_2)\beta h_3=h_1\alpha(h_2\beta h_3)$ for all $h_1,h_2,h_3\in H$, and $\alpha,\beta\in\Gamma$."

Definition 2.1.2. [68] "(Commutative Γ-Ring) A Γ-Ring H is said to be commutative if $h\gamma k=k\gamma h$ for all $h,k\in H,\gamma\in\Gamma$."

Example 2.1.3. [8,49] "**(1)** Let us take H = $\{[a_{ij}]: a_{ij} \in Z, i = 1,2,\ldots,m; j = 1,2,\ldots,n\}$, the set of $(m \times n)$ matrices whose entries are from Z and $\Gamma = \{[a_{ij}]: a_{ij} \in Z, i = 1,2,\ldots,n; j = 1,2,\ldots,m\}$, the set of $(n \times m)$ matrices whose entries are from Z, then H will become a Γ-Ring.

(2) Consider $H = Z_2 \times Z_2 = \{(0,0),(1,0),(0,1),(1,1)\}, \Gamma = \{(0,0),(1,1)\}$. Clearly, H and Γ are additive Abelian groups, and that H is Γ-Ring.

(3) If R and R' are two additive Abelian groups, $H = Hom(R, R')$, $\Gamma = Hom(R', R)$ then H will be a Γ-Ring w.r.t. pointwise addition and composition of mappings."

Definition 2.1.4. [8,49] "(Ideal in Γ-Ring) A subset N of a Γ-Ring H is a left (right) ideal of H if N is an additive subgroup of H and $H\Gamma N = \{h\alpha k | h \in H, \alpha \in \Gamma, k \in N\}$, $(N\Gamma H)$ is contained in N. If N is both a left and a right ideal then N is a two-sided ideal, or simply an ideal of H.

Example 2.1.5. **(1)** Let us take $H = Z_2 \times Z_2 = \{(0,0),(1,0),(0,1),(1,1)\}$, $\Gamma = \{(0,0),(1,1)\}$ and $K = Z_2 \times \{0\} = \{(1,0),(0,0)\}$. Clearly, H and Γ are additive Abelian groups, and that H is Γ-Ring. Also, here K is the Γ-ideal of H.

(2) Let Z be the set of all integers. Take H = $\Gamma = Z$. Then Z is a Γ-Ring. Let $a, b \in H, \alpha \in \Gamma$. Suppose $a\alpha b \in H$ is the product of a, α, and b. Then H is a Γ-Ring. Take $N = 2Z$ be a subset of H. Then N is an ideal of H."

Definition 2.1.6. [8,49] "(Prime Ideal in Γ-Ring) Let H be a Γ-Ring. A proper ideal L of H is called prime if, for all pair of ideals S and T of H, $S\Gamma T \subseteq L$ implies that $S \subseteq L$ or $T \subseteq L$.

Remark 2.1..7. If L is an ideal of a Γ-Ring H. Then L is a PI iff $a \notin L$, $b \notin L$ implies $\exists$ $\gamma \in \Gamma$ such that $a\gamma b \notin L$."

Theorem 2.1.8. [8] " *If L is an ideal of a Γ-Ring H, the following conditions are equivalent:*

1. *L is a prime ideal of H*

2. If $a, b \in H$ and $a\Gamma H\Gamma b \subseteq L$ then $a \in L$ or $b \in L$."

Definition 2.1.9. [49] "(Semi-prime ideal in Γ-Ring) Let H be a Γ-Ring. A proper ideal L of H is called semi-prime if, for any ideal S of H, $S\Gamma S \subseteq L$ implies that $S \subseteq L$.

Remark 2.1.10. For an ideal L of a Γ-Ring H, L is SPI iff $a \notin L$ implies there exists $\gamma \in \Gamma$ such that $a\gamma a \notin L$."

Theorem 2.1.11. *"If L is an ideal of a Γ-Ring H, the following conditions are equivalent:*

1. *L is a SPI of H*

2. If $a \in H$ s.t. $a\Gamma H\Gamma a \subseteq L$, then $a \in L$."

Definition 2.1.12. [8,49,68] "Let H be a Γ-Ring. Then the radical of an ideal K of H is denoted by $\sqrt{K}$ and is defined as the set

$$\sqrt{K} = \{h \in H: (h\gamma)^{n-1}h \in K, \text{ for some } n \in \mathbf{N} \text{ and for all } \gamma \in \Gamma \}$$

where $(h\gamma)^{n-1}h = h$ for $n = 1$."

Definition 2.1.13. [8,49,68] "An ideal K of a commutative Γ-Ring H is said to be primary if, for any two ideals M and J of H, $M\Gamma J \subseteq K$ implies either $M \subseteq K$ or $J \subseteq \sqrt{K}$, where $\sqrt{K}$ is the prime radical of K."

Definition 2.1.14. [6] "A proper ideal M of Γ-Ring H is called *the* 2-absorbing ideal of H if whenever $h_1, h_2, h_3 \in H, \gamma_1, \gamma_2 \in \Gamma$ and $h_1\gamma_1 h_2\gamma_2 h_3 \in M$, then $h_1\gamma_1 h_2 \in M$ or $h_1\gamma_2 h_3 \in M$ or $h_2\gamma_2 h_3 \in M$."

Definition 2.1.15. [7] "A proper ideal M of Γ-Ring H is called 2-absorbing primary ideal of H if whenever $h_1, h_2, h_3 \in H, \gamma_1, \gamma_2 \in \Gamma$ and $h_1\gamma_1 h_2\gamma_2 h_3 \in M$, then $h_1\gamma_1 h_2 \in M$ or $h_1\gamma_2 h_3 \in \sqrt{M}$ or $h_2\gamma_2 h_3 \in \sqrt{M}$.

Remark 2.1.16. Every 2-absorbing ideal in H is a $2-$APrI in H.

However, the converse of the above remark does not hold.

For example: Consider $H = \mathbb{Z}, \Gamma = 5\mathbb{Z}$. Then H is a Γ-Ring. Consider $M = 12\mathbb{Z}$. Take $\gamma_1, \gamma_2 \in \Gamma$ such that $2\gamma_1 2\gamma_2 3 \in M$ implies $2\gamma_1 2 \notin M$, but $2\gamma_2 3 \in \sqrt{M}$. Thus M is a

$2-$APrI of H, however, M is not the 2-absorbing ideal of H, for $2\gamma_1 2\gamma_2 3 \in M$, but $2\gamma_1 2 \notin M$ and $2\gamma_2 3 \notin M$."

Definition 2.1.17. [8] "A function $\sigma: H_1 \to H_2$, where H_1 and H_2 are Γ-Rings, is said to be a Γ-homomorphism if for all $h, k \in H_1, \gamma \in \Gamma$, the following holds

1. $\sigma(h+k) = \sigma(h) + \sigma(k)$
2. $\sigma(h\gamma k) = \sigma(h)\gamma\sigma(k)$.

A surjective Γ-homomorphism $\sigma: H \to H$ is called a Γ-endomorphism and an injective Γ-endomorphism is called a Γ-automorphism. The set of all Γ-automorphisms is denoted by $Aut(H)$."

Definition 2.1.18. ([39,56]) "An ideal M of a Γ-Ring H is called a characteristic ideal of H if $f(M) = M$, for all $f \in Aut(H)$."

Definition 2.1.19. ([39,56]) "Let for a Γ-Ring H. Let us signify a relation σ on $H \times \Gamma$ as given below:

$$(h,\alpha)\sigma(k,\beta) \text{ iff } h\alpha m = k\beta m, \forall m \in H \text{ and } \gamma h\alpha = \gamma k\beta, \forall \gamma \in \Gamma.$$

Thus σ is an equivalence relation on $H \times \Gamma$. Set $[h,\alpha]$ be the equivalence class containing (h,α). Let $L = \{[h,\alpha]: h \in H, \alpha \in \Gamma\}$. Then L is a ring with respect to the compositions

$$[h,\alpha] + [k,\alpha] = [h+k,\alpha] \text{ ; } [h,\alpha] + [h,\beta] = [h,\alpha+\beta] \text{ ;}$$

$$\sum_i [h_i,\alpha_i] \sum_j [k_j,\beta_j] = \sum_{i,j} [h_i\alpha_i k_j, \beta_j].$$

This ring L is called the left operator ring of Γ-Ring H. Dually the right operator ring R of Γ-Ring H is formed where the compositions on R are defined as:

$$[\alpha,h] + [\beta,h] = [\alpha+\beta,h]; [\alpha,h] + [\alpha,k] = [\alpha,h+k];$$

$$\sum_i [\alpha_i,h_i] \sum_j [\beta_j,k_j] = \sum_{i,j} [\alpha_i, h_i\beta_j k_j].$$

Remark 2.1.20. [56]

(1) If there exists an element $1_L = \sum_i [e_i,\delta_i] \in L$ (or $1_R = \sum_i [\gamma_i,a_i] \in R$) such that $\sum_i e_i\,\delta_i h = h$ (resp. $\sum_i h\,\gamma_i a_i = h$) for all $h \in H$ then $\sum_i [e_i,\delta_i]$ (resp. $\sum_i [\gamma_i,a_i]$) is called the left (resp. right) unity of H.

(2) If we define a mapping $L \times H \to H$ by $(\sum_i [h_i,\alpha_i], k) \to \sum_i h_i\,\alpha_i k$, then we can show that the above mapping is well defined and H is a left L-module, and we call L the left

operator ring of the Γ-Ring H. Similarly, we can construct a right operator ring R of H so that H is a right R-module.

Let H be a Γ-Ring with the left operator ring L. For $P \subseteq L$ and $Q \subseteq H$, we define $P^{+} = \{h \in H: [h,\alpha] \in P, \forall \alpha \in \Gamma\}$ and $Q^{+'} = \{[h,\alpha] \in L: h\alpha k \in Q, \forall k \in H\}$. Similarly, if H is a Γ-Ring with right operator ring R. For $P \subseteq R$ and $Q \subseteq H$, we define $P^{*} = \{h \in H: [\alpha,h] \in P, \forall \alpha \in \Gamma\}$ and $Q^{*'} = \{[\alpha,h] \in R: k\alpha h \in Q, \forall k \in H\}$.

Then in [10], it was shown that if P (resp. Q) is a right ideal of L (resp. H), then P^{+} (resp. $Q^{+'}$) is a right ideal of H (resp. L) and there exists an inclusion preserving mapping $Q \to Q^{+'}$. Also if P (resp. Q) is a left ideal of R (resp. H), then P^{*} (resp. $Q^{*'}$) is a left ideal of H (resp. R) and there exists an inclusion preserving mapping $Q \to Q^{*'}$."

Definition 2.1.21. [56] "Let H be a Γ-Ring and L be the left operator ring of H. Then the bijection $f: L \to L$ is said to be automorphism if for all $[h,\alpha]$, $[h,\beta], [k,\alpha], [k,\beta] \in L$

1. $f([h,\alpha]+[k,\alpha]) = f([h,\alpha]) + f([k,\alpha])$ and $f([h,\alpha]+[h,\beta]) = f([h,\alpha]) + f([h,\beta])$,
2. $f\left(\sum_i [h_i,\alpha_i] \sum_j [k_j,\beta_j]\right) = f(\sum_i [h_i,\alpha_i]) f\left(\sum_j [k_j,\beta_j]\right)$,
3. $f(\sum_i [e_i,\delta_i]) = \sum_i [e_i,\delta_i]$, if $\sum_i [e_i,\delta_i]$ is the left unity of H,
4. $f(\sum_i [a_i,\gamma_i]) = \sum_i [a_i,\gamma_i]$, if $\sum_i [a_i,\gamma_i]$ is the right unity of H.

Similarly, we can define the automorphism on the right operator ring R of the Γ-Ring H."

Proposition 2.1.22. *([43]) "Every left (or right) ideal of Γ-Ring H defines a left (or right) ideal of the right operator ring R and conversely."*

2.2 Intuitionistic Fuzzification Of Some Results In Γ-Ring

This section contains some definitions and results on IFSs on Γ-Ring which are mainly taken from [4,5,34,40,42,43,46,50].

Definition 2.2.1. [4,5] "(Intuitionistic Fuzzy Set) An IFS G in X can be represented as an object of the form $G = \{< x, \mu_G(x), \nu_G(x) >: x \in X\}$, where the functions $\mu_G: X \to [0,1]$ and $\nu_G: X \to [0,1]$ denote the degree of membership (namely $\mu_G(x)$) and the degree of non-membership (namely $\nu_G(x)$) of each element $x \in X$ to G respectively and $0 \leq \mu_G(x) + \nu_G(x) \leq 1$ for each $x \in X$."

Remark 2.2.4. [4,5,71]"1. When $\mu_G(x) + \nu_G(x) = 1$, i.e., $\nu_G(x) = 1 - \mu_G(x) = \mu_{G^c}(x), \forall\, x \in X$. Then G is called a fuzzy set.

2. An IFS $G = \{< x, \mu_G(x), \nu_G(x) >: x \in X\}$ is shortly denoted by $G(x) = \left(\mu_G(x), \nu_G(x)\right)$, for all $x \in X$.

3. The set of all IFS on X is denoted by $IFS(X)$."

"If $\mathbb{G}_1, \mathbb{G}_2 \in IFS(X)$, then $\mathbb{G}_1 \subseteq \mathbb{G}_2$ if and only if $\mu_{\mathbb{G}_1}(x) \leq \mu_{\mathbb{G}_2}(x)$ and $\nu_{\mathbb{G}_1}(x) \geq \nu_{\mathbb{G}_2}(x)\ \forall\, x \in X$ and $\mathbb{G}_1 = \mathbb{G}_2 \Leftrightarrow \mathbb{G}_1 \subseteq \mathbb{G}_2\ and\ \mathbb{G}_2 \subseteq \mathbb{G}_1$. For any subset Y of X, the IFCF χ_Y is an IFS of X, defined as $\chi_Y(x) = (1,0), \forall x \in Y$ and $\chi_Y(x) = (0,1), \forall x \in X\backslash Y$. Let $\eta, \theta \in [0,1]$ with $\eta + \theta \leq 1$. Then the crisp set $G_{(\eta,\theta)} = \{x\epsilon X: \mu_G(x) \geq \eta\ and\ \nu_G(x) \leq \theta\}$ is called the $(\eta,\theta) -$ level cut subset of G. Also the IFS $x_{(\eta,\theta)}$ of X defined as $x_{(\eta,\theta)}(y) = (\eta,\theta), if\ y = x$, otherwise (0, 1) is called intuitionistic fuzzy point (IFP) in X with support x. By $x_{(\eta,\theta)} \in G$ we mean $\mu_G(x) \geq \eta$ and $\nu_G(x) \leq \theta$. Further if $f: X \to Y$ is a mapping and $\mathbb{G}_1, \mathbb{G}_2$be respectively IFS of X and Y, then the image f ($\mathbb{G}_1$) is an IFS of Y is defined as $\mu_{f(\mathbb{G}_1)}(y)$ = Sup $\{\mu_{\mathbb{G}_1}(x): f(x) = y\}, \nu_{f(\mathbb{G}_1)}(x)$: f(x) = y}, for all y $\in Y$ and the inverse image $f^{-1}(\mathbb{G}_2)$ is an IFS of X is defined as $\mu_{f^{-1}(\mathbb{G}_2)} = \mu_{\mathbb{G}_2}(f(x))$, $\nu_{f^{-1}(\mathbb{G}_2)}(x) = \nu_{\mathbb{G}_2}(f(x))$ for all $x \in X, i.e.,$ $f^{-1}(\mathbb{G}_2)$(x) =B(f(x)), for all x$\in X$. Also the IFS $\mathbb{G}_1$ of X is said to be $\mathrm{f} -$ invariant if for any x, y $\in$ X, whenever f(x) =f(y) implies $\mathbb{G}_1(\mathrm{x}) = \mathbb{G}_1(\mathrm{y})$"

Definition 2.2.3. [34,42,50] "Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be two IFSs of a Γ-Ring H and $\gamma \in \Gamma$. Then the product $\mathbb{G}_1 \Gamma \mathbb{G}_2$ and the composition $\mathbb{G}_1 \circ \mathbb{G}_2$ of $\mathbb{G}_1$ and $\mathbb{G}_2$ are defined by

$$\left(\mu_{\mathbb{G}_1 \Gamma \mathbb{G}_2}(h), \nu_{\mathbb{G}_1 \Gamma \mathbb{G}_2}(h)\right)$$
$$= \begin{cases} (\vee_{h=k\gamma p} \left(\mu_{\mathbb{G}_1}(k) \wedge \mu_{\mathbb{G}_2}(p)\right), \wedge_{h=k\gamma p} \left(\nu_{\mathbb{G}_1}(k) \vee \nu_{\mathbb{G}_2}(p)\right), & \text{if } h = k\gamma p \\ (0,1), & \text{otherwise} \end{cases}$$

and

$$\left(\mu_{\mathbb{G}_1\circ\mathbb{G}_2}(h),\nu_{\mathbb{G}_1\circ\mathbb{G}_2}(h)\right)$$

$$=\begin{cases}\left(\vee_{h=\sum_{i=1}^{n}k_i\gamma p_i}\left(\mu_{\mathbb{G}_1}(k_i)\wedge\mu_{\mathbb{G}_2}(p_i)\right),\wedge_{h=\sum_{i=1}^{n}k_i\gamma p_i}\left(\nu_{\mathbb{G}_1}(k_i)\vee\nu_{\mathbb{G}_2}(p_i)\right)\right), & \text{if } h=\sum_{i=1}^{n}k_i\,\gamma p_i\\ (0,1), & \text{otherwise}\end{cases}$$

”

Remark 2.2.4. [42] “If $\mathbb{G}_1$ and $\mathbb{G}_2$ are two IFSs of a Γ-Ring H, then $\mathbb{G}_1\Gamma\mathbb{G}_2\subseteq\mathbb{G}_1\circ\mathbb{G}_2\subseteq\mathbb{G}_1\cap\mathbb{G}_2$.”

Definition 2.2.5. [34,42] “Let G be an IFS of a Γ-Ring H, then G is called an IFI of H if for all $r,n\in H,\gamma\in\Gamma$, the following are satisfied:

1. $\mu_G(r-n)\geq\mu_G(r)\wedge\mu_G(n)$;
2. $\mu_G(r\gamma n)\geq\mu_G(r)\vee\mu_G(n)$;
3. $\nu_G(r-n)\leq\nu_G(r)\vee\nu_G(n)$;
4. $\nu_G(r\gamma n)\leq\nu_G(r)\wedge\nu_G(n)$.

The set of all IFI of Γ-Ring H is denoted by $IFI(H)$. Note that if $G\in IFI(H)$, then $\mu_G(0_H)\geq\mu_G(h)$ and $\nu_G(0_H)\leq\nu_G(h),\forall h\in H$.”

Definition 2.2.6. [46] “(Intuitionistic fuzzy prime ideal) Let H be a Γ-Ring. A non-constant IFI P of H is called an IFPI of H, if for all pair of IFIs $\mathbb{G}_1,\mathbb{G}_2$ of H, $\mathbb{G}_1\Gamma\mathbb{G}_2\subseteq P$ implies that $\mathbb{G}_1\subseteq P$ or $\mathbb{G}_2\subseteq P$.”

Theorem 2.2.7. ([46,50]) *“Let H be a commutative Γ-Ring and G be an IFI of H, then the following are equivalent:*

(i) $h_{(\eta,\theta)}\Gamma k_{(\beta,\vartheta)}\subseteq G\Rightarrow h_{(\eta,\theta)}\subseteq G$ or $k_{(\beta,\vartheta)}\subseteq G$, where $h_{(\eta,\theta)},k_{(\beta,\vartheta)}\in IFP(H)$.

(ii) G is an IFPI of H.”

Theorem 2.2.8. ([42,43,50]) *“Let G be an IFI of Γ-Ring H. Then each (η,θ)-level cut set $G_{(\eta,\theta)}$ is either empty or an ideal of H, where $\eta\leq\mu_G(0_H)$ and $\theta\geq\nu_G(0_H)$. In particular, $G_{(1,0)}$ which is denoted by G_*, i.e., the set $G_*=\{h\in H:\mu_G(h)=\mu_G(0_H)$ and $\nu_G(h)=\nu_G(0_H)\}$ is ideal of H. If $G\in IFPI(H)$, then G_* is a prime ideal of H.”*

Theorem 2.2.9. [46,50] *“If P is an IFPI of a Γ-Ring H, then the following conditions hold:*

1.$P(0_H)=(1,0)$,

2. P_* is a prime ideal of H,

3. $Img(P) = \{(1,0), (\lambda, \zeta)\}$, where $\lambda, \zeta \in [0,1)$ such that $\lambda + \zeta \leq 1$."

Definition 2.2.10. [46,50] "(Intuitionistic fuzzy semi-prime ideal) A non-constant IFI P of a Γ-Ring H is said to be an IFSPI if for any IFI G of H, $G\Gamma G \subseteq P$, implies that $G \subseteq P$."

Proposition 2.2.11. [46] *"Let P be a non-constant IFI of a Γ-Ring H, then the following conditions are equivalent:*

(i) P is an IFSPI of H

(ii)For any $a \in H$, $Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_P(a\gamma_1 r\gamma_2 a)\} = \mu_P(a)$ and $Sup_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_P(a\gamma_1 r\gamma_2 a)\} = \nu_P(a)$."

Proof. (i) ⇒ (ii) Let P be an IFSPI of H. Since P is IFI of H, it follows that $\mu_P(a\gamma_1 m\gamma_2 a) \geq \mu_P(a)$ and $\nu_P(a\gamma_1 r\gamma_2 a) \leq \nu_P(a), \forall r \in H, \gamma_1, \gamma_2 \in \Gamma$. If possible let us suppose that $\mu_P(a\gamma_1 r\gamma_2 a) > \mu_P(a)$ and $\nu_P(a\gamma_1 r\gamma_2 a) < \nu_P(a)$, for some $a \in H$. Let $< a >$ be the ideal generated a. Define the IFS C on H by

$$\mu_C(h) = \begin{cases} t, & \text{if } h \in < a > \\ 0, & \text{otherwise} \end{cases};$$

$$\nu_C(h) = \begin{cases} s, & \text{if } h \in < a > \\ 1, & \text{otherwise.} \end{cases}$$

Where, $t, s \in (0,1)$ such that $t + s \leq 1$. Then C is an IFI of H. Consider $h \in H$ s.t. $h \neq u\gamma v$, for some $u, v \in < a >$, then $C\Gamma C(h) = (0,1)$ and $C\Gamma C(h) = \left(Sup_{h=u\gamma v,u,v\in<a>}\{\mu_C(u) \wedge \mu_C(v)\}, Inf_{h=u\gamma v,u,v\in<a>}\{\mu_C(u) \vee \mu_C(v)\}\right)$.

Now any $u \in < a >$ is of the form $u = \sum_{i=1}^{p} r_i' \gamma_i' a \gamma_i'' r_i'', r_i', r_i'' \in H, \gamma_i', \gamma_i'' \in \Gamma$ and $p \in Z^+$. Similarly,$v = \sum_{j=1}^{q} r_j' \gamma_j' a \gamma_j'' r_j'', r_j', r_j'' \in H, \gamma_j', \gamma_j'' \in \Gamma$ and $q \in Z^+$.

Now, $u\gamma v = \left(\sum_{i=1}^{p} r_i' \gamma_i' a \gamma_i'' r_i''\right)\left(\sum_{j=1}^{q} r_j' \gamma_j' a \gamma_j'' r_j'', r_j'\right)$. Since P is an IFI of H, it follows that

$\mu_P(h) = \mu_P(u\gamma v) \geq \mu_P(a\xi_1 r' \xi_2 a) \geq Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_P(a\gamma_1 r\gamma_2 a)\} > t = \mu_{C\Gamma C}(h)$, for some $r' \in H$. Similarly, we can show $\nu_P(h) < \nu_{C\Gamma C}(h)$. So, we get $C\Gamma C \subseteq P$. As P is an IFSPI of H, it follows that $C \subseteq P$.

Hence $t = \mu_C(a) \leq \mu_P(a)$ and $s = \nu_C(a) \geq \nu_P(a)$, a contradiction. Consequently we have $Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_P(a\gamma_1 r\gamma_2 a)\} = \mu_P(a)$ and $Sup_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_P(a\gamma_1 r\gamma_2 a)\} = \nu_P(a)$.

$(ii) \Rightarrow (i)$, Let us assume that P be an IFI of H satisfying for any $a \in H$, $Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_P(a\gamma_1 r\gamma_2 a)\} = \mu_P(a)$ and $Sup_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_P(a\gamma_1 r\gamma_2 a)\} = \nu_P(a)$.
Let C be an IFI of H s.t. $C\Gamma C \subseteq P$ and $C \nsubseteq P$. Then there exist $b \in H$ s.t. $\mu_C(b) > \mu_P(b)$ and $\nu_C(b) < \nu_P(b)$.

Now $\mu_P(b\gamma_1 r\gamma_2 b) \geq \mu_{C\Gamma C}(b\gamma_1 r\gamma_2 b) \geq \mu_C(b)$ and $\nu_P(b\gamma_1 r\gamma_2 b) \leq \nu_{C\Gamma C}(b\gamma_1 r\gamma_2 b) \leq \nu_C(b)$, for all $r \in H, \gamma_1, \gamma_2 \in \Gamma$. Therefore $Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_P(b\gamma_1 r\gamma_2 b)\} \geq \mu_C(b)$ and $Sup_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_P(b\gamma_1 r\gamma_2 b)\} \leq \nu_C(b)$. Thus

$\mu_P(b) = Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_P(b\gamma_1 r\gamma_2 b)\} \geq \mu_C(b) > \mu_P(b)$ and $\nu_P(b) = Sup_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_P(b\gamma_1 r\gamma_2 b)\} \leq \nu_C(b) < \nu_P(b)$, a contradiction. So P is an IFSPI of H."

Definition 2.2.12. ([40,57]) "Let Q be a non-constant IFI of a Γ-Ring H. Then Q is called an IF2 $-$AI of H if for any $IFPs$ $h_{(\eta,\theta)}, k_{(\beta,\vartheta)}, p_{(\tau,\omega)}$ of H and $\gamma_1, \gamma_2 \in \Gamma$ such that $h_{(\eta,\theta)}\gamma_1 k_{(\beta,\vartheta)}\gamma_2 p_{(\tau,\omega)} \subseteq Q$ implies that either $h_{(\eta,\theta)}\gamma_1 k_{(\beta,\vartheta)} \subseteq Q$ or $h_{(\eta,\theta)}\gamma_2 p_{(\tau,\omega)} \subseteq Q$ or $k_{(\beta,\vartheta)}\gamma_2 p_{(\tau,\omega)} \subseteq Q$."

Theorem 2.2.13. ([34]) "*Let J be a subset of a Γ-Ring H, then the IFCF χ_J be an IFI of H iff J is an ideal of H.*"

Theorem 2.2.14. [50] "*A Γ-Ring H is Noetherian iff the set of values of any IFI of H is a well-ordered subset of* $[0,1]$*.*"

Theorem 2.2.15. [50] "*Let every decreasing chain of ideals terminate at a finite step in Γ-Ring H. For an IFI G of H, G has a finite number of intuitionistic values, that is, μ_G and ν_G have a finite number of value.*"

Chapter 3

On Intuitionistic Fuzzy Characteristic Ideal Of A Γ-Ring

3.1 Introduction

The significance of characteristic ideals stands out prominently in ring theory, constituting a distinct class among various types of ideals. These ideals exhibit invariance under any automorphism, highlighting their fundamental role. This chapter introduces and examines the concept of IFCI in a Γ-Ring, delving into its properties and discussing its various attributes. Additionally, it explores the relationship between the IFCI of a Γ-Ring and its level cut sets. Furthermore, it delineates a connection between the Aut(H) and the corresponding Aut(OR). Lastly, it delves into the correspondence between IFCI(H) and IFCI(OR), thoroughly investigating their interrelation.

3.2 Intuitionistic Fuzzy Characteristic Ideal Of A Γ-Ring

Definition 3.2.1. Suppose for an IFS G in a Γ-Ring H, $\sigma: H \to H$ be a Γ-endomorphism, then G^σ is an IFS on H defined as $G^\sigma(\hbar) = G(\sigma(\hbar))$, $\forall\ \hbar \in H$, i.e., $\mu_{G^\sigma}(\hbar) = \mu_G(\sigma(\hbar))$ and $\nu_{G^\sigma}(\hbar) = \nu_G(\sigma(\hbar))$, for all $\hbar \in H$.

Theorem 3.2.2. *Let G be an IFI of Γ-Ring H and σ be a Γ-endomorphism, then G^σ is also an IFI of H.*

Proof. Let G be an IFI of Γ-Ring H. Let $h_1, h_2 \in H, \alpha \in \Gamma$. Then

$$
\begin{aligned}
\mu_{G^\sigma}(h_1 - h_2) &= \mu_G\big(\sigma(h_1 - h_2)\big) \\
&= \mu_G\big(\sigma(h_1) - \sigma(h_2)\big) \\
&\geq \mu_G\big(\sigma(h_1)\big) \wedge \mu_G\big(\sigma(h_2)\big) \\
&= \mu_{G^{\sigma(h_1)}} \wedge \mu_{G^\sigma}(h_2).
\end{aligned}
$$

Thus, $\mu_{G^\sigma}(h_1 - h_2) \geq \mu_{G^{\sigma(h_1)}} \wedge \mu_{G^\sigma}(h_2)$. Similarly, we can prove $\nu_{G^\sigma}(h_1 - h_2) \leq \nu_{G^{\sigma(h_1)}} \vee \nu_{G^\sigma}(h_2)$. Also,

$$
\begin{aligned}
\mu_{G^\sigma}(h_1 \alpha h_2) &= \mu_G\big(\sigma(h_1 \alpha h_2)\big) \\
&= \mu_G\big(\sigma(h_1)\alpha\sigma(h_2)\big) \\
&\geq \mu_G\big(\sigma(h_1)\big) \vee \mu_G\big(\sigma(h_2)\big) \\
&= \mu_{G^{\sigma(h_1)}} \vee \mu_{G^\sigma}(h_2).
\end{aligned}
$$

i.e., $\mu_{G^\sigma}(h_1 \alpha h_2) \geq \mu_{G^{\sigma(h_1)}} \vee \mu_{G^\sigma}(h_2)$. Similarly, we can prove $\nu_{G^\sigma}(h_1 \alpha h_2) \leq \nu_{G^{\sigma(h_1)}} \wedge \mu_{G^\sigma}(h_2)$.

Hence G^σ is an IFI of Γ-Ring H.

Definition 3.2.3. An IFI G of Γ-Ring H is said to be an IFCI if $G^\sigma(\hbar) = G(\hbar), \forall \hbar \in H$ and $\forall\ \sigma \in Aut(H)$, i.e., $\mu_{G^\sigma}(\hbar) = \mu_G(\hbar)$ & $\nu_{G^\sigma}(\hbar) = \nu_G(\hbar)$ $\forall\ \hbar \in H$ and $\forall\ \sigma \in Aut(H)$.

Example 3.2.4. [62] "Consider the Γ-Ring H, where $H = \mathbb{Z}$, the ring of integers, and $\Gamma = 2\mathbb{Z}$, the ring of even integers, and $h_1\gamma h_2$ denote the usual product of integers" $h_1, h_2 \in H$, $\gamma \in \Gamma$. Let $G = (\mu_G, \nu_G)$ be an IF subset of H defined by

$$
\mu_G(h_1) = \begin{cases} 1, & \text{if } h_1 \text{ is even integer} \\ 0.5, & \text{if } h_1 \text{ is odd integer} \end{cases}; \quad \nu_G(h_1) = \begin{cases} 0, & \text{if } h_1 \text{ is even integer} \\ 0.3, & \text{if } h_1 \text{ is odd integer.} \end{cases}
$$

It can be easily checked that G is an IFCI of Γ-Ring H.

Example 3.2.5. [62] "Consider the Γ-Ring H, where $H = \{[a_{ij}]: a_{ij} \in \mathbb{Z}, i = 1,2, j = 1,2,3\}$, the set of (2×3) matrices and $\Gamma = \{[a_{ij}]: a_{ij} \in \mathbb{Z}, i = 1,2,3, j = 1,2\}$, the set of (3×2) matrices whose entries are from the ring of integers $\mathbb{Z}$." Let $G = (\mu_G, \nu_G)$ be an IFS of H defined by

$$G([a_{ij}]) = \begin{cases} (0.7,0.2), & \text{if } a_{ij} = 0, \forall i, j \\ (0.3,0.5), & \text{if } a_{ij} \neq 0 \text{ for at least one } i \text{ and } j \end{cases}$$

Then it can be easily checked that G is an IFCI of Γ-Ring H.

Example 3.2.6. [62] "Consider "$H = \mathbf{Z}_2 \times \mathbf{Z}_2 = \{(0,0), (1,0), (0,1), (1,1)\}$, $\Gamma = \{(0,0), (1,1)\}$ and $\ddot{\mathrm{W}} = \mathbf{Z}_2 \times \{0\} = \{(1,0), (0,0)\}$, where $\mathbf{Z}_2$ be the ring of integers modulo 2." Clearly, H and Γ are additive abelian groups and H is Γ-Ring. Also, here $\ddot{\mathrm{W}}$ is Γ-ideal of H. Consider the IFS G defined on H a

$$\mu_G(\ddot{\varpi}) = \begin{cases} 1, & \text{if } \ddot{\varpi} \in \ddot{\mathrm{W}} \\ 0.5, & \text{if } \ddot{\varpi} \notin \ddot{\mathrm{W}} \end{cases}; \quad \nu_G(\ddot{\varpi}) = \begin{cases} 0, & \text{if } \ddot{\varpi} \in \ddot{\mathrm{W}} \\ 0.3, & \text{if } \ddot{\varpi} \notin \ddot{\mathrm{W}} \end{cases}$$

It can be verified with ease that G is an IFI of Γ-Ring H, but it is not an IFCI, as there exists a Γ-automorphism $\sigma: H \to H$ defined by $\sigma(\ddot{\varpi}, \hat{\omega}) = (\hat{\omega}, \ddot{\varpi})$, for all $(\ddot{\varpi}, \hat{\omega}) \in H$ s.t. $G^{\sigma}((\ddot{\varpi}, \hat{\omega})) \neq G((\ddot{\varpi}, \hat{\omega}))$, for all $(\ddot{\varpi}, \hat{\omega}) \in H$.

For example $G^{\sigma}((1,0)) = (0.5,0.3) \neq (1,0) = G((1,0))$.

Theorem 3.2.7. *Suppose G is an IFCI of Γ-Ring H. Then for each $\eta, \theta \in [0,1]$ s.t. $\eta + \theta \leq 1$ the level cut set $G_{(\eta,\theta)}$ is a CI of Γ-Ring H.*

Proof. Assume that G is an IFCI of Γ-Ring H. We want to prove that $\sigma(G_{(\eta,\theta)}) = G_{(\eta,\theta)}$ i.e. image of level cut set under σ is equal to level cut set $\forall\, \eta, \theta \in [0,1]$ s.t. $\eta + \theta \leq 1$.

Let $h \in G_{(\eta,\theta)}$. Since G be an IFCI of Γ-Ring H, we have $\mu_{G^{\sigma}}(h) = \mu_G(h) \geq \eta$ and $\nu_{G^{\sigma}}(h) = \nu_G(h) \leq \theta$ implies $\mu_G(\sigma(h)) \geq \eta$ and $\nu_G(\sigma(h)) \leq \theta$, i.e., $\sigma(h) \in G_{(\eta,\theta)}$. Thus $\sigma(G_{(\eta,\theta)}) \subseteq G_{(\eta,\theta)}$.

For the reverse inclusion, let $j \in G_{(\eta,\theta)}$ and let $h \in H$ be s.t. $\sigma(h) = j$. Then $\mu_G(h) = \mu_{G^{\sigma}}(h) = \mu_G(\sigma(h)) = \mu_G(j) \geq \eta$. In the same manner, it can be shown that $\nu_G(h) \leq \theta$ implies $h \in G_{(\eta,\theta)}$ and so $j = \sigma(h) \in \sigma(G_{(\eta,\theta)})$ gives that $G_{(\eta,\theta)} \subseteq \sigma(G_{(\eta,\theta)})$. Therefore by using the above two equations it can be seen that $\sigma(G_{(\eta,\theta)}) = G_{(\eta,\theta)}$. Therefore $G_{(\eta,\theta)}$ is a characteristic ideal of Γ-Ring H.

Lemma 3.2.8. *Let G be an IFI of Γ-Ring H and let $h_1 \in H$. Then $G(h_1) = (\eta, \theta)$ iff $h_1 \in G_{(\eta,\theta)}$ and $h_1 \notin G_{(c,d)}$ $\forall$ $c > \eta$ and $d < \theta$.*

Proof. Directly can be proved with the help of above stated theorem (3.2.7.) Converse of Theorem (3.2.7) can be seen in theorem (3.2.9.)

Theorem 3.2.9. *Suppose G is an IFI of Γ-Ring H. If for each $\eta, \theta \in [0,1]$ s.t. $\eta + \theta \leq 1$ the level cut set $G_{(\eta,\theta)}$ is a CI of H, then G is an IFCI of Γ-Ring H.*

Proof. Suppose G be an IFI of Γ-Ring H. Let $h \in H$, $\sigma \in Aut(H)$ and $G(h) = (\eta, \theta)$. By Lemma (3.2.8), $h \in G_{(\eta,\theta)}$ and $h \notin G_{(c,d)}$ $\forall$ $c > \eta$ and $d < \theta$.

From hypothesis it follows that $\sigma\, image\, of\, level\, cut\, set\, is\, equals\, to\, level\, cut\, set$. Therefore $\sigma(h) \in \sigma\left(G_{(\eta,\theta)}\right) = G_{(\eta,\theta)}$, and so $\mu_G\left(\sigma(h)\right) \geq \eta$, $\nu_G\left(\sigma(h)\right) \leq \theta$.

Suppose $\mu_G\left(\sigma(h)\right) = c$ and $\nu_G\left(\sigma(h)\right) = d$ and we assume that $c > \eta$ and $d < \theta$. Then $\sigma(h) \in G_{(c,d)} = \sigma\left(G_{(c,d)}\right)$. Since σ is one one implies $h \in G_{(c,d)}$. This is a contradiction. Therefore $\mu_{G^\sigma}(h) = \mu_G\left(\sigma(h)\right) = \eta = \mu_G(h)$ and $\nu_{G^\sigma}(h) = \nu_G\left(\sigma(h)\right) = \theta = \nu_G(h)$, showing that G is an IFCI of Γ-Ring H.

Theorem 3.2.10. *Suppose $\ddot{\mathbb{W}}$ is a non-empty subset which is also a characteristic ideal of a Γ-Ring H then its IFCF $\chi_{\ddot{\mathbb{W}}}$ is an IFCI of Γ-Ring H and the converse is also true.*

Proof. Suppose $\ddot{\mathbb{W}}$ is a CI of Γ-Ring H. According to definition $\sigma(\ddot{\mathbb{W}}) = \ddot{\mathbb{W}}$, $\forall \sigma \in Aut(H)$. Let $\chi_{\ddot{\mathbb{W}}}$ be the IFCF w.r.t. $\ddot{\mathbb{W}}$. Then by Theorem (2.2.13) $\chi_{\ddot{\mathbb{W}}}$ be an IFI of Γ-Ring H.

If $h \in \ddot{\mathbb{W}}$ then $\sigma(h) \in \sigma(\ddot{\mathbb{W}}) = \ddot{\mathbb{W}}$ and so $\chi_{\ddot{\mathbb{W}}}\left(\sigma(h)\right) = (1,0) = \chi_{\ddot{\mathbb{W}}}(h)$.

If $h \notin \ddot{\mathbb{W}}$ then $\sigma(h) \notin \sigma(\ddot{\mathbb{W}}) = \ddot{\mathbb{W}}$ and so $\chi_{\ddot{\mathbb{W}}}\left(\sigma(h)\right) = (0,1) = \chi_{\ddot{\mathbb{W}}}(h)$.

Thus we see that $\chi_{\ddot{\mathbb{W}}}\left(\sigma(h)\right) = \chi_{\ddot{\mathbb{W}}}(h), \forall h \in H, \forall \sigma \in Aut(H)$, i.e., $\mu_{\chi^\sigma_{\ddot{\mathbb{W}}}}(h) = \mu_{\chi_{\ddot{\mathbb{W}}}}(h)$ and $\nu_{\chi^\sigma_{\ddot{\mathbb{W}}}}(h) = \nu_{\chi_{\ddot{\mathbb{W}}}}(h), \forall h \in H, \forall \sigma \in Aut(H)$. Hence $\chi_{\ddot{\mathbb{W}}}$ is an IFCI of Γ-Ring H.

Conversely, let us suppose that $\chi_{\ddot{\mathbb{W}}}$ is an IFCI of Γ-Ring H. Using Theorem (2.2.13) $\ddot{\mathbb{W}}$ is an Γ-ideal of H. So, we need only to show that $\sigma(\ddot{\mathbb{W}}) = \ddot{\mathbb{W}}\ \forall \sigma \in Aut(H)$. Let $\sigma \in Aut(H)$ and $h \in \ddot{\mathbb{W}}$, then $\mu_{\chi^\sigma_{\ddot{\mathbb{W}}}}(h) = \mu_{\chi_{\ddot{\mathbb{W}}}}(h) = 1$ and $\nu_{\chi^\sigma_{\ddot{\mathbb{W}}}}(h) = \nu_{\chi_{\ddot{\mathbb{W}}}}(h) = 0$ implies $\mu_{\chi_{\ddot{\mathbb{W}}}}\left(\sigma(h)\right) = 1$ and $\nu_{\chi_{\ddot{\mathbb{W}}}}\left(\sigma(h)\right) = 0$ implies $\sigma(h) \in \ddot{\mathbb{W}}$. Thus, we obtain $\sigma(\ddot{\mathbb{W}}) \subseteq \ddot{\mathbb{W}}$, for all $\sigma \in Aut(H)$. Since $\sigma \in Aut(H)$ implies $\sigma^{-1} \in Aut(H)$ and so $\sigma^{-1}(\ddot{\mathbb{W}}) \subseteq \ddot{\mathbb{W}}$. Hence $\ddot{\mathbb{W}} \subseteq \sigma(\ddot{\mathbb{W}})$ and so by using the above two equations we have $\sigma(\ddot{\mathbb{W}}) = \ddot{\mathbb{W}}$, i.e., $\ddot{\mathbb{W}}$ is CI of H.

3.3 Operator Rings And Corresponding IFI Of Γ-Ring

In this section L is used for left operator ring (OR) and R is used for right operator ring (OR) of Γ-Ring H.

Definition 3.3.1. For any fixed IFS G of L (or R) and any fixed IFS B of H we define IFSs G^+, G^* of H and $B^{+'}$ of L, $B^{*'}$ of R by

(i) $\mu_{G^+}(h) = Inf_{\alpha\in\Gamma}\big(\mu_G([h,\alpha])\big)$ and $\nu_{G^+}(h) = Sup_{\alpha\in\Gamma}\big(\mu_G([h,\alpha])\big)$, where $h \in H$.

(ii) $\mu_{G^*}(h) = Inf_{\alpha\in\Gamma}\big(\mu_G([\alpha,h])\big)$ and $\nu_{G^*}(h) = Sup_{\alpha\in\Gamma}\big(\mu_G([\alpha,h])\big)$, where $h \in H$.

(iii) $\mu_{B^{+'}}(\sum_i [h_i,\alpha_i]) = Inf_{r\in H}\big(\mu_B(\sum_i h_i\, \alpha_i r)\big)$ and $\nu_{B^{+'}}(\sum_i [h_i,\alpha_i]) = Sup_{r\in H}\big(\mu_B(\sum_i h_i\, \alpha_i r)\big)$, where $[h_i,\alpha_i] \in L$.

(iv) $\mu_{B^{*'}}(\sum_i [\alpha_i,h_i]) = Inf_{r\in H}\big(\mu_B(\sum_i r\, \alpha_i h_i)\big)$ and $\nu_{B^{*'}}(\sum_i [\alpha_i,h_i]) = Sup_{r\in H}\big(\mu_B(\sum_i \mathrm{r}\, \alpha_i h_i)\big)$, where $[\alpha_i,h_i] \in R$.

Proposition 3.3.2. *Let G is an IFI of L of a Γ-Ring H then G^+ is an IFI of H.*

Proof. Here $\mu_G(0_L) = 1, \nu_G(0_L) = 0$ as G is an IFI of L..

Now $\mu_{G^+}(0_H) = Inf_{\alpha\in\Gamma}\big(\mu_G([0_H,\alpha])\big) = Inf_{\alpha\in\Gamma}\big(\mu_G(0_L)\big) = 1$. Similarly, we can show that $\nu_{G^+}(0_H) = 0$. So G^+ is non-empty.

Let $h_1, h_2 \in H, \alpha, \beta \in \Gamma$ be any elements, then we have

$$\begin{aligned}\mu_{G^+}(h_1-h_2) &= Inf_{\alpha\in\Gamma}\big(\mu_G([h_1-h_2,\alpha])\big)\\ &= Inf_{\alpha\in\Gamma}\big(\mu_G([h_1,\alpha]-[h_2,\alpha])\big)\\ &\geq Inf_{\alpha\in\Gamma}\{\mu_G([h_1,\alpha])\wedge\mu_G([h_2,\alpha])\}\\ &= Inf_{\alpha\in\Gamma}\big(\mu_G([h_1,\alpha])\big)\wedge Inf_{\alpha\in\Gamma}\big(\mu_G([h_2,\alpha])\big)\\ &= \mu_{G^+}(h_1)\wedge\mu_{G^+}(h_2).\end{aligned}$$

Thus $\mu_{G^+}(h_1-h_2)\geq\mu_{G^+}(h_1)\wedge\mu_{G^+}(h_2)$. In the same manner it can be shown that $\nu_{G^+}(h_1-h_2)\leq\nu_{G^+}(h_1)\vee\nu_{G^+}(h_2)$. Also,

$$\begin{aligned}\mu_{G^+}(h_1\beta h_2) &= Inf_{\alpha\in\Gamma}\big(\mu_G([h_1\beta h_2,\alpha])\big)\\ &= Inf_{\alpha\in\Gamma}\big(\mu_G([h_1,\beta][h_2,\alpha])\big)\\ &\geq Inf_{\alpha\in\Gamma}\big(\mu_G([h_1,\beta])\big)\big[\text{ and }\geq Inf_{\alpha\in\Gamma}\big(\mu_G([h_2,\alpha])\big)\big]\\ &= Inf_{\beta\in\Gamma}\big(\mu_G([h_1,\beta])\big)\vee Inf_{\alpha\in\Gamma}\big(\mu_G([h_2,\alpha])\big)\\ &= \mu_{G^+}(h_1)\vee\mu_{G^+}(h_2).\end{aligned}$$

Thus $\mu_{G^+}(h_1\alpha h_2)\geq\mu_{G^+}(h_1)\vee\mu_{G^+}(h_2)$. In the same manner it can be shown that $\nu_{G^+}(h_1\alpha h_2)\leq\nu_{G^+}(h_1)\wedge\nu_{G^+}(h_2)$. Hence G^+ is an IFI of H.

Proposition 3.3.3. *Let B is an IFI of H Then $B^{+'}$ is an IFI of L.*

Proof. Let B be an IFI of H. Then $\mu_B(0_H)=1,\nu_B(0_H)=0$.

Now $\mu_{B^{+'}}([0_H,\alpha])=Inf_{r\in H}\big(\mu_B(0_H\alpha r)\big)=\mu_B(0_H)=1$. Similarly, we can show that

$\nu_{B^{+'}}([0_H,\alpha])=0$. So $B^{+'}$ is non-empty.

Let $\sum_i[h_i,\alpha_i],\sum_j[k_j,\beta_j]\in L$, $r\in H,\alpha_i,\beta_j\in\Gamma$ be any elements, then we have

$$\begin{aligned}\mu_{B^{+'}}\big(\textstyle\sum_i[h_i,\alpha_i]-\sum_j[k_j,\beta_j]\big) &= Inf_{r\in H}\Big(\mu_B\big(\textstyle\sum_i h_i\,\alpha_i r-\sum_j k_j\,\beta_j r\big)\Big)\\ &\geq Inf_{r\in H}\{\mu_B(\textstyle\sum_i h_i\,\alpha_i r)\wedge\mu_B\big(\sum_j k_j\,\beta_j r\big)\}\\ &= \Big(Inf_{r\in H}\big(\mu_B(\textstyle\sum_i h_i\,\alpha_i r)\big)\Big)\wedge\Big(Inf_{r\in H}\Big(\mu_B\big(\sum_j k_j\,\beta_j r\big)\Big)\Big)\\ &= \mu_{B^{+'}}(\textstyle\sum_i[h_i,\alpha_i])\wedge\mu_{B^{+'}}\big(\sum_j[k_j,\beta_j]\big).\end{aligned}$$

Thus $\mu_{B^{+'}}\left(\sum_i [h_i,\alpha_i]-\sum_j [k_j,\beta_j]\right)\geq \mu_{B^{+'}}(\sum_i [h_i,\alpha_i])\wedge \mu_{B^{+'}}\left(\sum_j [k_j,\beta_j]\right)$. Similarly, we can show $\nu_{B^{+'}}\left(\sum_i [h_i,\alpha_i]-\sum_j [k_j,\beta_j]\right)\leq \nu_{B^{+'}}(\sum_i [h_i,\alpha_i])\vee \mu_{B^{+'}}\left(\sum_j [k_j,\beta_j]\right)$ Also

$$\begin{aligned}
\mu_{B^{+'}}\left(\sum_i [h_i,\alpha_i]\sum_j [k_j,\beta_j]\right) &= \mu_{B^{+'}}\left(\sum_{i,j} [h_i\alpha_i k_j,\beta_j]\right)\\
&= Inf_{r\in H}\left(\mu_B\left(\sum_{i,j} h_i\, \alpha_i k_j\beta_j r\right)\right)\\
&= Inf_{r\in H}\left(\mu_B\left(\sum_{i,j} (h_i\alpha_i)\left(k_j\beta_j r\right)\right)\right)\\
&= Inf_{r_j'\in H}\left(\mu_B\left(\sum_{i,j} h_i\, \alpha_i r_j'\right)\right)\left[\text{ where } r_j' = y_j\beta_j r\in H\right]\\
&= Inf_{r_j'\in H}[\mu_B(\textstyle\sum_i h_i\, \alpha_i r_1'+\sum_i h_i\, \alpha_i r_2'+\ldots)]\\
&\geq Inf_{r_j'\in H}\left[\vee_j\, \mu_B\left(\textstyle\sum_i h_i\, \alpha_i r_j'\right)\right]\\
&=\vee_j\left[Inf_{r_j'\in H}\left(\textstyle\sum_i h_i\, \alpha_i r_j'\right)\right]\\
&=\vee_j\left[\mu_{B^{+'}}(\textstyle\sum_i [h_i,\alpha_i])\right]\\
&= \mu_{B^{+'}}(\textstyle\sum_i [h_i,\alpha_i])
\end{aligned}$$

.

Also, we can prove that $\mu_{B^{+'}}\left(\sum_i [h_i,\alpha_i]\sum_j [k_j,\beta_j]\right)\geq \mu_{B^{+'}}\left(\sum_j [k_j,\beta_j]\right)$. Thus we have $\mu_{B^{+'}}\left(\sum_i [h_i,\alpha_i]\sum_j [k_j,\beta_j]\right)\geq \mu_{B^{+'}}(\sum_i [h_i,\alpha_i])\vee \mu_{B^{+'}}\left(\sum_j [k_j,\beta_j]\right)$. In the same manner, it can be shown that $\nu_{B^{+'}}\left(\sum_i [h_i,\alpha_i]\sum_j [k_j,\beta_j]\right)\leq \nu_{B^{+'}}(\sum_i [h_i,\alpha_i])\wedge \nu_{B^{+'}}\left(\sum_j [k_j,\beta_j]\right)$. Hence $B^{+'}$ is an IFI of L.

Using the same logic following propositions can be proved.

Proposition 3.3.4. *Let G be an IFI of R of a Γ-Ring H then G^*an IFI of H.*

Proposition 3.3.5. *Let B an IFI of H. Then $B^{*'}$ an IFI of R.*

Theorem 3.3.6. *Suppose H is a Γ-Ring having unities & L is its left operator ring. Then $\exists$ an inclusion preserving one-to-one map $G\rightarrow G^{+'}$ between IFI(H) and the IFI(L).*

Proof. First we show that $((G^+)')^+ = G$, where G is an IFI of H. Let $h\in H$. Then

$$\begin{aligned}\mu_{\left((G^+)'\right)^+}(h) &= Inf_{\alpha\in\Gamma}\left(\mu_{(G^+)'}([h,\alpha])\right)\\ &= Inf_{\alpha\in\Gamma}\left[Inf_{r\in H}\left(\mu_G(h\alpha r)\right)\right]\\ &\geq Inf_{\alpha\in\Gamma}\left[Inf_{r\in H}\left(\mu_G(h)\right)\right]\\ &= \mu_G(h).\end{aligned}$$

Thus $\mu_{\left((G^+)'\right)^+}(h) \geq \mu_G(h)$.In the same manner, it can be shown that $\nu_{\left((G^+)'\right)^+}(h) \leq \nu_G(h)$. Thus $G \subseteq ((G^+)')^+$.

Suppose $\sum_i [\gamma_i,\ a_i]$ be the right unity of H. Then $\sum_i h\,\gamma_i a_i = h, \forall\ h \in H$. Now,

$$\begin{aligned}\mu_G(h) &= \mu_G\left(\sum_i h\,\gamma_i a_i\right)\\ &\geq Inf_i[\mu_i(h\gamma_i a_i)]\\ &\geq Inf_{\gamma\in\Gamma}\left[Inf_{r\in H}\left(\mu_G(h\gamma r)\right)\right]\\ &= Inf_{\gamma\in\Gamma}\left(\mu_{(G^+)'}([h,\gamma])\right)\\ &= \mu_{\left((G^+)'\right)^+}(h)\end{aligned}$$

In the same manner, it can be shown that $\nu_G(h) \leq \nu_{\left((G^+)'\right)^+}(h)$. So $((G^+)')^+ \subseteq G$. Hence $G = ((G^+)')^+$.

Again, let G be an IFI of L. Now,

$$\mu_{\left((G^+)^+\right)'}\left(\sum_i [h_i,\alpha_i]\right) = Inf_{r\in H}\left(\mu_{G^+}\left(\sum_i h_i\,\alpha_i r\right)\right)$$

$$= Inf_{r\in H}\left[Inf_{\beta\in\Gamma}\left(\mu_G\left(\left[\sum_i h_i\,\alpha_i r,\beta\right]\right)\right)\right]$$

$$= Inf_{r\in H}\left[Inf_{\beta\in\Gamma}\left(\mu_G\left(\sum_i [h_i,\alpha_i]\,[r,\beta]\right)\right)\right]$$

$$\geq \mu_G\left(\sum_i [h_i,\alpha_i]\right).$$

Thus $\mu_{\left((G^+)^+\right)'}(\sum_i [h_i,\alpha_i] \geq \mu_G(\sum_i [h_i,\alpha_i])$. Similarly, we can prove $\nu_{\left((G^+)^+\right)'}(\sum_i [h_i,\alpha_i] \leq \nu_G(\sum_i [h_i,\alpha_i])$. So $G \subseteq ((G^+)^+)'$.

Let $\sum_j \left[a_j,\gamma_j\right]$ be the right unity of H, then

$$\mu_G\left(\sum_i [h_i,\alpha_i]\right) = \mu_G\left(\sum_i [h_i,\alpha_i]\sum_j \left[a_j,\gamma_j\right]\right)$$

$$\geq \wedge_j \left[\mu_G\left(\sum_i [h_i,\alpha_i]\left[a_j,\gamma_j\right]\right)\right]]$$

$$\geq Inf_{r\in H}\left[Inf_{\gamma\in\Gamma}\left(\mu_G\left([h_i,\alpha_i]\left[a_j,\gamma_j\right]\right)\right)\right]$$

$$= \mu_{\left((G^+)^+\right)'}\left(\sum_i [h_i,\alpha_i]\right).$$

Thus $\mu_G(\sum_i [h_i,\alpha_i]) \geq \mu_{\left((G^+)^+\right)'}(\sum_i [h_i,\alpha_i])$. Similarly, we can prove $\nu_G(\sum_i [h_i,\alpha_i]) \leq \nu_{\left((G^+)^+\right)'}(\sum_i [h_i,\alpha_i])$ and so $((G^+)^+)' \subseteq G$ and hence $G = ((G^+)^+)'$.

Thus, the correspondence $G \to G^{+'}$ is a bijection. Now let $₲_1$, $₲_2$ be IFI of H s.t. $₲_1 \subseteq ₲_2$. Then $\forall \sum_i [h_i,\alpha_i] \in L$, we have

$$\begin{aligned}\mu_{\mathbb{G}_1^{+'}}\left(\sum_i [h_i,\alpha_i]\right) &= Inf_{r\in H}\left(\mu_{\mathbb{G}_1}\left(\sum_i h_i\,\alpha_i r\right)\right)\\ &\le Inf_{r\in H}\left(\mu_{\mathbb{G}_2}\left(\sum_i h_i\,\alpha_i r\right)\right)\\ &= \mu_{\mathbb{G}_2^{+'}}\left(\sum_i [h_i,\alpha_i]\right).\end{aligned}$$

Thus $\mu_{\mathbb{G}_1^{+'}}(\sum_i [h_i,\alpha_i]) \le \mu_{\mathbb{G}_2^{+'}}(\sum_i [h_i,\alpha_i])$. Similarly, we can show $\nu_{\mathbb{G}_1^{+'}}(\sum_i [h_i,\alpha_i]) \ge \nu_{\mathbb{G}_2^{+'}}(\sum_i [h_i,\alpha_i])$. Thus $\mathbb{G}_1^{+'} \subseteq \mathbb{G}_2^{+'}$. Similarly, we can show that if $\mathbb{G}_1$ and $\mathbb{G}_2$ are IFIs of L s.t. $\mathbb{G}_1 \subseteq \mathbb{G}_2$, then $\mathbb{G}_1^{+} \subseteq \mathbb{G}_2^{+}$. Hence $G \to G^{+'}$ is an inclusion-preserving one to one map.

Theorem 3.3.7. *For R of a Γ-Ring H with unities, ∃ an inclusion preserving one-to-one map* $B \to B^{*'}$ *between the IFIs(H) and the IFIs(R).*

Proof. The proof of the theorem directly follows from theorem (3.3.6.)

Lemma 3.3.8. *Let K be an ideal of L of a Γ-Ring H. Then* $(\chi_K)^+ = \chi_{K^+}$, *where* χ_K *denotes the IFCF of* K.

Proof. Let $h_1 \in K^+$. Then $[h_1,\alpha] \in K$ for all $\alpha \in \Gamma$. This mean $Inf_{\alpha\in\Gamma}\left(\mu_{\chi_K}([h_1,\alpha])\right) = 1$ and $Sup_{\alpha\in\Gamma}\left(\nu_{\chi_K}([h_1,\alpha])\right) = 0$. Also $\mu_{\chi_{K^+}}(h_1) = 1$ and $\nu_{\chi_{K^+}}(h_1) = 0$. Thus $Inf_{\alpha\in\Gamma}\left(\mu_{\chi_K}([h_1,\alpha])\right) = \mu_{\chi_{K^+}}(h_1)$ and $Sup_{\alpha\in\Gamma}\left(\nu_{\chi_K}([h_1,\alpha])\right) = \nu_{\chi_{K^+}}(h_1), \forall\, h_1 \in K^+$, i.e., $(\chi_K)^+(h_1) = \chi_{K^+}(h_1), \forall\, h_1 \in K^+$.

Now suppose $h_1 \notin K^+$. Then $\exists\, \beta \in \Gamma$ s.t. $[h_1,\beta] \notin K$. Therefore $\mu_{\chi_K}([h_1,\beta]) = 0, \nu_{\chi_K}([h_1,\beta]) = 1$ and so $Inf_{\alpha\in\Gamma}\left(\mu_{\chi_K}([h_1,\alpha])\right) = 0$ and $Sup_{\alpha\in\Gamma}\left(\nu_{\chi_K}([h_1,\alpha])\right) = 1$. Thus $Inf_{\alpha\in\Gamma}\left(\mu_{\chi_K}([h_1,\alpha])\right) = \mu_{\chi_{K^+}}(h_1)$ and $Sup_{\alpha\in\Gamma}\left(\nu_{\chi_K}([h_1,\alpha])\right) = \nu_{\chi_{K^+}}(h_1), \forall\, h_1 \notin K^+$, i.e., $(\chi_K)^+(h_1) = \chi_{K^+}(h_1), \forall\, h_1 \notin K^+$. Hence $(\chi_K)^+ = \chi_{K^+}$.

Lemma 3.3.9. *Suppose for an ideal K of L of a Γ-Ring H. Then* $(\chi_K)^{+'} = \chi_{K^{+'}}$.

Proof. Let $\sum_i [h_i,\alpha_i] \in K^{+'}$. Then $\sum_i h_i\,\alpha_i r \in K, \forall r \in H$.

This means $Inf_{r\in H}\mu_{\chi_K}(\sum_i h_i\,\alpha_i r) = 1$ and $Sup_{r\in H}\nu_{\chi_K}(\sum_i h_i\,\alpha_i r) = 0$, i.e., $\mu_{(\chi_K)^{+'}}(\sum_i [h_i, \alpha_i]) = 1$ and $\nu_{(\chi_K)^{+'}}(\sum_i [h_i, \alpha_i]) = 0$.

Also $\mu_{\left(\chi_{K^{+'}}\right)}(\sum_i [h_i, \alpha_i]) = 1$ and $\nu_{\left(\chi_{K^{+'}}\right)}(\sum_i [h_i, \alpha_i]) = 0$. Then

$\mu_{\left(\chi_{K^{+'}}\right)}(\sum_i [h_i, \alpha_i]) = \mu_{(\chi_K)^{+'}}(\sum_i [h_i, \alpha_i])$ and $\nu_{\left(\chi_{K^{+'}}\right)}(\sum_i [h_i, \alpha_i]) = \nu_{(\chi_K)^{+'}}(\sum_i [h_i, \alpha_i])$. So $(\chi_K)^{+'}(\sum_i [h_i, \alpha_i]) = \left(\chi_{K^{+'}}\right)(\sum_i [h_i, \alpha_i])$.

Let $\sum_i [h_i, \alpha_i] \notin K^{+'}$. Then $\sum_i h_i \alpha_i r \notin K, \forall r \in H$.

This means $Inf_{r\in H}\mu_{\chi_K}(\sum_i h_i \alpha_i r) = 0$ and $Sup_{r\in H}\nu_{\chi_K}(\sum_i h_i \alpha_i r) = 1$,

i.e., $\mu_{(\chi_K)^{+'}}(\sum_i [h_i, \alpha_i]) = 0$ and $\nu_{(\chi_K)^{+'}}(\sum_i [h_i, \alpha_i]) = 1$.

Also $\mu_{\left(\chi_{K^{+'}}\right)}(\sum_i [h_i, \alpha_i]) = 0$ and $\nu_{\left(\chi_{K^{+'}}\right)}(\sum_i [h_i, \alpha_i]) = 1$. Thus we have

$\mu_{\left(\chi_{K^{+'}}\right)}(\sum_i [h_i, \alpha_i]) = \mu_{(\chi_K)^{+'}}(\sum_i [h, \alpha_i])$ and $\nu_{\left(\chi_{K^{+'}}\right)}(\sum_i [h_i, \alpha_i]) = \nu_{(\chi_K)^{+'}}(\sum_i [h_i, \alpha_i])$.

So $(\chi_K)^{+'}(\sum_i [h_i, \alpha_i]) = \left(\chi_{K^{+'}}\right)(\sum_i [h_i, \alpha_i])$.

Thus from both cases, we get $(\chi_K)^{+'} = \chi_{K^{+'}}$.

Remark 3.3.10. Similar results can be seen for R of Γ-Ring H by using an analogy that follows in previously mentioned Lemmas.

Theorem 3.3.11. *Suppose H is a Γ-Ring with unities. Then ∃ an inclusion preserving one-to-one between I(H) and that of its I(L) via the mapping* $K \to K^{+'}$.

Proof. Suppose $\phi: K \to K^{+'}$ is the mapping. This is a mapping that is used in Proposition (3.3.5). Let $\phi(K_1) = \phi(K_2)$. So $K_1^{+'} = K_2^{+'}$. This implies $\chi_{K_1^{+'}} = \chi_{K_2^{+'}}$ (where χ_K is the IFCF of K). Hence by Lemma (3.3.9), $\left(\chi_{K_1}\right)^{+'} = \left(\chi_{K_2}\right)^{+'}$. This together with Theorem (3.3.6) gives $\chi_{K_1} = \chi_{K_2}$, hence $K_1 = K_2$. Consequently, ϕ is one-to-one.

Let K be an ideal of L. Then its IFCF χ_K is an IFI of L. Hence by Theorem (3.3.6), $((\chi_K)^+)^{+'} = \chi_K$. This implies that $\chi_{(K^+)^{+'}} = \chi_K$ [by Lemma (3.3.8) and (3.3.9)]. Hence $(K^+)^{+'} = K$, i.e., $\phi(K^+) = K$. Now since K^+ is an ideal of H, then it states that ϕ is onto. Let K_1, $and\, K_2$ be two ideals of H with $K_1 \subseteq K_2$. Then $\chi_{K_1} \subseteq \chi_{K_2}$. Hence by Theorem (3.3.6), we see that $\left(\chi_{K_1}\right)^{+'} \subseteq \left(\chi_{K_2}\right)^{+'}$, i.e., $\chi_{K_1^{+'}} \subseteq \chi_{K_2^{+'}}$ [by Lemma (3.3.9)] which gives $K_1^{+'} \subseteq K_2^{+'}$.

Remark 3.3.12. We can prove the same for R that $()^{*'}$ is an inclusion preserving one-to-one map (with $()^*$ as above) between the I(H) and that of I(R) using Lemmas (3.3.8.) and (3.3.9), Remark (3.3.10) and Theorem (3.3.11)

Definition 3.3.13. For L of a Γ-Ring H and $\sigma \in Aut(H)$, we define $\sigma^{+'}: L \to L$ by $\sigma^{+'}(\sum_i [h_i, \alpha_i]) = \sum_i [\sigma(h_i), \alpha_i]$.

We first show that the map $\sigma^{+'}$ is well-defined.

Suppose $\sum_i \left[h_{1_i}, \alpha_i\right] = \sum_j \left[h_{2_j}, \beta_j\right]$, then $\left[h_{1_i}, \alpha_i\right] = \left[h_{2_j}, \beta_j\right]$, so, $h_{1_i}\alpha_i r = h_{2_j}\beta_j r, \forall r \in H$. Thus $\sum_i h_{1_i}\, \alpha_i r = \sum_j h_{2_j}\, \beta_j r$. This implies $\sigma\left(\sum_i h_{1_i}\, \alpha_i r\right) = \sigma\left(\sum_j h_{2_j}\, \beta r\right), \forall\, r \in H$.

Now for $a \in H$, we have $\sigma\left(h_{1_i}\right)\alpha_i a = \sigma\left(h_{1_i}\right)\alpha_i \sigma(a')$ [As σ is onto so $\exists\ a' \in H$ s.t. $\sigma(a') = a$] $= \sigma\left(h_{1_i}\alpha_i a'\right) = \sigma\left(h_{2_j}\beta_j a'\right) = \sigma\left(h_{2_j}\right)\beta_j \sigma(a') = \sigma\left(h_{2_j}\right)\beta_j a$. This implies $\sigma\left(h_{1_i}\right)\alpha_i a = \sigma\left(h_{2_j}\right)\beta_j a$. So $\left[\sigma\left(h_{1_i}\right), \alpha_i\right] = \left[\sigma\left(h_{2_j}\right), \beta_j\right] \Rightarrow \sum_i \left[\sigma\left(h_{1_i}\right), \alpha_i\right] = \sum_j \left[\sigma\left(h_{2_j}\right), \beta_j\right]$. Hence $\sigma^{+'}\left(\sum_i \left[h_{1_i}, \alpha_i\right]\right) = \sigma^{+'}\left(\sum_j \left[h_{2_j}, \beta_j\right]\right)$. Therefore, the map $\sigma^{+'}$ is well-defined.

Proposition 3.3.14. For L of a Γ-Ring H *let* $\sigma \in Aut(H)$. *Then* $\sigma^{+'} \in Aut(L)$.

Proof. Let $\sigma \in Aut(H)$ and $[h_1, \alpha], [h_2, \alpha], [h_1, \beta] \in L$. Then

$\sigma^{+'}([h_1, \alpha] + [h_2, \alpha]) = \sigma^{+'}([h_1 + h_2, \alpha]) = [\sigma(h_1 + h_2), \alpha] = [\sigma(h_1) + \sigma(h_2), \alpha] = [\sigma(h_1), \alpha] + [\sigma(h_2), \beta]$

$\sigma^{+'}([h_1, \alpha] + [h_1, \beta]) = \sigma^{+'}([h_1, \alpha + \beta]) = [\sigma(h_1), \alpha + \beta] = [\sigma(h_1), \alpha] + [\sigma(h_1), \beta]$.

$$\begin{aligned}\sigma^{+'}\left(\sum_i \left[h_{1_i},\alpha_i\right].\sum_j \left[h_{2_j},\beta_j\right]\right) &= \sigma^{+'}\left(\sum_{i,j} \left[h_{1_i}\alpha_i h_{2_j},\beta_j\right]\right)\\ &= \sum_{i,j} \left[\sigma\left(h_{1_i}\alpha_i h_{2_j}\right),\beta_j\right]\\ &= \sum_{i,j} \left[\sigma(h_{1_i})\alpha_i\sigma\left(h_{2_j}\right),\beta_j\right]\\ &= \sum_i \left[\sigma(h_{1_i}),\alpha_i\right].\sum_j \left[\sigma\left(h_{2_j}\right),\beta_j\right]\\ &= \sigma^{+'}\left(\sum_i \left[h_{1_i},\alpha_i\right]\right)\sigma^{+'}(\sum_j \left[h_{2_j},\beta_j\right]\end{aligned}$$

Hence $\sigma^{+'}$ is an endomorphism of L. As $\sigma^{+'}$ is well-defined implies $\sigma^{+'}$ is one to one map. Further, let $\sum_i \left[h_{1_i},\alpha_i\right] \in L$. Then $\exists$, $h_{1_i}' \in H$ s.t. $\sigma\left(h_{1_i}'\right) = h_{1_i}$. So $\sum_i \left[h_{1_i}',\alpha_i\right] \in L$ s.t. $\sigma^{+'}\left(\sum_i \left[h_{1_i}',\alpha_i\right]\right) = \sum_i \left[\sigma\left(h_{1_i}'\right),\alpha_i\right] = \sum_i \left[h_{1_i},\alpha_i\right]$. Consequently, $\sigma^{+'}$ is onto. Suppose L has the left unity $\sum_i [e_i,\delta_i]$. Then for any $\alpha_i \in \Gamma$, we have $\sigma^{+'}\sum_i [e_i,\alpha_i] = \sum_i [\sigma(e_i),\alpha_i] = \sum_i [e_i,\alpha_i]$. Again if H has the right unity $\sum_i [\gamma_i,\ a_i]$. Then for any $\alpha_i \in \Gamma$, we have $\sigma^{+'}\sum_i [\gamma_i,\alpha_i] = \sum_i [\sigma(\gamma_i),\alpha_i] = \sum_i [\gamma_i,\alpha_i]$. Hence $\sigma^{+'} \in Aut(L)$.

We use the Remark (3.3.10) to frame the following precision and also to demonstrate the subsequent Propositions.

Definition 3.3.15. Let H be a Γ-Ring with right unity $\sum_i [\gamma_i,\ a_i]$ and L be its left operator ring. Then for $\sigma \in Aut(L)$, we set $\sigma^+ : H \to H$ by $\sigma^+(h) = \sum_i \sigma([h,\gamma_i])a_i$.

We first show that the map σ^+ is well-defined. Let $h_1, h_2 \in H, \gamma_i, \beta_i \in \Gamma$ be s.t. $\sigma^+(h_1) = \sigma^+(h_2)$, then $\sum_i \sigma([h_1,\gamma_i])a_i = \sum_i \sigma([h_2,\gamma_i])a_i$

$\Rightarrow \sum_i [\sigma([h_1,\gamma_i])a_i,\gamma_i] = \sum_i [\sigma([h_2,\gamma_i])a_i,\gamma_i]$

$\Rightarrow \sum_i ([h_1,\gamma_i]).\sum_i [a_i,\gamma_i] = \sum_i ([h_2,\gamma_i]).\sum_i [a_i,\gamma_i]$

$\Rightarrow \sum_i \sigma([h_1,\gamma_i]).\sigma(\sum_i [a_i,\gamma_i]) = \sum_i \sigma([h_2,\gamma_i]).\sigma(\sum_i [a_i,\gamma_i])$ [Using Definition (2.1.21)]

$\Rightarrow \sigma(\sum_i [h_1,\gamma_i].\sum_i [a_i,\gamma_i]) = \sigma(\sum_i [h_2,\gamma_i].\sum_i [a_i,\gamma_i])$

$\Rightarrow \sigma(\sum_i [h_1\gamma_i a_i,\gamma_i]) = \sigma(\sum_i [h_2\gamma_i a_i,\gamma_i])$

$\Rightarrow \sum_i [h_1\gamma_i a_i,\gamma_i] = \sum_i [h_2\gamma_i a_i,\gamma_i]$ [Since σ is one to one]

$\Rightarrow \sum_i [h_1, \gamma_i] . \sum_i [a_i, \gamma_i] = \sum_i [h_2, \gamma_i] . \sum_i [a_i, \gamma_i]$

$\Rightarrow \sum_i [h_1, \gamma_i] = \sum_i [h_2, \gamma_i]$

$\Rightarrow [h_1, \gamma_i] = [h_2, \gamma_i] \Rightarrow h_1 \gamma_i r = h_2 \gamma_i r, \forall r \in H.$

In particular, take $r = a_i$, we get $\sum_i h_1 \gamma_i a_i = \sum_i h_2 \gamma_i a_i \Rightarrow h_1 = h_2$. Hence σ^+ is well-defined.

Proposition 3.3.16. *Let H be a Γ-Ring with right unity* $\sum_i [\gamma_i,\ a_i]$ *and L be its left operator ring. Assume* $\sigma \in Aut(L)$*, then* $\sigma^+ \in Aut(H)$.

Proof. Let $h_1, h_2 \in H, \eta \in \Gamma$. Then

$$\begin{aligned}
\sigma^+(h_1 + h_2) &= \sum_i \sigma([h_1 + h_2, \gamma_i])a_i \\
&= \sum_i \sigma([h_1, \gamma_i] + [h_2, \gamma_i])a_i \\
&= \sum_i (\sigma([h_1, \gamma_i])a_i + \sigma([h_2, \gamma_i])a_i)) \\
&= \sum_i \sigma([h_1, \gamma_i])a_i + \sum_i \sigma([h_2, \gamma_i])a_i \\
&= \sigma^+(h_1) + \sigma^+(h_2)
\end{aligned}$$

$$\begin{aligned}
\sigma^+(h_1 \eta h_2) &= \sum_i \sigma([h_1 \eta h_2, \gamma_i])a_i = \sum_i \sigma([h_1, \eta][h_2, \gamma_i])a_i \\
&= \sum_i \sigma([h_1, \eta]). \sum_i \sigma([h_2, \gamma_i])a_i = \sum_i \sigma([h_1 \gamma_i a_i, \eta]). \sum_i \sigma([h_2, \gamma_i])a_i \\
&= \sum_i \sigma([h_1, \gamma_i][a_i, \eta]). \sum_i \sigma([h_2, \gamma_i])a_i = \sum_i \sigma([h_1, \gamma_i]). \sum_i \sigma([a_i, \eta]). \sum_i \sigma([h_2, \gamma_i]) \\
&= \sum_i \sigma([h_1, \gamma_i]). \sum_i [a_i, \eta] . \sum_i \sigma([h_2, \gamma_i])a_i = \sum_i \sigma([h_1, \gamma_i])a_i \eta \sum_i \sigma([h_2, \gamma_i])a_i \\
&= \left(\sum_i \sigma([h_1, \gamma_i])a_i\right) \eta \left(\sum_i \sigma([h_2, \gamma_i])a_i\right) \\
&= \sigma^+(h_1) \eta \sigma^+(h_2).
\end{aligned}$$

Hence σ^+ is an endomorphism of H. As σ^+ is well-defined implies that σ^+ is one to one map.

Further, let $h_2 \in H$. Since $\sigma: L \to L$ is onto, $\exists \sum_i [h_1, \gamma_i] \in L$ s.t. $\sigma(\sum_i [h_1, \gamma_i]) = \sum_i [h_2, \gamma_i]$.

$$\begin{aligned}
\sigma^+(h_1) &= \sum_i \sigma([h_1, \gamma_i]) a_i = \sum_i \sigma([h_1 \gamma_i a_i, \gamma_i]) a_i \\
&= \sum_i \sigma([h_1, \gamma_i].[a_i, \gamma_i]) a_i = \sum_i \sigma([h_1, \gamma_i]). \sum_i \sigma([a_i, \gamma_i]) a_i \\
&= \sum_i [h_2, \gamma_i]. \sum_i [a_i, \gamma_i]\, a_i = \sum_i [h_2, \gamma_i].[a_i, \gamma_i] a_i \\
&= \sum_i [h_2 \gamma_i a_i, \gamma_i]\, a_i = \sum_i [h_2, \gamma_i]\, a_i \\
&= \sum_i h_2\, \gamma_i a_i = h_2
\end{aligned}$$

Hence σ^+ is onto. Again if $\sum_i [e_i, \delta_i]$ is the left unity of H then

$\sigma^+(e) = \sum_i \sigma([e, \delta_i]) a_i = \sum_i [e, \delta_i]\, a_i = \sum_i e\, \delta_i a_i = e$. Consequently, $\sigma^+ \in Aut(H)$.

Proposition 3.3.17. *Let H be a Γ-Ring with left unity* $\sum_i [e_i, \delta_i]$ *and right unity* $\sum_i [\gamma_i,\, a_i]$ *and L be its left operator ring. Assume* $\sigma \in Aut(L)$, *then* $\left(\sigma^{+'}\right)^+ = \sigma$.

Proof. By Proposition (3.3.14), $\sigma^{+'} \in Aut(L)$ whence by Proposition (3.3.16), $\left(\sigma^{+'}\right)^+ \in Aut(H)$. Let $h \in H$. Then $\left(\sigma^{+'}\right)^+(h) = \sigma^{+'}(\sum_i [h, \gamma_i]) a_i = \sum_i [\sigma(h), \gamma_i]\, a_i = \sum_i \sigma(h) \gamma_i a_i = \sigma(h)$.

Hence $\left(\sigma^{+'}\right)^+ = \sigma$.

Proposition 3.3.18. *Let H be a Γ-Ring with left unity* $\sum_i [e_i, \delta_i]$ *and right unity* $\sum_i [\gamma_i,\, a_i]$ *and L be its left operator ring. Let* $\sigma \in Aut(H)$. *Then* $(\sigma^+)^{+'} = \sigma$.

Proof. By Proposition (3.3.16), $\sigma^+ \in Aut(H)$ whence by Proposition (3.3.14), $(\sigma^+)^{+'} \in Aut(L)$. Let $\sum_i [h_i, \alpha_i] \in L$. Then

$$
\begin{aligned}
(\sigma^+)^{+'}\left(\sum_i [h_i,\alpha_i]\right) &= \sum_i [\sigma^+(h_i),\alpha_i] = \sum_i [\sigma([h_i,\gamma_i])a_i,\alpha_i] \\
&= \sum_i \sigma([h_i,\gamma_i]) \sum_i [a_i,\alpha_i] = \sum_i \sigma([h_i,\gamma_i])\sigma\left(\sum_i [a_i,\alpha_i]\right) \\
&= \sum_i \sigma([h_i,\gamma_i][a_i,\alpha_i]) = \sum_i \sigma([h_i\gamma_i a_i,\alpha_i]) = \sum_i \sigma([h_i,\alpha_i]) \\
&= \sigma\left(\sum_i [h_i,\alpha_i]\right).
\end{aligned}
$$

Hence $(\sigma^+)^{+'} = \sigma$.

Theorem 3.3.19. *For L of a Γ-Ring H there exists a bijection between the Aut(H) and the Aut(L).*

Proof. Let us define the map $\phi: Aut(H) \to Aut(L)$ by $\phi(\sigma) = \sigma^{+'}, \forall \sigma \in Aut(H)$.

Consider $\sigma, \tau \in Aut(H)$ s.t. $\phi(\sigma) = \phi(\tau)$. Then $\sigma^{+'} = \tau^{+'}$

$\Rightarrow \sigma^{+'}(\sum_i [h_i,\alpha_i]) = \tau^{+'}(\sum_i [h_i,\alpha_i]), \forall \sum_i [h_i,\alpha_i] \in L \Rightarrow \sum_i [\sigma(h_i),\alpha_i] = \sum_i [\tau(h_i),\alpha_i]$ $\Rightarrow \sigma(h_i)\alpha_i r = \tau(h_i)\alpha_i r, \forall r \in H, \alpha_i \in \Gamma$. In particular, $\sigma(h_i)\gamma_i a_i = \tau(h_i)\gamma_i a_i \Rightarrow \sigma(h_i) = \tau(h_i)$.

So $\sigma = \tau$. Hence ϕ is one to one.

Suppose $\sigma \in Aut(L)$. Then by Proposition (3.3.16), $\sigma^+ \in Aut(H)$. Now $\phi(\sigma^+) = \sigma^{+'} = \sigma$ (by Proposition (3.3.18)). Consequently, ϕ is onto. Hence ϕ is a bijection.

Proposition 3.3.20. *For L of a Γ-Ring H with unities and G be an IFCI of L. Then G^+ is an IFCI of H, where G^+ is explained in Definition (3.3.1).*

Proof. By Proposition (3.3.2), G^+ is an IFI of Γ-Ring H. Let $h \in H$ and $\sigma \in Aut(H)$. Then by Proposition (3.3.14), $\sigma^{+'} \in Aut(L)$. Hence by using Definition (3.3.1) and (3.3.13) we obtain

$$
\begin{aligned}
\mu_{(G^+)^\sigma}(h) &= \mu_{G^+}(\sigma(h)) = Inf_{\alpha\in\Gamma}(\mu_G([\sigma(h),\alpha])) \\
&= Inf_{\alpha\in\Gamma}\left(\mu_i(\sigma^+([h,\alpha]))\right) = Inf_{\alpha\in\Gamma}(\mu_i([h,\alpha])) \\
&= \mu_{G^+}(h).
\end{aligned}
$$

Similarly, we can prove $\nu_{(G^+)^\sigma}(h) = \nu_{G^+}(h)$, i.e., $(G^+)^\sigma(h) = G^+(h), \forall \sigma \in Aut(H)$.

Hence G^+ is an IFCI of H.

Proposition 3.3.21. *For L of a Γ-Ring H with unities and B be an IFCI of H. Then $B^{+'}$ is an IFCI of L, where $B^{+'}$ is explained in Definition (3.3.1).*

Proof. By Proposition (3.3.3), $B^{+'}$ is an IFI of L. Let $\sum_i [h_i, \alpha_i] \in L$ and $\tau \in Aut(L)$. Then by Theorem (3.3.19) $\exists, \sigma \in Aut(H)$ s.t. $\sigma^{+'} = \tau$. Now

$$\begin{aligned}
\mu_{(B^{+'})^\tau}(\textstyle\sum_i [h_i, \alpha_i]) &= \mu_{B^{+'}}\left(\tau(\textstyle\sum_i [h_i, \alpha_i])\right) = \mu_{B^{+'}}\left(\sigma^{+'}(\textstyle\sum_i [h_i, \alpha_i])\right) \\
&= \mu_{B^{+'}}(\textstyle\sum_i [\sigma(h_i), \alpha_i]) = Inf_{r\in H}\left(\mu_B(\textstyle\sum_i \sigma(h_i)\alpha_i r)\right) \\
&= nf_{n\in H}(\mu_B\left(\textstyle\sum_i \sigma\left(h_i \alpha_i \sigma(n)\right)\right)) \text{[As } \sigma \text{ is a bijection so } \sigma(n) = r \text{]} \\
&= Inf_{n\in H}\left(\mu_B\left(\textstyle\sum_i \sigma(h_i \alpha_i n)\right)\right) = Inf_{r\in H}\left(\mu_B(\textstyle\sum_i h_i\, \alpha_i n)\right) \text{[As } B \text{ is IFCI of H]} \\
&= \mu_{B^{+'}}(\textstyle\sum_i [h_i, \alpha_i]).
\end{aligned}$$

Similarly, we can prove $\nu_{(B^{+'})^\tau}(\sum_i [h_i, \alpha_i]) = \nu_{B^{+'}}(\sum_i [h_i, \alpha_i])$, i.e.,

$\left(B^{+'}\right)^\tau(\sum_i [h_i, \alpha_i]) = B^{+'}(\sum_i [h_i, \alpha_i]), \forall \tau \in Aut(L)$. Hence $B^{+'}$ is an IFCI of L.

Theorem 3.3.22. *For L of a Γ-Ring H with unities $\exists$ a one-to-one map between the IFCIs(H) and the IFCIs(L).*

Proof. Let ϕ be a mapping from the IFCIs(H) to that of L. Let D be an IFCI of H. Let us define $\phi(D) = D^{+'}$. Then by Proposition (3.3.21), $\phi(D)$ is an IFCI of L. Let G be an IFCI of L. Then by Proposition (3.3.20), G^+ is an IFCI of H. Then by Theorem (3.3.6), $(G^+)^{+'} = G$, i.e., $\phi(G^+) = G$. Thus ϕ is onto. Again if for D_1, D_2 of H s.t. $\phi(D_1) = \phi(D_2)$ then $D_1^{+'} = D_2^{+'} \Rightarrow \left(D_1^{+'}\right)^+ = \left(D_2^{+'}\right)^+ \Rightarrow D_1 = D_2$ (by Theorem (3.3.6)). Therefore ϕ is one-to-one, hence the proof.

Proposition 3.3.23. *For L of a Γ-Ring H with left unity $\sum_i [e_i, \delta_i]$, right unity $\sum_i [\gamma_i,\ a_i]$ let $\ddot{W}$ be a CI of L. Then $\ddot{W}^{+}$ is a CI of H.*

Proof. Let $\sigma \in Aut(H)$. Then by Proposition (3.3.14), $\sigma^{+'} \in Aut(L)$. Hence $\sigma^{+'}(\ddot{W}) = \ddot{W}$. Let $\sigma(h) \in \sigma(\ddot{W}^{+})$, where $h \in \ddot{W}^{+}$. Then $[h, \alpha] \in \ddot{W}, \forall \alpha \in \Gamma$. Hence

$\sigma^{+'}([h,\alpha]) \in \sigma^{+'}(\ddot{\bar{W}})$, for all $\alpha \in \Gamma \Rightarrow [\sigma(h),\alpha] \in \ddot{\bar{W}}, \forall\, \alpha \in \Gamma \Rightarrow \sigma(h) \in \ddot{\bar{W}}^{+}$. Thus $\sigma(\ddot{\bar{W}}^{+}) \subseteq \ddot{\bar{W}}^{+}$. Hence $\sigma^{-1}(\ddot{\bar{W}}^{+}) \subseteq \ddot{\bar{W}}^{+}$ (since $\sigma \in Aut(H) \Rightarrow \sigma^{-1} \in Aut(H) \Rightarrow$ $\ddot{\bar{W}}^{+} \subseteq \sigma(\ddot{\bar{W}}^{+})$. Hence $\sigma(\ddot{\bar{W}}^{+}) = \ddot{\bar{W}}^{+}$. Consequently, $\ddot{\bar{W}}^{+}$ is a CI of H.

Theorem 3.3.24. *For L of a Γ-Ring H with unities ∃ an inclusion preserving one-to-one between the CI(H) and the CI(L) via the mapping* $\ddot{\bar{W}} \rightarrow \ddot{\bar{W}}^{+'}$.

Proof. Suppose we define the mapping $\psi: \ddot{\bar{W}} \rightarrow \ddot{\bar{W}}^{+'}$. Let $\ddot{\bar{W}}, \hat{W}$ be two characteristic ideals of H s.t. $\psi(\ddot{\bar{W}}) = \psi(\hat{W})$. Then $\ddot{\bar{W}}^{+'} = \hat{W}^{+'} \Rightarrow \left(\ddot{\bar{W}}^{+'}\right)^{+} = \left(\hat{W}^{+'}\right)^{+} \Rightarrow \ddot{\bar{W}} = \hat{W}$ (by Theorem (3.3.11). So ψ is one-one.

Let $\ddot{\bar{W}}$ be a CI of L, then by proposition (3.3.23), $\ddot{\bar{W}}^{+}$ is a CI of H. Also $\left(\ddot{\bar{W}}^{+'}\right)^{+'} = \ddot{\bar{W}}$. Thus $\psi(\ddot{\bar{W}}^{+}) = (\ddot{\bar{W}}^{+})^{+'} = \ddot{\bar{W}}$. Hence ψ is onto. From Theorem (3.3.11), it follows that ψ is inclusion preserving.

3.4 Conclusion

This chapter, explores the concept of IFCI in a Γ-Ring, examining specific examples to illustrate instances where an IFI is not an IFCI. The relationship between IFCI and its level cut sets is thoroughly analyzed. Furthermore, the connections between Aut(H) and the corresponding Aut(OR) are investigated. The chapter establishes a one-to-one mapping between IFCI(H) and IFCI(OR). These structures play a crucial role in the development of concepts such as IFPIs, IFPrIs, and IFSPIs in a Γ-Ring framework.

Chapter 4

Intuitionistic Fuzzy Prime Radicals, Intuitionistic Fuzzy Primary Ideals And Intuitionistic Fuzzy 2-Absorbing Primary Ideals Of Γ-Ring

4.1 Introduction

Primary ideals hold significance in commutative Γ-Ring theory, primarily due to the fact that every ideal of a Noetherian Γ-Ring can be decomposed into primary ideals, a principle known as the Lasker-Noether theorem, initially established by Z.K. Warsi in [66]. The first section of this chapter introduces and investigates the concept of IFPR in a Γ-Ring, which subsequently serves as the basis for defining IFPrI in the next section. Numerous characterizations associated with these concepts are derived and explored.

4.2 Intuitionistic Fuzzy Prime Radical Of An Intuitionistic Fuzzy Ideal Of A Γ-Ring

While discussing this paper we will consider H as a commutative Γ-Ring with unity.

Definition 4.2.1 Suppose $G \neq \emptyset$ IFS of a Γ-Ring H. Define a set $\wp(G)$ of all IFPI of H which contains G, i.e.,

$$\wp(G) = \{B : B \in IFPI(H), G \subseteq B\}.$$

Proposition 4.2.2. *Consider* $₲_1$ *and* $₲_2$ *to be two non-empty IFSs in a* Γ*-Ring H, then:*

(i) $₲_1 \subseteq ₲_2$ *implies that* $\wp(₲_2) \subseteq \wp(₲_1)$*;*

(ii) $\wp(₲_1) \cup \wp(₲_2) \subseteq \wp(₲_1 \cap ₲_2)$*;*

(iii) $\wp(₲_1) \cup \wp(₲_2) = \wp(₲_1 \Gamma ₲_2)$*, if* $₲_1, ₲_2$ *are two IFIs of H;*

(iv) $\wp(₲_1) \cup \wp(₲_2) = \wp(₲_1 \circ ₲_2)$*, if* $₲_1, ₲_2$ *are two IFIs of H*

(v) $\wp(\chi_{ɨ}) \cup \wp(\chi_{ɉ}) = \wp(\chi_{ɨ \cap ɉ})$ *if* ɨ *and* ɉ *are ideals of H.*

Proof. **(i)** Let $B \in \wp(₲_2)$. So B will be an IFPI of H and $₲_2 \subseteq B$. Since $₲_1 \subseteq ₲_2$, $₲_1 \subseteq B$. So $B \in \wp(₲_1)$. Hence $\wp(₲_2) \subseteq \wp(₲_1)$.

(ii) Since $₲_1 \cap ₲_2 \subseteq ₲_1$ and $₲_1 \cap ₲_2 \subseteq ₲_2$. Therefore by (i) we have $\wp(₲_1) \subseteq \wp(₲_1 \cap ₲_2)$ and $\wp(₲_2) \subseteq \wp(₲_1 \cap ₲_2)$. Thus $\wp(₲_1) \cup \wp(₲_2) \subseteq \wp(₲_1 \cap ₲_2)$.

(iii) Since $₲_1$ and $₲_2$ are IFIs of the Γ-Ring H, then $₲_1 \Gamma ₲_2 \subseteq ₲_1 \cap ₲_2$ [by Remark (2.2.4)]. Therefore by (i), we have $\wp(₲_1 \cap ₲_2) \subseteq \wp(₲_1 \Gamma ₲_2)$. So by (ii) we have $\wp(₲_1) \cup \wp(₲_2) \subseteq \wp(₲_1 \Gamma ₲_2)$.

Again, let $B \in \wp(₲_1 \Gamma ₲_2)$. Then $₲_1 \Gamma ₲_2 \subseteq B$ and $B \in \mathrm{IFPI}(H)$, so either $₲_1 \subseteq B$ or $₲_2 \subseteq B$. Therefore $\wp(B) \subseteq \wp(₲_1)$ or $\wp(B) \subseteq \wp(₲_2)$.

Now $B \in \mathrm{IFPI}(H)$ and $B \subseteq B$ so $B \in \wp(B)$. Therefore $B \in \wp(₲_1)$ or $B \in \wp(₲_2)$. Therefore $B \in \wp(₲_1) \cup \wp(₲_2)$. Hence $\wp(₲_1 \Gamma ₲_2) \subseteq \wp(₲_1) \cup \wp(₲_2)$. Hence $\wp(₲_1) \cup \wp(₲_2) = \wp(₲_1 \Gamma ₲_2)$.

(iv) Since $₲_1$ and $₲_2$ are IFIs of the Γ-Ring H, then $₲_1 \Gamma ₲_2 \subseteq ₲_1 \circ ₲_2$ [by Remark (2.2.4)]. Then by (i) we have $\wp(₲_1 \circ ₲_2) \subseteq \wp(₲_1 \Gamma ₲_2)$.

Again, let $B \in \wp(₲_1 \Gamma ₲_2)$. Then $₲_1 \Gamma ₲_2 \subseteq B$ and $B \in \mathrm{IFPI}(H)$. This implies that $₲_1 \circ ₲_2 \subseteq B$, $B \in \mathrm{IFPI}(H)$ [by Remark (2.2.4)]. So $B \in \wp(₲_1 \circ ₲_2)$. Thus $\wp(₲_1 \Gamma ₲_2) \subseteq \wp(₲_1 \circ ₲_2)$. Thus $\wp(₲_1 \Gamma ₲_2) = \wp(₲_1 \circ ₲_2)$. Hence from (iii) we get

$\wp(\mathbb{G}_1) \cup \wp(\mathbb{G}_2) = \wp(\mathbb{G}_1 \circ \mathbb{G}_2)$.

(v) Assume that Ɨ and Ɉ are two ideals of the Γ-Ring H. Clearly $\chi_{Ɨ} \cap \chi_{Ɉ} = \chi_{Ɨ\cap Ɉ}$. Thus $\wp(\chi_{Ɨ}) \cup \wp(\chi_{Ɉ}) \subseteq \wp(\chi_{Ɨ} \cap \chi_{Ɉ}) \subseteq \wp(\chi_{Ɨ\cap Ɉ})$.

Again, let $B \in \wp(\chi_{Ɨ\cap Ɉ})$. Then $\chi_{Ɨ\cap Ɉ} \subseteq B$. So $\chi_{Ɨ}\Gamma\chi_{Ɉ} \subseteq \chi_{Ɨ} \cap \chi_{Ɉ} = \chi_{Ɨ\cap Ɉ} \subseteq B$.

Since $B \in IFPI(H)$, we have $\chi_{Ɨ} \subseteq B$ or $\chi_{Ɉ} \subseteq B$. Thus $B \subseteq \wp(\chi_{Ɨ})$ or $B \subseteq \wp(\chi_{Ɉ})$. Therefore,$B \subseteq \wp(\chi_{Ɨ}) \cup \wp(\chi_{Ɉ})$.Thus $\wp(\chi_{Ɨ\cap Ɉ}) \subseteq \wp(\chi_{Ɨ}) \cup \wp(\chi_{Ɉ})$.

Hence $\wp(\chi_{Ɨ}) \cup \wp(\chi_{Ɉ}) = \wp(\chi_{Ɨ\cap Ɉ})$.

Definition 4.2.3. Consider an IFI G in a Γ-Ring H. Then the IFS $\sqrt{G}$ of H defined by

$$\sqrt{G} = \cap\left(\wp(G)\right) = \cap\{B : B \in IFPI(H); G \subseteq B\}$$

is said to be the IFPR of G.

Proposition 4.2.4. *Consider an IFI G in a Γ-Ring H. So $\sqrt{G}$ is a non-constant IFI of H with $\sqrt{G}(0_H) = (1,0)$.*

Proof. Consider an IFI G in a Γ-Ring H. Therefore

$$\begin{aligned}\mu_{\sqrt{G}}(0_H) &= \mu_{\cap(\wp(G))}(0_H) \\ &= Inf\{\mu_B(0_H) : B \in IFPI(H); G \subseteq B\} \\ &= 1.\end{aligned}$$

Similarly, we can show $\nu_{\sqrt{G}}(0_H) = 0$. Thus $\sqrt{G}(0_H) = (1,0)$.

Let $B \in IFPI(H)$. So ∃ at least one $h \in H$ s.t. $B(h_H) \neq (1,0)$. Therefore $\sqrt{G}(h_H) \neq (1,0)$. Thus $\sqrt{G}$ is a non-constant IFS of H. Now ∀ $\hat{h}, ħ \in H$, we have

$$\begin{aligned}\mu_{\sqrt{G}}(\hat{h} - ħ) &= \mu_{\cap(\wp(G))}(\hat{h} - ħ) = Inf\{\mu_B(\hat{h} - ħ) : B \in IFPI(H); G \subseteq B\} \\ &\geq Inf\{\mu_B(\hat{h}) \wedge \mu_B(ħ) : B \in IFPI(H); G \subseteq B\} \\ &= \left(Inf\{\mu_B(\hat{h}) : B \in IFPI(H); G \subseteq B\}\right) \wedge \left(Inf\{\mu_B(ħ) : B \in IFPI(H); G \subseteq B\}\right) \\ &= \mu_{\cap(\wp(G))}(\hat{h}) \wedge \mu_{\cap(\wp(G))}(ħ) \\ &= \mu_{\sqrt{G}}(\hat{h}) \wedge \mu_{\sqrt{G}}(ħ).\end{aligned}$$

Thus $\mu_{\sqrt{G}}(\hat{h} - \hbar) \geq \mu_{\sqrt{G}}(\hat{h}) \wedge \mu_{\sqrt{G}}(\hbar)$. Similarly, we can prove $\nu_{\sqrt{G}}(\hat{h} - \hbar) \leq \mu_{\sqrt{G}}(h_1) \vee \nu_{\sqrt{G}}(h_2)$.

Again for any $\hat{h}, \hbar \in H$ and $\gamma \in \Gamma$, we have

$$\begin{aligned} \mu_{\sqrt{G}}(\hat{h}\gamma\hbar) &= \mu_{\cap(\wp(G))}(\hat{h}\gamma\hbar) = Inf\{\mu_B(\hat{h}\gamma\hbar): B \in IFPI(H); G \subseteq B\} \\ &\geq Inf\{\mu_B(\hat{h}): B \in IFPI(H); G \subseteq B\} \\ &= \mu_{\cap(\wp(G))}(\hat{h}) \\ &= \mu_{\sqrt{G}}(\hat{h}). \end{aligned}$$

Similarly, we can show $\mu_{\sqrt{G}}(\hat{h}\gamma\hbar) \geq \mu_{\sqrt{G}}(\hbar)$. Thus, we have $\mu_{\sqrt{G}}(\hat{h}\gamma\hbar) \geq \mu_{\sqrt{G}}(\hat{h}) \vee \mu_{\sqrt{G}}(\hbar)$.

Similarly, we can prove $\nu_{\sqrt{G}}(\hat{h}\gamma\hbar) \leq \nu_{\sqrt{G}}(\hat{h}) \wedge \nu_{\sqrt{G}}(\hbar)$. Hence $\sqrt{G}$ is a non-constant IFI of H.

Proposition 4.2.5. *Consider an IFI G in a Γ-Ring H. So $\sqrt{G}$ is an IFSPI of H.*

Proof. We have already shown that $\sqrt{G}$ is a non-constant IFI of H. Now $\forall\, r \in H$, we have

$$\begin{aligned} Inf\{\mu_{\sqrt{G}}(r\gamma_1 h\gamma_2 r): h \in H, \gamma_1, \gamma_2 \in \Gamma\} &= Inf\{\mu_{\cap(\wp(G))}(r\gamma_1 h\gamma_2 r): h \in H, \gamma_1, \gamma_2 \in \Gamma\} \\ &= Inf\{Inf\{\mu_B(r\gamma_1 h\gamma_2 r): B \in IFPI(H); G \subseteq B\}, h \in H, \gamma_1, \gamma_2 \in \Gamma\} \\ &= Inf\{\mu_B(r): B \in IFPI(H); G \subseteq B\} [\text{ As } B \in IFPI(H)] \\ &= \mu_{\cap(\wp(G))}(r) \\ &= \mu_{\sqrt{G}}(r). \end{aligned}$$

$\mu_{\sqrt{G}}(r)$

Similarly, we can prove $Sup\{\nu_{\sqrt{G}}(r\gamma_1 h\gamma_2 r): h \in H, \gamma_1, \gamma_2 \in \Gamma\} = \nu_{\sqrt{G}}(r)$.

Hence $\sqrt{G}$ is an IFSPI of H (by Proposition (2.2.11)).

Proposition 4.2.6. *Suppose $\mathbb{G}_1$ and $\mathbb{G}_2$ be two IFIs of a Γ-Ring H. Then*

(i) $\sqrt{\mathbb{G}_1}(h) = (1,0)$ *if* $h \in \left(\sqrt{\mathbb{G}_1}\right)_*$

(ii) $\mathbb{G}_1 \subseteq \sqrt{\mathbb{G}_1}$

(iii) If $\mathbb{G}_1 \subseteq \mathbb{G}_2$ *then* $\sqrt{\mathbb{G}_1} \subseteq \sqrt{\mathbb{G}_2}$

(iv) $\sqrt{\sqrt{\mathbb{G}_1}} = \sqrt{\mathbb{G}_1}$

(v) $\sqrt{\mathbb{G}_1 \oplus \mathbb{G}_2} = \sqrt{\sqrt{\mathbb{G}_1} \oplus \sqrt{\mathbb{G}_2}}$, *where* $\mathbb{G}_1(0_H) = \mathbb{G}_2(0_H) = (1,0)$.

Proof. **(i)** Let $h \in \left(\sqrt{\mathbb{G}_1}\right)_*$. Then

$$\begin{aligned} \mu_{\sqrt{\mathbb{G}_1}}(h) &= \mu_{\sqrt{\mathbb{G}_1}}(0_H) = \mu_{\cap(\wp(\mathbb{G}_1))}(0_H) \\ &= Inf\{\mu_{\mathrm{B}}(0_H): \mathrm{B} \in IFPI(H); \mathbb{G}_1 \subseteq \mathrm{B}\} \\ &= 1. \end{aligned}$$

In the same manner, it can be shown that $\nu_{\sqrt{\mathbb{G}_1}}(h) = 0$. Thus $\sqrt{\mathbb{G}_1}(h) = (1,0)$.

(ii) For any $h \in H$

$$\begin{aligned} \mu_{\sqrt{\mathbb{G}_1}}(h) &= \mu_{\cap(\wp(\mathbb{G}_1))}(h) \\ &= Inf\{\mu_{\mathbb{G}_2}(h): \mathbb{G}_2 \in IFPI(H); \mathbb{G}_1 \subseteq \mathbb{G}_2\} \\ &\geq \mu_{\mathbb{G}_1}(h). \end{aligned}$$

In the same manner, it can be shown that $\nu_{\sqrt{\mathbb{G}_1}}(h) \leq \nu_{\mathbb{G}_1}(h)$. Thus $\mathbb{G}_1 \subseteq \sqrt{\mathbb{G}_1}$.

(iii) Consider two IFIs $\mathbb{G}_1$ and $\mathbb{G}_2$ in a Γ-Ring H s.t. $\mathbb{G}_1 \subseteq \mathbb{G}_2$. Then $\wp(\mathbb{G}_2) \subseteq \wp(\mathbb{G}_1)$. Thus $\cap\left(\wp(\mathbb{G}_1)\right) \subseteq \cap\left(\wp(\mathbb{G}_2)\right)$, i.e., $\sqrt{\mathbb{G}_1} \subseteq \sqrt{\mathbb{G}_2}$.

(iv) Since $\mathbb{G}_1 \subseteq \sqrt{\mathbb{G}_1}$, it follows that $\sqrt{\mathbb{G}_1} \subseteq \sqrt{\sqrt{\mathbb{G}_1}}$ and $\wp(\mathbb{G}_1) \subseteq \wp\left(\sqrt{\mathbb{G}_1}\right)$. Thus $\cap\left(\wp\left(\sqrt{\mathbb{G}_1}\right)\right) \subseteq \cap\left(\wp(\mathbb{G}_1)\right)$, i.e., $\sqrt{\sqrt{\mathbb{G}_1}} \subseteq \sqrt{\mathbb{G}_1}$. Hence $\sqrt{\sqrt{\mathbb{G}_1}} = \sqrt{\mathbb{G}_1}$.

(v) Since $\mathbb{G}_1 \subseteq \sqrt{\mathbb{G}_1}$ and $\mathbb{G}_2 \subseteq \sqrt{\mathbb{G}_2}$, so $\mathbb{G}_1 \oplus \mathbb{G}_2 \subseteq \sqrt{\mathbb{G}_1} \oplus \sqrt{\mathbb{G}_2}$. Thus $\sqrt{\mathbb{G}_1 \oplus \mathbb{G}_2} \subseteq \sqrt{\sqrt{\mathbb{G}_1} \oplus \sqrt{\mathbb{G}_2}}$.

Again $\mathbb{G}_1 \subseteq \mathbb{G}_1 \oplus \mathbb{G}_2$ and $\mathbb{G}_2 \subseteq \mathbb{G}_1 \oplus \mathbb{G}_2$ so $\sqrt{\mathbb{G}_1} \subseteq \sqrt{\mathbb{G}_1 \oplus \mathbb{G}_2}$ and $\sqrt{\mathbb{G}_2} \subseteq \sqrt{\mathbb{G}_1 \oplus \mathbb{G}_2}$ implies $\sqrt{\mathbb{G}_1} \oplus \sqrt{\mathbb{G}_2} \subseteq \sqrt{\mathbb{G}_1 \oplus \mathbb{G}_2}$. Thus $\sqrt{\sqrt{\mathbb{G}_1} \oplus \sqrt{\mathbb{G}_2}} \subseteq \sqrt{\sqrt{\mathbb{G}_1 \oplus \mathbb{G}_2}} = \sqrt{\mathbb{G}_1 \oplus \mathbb{G}_2}$.

Hence $\sqrt{\mathbb{G}_1 \oplus \mathbb{G}_2} = \sqrt{\sqrt{\mathbb{G}_1} \oplus \sqrt{\mathbb{G}_2}}$.

Proposition 4.2.7. *Let G be an $IFPI$ of a Γ-Ring H. Therefore $\sqrt{G} = G$ and so every $IFPI$ is IFSPI.*

Proof. Assume that G is an $IFPI$ of Γ-Ring H. Therefore $G \in IFPI(H)$.

$\sqrt{G} = \cap\left(\wp(G)\right) = \cap\{B: B \in IFPI(H); G \subseteq B\} \subseteq G$. Again $G \subseteq \sqrt{G}$. So $\sqrt{G} = G$.

The second assertion follows from Proposition (4.2.5).

Lemma 4.2.8. *Consider an IFI G in H s.t. $G(0_H) = (1,0)$, then $\sqrt{G_*} \subseteq \left(\sqrt{G}\right)_*$, where* $\sqrt{G_*} = \cap\{L: L$ is a PI of H s.t. $G_* \subseteq L\}$.

Proof. Let $h \in \sqrt{G_*}$. So $h \in L\ \forall$ PI L of H s.t. $G_* \subseteq L$. Suppose B is an IFPI of H s.t. $G \subseteq B$. Let $r \in G_*$. Then $\mu_G(r) = \mu_G(0_H) = 1 = \mu_B(r)$ and $\nu_G(r) = \nu_G(0_H) = 0 = \nu_B(r)$. So $r \in B_*$. Hence $G_* \subseteq B_*$. As B is an IFPI of H, and B_* is a PI of H (By Theorem (2.2.9)). Also $G_* \subseteq B_*$ so $h \in B_*$. Hence $B(h) = B(0_H) = (1,0)$. Now

$$\begin{aligned}
\mu_{\sqrt{G}}(h) &= \mu_{\cap(\wp(G))}(h) \\
&= Inf\{\mu_B(h): B \in IFPI(H); G \subseteq B\} \\
&= 1 = \mu_{\sqrt{G}}(0_H).
\end{aligned}$$

Similarly, we can prove that $\nu_{\sqrt{G}}(h) = \nu_{\sqrt{G}}(0_H)$. So $h \in \left(\sqrt{G}\right)_*$. Thus $\sqrt{G_*} \subseteq \left(\sqrt{G}\right)_*$.

Lemma 4.2.9. *If G is an IFI of H s.t. $|Img(G)| = 2 = \{(1,0),(\lambda,\zeta)\}$, where $0 \le \lambda, \zeta < 1$ s.t. $\lambda + \zeta \le 1$. Then $\left(\sqrt{G}\right)_* \subseteq \sqrt{G_*}$.*

Proof. Let $h \in \left(\sqrt{G}\right)_*$. Then $\mu_{\sqrt{G}}(h) = \mu_{\sqrt{G}}(0_H) = 1$ and $\nu_{\sqrt{G}}(h) = \nu_{\sqrt{G}}(0_H) = 0$. Therefore, $\sqrt{G}(h) = (1,0)$. This implies that $P(h) = (1,0)$ for all IFPI P with the condition that $G \subseteq P$. Thus $h \in P_*$ whenever $P \in IFPI(H)$, $G \subseteq P$.

Let Ω be a PI of H s.t. $G_* \subseteq \Omega$. Now we define an IFS B of H as

$$\mu_B(\gamma) = \begin{cases} 1, & \text{if } \gamma \in \Omega \\ \lambda_1, & \text{if } \gamma \in H \setminus \Omega \end{cases}; \quad \nu_B(\gamma) = \begin{cases} 0, & \text{if } \gamma \in \Omega \\ \zeta_1, & \text{if } \gamma \in H \setminus \Omega. \end{cases}$$

where $\lambda_1, \zeta_1 \in (0,1)$ such that $\lambda_1 > \lambda$ and $\zeta_1 < \zeta$. Then B is an IFPI of H [by Theorem (2.2.9)] s.t. G is contained in B. Hence $h \in B_* = \Omega$. So $h \in \bigcap\{\Omega\colon \Omega$ is a PI of H s.t. $G_* \subseteq \Omega\}$. Hence h belongs to radical of G_*. Thus we have $\left(\sqrt{G}\right)_* \subseteq \sqrt{G_*}$.

4.3 Intuitionistic Fuzzy Primary Ideal Of A *Γ*-Ring

Definition 4.3.1. Consider G to be any IFI in Γ-Ring H. Then IFS $\sqrt{G}$ which is defined as

$\mu_{\sqrt{G}}(h) = \vee \{\mu_G((h\gamma)^{n-1}h) : n \in \mathbf{N}\}$ and $\nu_{\sqrt{G}}(h) = \wedge \{\nu_G((h\gamma)^{n-1}h) : n \in \mathbf{N})\}$ is called the IFPR of G, where $(h\gamma)^{n-1}h = h$, for $n = 1, \gamma \in \Gamma$.

Proposition 4.3.2. *∀ IFIs ₲ and Ğ of Γ-Ring H, we have*

(i) $₲ \subseteq \sqrt{₲}$*;*

(ii) $₲ \subseteq Ğ \Rightarrow \sqrt{₲} \subseteq \sqrt{Ğ}$*;*

(iii) $\sqrt{\sqrt{₲}} = \sqrt{₲}$.

Proof. Straightforward.

Theorem 4.3.3. *For any IFI G of Γ-Ring H,* $\sqrt{G}$ *is an IFI of H.*

Proof. Let $h_1, h_2 \in H, \gamma \in \Gamma$.

$$
\begin{aligned}
\mu_{\sqrt{G}}(h_1+h_2) &= \vee_{k\geq 1}\left[\mu_G\{\left((h_1+h_2)\gamma\right)^k(h_1+h_2)\}\right] \\
&\geq \mu_G\{\left((h_1+h_2)\gamma\right)^{\mathfrak{m}+\mathfrak{n}}(h_1+h_2)\} \\
&= \mu_G\{(h_1\gamma)^{\mathfrak{m}+\mathfrak{n}}h_1\} \wedge \mu_G\{(h_2\gamma)^{\mathfrak{m}+\mathfrak{n}}h_2\} \wedge_{p+q=\mathfrak{m}+\mathfrak{n}} \mu_G\{(h_1\gamma)^p(h_2\gamma)^q h_1\} \\
&\qquad \wedge_{p+q=\mathfrak{m}+\mathfrak{n}} \mu_G\{(h_2\gamma)^p(h_1\gamma)^q h_2\} \\
&\geq \mu_G\{(h_1\gamma)^{\mathfrak{n}}h_1\} \wedge \mu_G\{(h_2\gamma)^{\mathfrak{n}}h_2\} \\
&= \mu_{\sqrt{G}}(h_1) \wedge \mu_{\sqrt{G}}(h_2).
\end{aligned}
$$

[As $\left((h_1+h_2)\gamma\right)^{\mathfrak{m}+\mathfrak{n}}(h_1+h_2)$ may be seen as the sum of the terms of the forms $(h_1\gamma)^{\mathfrak{m}+\mathfrak{n}}h_1$, $(h_2\gamma)^{\mathfrak{m}+\mathfrak{n}}h_2$, $(h_1\gamma)^p(h_2\gamma)^q h_1$, and $(h_2\gamma)^p(h_1\gamma)^q h_2$, for some $p, q \in \mathbf{N}$ s.t. $p+q=\mathfrak{m}+\mathfrak{n}$.]

In the same manner it can be shown that $\nu_{\sqrt{G}}(h_1+h_2) \leq \nu_{\sqrt{G}}(h_1) \vee \nu_{\sqrt{G}}(h_2)$. Further, since

$$
\begin{aligned}
\mu_G\{(h_1\gamma)^{\mathfrak{n}}h_1\} \vee \mu_G\{(h_2\gamma)^{\mathfrak{n}}h_2\} &\leq \mu_G\{(h_1\gamma)^{\mathfrak{n}}h_1\gamma(h_2\gamma)^{\mathfrak{n}}h_2\} \\
&\leq \vee_{k\geq 1}\left[\mu_G\{(h_1\gamma h_2)^k h_1\gamma h_2\}\right] \\
&= \mu_{\sqrt{G}}(h_1\gamma h_2).
\end{aligned}
$$

Thus $\mu_{\sqrt{G}}(h_1\gamma h_2) \geq \mu_G\{(h_1\gamma)^{\mathfrak{n}}h_1\} \vee \mu_G\{(h_2\gamma)^{\mathfrak{n}}h_2\}$. Similarly, we can show that $\nu_{\sqrt{G}}(h_1\gamma h_2) \leq \nu_G\{(h_1\gamma)^{\mathfrak{n}}h_1\} \wedge \nu_G\{(h_2\gamma)^{\mathfrak{n}}h_2\}$. Hence $\sqrt{G}$ is an IFI of H.

Proposition 4.3.4. *Let $\mathbb{G}_1$, and $\mathbb{G}_2$ be two IFIs of a Γ-Ring H. Then*

$$\sqrt{\mathbb{G}_1\Gamma\mathbb{G}_2} = \sqrt{\mathbb{G}_1 \cap \mathbb{G}_2} = \sqrt{\mathbb{G}_1} \cap \sqrt{\mathbb{G}_2}$$

Proof. Since $\mathbb{G}_1 \cap \mathbb{G}_2 \subseteq \mathbb{G}_1$ and $\mathbb{G}_1 \cap \mathbb{G}_2 \subseteq \mathbb{G}_2$ implies $\sqrt{\mathbb{G}_1 \cap \mathbb{G}_2} \subseteq \sqrt{\mathbb{G}_1}$ and $\sqrt{\mathbb{G}_1 \cap \mathbb{G}_2} \subseteq \sqrt{\mathbb{G}_2}$ and so $\sqrt{\mathbb{G}_1 \cap \mathbb{G}_2} \subseteq \sqrt{\mathbb{G}_1} \cap \sqrt{\mathbb{G}_2}$.

For the reverse inclusion, let $h \in H, \gamma \in \Gamma$ be any element. Now

$$
\begin{aligned}
\mu_{\sqrt{\mathbb{G}_1}\cap\sqrt{\mathbb{G}_2}}(h) &= \mu_{\sqrt{\mathbb{G}_1}}(h) \wedge \mu_{\sqrt{\mathbb{G}_2}}(h) \\
&= \left[\vee \{\mu_{\mathbb{G}_1}((h\gamma)^{\mathfrak{m}}h) : \mathfrak{m} > 0\}\right] \wedge \left[\vee \{\mu_{\mathbb{G}_2}((h\gamma)^{\mathfrak{n}}h) : \mathfrak{n} > 0\}\right] \\
&= \vee \{\mu_{\mathbb{G}_1}((h\gamma)^{\mathfrak{m}}h) \wedge \mu_{\mathbb{G}_2}((h\gamma)^{\mathfrak{n}}) : \mathfrak{m}, \mathfrak{n} > 0\} \\
&\leq \vee \{\mu_{\mathbb{G}_1}((h\gamma)^{\mathfrak{m}+\mathfrak{n}}h) \wedge \mu_{\mathbb{G}_2}((h\gamma)^{\mathfrak{m}+\mathfrak{n}}h) : \mathfrak{m} + \mathfrak{n} > 0\} \\
&= \vee \{\mu_{\mathbb{G}_1\cap\mathbb{G}_2}((h\gamma)^{\mathfrak{m}+\mathfrak{n}}h) : \mathfrak{m} + \mathfrak{n} > 0\} \\
&= \mu_{\sqrt{\mathbb{G}_1\cap\mathbb{G}_2}}(h).
\end{aligned}
$$

Similarly, we can show that $\nu_{\sqrt{\mathbb{G}_1}\cap\sqrt{\mathbb{G}_2}}(h) \geq \nu_{\sqrt{\mathbb{G}_1\cap\mathbb{G}_2}}(h)$. Thus $\sqrt{\mathbb{G}_1} \cap \sqrt{\mathbb{G}_2} \subseteq \sqrt{\mathbb{G}_1 \cap \mathbb{G}_2}$.

Hence $\sqrt{\mathbb{G}_1 \cap \mathbb{G}_2} = \sqrt{\mathbb{G}_1} \cap \sqrt{\mathbb{G}_2}$.

Further, as $\mathbb{G}_1 \Gamma \mathbb{G}_2 \subseteq \mathbb{G}_1 \cap \mathbb{G}_2$ implies $\sqrt{\mathbb{G}_1 \Gamma \mathbb{G}_2} \subseteq \sqrt{\mathbb{G}_1 \cap \mathbb{G}_2}$. For the other inclusion, let $h \in H, \gamma \in \Gamma$ be any element. Now

$$
\begin{aligned}
\mu_{\sqrt{\mathbb{G}_1}\cap\sqrt{\mathbb{G}_2}}(h) &= \mu_{\sqrt{\mathbb{G}_1}}(h) \wedge \mu_{\sqrt{\mathbb{G}_2}}(h) \\
&= \left[\vee \{\mu_{\mathbb{G}_1}((h\gamma)^{\mathfrak{m}}h) : \mathfrak{m} > 0\}\right] \wedge \left[\vee \{\mu_{\mathbb{G}_2}((h\gamma)^{\mathfrak{n}}h) : \mathfrak{n} > 0\}\right] \\
&= \vee \{\mu_{\mathbb{G}_1}((h\gamma)^{\mathfrak{m}}h) \wedge \mu_{\mathbb{G}_2}((h\gamma)^{\mathfrak{n}}) : \mathfrak{m}, \mathfrak{n} > 0\} \\
&\leq \vee \{\mu_{\mathbb{G}_1}((h\gamma)^{\mathfrak{m}+\mathfrak{n}}h) \wedge \mu_{\mathbb{G}_2}((h\gamma)^{\mathfrak{m}+\mathfrak{n}}h) : \mathfrak{m} + \mathfrak{n} > 0\} \\
&= \vee \{\mu_{\mathbb{G}_1\cap\mathbb{G}_2}((h\gamma)^{\mathfrak{m}+\mathfrak{n}}h) : \mathfrak{m} + \mathfrak{n} > 0\} \\
&= \mu_{\sqrt{\mathbb{G}_1\cap\mathbb{G}_2}}(h).
\end{aligned}
$$

In the same manner, it can be shown that $\nu_{\sqrt{\mathbb{G}_1\cap\mathbb{G}_2}}(h) \geq \nu_{\sqrt{\mathbb{G}_1\Gamma\mathbb{G}_2}}(h)$. Thus $\sqrt{\mathbb{G}_1 \cap \mathbb{G}_2} \subseteq \sqrt{\mathbb{G}_1 \Gamma \mathbb{G}_2}$.

Thus $\sqrt{\mathbb{G}_1 \cap \mathbb{G}_2} = \sqrt{\mathbb{G}_1 \Gamma \mathbb{G}_2}$. Hence $\sqrt{\mathbb{G}_1 \Gamma \mathbb{G}_2} = \sqrt{\mathbb{G}_1 \cap \mathbb{G}_2} = \sqrt{\mathbb{G}_1} \cap \sqrt{\mathbb{G}_2}$.

Corollary 4.3.5. *If* $\{\mathbb{G}_i : 1 \leq i \leq \mathfrak{n}\}$ *is a finite number of IFIs of a Γ-Ring H, then*

$\sqrt{\mathbb{G}_1 \Gamma \mathbb{G}_2 \Gamma \mathbb{G}_3 \ldots\ldots \Gamma \mathbb{G}_n} = \sqrt{\mathbb{G}_1 \cap \mathbb{G}_2 \cap \mathbb{G}_3 \cap \ldots\ldots \cap \mathbb{G}_n} = \sqrt{\mathbb{G}_1} \cap \sqrt{\mathbb{G}_2} \cap \sqrt{\mathbb{G}_3} \cap \ldots\ldots \cap \sqrt{\mathbb{G}_n}$.

Definition 4.3.6. A non-constant IFI Q in a Γ-Ring H is called an IFPrI of H if, for any two IFIs $\mathbb{G}$ and $\breve{G}$ of H s.t. $\mathbb{G} \Gamma \breve{G} \subseteq Q \Rightarrow$ either $\mathbb{G} \subseteq Q$ or $\breve{G} \subseteq \sqrt{Q}$.

Theorem 4.3.7. Let $Q \in IFI(H)$. *Then* Q *is an IFPrI of H iff* Q *is non-constant and* $\mathbb{G} \circ \breve{G} \subseteq Q \Rightarrow$ *either* $\mathbb{G} \subseteq Q$ *or* $\breve{G} \subseteq \sqrt{Q}$, *where* $\mathbb{G}, \breve{G} \in IFI(H)$.

Proof. By using Remark (2.2.4) the proof is straightforward, since $\mathbb{G} \circ \breve{G} \subseteq Q$ iff $\mathbb{G}\Gamma\breve{G} \subseteq Q$, where $\mathbb{G}, \breve{G} \in IFI(H)$.

Theorem 4.3.8. *Let Q be an IFI of comm. Γ-Ring H. Then for any two IFPs $h_{(\eta,\theta)}, k_{(\beta,\vartheta)} \in IFP(H)$ the following are equivalent to each other:*

(i) Q is an IFPrI of H

(ii) $h_{(\eta,\theta)}\Gamma k_{(\beta,\vartheta)} \subseteq Q$ implies $h_{(\eta,\theta)} \subseteq Q$ or $k_{(\beta,\vartheta)} \subseteq \sqrt{Q}$.

Proof. (i) $implies$ (ii) Let Q is an IFPrI of H.

Let $h_{(\eta,\theta)}, k_{(\beta,\vartheta)} \in IFP(H)$ s.t. $h_{(\eta,\theta)}\Gamma k_{(\beta,\vartheta)} \subseteq Q$. This implies $(h\Gamma k)_{(\eta\wedge\beta,\theta\vee\vartheta)} \subseteq Q$, i.e., $\mu_Q(h\gamma k) \geq \eta \wedge \beta$ and $\nu_Q(h\gamma k) \leq \vartheta \wedge s$, for every $\gamma \in \Gamma$.

Define two IFSs $\mathbb{G}_1,\ and\ \mathbb{G}_2$ of H as follows

$$\mathbb{G}_1(p) = \begin{cases} (\eta,\theta), & \text{if } p \in < h > \\ (0,1), & \text{otherwise} \end{cases}; \quad \mathbb{G}_2(p) = \begin{cases} (\beta,\vartheta), & \text{if } p \in < k > \\ (0,1), & \text{otherwise} \end{cases}$$

Clearly $\mathbb{G}_1, \mathbb{G}_2$ are IFIs of H and $h_{(\eta,\theta)} \subseteq \mathbb{G}_1$ and $k_{(\beta,\vartheta)} \subseteq \mathbb{G}_2$. Now $\mu_{\mathbb{G}_1\Gamma\mathbb{G}_2}(p) = \vee_{p=u\gamma v}\left[\mu_{\mathbb{G}_1}(u) \wedge \mu_{\mathbb{G}_2}(v)\right] = \eta \wedge \beta$ and $\nu_{\mathbb{G}_1\Gamma\mathbb{G}_2}(p) = \wedge_{p=u\gamma v}\left[\nu_{\mathbb{G}_1}(u) \vee \nu_{\mathbb{G}_2}(v)\right] = \theta \vee \vartheta$, where $u \in < h >$, $v \in < k >$. Thus $\mu_{\mathbb{G}_1\Gamma\mathbb{G}_2}(p) = \eta \wedge \beta \leq \mu_Q(h\gamma k)$ and $\nu_{\mathbb{G}_1\Gamma\mathbb{G}_2}(p) = \theta \vee \vartheta \geq \nu_Q(h\gamma k)$.

Thus when $p = u\gamma v$, where $u \in < h >$, $v \in < k >$. $(\mathbb{G}_1\Gamma\mathbb{G}_2)(p) \subseteq Q(p)$ otherwise $(\mathbb{G}_1\Gamma\mathbb{G}_2)(p) = (0,1)$. Thus get $\mathbb{G}_1\Gamma\mathbb{G}_2 \subseteq Q$. As Q is IFPrI of H, so either $\mathbb{G}_1 \subseteq Q$ or $\mathbb{G}_2 \subseteq \sqrt{Q}$. Thus we have $h_{(\eta,\theta)} \subseteq \mathbb{G}_1 \subseteq Q$ or $k_{(\beta,\vartheta)} \subseteq \mathbb{G}_2 \subseteq \sqrt{Q}$, i.e., $h_{(\eta,\theta)} \subseteq Q$ or $k_{(\beta,\vartheta)} \subseteq \sqrt{Q}$.

$(ii) \Rightarrow (i)$, Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be two IFIs of Γ-Ring H s.t. $\mathbb{G}_1\Gamma\mathbb{G}_2 \subseteq Q$. Suppose that $\mathbb{G}_1 \nsubseteq Q$. Then $\exists h \in H$ s.t. $\mu_{\mathbb{G}_1}(h) > \mu_Q(h)$ and $\nu_{\mathbb{G}_1}(h) < \nu_Q(h)$. Let $\mu_{\mathbb{G}_1}(h) = \mathfrak{m}, \nu_{\mathbb{G}_1}(h) = \mathfrak{n}$.

Let $k \in H$ and $\mu_{\mathbb{G}_2}(k) = \tau, \nu_{\mathbb{G}_2}(k) = \omega$.

If $p = h\gamma k$ for some $\gamma \in \Gamma$, then $\left(h_{(\mathfrak{m},\mathfrak{n})}\Gamma k_{(\tau,\omega)}\right)(p) = (\mathfrak{m} \wedge \tau, \mathfrak{n} \vee \omega)$. Hence

$\mu_Q(p) = \mu_Q(h\gamma k) \geq \mu_{\mathbb{G}_1\Gamma\mathbb{G}_2}(h\gamma k) \geq \left[\mu_{\mathbb{G}_1}(h) \wedge \mu_{\mathbb{G}_2}(k)\right] = \mathfrak{m} \wedge \tau = \mu_{x_{(\mathfrak{m},\mathfrak{n})}\Gamma y_{(\tau,\omega)}}(h\gamma k) = \mu_{h_{(\mathfrak{m},\mathfrak{n})}\Gamma y_{(\tau,\omega)}}(p)$

$\nu_Q(p) = \nu_Q(h\gamma k) \leq \nu_{\mathbb{C}_1 \Gamma \mathbb{C}_2}(h\gamma k) \leq [\nu_{\mathbb{C}_1}(h) \vee \nu_{\mathbb{C}_2}(k)] = \eta \vee \omega = \nu_{h_{(\mathfrak{m},\eta)}\Gamma y_{(\tau,\omega)}}(h\gamma k) = \nu_{h_{(\mathfrak{m},\eta)}\Gamma y_{(\tau,\omega)}}(p)$.

If $\mu_{h_{(\mathfrak{m},\eta)}\Gamma y_{(\tau,\omega)}}(p) = 0, \nu_{h_{(\mathfrak{m},\eta)}\Gamma y_{(\tau,\omega)}}(p) = 1$, then $\mu_Q(p) \geq \mu_{h_{(\mathfrak{m},\eta)}\Gamma y_{(\tau,\omega)}}(p), \nu_Q(p) \leq \nu_{h_{(\mathfrak{m},\eta)}\Gamma k_{(\tau,\omega)}}(p)$. Hence $h_{(\mathfrak{m},\eta)}\Gamma k_{(\tau,\omega)} \subseteq Q$. By (i) either $h_{(\mathfrak{m},\eta)} \subseteq Q$ or $k_{(\tau,\omega)} \subseteq \sqrt{Q}$.

i.e., either $\mu_Q(h) \geq \mathfrak{m}, \nu_Q(h) \leq \eta$ or $\mu_{\sqrt{Q}}(k) \geq \tau, \nu_{\sqrt{Q}}(k) \leq \omega$.

Since $\mathfrak{m} \nleq \mu_Q(h), \eta \ngeq \nu_Q(h)$ implies that $h_{(\mathfrak{m},\eta)} \nsubseteq Q$ and so $k_{(\tau,\omega)} \subseteq \sqrt{Q}$. This implies that $\mu_{\sqrt{Q}}(k) \geq \tau = \mu_{\mathbb{C}_2}(k)$ and $\nu_{\sqrt{Q}}(k) \leq \omega = \nu_{\mathbb{C}_2}(k), \forall k \in H$. Which implies that $\mathbb{C}_2 \subseteq \sqrt{Q}$. Hence Q is an IFPrI of H.

Proposition 4.3.9. *Let Q be an IFI in a Γ-Ring H. If Q is an IFPrI of H, then for all $h_1, h_2 \in H, \gamma \in \Gamma$ such that $\mu_Q(h_1\gamma h_2) > \mu_Q(h_1), \nu_Q(h_1\gamma h_2) < \nu_Q(h_1)$ implies that $\mu_Q(h_1\gamma h_2) < \mu_{\sqrt{Q}}(h_2), \nu_Q(h_1\gamma h_2) > \nu_{\sqrt{Q}}(h_2)$.*

Proof. $\mu_Q(h_1\gamma h_2) = r > \mu_Q(h_1), \nu_Q(h_1\gamma h_2) = s < \nu_Q(h_1)$. Then $(h_1\gamma h_2)_{(r,s)} \in Q$ and $h_{1(r,s)} \notin Q$. Since Q is an IFPrI of H then $h_{2(r,s)} \in \sqrt{Q}$. Thus $\mu_{\sqrt{Q}}(h_2) \geq r = \mu_Q(h_1\gamma h_2)$ and $\nu_{\sqrt{Q}}(h_2) \leq s = \nu_Q(h_1\gamma h_2)$. This completes the proof.

Theorem 4.3.10. *Assume that Q is an IFPrI of Γ-Ring H. Then $Q_* = \{h \in H: \mu_Q(h) = \mu_Q(0_H)$ and $\nu_Q(h) = \nu_Q(0_H)\}$ will be a PrI of H.*

Proof. Suppose $h_1, h_2 \in Q_*$. So $\mu_Q(h_1) = \mu_Q(h_2) = \mu_Q(0_H)$ and $\nu_Q(h_1) = \nu_Q(h_2) = \nu_Q(0_H)$. Now

$\mu_Q(h_1 - h_2) \geq \mu_Q(h_1) \wedge \mu_Q(h_2) = \mu_Q(0_H)$ and $\nu_Q(h_1 - h_2) \leq \nu_Q(h_1) \vee \nu_Q(h_2) = \nu_Q(0_H)$ implies that $\mu_Q(h_1 - h_2) = \mu_Q(0_H)$ and $\nu_Q(h_1 - h_2) = \nu_Q(0_H)$. So $h_1 - h_2 \in Q_*$.

Further, let $h_1 \in H$ and $h_2 \in Q_*$, then $\mu_Q(h_2) = \mu_Q(0_H)$ and $\nu_Q(h_2) = \nu_Q(0_H)$.

Let $\gamma \in \Gamma$ be any element, then $\mu_Q(h_1\gamma h_2) \geq \mu_Q(h_1) \vee \mu_Q(h_2) = \mu_Q(h_1) \vee \mu_Q(0_H) = \mu_Q(0_H)$.

But $\mu_Q(0_H) \geq \mu_Q(h_1\gamma h_2)$ always implies $\mu_Q(h_1\gamma h_2) = \mu_Q(0_H)$. Similarly, $\nu_Q(h_1\gamma h_2) = \nu_Q(0_H)$.

Thus $h_1\gamma h_2 \in Q_*$. This shows that Q_* is the right ideal of Γ-Ring H. In the same manner, it can be shown that Q_* is a left ideal of Γ-Ring H. Thus Q_* is an ideal of Γ-Ring H.
Further, let $h_1, h_2 \in H, \gamma \in \Gamma$ s.t. $h_1\gamma h_2 \in Q_*$,i.e., $\mu_Q(h_1\gamma h_2) = \mu_Q(0_H)$ and $\nu_Q(h_1\gamma h_2) = \nu_Q(0_H)$. Suppose that $h_1 \notin Q_*$,then we claim that $h_2 \in \sqrt{Q_*}$, i.e.,$\exists$ some $m \in \mathbf{N}$ and $\gamma \in \Gamma$ s.t. $(h_2\gamma)^m h_2 \in Q_*$.

As $h_1 \notin Q_* \Rightarrow \mu_Q(h_1) < \mu_Q(0_H)$ and $\nu_Q(h_1) > \mu_Q(0_H)$. Thus we have $\mu_Q(h_1\gamma h_2) > \mu_Q(h_1), \nu_Q(h_1\gamma h_2) < \nu_Q(h_1)$. Then by above proposition (4.3.9) we have $\mu_Q(h_1\gamma h_2) < \mu_{\sqrt{Q}}(h_2), \nu_Q(h_1\gamma h_2) > \nu_{\sqrt{Q}}(h_2)$, i.e., $\mu_{\sqrt{Q}}(h_2) > \mu_Q(0_H), \nu_{\sqrt{Q}}(h_2) < \nu_Q(0_H)$ implies that $\vee\{\mu_Q((h_2\gamma)^m h_2): m > 0\} > \mu_Q(0_H), \wedge\{\nu_Q((h_2\gamma)^m h_2): m > 0\} < \nu_Q(0_H)$.Thus $\exists$ some $m \in \mathbf{N}, \gamma \in \Gamma$ such that $\mu_Q((h_2\gamma)^m h_2) > \mu_Q(0_H)$ and $\nu_Q((h_2\gamma)^m h_2) < \nu_Q(0_H)$, i.e., $\mu_Q((h_2\gamma)^m h_2) = \mu_Q(0_H)$ and $\nu_Q((h_2\gamma)^m h_2) = \nu_Q(0_H)$ and so $(h_2\gamma)^m h_2 \in Q_*$. Thus $h_2 \in \sqrt{Q_*}$. This complete the proof.

Theorem 4.3.11. *Let Q be an IFS of a Γ-Ring H. If $Q(0_H) = (1,0)$, Q_* is a PrI of H and $Img(Q) = \{(1,0),(\lambda,\zeta)\}$, where $\lambda, \zeta \in [0,1)$ s.t. $\lambda + \zeta \leq 1$. Then Q is an IFPrI of H.*

Proof. Q is a non-constant IFI of H as Q_* is an ideal of H. Assume that $\mathbb{G}_1$, $\mathbb{G}_2 \in IFI(H)$ s.t. $\mathbb{G}_1\Gamma\mathbb{G}_2 \subseteq Q$. Suppose $\mathbb{G}_1 \nsubseteq Q$ and $\mathbb{G}_2 \nsubseteq \sqrt{Q}$. Then $\exists$, $h_1, h_2 \in H$ s.t. $\mu_{\mathbb{G}_1}(h_1) > \mu_Q(h_1)$, $\nu_{\mathbb{G}_1}(h_1) < \nu_Q(h_1)$ and $\mu_{\mathbb{G}_2}(h_2) > \mu_{\sqrt{Q}}(h_2)$, $\nu_{\mathbb{G}_2}(h_2) < \nu_{\sqrt{Q}}(h_2)$. Since $Q(0_H) = (1,0) = \sqrt{Q}(0_H)$ gives that $h_1 \notin Q_*$ and $h_2 \notin \left(\sqrt{Q}\right)_*$. Again since $\sqrt{Q_*} \subseteq \left(\sqrt{Q}\right)_*$, so $h_2 \notin \sqrt{Q_*}$.

Therefore $h_1\Gamma H\Gamma h_2 \nsubseteq Q_*$ (by Theorem 9,[8]) as Q_* is a PrI of H.

Therefore $\mu_Q(h_1\gamma_1 m\gamma_2 h_2) = \lambda \neq 1$, $\nu_Q(h_1\gamma_1 m\gamma_2 h_2) = \zeta \neq 0$, for some $\gamma_1, \gamma_2 \in \Gamma$, $m \in H$.

Since $h_1 \notin Q_*$, $\mu_Q(h_1) \neq \mu_Q(0_H) = 1$, $\nu_Q(h_1) \neq \nu_Q(0_H) = 0$. So $\mu_Q(h_1) = \lambda, \nu_Q(h_1) = \zeta$. Thus $\mu_{\mathbb{G}_1}(h_1) > \mu_Q(h_1) = \lambda$, $\nu_{\mathbb{G}_1}(h_1) < \nu_Q(h_1) = \zeta$.

Again since $\mu_Q(h_2) \leq \mu_{\sqrt{Q}}(h_2) < \mu_{\mathbb{G}_2}(h_2)$ and $\nu_Q(h_2) \geq \nu_{\sqrt{Q}}(h_2) > \nu_{\mathbb{G}_2}(h_2)$, $Q(h_2) \neq (1,0)$. So $\lambda = \mu_Q(h_2) < \mu_{\mathbb{G}_2}(h_2)$ and $\zeta = \nu_Q(h_2) > \nu_{\mathbb{G}_2}(h_2)$. Now we will have

$$\begin{aligned} \lambda \quad &= \mu_Q(h_1\gamma_1 m\gamma_2 h_2) \\ &\geq \mu_{\mathbb{G}_1\Gamma\mathbb{G}_2}(h_1\gamma_1 m\gamma_2 h_2) \\ &\geq \mu_{\mathbb{G}_1}(h_1) \wedge \mu_{\mathbb{G}_2}(h_2) \\ &> \lambda. \end{aligned}$$

which is not possible as per our supposition. Therefore Q is an IFPrI of H.

Example 4.3.12. For a PrI W of Γ-Ring H, the IFCF χ_W is an IFPrI of H.

Proof. Here we have

$$\mu_{\chi_W}(h) = \begin{cases} 1, & \text{if } h \in W \\ 0, & \text{otherwise} \end{cases}; \quad \nu_{\chi_W}(h) = \begin{cases} 0, & \text{if } h \in W \\ 1, & \text{otherwise.} \end{cases}$$

Clearly, $\mu_{\chi_W}(0_H) = 1, \nu_{\chi_W}(0_H) = 0$ and $(\chi_W)_* = W$ is a PrI of H. Hence χ_W is an IFPrI of H.

Proposition 4.3.13. *Assume that Q is a non-constant IFPrI in Γ-Ring H. Then there exists an IFPI P of H s.t. $P \in \wp(Q)$.*

Proof. As Q is non-constant, then $\exists$ ɱ $\in H$ s.t. $\mu_Q(ɱ) \neq \mu_Q(0_H)$ and $\nu_Q(ɱ) \neq \nu_Q(0_H)$. Let $\mu_Q(ɱ) < \tau < \mu_Q(0_H)$ and $\nu_Q(ɱ) > \omega > \nu_Q(0_H)$. Then $Q_{(\tau,\omega)} \neq H$, and $Q_{(\tau,\omega)}$ is an ideal of H. So $\exists$ a prime $\ddot{\bar{W}}$ of H s.t. $Q_{(\tau,\omega)} \subset \ddot{\bar{W}} \subset H$.

Let P be an IFS on H which is defined as

$$\mu_P(\ddot{\varpi}) = \begin{cases} 1, & \text{if } \ddot{\varpi} \in \ddot{\bar{W}} \\ \tau, & \text{otherwise} \end{cases}; \quad \nu_P(\ddot{\varpi}) = \begin{cases} 0, & \text{if } \ddot{\varpi} \in \ddot{\bar{W}} \\ \omega, & \text{otherwise.} \end{cases}$$

Then P is an IFPI of H (by Theorem (2.2.9))

Let $\ddot{\varpi} \in H$. Then either $\mu_Q(\ddot{\varpi}) \geq \tau, \nu_Q(\ddot{\varpi}) \leq \omega$ or $\mu_Q(\ddot{\varpi}) > \tau, \nu_Q(\ddot{\varpi}) < \omega$.

In the second case we get $\mu_Q(\ddot{\varpi}) \leq \mu_P(\ddot{\varpi}), \mu_Q(\ddot{\varpi}) \geq \mu_P(\ddot{\varpi})$.

In the first case, we get $\ddot{\varpi} \in Q_{(\tau,\omega)} \subset \ddot{\bar{W}}$, so $\mu_P(\ddot{\varpi}) = 1, \nu_P(\ddot{\varpi}) = 0$. Hence in both cases, we get same result. Thus $Q \subseteq P$. Hence $P \in \wp(Q)$.

Proposition 4.3.14. *Let H be a Γ-Ring and* $\sum_i^n [e_i, \delta_i]$ *,* $e_i \in H, \delta_i \in \Gamma$*, for* $i = 1,2,3 \ldots, \eta$ *be the left unity of H and G be a non-constant IFI of H. Let $r \in H$ be s.t.* $min\{\mu_G(e_i): i = 1,2, \ldots \eta\} < \mu_G(r)$ *and* $max\{\nu_G(e_i): i = 1,2, \ldots \eta\} > \nu_G(r)$*. Then* $\exists$ $e \in \{e_i: i = 1,2, \ldots \eta\}$ *s.t.* $\mu_{\sqrt{G}}(e) < \mu_G(r)$ *and* $\nu_{\sqrt{G}}(e) > \nu_G(r)$*.*

Proof. Let $\mu_G(r) = s_1, \nu_G(r) = s_2$ and $min\{\mu_G(e_i): i = 1,2, \ldots \eta\} = t_1 = \mu_G(e')$, $max\{\nu_G(e_i): i = 1,2, \ldots \eta\} = t_2 = \nu_G(e')$, where $e' \in \{e_i: i = 1,2, \ldots \eta\}$.

Suppose that $r_1, r_2 \in [0,1)$ s.t. $t_1 < r_1 < s_1$ and $t_2 > r_2 > s_2$. Then (r_1, r_2)-cut set $G_{(r_1,r_2)}$ is an ideal of H. Since $e' \notin G_{(r_1,r_2)}$. Let L be a PI of H s.t. $G_{(r_1,r_2)} \subseteq L$, and $L \neq H$.

Let B be an IFS of H which is defined as

$$\mu_B(l) = \begin{cases} 1, & \text{if } l \in L \\ r_1, & \text{if } l \notin L \end{cases}; \quad \nu_B(l) = \begin{cases} 0, & \text{if } l \in L \\ r_2, & \text{if } l \notin L. \end{cases}$$

Then by proposition (4.3.13), we can prove that $B \in \wp(G)$. Now as L is a proper ideal of H, $\exists$ at least one $e \in \{e_i: i = 1,2, \ldots \eta\}$ s.t. $e \notin L$, for if $e_i \in L$ for all $i = 1,2,3, \ldots, \eta$, then $h = \sum_i e_i \delta_i h \; \forall \; h \in H$ that is $L = H$, a contradiction.

Hence $\mu_B(e) = r_1$ and $\nu_B(e) = r_2$. As $B \in \wp(G)$, $\sqrt{G} \subseteq B$,

Now $\mu_{\sqrt{G}}(e) \leq \mu_B(e) = r_1 < \mu_G(r)$ and $\nu_{\sqrt{G}}(e) \geq \nu_B(e) = r_2 > \nu_G(r)$.

This completes the result.

Now we have the converse of Theorem (4.3.11)

Theorem 4.3.15. *Let H be a Γ-Ring and Q be an IFPrI of H. Then* $Q(0_H) = (1,0)$, $|Img(Q)| = 2$*, and Q_* is a PrI of H.*

Proof. Let us assume that $\mu_Q(0_H) = \lambda < 1$ and $\nu_Q(0_H) = \zeta > 0$.

Let $min_i\{\mu_Q(e_i)\} = \alpha < \mu_Q(0_H)$ and $max_i\{\nu_Q(e_i)\} = \beta > \nu_Q(0_H)$. Then $\exists$ $e \in \{e_i: i = 1,2, \ldots \eta\}$ s.t. $\mu_{\sqrt{Q}}(e) = \lambda_1 < \lambda$ and $\nu_{\sqrt{Q}}(e) = \zeta_1 < \zeta$ (by Proposition (4.3.14)).

Let $\lambda < p \leq 1$ and $\zeta > q \geq 0$. Then $\alpha < \lambda_1 < p \leq 1$ and $\beta > \zeta_1 > \zeta \geq 0$.

Let $\mathbb{G}_1$, and $\mathbb{G}_2$ be two IFSs on H defined by

$$\mu_{\mathbb{G}_1}(h) = \begin{cases} p, & \text{if } h \in Q_* \\ \alpha, & \text{if } h \notin Q_* \end{cases}; \quad \nu_{\mathbb{G}_1}(h) = \begin{cases} q, & \text{if } h \in Q_* \\ \beta, & \text{if } h \notin Q_*. \end{cases}$$

and $\mathbb{G}_2(h) = (\lambda, \zeta)$, $\forall\, h \in H$. Then $\mathbb{G}_1$ and $\mathbb{G}_2$ are IFIs of H. Let $h_1 \in H$ be any element.

If $h_1 \in Q_*$, then $Q(h_1) = G_2(h_1) = (\lambda, \zeta)$ and so $\mu_{\mathbb{G}_1 \Gamma \mathbb{G}_2}(h_1) = \vee_{h_1 = h_2 \gamma h_3} \left[\mu_{\mathbb{G}_1}(h_2) \wedge \mu_{\mathbb{G}_2}(h_3)\right] \leq \lambda = \mu_Q(h_1)$ and $\nu_{\mathbb{G}_1 \Gamma \mathbb{G}_2}(h_1) = \wedge_{h_1 = h_2 \gamma h_3} \left[\nu_{\mathbb{G}_1}(h_2) \vee \nu_{\mathbb{G}_2}(h_3)\right] \geq \zeta = \nu_Q(h_1)$.

If $h_1 \notin Q_*$, then $\mathbb{G}_1(h_1) = (\alpha, \beta)$, then $\mu_{\mathbb{G}_1 \Gamma \mathbb{G}_2}(h_1) = \alpha = min_i\{\mu_Q(e_i)\} \leq \mu_Q(h_1)$ and $\nu_{\mathbb{G}_1 \Gamma \mathbb{G}_2}(h_1) = \beta = max_i\{\nu_Q(e_i)\} \geq \nu_Q(h_1)$. So $\mathbb{G}_1 \Gamma \mathbb{G}_2 \subseteq Q$.

Also $\mu_{\mathbb{G}_1}(0_H) = p > \lambda = \mu_Q(0_H)$ and $\nu_{\mathbb{G}_1}(0_H) = q < \zeta = \nu_Q(0_H)$. So $\mathbb{G}_1 \nsubseteq Q$.

Again for some $e \in \{e_i : i = 1,2, \ldots, \eta\}$, $\mu_{G_2}(e) = \lambda > \lambda_1 = \mu_{\sqrt{Q}}(e)$ and $\nu_{\mathbb{G}_2}(e) = \zeta < \zeta_1 = \nu_{\sqrt{Q}}(e)$ implies that $\mathbb{G}_2 \nsubseteq \sqrt{Q}$. This is a contradiction since Q is an IFPrI of H.

Hence $\mu_Q(0_H) = 1$ and

$\nu_Q(0_H) = 0$, i.e., $Q(0_H) = (1,0)$.

Since Q is non-constant, so $|Img(Q)| \geq 2$. Suppose that $|Img(Q)| \geq 3$. Let $min_i\{\mu_Q(e_i)\} = \alpha$ and $max_i\{\nu_Q(e_i)\} = \beta$. Then $\exists\, (\lambda, \zeta) \in Img(Q)$ s.t. $\alpha < \lambda < 1$ and $\beta > \zeta > 0$. Let $r \in H$ be s.t. $\mu_Q(r) = \lambda, \nu_Q(r) = \zeta$. Then $\exists\; e \in \{e_i : i = 1,2, \ldots . \eta\}$ s.t. $\mu_{\sqrt{Q}}(e) < \mu_Q(r), \nu_{\sqrt{Q}}(e) > \nu_Q(r)$.

Let $\mathbb{G}_1,\ and\ \mathbb{G}_2$ be two IFSs of H s.t.

$$\mu_{\mathbb{G}_1}(h) = \begin{cases} 1, & \text{if } h \in Q_{(\lambda,\zeta)} \\ \alpha, & \text{if } h \notin Q_{(\lambda,\zeta)} \end{cases}; \quad \nu_{\mathbb{G}_1}(h) = \begin{cases} 0, & \text{if } h \in Q_{(\lambda,\zeta)} \\ \beta, & \text{if } h \notin Q_{(\lambda,\zeta)}. \end{cases}$$

and $\mathbb{G}_2(h) = (\lambda, \zeta)$, for all $h \in H$. Then $\mathbb{G}_1$ and $\mathbb{G}_2$ are IFIs of H and $\mathbb{G}_1 \Gamma \mathbb{G}_2 \subseteq Q$.

Now $\mu_{\mathbb{G}_1}(r) = 1 > \lambda = \mu_Q(r)$ and $\nu_{\mathbb{G}_1}(r) = 0 < \zeta = \nu_Q(r)$. Thus $\mathbb{G}_1 \nsubseteq Q$. Also for some $e \in \{e_i : i = 1,2, \ldots, \eta\}$ $\mu_{\mathbb{G}_2}(e) = \lambda = \mu_Q(r) > \mu_{\sqrt{Q}}(e)$ and $\nu_{\mathbb{G}_2}(e) = \zeta = \nu_Q(r) < \mu_{\sqrt{Q}}(e)$. Hence $\mathbb{G}_2 \nsubseteq \sqrt{Q}$. Thus we see that $\mathbb{G}_1 \nsubseteq Q$ and $\mathbb{G}_2 \nsubseteq \sqrt{Q}$, which is a contradiction. Hence $|Q(H)| = 2$.

Let $Img(Q) = \{(1,0), (\lambda, \zeta)\}$, where $\lambda, \zeta \in [0,1)$ such that $\lambda + \zeta \leq 1$. Let Ɨ, Ɉ be two ideals of H s.t. ƗΓɈ $\subseteq Q_*$. Let ₲$_1 = \chi_{Ɨ}$, ₲$_2 = \chi_{Ɉ}$. Then ₲$_1 \Gamma$₲$_2 \subseteq Q$. Since Q is IFPrI, either ₲$_1 \subseteq Q$ or ₲$_2 \subseteq \sqrt{Q}$.

If ₲$_1 \subseteq Q$, then Ɨ $\subseteq Q_*$, and if ₲$_2 \subseteq \sqrt{Q}$, then Ɉ $\subseteq \left(\sqrt{Q}\right)_* \subseteq \sqrt{Q_*}$ (by Lemma (4.2.9)). Hence Q_* is PrI of H.

Corollary 4.3.16. *Assume that* Ɨ *is an ideal of the Γ-Ring H s.t.* $\chi_{Ɨ}$ *is an IFPrI of H, then* Ɨ *is a PrI of H.*

Proof. As $\chi_{Ɨ}$ is an IFPrI of H, so Ɨ $= (\chi_{Ɨ})_* = \chi_{Ɨ_*}$ is a PrI of H.

From Theorem (4.3.11) and Theorem (4.3.15) we have

Theorem 4.3.17. *If Q is an IFPrI of a Γ-Ring H, then the following conditions hold:*

(i) $Q(0_H) = (1,0)$,

(ii) Q_* *is a primary ideal of H,*

(iii) $Img(Q) = \{(1,0), (\lambda, \zeta)\}$, *where* $\lambda, \zeta \in [0,1)$ *such that* $\lambda + \zeta \leq 1$.

Example 4.3.18. Consider $H = \Gamma = Z$, the ring of integers. Then H is a Γ-Ring. Let us take IFS Q on H which is defined as

$$\mu_Q(h) = \begin{cases} 1, & \text{if } h \in < p^n > \\ \lambda, & \text{if } h \notin < p^n > \end{cases}; \quad \nu_Q(h) = \begin{cases} 0, & \text{if } h \in < p^n > \\ \zeta, & \text{if } h \notin < p^n >. \end{cases}$$

where p is a prime number and $n > 1$ a positive integer, $\lambda, \zeta \in [0,1)$ s.t. $\lambda + \zeta \leq 1$. So it can be easily verified that Q is an IFPrI of H.

Remark 4.3.19. Every IFPI of a Γ-Ring H is an IFPrI but the converse is not true.

Proof. It follows from definition (4.3.6) and Proposition (4.2.7). For the converse part, the IFS Q as defined in Example (4.3.18) is an IFPrI but it is not an IFPI (as $Q_* =< p^n >$ is not a PI of H).

Theorem 4.3.20. *For an IFPrI Q of a Γ-Ring H $\sqrt{Q}$ will be an IFPI of H.*

Proof. As Q is an IFPrI of H, $Q(0_H) = (1,0)$, Q_* is a PrI of H and $Img(Q) = \{(1,0), (\lambda, \zeta)\}$, where $\lambda, \zeta \in [0,1)$ s.t. $\lambda + \zeta \leq 1$. (by Theorem (4.3.15)). Now support of radical of Q is equals to radical of support of Q is a PI of H and $\sqrt{Q}(h) = (1,0)$ for $h \in Q_*$.

Let G be an IFS of H s.t.

$$\mu_G(h) = \begin{cases} 1, & \text{if } h \in \left(\sqrt{Q}\right)_* \\ \lambda, & \text{if } h \notin \left(\sqrt{Q}\right)_* \end{cases}; \quad \nu_G(h) = \begin{cases} 0, & \text{if } h \in \left(\sqrt{Q}\right)_* \\ \zeta, & \text{if } h \notin \left(\sqrt{Q}\right)_*. \end{cases}$$

Then $G \in \wp(G)$ and $G_* = \left(\sqrt{Q}\right)_* = \sqrt{Q_*}$.

Let $h \notin \left(\sqrt{Q}\right)_*$. Then

$\lambda = \mu_Q(h) \le \mu_{\sqrt{Q}}(h) \le \mu_G(h) = \lambda$ and $\zeta = \nu_Q(h) \ge \nu_{\sqrt{Q}}(h) \ge \nu_G(h) = \zeta$.

Thus $Img\left(\sqrt{Q}\right) = \{(1,0), (\lambda, \zeta)\}$, where $\lambda, \zeta \in [0,1)$ such that $\lambda + \zeta \le 1$, $\sqrt{Q}(0_H) = (1,0)$ and $\left(\sqrt{Q}\right)_*$ is a PI of H. Hence $\sqrt{Q}$ is an IFPI of H (By Theorem (2.2.9)).

4.4 Homomorphic Behaviour Of Intuitionistic Fuzzy Primary Ideals And Intuitionistic Fuzzy Prime Radical Of *Γ*-Ring

Lemma 4.4.1. *If σ be a $Hom_{H_1}^{H_2}$ and G is an σ-invariant IFI of H, then $\sigma(G_*) = \left(\sigma(G)\right)_*$.*

Proof. Clearly, $\mu_{\sigma(G)}\left(0_{H_2}\right) = Sup\{\mu_G(h_1) : \sigma(h_1) = 0_{H_2}\} = Sup\{\mu_G(h_1) : \sigma(h_1) = \sigma\left(0_{H_1}\right)\} = \mu_G\left(0_{H_1}\right)$. Similarly, we can show that $\nu_{\sigma(G)}\left(0_{H_2}\right) = \nu_G\left(0_{H_1}\right)$. Thus $\sigma(G)\left(0_{H_2}\right) = G\left(0_{H_1}\right)$.

Let $h_2 \in \sigma(G_*)$. Then $h_2 = \sigma(h_1)$ for some $h_1 \in G_*$. Hence $G(h_1) = G\left(0_{H_1}\right) = \sigma(G)\left(0_{H_2}\right)$.

$$\begin{aligned} \mu_{\sigma(G)}(h_2) &= Sup\{\mu_G(z) : \sigma(z) = h_2\} = Sup\{\mu_G(z) : \sigma(z) = \sigma(h_1)\} = \mu_G(h_1) \\ &= \mu_{\sigma(G)}\left(0_{H_2}\right). \end{aligned}$$

In the same manner, it can be seen that $\nu_{\sigma(G)}(h_2) = \nu_{\sigma(G)}\left(0_{H_2}\right)$. So $h_2 \in \left(\sigma(G)\right)_*$. Hence $\sigma(G_*) \subseteq \left(\sigma(G)\right)_*$. Again let $\sigma(h_1) \in \left(\sigma(G)\right)_*$.

$\mu_{\sigma(G)}(0_{H_2}) = \mu_{\sigma(G)}(\sigma(h_1)) = Sup\{\mu_G(t): \sigma(t) = \sigma(h_1)\} = \mu_G(h_1)$. In the same manner, it can be shown that $\nu_{\sigma(G)}(0_{H_2}) = \nu_G(h_1)$. So $G(h_1) = (\sigma(G))(0_{H_2}) = G(0_{H_1})$ implies that $h_1 \in G_*$, i.e., $\sigma(h_1) \in \sigma(G_*)$. Thus $(\sigma(G))_* \subseteq \sigma(G_*)$. Hence the result proves.

Lemma 4.4.2. ([34]) *Let σ be a $Hom_{H_1}^{H_2}$. If G is an σ-invariant IFI of H_1, then $\sigma(G)$ is an IFI of H_2.*

Theorem 4.4.3. *Let σ be a $Hom_{H_1}^{H_2}$. If G is an σ-invariant IFPrI of H_1, then $\sigma(G)$ is an IFPrI of H_2.*

Proof. Let G be an σ-invariant IFPrI of H_1. Then $\sigma(G)$ is IFI of H_2 (by Lemma (4.4.2)). Since G is IFPrI of H_1, then $G(0_{H_1}) = (1,0)$, G_* is a PrI of H_1 and $G(H_1) = \{(1,0), (\lambda, \zeta)\}$, where $\lambda, \zeta \in [0,1)$ s.t. $\lambda + \zeta \leq 1$. From the proof of the Lemma (4.4.1), we have $\sigma(G)(0_{H_2}) = G(0_{H_1}) = (1,0)$. Also $(\sigma(G))_* = \sigma(G_*)$ is a PrI of H_2. Now we prove $\sigma(G(H_1)) = \{(1,0), (\lambda, \zeta)\}$ where $\lambda, \zeta \in [0,1)$ s.t. $\lambda + \zeta \leq 1$.

Assume that $h \in H_1$ be s.t. $\mu_G(h) = \lambda, \nu_G(h) = \zeta$. Then $\mu_{\sigma(G)}(\sigma(h)) = Sup\{\mu_G(z): \sigma(z) = \sigma(h)\} = \mu_G(h) = \lambda$ and $\nu_{\sigma(G)}(\sigma(h)) = Inf\{\nu_G(z): \sigma(z) = \sigma(h)\} = \nu_G(h) = \zeta$. As G is σ-invariant also $\sigma(G)(0_{H_2}) = (1,0)$. So $\sigma(G(H_1)) = \{(1,0), (\lambda, \zeta)\}$. By Theorem (4.3.11) it follows that $\sigma(G)$ is an IFPrI of H_1.

Example 4.4.4. Assume that $H = \Gamma = Z$, the ring of integers, and σ be a Γ-homomorphism from H to H defined by $\sigma(h) = 2h$, and let

$$\mu_G(h) = \begin{cases} 1, & \text{if } h \in 3Z \\ 0.2, & \text{if otherwise} \end{cases}; \quad \nu_G(x) = \begin{cases} 0, & \text{if } h \in 3Z \\ 0.7, & \text{otherwise.} \end{cases}$$

be an IFPrI of H. Then

$\sigma(G)(0) = (Sup\{\mu_G(h): \sigma(n) = 0\}, Inf\{\nu_G(h): \sigma(n) = 0\}) = (\mu_G(0), \nu_G(0)) = (1,0)$ and $\sigma(G)(1) = (Sup\{\mu_G(h): \sigma(n) = 1\}, Inf\{\nu_G(h): \sigma(n) = 1\}) = (0,1)$ [As $\sigma^{-1}(1) = \emptyset$]. Similarly, we can find that $\sigma(G)(3) = \sigma(G)(5) = (0,1)$ and $\sigma(G)(2) = \sigma(G)(4) = (0.2, 0.7)$ and so on we get

$$\mu_{\sigma(G)}(h)=\begin{cases}1, & \text{if } h\in 6Z\\ 0.2, & \text{if } h\in 2Z-6Z\\ 0, & \text{if } h\in Z-2Z\end{cases};\quad \nu_{\sigma(G)}(x)=\begin{cases}0, & \text{if } h\in 6Z\\ 0.7, & \text{if } h\in 2Z-6Z\\ 1, & \text{if } h\in Z-2Z,\end{cases}$$

is not an IFPrI of H (As $|Img(G)|=3\neq 2$). This shows that the assumption that σ is an epimorphism in Theorem (4.4.3) cannot be dropped.

Lemma 4.4.5. *Assume that σ be a $Hom_{H_1}^{H_2}$. If B is an IFI of H_2, then $\left(\sigma^{-1}(B)\right)_*=\sigma^{-1}(B_*)$.*

Proof. Assume that $y\in\left(\sigma^{-1}(B)\right)_*\Leftrightarrow\left(\sigma^{-1}(B)\right)(y)=\left(\sigma^{-1}(B)\right)\left(0_{H_1}\right)$

$\Leftrightarrow B\left(\sigma(y)\right)=B\left(\sigma\left(0_{H_1}\right)\right)=B\left(0_{H_2}\right)=(1,0)$

$\Leftrightarrow \sigma(y)\in B_*\Leftrightarrow y\in\sigma^{-1}(B_*)$.

Hence $\left(\sigma^{-1}(B)\right)_*=\sigma^{-1}(B_*)$.

Lemma 4.4.6. ([34,43]) *Assume that σ be a $Hom_{H_1}^{H_2}$. If B is an IFI of H_2, then $\sigma^{-1}(B)$ is an IFI of H_1.*

Theorem 4.4.7. *Let σ be a $Hom_{H_1}^{H_2}$. If B is an IFPrI of H_2, then its inverse image will be an IFPrI of H_1.*

Proof. By lemma (4.4.6) $\sigma^{-1}(B)$ is an IFI of H_1. Also $(\sigma^{-1}(B)\left(0_{H_1}\right)=B\left(\sigma\left(0_{H_1}\right)\right)=B\left(0_{H_2}\right)=(1,0)$. As B is an IFPrI of H_2. Now $B(H_2)=\{(1,0),(\lambda,\zeta)\}$, where $\lambda,\zeta\in[0,1)$ s.t. $\lambda+\zeta\leq 1$. Let $h_2\in H_2$ be s.t. $\mu_B(h_2)=\lambda,\nu_B(h_2)=\zeta$, then $\exists$ $h_1\in H_1$ s.t. $\sigma(h_1)=h_2$. Now $\sigma^{-1}(B)(h_1)=B\left(\sigma(h_1)\right)=(\lambda,\zeta)$. Thus $\sigma^{-1}\left(B(H_1)\right)=\{(1,0),(\lambda,\zeta)\}$, where $\lambda,\zeta\in[0,1)$ s.t. $\lambda+\zeta\leq 1$. Also, by lemma (4.4.5) we have $\left(\sigma^{-1}(B)\right)_*=\sigma^{-1}(B_*)$ is a PrI of H_1. Hence by Theorem (4.3.11), $\sigma^{-1}(B)$ is an IFPrI of H_1.

Theorem 4.4.8. *Let σ be a $Hom_{H_1}^{H_2}$. If G is an IFI of H_1 s.t. G is constant on $Ker\sigma$, then* $\sqrt{\sigma(G)}=\sigma(\sqrt{G})$.

Proof. Clearly, $\sqrt{\sigma(G)}$ and $\sigma\left(\sqrt{G}\right)$ are IFIs of H_2. Let $h_2\in H_2,\gamma\in\Gamma$ be any element, as σ is onto so $\exists$ some $h_1\in H_1$ s.t. $\sigma(h_1)=h_2$. Now $\sigma((h_1\gamma)^r h_1)=(y\gamma)^r h_2$.

$$
\begin{aligned}
\mu_{\sigma(\sqrt{G})}(h_2) &= Sup\{\mu_{\sqrt{G}}(h_1): h_1 \in \sigma^{-1}(h_2)\} \\
&= Sup\{\vee\{\mu_G((h_1\gamma)^r h_1): r > 0\}: h_1 \in \sigma^{-1}(v)\} \\
&= \vee\{Sup\{\mu_G((h_1\gamma)^r h_1): h_1 \in \sigma^{-1}(h_2)\}: r > 0\} \\
&\leq \vee\{Sup\{\mu_G((h_1\gamma)^r h_1): (h_1\gamma)^r h_1 \in \sigma^{-1}((h_2\gamma)^r h_2)\}: r > 0\} \\
&= \vee\{Sup\{\mu_G((h_1\gamma)^r h_1): (h_1\gamma)^r h_1 \in \sigma^{-1}((h_2\gamma)^r h_2)\}: r > 0\} \\
&= \vee\{\mu_{\sigma(G)}((h_2\gamma)^r h_2): r > 0\} \\
&= \mu_{\sqrt{\sigma(G)}}(h_2).
\end{aligned}
$$

In the same manner it can be shown that $\nu_{\sigma(\sqrt{G})}(h_2) \geq \nu_{\sqrt{\sigma(G)}}(h_2)$. Thus $\sigma(\sqrt{G}) \subseteq \sqrt{\sigma(G)}$.

Further, if G is constant on $Ker\sigma$ and $h_{1_0} \in \sigma^{-1}(h_2)$ be a fixed element of H. Then $\mu_G((x\gamma)^r h_1) = \mu_G\left(\left(h_{1_0}\gamma\right)^r h_{1_0}\right)$ and $\nu_G((h_1\gamma)^r h_1) = \nu_G\left(\left(h_{1_0}\gamma\right)^r h_{1_0}\right)$ for all $h_1 \in \sigma^{-1}(y), \gamma \in \Gamma,\ m \in \mathbf{N}$ and $\mu_G(h_1) = \mu_G\left(\left(h_{1_0}\gamma\right)^r h_{1_0}\right)$ and $\nu_G(h_1) = \nu_G\left(\left(h_{1_0}\gamma\right)^r h_{1_0}\right)$ for all $h_1 \in \sigma^{-1}(h_2), \gamma \in \Gamma, m \in \mathbf{N}$. Hence

$$
\begin{aligned}
\mu_{\sqrt{\sigma(G)}}(h_2) &= \vee\{\mu_{\sigma(G)}((h_2\gamma)^r h_2): r > 0\} \\
&= \vee\{Sup\{\mu_G((h_1\gamma)^r h_1): (h_1\gamma)^r h_1 \in \sigma^{-1}((h_2\gamma)^r h_2)\}: r > 0\} \\
&= Sup\{\vee\{\mu_G((h_1\gamma)^r h_1): r > 0\}: (x\gamma)^r x \in \sigma^{-1}((h_2\gamma)^r h_2)\} \\
&\geq Sup\{\vee\{\mu_G\left(\left(h_{1_0}\gamma\right)^r h_{1_0}\right): r > 0\}: h_1 \in \sigma^{-1}(h_2)\} \\
&= Sup\{\vee\{\mu_G((h_1\gamma)^r h_1): r > 0\}: h_1 \in \sigma^{-1}(h_2)\} \\
&= Sup\{\mu_{\sqrt{G}}(h_1): h_1 \in \sigma^{-1}(h_2)\} \\
&= \mu_{\sigma(\sqrt{G})}(h_2) \qquad .
\end{aligned}
$$

Similarly, we can show that $\nu_{\sqrt{\sigma(G)}}(h_2) \leq \nu_{\sigma(\sqrt{G})}(h_2)$. Thus $\sqrt{\sigma(G)} \subseteq \sigma(\sqrt{G})$.

Hence by using above two equations $\sqrt{\sigma(G)} = \sigma(\sqrt{G})$ is proved.

Theorem 4.4.9. *Let σ be a $Hom_{H_1}^{H_2}$. If G is an IFI of H_1, then* "$\sqrt{\sigma^{-1}(G)} = \sigma^{-1}(\sqrt{G})$".

Proof. Clearly, $\sqrt{\sigma^{-1}(G)}$ and $\sigma^{-1}(\sqrt{G})$ are IFIs of H_1. Let $h \in H_1, \gamma \in \Gamma$ be any element, then

$$\begin{aligned}\mu_{\sigma^{-1}(\sqrt{G})}(h) &= \mu_{\sqrt{G}}(\sigma(h)) = \vee \{\mu_G((\sigma(h)\gamma)^r\sigma(h)) : r > 0\} \\ &= \vee \{\mu_G(\sigma((h\gamma)^r h)) : r > 0\} \\ &= \vee \{\mu_{\sigma^{-1}(G)}((h\gamma)^r h) : r > 0\} \\ &= \mu_{\sqrt{\sigma^{-1}(G)}}(h) .\end{aligned}$$

In the same manner, it can be shown that $\nu_{\sigma^{-1}(G)}(h) = \nu_{\sqrt{\sigma^{-1}(G)}}(h)$, $\forall\ h \in H_1, \gamma \in \Gamma$.

Hence $\sqrt{\sigma^{-1}(B)} = \sigma^{-1}(\sqrt{B})$.

4.5 Intuitionistic Fuzzy 2-Absorbing Primary Ideals Of A *Γ*-Ring

The notion of a 2-absorbing ideal, an extension of the PI, was pioneered by Badawi in [6], while the concept of a 2-APrI, a generalization of the PrI, was introduced and analyzed by Badawi in [7]. Presently, research on 2-absorbing ideal theory is rapidly advancing. Elkettani and Kasem [20] have unified the concepts of 2-AIs and 2A-PrI into 2-Aδ-PrI within the realm of Γ-Rings, yielding numerous compelling findings. Yavuza, Onara, and Ersoya in [69, 70] investigated IF2-APrI and IF2-SPrI within commutative rings. In this section, the notion of IF2-APrIs is extended to Γ-Rings.

Definition 4.5.1. For a non-constant IFI Q in a Γ-Ring H to be an IF2 $-$APrI of H the condition is as follows that for any $IFPs\ h_{(\eta,\theta)}, k_{(\beta,\vartheta)}, p_{(\tau,\omega)}$ of H and $\gamma_1, \gamma_2 \in \Gamma$ such that $h_{(\eta,\theta)}\gamma_1 k_{(\beta,\vartheta)}\gamma_2 p_{(\tau,\omega)} \subseteq Q$ implies that

either $h_{(\eta,\theta)}\gamma_1 k_{(\beta,\vartheta)} \subseteq Q$ or $h_{(\eta,\theta)}\gamma_2 p_{(\tau,\omega)} \subseteq \sqrt{Q}$ or $k_{(\beta,\vartheta)}\gamma_2 p_{(\tau,\omega)} \subseteq \sqrt{Q}$.

Proposition 4.5.2. *Every IFPrI in a Γ-Ring H will be an IF2 $-$APrI of H.*

Proof. The proof is straightforward.

Theorem 4.5.3. *Suppose Q is an IFI in a Γ-Ring H. If Q is an IF2 $-$APrI of H then $Q_{(\eta,\theta)}$ is a 2-APrI of Γ-Ring H for all $\eta \in [0, \mu_Q(0)]$, and $\theta \in [\nu_Q(0), 1]$ with $\eta + \theta \leq 1$ and $Q_{(\eta,\theta)} \neq H$.*

Proof. Let Q be an IF2 $-$APrI of H and suppose that $h,k,p \in H, \gamma_1,\gamma_2 \in \Gamma$ are such that $h\gamma_1 k\gamma_2 p \in Q_{(\eta,\theta)}$ for all $\eta \in [0,\mu_Q(0)]$ and $\theta \in [\nu_Q(0),1]$ with $\eta+\theta \leq 1$ and $Q_{(\eta,\theta)} \neq H$. Then

$\mu_Q(h\gamma_1 k\gamma_2 p) \geq \eta$, $\nu_Q(h\gamma_1 k\gamma_2 p) \leq \theta$ implies $\mu_{(h\gamma_1 k\gamma_2 p)_{(\eta,\theta)}}(h\gamma_1 k\gamma_2 p) = \eta \leq \mu_Q(h\gamma_1 k\gamma_2 p)$ and $\nu_{(h\gamma_1 k\gamma_2 p)_{(\eta,\theta)}}(h\gamma_1 k\gamma_2 p) = \theta \geq \nu_Q(h\gamma_1 k\gamma_2 p)$ and so we have

$(h\gamma_1 k\gamma_2 p)_{(\eta,\theta)} \subseteq Q$, i.e., $h_{(\eta,\theta)}\gamma_1 k_{(\eta,\theta)}\gamma_2 p_{(\eta,\theta)} \subseteq Q$. Since Q is an IF2-APrI of Γ-Ring H, we have

$h_{(\eta,\theta)}\gamma_1 k_{(\eta,\theta)} \subseteq Q$ or $h_{(\eta,\theta)}\gamma_2 p_{(\eta,\theta)} \subseteq \sqrt{Q}$ or $k_{(\eta,\theta)}\gamma_2 p_{(\eta,\theta)} \subseteq \sqrt{Q}$.

i.e., $(h\gamma_1 k)_{(\eta,\theta)} \subseteq Q$ or $(h\gamma_2 p)_{(\eta,\theta)} \subseteq \sqrt{Q}$ or $(k\gamma_2 p)_{(\eta,\theta)} \subseteq \sqrt{Q}$.

Thus $h\gamma_1 k \in Q_{(\eta,\theta)}$ or $h\gamma_2 p \in \left(\sqrt{Q}\right)_{(\eta,\theta)} = \sqrt{Q_{(\eta,\theta)}}$ or $k\gamma_2 p \in \sqrt{Q_{(\eta,\theta)}}$.

Therefore $Q_{(\eta,\theta)}$ is a 2 $-$APrI of Γ-Ring H.

The non-validation of the converse of the above-stated theorem is justified with the help of the following example.

Example 4.5.4. Let $H = \mathbb{Z}$ and $\Gamma = 2\mathbb{Z}$, so that H is a Γ-Ring. Define the IFI Q of H by

$$\mu_Q(h) = \begin{cases} 1, & \text{if } h = 0 \\ 1/3, & \text{if } h \in 15\mathbb{Z} - \{0\} \\ 0, & \text{if } h \in \mathbb{Z} - 15\mathbb{Z} \end{cases}; \quad \nu_Q(h) = \begin{cases} 0, & \text{if } h = 0 \\ 1/2, & \text{if } h \in 15\mathbb{Z} - \{0\} \\ 1, & \text{if } h \in \mathbb{Z} - 15\mathbb{Z}. \end{cases}$$

Since $Q_{(0,1)} = \mathbb{Z}$, $Q_{(1/3,1/2)} = 15\mathbb{Z}$, $Q_{(1,0)} = \{0\}$, then we get $Q_{(\eta,\theta)}$ is a 2-APrI of Γ-Ring H. But for $\gamma_1,\gamma_2 \in 2\mathbb{Z}$, we get

$3_{(1/2,1/3)}\gamma_1 5_{(1/2,1/3)}\gamma_2 1_{(1/3,1/2)} = (3\gamma_1 5\gamma_2 1)_{(1/2\wedge 1/2\wedge 1/3, 1/3\vee 1/3\vee 1/2)} = (3\gamma_1 5\gamma_2 1)_{(1/3,1/2)} \subseteq Q$ and $\mu_{3_{(1/2,1/3)}\gamma_1 5_{(1/2,1/3)}}(3\gamma_1 5) = \mu_{(3\gamma_1 5)_{(1/2,1/3)}}(3\gamma_1 5) = 1/2 > 1/3 = \mu_Q(3\gamma_1 5)$.

Similarly, we get $\nu_{3_{(1/2,1/3)}\gamma_1 5_{(1/2,1/3)}}(3\gamma_1 5) < \nu_Q(3\gamma_1 5)$. This implies that

$3_{(1/2,1/3)}\gamma_1 5_{(1/2,1/3)} \nsubseteq Q$.

$\mu_{3_{(1/2,1/3)}\gamma_2 1_{(1/3,1/2)}}(3\gamma_2 1) = \mu_{(3\gamma_2 1)_{(1/3,1/2)}}(3\gamma_2 1) = 1/3 > 0 = \mu_{\sqrt{Q}}(3\gamma_2 1)$.

Similarly, $\nu_{3_{(1/2,1/3)}\gamma_2 1_{(1/3,1/2)}}(3\gamma_2 1) < \nu_{\sqrt{Q}}(3\gamma_2 1)$. This implies $3_{(1/2,1/3)}\gamma_2 1_{(1/3,1/2)} \nsubseteq \sqrt{Q}$. In the same way, we can show that $5_{(1/2,1/3)}\gamma_2 1_{(1/3,1/2)} \nsubseteq \sqrt{Q}$. Thus Q is not an IF2-APrI of Γ-Ring H.

Corollary 4.5.5. *If Q is an IF$2-$APrI of Γ-Ring H, then $Q_* = \{h \in H: \mu_Q(h) = \mu_Q(0)$ and $\nu_Q(h) = \nu_Q(0)\}$ is a 2-APrI of Γ-Ring H.*

Proof. Since Q is a non-constant IFI of Γ-Ring H, then $Q_* \neq H$. The proof is straightforward by using the above theorem.

In the sequel of the paper, for the sake of simplicity, we denote $h^r = h\gamma_1 h\gamma_2 h \ldots\ldots \gamma_{r-1}h$ for some $\gamma_1, \gamma_2, \ldots\ldots, \gamma_{r-1} \in \Gamma$ and for some $r \in \mathbb{Z}^+$.

Theorem 4.5.6. *Suppose $\ddot{\mathbb{W}}$ be a $2-$APrI of Γ-Ring H. Then the IFCF $\chi_{\ddot{\mathbb{W}}}$ w.r.t. $\ddot{\mathbb{W}}$ defined by*

$$\mu_{\chi_{\ddot{\mathbb{W}}}}(h) = \begin{cases} 1, & \text{if } h \in \ddot{\mathbb{W}} \\ 0, & \text{otherwise} \end{cases}; \quad \nu_{\chi_{\ddot{\mathbb{W}}}}(h) = \begin{cases} 0, & \text{if } h \in \ddot{\mathbb{W}} \\ 1, & \text{otherwise} \end{cases}.$$

is an IF$2-$APrI of Γ-Ring H.

Proof. We have $\ddot{\mathbb{W}} \neq H$ and so $Q = \chi_{\ddot{\mathbb{W}}}$ is non-constant because $\ddot{\mathbb{W}}$ is a 2-APrI of H. Assume that $h_{(\eta,\theta)}\gamma_1 k_{(\beta,\vartheta)}\gamma_2 p_{(\tau,\omega)} \subseteq Q$, but $h_{(\eta,\theta)}\gamma_1 k_{(\beta,\vartheta)} \nsubseteq Q$ or $h_{(\eta,\theta)}\gamma_2 p_{(\tau,\omega)} \nsubseteq \sqrt{Q}$ or $k_{(\beta,\vartheta)}\gamma_2 p_{(\tau,\omega)} \nsubseteq \sqrt{Q}$, where $h_{(\eta,\theta)}, k_{(\beta,\vartheta)}, p_{(\tau,\omega)}$ are $IFPs$ of H and $\gamma_1, \gamma_2 \in \Gamma$. Then

$\mu_Q(h\gamma_1 k) < \eta \wedge \beta$, $\nu_Q(h\gamma_1 k) > \theta \vee \vartheta$ and

$\mu_Q\{(h\gamma_2 p)^r\} < \mu_{\sqrt{Q}}(h\gamma_2 p) = \eta \wedge \tau$, $\nu_Q\{(h\gamma_2 p)^r\} > \nu_{\sqrt{Q}}(h\gamma_2 p) = \theta \vee \omega$ and

$\mu_Q\{(k\gamma_2 p)^r\} < \mu_{\sqrt{Q}}(k\gamma_2 p) = \beta \wedge \tau$, $\nu_Q\{(k\gamma_2 p)^r\} > \mu_{\sqrt{Q}}(k\gamma_2 p) = \vartheta \vee \omega$ for all $r \in \mathbb{Z}$.

Hence $\mu_Q(h\gamma_1 k) = 0, \nu_Q(h\gamma_1 k) = 1$ and so $h\gamma_1 k \notin \ddot{\mathbb{W}}$;

$\mu_Q\{(h\gamma_2 p)^r\} = 0, \nu_Q\{(h\gamma_2 p)^r\} = 1$ and so $(h\gamma_2 p)^r \notin Q$ implies that $h\gamma_2 p \notin \sqrt{Q}$;

$\mu_Q\{(k\gamma_2 p)^r\} = 0, \nu_Q\{(k\gamma_2 p)^r\} = 1$ and so $(k\gamma_2 p)^r \notin Q$ implies that $k\gamma_2 p \notin \sqrt{Q}$.

Since $\ddot{\mathbb{W}}$ is a $2-$AI of H, we have $h\gamma_1 k\gamma_2 p \notin \ddot{\mathbb{W}}$ and so $\mu_Q(h\gamma_1 k\gamma_2 p) = 0, \nu_Q(h\gamma_1 k\gamma_2 p) = 1 \; \forall \; h, k, p \in H$ and $\gamma_1, \gamma_2 \in \Gamma$.

By our hypothesis, we have $(h\gamma_1 k\gamma_2 p)_{(\eta\wedge\beta\wedge\tau,\theta\vee\vartheta\vee\omega)} = h_{(\eta,\theta)}\gamma_1 k_{(\beta,\vartheta)}\gamma_2 p_{(\tau,\omega)} \subseteq Q$ and $\eta\wedge\beta\wedge\tau < \mu_Q(h\gamma_1 k\gamma_2 p) = 0$, $\theta\vee\vartheta\vee\omega > \nu_Q(h\gamma_1 k\gamma_2 p) = 1$. Hence $\eta\vee\beta = 0, \theta\vee\vartheta = 1$ or $\eta\vee\tau = 0, \theta\vee\omega = 1$ or $\beta\vee\tau = 0, \vartheta\vee\omega = 1$, which is a contradiction. Hence $h_{(\eta,\theta)}\gamma_1 k_{(\beta,\vartheta)} \subseteq Q$ or $h_{(\eta,\theta)}\gamma_2 p_{(\tau,\omega)} \subseteq \sqrt{Q}$ or $k_{(\beta,\vartheta)}\gamma_2 p_{(\tau,\omega)} \subseteq \sqrt{Q}$ and $Q = \chi_{\bar{W}}$ is an IF2 −APrI of Γ-Ring H.

Theorem 4.5.7. *Every IF2 −AI of Γ-Ring H is an IF2 −APrI of H.*

Proof. The proof is straightforward.

The non-validation of the converse of the above-stated theorem may be seen using the following example.

Example 4.5.8. Let $H = \mathbb{Z}$ and $\Gamma = 5\mathbb{Z}$, so H is a Γ-Ring. Let $Q = \chi_{12\mathbb{Z}}$. Then Q is an IFI of Γ-Ring H. It can be easily verified that Q is an IF2 −APrI of H, but it is not an IF2 −AI of H for $\gamma_1, \gamma_2 \in \Gamma$ such that $2_{(\eta,\theta)}\gamma_1 2_{(\beta,\vartheta)}\gamma_2 3_{(\tau,\omega)} = (2\gamma_1 2\gamma_2 3)_{(\eta\wedge\beta\wedge\tau,\theta\vee\vartheta\vee\omega)} \subseteq Q$ implies $2_{(\eta,\theta)}\gamma_1 2_{((\beta,\vartheta))} = (2\gamma_1 2)_{(\eta\wedge\beta,\theta\vee\vartheta)} \nsubseteq Q$, $2_{(\eta,\theta)}\gamma_2 3_{(\tau,\omega)} = (2\gamma_2 3)_{(\eta\wedge\tau,\theta\vee\omega)} \nsubseteq Q$, $2_{(\beta,\vartheta)}\gamma_2 3_{(\tau,\omega)} = (2\gamma_2 3)_{(\beta\wedge\tau,\vartheta\vee\omega)} \nsubseteq Q$.

Proposition 4.5.9. *$\sqrt{Q}$ will be an IF2 −AI of H if Q is an IF2 −APrI of Γ-Ring H.*

Proof. Suppose that $h_{(\eta,\theta)}\gamma_1 k_{(\beta,\vartheta)}\gamma_2 p_{(\tau,\omega)} \subseteq \sqrt{Q}$ and $h_{(\eta,\theta)}\gamma_1 k_{(\beta,\vartheta)} \nsubseteq \sqrt{Q}$, where $h_{(\eta,\theta)}, k_{(\beta,\vartheta)}, p_{(\tau,\omega)} \in IFPs(H)$ and $\gamma_1, \gamma_2 \in \Gamma$.

Since $h_{(\eta,\theta)}\gamma_1 k_{(\beta,\vartheta)}\gamma_2 p_{(\tau,\omega)} = (h\gamma_1 k\gamma_2 p)_{(\eta\wedge\beta\wedge\tau,\theta\vee\vartheta\vee\omega)} \subseteq \sqrt{Q}$

$\Rightarrow \mu_{\sqrt{Q}}(h\gamma_1 k\gamma_2 p) \geq \eta\wedge\beta\wedge\tau$ and $\nu_{\sqrt{Q}}(h\gamma_1 k\gamma_2 p) \leq \theta\vee\vartheta\vee\omega$

From the definition of $\sqrt{Q}$, we have

$\mu_{\sqrt{Q}}(h\gamma_1 k\gamma_2 p) = Inf\{\mu_Q((h\gamma_1 k\gamma_2 p)^m) : m \in \mathbb{N}\} \geq Inf\{\mu_Q(h^m\gamma_3 k^m\gamma_4 p^m) : m \in \mathbb{N}\} \geq \eta\wedge\beta\wedge\tau$, for some $\gamma_3, \gamma_4 \in \Gamma$. Similarly, we can show that $\nu_{\sqrt{Q}}(h\gamma_1 k\gamma_2 p) \leq \theta\vee\vartheta\vee\omega$.

Then $\exists\ n \in \mathbb{Z}^+$ s.t. for some $\gamma_1', \gamma_2' \in \Gamma$,

$\mu_Q((h\gamma_1 k\gamma_2 p)^n) \geq \mu_Q(h\gamma_1 k\gamma_2 p) \geq \eta\wedge\beta\wedge\tau$ and $\nu_Q((h\gamma_1 k\gamma_2 p)^n) \leq \nu_Q(h\gamma_1 k\gamma_2 p) \leq \theta\vee\vartheta\vee\omega$. This implies that $\left(h_{(\eta,\theta)}\gamma_1 k_{(\beta,\vartheta)}\gamma_2 p_{(\tau,\omega)}\right)^n \in Q$. If $h_{(\eta,\theta)}\gamma_1 k_{(\beta,\vartheta)} \notin \sqrt{Q}$, then for all $n \in \mathbb{Z}^+$ and for some $\gamma \in \Gamma$, we have $\mu_Q\left(h_{(\eta,\theta)}\gamma_1 k_{(\beta,\vartheta)}\right)^n \geq \mu_Q\left(h^n_{(\eta,\theta)}\gamma k^n_{(\beta,\vartheta)}\right)$ and

$\nu_Q\left(h_{(\eta,\theta)}\gamma_1 k_{(\text{ß},\vartheta)}\right)^n \leq \nu_Q\left(h^n_{(\eta,\theta)}\gamma k^n_{(\text{ß},\vartheta)}\right)$ implies that $h_{(\eta,\theta)}\gamma_1 k_{(\text{ß},\vartheta)} \nsubseteq \sqrt{Q}$. Since Q is an IF2 $-$APrI of H, then $h_{(\eta,\theta)}\gamma_2 p_{(\tau,\omega)} \subseteq \sqrt{Q}$ or $k_{(\text{ß},\vartheta)}\gamma_2 p_{(\tau,\omega)} \subseteq \sqrt{Q}$. Hence $\sqrt{Q}$ is an IF2 $-$AI of H.

Definition 4.5.10. Suppose Q is an IF2 $-$APrI of Γ-Ring H and $P = \sqrt{Q}$ which is an IF2 $-$AI of H. Then Q is called an IF$P - 2$ $-$APrI of H.

Theorem 4.5.11. *Assume that $Q_1, Q_2, \ldots\ldots, Q_n$ be IF$P - 2$ $-$APrIs of Γ-Ring H for some IF2 $-$AI P of H. Then $Q = \bigcap_{i=1}^n Q_i$ is an IF$P - 2$ $-$APrI of H.*

Proof. Assume that $h_{(\eta,\theta)}\gamma_1 k_{(\text{ß},\vartheta)}\gamma_2 p_{(\tau,\omega)} \subseteq Q$ and $h_{(\eta,\theta)}\gamma_1 k_{(\text{ß},\vartheta)} \nsubseteq Q$, for any $h_{(\eta,\theta)}, k_{(\text{ß},\vartheta)}, p_{(\tau,\omega)} \in IFP(H)$ and $\gamma_1, \gamma_2 \in \Gamma$. Then $h_{(\eta,\theta)}\gamma_1 k_{(\text{ß},\vartheta)} \nsubseteq Q_j$, for some $j \in \{1,2,\ldots\ldots,n\}$ and $h_{(\eta,\theta)}\gamma_1 k_{(\text{ß},\vartheta)}\gamma_2 p_{(\tau,\omega)} \subseteq Q_j$, for all $j \in \{1,2,\ldots\ldots,n\}$. Since Q_j is an IF$P - 2$ $-$APrIs of H, we have $k_{(\text{ß},\vartheta)}\gamma_2 p_{(\tau,\omega)} \subseteq \sqrt{Q_j} = P = \bigcap_{i=1}^n \sqrt{Q_i} = \sqrt{\bigcap_{i=1}^n Q_i} = \sqrt{Q}$ or $h_{(\eta,\theta)}\gamma_2 p_{(\tau,\omega)} \subseteq \sqrt{Q_j} = P = \bigcap_{i=1}^n \sqrt{Q_i} = \sqrt{\bigcap_{i=1}^n Q_i} = \sqrt{Q}$. Thus Q is an IF$P - 2$ $-$APrIs of *H*.

In the following example, we show that if Q_1 *and* Q_2 are two IF2 $-$APrIs of a Γ-Ring H, then $Q_1 \cap Q_2$ need not be an IF2 $-$APrI of H.

Example 4.5.12. Let $H = \mathbb{Z}$ and $\Gamma = p\mathbb{Z}$, where $p > 5$ is a prime integer. So that H is a Γ-Ring. Take $Q_1 = \chi_{50\mathbb{Z}}, Q_2 = \chi_{75\mathbb{Z}}$. Clearly Q_1 *and* Q_2 are IF2 $-$APrIs of H. But $Q_1 \cap Q_2 = \chi_{150\mathbb{Z}}$ and as such $\sqrt{Q_1 \cap Q_2} = \chi_{30\mathbb{Z}}$, then for $\gamma_1, \gamma_2 \in \Gamma$ s.t. $25_{(\eta,\theta)}\gamma_1 3_{(\text{ß},\vartheta)}\gamma_2 2_{(\tau,\omega)} \subseteq Q_1 \cap Q_2$, but $25_{(\eta,\theta)}\gamma_1 3_{(\text{ß},\vartheta)} \nsubseteq Q_1 \cap Q_2$, $25_{(\eta,\theta)}\gamma_2 2_{(\tau,\omega)} \nsubseteq \sqrt{Q_1 \cap Q_2}$ and $3_{(\text{ß},\vartheta)}\gamma_2 2_{(\tau,\omega)} \nsubseteq \sqrt{Q_1 \cap Q_2}$. Therefore, $Q_1 \cap Q_2$ is not an IF2 $-$APrI of H.

Theorem 4.5.13. *Assume that Q is an IFI of a Γ-Ring H. If $\sqrt{Q}$ is an IFPI of H, then Q is an IF2 $-$APrI of H.*

Proof. Suppose that $h_{(\eta,\theta)}\gamma_1 k_{(\text{ß},\vartheta)}\gamma_2 p_{(\tau,\omega)} \subseteq Q$ and $h_{(\eta,\theta)}\gamma_1 k_{(\text{ß},\vartheta)} \nsubseteq Q$, for any $h_{(\eta,\theta)}, k_{(\text{ß},\vartheta)}, p_{(\tau,\omega)} \in IFP(H)$ and $\gamma_1, \gamma_2 \in \Gamma$.

Since $h_{(\eta,\theta)}\gamma_1 k_{(\text{ß},\vartheta)}\gamma_2 p_{(\tau,\omega)} \in Q$ and H is commutative Γ-Ring, we have

$h_{(\eta,\theta)}\gamma_1 k_{(\beta,\vartheta)}\gamma_2 p_{(\tau,\omega)}\gamma_2 p_{(\tau,\omega)} = (h_{(\eta,\theta)}\gamma_1 p_{(\tau,\omega)})\gamma_2(k_{(\beta,\vartheta)}\gamma_2 p_{(\tau,\omega)}) \subseteq Q \subseteq \sqrt{Q}$. Thus $h_{(\eta,\theta)}\gamma_1 p_{(\tau,\omega)} \subseteq \sqrt{Q}$ or $k_{(\beta,\vartheta)}\gamma_2 p_{(\tau,\omega)} \subseteq \sqrt{Q}$. Since $\sqrt{Q}$ is an IFPI of *H*. Therefore we conclude that Q is an IF2-APrI of *H*.

Theorem 4.5.14. *Let σ be a surjective Γ-$Hom_{H_1}^{H_2}$. If Q is an IF2 $-$APrI of H_1 which is constant on $Ker\sigma$, then $\sigma(Q)$ is an IF2 $-$APrI of H_2.*

Proof. Suppose that $h_{(\eta,\theta)}\gamma_1 k_{(\beta,\vartheta)}\gamma_2 p_{(\tau,\omega)} = (h\gamma_1 k\gamma_2 p)_{(\eta\wedge\beta\wedge\tau,\theta\vee\vartheta\vee\omega)} \subseteq \sigma(Q)$, where $h_{(\eta,\theta)}, k_{(\beta,\vartheta)}, p_{(\tau,\omega)} \in IFP(H_2)$ and $\gamma_1,\gamma_2 \in \Gamma$. Since σ is a surjective Γ-homomorphism, then $\exists\ a,b,c \in H_1$ s.t. $\sigma(a) = h, \sigma(b) = k, \sigma(c) = p$. Thus

$$\begin{aligned}
\mu_{a_{(\eta,\theta)}\gamma_1 b_{(\beta,\vartheta)}\gamma_2 c_{(\tau,\omega)}}(a\gamma_1 b\gamma_2 c) &= \mu_{(a\gamma_1 b\gamma_2 c)_{(\eta\wedge\beta\wedge\tau,\theta\vee\vartheta\vee\omega)}}(a\gamma_1 b\gamma_2 c)\\
&= \eta\wedge\beta\wedge\tau\\
&\le \mu_{\sigma(Q)}(h\gamma_1 k\gamma_2 p))\\
&= \mu_{\sigma(Q)}\big(\sigma(a)\gamma_1\sigma(b)\gamma_2\sigma(c)\big)\\
&= \mu_{\sigma(Q)}\big(\sigma(a\gamma_1 b\gamma_2 c)\big)\\
&= \mu_{\sigma^{-1}(\sigma(Q))}(a\gamma_1 b\gamma_2 c)\big[\text{ As } Q \text{ is constant on } Ker\sigma, \text{ so } \sigma^{-1}\big(\sigma(Q)\big) = Q\big]\\
&= \mu_Q(a\gamma_1 b\gamma_2 c)
\end{aligned}$$

Thus $\mu_{a_{(\eta,\theta)}\gamma_1 b_{(\beta,\vartheta)}\gamma_2 c_{(\tau,\omega)}}(a\gamma_1 b\gamma_2 c) \le \mu_Q(a\gamma_1 b\gamma_2 c)$. Similarly, we can show that $\nu_{a_{(\eta,\theta)}\gamma_1 b_{((\beta,\vartheta))}\gamma_2 c_{(u,v)}}(a\gamma_1 b\gamma_2 c) \ge \nu_Q(a\gamma_1 b\gamma_2 c)$. Then we get $a_{(\eta,\theta)}\gamma_1 b_{(\vartheta,\vartheta)}\gamma_2 c_{(\tau,\omega)} \subseteq Q$.

Since Q is an IF2 $-$APrI of H_1, then

$a_{(\eta,\theta)}\gamma_1 b_{(\beta,\vartheta)} \subseteq Q$ or $a_{(\eta,\theta)}\gamma_2 c_{(\tau,\omega)} \subseteq \sqrt{Q}$ or $b_{(\beta,\vartheta)}\gamma_2 c_{(\tau,\omega)} \subseteq \sqrt{Q}$. Thus

$$\begin{aligned}
\eta\wedge\beta &\le \mu_Q(a\gamma_1 b) = \mu_{\sigma(Q)}\big(\sigma(a\gamma_1 b)\big)\\
&= \mu_{\sigma(Q)}\big(\sigma(a)\gamma_1\sigma(b)\big)\\
&= \mu_{\sigma(Q)}(h\gamma_1 k).
\end{aligned}$$

Similarly, we can show that $\theta\vee\vartheta \ge \mu_{\sigma(Q)}(h\gamma_1 k)$ and so $(h\gamma_1 k)_{(\eta\wedge\beta,\theta\vee\vartheta)} \subseteq \sigma(Q)$.

Thus $h_{(\eta,\theta)}\gamma_1 k_{(\beta,\vartheta)} \subseteq \sigma(Q)$ or

$$\begin{aligned} \eta \wedge \tau &\leq \mu_{\sqrt{Q}}(a\gamma_2 c) = \mu_{\sigma(\sqrt{Q})}\big(\sigma(a\gamma_2 c)\big) \\ &= \mu_{\sigma(\sqrt{Q})}\big(\sigma(a)\gamma_2\sigma(c)\big) \\ &= \mu_{\sigma(\sqrt{Q})}(h\gamma_2 p). \end{aligned}$$

Similarly, we can show that $\theta \vee \omega \geq \nu_{\sigma(\sqrt{Q})}(h\gamma_2 p)$ and so $(h\gamma_2 p)_{(\eta\wedge\tau,\theta\vee\omega)} \subseteq \sigma\big(\sqrt{Q}\big)$. Thus $h_{(\eta,\theta)}\gamma_2 p_{(\tau,\omega)} \subseteq \sigma\big(\sqrt{Q}\big)$ or

$$\begin{aligned} ϐ \wedge \tau &\leq \mu_{\sqrt{Q}}(b\gamma_2 c) = \mu_{\sigma(\sqrt{Q})}\big(\sigma(b\gamma_2 c)\big) \\ &= \mu_{\sigma(\sqrt{Q})}\big(\sigma(b)\gamma_2\sigma(c)\big) \\ &= \mu_{\sigma(\sqrt{Q})}(k\gamma_2 p). \end{aligned}$$

Similarly, we can show that $\vartheta \vee \omega \geq \nu_{\sigma(\sqrt{Q})}(k\gamma_2 p)$ and so $(k\gamma_2 p)_{(ϐ\wedge\tau,\vartheta\vee\omega)} \subseteq \sigma\big(\sqrt{Q}\big)$. Thus $k_{(ϐ,\vartheta)}\gamma_2 p_{(\tau,\omega)} \subseteq \sigma\big(\sqrt{Q}\big)$. Hence $\sigma(Q)$ is an IF2 $-$APrI of H_2.

Corollary 4.5.15. *Let σ be a surjective Γ-$Hom_{H_1}^{H_2}$. If Q is an IF2 $-$APrI of H_1 which is constant on $Ker\sigma$, then $\sigma\big(\sqrt{Q}\big)$ is an IF2 $-$AI of H_2.*

Proof. The result follows from Proposition (4.5.9), Theorem (4.5.14), and Theorem (4.4.9).

Theorem 4.5.16. *Let σ be a Γ-$Hom_{H_1}^{H_2}$. If Q' is an IF2 $-$APrI of H_2, then $\sigma^{-1}(Q')$ is an IF2 $-$APrI of H_1.*

Proof. Suppose that $h_{(\eta,\theta)}\gamma_1 k_{(ϐ,\vartheta)}\gamma_2 p_{(\tau,\omega)} \subseteq \sigma^{-1}(Q')$, where $h_{(\eta,\theta)}, k_{(ϐ,\vartheta)}, p_{(\tau,\omega)} \in IFP(H)$ and $\gamma_1, \gamma_2 \in \Gamma$.

$$\begin{aligned} \eta \wedge ϐ \wedge \tau &\leq \mu_{\sigma^{-1}(Q')}(h\gamma_1 k\gamma_2 p) \\ &= \mu_{T'}\big(\sigma(h\gamma_1 k\gamma_2 p)\big) \\ &= \mu_{Q'}\big(\sigma(h)\gamma_1\sigma(k)\gamma_2\sigma(p)\big) \end{aligned}$$

$\eta \wedge ϐ \wedge \tau \leq \mu_{Q'}\big(\sigma(h)\gamma_1\sigma(k)\gamma_2\sigma(p)\big)$. Similarly, we can show that $\theta \vee \vartheta \vee \omega \geq \nu_{Q'}\big(\sigma(h)\gamma_1\sigma(k)\gamma_2\sigma(p)\big)$. Let $\sigma(h) = a, \sigma(k) = b, \sigma(p) = c$. Hence we have that $\eta \wedge ϐ \wedge \tau \leq \mu_{Q'}(a\gamma_1 b\gamma_2 c)$ and $\theta \vee \vartheta \vee \omega \geq \nu_{Q'}(a\gamma_1 b\gamma_2 c)$ and as such

$a_{(\eta,\theta)}\gamma_1 b_{(\beta,\vartheta)}\gamma_2 c_{(\tau,\omega)} \subseteq Q'$. Since Q' is an IF2 $-$APrI of H_1, then $a_{(\eta,\theta)}\gamma_1 b_{(\beta,\vartheta)} \subseteq Q'$ or $a_{(\eta,\theta)}\gamma_2 c_{(\tau,\omega)} \subseteq \sqrt{Q'}$ or $b_{(\beta,\vartheta)}\gamma_2 c_{(\tau,\omega)} \subseteq \sqrt{Q'}$. If $a_{(\eta,\theta)}\gamma_1 b_{(\beta,\vartheta)} \subseteq Q'$, then

$$\begin{aligned} \eta \wedge \beta \quad &\leq \mu_{Q'}(a\gamma_1 b) = \mu_{Q'}\big(\sigma(h)\gamma_1\sigma(k)\big) \\ &= \mu_{Q'}\big(\sigma(h\gamma_1 k)\big) \\ &= \mu_{\sigma^{-1}(Q')}(h\gamma_1 k). \end{aligned}$$

i.e., $\eta \wedge \beta \leq \mu_{\sigma^{-1}(Q')}(h\gamma_1 k)$. Similarly, we can show that $\theta \vee \vartheta \geq \nu_{\sigma^{-1}(Q')}(h\gamma_1 k)$. Thus we get $h_{(\eta,\theta)}\gamma_1 k_{(\beta,\vartheta)} = (h\gamma_1 k)_{(\eta\wedge\beta,\theta\vee\vartheta)} \subseteq \sigma^{-1}(Q')$. If $a_{(\eta,\theta)}\gamma_2 c_{(\tau,\omega)} \subseteq \sqrt{Q'}$, then

$$\begin{aligned} \eta \wedge \tau \quad &\leq \mu_{\sqrt{Q'}}(a\gamma_2 c) = \mu_{\sqrt{Q'}}\big(\sigma(h)\gamma_2\sigma(p)\big) \\ &= \mu_{\sqrt{Q'}}\big(\sigma(h\gamma_2 p)\big) \\ &= \mu_{\sigma^{-1}\left(\sqrt{Q'}\right)}(h\gamma_2 p). \end{aligned}$$

i.e., $\eta \wedge \tau \leq \mu_{\sigma^{-1}\left(\sqrt{Q'}\right)}(h\gamma_2 p)$. Similarly, we can show that $\theta \vee \omega \geq \nu_{\sigma^{-1}\left(\sqrt{Q'}\right)}(h\gamma_2 p)$. Thus we get $h_{(\eta,\theta)}\gamma_2 p_{(\tau,\omega)} = (h\gamma_2 p)_{(\eta\wedge\tau,\theta\vee\omega)} \subseteq \sigma^{-1}\left(\sqrt{Q'}\right)$. If $b_{(\beta,\vartheta)}\gamma_2 c_{(\tau,\omega)} \subseteq \sqrt{Q'}$, then

$$\begin{aligned} \beta \wedge \tau \quad &\leq \mu_{\sqrt{Q'}}(b\gamma_2 c) = \mu_{\sqrt{Q'}}\big(\sigma(k)\gamma_2\sigma(p)\big) \\ &= \mu_{\sqrt{Q'}}\big(\sigma(k\gamma_2 p)\big) \\ &= \mu_{\sigma^{-1}\left(\sqrt{Q'}\right)}(k\gamma_2 p). \end{aligned}$$

i.e., $\beta \wedge \tau \leq \mu_{\sigma^{-1}\left(\sqrt{Q'}\right)}(k\gamma_2 p)$. Similarly, we can show that $\vartheta \vee \omega \geq \nu_{\sigma^{-1}\left(\sqrt{Q'}\right)}(k\gamma_2 p)$. Thus we get $k_{(\beta,\vartheta)}\gamma_2 p_{(\tau,\omega)} = (k\gamma_2 p)_{(\beta\wedge\tau,\vartheta\vee\omega)} \subseteq \sigma^{-1}\left(\sqrt{Q'}\right)$. Therefore, we see that $\sigma^{-1}(Q')$ is an IF2-APrI of H_1.

Corollary 4.5.17. *Suppose $\sigma: H_1 \to H_2$ be a Γ-homomorphism. If Q' is an IF2 $-$APrI of H_2, then $\sigma^{-1}\left(\sqrt{Q'}\right)$ is an IF2-AI of H_1.*

Proof. The proof of the corollary comes from Proposition (4.5.9), Theorem (4.5.16), and Theorem (4.4.9).

4.6 Conclusion

In this chapter, the foundational concepts of IFPrI and IFPR in Γ-Ring H are thoroughly examined. It has been demonstrated that IFPrI of a Γ-Ring forms a two-valued IFS, with the base set defined as the primary ideal (The base set of IFS Q is defined as the set $\{h \in H: \mu_Q(h) = 1, \nu_Q(h) = 0\}$). The concept of IFPR in Γ-Ring H has been introduced, establishing that the IFPR of an IFPrI yields an IFPI. The homeomorphic characteristics of IFPrI and IFPR in Γ-Ring are investigated. The findings presented in this paper represent a significant advancement beyond classical ring theory within the IF framework. Furthermore, these results not only enhance prior research but also lay the groundwork for more robust future investigations, such as the decomposition of ideals into primary ideals within the IF environment—a generalization akin to prime factorization in number theory.

Chapter 5

Decomposition Of Intuitionistic Fuzzy Primary Ideal Of Γ-Ring

5.1 Introduction

An ideal decomposition in terms of primary ideals serves as a fundamental aspect of ideal theory, providing the algebraic groundwork for breaking down an algebraic variety into its irreducible components. Alternatively, it offers a broader perspective akin to the factorization of an integer into prime powers. An ideal K in a ring H undergoes a primary decomposition if $K = \bigcap_{i=1}^{k} T_i$, where each T_i represents a primary ideal in H. Moreover, if no $T_j \supset \bigcap_{i=1, j\neq i}^{n} T_i, \forall j, 1 \leq j \leq k$, and if the prime ideals $P_i = \sqrt{T_i}$ are all distinct, then the primary decomposition is termed minimal, and the set $Ass(K) = \{P_1, P_2, \ldots\ldots, P_k\}$ is identified as the set of associated prime ideals of K. (For further details, refer to [14, 55]). This chapter delves into the study of IF primary decomposition and minimal IF primary decomposition of an IFI within a Noetherian Γ-Ring

5.2 Intuitionistic Fuzzy Irreducible Ideals

In this section, the irreducibility of an IFI has been studied and some relations between IFPIs, IFIrIs, and IFPrIs has been proved. Firstly it has been proved that every IFI in a Noetherian Γ-ring can be written as a finite intersection of IFIrIs, where the IFI takes only two values.

Definition 5.2.1. Let G be an IFI of a Γ-Ring H. We say that G is an IFIrI if G cannot be expressed as the intersection of two IFIs of H properly containing G; otherwise, G is called reducible.

Thus G is an IFIrI iff, whenever $G = ₲_1 \cap ₲_2$ with $₲_1$, $₲_2$ IFIs of H, then either $G = ₲_1$ or $G = ₲_2$.

Proposition 5.2.2. *Let G be a non-constant IFI of a Γ-Ring H. Then G is an IFIrI of H if and only if the following hold:*

1. *G_* is an IrI of H*
2. $Im(G) = \{(1,0), (\lambda, \zeta)\}$, *where* $\lambda, \zeta \in [0,1)$ *such that* $\lambda + \zeta \leq 1$.
3. *G is of the form*

$$\mu_G(h) = \begin{cases} 1, & if\, h \in G_* \\ \lambda, & if\, h \in H \backslash G_* \end{cases}; \quad \nu_G(h) = \begin{cases} 0, & if\, h \in G_* \\ \zeta, & if\, h \in H \backslash G_*. \end{cases}$$

Proof. Firstly suppose that G is an IFIrI of H. Let $G_* = ł_1 \cap ł_2$ for some ideals $ł_1$, $ł_2$ of H. We have $G_* \subseteq ł_1$ and $G_* \subseteq ł_2$. If possible, let $G_* \neq ł_1$ and $G_* \neq ł_2$.

Then $(ł_1 \backslash G_*) \cap (ł_2 \backslash G_*)$ is empty. Let us define two IFSs $₲_1$ and $₲_2$ as follows:

$$\mu_{₲_1}(h) = \begin{cases} 1, & \text{if } h \in G_* \\ \lambda_1, & \text{if } h \in ł_1 \backslash G_* \\ \lambda_2, & \text{if } h \in H \backslash ł_1 \end{cases}; \quad \nu_{₲_1}(h) = \begin{cases} 0, & \text{if } h \in G_* \\ \zeta_1, & \text{if } h \in ł_1 \backslash G_* \\ \zeta_2, & \text{if } h \in H \backslash ł_1. \end{cases}$$

and

$$\mu_{₲_2}(h) = \begin{cases} 1, & \text{if } h \in G_* \\ \lambda_1, & \text{if } h \in ł_2 \backslash G_* \\ \lambda_2, & \text{if } h \in H \backslash ł_2 \end{cases}; \quad \nu_{₲_2}(h) = \begin{cases} 0, & \text{if } h \in G_* \\ \zeta_1, & \text{if } h \in ł_2 \backslash G_* \\ \zeta_2, & \text{if } h \in H \backslash ł_2. \end{cases}$$

Now, it is a straightforward case study to verify that $₲_1$ and $₲_2$ are IFIs of H and $G = ₲_1 \cap ₲_2$. Though we have $G \neq ₲_1$ and $G \neq ₲_2$. This contradicts the fact that G is an IFIrI of H. Consequently, $G_* = ł_1$ or $G_* = ł_2$, and hence G_* is an irreducible ideal of H.

Next, we show that $(1,0) \in Im(G)$. If possible, suppose that $(1,0) \notin Im(G)$. Then $\mu_G(0) < 1, \nu_G(0) > 0$. Let us define two IFSs $₲_3$ and $₲_4$ as follows:

$$\mu_{₲_3}(h) = \begin{cases} 1, & \text{if } h \in G_* \\ \mu_G(0), & \text{if otherwise} \end{cases}; \quad \nu_{₲_3}(h) = \begin{cases} 0, & \text{if } h \in G_* \\ \nu_G(0), & \text{if otherwise.} \end{cases}$$

and $₲_4(h) = G(0), \forall\, h \in H$. It is easy to verify that $₲_3$ and $₲_4$ are IFIs of H s.t. $G = ₲_3 \cap ₲_4$. But $G \subset ₲_3$ and $G \subset ₲_4$. Thus we arrive at a contradiction since G is an IFIrI of H. Consequently $(1,0) \in Im(G)$.

Further, to show that $|Im(G)| = 2$. It is sufficient to show that the chain of the level-cut set ideals is given by $G_* \subseteq H$. If possible, let the chain of the level-cut set ideals be $G_* \subseteq G_{(\lambda_1,\zeta_1)} \subseteq H$, where $\lambda_1, \zeta_1 \in (0,1)$ with $\lambda_1 + \zeta_1 \leq 1$. Then G is given by

$$\mu_G(h) = \begin{cases} 1, & \text{if } h \in G_* \\ \lambda_1, & \text{if } h \in G_{(\lambda_1,\zeta_1)} \backslash G_* \\ \lambda_2, & \text{if } h \in H \backslash G_{(\lambda_1,\zeta_1)} \end{cases}; \quad \nu_G(h) = \begin{cases} 0, & \text{if } h \in G_* \\ \zeta_1, & \text{if } h \in G_{(\lambda_1,\zeta_1)} \backslash G_* \\ \zeta_2, & \text{if } h \in H \backslash G_{(\lambda_1,\zeta_1)}. \end{cases}$$

where $\lambda_2 < \lambda_1$ and $\zeta_2 > \zeta_1$. Let us construct two IFSs G_5 and G_6 as follows:

$$\mu_{\mathbb{G}_5}(h) = \begin{cases} 1, & \text{if } h \in G_{(\lambda_1,\zeta_1)} \\ \mu_G(h), & \text{if } h \in H \backslash G_{(\lambda_1,\zeta_1)} \end{cases}; \quad \nu_{\mathbb{G}_5}(h) = \begin{cases} 0, & \text{if } h \in G_{(\lambda_1,\zeta_1)} \\ \nu_G(h), & \text{if } h \in H \backslash G_{(\lambda_1,\zeta_1)}. \end{cases}$$

and

$$\mu_{\mathbb{G}_6}(h) = \begin{cases} 1, & \text{if } h \in G_* \\ \lambda_1, & \text{if } h \in G_{(\lambda_1,\zeta_1)} \backslash G_* \\ \lambda_3, & \text{if } h \in H \backslash G_{(\lambda_1,\zeta_1)} \end{cases}; \quad \nu_{\mathbb{G}_6}(h) = \begin{cases} 0, & \text{if } h \in G_* \\ \zeta_1, & \text{if } h \in G_{(\lambda_1,\zeta_1)} \backslash G_* \\ \zeta_3, & \text{if } h \in H \backslash G_{(\lambda_1,\zeta_1)}. \end{cases}$$

where $\lambda_2 < \lambda_3 < \lambda_1$ and $\zeta_2 > \zeta_3 > \zeta_1$. It is a routine case study to check that $\mathbb{G}_5$ and $\mathbb{G}_6$ are IFIs of H and $G = \mathbb{G}_5 \cap \mathbb{G}_6$. But $G \subset \mathbb{G}_5$ and $G \subset \mathbb{G}_6$. It contradicts the fact that G is an IFIrI of H. Consequently the chain of level cut-set ideal is $G_* \subseteq H$ and hence G is given by

$$\mu_G(h) = \begin{cases} 1, & \text{if } h \in G_* \\ \lambda_1, & \text{if } h \in H \backslash G_* \end{cases}; \quad \nu_G(h) = \begin{cases} 0, & \text{if } h \in G_* \\ \zeta_1, & \text{if } h \in H \backslash G_*. \end{cases}$$

Hence $|Im(G)| = 2$.

Conversely, let the conditions hold. Let us consider that G is not an IFIrI of H. Suppose that $G = \mathbb{G}_7 \cap \mathbb{G}_8$ for some IFIs $\mathbb{G}_7$, $\mathbb{G}_8$ of H with $G \subset \mathbb{G}_7$ and $G \subset \mathbb{G}_8$. Then $\exists$ $h, k \in H$ s.t. $\mu_G(h) < \mu_{\mathbb{G}_7}(h), \nu_G(h) > \nu_{\mathbb{G}_7}(h)$ and $\mu_G(k) < \mu_{\mathbb{G}_8}(k), \nu_G(k) > \nu_{\mathbb{G}_8}(k)$. It follows that $h, k \notin G_*$. Now, if $h = k$, then $\mu_G(h) < \mu_{\mathbb{G}_7 \cap \mathbb{G}_8}(h)$ and $\nu_G(h) > \nu_{\mathbb{G}_7 \cap \mathbb{G}_8}(h)$, i.e., $G \subset \mathbb{G}_7 \cap \mathbb{G}_8$, which is a contradiction. So $h \neq k$ implies $G_* \subseteq \langle G_*, h \rangle$ and $G_* \subseteq \langle G_*, k \rangle$. Therefore $G_* \subseteq \langle G_*, h \rangle \cap \langle G_*, k \rangle$. Let $z \in \langle G_*, h \rangle \cap \langle G_*, k \rangle$, then $z = m + r_1\gamma_1 h = n + r_2\gamma_2 k$, for some $m, n \in G_*$, $r_1, r_2 \in H$, $\gamma_1, \gamma_2 \in \Gamma$.

Therefore, $\mu_G(m-n) = \mu_G(-r_1\gamma_1 h + r_2\gamma_2 k) = 1$ and $\nu_G(m-n) = \nu_G(-r_1\gamma_1 h + r_2\gamma_2 k) = 0$ implies that $\mu_{G_7}(-r_1\gamma_1 h + r_2\gamma_2 k) = \mu_{G_8}(-r_1\gamma_1 h + r_2\gamma_2 k) = 1$ and $\nu_{G_7}(-r_1\gamma_1 h + r_2\gamma_2 k) = \nu_{G_8}(-r_1\gamma_1 h + r_2\gamma_2 k) = 0$. This imply $\mu_{G_7}(r_1\gamma_1 h) = \mu_{G_7}(r_2\gamma_2 k), \nu_{G_7}(r_1\gamma_1 h) = \nu_{G_7}(r_2\gamma_2 k)$ and

$\mu_{G_8}(r_1\gamma_1 h) = \mu_{G_8}(r_2\gamma_2 k), \nu_{G_8}(r_1\gamma_1 h) = \nu_{G_8}(r_2\gamma_2 k)$.

But $\mu_{G_7}(r_1\gamma_1 h) \geq \mu_{G_7}(r_1) \vee \mu_{G_7}(h) \geq \mu_{G_7}(h) > \mu_G(h) = \alpha$.

Similarly $\nu_{G_7}(r_1\gamma_1 h) \leq \nu_{G_7}(r_1) \wedge \nu_{G_7}(h) \leq \nu_{G_7}(h) < \nu_G(h) = \beta$. This gives $r_1\gamma_1 h, r_2\gamma_2 k \in G_*$. Hence $z \in G_*$. Thus, we have $G_* = \langle G_*, h\rangle \cap \langle G_*, k\rangle$ with $G_* \subset \langle G_*, h\rangle$ and $G_* \subset \langle G_*, k\rangle$. This implies that G_* is not an IrI of H, which is a contradiction.

Corollary 5.2.3. *Let I_1 be an ideal of Γ-Ring H. Then I_1 is an IrI iff χ_{I_1} is an IFIrI of H.*

Corollary 5.2.4. *If G is an IFPI of Γ-Ring H. Then G is an IFIrI of H.*

Proof. By Theorem (2.2.9) and Proposition (5.2.2) and the fact that every PI in Γ-Ring is an IrI.

Note that the converse of Corollary (5.2.4) may not be true. See the following example:

Example 5.2.5. Consider $H = \Gamma = \mathbb{Z}$ to be the additive group of integers. Then H is a Γ-Ring. Consider the IFI G of H defined by

$$\mu_G(h) = \begin{cases} 1, & \text{if } h \in \langle 4\rangle \\ 0.4, & \text{if otherwise} \end{cases}; \quad \nu_G(h) = \begin{cases} 0, & \text{if } h \in \langle 4\rangle \\ 0.3, & \text{if otherwise.} \end{cases}$$

As in the above example it can be seen with ease that G is an IFIrI of H, but it is not an IFPI of H, as $G_* = \langle 4\rangle$ is not a PI in H.

Corollary 5.2.6. *If G is an IFIrI of a Noetherian Γ-Ring H, then G is an IFPrI in H.*

Proof. From [[68], Lemma(4.2)] we see that every IrI in a Noetherian Γ-Ring is a PrI. Then the result follows by Proposition (5.2.2) and Theorem (4.3.11).

Proposition 5.2.7. *Suppose G be an IFI of a Noetherian Γ-Ring H with $Img(G) = \{(1,0),(\lambda,\zeta)\}$, where $\lambda,\zeta \in [0,1)$ s.t. $\lambda + \zeta \leq 1$. Then G may be seen as a finite intersection of IFIrIs of H.*

Proof. By [[68], "Lemma(4.1)], every ideal in a Noetherian Γ-Ring is a finite intersection of IrIs." Therefore, suppose that $G_* = \bigcap_{i=1}^{\mathfrak{n}} J_i$, J_i be an IrI of H. Define the IFIs $\mathbb{G}_1, \mathbb{G}_2, \ldots, \mathbb{G}_n$ by

$$\mu_{\mathbb{G}_i}(h) = \begin{cases} 1, & \text{if } h \in J_i \\ \lambda, & \text{if } h \notin J_i \end{cases}; \quad \nu_{\mathbb{G}_i}(h) = \begin{cases} 0, & \text{if } h \in J_i \\ \zeta, & \text{if } h \notin J_i. \end{cases}$$

Where $\lambda, \zeta \in [0,1)$ s.t. $\lambda + \zeta \leq 1$. Then by Proposition (5.2.2), $\forall\ i = 1,2,\ldots,\eta$, $\mathbb{G}_i$ is an IFIrI of H and it can be also verified with ease that $G = \bigcap_{i=1}^{\eta} \mathbb{G}_i$.

Proposition 5.2.8. *Suppose G be an IFI of a Noetherian Γ-Ring H with $Img(G) = \{(1,0), (\lambda, \zeta)\}$, where $\lambda, \zeta \in [0,1)$ s.t. $\lambda + \zeta \leq 1$. Then G may be seen as a finite intersection of IFPrIs of H.*

Proof. This follows from Proposition (5.2.7) and Corollary (5.2.6).

5.3 Decomposition Of IFPrI Of *Γ*-Ring

In this section, the decomposability of an IFI in a Noetherian Γ-Ring will be studied, in terms of IFPrIs that the set of their respective IFRIs are independent of the particular decomposition.

To begin this section, we first recall the definition of the residual quotient $(\mathbb{G}_1 : \mathbb{G}_2)$ of an IFI $\mathbb{G}_1$ by an IFS $\mathbb{G}_2$ in a Γ-Ring H.

Definition 5.3.1. For any IFI $\mathbb{G}_1$ of a Γ-Ring H and any IFS $\mathbb{G}_2$ of H, the IF residual quotient of $\mathbb{G}_1$ by $\mathbb{G}_2$ is denoted by $(\mathbb{G}_1 : \mathbb{G}_2)$ and is defined as

$$(\mathbb{G}_1 : \mathbb{G}_2) = \bigcup \{h_{(\eta,\theta)} \in IFP(H) : h_{(\eta,\theta)} \Gamma \mathbb{G}_2 \subseteq \mathbb{G}_1\}$$

For any IFP $h_{(\eta,\theta)}$ of Γ-Ring H,

we use a streamlined notation $\left(G : h_{(\eta,\theta)}\right)$ for $\left(G : \langle h_{(\eta,\theta)} \rangle\right)$, where $\langle h_{(\eta,\theta)} \rangle = \bigcap\{C : C$ is an IFI of H s.t. $h_{(\eta,\theta)} \subseteq C\}$, be an IFI generated by $h_{(\eta,\theta)}$. There is no difficulty in seeing that $\left(G : h_{(\eta,\theta)}\right)$ is an IFI of H and $G \subseteq \left(G : h_{(\eta,\theta)}\right)$.

Proposition 5.3.2 *Let T be an IFP-PrI of Γ-Ring H, where $P = \sqrt{T}$. If $h_{(\eta,\theta)} \in IFP(H)$ be any IFP of H. Then*

(i) If $h_{(\eta,\theta)} \in T$, then $\left(T : h_{(\eta,\theta)}\right) = \chi_H$;

(ii) If $h_{(\eta,\theta)} \notin T$, then $\left(T : h_{(\eta,\theta)}\right)$ is an IFP $-$PrI and $\sqrt{\left(T : h_{(\eta,\theta)}\right)} = P$;

(iii) If $h_{(\eta,\theta)} \notin \sqrt{T}$*, then* $\left(T\colon h_{(\eta,\theta)}\right) = T$.

Proof. Let $h_{(\eta,\theta)} \in IFP(H)$, T be an IFPrI of H such that $P = \sqrt{T}$.

(i) If $h_{(\eta,\theta)} \in T$, then $\left(T\colon h_{(\eta,\theta)}\right) = \cup\{k_{(\beta,\vartheta)} \in IFP(H)\colon k_{(\beta,\vartheta)}\Gamma h_{(\eta,\theta)} \subseteq T\}$.

Now $\left(T\colon h_{(\eta,\theta)}\right) \subseteq \chi_H$ always. For other inclusion.

Let $k_{(\beta,\vartheta)} \in \chi_H$ then $k_{(\beta,\vartheta)}\Gamma h_{(\eta,\theta)} = (k\Gamma h)_{(\beta\wedge\eta,\vartheta\vee\theta)} \subseteq T$. This implies $k_{(\beta,\vartheta)} \in \left(T\colon h_{(\eta,\theta)}\right)$. Thus $\chi_H \subseteq \left(T\colon h_{(\eta,\theta)}\right)$. Hence $\left(T\colon h_{(\eta,\theta)}\right) = \chi_H$.

(ii) Obviously $T \subseteq \left(T\colon h_{(\eta,\theta)}\right)$. Let $k_{(\beta,\vartheta)} \in \left(T\colon h_{(\eta,\theta)}\right)$. So $k_{(\beta,\vartheta)}\Gamma h_{(\eta,\theta)} \subseteq T$. Since $h_{(\eta,\theta)} \notin T$ imply that$k_{(\beta,\vartheta)} \in \sqrt{T} = P$. This means that $T \subseteq \left(T\colon h_{(\eta,\theta)}\right) \subseteq P$ and so $\sqrt{T} \subseteq \sqrt{\left(T\colon h_{(\eta,\theta)}\right)} \subseteq \sqrt{P} = P$. This imply that $\sqrt{\left(T\colon h_{(\eta,\theta)}\right)} = P$.

Now we show that $\left(T\colon h_{(\eta,\theta)}\right)$ is an IFPrI of H. Assume that for any $\gamma_1 \in \Gamma$ such that $a_{(u_1,v_1)}\gamma_1 b_{(u_2,v_2)} \in \left(T\colon h_{(\eta,\theta)}\right)$ and $b_{(u_2,v_2)} \notin \sqrt{\left(T\colon h_{(\eta,\theta)}\right)}$, then $a_{(u_1,v_1)}\gamma_1 b_{(u_2,v_2)}\gamma_2 h_{(\eta,\theta)} \in T$, i.e., $\left(a_{(u_1,v_1)}\gamma_1 h_{(\eta,\theta)}\right)\gamma_2 b_{(u_2,v_2)} \in T$ and T is IFP −PrI of H, This implies that either $a_{(u_1,v_1)}\gamma_1 h_{(\eta,\theta)} \in T$ or $b_{(u_2,v_2)} \in \sqrt{T} = P = \sqrt{\left(T\colon h_{(\eta,\theta)}\right)}$. This imply $a_{(u_1,v_1)}\gamma_1 h_{(\eta,\theta)} \in T$. Thus $a_{(u_1,v_1)} \in \left(T\colon h_{(\eta,\theta)}\right)$. Hence $\left(T\colon h_{(\eta,\theta)}\right)$ is an IFPrI of H.

(iii) Since $T \supseteq h_{(\eta,\theta)} \cap T \supseteq h_{(\eta,\theta)}\Gamma T$, i.e., $h_{(\eta,\theta)}\Gamma T \subseteq T$. Therefore by the properties of the residual quotient, we have $T \subseteq \left(T\colon h_{(\eta,\theta)}\right)$. Further, $h_{(\eta,\theta)}\Gamma\left(T\colon h_{(\eta,\theta)}\right) \subseteq T$. Is T is an IFPrI of H and $h_{(\eta,\theta)} \notin \sqrt{T}$ implies that $\left(T\colon h_{(\eta,\theta)}\right) \subseteq T$. Hence $\left(T\colon h_{(\eta,\theta)}\right) = T$.

Proposition 5.3.3. *If* $T_1, T_2, \ldots\ldots, T_n$ *be IFIs of* Γ*-Ring* H *and* $h_{(\eta,\theta)} \in IFP(H)$*, then* $\left(\cap_{i=1}^{n} T_i\colon h_{(\eta,\theta)}\right) = \cap_{i=1}^{n}\left(T_i\colon h_{(\eta,\theta)}\right)$.

Proof. Now $k_{(\beta,\vartheta)} \in \left(\cap_{i=1}^{n} T_i\colon h_{(\eta,\theta)}\right)$

$\Leftrightarrow k_{(\beta,\vartheta)}\Gamma h_{(\eta,\theta)} \subseteq \cap_{i=1}^{n} T_i$

$\Leftrightarrow k_{(\beta,\vartheta)}\Gamma h_{(\eta,\theta)} \subseteq T_i, \forall i = 1,2, \dots, n$

$\Leftrightarrow k_{(\beta,\vartheta)} \in \left(T_i: h_{(\eta,\theta)}\right), \forall\ i = 1,2, \dots, n$

$\Leftrightarrow k_{(\beta,\vartheta)} \in \bigcap_{i=1}^{n}\left(T_i: h_{(\eta,\theta)}\right).$

Hence $\left(\bigcap_{i=1}^{n} T_i: h_{(\eta,\theta)}\right) = \bigcap_{i=1}^{n}\left(T_i: h_{(\eta,\theta)}\right).$

In the following example, we show that if $T_1\ and\ T_2$ are two IFPrIs of a Γ-Ring H, then $T_1 \cap T_2$ need not be an IFPrI of H.

Example 5.3.4. Suppose $H = \Gamma = \mathbb{Z}$, be the additive group of integers. Then H is a Γ-Ring. Let $I_1 = 2\mathbb{Z}, I_2 = 3\mathbb{Z}$. Clearly, $I_1\ and\ I_2$ are primary (in fact prime) ideal in H. Define $T_1 = \chi_{I_1}$, $T_2 = \chi_{I_2}$. Then by Example (4.3.12), $T_1,\ and\ T_2$ are IFPrIs of H. Also, $T_1 \cap T_2 = \chi_{I_1 \cap I_2} = \chi_{6\mathbb{Z}}$, which is not an IFPrI of H (by Example (4.3.12)).

Theorem 5.3.5. *Let $T_1, T_2, \dots\dots, T_n$ be IFP $-$PrIs of Γ-Ring H with $P = \sqrt{T_i}, \forall i = 1,2, \dots, n$, an IFPI of H. Then $T = \bigcap_{i=1}^{n} T_i$ is an IFP $-$PrI of H.*

Proof. Let $h_{(\eta,\theta)}, k_{(\beta,\vartheta)} \in IFP(H)$ be s.t. $h_{(\eta,\theta)}\Gamma k_{(\beta,\vartheta)} \subseteq T = \bigcap_{i=1}^{n} T_i$ and $h_{(\eta,\theta)} \notin T$. Then $h_{(\eta,\theta)} \notin T_j$, for few $j \in \{1,2, \dots\dots, n\}$ also $h_{(\eta,\theta)}\Gamma k_{(\beta,\vartheta)} \subseteq T_j$, $\forall\ j \in \{1,2, \dots\dots, n\}$.

Since each T_j is an IFP $-$PrI of H, we have

$k_{(\beta,\vartheta)} \in \sqrt{T_j} = P = \bigcap_{i=1}^{n}\sqrt{T_i} = \sqrt{\bigcap_{i=1}^{n} T_i} = \sqrt{T}.$

Hence T is an IFP $-$PrIs of H.

Definition 5.3.6. A primary decomposition of an IFI G in a Γ-Ring H is an expression of G as a finite intersection of IFPrIs T_i, say $G = \bigcap_{i=1}^{n} T_i$.

Definition 5.3.7. In IF primary decomposition of an IFI $G = \bigcap_{i=1}^{n} T_i$ of Γ-Ring H is called as minimal if:

1. all IFPrI T_i have distinct $\sqrt{T_i}$;
2. $\bigcap_{j \neq i=1}^{n} T_j \nsubseteq T_i$.

Remark 5.3.8. If IF primary decomposition $G = \bigcap_{i=1}^{n} T_i$ is not minimal, that is if $\sqrt{T_j} = \sqrt{T_k} = P$ for $j \neq k$, then we may achieve (1) of definition (5.3.7) by replacing T_j and T_k by $T' = T_j \cap T_k$ which is an IFP $-$PrI of H by Theorem (5.3.5). Repeating this process, we get will arrive at an IF primary decomposition in which all $\sqrt{T_i}$ are distinct. If

$\bigcap_{j\neq i=1}^{n} T_j \subseteq T_i$, we may simply omit T_i. Repeating this process, we will achieve (2) of definition (5.3.7).

Lemma 5.3.9. *Let $\mathbb{G}_1, \mathbb{G}_2, \dots, \mathbb{G}_n$ be IFIs of Γ-Ring H and let P be an IFPI of H. Then*

1. If $\bigcap_{i=1}^{n} \mathbb{G}_i \subseteq P$, then $\mathbb{G}_i \subseteq P$ for some i;

2. If $\bigcap_{i=1}^{n} \mathbb{G}_i = P$, then $\mathbb{G}_i = P$ for some i.

Proof. **(1)** Suppose $\mathbb{G}_i \nsubseteq P$ for all i. Then $\exists^s$, $(h_i)_{(p_i,q_i)} \in \mathbb{G}_i$ s.t. $(h_i)_{(p_i,q_i)} \notin P$ for $1 \leq i \leq n$. Therefore $(h_1)_{(p_1,q_1)}\Gamma(h_2)_{(p_2,q_2)}\Gamma \dots \Gamma(h_n)_{(p_n,q_n)} \subseteq \mathbb{G}_1\Gamma\mathbb{G}_2\Gamma \dots \Gamma\mathbb{G}_n \subseteq \bigcap_{i=1}^{n} \mathbb{G}_i \subseteq P$.

But, since P is an IFPI and $\mathbb{G}_1\Gamma\mathbb{G}_2\Gamma \dots \Gamma\mathbb{G}_n \subseteq P$, then $\mathbb{G}_i \subseteq P$ for some i.

(2) If $P = \bigcap_{i=1}^{n} \mathbb{G}_i$, then $P \subseteq \mathbb{G}_i$ for some i, and from part (1), $\mathbb{G}_i \subseteq P$ for some i. Hence $P = G_i$, for some i.

Definition 5.3.10. An IFPI P in a Γ-Ring H is called an IF-associated prime ideal of an IFI G if $P = \sqrt{(G: h_{(\eta,\theta)})}$ for some $h_{(\eta,\theta)} \in IFP(H)$.

Moreover, for an IFI G of a Γ-Ring H. We define $IF - ASS(G)$ to be the set of all IFPIs associated with the IFI G, i.e.,

$IF - ASS(G) = \{\sqrt{(G: h_{(\eta,\theta)})}: \sqrt{(G: h_{(\eta,\theta)})}$ is an IFPI of H, $h_{(\eta,\theta)} \in IFP(H)\}$.

Theorem 5.3.11. *For IFI G of a Noetherian Γ-Ring H Let $G = \bigcap_{i=1}^{n} T_i$, be a minimal IF primary decomposition of G. Let $P_i = \sqrt{T_i}$, $1 \leq i \leq n$. Then $IF - ASS(G) = \{P_i, i = 1,2, \dots, n\}$ and these, are independent of the particular decomposition.*

Proof. Let $G = \bigcap_{i=1}^{n} T_i$ with $P_i = \sqrt{T_i}$, $1 \leq i \leq n$ be the minimal IF primary decomposition of G. Consider any $h_{(\eta,\theta)} \in IFP(H)$, we have

$(G: h_{(\eta,\theta)}) = (\bigcap_{i=1}^{n} T_i: h_{(\eta,\theta)}) = \bigcap_{i=1}^{n} (T_i: h_{(\eta,\theta)})$. Hence $\sqrt{(G: h_{(\eta,\theta)})} = \bigcap_{i=1}^{n} \sqrt{(T_i: h_{(\eta,\theta)})}$.

Also, by Proposition (5.3.2), if $h_{(\eta,\theta)} \in T_j$ then $\sqrt{\left(T_j : h_{(\eta,\theta)}\right)} = \chi_H$ and if, $h_{(\eta,\theta)} \notin T_j$, then $\sqrt{\left(T_j : h_{(\eta,\theta)}\right)} = P_j$, be an IFPI of H. So

$$\sqrt{\left(G : h_{(\eta,\theta)}\right)} = \cap_{i=1}^{n} \sqrt{\left(T_i : h_{(\eta,\theta)}\right)} = \cap_{h_{(\eta,\theta)} \notin T_j} P_j$$

.

Now, suppose that $P \in IF - ASS(G)$, then $P = \sqrt{\left(G : h_{(\eta,\theta)}\right)}$ be an IFPI of H, for some $h_{(\eta,\theta)} \in IFP(H)$.

Since $\sqrt{\left(G : h_{(\eta,\theta)}\right)} = \cap_{h_{(\eta,\theta)} \notin T_j} P_j$, then by Lemma (5.3.9)(2) we have $\sqrt{\left(G : h_{(\eta,\theta)}\right)} = P_j$ for some j. So, $P \in \{P_i, i = 1,2,\dots,n\}$. Therefore, $IF - ASS(G) \subseteq \{P_i, i = 1,2,\dots,n\}$.

Conversely, as the decomposition is minimal so $\cap_{j \neq i=1}^{n} T_j \nsubseteq T_i$. Then $\forall\ i \in \{1,2,\dots,n\}$, $\exists$ $(h_i)_{(\eta_i,\theta_i)} \in \cap_{j \neq i=1}^{n} T_j$ and $(h_i)_{(\eta_i,\theta_i)} \notin T_i$, we have

$$\sqrt{(G : (h_i)_{(\eta_i,\theta_i)})} = \cap_{j=1}^{n} \sqrt{(T_j : \left(h_j\right)_{(\eta_j,\theta_j)})} = P_i$$

(Since all other's $\sqrt{\left(T_j : \left(h_j\right)_{(\eta_j,\theta_j)}\right)} = \chi_H$, for $j \neq i$ by Proposition (5.3.2)).

So, $P_i \in IF - ASS(G)$. Therefore, $\{P_i, i = 1,2,\dots,n\} \subseteq IF - ASS(G)$.

Hence, $IF - ASS(G) = \{P_i, i = 1,2,\dots,n\}$. Thus $IF - ASS(G)$ are independent of the particular decomposition.

Example 5.3.12. Let $H = \Gamma = Z_{p_1^{n_1}} \times Z_{p_2^{n_2}} \times \dots\dots \times Z_{p_k^{n_k}}$ be a comm. ring of order $n = p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$, where p_i are distinct primes. Then H is a Γ-Ring. Let $H = \langle h_1, h_2, \dots., h_k \rangle$ such that $o(h_i) = p_i^{n_i}$, for $1 \leq i \leq k$. Let $U_0 = \langle 0 \rangle$, $U_1 = \langle h_1 \rangle$, $U_2 = \langle h_1, h_2 \rangle$,........,$U_k = \langle h_1, h_2, \dots\dots\dots, h_k \rangle = H$ be the chain of ideals of H such that $U_0 \subset U_1 \subset \dots\dots\dots \subset U_{k-1} \subset U_k$.

Let G be any IFI of H defined by

$$\mu_G(h) = \begin{cases} 1 & \text{if } h \in U_0 \\ \alpha_1 & \text{if } h \in U_1 \setminus U_0 \\ \alpha_2 & \text{if } h \in U_2 \setminus U_1 \\ \ldots\ldots\ldots.. & \\ \alpha_k & \text{if } h \in U_k \setminus U_{k-1} \end{cases} ; \quad \nu_G(h) = \begin{cases} 0 & \text{if } h \in U_0 \\ \beta_1 & \text{if } h \in U_1 \setminus U_0 \\ \beta_2 & \text{if } h \in U_2 \setminus U_1 \\ \ldots\ldots\ldots\ldots & \\ \beta_k & \text{if } h \in U_k \setminus U_{k-1}. \end{cases}$$

where $1 = \alpha_0 \geq \alpha_1 \geq \ldots\ldots \geq \alpha_k$ and $0 = \beta_0 \leq \beta_1 \leq \ldots. \leq \beta_k$ and the pair (α_i, β_i) are called double pins, and the set $\wedge(G) = \{(\alpha_0, \beta_0), (\alpha_1, \beta_1), \ldots\ldots, (\alpha_k, \beta_k)\}$ is called the set of double pinned flags for the IFI G of H (by Theorem (2.2.15)).

Define IFSs $\mathbb{G}_i$ on H as follows:

$$\mu_{\mathbb{G}_i}(h) = \begin{cases} 1, & \text{if } h \in H_i \\ \alpha_{i+1}, & \text{if otherwise} \end{cases} ; \quad \nu_{\mathbb{G}_i}(x) = \begin{cases} 0, & \text{if } h \in H_i \\ \beta_{i+1}, & \text{otherwise.} \end{cases}$$

where $\alpha_i, \beta_i \in (0,1)$ s.t. $\alpha_i + \beta_i \leq 1$, for $1 \leq i \leq k$ and $\alpha_{k+1} = \alpha_1$, $\beta_{k+1} = \beta_1$ and $H_i = Z_{p_1^{n_1}} \times \ldots.. \times Z_{p_{i-1}^{n_{i-1}}} \times \langle 0 \rangle \times Z_{p_{i+1}^{n_{i+1}}} \times \ldots\ldots. \times Z_{p_k^{n_k}}$ is a PrI of H. $\mathbb{G}_i$ are IFPrI of H. It can be easily checked that $G = \cap_{i=1}^{n} \mathbb{G}_i$ is an IF primary decomposition of G.

Example 5.3.13. Consider $H = \Gamma = \prod_{i=1}^{\infty} \mathbf{Z_2}$, a direct product of infinitely many copies of the field $\mathbf{Z_2} = \{\overleftarrow{0}, \overleftarrow{1}\}$ be a boolean ring. Then H is a Γ-Ring, which is not a Noetherian ring, as the strictly ascending chain of ideals $\mathbf{0} \subset \mathbf{Z_2} \times \mathbf{0} \subset \mathbf{Z_2} \times \mathbf{Z_2} \times \mathbf{0} \subset \ldots..$ is not stationary.

For every $\lambda_i, \zeta_i \in [0,1)$ such that $\lambda_i + \zeta_i \leq 1$, define $G_i \in IFS(H)$ as

$$\mu_{\mathbb{G}_i}(h) = \begin{cases} 1, & \text{if } h = \prod_{i=1}^{\infty} \overleftarrow{0} \\ \lambda_i, & \text{if otherwise} \end{cases} ; \quad \nu_{\mathbb{G}_i}(h) = \begin{cases} 0, & \text{if } h = \prod_{i=1}^{\infty} \overleftarrow{0} \\ \zeta_i, & \text{if otherwise.} \end{cases}$$

for all $h \in H$. Then by Theorem (2.2.9), $\mathbb{G}_i$ is an IFPI and hence the primary ideal of H.

Consider the IFI G of H defined by $G(h) = (0,1), \forall h \in H$. Then G has no IF primary decomposition in H, i.e., $G \neq \bigcap_{i=1}^{n} \mathbb{G}_i$, for any $n \in \mathbb{N}$.

5.4 Conclusion

This chapter, introduces and investigates the irreducibility of an IFI in a Γ-Ring. It is demonstrated that every IFI in a Noetherian Γ-Ring can be expressed as an intersection of a finite number of IFIrIs. Moreover, the IF version of the Lasker-Noether theorem is established for a commutative Noetherian Γ-Ring, demonstrating that every IFI G in such a ring can be decomposed into a finite intersection of IFPrIs. This decomposition is termed an IF primary decomposition. Additionally, it is shown that in the case of a minimal IF primary decomposition of an IFI G, the set of all IF-associated PIs of G remains independent of the specific decomposition. The potential extension of the IF primary decomposition theorem to other algebraic structures beyond commutative Γ-Rings opens new avenues for research. In this context, our investigation of IF primary decomposition in a commutative Noetherian Γ-Ring establishes a new horizon and contributes to the advancement of further research endeavors.

Chapter 6

Intuitionistic Fuzzy Structure Space Of *Γ*-Ring

6.1 Introduction

This chapter, introduces a topology on IFPIs(H) of a commutative Γ-Ring H with identity, which results in a structure space named as IFSpec(H). The study further explores separation axioms, compactness, irreducibility, and connectedness in this structured space.

6.2 Intuitionistic Fuzzy Structure Space Of *Γ*-Ring

In this section, we introduce a topological structure on the collection $\mathcal{X}$ of all IFPI of Γ-Ring H and investigate some of its properties.

Remark 6.2.1. (i) $\mathcal{X} = \{P: P$ is an IFPI of Γ-Ring H$\}$

(ii) $\mathcal{V}(G) = \{P \in \mathcal{X}: G \subseteq P\}$, where G is any IFS of H.

(iii) $\mathcal{X}(G) = \mathcal{X} \backslash \mathcal{V}(G)$,the complement of $\mathcal{V}(G)$ in $\mathcal{X}$, i.e., = { P ∈ $\mathcal{X}$: G ⊈ P }

(iv) For any IFS B of H, $< B >$ denotes the IFI generated by B.

Theorem 6.2.2. *Let H be a Γ-Ring and* $\tau = \{\mathcal{X}(G)$: G is an IFPI of $H\} = \{P \in \mathcal{X}: G \nsubseteq P\}$. *Then, τ is a topology on $\mathcal{X}$ and the pair $(\mathcal{X}, \tau)$ is a topological space.*

Proof. Consider the trivial IFIs $G = \tilde{0}$ and $B = \tilde{1}$ of H. Then, $\mathcal{V}(G) = \mathcal{V}(\tilde{0}) = \mathcal{X}$ and $\mathcal{V}(B) = \mathcal{V}(\tilde{1}) = \emptyset$, so that $\mathcal{X}(\tilde{0}) = \emptyset$ and $\mathcal{X}(\tilde{1}) = \mathcal{X}$ implies $\emptyset, \mathcal{X} \in \tau$.

Next, let $\mathbb{G}_1$ and $\mathbb{G}_2$ be any two IFIs of H. Then

$B \in \mathcal{V}(\mathbb{G}_1) \cup \mathcal{V}(\mathbb{G}_2) \Rightarrow \mathbb{G}_1 \subseteq B$ or $\mathbb{G}_2 \subseteq B \Rightarrow \mathbb{G}_1 \cap \mathbb{G}_2 \subseteq B \Rightarrow B \in \mathcal{V}(\mathbb{G}_1 \cap \mathbb{G}_2)$ and

$B \in \mathcal{V}(\mathbb{G}_1 \cap \mathbb{G}_2) \Rightarrow \mathbb{G}_1 \cap \mathbb{G}_2 \subseteq B \Rightarrow \mathbb{G}_1 \Gamma \mathbb{G}_2 \subseteq B$ [As $\mathbb{G}_1 \Gamma \mathbb{G}_2 \subseteq \mathbb{G}_1 \cap \mathbb{G}_2$]

$\Rightarrow \mathbb{G}_1 \subseteq B$ or $\mathbb{G}_2 \subseteq B$ [As B is IFPI of H]

$\Rightarrow B \in \mathcal{V}(\mathbb{G}_1)$ or $B \in \mathcal{V}(\mathbb{G}_2) \Rightarrow B \in \mathcal{V}(\mathbb{G}_1) \cup \mathcal{V}(\mathbb{G}_2)$.

Hence, $\mathcal{V}(\mathbb{G}_1) \cup \mathrm{U}(\mathbb{G}_2) = \mathcal{V}(\mathbb{G}_1 \cap \mathbb{G}_2) \Rightarrow \mathcal{X} \backslash \big(\mathcal{V}(\mathbb{G}_1) \cup \mathcal{V}(\mathbb{G}_2)\big) = \mathcal{X} \backslash \mathcal{V}(\mathbb{G}_1 \cap \mathbb{G}_2) \Rightarrow \big(\mathcal{X} \backslash \mathcal{V}(\mathbb{G}_1)\big) \cap \big(\mathcal{X} \backslash \mathcal{V}(\mathbb{G}_2)\big) = \mathcal{X} \backslash \mathcal{V}(\mathbb{G}_1 \cap \mathbb{G}_2)$, i.e., $\mathcal{X}(\mathbb{G}_1) \cap \mathcal{X}(\mathbb{G}_2) = \mathcal{X}(\mathbb{G}_1 \cap \mathbb{G}_2)$.

From this, we conclude that τ is closed under finite intersections.

Now, suppose that $\{\mathbb{G}_i : i \in \Lambda\}$ be any family of IFIs of H. It can be verified that

$\cap \{\mathcal{V}(\mathbb{G}_i) : i \in \Lambda\} = \mathcal{V}(<\cup \{\mathbb{G}_i : i \in \Lambda\}>)$. In another way,

$\{\mathcal{X}(\mathbb{G}_i) : i \in \Lambda\} = \mathcal{X}(<\cup \{\mathbb{G}_i : i \in \Lambda\}>)$. Hence, τ is closed under arbitrary unions. Hence, τ defines a topology on $\mathcal{X}$.

Remark 6.2.3. The topological space (X, τ) defined in Theorem (6.2.2) is called the IF prime spectrum of H and is denoted by $IFSpec(H)$, or, for convenience, we denote it by $\mathcal{X}$ only.

Example 6.2.4. (1) Consider $H = \Gamma = \mathbb{Z}$, the ring of integers. Then H is a Γ-Ring. Suppose that $p \in \mathbb{Z}$ is a prime integer. Then for every $\lambda, \zeta \in [0,1)$ s.t. $\lambda + \zeta \leq 1$, define $P_{\lambda,\zeta} \in IFS(H)$ as

$$\mu_{P_{\lambda,\zeta}}(h) = \begin{cases} 1, & \text{if } h \in <p> \\ \lambda, & \text{if otherwise} \end{cases}; \quad \nu_{P_{\lambda,\zeta}}(x) = \begin{cases} 0, & \text{if } h \in <p> \\ \zeta, & \text{otherwise.} \end{cases}$$

for all $h \in H$. Then by Theorem (2.2.9), $P_{\lambda,\zeta}$ is an IFPI of H.

Thus, $IFSpec(H) = \{P_{\lambda,\zeta}$, where $\lambda, \zeta \in [0,1)$ s.t. $\lambda + \zeta \leq 1$ and p is prime element of $\mathbb{Z}\}$.

(2) Consider $H = \Gamma = \mathbf{Z_2}$, where $\mathbf{Z_2} = \{\overleftarrow{0}, \overleftarrow{1}\}$ be a boolean ring. Then H is a Γ-Ring and for every $\lambda, \zeta \in [0,1)$ such that $\lambda + \zeta \leq 1$, define $P_{\lambda,\zeta} \in IFS(H)$ as

$$\mu_{P_{\lambda,\zeta}}(h) = \begin{cases} 1, & \text{if } h = \overleftarrow{0} \\ \lambda, & \text{if } h = \overleftarrow{1} \end{cases}; \quad \nu_{P_{\lambda,\zeta}}(x) = \begin{cases} 0, & \text{if } h = \overleftarrow{0} \\ \zeta, & \text{if } h = \overleftarrow{1}. \end{cases}$$

for all $h \in H$. Then by Theorem (2.2.9), $P_{\lambda,\zeta}$ is an IFPI of H.

Thus, $IFSpec(H) = \{P_{\lambda,\zeta}$, where $\lambda, \zeta \in [0,1)$ such that $\lambda + \zeta \leq 1\}$.

Proposition 6.2.5. *If f is a $Hom_{H_1}^{H_2}$, then* $f\big(h_{(\beta,\vartheta)}\big) = \big(f(h)\big)_{(\beta,\vartheta)}, \forall h \in H_1, \beta, \vartheta \in (0,1]$ s.t. $\beta + \vartheta \leq 1$.

Proof. Let $k \in H_2$ be any element, then $f\left(h_{(\beta,\vartheta)}\right)(k) = \left(\mu_{f\left(h_{((\beta,\vartheta))}\right)}(k), \nu_{f\left(h_{(\beta,\vartheta)}\right)}(k)\right)$,

Where

$\mu_{f\left(h_{(\beta,\vartheta)}\right)}(k) = Sup\{\mu_{h_{(\beta,\vartheta)}}(p): f(p) = k\}$=$\begin{cases} \beta, if\ p = h\ (i.e., k = f(h)); \\ 0 \quad\quad otherwise \end{cases}$=$\mu_{(f(h))_{(\beta,\vartheta)}}(k)$

and

$\nu_{f\left(h_{(\beta,\vartheta)}\right)}(k) = Inf\{\nu_{h_{(\beta,\vartheta)}}(p): f(p) = y\}$ = $\begin{cases} \vartheta, if\ p = h\ (i.e., k = f(h)) \\ 1 \quad\quad otherwise \end{cases}$

=$\nu_{(f(h))_{(\beta,\vartheta)}}(k)$

Hence $f\left(h_{(\beta,\vartheta)}\right) = \left(f(h)\right)_{(\beta,\vartheta)}$.

Theorem 6.2.6. *Let H be a Γ-Ring and $h, k \in H$ and $\beta, \vartheta \in (0,1]$* s.t.$\beta+\vartheta \leq 1$. *Then the following statements are true:*

(i) $\mathcal{X}\left(h_{(\beta,\vartheta)}\right) \cap \mathcal{X}\left(k_{(\beta,\vartheta)}\right) = \mathcal{X}\left((h\gamma k)_{(\beta,\vartheta)}\right)$, *for all* $\gamma \in \Gamma$.

(ii) $\mathcal{X}\left(h_{(\beta,\vartheta)}\right) = \emptyset$ *iff h is nilpotent.*

(iii) $\mathcal{X}\left(h_{(\beta,\vartheta)}\right) = \mathcal{X}$ *if h is a unit in H.*

Proof. (i) Let $h, k \in H, \gamma \in \Gamma$ and *and* $\beta, \vartheta \in (0,1]$ s.t .$\beta+\vartheta \leq 1$. Let $P \in \mathcal{X}$. Then $\mu_P(0_H) = 1, \nu_P(0_H) = 0, Img(P) = \{(1,0), (\lambda, \zeta)\}$, where $\lambda, \zeta \in [0,1)$ such that $\lambda + \zeta \leq 1$, P_* is a PI of H (by Theorem (2.2.9)).

Suppose $P \in \mathcal{X}\left(h_{(\beta,\vartheta)}\right) \cap \mathcal{X}\left(k_{(\beta,\vartheta)}\right)$, then $P \in \mathcal{X}\left(h_{(\beta,\vartheta)}\right)$ and $P \in \mathcal{X}\left(k_{(\beta,\vartheta)}\right)$

$\Leftrightarrow h_{(\beta,\vartheta)} \nsubseteq P$, $k_{(\beta,\vartheta)} \nsubseteq P \Leftrightarrow \mu_P(h) < \beta, \nu_P(h) > \vartheta$ and $\mu_P(k) < \beta, \nu_P(k) > \vartheta$

$\Leftrightarrow \beta = \mu_{h_{(\beta,\vartheta)}}(h) > \mu_P(h)$, $\vartheta = \nu_{h_{(\beta,\vartheta)}}(h) < \nu_P(h)$ and $\beta = \mu_{k_{(\beta,\vartheta)}}(k) > \mu_P(k)$, $\vartheta = \nu_{k_{(\beta,\vartheta)}}(k) < \nu_P(k)$

$\Leftrightarrow h, k \notin P_*$, for if $h, k \in P_*$, then $\beta > \mu_P(h) = \mu_P(k) = 1$ and $\vartheta < \nu_P(h) = \nu_P(k) = 0$

$\Leftrightarrow h\gamma k \notin P_*$, for all $\gamma \in \Gamma$, as P_* is a PI of H.

$\Leftrightarrow \beta > \mu_P(h\gamma k)$ and $\vartheta < \nu_P(h\gamma k)$, since $Img(P) = \{(1,0), (\lambda, \zeta)\}$, $\lambda, \zeta \in [0,1)$ such that $\lambda + \zeta \leq 1$

$\Leftrightarrow (h\gamma k)_{(\beta,\vartheta)} \nsubseteq P \Leftrightarrow P \in \mathcal{X}\big((h\gamma k)_{(\beta,\vartheta)}\big)$.

This proves that $\mathcal{X}\big(h_{(\beta,\vartheta)}\big) \cap \mathcal{X}\big(k_{(\beta,\vartheta)}\big) = \mathcal{X}\big((h\gamma k)_{(\beta,\vartheta)}\big)$, for all $\gamma \in \Gamma$.

(ii) Suppose J be any PI of H and χ_J be the IFCF of J, then from Theorem (2.2.9) we have $\chi_J \in \mathcal{X}$. Further, if $\mathcal{X}\big(h_{(\beta,\vartheta)}\big) = \emptyset$ then $\mathcal{V}\big(h_{(\beta,\vartheta)}\big) = \mathcal{X}$ that implies $h_{(\beta,\vartheta)} \subseteq \chi_J$ and therefore, $\mu_{\chi_J}(h) \geq \beta > 0$ and $\nu_{\chi_J}(h) \leq \vartheta < 1$ so that $\mu_{\chi_J}(h) = 1$ and $\nu_{\chi_J}(h) = 0$ and so $h \in J$. Thus $h \in \cap \{J$: J is PI of H }. As the prime radical is subset of the nil radical so h is nilpotent.

Conversely, assume that x is nilpotent, then for every $\gamma \in \Gamma, \exists\ n \in \mathbb{N}$ depending on γ so that

$(h\gamma)^n h = 0_H$. Let $P \in \mathcal{X}$ be any element. Then $\mu_P((h\gamma)^n h) = \mu_P(0_H) = 1$ and $\nu_P((h\gamma)^n h) = \nu_P(0_H) = 0$. Therefore $1 = \mu_P((h\gamma)^n h) \geq \mu_P(h)$ and $0 = \nu_P((h\gamma)^n h) \leq \nu_P(h)$ implies that $\mu_P(h) = 1$ and $\nu_P(h) = 0$. So $h \in P_*$. But P_* is a PI of H. Hence $\beta = \mu_{h_{(\beta,\vartheta)}}(h) \leq \mu_P(h)$ and $\vartheta = \nu_{h_{(\beta,\vartheta)}}(h) \geq \nu_P(h)$, whence $h_{(\beta,\vartheta)} \subseteq P, \forall P \in \mathcal{X}$. Thus $\mathcal{V}\left(h_{((\beta,\vartheta))}\right) = \mathcal{X}$, i.e., $\mathcal{X}\big(h_{(\beta,\vartheta)}\big) = \emptyset$.

(iii) Suppose J and χ_J be same as in part (ii). Now if $\mathcal{X}\big(h_{(\beta,\vartheta)}\big) = \mathcal{X}$ then $\mathcal{V}\big(h_{(\beta,\vartheta)}\big) = \emptyset$ that implies $h_{(\beta,\vartheta)} \nsubseteq \chi_J$ and thus $\mu_{\chi_J}(h) < \beta$ and $\nu_{\chi_J}(h) > \vartheta$ so that $h \notin J$. Hence $h \notin \cup \{J$: J is a PI of H }. This shows that h is a unit.

The following example shows that the converse of Theorem (6.2.6)(iii) is not true in general. This is a deviation of the result from the crisp theory.

Example 6.2.7. Consider H, Γ, and $\mathcal{X} = IFSpec(H)$ as in Example (6.2.4)(1).

Define $G \in \mathcal{X}$ as follows:

$$\mu_G(h) = \begin{cases} 1, & \text{if } h \in <2> \\ 0.6, & \text{if otherwise} \end{cases}; \quad \nu_G(h) = \begin{cases} 0, & \text{if } h \in <2> \\ 0.3, & \text{otherwise.} \end{cases}$$

Take $\beta = 0.5, \vartheta = 0.4$ and $h = 1$. Then we see that IFP $h_{(\beta,\vartheta)} \subseteq G$, hence $G \notin \mathcal{X}\big(h_{(\beta,\vartheta)}\big)$, and consequently $\mathcal{X} \neq \mathcal{X}\big(h_{(\beta,\vartheta)}\big)$.

Proposition 6.2.8. *The subfamily* $\{\mathcal{X}\big(h_{(\beta,\vartheta)}\big): h \in H, \beta, \vartheta \in (0,1]$ *s.t.* $\beta + \vartheta \leq 1\}$ *of* τ *is a base for* τ.

Proof. Let $\mathcal{X}(G) \in \tau$, where G is an IFI of H. Let $B \in \mathcal{X}(G)$. Then $G \nsubseteq B \Rightarrow$ there exit $h \in H$ s.t. $\mu_G(h) > \mu_B(h)$ and $\nu_G(h) < \nu_B(h)$. Thus $h \notin B_*$ and hence $\mu_B(h) = \lambda$ and $\nu_B(h) = \zeta$, for some $\lambda, \zeta \in [0,1)$ with $\lambda + \zeta \leq 1$.

Let $\mu_G(h) = \alpha > 0, \nu_G(h) = \beta < 1$. Clearly $h_{(\beta,\vartheta)} \nsubseteq B$ and so $B \in \mathcal{X}\big(h_{(\beta,\vartheta)}\big)$.

Now, $\mathcal{V}(G) \subseteq \mathcal{V}\big(h_{(\beta,\vartheta)}\big)$, because if $P \in \mathcal{V}(G)$ then $G \subseteq P$ and so $\mu_{h_{(\beta,\vartheta)}}(h) = \beta = \mu_G(h) < \mu_P(h)$ and $\nu_{h_{(\beta,\vartheta)}}(h) = \vartheta = \nu_G(h) > \nu_P(h)$. This implies that $h_{(\beta,\vartheta)} \subseteq P$ and thus $P \in \mathcal{V}\big(h_{(\beta,\vartheta)}\big)$. Hence $\mathcal{X}\big(h_{(\beta,\vartheta)}\big) \subseteq \mathcal{X}(G)$. Thus $B \in \mathcal{X}\big(h_{(\beta,\vartheta)}\big) \subseteq \mathcal{X}(G)$. Hence the subfamily $\{\mathcal{X}\big(h_{(\beta,\vartheta)}\big): h \in H, \beta, \vartheta \in (0,1]$ s.t. $\beta + \vartheta \leq 1\}$ is a base for τ.

Proposition 6.2.9. The subset $\mathcal{Y} = \{\mathrm{P} \in \mathcal{X}: \mathrm{Img(P)} = \{(1,0), (\lambda, \zeta)\}$, where $\lambda, \zeta \in [0,1)$ s.t. $\lambda + \zeta \leq 1\}$, is compact w.r.t. the subspace topology.

Proof. The family $\{\mathcal{X}\left(h_{((\beta,\vartheta))}\right) \cap \mathcal{Y}: h \in H$, and $\beta \in (\lambda, 1]$ and $\vartheta \in [0,\zeta)$ such that $\beta + \vartheta \leq 1\}$ forms a base for $\mathcal{Y}$ in the same way as explained in previous Theorem. Now, let us consider that $\{\mathcal{X}\big((h_i)_{(\mathfrak{m},\mathfrak{n})}\big) \cap \mathcal{Y}: i \in \Lambda$ and $(\mathfrak{m}, \mathfrak{n}) \in K \times S \subseteq (\lambda, 1] \times [0, \zeta)\}$ is a covering of $\mathcal{Y}$ taken from the basic open sets. Suppose $\beta = Sup\{\mathfrak{m}: \mathfrak{m} \in K\}$ and $\vartheta = Inf\{\mathfrak{n}: \mathfrak{n} \in S\}$. Then the family $\{\mathcal{X}\big((h_i)_{(\beta,\vartheta)}\big) \cap \mathcal{Y}: i \in \Lambda\}$ also covers $\mathcal{Y}$. Now,

$$
\begin{aligned}
\mathcal{Y} &= \cup \{\mathcal{X}\big((h_i)_{(\beta,\vartheta)}\big) \cap \mathcal{Y}: i \in \Lambda\} \\
&= \big(\cup \{\mathcal{X}\big((h_i)_{(\beta,\vartheta)}\big): i \in \Lambda\}\big) \cap \mathcal{Y} \\
&= \Big(\mathcal{X} \backslash \mathcal{V}\big(\cup \{(h_i)_{(\beta,\vartheta)}: i \in \Lambda\}\big)\Big) \cap \mathcal{Y} \\
&= (\mathcal{X} \cap \mathcal{Y}) \setminus \big(\mathcal{V}\big(\cup \{(h_i)_{(\beta,\vartheta)}: i \in \Lambda\}\big) \cap \mathcal{Y}\big) \\
&= \mathcal{Y} \setminus \big(\mathcal{V}\big(\cup \{(h_i)_{(\beta,\vartheta)}: i \in \Lambda\}\big) \cap \mathcal{Y}\big).
\end{aligned}
$$

This shows that $\mathcal{V}\big(\cup \{(h_i)_{(\beta,\vartheta)}: i \in \Lambda\}\big) \cap \mathcal{Y} = \emptyset$. Further, suppose that J be any PI of Γ-Ring H. Consider an IFI G of H given by

$$
\mu_G(h) = \begin{cases} 1, & \text{if } h \in J \\ \alpha, & \text{if otherwise} \end{cases}; \quad \nu_G(h) = \begin{cases} 0, & \text{if } h \in J \\ \beta, & \text{if otherwise} \end{cases}.
$$

Clearly, G is an IFPI of H and $G \in \mathcal{Y}$. So $G \notin \mathcal{V}\big(\cup \{(h_i)_{(\beta,\vartheta)}: i \in \Lambda\}\big)$. Hence $\big(h_j\big)_{(\beta,\vartheta)}$is not proper subset of G for some $j \in \Lambda$. Thus $\gamma > \mu_G\big(h_j\big)$ and $\delta < \nu_G\big(h_j\big)$ for some $j \in \Lambda$. As a result, $h_j \notin J$. This proves that there is no PI of H containing the set $\{h_i : i \in \Lambda\}$. So this implies, $< \{h_i : i \in \Lambda\} >= H$. Let $\sum_{l=1}^{n} [\delta_l, e_l]$ be the right unity of Γ-Ring H, where $\delta_l \in \Gamma$, $e_l \in H$ for all $l = 1,2,\ldots.,n$ and $e_l = \sum_{\theta=1}^{n_l} m_{\theta_l} \gamma_{\theta_l} h_{\theta_l}$, where n_l is a finite positive integer, $m_{\theta_l} \in H$, $h_{\theta_l} \in \{h_j : J \in \Lambda\}$, $\gamma_{q_l} \in \Gamma$ for all $\theta = 1,2,\ldots,n_l$ and $l = 1,2,\ldots.n$. Now we claim that $\mathcal{V}\left(\cup_{l=1}^{n} \cup_{\theta=1}^{n_l} \left(h_{\theta_l}\right)_{(\beta,\vartheta)}\right) \cap \mathcal{Y} = \emptyset$, as $G \in \mathcal{V}\left(\cup_{l=1}^{n} \cup_{\theta=1}^{n_l} \left(h_{\theta_l}\right)_{(\beta,\vartheta)}\right) \cap \mathcal{Y}$ implies $\cup_{l=1}^{n} \cup_{\theta=1}^{n_l} \left(h_{\theta_l}\right)_{(\beta,\vartheta)} \subseteq G$ and $Img(G) = \{(1,0), (\alpha, \beta)\}$. This imply

$\beta = \mu_{\left(h_{\theta_l}\right)_{(\beta,\vartheta)}}\left(h_{\theta_l}\right) \le \mu_G\left(h_{\theta_l}\right)$ and $\vartheta = \nu_{\left(h_{\theta_l}\right)_{(\beta,\vartheta)}}\left(h_{\theta_l}\right) \ge \nu_G\left(h_{\theta_l}\right), \forall\, \theta = 1,2,..,n_l, l = 1,2,..,n.$

$\Rightarrow \mu_G\left(h_{\theta_l}\right) = 1, \nu_G\left(h_{\theta_l}\right) = 0, \forall\, \theta = 1,2,\ldots,n_l, l = 1,2,\ldots,n$, since $\beta > \alpha, \vartheta < \beta$.

$\Rightarrow h_{\theta_l} \in G_* \;\forall\, \theta = 1,2,\ldots,n_l, l = 1,2,\ldots,n$

$\Rightarrow e_l \in G_* \;\forall\, l = 1,2,\ldots,n$

$\Rightarrow h_j = \sum_{l=1}^{n} h_j\, \delta_l e_l \in G_* = J$, which is a contradiction. Thus we have

$$
\begin{aligned}
\mathcal{Y} &= \mathcal{Y} \backslash \left(\mathcal{V}\left(\cup_{l=1}^{n} \cup_{\theta=1}^{n_l} \left(h_{\theta_l}\right)_{(\beta,\vartheta)}\right) \cap \mathcal{Y}\right) \\
&= (\mathcal{X} \cap \mathcal{Y}) \backslash \left(\mathcal{V}\left(\cup_{l=1}^{n} \cup_{\theta=1}^{n_l} \left(h_{\theta_l}\right)_{(\beta,\vartheta)}\right) \cap \mathcal{Y}\right) \\
&= \left(\mathcal{X} \backslash \mathcal{V}\left(\cup_{l=1}^{n} \cup_{\theta=1}^{n_l} \left(h_{\theta_l}\right)_{(\beta,\vartheta)}\right)\right) \cap \mathcal{Y} \\
&= \left(\cup_{l=1}^{n} \cup_{\theta=1}^{n_l} \mathcal{X}\left(h_{\theta_l}\right)_{(\beta,\vartheta)}\right) \cap \mathcal{Y} \\
&= \cup_{l=1}^{n} \cup_{q=1}^{n_l} \left(\mathcal{X}\left(h_{\theta_l}\right)_{(\beta,\vartheta)} \cap \mathcal{Y}\right).
\end{aligned}
$$

This proves that $\{\mathcal{X}\left(\left(h_{\theta_l}\right)_{(\beta,\vartheta)}\right) \cap \mathcal{Y} : \theta = 1,2,\ldots,n_l, l = 1,2,\ldots, n\}$ covers $\mathcal{Y}$. Hence $\mathcal{Y}$ is compact.

6.3 Separation Axioms Of IF Spec(H)

In this section the conditions for a topological space $\mathcal{X}$ to be a T_0 space and T_1 space in intuitionistic fuzzy environment are discussed.

Proposition 6.3.1. *The topological space* $\mathcal{X}$ *is* T_0.

Proof. Suppose $\mathbb{G}_1, \mathbb{G}_2 \in \mathcal{X}$ s.t. $\mathbb{G}_1 \neq \mathbb{G}_2$. Then either $\mathbb{G}_1 \nsubseteq \mathbb{G}_2$ or $\mathbb{G}_2 \nsubseteq \mathbb{G}_1$. Let $\mathbb{G}_2 \nsubseteq \mathbb{G}_1$. Then $\mathbb{G}_2 \in \mathcal{X}(\mathbb{G}_1)$. Also, $\mathbb{G}_1 \notin \mathcal{X}(\mathbb{G}_1)$ and $\mathcal{X}(\mathbb{G}_1)$ are open. Therefore, $\mathcal{X}$ is T_0 space.

In the following examples, we depict that $\exists s$ some element of the basis of $\mathcal{X}$ that is not closed, and it is even possible that $\mathcal{X}$ is not T_1 and hence not T_2. These results are also deviations from the results in crisp theory.

Example 6.3.2. Consider H and Γ as in Example (6.2.4)(2).

Then $\mathcal{X} = \{P_{\lambda,\zeta}$, where $\lambda, \zeta \in [0,1)$ such that $\lambda + \zeta \leq 1\}$, where $P_{\lambda,\zeta}$ is defined as

$$\mu_{P_{\lambda,\zeta}}(h_1) = \begin{cases} 1, & \text{if } h_1 = \overleftarrow{0} \\ \lambda, & \text{if } h_1 = \overleftarrow{1} \end{cases}; \quad \nu_{P_{\lambda,\zeta}}(h_1) = \begin{cases} 0, & \text{if } h_1 = \overleftarrow{0} \\ \zeta, & \text{if } h_1 = \overleftarrow{1}. \end{cases}$$

$\forall\ h_1 \in H$. Now we show that if $h_1 = \overleftarrow{1}$ and $ϐ = 0.6, \vartheta = 0.3$, then $\mathcal{X}\left(\overleftarrow{1}_{(ϐ,\vartheta)}\right)$ is not closed. Suppose on the contrary that $\mathcal{X}\left(\overleftarrow{1}_{(ϐ,\vartheta)}\right)$ is closed. Then $\exists$ subset $K \times S$ of $[0,1] \times [0,1]$ s.t. $\mathcal{X}\left(\overleftarrow{1}_{(ϐ,\vartheta)}\right) = \cap \{\mathcal{V}\left(k_{(ɱ,ɳ)}\right): (ɱ, ɳ) \in K \times S, k \in \mathbf{Z_2}\}$. If $k = \overleftarrow{1}$ and $(ɱ, ɳ) \in K \times S = (ϐ, 1] \times [0, \vartheta)$ s.t. $ɱ + ɳ \leq 1$, then it is not difficult to check that $\mathcal{X}\left(\overleftarrow{1}_{(ϐ,\vartheta)}\right) \nsubseteq \mathcal{V}\left(\overleftarrow{1}_{(ɱ,ɳ)}\right)$ and if $k = \overleftarrow{1}$ and $ɱ = 0, ɳ = 1$ or $k = \overleftarrow{0}, (ɱ, ɳ) \in [0,1] \times [0,1]$, then it is seen that $\mathcal{V}\left(k_{(ɱ,ɳ)}\right) = \mathcal{X}$. Thus $\mathcal{X}\left(\overleftarrow{1}_{(ϐ,\vartheta)}\right)$ must be equal to $\mathcal{X}$, which is a contradiction. Therefore $\mathcal{X}\left(\overleftarrow{1}_{(ϐ,\vartheta)}\right)$ is not closed.

Example 6.3.3. Consider the space $\mathcal{X}$ as in Example (6.3.2). Choose $P_{0.6,0.3}, P_{0.5,0.4} \in \mathcal{X}$. Let W be an open set containing $P_{0.6,0.3}$. Then $W = \cap \{\mathcal{X}\left(\overleftarrow{1}_{(ɱ,ɳ)}\right): (ɱ, ɳ) \in K \times S\}$ for some $K \times S \subseteq (0,1] \times (0,1]$. Thus there exists $(ɱ, ɳ) \in K \times S$ such that $P_{0.6,0.3} \in \mathcal{X}\left(\overleftarrow{1}_{(ɱ,ɳ)}\right)$. So $ɱ > 0.6 > 0.5$ and $ɳ < 0.3 < 0.4$. Consequently $P_{0.5,0.4} \in \mathcal{X}\left(\overleftarrow{1}_{(ɱ,ɳ)}\right) \subseteq W$. In other words, any open neighborhood of $P_{0.6,0.3}$ also contains $P_{0.5,0.4}$. Thus $\mathcal{X}$ is not T_1.

Proposition 6.3.4. *Let H be a Γ-Ring and $\mathbb{G}_1 \in \mathcal{X}$ then $\mathcal{V}(\mathbb{G}_1) = cl\{\mathbb{G}_1\}$, the closure of $\mathbb{G}_1$ in $\mathcal{X}$. Further $\mathbb{G}_2 \in cl\{\mathbb{G}_1\}$ iff $\mathbb{G}_1 \subseteq \mathbb{G}_2$, where $\mathbb{G}_1, \mathbb{G}_2 \in \mathcal{X}$.*

Proof. Since $\mathcal{V}(\mathbb{G}_1)$ is a closed subset of $\mathcal{X}$ containing $\mathbb{G}_1$. Therefore $cl\{\mathbb{G}_1\} \subseteq \mathcal{V}(\mathbb{G}_1)$

For the reverse inclusion, consider $\mathbb{G}_2 \in \mathcal{X}$ s.t. $\mathbb{G}_2 \notin cl\{\mathbb{G}_1\}$. Then, $\exists$ an open set $\mathcal{X}(C)$ where C is an IFI of H containing $\mathbb{G}_2$ but not $\mathbb{G}_1$. Therefore, $C \nsubseteq \mathbb{G}_2$ but $C \subseteq \mathbb{G}_1$. So $\mathbb{G}_1 \nsubseteq \mathbb{G}_2$ and hence $\mathbb{G}_2 \notin \mathcal{V}(\mathbb{G}_1)$. Thus $\mathcal{V}(\mathbb{G}_1) \subseteq cl\{\mathbb{G}_1\}$. Hence $\mathcal{V}(\mathbb{G}_1) = cl\{\mathbb{G}_1\}$.

Further, $\mathbb{G}_2 \in cl\{\mathbb{G}_1\}$ iff $\mathbb{G}_2 \in \mathcal{V}(\mathbb{G}_1)$, which is equivalent to $\mathbb{G}_1 \subseteq \mathbb{G}_2$.

Proposition 6.3.5. *Let $\mathcal{Y}$ be the same as in proposition (6.2.9). If $\mathbb{G}_1 \in \mathcal{Y}$, then $\{\mathbb{G}_1\}$ is closed in $\mathcal{Y}$ iff $\mathbb{G}_1$ is an IFMI of H. (In other words, $\mathcal{Y}$ is T_1 iff every singleton element of $\mathcal{Y}$ is an IFMI of H).*

Proof. Let $\mathbb{G}_1 \in \mathcal{Y}$ and $\{\mathbb{G}_1\}$ be closed. Then $\mathcal{V}(\mathbb{G}_1) = cl\{\mathbb{G}_1\} = \{\mathbb{G}_1\}$. Hence $\mathcal{V}(\mathbb{G}_1) \cap \mathcal{Y} = \{\mathbb{G}_1\}$, by proposition (6.3.4). Now, we show that $\mathbb{G}_1$ is an IFMI. As $\mathbb{G}_1 \in \mathcal{Y}$, $Img(\mathbb{G}_1) = \{(1,0), (\lambda, \zeta)\}$. So it is left to prove that the ideal ${\mathbb{G}_1}_* = \{h \in H: \mu_{\mathbb{G}_1}(h) = 1 \text{ and } \nu_{\mathbb{G}_1}(h) = 0\}$ is maximal. For this, it is enough to show that there is no PI of H properly containing ${\mathbb{G}_1}_*$. Let J be a PI of H properly containing ${\mathbb{G}_1}_*$.

Let $\mathbb{G}_2$ be an IFI of H defined by

$$\mu_{\mathbb{G}_2}(h) = \begin{cases} 1, & \text{if } h \in J \\ \lambda, & \text{if otherwise} \end{cases}; \quad \nu_{\mathbb{G}_2}(h) = \begin{cases} 0, & \text{if } h \in J \\ \zeta, & \text{if otherwise} \end{cases}, \text{where } \lambda + \zeta \leq 1.$$

Then $\mathbb{G}_2 \in \mathcal{Y}$ and $\mathbb{G}_1$ are properly contained in $\mathbb{G}_2$. So this cannot happen that $\mathcal{V}(\mathbb{G}_1) \cap \mathcal{Y} = \{\mathbb{G}_1\}$. This proves that ${\mathbb{G}_1}_*$ is a MI of H and so $\mathbb{G}_1$ is an IFMI of H.

Conversely, let $\mathbb{G}_1 \in \mathcal{Y}$ and $\mathbb{G}_1$ be an IFMI. Then the ideal ${\mathbb{G}_1}_* = \{h \in H: \mu_{\mathbb{G}_1}(h) = 1 \text{ and } \mu_{\mathbb{G}_1}(h) = 0\}$ is the MI of H. We claim that $\mathcal{V}(\mathbb{G}_1) \cap \mathcal{Y} = \{\mathbb{G}_1\}$. Clearly, $\{\mathbb{G}_1\} \subseteq \mathcal{V}(\mathbb{G}_1) \cap \mathcal{Y}$. Next

$\mathbb{G}_2 \in \mathcal{V}(\mathbb{G}_1) \cap \mathcal{Y} \Rightarrow G_* \subseteq {\mathbb{G}_2}_* \Rightarrow {\mathbb{G}_1}_* = {\mathbb{G}_2}_*$

since ${\mathbb{G}_1}_*$ is a maximal ideal. Thus we have $\mathbb{G}_1 = \mathbb{G}_2$, since $Img(\mathbb{G}_1) = Img(\mathbb{G}_2) = \{(1,0), (\lambda, \zeta)\}$. Therefore, $\mathcal{V}(\mathbb{G}_1) \cap \mathcal{Y} = \{\mathbb{G}_1\}$. Consequently, $\{\mathbb{G}_1\}$ is a closed subset of $\mathcal{Y}$.

We know that a topological space $\mathcal{X}$ is Hausdorff (or T_2 space), if and only if for $h \neq k$ be two points of $\mathcal{X}$, then $\exists$ two disjoint open sets one containing x and another containing y.

Theorem 6.3.6. *Let H be a Γ-Ring whose every PI is MI. Then the space* $\mathcal{X} = IFSpec(H)$ *is not* T_2.

Proof. For the proof, we show that $\exists$ two distinct elements $\mathbb{G}_1$, *and* $\mathbb{G}_2$ of $\mathcal{X} = IFSpec(H)$ cannot be separated by two disjoint basic open sets.

Consider a prime ideal J and two IFPI $\mathbb{G}_1$ and $\mathbb{G}_2$ of H as follows

$$\mu_{\mathbb{G}_1}(h) = \begin{cases} 1, & \text{if } h \in J \\ 0.1, & \text{if otherwise} \end{cases}; \quad \nu_{\mathbb{G}_1}(h) = \begin{cases} 0, & \text{if } h \in J \\ 0.2, & \text{if otherwise} \end{cases};$$

$$\mu_{\mathbb{G}_2}(h) = \begin{cases} 1, & \text{if } h \in J \\ 0.3, & \text{if otherwise} \end{cases}; \quad \nu_{\mathbb{G}_2}(h) = \begin{cases} 0, & \text{if } h \in J \\ 0.4, & \text{if otherwise} \end{cases}.$$

Consider $\mathcal{X}(h_{(\beta,\vartheta)})$, and $\mathcal{X}(k_{(\beta,\vartheta)})$ be two basic open sets in $\mathcal{X}$ containing $\mathbb{G}_1$ and $\mathbb{G}_2$ respectively, where $h, k \in H$ and $\beta,\vartheta \in (0,1]$ s.t. $\beta+\vartheta \leq 1$. Then $h_{(\beta,\vartheta)} \not\subseteq \mathbb{G}_1$ and $k_{(\beta,\vartheta)} \not\subseteq \mathbb{G}_2$ and so $h \notin {\mathbb{G}_1}_* = J$ and $k \notin {\mathbb{G}_2}_* = J$. Since J is a PI in H, so $h\gamma k \notin J$, for every $\gamma \in \Gamma$. Then $h\gamma k$ is not nilpotent and so by Theorem (6.2.6) (i) and (ii) we have $\mathcal{X}(h_{(\beta,\vartheta)}) \cap \mathcal{X}(k_{(\beta,\vartheta)}) = X((h\gamma k)_{(\beta,\vartheta)}) \neq \emptyset$. Hence $\mathcal{X}$ is not T_2.

Theorem 6.3.7. *Let H be a Boolean Γ-Ring with unity e. Let* $\lambda, \zeta \in [0,1)$ *be s.t.* $\lambda + \zeta \leq 1$ *and suppose* $\mathcal{Y} = \{P \in \mathcal{X}: Img(P) = \{(1,0), (\lambda, \zeta)\}\}, h, k \in H,$ *and* $\beta, \vartheta \in (0,1]$ *s.t.* $\beta + \vartheta \leq 1$. *Then:*

(i) The set $\mathcal{X}(h_{(\beta,\vartheta)}) \cap \mathcal{Y}$ *is a clopen set in* $\mathcal{Y}$, *provided* $\beta > \lambda$ *and* $\vartheta < \zeta$.

(ii) $\mathcal{X}(h_{(\beta,\vartheta)}) \cup \mathcal{X}(k_{(\beta,\vartheta)}) = \mathcal{X}(p_{(\beta,\vartheta)})$ *for some* $p \in H$.

(iii) $\mathcal{Y}$ *is* T_2 *space*.

Proof. (i) As $\mathcal{X}(h_{(\beta,\vartheta)})$ is an open set in $\mathcal{X}$, then $\mathcal{X}(h_{(\beta,\vartheta)}) \cap \mathcal{Y}$ will also be an open set in $\mathcal{Y}$. We now show that $\mathcal{X}(h_{(\beta,\vartheta)}) \cap \mathcal{Y} = \mathcal{V}((\mathrm{e} - h)_{(\beta,\vartheta)}) \cap \mathcal{Y}$. [This would simply imply that $\mathcal{X}(h_{(\beta,\vartheta)})$ is closed set in $\mathcal{Y}$.

If $G \in \mathcal{X}(h_{(\beta,\vartheta)}) \cap \mathcal{Y}$ then $\mu_G(h) < \beta, \nu_G(h) > \vartheta$, but $Img(G) = \{(1,0),(\lambda,\zeta)\}$ so that $\mu_G(h) = \lambda, \nu_G(h) = \zeta$. Hence $\beta > \lambda$ and $\vartheta < \zeta$ and $x \notin G_*$. This implies that $\beta > \lambda$ and $\vartheta < \zeta$ and $\mathrm{e} - h \in G_*$, since $h\Gamma(\mathrm{e} - h) = h\Gamma\mathrm{e} - h\Gamma h = h - h = 0 \in G_*$ and the ideal G_* is prime implies that $(\mathrm{e} - h) \in G_*$. Is a result, $\mu_G(\mathrm{e} - h) = 1$ and $\nu_G(\mathrm{e} - h) = 0$ so that $(\mathrm{e} - h)_{(\beta,\vartheta)} \subseteq G$ and thus $G \in \mathcal{V}\left((\mathrm{e} - h)_{((\beta,\vartheta))}\right) \cap \mathcal{Y}$.

Conversely, let $G \in \mathcal{V}((\mathrm{e} - h)_{(\beta,\vartheta)}) \cap \mathcal{Y}$ then $(\mathrm{e} - h)_{(\beta,\vartheta)} \subseteq G$ and $Img(G) = \{(1,0),(\lambda,\zeta)\}$ which implies that $\eta \leq \mu_G(\mathrm{e} - h)$ and $\theta \geq \nu_G(\mathrm{e} - h)$. Hence $\lambda < \mu_G(\mathrm{e} - h)$ and $\zeta > \mu_G(\mathrm{e} - h)$ and thus $\mu_G(\mathrm{e} - h) = 1$ and $\nu_G(\mathrm{e} - h) = 0$. It follows that $\mathrm{e} - h \in G_*$ and hence $h \in G_*$ so that $\mu_G(h) = \lambda < \beta$ and $\nu_G(h) = \zeta > \vartheta$. This means that $h_{(\beta,\vartheta)} \nsubseteq G$ and thus $G \in \mathcal{X}(h_{(\beta,\vartheta)}) \cap \mathcal{Y}$. Hence $\mathcal{X}(h_{(\beta,\vartheta)}) \cap \mathcal{Y} = \mathcal{V}((\mathrm{e} - h)_{(\beta,\vartheta)}) \cap \mathcal{Y}$.

(ii) If $G \in \mathcal{X}(h_{(\beta,\vartheta)}) \cup \mathcal{X}(k_{(\beta,\vartheta)})$ then $h_{(\beta,\vartheta)} \nsubseteq G$ or $k_{(\beta,\vartheta)} \nsubseteq G$ (which mean that $\mu_G(h) < \beta$ and $\nu_G(h) > \vartheta$ or $\mu_G(k) < \beta$ and $\nu_G(k) > \vartheta$). This implies that $h \notin G_*$ or $k \notin G_*$ and thus $\mathrm{e} - h \notin G_*$ or $\mathrm{e} - k \notin G_*$. As a result, $(\mathrm{e} - h)\Gamma(\mathrm{e} - k) = \mathrm{e} - h - k + h\Gamma k \notin G_*$, so that $h + k - h\Gamma k \notin G_*$. Hence $G \in \mathcal{X}(p_{(\eta,\theta)})$, where $p = h + k - h\Gamma k$.

(iii) Let $\mathbb{G}_1, \mathbb{G}_2 \in \mathcal{X}, \mathbb{G}_1 \neq \mathbb{G}_2$. Then $\mathbb{G}_1$ and $\mathbb{G}_2$ are IFPIs of H and $Img(\mathbb{G}_1) = Img(\mathbb{G}_2)$ $= \{(1,0),(\lambda,\zeta)\}$. As we know that every PI in a Boolean Γ-Ring is MI. It follows that ${\mathbb{G}_1}_*, {\mathbb{G}_2}_*$ are maximal ideals of H. So ${\mathbb{G}_1}_* \nsubseteq {\mathbb{G}_2}_*$, since $\mathbb{G}_1 \neq \mathbb{G}_2$. Choose $h \in {\mathbb{G}_1}_*$ and $h \notin {\mathbb{G}_2}_*$. Then $\mathrm{e} - h \in {\mathbb{G}_2}_*$ and $\mathrm{e} - h \notin {\mathbb{G}_1}_*$. Now, $\mu_{\mathbb{G}_2}(h) = \mu_{\mathbb{G}_1}(\mathrm{e} - h) = \lambda$ and $\nu_{\mathbb{G}_2}(h) = \nu_{\mathbb{G}_1}(\mathrm{e} - h) = \zeta$ and $\mu_{\mathbb{G}_1}(h) = 1 = \mu_{\mathbb{G}_2}(\mathrm{e} - h)$ and $\nu_{\mathbb{G}_1}(h) = 0 = \nu_{\mathbb{G}_2}(\mathrm{e} - h)$. Let $\beta \in (\lambda, 1)$ and $\vartheta \in (0, \zeta)$ s.t. $\beta + \vartheta \leq 1$. Then $\mu_{h_{(\beta,\vartheta)}}(h) = \beta > \lambda = \mu_B(h)$ and $\nu_{h_{(\beta,\vartheta)}}(h) = \vartheta < \zeta = \nu_{\mathbb{G}_2}(h)$ so that $h_{(\beta,\vartheta)} \nsubseteq \mathbb{G}_2$. Hence $\mathbb{G}_2 \in \mathcal{X}(h_{(\beta,\vartheta)})$. Also, $\mu_{(\mathrm{e}-h)_{(\beta,\vartheta)}}(\mathrm{e} - h) = \beta > \lambda = \mu_{\mathbb{G}_1}(\mathrm{e} - h)$ and $\nu_{(\mathrm{e}-h)_{(\beta,\vartheta)}}(\mathrm{e} - h) = \vartheta < \zeta = \nu_{\mathbb{G}_1}(\mathrm{e} - h)$, so that $(\mathrm{e} - h)_{\beta,\vartheta} \nsubseteq \mathbb{G}_1$. Hence $\mathbb{G}_1 \in \mathcal{X}((\mathrm{e} - h)_{(\beta,\vartheta)})$. Then, by theorem (6.2.6)(i), we have $\mathcal{X}(h_{(\beta,\vartheta)}) \cap$

$\mathcal{X}((\mathrm{e}-h)_{(\beta,\vartheta)}) = \mathcal{X}\left(\left(h\Gamma(\mathrm{e}-h)\right)_{(\beta,\vartheta)}\right) = \mathcal{X}((0)_{(\beta,\vartheta)}) = \emptyset$ [As H is Boolean Γ-Ring]. Consequently, $\mathcal{Y}$ is Hausdorff.

Theorem 6.3.8. *If H is Boolean Γ-Ring, $\lambda, \zeta \in [0,1)$ s.t. $\lambda + \zeta \leq 1$ and $\mathcal{Y} = \{P \in \mathcal{X}: Img(P) = \{(1,0), (\lambda, \zeta)\}\}$, then the space $\mathcal{Y}$ is compact, Hausdorff and zero-dimensional.*

Proof. For proof of the Theorem refer Proposition (6.2.9) and Theorem (6.3.7)(i),(iii).

6.4 Intuitionistic Fuzzy Prime Radical And Algebraic Nature Of Intuitionistic Fuzzy Prime Ideal Under Γ-Homomorphism

Definition 6.4.1. ([64]) Let H be a Γ-Ring. For any IFI G of H. The IFS $\sqrt{G}$ defined by $\mu_{\sqrt{G}}(h) = \vee\{\mu_G((h\gamma)^{n-1}h): n \in \mathbf{N}\}$ and $\nu_{\sqrt{G}}(h) = \wedge\{\nu_G((h\gamma)^{n-1}): n \in \mathbf{N})\}$ is called the IFPR of G, where $(h\gamma)^{n-1}h = h$, for $n = 1, \gamma \in \Gamma$. Further, $\sqrt{G}$ is the smallest IFSPI of H containing G.

Proposition 6.4.2. ([64]) *Let G be an $IFPI$ of a Γ-Ring H. Then $\sqrt{G} = G$ and hence every $IFPI$ is IFSPI.*

Theorem 6.4.3. *Let $\mathbb{G}_1$ be any IFI of a Γ-Ring H. Then*

(i) $\mathcal{V}(\mathbb{G}_1) = \mathcal{V}(\sqrt{\mathbb{G}_1})$

(ii) $\mathcal{X}(h_{(\beta,\vartheta)}) = \mathcal{X}(k_{(\beta,\vartheta)})$ *iff* $\sqrt{< h_{(\beta,\vartheta)} >} = \sqrt{< k_{(\beta,\vartheta)} >}$, *where* $\beta, \vartheta \in (0,1]$ *with* $\beta + \vartheta \leq 1$.

Proof. (i) Let $\mathbb{G}_2 \in \mathcal{V}(\mathbb{G}_1)$ be any element. Then $\mathbb{G}_1 \subseteq \mathbb{G}_2$, where $\mathbb{G}_2$ is an IFPI of H, from proposition (4.3.2) we have $\sqrt{\mathbb{G}_2} = \mathbb{G}_2$, therefore we have $\mathbb{G}_1 \subseteq \sqrt{\mathbb{G}_2}$. Hence $\mathbb{G}_2 \in \mathcal{V}(\sqrt{\mathbb{G}_1})$, so that $\mathcal{V}(\mathbb{G}_1) \subseteq \mathcal{V}(\sqrt{\mathbb{G}_1})$. The reverse inclusion is clear-cut.

(ii)If $\mathcal{X}(h_{(\beta,\vartheta)}) = \mathcal{X}(k_{(\beta,\vartheta)})$, then $\mathcal{V}(h_{(\beta,\vartheta)}) = \mathcal{V}(k_{(\beta,\vartheta)})$ which implies $\mathcal{V}(< h_{(\beta,\vartheta)} >) = \mathcal{V}(< k_{(\beta,\vartheta)} >)$. This mean $\cap\{\mathbb{G}_2: \mathbb{G}_2 \in \mathcal{V}(< h_{(\beta,\vartheta)} >)\} = \cap\{\mathbb{G}_2: \mathbb{G}_2 \in \mathcal{V}(< k_{(\beta,\vartheta)} >)\}$ and therefore, $\sqrt{< h_{(\beta,\vartheta)} >} = \sqrt{< k_{(\beta,\vartheta)} >}$.

Conversely, let $\sqrt{< h_{(\beta,\vartheta)} >} = \sqrt{< k_{(\beta,\vartheta)} >}$. Then

$$\begin{aligned}
\mathbb{G}_2 \in \mathcal{V}\big(h_{(\beta,\vartheta)}\big) \quad &\Leftrightarrow h_{(\beta,\vartheta)} \subseteq \mathbb{G}_2 \\
&\Leftrightarrow < h_{(\beta,\vartheta)} > \subseteq \mathbb{G}_2 \\
&\Leftrightarrow \sqrt{< k_{(\beta,\vartheta)} >} \subseteq \mathbb{G}_2 \\
&\Leftrightarrow \sqrt{< k_{(\beta,\vartheta)} >} \subseteq \mathbb{G}_2 \\
&\Leftrightarrow k_{(\beta,\vartheta)} \subseteq \mathbb{G}_2 \text{ as before} \\
&\Leftrightarrow \mathbb{G}_2 \in \mathcal{V}\big(k_{(\beta,\vartheta)}\big).
\end{aligned}$$

Hence $\mathcal{V}\big(h_{(\beta,\vartheta)}\big) = \mathcal{V}\big(k_{(\beta,\vartheta)}\big)$ so that $\mathcal{X}\big(h_{(\beta,\vartheta)}\big) = \mathcal{X}\big(k_{(\beta,\vartheta)}\big)$.

Definition 6.4.4. ([46]) Let H_1 and H_2 be any sets and let $f\colon H_1 \to H_2$ be a function. An IFS G of H_1 is called an f - invariant if $f(h) = f(k) \Rightarrow G(h) = G(k)$, i.e., $\mu_G(h) = \mu_G(k)$ and $\nu_G(h) = \nu_G(k)$, where $h, k \in H_1$.

For any f - invariant IFS G of H_1, we have $f^{-1}\big(f(G)\big) = G$.

Theorem 6.4.5. ([46]) *Let f be an onto Γ-$\boldsymbol{Hom}_{H_1}^{H_2}$. Let $\mathbb{G}_1$ be any f - invariant IFPI of H_1 and $\mathbb{G}_2$ be any IFPI of H_2. Then $f(\mathbb{G}_1)$ and $f^{-1}(\mathbb{G}_2)$ are IFPI of H_2 and H_1 respectively.*

Theorem 6.4.6. *Let f be an onto $\Gamma - \boldsymbol{Hom}_{H_1}^{H_2}$ and $\mathcal{X} = IFSpec(H_1)$, $\mathcal{X}' = IFSpec(H_2)$, $\mathcal{X}^* = \{\mathbb{G}_1 \in \mathcal{X}\colon \mathbb{G}_1$ is f- invariant $\}$, $\mathcal{X}'(\mathbb{G}_2) = \mathcal{X}' \backslash \mathcal{V}(\mathbb{G}_2)$, where $\mathbb{G}_2$ is any IFI of H_2, and ξ be a map from $\mathcal{X}'$ to $\mathcal{X}^*$ defined by $\xi\big(\mathbb{G}_1{}'\big) = f^{-1}\big(\mathbb{G}_1{}'\big)$, $\mathbb{G}_1{}' \in \mathcal{X}'$. Then the following statements are equivalent*

(i) ξ is continuous

(ii) ξ is open, and

(iii) ξ is a homeomorphism of $\mathcal{X}'$ onto $\mathcal{X}^*$ in other words the map ξ is an embedding that maps $\mathcal{X}'$ onto $\mathcal{X}^*$.

Proof. (i) Suppose $\mathbb{G}_1{}' \in \mathcal{X}'$. Then by using Theorem (6.4.5) $f^{-1}\big(\mathbb{G}_1{}'\big) \in \mathcal{X}$.

Also, $f^{-1}\big(\mathbb{G}_1{}'\big)$ is f -invariant, since for all $a\ b \in H$, if $f(a) = f(b)$, then

$\mu_{\mathbb{G}_1{}'}(f(a)) = \mu_{\mathbb{G}_1{}'}(f(b))$ and $\nu_{\mathbb{G}_1{}'}(f(a)) = \nu_{\mathbb{G}_1{}'}(f(b)) \Rightarrow \mu_{f^{-1}(\mathbb{G}_1{}')}(a) = \mu_{f^{-1}(\mathbb{G}_1{}')}(b)$ and $\nu_{f^{-1}(\mathbb{G}_1{}')}(a) = \nu_{f^{-1}(\mathbb{G}_1{}')}(b)$, i.e., $f^{-1}(\mathbb{G}_1{}')(a) = f^{-1}(\mathbb{G}_1{}')(b)$. Hence $\xi(G') = f^{-1}(G') \in \mathcal{X}^*$.

Next we show that $\xi^{-1}(\mathcal{X}(h_{(\beta,\vartheta)}) \cap \mathcal{X}^*) = \mathcal{X}'\left((f(h))_{(\beta,\vartheta)}\right)$.

Since $\mathbb{G}_1{}' \in \xi^{-1}(\mathcal{X}\left(h_{((\beta,\vartheta))}\right) \Leftrightarrow \xi(\mathbb{G}_1{}') \in \mathcal{X}(h_{(\beta,\vartheta)})$

$\Leftrightarrow h_{(\beta,\vartheta)} \not\subseteq \xi(\mathbb{G}_1{}') = f^{-1}(\mathbb{G}_1{}') \Leftrightarrow (f(h))_{(\beta,\vartheta)} = f(h_{(\beta,\vartheta)}) \not\subseteq \mathbb{G}_1{}'$, by proposition (6.2.5)

$\Leftrightarrow \mathbb{G}_1{}' \in \mathcal{X}'\left((f(h))_{((\beta,\vartheta))}\right)$.

This shows that the inverse image of any basic open set in $\mathcal{X}^*$ is an open set in $\mathcal{X}'$. Hence ξ is continuous.

(ii) $Let\ \mathcal{X}'\left((f(h))_{(\beta,\vartheta)}\right), h \in H_1 and\ (\beta,\vartheta) \in (0,1] s.t.\ \beta + \vartheta \leq 1, be\ any\ basic$ open set in $\mathcal{X}'. Let\ \mathbb{G}_2 \in \mathcal{X}'\left((f(h))_{(\beta,\vartheta)}\right). Then\ \mathbb{G}_2 = \xi(\mathbb{G}_1{}') = f^{-1}(\mathbb{G}_1{}')$ for some $\mathbb{G}_1{}' \in \mathcal{X}'\ such\ that\ (f(h))_{(\beta,\vartheta)} \not\subseteq \mathbb{G}_1{}'. As\ in\ part\ (1) we\ can\ show\ that\ \mathbb{G}_2\ is\ f-invariant.$ $Next, \xi\left(\mathcal{X}'\left((f(h))_{(\beta,\vartheta)}\right)\right) = \mathcal{X}(h_{(\beta,\vartheta)}) \cap \mathcal{X}^*, because$

$\mathbb{G}_1 \in \xi\left(X'\left((f(h))_{(\beta,\vartheta)}\right)\right) \Leftrightarrow \xi^{-1}(\mathbb{G}_1) \in X'\left((f(h))_{(\beta,\vartheta)}\right)\ and\ \mathbb{G}_1\ is\ f\ -invariant$

$\Leftrightarrow f(h_{(\beta,\vartheta)}) = (f(h))_{(\beta,\vartheta)} \not\subseteq \xi^{-1}(\mathbb{G}_1) = f(\mathbb{G}_1)$

$\Leftrightarrow h_{(\beta,\vartheta)} \not\subseteq f^{-1}(f(\mathbb{G}_1)) = \mathbb{G}_1, since\ \mathbb{G}_1\ is\ f-invariant$

$\Leftrightarrow \mathbb{G}_1 \in \mathcal{X}(h_{(\beta,\vartheta)}) \cap \mathcal{X}^*$.

Hence the direct image of every basic open set in $\mathcal{X}'$ is open in $\mathcal{X}^*$ and so ξ is open.

(iii) In the light of part (i) and part (ii), it is enough to prove that h is one-one and onto.

Let $\mathbb{G}_1, \mathbb{G}_2{}' \in \mathcal{X}'$. Then $\xi(\mathbb{G}_1{}') = \xi(\mathbb{G}_2{}') \Rightarrow f^{-1}(\mathbb{G}_1{}') = f^{-1}(\mathbb{G}_2{}') \Rightarrow f\left(f^{-1}(\mathbb{G}_1{}')\right) = f\left(f^{-1}(\mathbb{G}_2{}')\right)$. As f is onto, therefore, we get $\mathbb{G}_1{}' = \mathbb{G}_2{}'$. Thus f is one-one. Finally, let $\mathbb{G}_1 \in \mathcal{X}^*$. Then $\mathbb{G}_1$ is an f -invariant IFPI of H_1 and Therefore by Theorem (6.4.5), $f(\mathbb{G}_1)$

is an IFPI of H_2. Further, $\xi\big(f(\mathbb{G}_1)\big) = f^{-1}\big(f(\mathbb{G}_1)\big) = \mathbb{G}_1$. Since $\mathbb{G}_1$ is f -invariant. Therefore ξ is onto.

6.5 Irreducibility And Connectedness Of IF Spec(H)

In this section the conditions for irreducibility and connectedness of topological space $\mathcal{X}$ are discussed.

Definition 6.5.1. The intersection of all IFPI of H is called the IF nil radical of Γ-Ring H and is written as $IFnil(H)$.

Theorem 6.5.2. *The space $\mathcal{X}$ is irreducible iff $IFnil(H) \in \mathcal{X}$.*

Proof. Let $\mathcal{X}$ be irreducible and let $\mathcal{N}$ be the nil radical of Γ-Ring H. Then

$$\mu_{IFnil(H)}(x) = \begin{cases} 1, & \text{if } h \in \mathcal{N} \\ 0, & \text{if } H\backslash\mathcal{N} \end{cases}; \quad \nu_{IFnil(H)}(x) = \begin{cases} 0, & \text{if } h \in \mathcal{N} \\ 1, & \text{if } H\backslash\mathcal{N} \end{cases}.$$

Next, let $h, k \in H$ and let $\mathcal{b}, \vartheta \in (0,1]$ s.t. $\mathcal{b} + \vartheta \leq 1$. Then $h\gamma k \in \mathcal{N} \Rightarrow h\gamma k$ is nilpotent and thus $\mathcal{X}\big((h\gamma k)_{(\mathcal{b},\vartheta)}\big) = \emptyset$ by Theorem (6.2.6)(ii). Therefore, $\mathcal{X}\big(h_{(\mathcal{b},\vartheta)}\big) \cap \mathcal{X}\big(k_{(\mathcal{b},\vartheta)}\big) = \emptyset$, since $\mathcal{X}$ is irreducible. Hence either h or k is nilpotent, and thus $h \in \mathcal{N}$ or $k \in \mathcal{N}$. Consequently, $\mathcal{N}$ is the prime ideal of H, whence it follows from Theorem (2.2.9) that $IFnil(H) \in \mathcal{X}$.

Conversely, suppose that $IFnil(H) \in \mathcal{X}$. Then $\mathcal{N}$ is the PI of H. Let $h, k \in H$ and let $\mathcal{b}, \vartheta \in (0,1]$ s.t. $\mathcal{b} + \vartheta \leq 1$. Then $\mathcal{X}\big(h_{(\mathcal{b},\vartheta)}\big) \cap \mathcal{X}\big(k_{(\mathcal{b},\vartheta)}\big) = \emptyset$ implies that $\mathcal{X}\big((h\Gamma k)_{(\mathcal{b},\vartheta)}\big) = \emptyset$, by Theorem (6.2.6)(i), and thus $h\gamma k$ is nilpotent for every $\gamma \in \Gamma$, by Theorem (6.2.2)(ii). Then $h\gamma k \in \mathcal{N}$ and so $h \in \mathcal{N}$ or $k \in \mathcal{N}$, which means x is nilpotent or y is nilpotent. Hence $\mathcal{X}\big(h_{(\mathcal{b},\vartheta)}\big) = \emptyset$ or $\mathcal{X}\big(k_{(\mathcal{b},\vartheta)}\big) = \emptyset$, by Theorem (6.2.6)(ii). This shows that the intersection of any two non-empty basic open sets is non-empty. Hence, $\mathcal{X}$ is irreducible.

Theorem 6.5.3. *The space $\mathcal{X}$ is disconnected iff H has a non-trivial idempotent element.*

Proof. Let $\mathcal{X}$ be disconnected. Then $\exists$ IFIs $\mathbb{G}_1$ and $\mathbb{G}_2$ of H s.t. $\mathcal{X} = \mathcal{V}(\mathbb{G}_1) \cup \mathcal{V}(\mathbb{G}_2), \mathcal{V}(\mathbb{G}_1), \mathcal{V}(\mathbb{G}_2) \neq \emptyset, \mathcal{V}(\mathbb{G}_1) \cap \mathcal{V}(\mathbb{G}_2) = \emptyset$.

Now, $\mathcal{V}(\mathbb{G}_1) \cap \mathcal{V}(\mathbb{G}_2) = \emptyset$ implies $\mathcal{V}(\mathbb{G}_1 \oplus \mathbb{G}_2) = \emptyset$ so that $\mu_{\mathbb{G}_1 \oplus \mathbb{G}_2}(x) = 1$ and $\nu_{\mathbb{G}_1 \oplus \mathbb{G}_2}(x) = 0$; for all $x \in H$. So, $Sup_{\mathrm{e}=m+n}\{max\{\mu_{\mathbb{G}_1}(m), \mu_{\mathbb{G}_2}(n)\}\} = 1$ and $Inf_{\mathrm{e}=m+n}\{min\{\nu_{\mathbb{G}_1}(m), \nu_{\mathbb{G}_2}(n)\}\} = 0$, where e is the unity of H $\Rightarrow \mu_{\mathbb{G}_1}(m) = \mu_{\mathbb{G}_2}(n) = 1$ and $\nu_{\mathbb{G}_1}(m) = \nu_{\mathbb{G}_2}(n) = 0$, for all $m, n \in H$ s.t. $\mathrm{e} = m + n$. Let $I = \mathbb{G}_{1*}$ and $J = \mathbb{G}_{2*}$. Let K be the prime ideal of H and χ_K be its IFCF. Then $\chi_K \in \mathcal{X}$. Since $\mathcal{X} = \mathcal{V}(\mathbb{G}_1) \cup \mathcal{V}(\mathbb{G}_2) = \mathcal{V}(\mathbb{G}_1 \cap \mathbb{G}_2)$, it follows that $\mathbb{G}_1 \cap \mathbb{G}_2 \subseteq \chi_K$.

Next, if $h \in I \cap J$, then $\mu_{\mathbb{G}_1 \cap \mathbb{G}_2}(h) = 1$ and $\nu_{\mathbb{G}_1 \cap \mathbb{G}_2}(h) = 0 \Rightarrow \mu_{\chi_K}(h) = 1$ and $\nu_{\chi_K}(h) = 0$ and then $h \in K$. Thus $h \in \cap \{K : K \text{ is a PI of } H\} \Rightarrow x$ is a nilpotent element. This shows that every element of $I \cap J$ is nilpotent.

Clearly, $H/(I \cap J) = I/(I \cap J) \oplus J/(I \cap J)$, Therefore, $\mathrm{e} + (I \cap J) = i + (I \cap J) + j + (I \cap J)$, for some $i \in I, j \in J$. So that $i\gamma(\mathrm{e} - i) \in (I \cap J)$ for every $\gamma \in \Gamma$ and hence $i\gamma(\mathrm{e} - i)$ is nilpotent. Thus $(i\gamma(\mathrm{e} - i)\gamma)^m i\gamma(\mathrm{e} - i) = 0$ for some $m \in Z^+$. Consequently, $(i\gamma(\mathrm{e} - i)\gamma)^m = (i\gamma(\mathrm{e} - i)\gamma)^{m+1} Q\left(\left(i\gamma(\mathrm{e} - i)\right)\right)$, for some polynomial $Q\left(i\gamma(\mathrm{e} - i)\right)$ in $\left(i\gamma(\mathrm{e} - i)\right)$. Let $x = (i\gamma(\mathrm{e} - i)\gamma)^m Q\left(i\gamma(\mathrm{e} - i)\right)$. It is now a simple matter to verify that $h \neq 0, h \neq \mathrm{e}$, and $h\gamma h = h$.

Conversely, for any non-trivial idempotent element x of H, it can be easily verified that $\mathcal{X} = \mathcal{V}(h_{(\beta,\vartheta)}) \cup \mathcal{V}(\left(e - h\right)_{(\beta,\vartheta)}), \mathcal{V}(h_{(\beta,\vartheta)}) \neq \emptyset, \mathcal{V}\left(e - h\right)_{(\beta,\vartheta)}) \neq \emptyset, \mathcal{V}(h_{(\alpha,\beta)}) \cap \mathcal{V}(\left(e - h\right)_{(\beta,\vartheta)}) = \emptyset$ where $\beta, \vartheta \in (0,1]$ s.t. $\beta + \vartheta \leq 1$.

This establishes that $\mathcal{X}$ is disconnected.

Corollary 6.5.4. The space $\mathcal{X}$ is connected iff 0_H and e are the only idempotent in H.

6.6 Conclusion

This chapter, establishes a topology on $\mathcal{X}$ = IFSpec(H), representing the collection of all IFPIs of a commutative Γ-Ring H with unity, known as the Zariski topology. Using the bases for the Zariski topology, it is demonstrated that the subspace $\mathcal{Y}$ of $\mathcal{X}$ is compact. Furthermore, it is shown that space $\mathcal{X}$ is always T_0 but not T_2; however, when H is a Boolean Γ-Ring, it becomes a T_2 space. It is also proven that subspace $\mathcal{Y}$ is T_1 iff every singleton element of $\mathcal{Y}$ is an IFMI of H. For f which is a $Hom_{H_1}^{H_2}$, it is established that $\mathcal{X}' = IFSpec(H_2)$ is homeomorphic to the subset $\mathcal{X}^* = \{G \in \mathcal{X}: G \text{ is } f\text{- invariant }\}$, consisting of f-invariant elements of $\mathcal{X} = IFSpec(H_1)$. Additionally, the space $\mathcal{X}$ is irreducible iff the intersection of all elements of $\mathcal{X}$ is also an element of $\mathcal{X}$. However, the space $\mathcal{X}$ is connected iff 0_H and e are the only idempotent elements in H.

Chapter 7

On Intuitionistic Fuzzy f-Primary Ideals Of Commutative Γ-Rings

7.1 Introduction

In the first section, the chapter introduces the concept of IFI expansion and defines IFPrIs concerning such an expansion. Alongside well-established expansions, a novel expansion denoted as $\mathcal{M}$, defined through IFMIs, is explored. Additionally, IFI expansions meeting certain additional conditions are examined, and further properties of generalized IFPrIs concerning such expansions are investigated.

In the second section, the concept of IF2-AI expansion is introduced, and IF2-APrIs regarding such an expansion are defined. In addition to familiar expansions, a new expansion denoted as $\mathcal{H}$, defined by IFMIs, is studied. Moreover, IF2-AI expansions fulfilling specific additional conditions are explored, and more properties of generalized IF2-APrIs concerning such expansions are investigated.

7.2 Intuitionistic Fuzzy f-Primary Ideals Of Γ-Rings

The notion of expansion of IFIs of a commutative Γ-Ring has been introduced in this section, and using this concept, we developed the notion of IFf-PrIs, where f is a map satisfying additional conditions, and proved more results w.r.t. such expansions.

Definition 7.2.1. Let $\mathcal{G}(H)$ denote the set of all IFIs of Γ-Ring H. Then the map $f: \mathcal{G}(H) \to \mathcal{G}(H)$ is called an expansion of IFIs of H (or briefly as IFI expansion) if following properties are satisfied:

(i) $G \subseteq f(G), \forall G \in \mathcal{G}(H)$

(ii) $\mathbb{G}_1 \subseteq \mathbb{G}_2 \Rightarrow f(\mathbb{G}_1) \subseteq f(\mathbb{G}_2), \forall \mathbb{G}_1, \mathbb{G}_2 \in \mathcal{G}(H)$.

Example 7.2.2.

(1) The identity map $i: \mathcal{G}(H) \to \mathcal{G}(H)$ defined by $i(G) = G$ is an expansion of IFIs of H.

(2) The map $f: \mathcal{G}(H) \to \mathcal{G}(H)$ defined by $f(G) = \sqrt{G}$ is an expansion of IFIs of H.

(3) Denote $\mathcal{M}(G) = \cap\{Q: Q \supseteq G$ and Q is an IFMI of $H\}$. Then the map $g: \mathcal{G}(H) \to \mathcal{G}(H)$ defined by $g(G) = \mathcal{M}(G)$ is an expansion of IFIs of H.

(4) The constant map $c: \mathcal{G}(H) \to \mathcal{G}(H)$ defined as $c(G) = \chi_H$ = (1,0) $\forall\, h \in H$ and $(0,1)\ \forall\, h \notin H$ is an expansion of IFIs of H.

Definition 7.2.3. Given an expansion f of IFIs of H. An IFI $G \in \mathcal{G}(H)$ is said to be an IF f-primary if it satisfies the condition

$$h_{(\eta,\theta)}\gamma k_{(t,s)} \subseteq G \Rightarrow h_{(\eta,\theta)} \subseteq G \text{ or } k_{(t,s)} \subseteq f(G), \forall h_{(\eta,\theta)}, k_{(t,s)} \in IFP(H), \gamma \in \Gamma.$$

Example 7.2.4. Every IFI $G \in \mathcal{G}(H)$ is an IF c-primary, where c is a constant expansion of IFIs of H.

Theorem 7.2.5. *Let f, g be two expansions of IFIs of Γ-Ring H. If $f(G) \subseteq g(G), \forall G \in \mathcal{G}(H)$, then every IFf-PrI is also an IFg $-$PrI.*

Proof. Let $G \in \mathcal{G}(H)$ be an IFfPrI of Γ-Ring H. Let $h_{(\eta,\theta)}, k_{(t,s)} \in IFP(H), \gamma \in \Gamma$ s.t. $h_{(\eta,\theta)}\gamma k_{(t,s)} \subseteq G, h_{(\eta,\theta)} \nsubseteq G$ implies that $k_{(t,s)} \subseteq f(G) \subseteq g(G)$, by using assertion. Hence G is an IFg-PrI of H.

Theorem 7.2.6. *Let f_1, and f_2 be two expansions of IFIs of Γ-Ring H. Let $f: \mathcal{G}(H) \to \mathcal{G}(H)$ defined by $f(G) = f_1(G) \cap f_2(G), \forall G \in \mathcal{G}(H)$. Then f is an IFI expansion of H.*

Proof. $\forall G \in \mathcal{G}(H)$, using definition $G \subseteq f_1(G)$and $G \subseteq f_2(G)$ and so $G \subseteq f_1(G) \cap f_2(G) = f(G)$. Thus $G \subseteq f(G)$. Further let $B, C \in \mathcal{G}(H)$ s.t. $B \subseteq C$. Then $f_1(B) \subseteq f_1(C)$ and $f_2(B) \subseteq f_2(C)$ and so $f(B) = f_1(B) \cap f_2(B) \subseteq f_1(C) \cap f_2(C) = f(C)$, i.e., $f(B) \subseteq f(C)$. Hence f is an IFI expansion of Γ-Ring H.

Theorem 7.2.7. *Let f be an expansion of IFIs of Γ-Ring H. For any subset S of H. Denote*

$\mathcal{G}_f(S) = \cap\{Q: Q$ is an IFf-PrI of H s.t. $\chi_S \subseteq Q\}$. *Then the map*

$\xi: \mathcal{G}(H) \to \mathcal{G}(H)$ defined by $\xi(G) = \mathcal{G}_f(G_*)$, $\forall G \in \mathcal{G}(H)$ is an expansion of IFIs of H.

Proof. Obviously $G \subseteq \mathcal{G}_f(G_*) = \xi(G), \forall G \in \mathcal{G}(H)$.

Let $\mathbb{G}, \breve{G} \in \mathcal{G}(H)$ s.t. $\mathbb{G} \subseteq \breve{G}$. Then

$$\begin{aligned}\xi(\mathbb{G}) &= \mathcal{G}_f(\mathbb{G}_*) = \bigcap\{Q: Q \in \mathcal{G}(H) \text{ s.t. } \chi_{\mathbb{G}_*} \subseteq Q \text{ and } Q \text{ is an IF } f\text{-primary Ideal}\}\\ &\subseteq \bigcap\{Q: Q \in \mathcal{G}(H) \text{ s.t. } \chi_{\breve{G}_*} \subseteq Q \text{ and } Q \text{ is an IF } f\text{-primary }\}\\ &= \mathcal{G}_f(\breve{G}_*)\\ &= \xi(\breve{G}).\end{aligned}$$

Hence ξ is an expansion of IFIs of H.

Theorem 7.2.8. *Let f be an expansion of IFIs of Γ-Ring H. If $\{\mathbb{G}_i: i \in \Lambda\}$ is a directed collection of IFf-PrIs of H, where Λ is an index set, then $G = \bigcup_{i\in\Lambda}\mathbb{G}_i$ is an IFf-PrI of H.*

Proof. Let $h_{(\eta,\theta)}, k_{(t,s)} \in IFI(H), \gamma \in \Gamma$ be s.t. $h_{(\eta,\theta)}\gamma k_{(t,s)} \subseteq G$ and $h_{(\eta,\theta)} \nsubseteq G = \bigcup_{i\in\Lambda}\mathbb{G}_i$. Then $\exists \mathbb{G}_i$ s.t. $h_{(\eta,\theta)}\gamma k_{(t,s)} \subseteq \mathbb{G}_i$ and $h_{(\eta,\theta)} \nsubseteq \mathbb{G}_i$. As each $\mathbb{G}_i$ is an IFf-PrI and $\mathbb{G}_i \subseteq G$. It follows that $k_{(t,s)} \subseteq f(\mathbb{G}_i) \subseteq f(G)$. Hence G will be an IFf-PrI of H.

Theorem 7.2.9. *Let f be an expansion of IFIs of Γ-Ring H. If Q is an IFf-PrI of H, then for every $\mathbb{G}_1$, $\mathbb{G}_2 \in \mathcal{G}(H)$ s.t. $\mathbb{G}_1\Gamma\mathbb{G}_2 \subseteq Q$ and $\mathbb{G}_1 \nsubseteq Q$ implies that $\mathbb{G}_2 \subseteq f(Q)$.*

Proof. Let us suppose Q is an IFf-PrI of H and let $\mathbb{G}_1$, $\mathbb{G}_2 \in \mathcal{G}(H)$ s.t. $\mathbb{G}_1\Gamma\mathbb{G}_2 \subseteq Q$, and $\mathbb{G}_1 \nsubseteq Q$. Suppose that $\mathbb{G}_2 \nsubseteq f(Q)$. Then $\exists\, h, k \in H$ s.t. $\mu_{\mathbb{G}_1}(h) > \mu_Q(h), \nu_{\mathbb{G}_1}(h) < \nu_Q(h)$ and $\mu_{\mathbb{G}_2}(k) > \mu_{f(Q)}(k), \nu_{\mathbb{G}_2}(k) < \nu_{f(Q)}(k)$. Let $\mu_{\mathbb{G}_1}(h) = \eta, \nu_{\mathbb{G}_1}(h) = \theta$ and $\mu_{\mathbb{G}_2}(h) = t, \nu_{\mathbb{G}_2}(h) = s$. Then $\mu_Q(h) < \eta, \nu_Q(h) > \theta$ and $\mu_{f(Q)}(k) < t, \nu_{f(Q)}(k) > s$. This implies that $h_{(\eta,\theta)} \subseteq \mathbb{G}_1$ and $k_{(t,s)} \subseteq \mathbb{G}_2$, but $h_{(\eta,\theta)} \nsubseteq Q$ and $k_{(t,s)} \nsubseteq f(Q)$. Now

$$\mu_Q(h\gamma k) \geq \mu_{\mathbb{G}_1\Gamma\mathbb{G}_2}(h\gamma k) \geq \{\mu_{\mathbb{G}_1}(h) \wedge \mu_{\mathbb{G}_2}(k)\} = \eta \wedge t = \mu_{h_{(\eta,\theta)}\gamma k_{(t,s)}}(x\gamma y) \text{ and}$$

$\nu_Q(h\gamma k) \leq \nu_{\mathbb{G}_1\Gamma\mathbb{G}_2}(h\gamma k) \leq \{\nu_{\mathbb{G}_1}(h) \vee \nu_{\mathbb{G}_2}(k)\} = \theta \vee s = \nu_{h_{(\eta,\theta)}\gamma k_{(t,s)}}(h\gamma k)$. Hence $h_{(\eta,\theta)}\gamma k_{(t,s)} \subseteq Q$. But $h_{(\eta,\theta)} \nsubseteq Q$ and $k_{(t,s)} \nsubseteq f(Q)$. This contradicts the assumption that Q is IFf-PrI of H. Consequently the result is valid.

Remark 7.2.10. In the definition of IFf-PrIs, the statement "$\mathbb{G}_1 \Gamma \mathbb{G}_2 \subseteq Q$" and $\mathbb{G}_1 \nsubseteq Q$ implies that $\mathbb{G}_2 \subseteq f(Q)$. In Theorem (7.2.9) this can be replaced as "$\mathbb{G}_1 \Gamma \mathbb{G}_2 \subseteq Q$" and $\mathbb{G}_1 \nsubseteq f(Q)$ implies that $\mathbb{G}_2 \subseteq Q$.

For any IFI $\mathbb{G}_1$ of a Γ-Ring H and any IFS $\mathbb{G}_2$ of H, the IF residual quotient of $\mathbb{G}_1$ by $\mathbb{G}_2$ is denoted by $(\mathbb{G}_1 : \mathbb{G}_2) = \cup\{h_{(\eta,\theta)} \in IFP(H) : h_{(\eta,\theta)} \Gamma \mathbb{G}_2 \subseteq \mathbb{G}_1\}$. It can be easily seen that $(\mathbb{G}_1 : \mathbb{G}_2)$ is an IFI of H s.t. $\mathbb{G}_1 \subseteq (\mathbb{G}_1 : \mathbb{G}_2)$.

Theorem 7.2.11. *Suppose f be an expansion of IFIs of Γ-Ring H. Then*

(i) If Q is an IFf-PrI and G is an IFI of H s.t. $G \nsubseteq f(Q)$, then $(Q: G) = Q$.

(ii) For any IFf-PrI Q and any subset N of H, $(Q: \chi_N)$ is also an IFf-PrI.

Proof. **(i)** Since $Q \supseteq G \cap Q \supseteq G\Gamma Q$, i.e., $G\Gamma Q \subseteq Q$, so $Q \subseteq (Q: G)$. Also by definition, we have $G\Gamma(Q: G) \subseteq Q$. Since $G \nsubseteq f(Q)$ we have $(Q: G) \subseteq Q$ [Using Remark 7.2.10]. Therefore $(Q: G) = Q$.

(ii) Let $h_{(\eta,\theta)} \Gamma k_{(t,s)} \subseteq (Q: \chi_N)$ and $h_{(\eta,\theta)} \nsubseteq (Q: \chi_N)$. Then $h_{(\eta,\theta)} \Gamma \chi_N \nsubseteq Q$. Therefore $\exists$, $n \in N, \gamma_1 \in \Gamma$ s.t. $\mu_{h_{(\eta,\theta)} \Gamma \chi_N}(h\gamma_1 n) > \mu_Q(h\gamma_1 n)$ and $\nu_{h_{(\eta,\theta)} \Gamma \chi_N}(h\gamma_1 n) < \nu_Q(h\gamma_1 n)$, i.e., $\eta > \mu_Q(h\gamma_1 n)$ and $\theta < \nu_Q(h\gamma_1 n)$ and so $(h\gamma_1 n)_{(\eta,\theta)} \nsubseteq Q$, i.e., $h_{(\eta,\theta)} \gamma_1 n_{(\eta,\theta)} \nsubseteq Q$. But $h_{(\eta,\theta)} \gamma_1 n_{(\eta,\theta)} \gamma_2 k_{(t,s)} = (h\gamma_1 n \gamma_2 k)_{(\eta \wedge t, \theta \vee s)} = (h\gamma_3 k)_{(\eta \wedge t, \theta \vee s)} \subseteq Q$, where $\gamma_3 = \gamma_1 n \gamma_2$. As Q is an IFf-PrI so $k_{(t,s)} \subseteq f(Q) \subseteq f\big((Q: \chi_N)\big)$. Hence $(Q: \chi_N)$ is an IFf-PrI.

Definition 7.2.12. Let f be an expansion of IFIs of Γ-Ring H. Then f is said to be intersection preserving if it satisfies "$f(\mathbb{G}_1 \cap \mathbb{G}_2) = f(\mathbb{G}_1) \cap f(\mathbb{G}_2)$", for every $\mathbb{G}_1$, $\mathbb{G}_2 \in \mathcal{G}(H)$.

Also, f is said to be global if for each σ which is $\Gamma\text{-}Hom_{H_1}^{H_2}$, the following hold:

$$f\big(\sigma^{-1}(G)\big) = \sigma^{-1}\big(f(G)\big) \ \forall \ G \in \mathcal{G}(H_2).$$

Note that an expansion i of IFIs of Γ-Ring H in example (7.2.2) (i) is both intersection preserving as well as global.

Theorem 7.2.13. $\forall G \in \mathcal{G}(H)$, *let* $\mathcal{P}(G) := \bigcap\{B : B \supseteq G$ and B is IFPI of $H\}$. *Then the map* $f : \mathcal{G}(H) \to \mathcal{G}(H)$ *given by* $f(G) = \mathcal{P}(G)$ *is an intersection preserving expansion of IFIs of* Γ*-Ring H.*

Proof. Obviously, f is an expansion of IFIs of Γ-Ring H. For every $\mathbb{G}_1, \mathbb{G}_2 \in \mathcal{G}(H)$, let us denote

$\mathcal{P}_1 := \{P : P \supseteq \mathbb{G}_1 \cap \mathbb{G}_2, P$ is IFPI of $H\}$; $\mathcal{P}_2 := \{P : P \supseteq \mathbb{G}_1$ or $P \supseteq \mathbb{G}_2$, P is IFPI of $H\}$.

Then $\bigcap \mathcal{P}_1 = \mathcal{P}(\mathbb{G}_1 \cap \mathbb{G}_2)$ and $\bigcap \mathcal{P}_2 = \mathcal{P}(\mathbb{G}_1) \cap \mathcal{P}(\mathbb{G}_2)$. Obviously $\mathcal{P}_2 \subseteq \mathcal{P}_1$. If $P \in \mathcal{P}_1$ then $\mathbb{G}_1 \Gamma \mathbb{G}_2 \subseteq \mathbb{G}_1 \cap \mathbb{G}_2 \subseteq P$. As P is IFPI, so $\mathbb{G}_1 \subseteq P$ or $\mathbb{G}_2 \subseteq P$. i.e., $P \in \mathcal{P}_2$ and so $\mathcal{P}_1 \subseteq \mathcal{P}_2$, then $\mathcal{P}_1 = \mathcal{P}_2$. Thus $f(\mathbb{G}_1 \cap \mathbb{G}_2) = \mathcal{P}(\mathbb{G}_1 \cap \mathbb{G}_2) = \bigcap \mathcal{P}_1 = \bigcap \mathcal{P}_2 = \mathcal{P}(\mathbb{G}_1) \cap \mathcal{P}(\mathbb{G}_2) = f(\mathbb{G}_1) \cap f(\mathbb{G}_2)$. Hence proved.

Theorem 7.2.14. *Let* f *be an expansion of IFIs of* Γ*-Ring H which is intersection preserving. If* $\mathbb{G}_1, \mathbb{G}_2, \ldots\ldots.\mathbb{G}_n$ *are IFf-PrIs of H and* $B = f(\mathbb{G}_k)\ \forall\ k = 1,2,\ldots\ldots.n$, *then* $G := \bigcap_{k=1}^{n} \mathbb{G}_k$ *is an IFf-PrI of H.*

Proof. Obviously, $G := \bigcap_{k=1}^{n} \mathbb{G}_k$ is an IFI of H. Let C, D are IFIs of H s.t. $C\Gamma D \subseteq G$ and $C \nsubseteq G$. Then $C \nsubseteq \mathbb{G}_k$ for some $\mathbb{G}_k$, where $k \in \{1,2,\ldots,n\}$. But $C\Gamma D \subseteq G \subseteq \mathbb{G}_k$ and $\mathbb{G}_k$ are IFf-PrI of H, which imply that $D \subseteq f(\mathbb{G}_k)$. Since f is intersection preserving, so

$$f(G) = f(\bigcap_{k=1}^{n} \mathbb{G}_k) = \bigcap_{k=1}^{n} f(\mathbb{G}_k) = B = f(\mathbb{G}_k)$$

and so $D \subseteq f(G)$. Therefore G is an IFf-PrI of H.

Let σ *be a* Γ*-*$Hom_{H_1}^{H_2}$. Note that if G is an IFI of H_2, then $\sigma^{-1}(G)$ is an IFI of H_1 and that if σ is surjective and G is an IFI of H_1, then $\sigma(G)$ is an IFI of H_2.

Theorem 7.2.15. *Let* f *be an expansion of IFIs which is global and let* σ *be a* Γ*-*$Hom_{H_1}^{H_2}$. *If* B *is an IFf-PrI of* H_2. *then* $\sigma^{-1}(B)$ *is an IFf-PrI of* H_1.

Proof. Let $\mathbb{G}, \breve{G}$ be two IFIs of H_1 s.t. $\mathbb{G}\Gamma\breve{G} \subseteq \sigma^{-1}(B)$ and $\mathbb{G} \nsubseteq \sigma^{-1}(B)$. Then $\sigma(\mathbb{G})\Gamma\sigma(\breve{G}) = \sigma(\mathbb{G}\Gamma\breve{G}) \subseteq B$ and $\sigma(\mathbb{G}) \nsubseteq B$, which implies that $\sigma(\breve{G}) \subseteq f(B)$. Since f is global, it follows that $\breve{G} \subseteq \sigma^{-1}(f(B)) = f(\sigma^{-1}(B))$. Hence $\sigma^{-1}(B)$ is an IFf-PrI of H_1.

By using the same argument it may be easily seen that if σ *be a* Γ-$Hom_{H_2}^{H_1}$, then $\sigma^{-1}(\sigma(G)) = G$ for each $G \in \mathcal{G}(H_1)$ that contains $Ker(\sigma)$.

Theorem 7.2.16. *Let* σ *be a surjective* Γ-$Hom_{H_1}^{H_2}$ *and let* G *be an IFI of* H_1 *that contains* $Ker(\sigma)$. *Then* G *is an IFf-PrI of* H_1 *iff* $\sigma(G)$ *is an IFf-PrI of* H_2, *where* f *is a global IFI expansion.*

Proof. If $\sigma(G)$ is an IFf-PrI of H_2, then G is an IFf-PrI of H, by Theorem (7.2.15) and $G = \sigma^{-1}(\sigma(G))$. Suppose that G is an IFf-PrI of H_1 and let B, C be IFIs of H_2 s.t. $B\Gamma C \subseteq \sigma(G)$ and $B \nsubseteq \sigma(G)$. Since σ is surjective we have $\sigma(D) = B$ and $\sigma(E) = C$ for some IFIs D and E in H_1. Then $\sigma(D\Gamma E) = \sigma(D)\Gamma\sigma(E) = B\Gamma C \subseteq \sigma(G)$ and $\sigma(D) = B \nsubseteq \sigma(G)$, which imply that $D\Gamma E \subseteq \sigma^{-1}(\sigma(G)) = G$ and $D \nsubseteq \sigma^{-1}(\sigma(G)) = G$. Since G is an IFf-PrI of H_1, it follows that $E \subseteq f(G)$ so that $C = \sigma(E) \subseteq \sigma(f(G))$. Using the fact that f is global, we have

$$f(G) = f\left(\sigma^{-1}(\sigma(G))\right) = \sigma^{-1}(f(\sigma(G))$$

and so $\sigma(f(G)) = \sigma\left(\sigma^{-1}\left(f(\sigma(G))\right)\right) = f(\sigma(G))$. Since σ is surjective, therefore $C \subseteq f(\sigma(G))$ and so $\sigma(G)$ is an IFf-PrI of H_2. This completes the proof.

7.3 Intuitionistic Fuzzy 2-Absorbing f −Primary Ideals Of Γ-Ring

In this section, we investigated IF2 −Af −PrIs of Γ-Ring, where f is an expansion of IFIs of Γ-Ring H.

Definition 7.3.1. Given an expansion f of IFIs of H. An IFI $G \in \mathcal{G}(H)$ is said to be IF2-Af −PrI if for any *IFPs* $h_{(\eta,\theta)}, k_{(\beta,\vartheta)}, p_{(\tau,\omega)}$ of H and $\gamma_1, \gamma_2 \in \Gamma$ s.t.

$$h_{(\eta,\theta)}\gamma_1 k_{(\beta,\vartheta)}\gamma_2 p_{(\tau,\omega)} \subseteq G \Rightarrow h_{(\eta,\theta)}\gamma_1 k_{(\beta,\vartheta)} \subseteq G \quad \text{or} \quad h_{(\eta,\theta)}\gamma_2 p_{(\tau,\omega)} \subseteq f(G) \quad \text{or} \quad k_{(\beta,\vartheta)}\gamma_2 p_{(\tau,\omega)} \subseteq f(G)$$

.

Example 7.3.2. (1) The map defined in example (1) of (7.2.2), IF2 $-$Af $-$PrI is just IF2 $-$AI as defined in definition (2.2.12).

(2) The map defined in example (2) of (7.2.2), IF2 $-$Af $-$PrI is just IF2 $-$APrI as defined in definition (4.5.1).

In the following we will give a list of results, they are an extension of some results.

Theorem 7.3.3. *Let f, g be two expansions of IFIs of Γ-Ring H. If $f(G) \subseteq g(G), \forall G \in \mathcal{G}(H)$, then every IF2 $-Af$ $-$PrI is also an IF2 $-Ag$ $-$PrI.*

Proof. Let $G \in \mathcal{G}(H)$ be IF2 $-$Af $-$PrI of Γ-Ring H. Let $h_{(\eta,\theta)}, k_{(\beta,\vartheta)}, p_{(\tau,\omega)}$ of H and $\gamma_1, \gamma_2 \in \Gamma$ s.t. $h_{(\eta,\theta)}\gamma_1 k_{(\beta,\vartheta)}\gamma_2 p_{(\tau,\omega)} \subseteq G \Rightarrow h_{(\eta,\theta)}\gamma_1 k_{(\beta,\vartheta)} \subseteq G$ or $h_{(\eta,\theta)}\gamma_2 p_{(\tau,\omega)} \subseteq f(G) \subseteq g(G)$ or $k_{(\beta,\vartheta)}\gamma_2 p_{(\tau,\omega)} \subseteq f(G) \subseteq g(G)$, by assertion. Hence G is IF2 $-$Ag $-$PrI of H.

Theorem 7.3.4. *Let f be an expansion of IFIs of Γ-Ring H. For any subset S of H. Denote*

$\mathcal{G}_f(S) = \bigcap\{Q: Q$ is an IF2 $-$ Af $-$ PrI of H s.t. $\chi_S \subseteq Q\}$. *Then the map $\xi: \mathcal{G}(H) \to \mathcal{G}(H)$ defined by $\xi(G) = \mathcal{G}_f(G_*)$, $\forall G \in \mathcal{G}(H)$ is an expansion of IFIs of H.*

Proof. Obviously $G \subseteq \mathcal{G}_f(G_*) = \xi(G), \forall G \in \mathcal{G}(H)$.

Let $\mathbb{G}, \breve{G} \in \mathcal{G}(H)$ s.t. $\mathbb{G} \subseteq \breve{G}$. Then

$$\begin{aligned}
\xi(\mathbb{G}) &= \mathcal{G}_f(\mathbb{G}_*) = \bigcap\{Q: Q \in \mathcal{G}(H) \text{ s.t. } \chi_{\mathbb{G}_*} \subseteq Q \text{ and } Q \text{ is IF2} - \text{A}f - \text{PrI}\} \\
&\subseteq \bigcap\{Q: Q \in \mathcal{G}(H) \text{ s.t. } \chi_{\breve{G}_*} \subseteq Q \text{ and } Q \text{ is IF2} - \text{A}f - \text{PrI}\} \\
&= \mathcal{G}_f(\breve{G}_*) \\
&= \xi(\breve{G}).
\end{aligned}$$

Hence ξ is an expansion of IFIs of H.

Theorem 7.3.5. *Let f be an expansion of IFIs of Γ-Ring H. If $\{\mathbb{G}_i: i \in \Lambda\}$ is a directed collection of IF2 $-Af$ $-$PrIs of H, where Λ is an index set, then $G = \bigcup_{i\in\Lambda}\mathbb{G}_i$ is IF2-Af-PrI of H.*

Proof. Let $h_{(\eta,\theta)}, k_{(\beta,\vartheta)}, p_{(\tau,\omega)}$ of H and $\gamma_1, \gamma_2 \in \Gamma$ s.t. $h_{(\eta,\theta)}\gamma_1 k_{(\beta,\vartheta)}\gamma_2 p_{(\tau,\omega)} \subseteq G$. Then $\exists\ i \in \Lambda$ s.t. $h_{(\eta,\theta)}\gamma_1 k_{(\beta,\vartheta)}\gamma_2 p_{(\tau,\omega)} \subseteq \mathbb{G}_i$. Since each $\mathbb{G}_i$ is IF2 $-$Af $-$PrI and $\mathbb{G}_i \subseteq G$. It follows that $h_{(\eta,\theta)}\gamma_1 k_{(\beta,\vartheta)} \subseteq \mathbb{G}_i$ or $h_{(\eta,\theta)}\gamma_2 p_{(\tau,\omega)} \subseteq f(\mathbb{G}_i)$ or $k_{(\beta,\vartheta)}\gamma_2 p_{(\tau,\omega)} \subseteq f(\mathbb{G}_i)$. Since

$\Bbbk_i \subseteq f(\Bbbk_i) \subseteq f(G)$, $h_{(\eta,\theta)}\gamma_1 k_{(\beta,\vartheta)} \subseteq G$ or $h_{(\eta,\theta)}\gamma_2 p_{(\tau,\omega)} \subseteq f(G)$ or $k_{(\beta,\vartheta)}\gamma_2 p_{(\tau,\omega)} \subseteq f(G)$, so that G is IF2 $-$Af $-$PrI of H.

Theorem 7.3.6. *Let f be an expansion of IFIs of Γ-Ring H which is intersection preserving. If $\Bbbk_1, \Bbbk_2, \ldots\ldots. \Bbbk_n$ are IF2 $-Af$ $-$PrIs of H and $B = f(\Bbbk_m)$ for all $m = 1,2, \ldots\ldots. n$, then $G := \bigcap_{m=1}^{n} \Bbbk_m$ is an IF2 $-Af$-PrI of H.*

Proof. Obviously, $G := \bigcap_{m=1}^{n} \Bbbk_m$ is an IFI of H. Let $h_{(\eta,\theta)}, k_{(\beta,\vartheta)}, p_{(\tau,\omega)} \in IFI(H)$ and $\gamma_1, \gamma_2 \in \Gamma$ such that $h_{(\eta,\theta)}\gamma_1 k_{(\beta,\vartheta)}\gamma_2 p_{(\tau,\omega)} \subseteq G$ and $h_{(\eta,\theta)}\gamma_1 k_{(\beta,\vartheta)} \nsubseteq G$. Then $h_{(\eta,\theta)}\gamma_1 k_{(\beta,\vartheta)} \nsubseteq \Bbbk_m$ for some $m \in \{1,2,\ldots,n\}$. But $h_{(\eta,\theta)}\gamma_1 k_{(\beta,\vartheta)}\gamma_2 p_{(\tau,\omega)} \subseteq G \subseteq \Bbbk_m$ and $\Bbbk_m$ is an IF2 $-$Af $-$PrI of H, which imply that $h_{(\eta,\theta)}\gamma_2 p_{(\tau,\omega)} \subseteq f(\Bbbk_m)$ or $k_{(\beta,\vartheta)}\gamma_2 p_{(\tau,\omega)} \subseteq f(\Bbbk_m)$. Since f is intersecting preserving, so

$$f(G) = f(\bigcap_{m=1}^{n} \Bbbk_m) = \bigcap_{m=1}^{n} f(\Bbbk_m) = B = f(\Bbbk_m)$$

and so $h_{(\eta,\theta)}\gamma_2 p_{(\tau,\omega)} \subseteq f(G)$ or $k_{(\beta,\vartheta)}\gamma_2 p_{(\tau,\omega)} \subseteq f(G)$. Therefore G is an IF2 $-$Af $-$PrI of H.

Theorem 7.3.7. *Let f be an expansion of IFIs which is global and let σ is Γ-$Hom_{H_1}^{H_2}$. If $\breve{G}$ is an IF2 $-Af$-PrI of H_2, then $\sigma^{-1}(\breve{G})$ is an IF2 $-Af$ $-$PrI of H_1.*

Proof. Let $h_{(\eta,\theta)}, k_{(\beta,\vartheta)}, p_{(u,v)} \in IFP(H_1)$ and $\gamma_1, \gamma_2 \in \Gamma$ s.t. $h_{(\eta,\theta)}\gamma_1 k_{(\beta,\vartheta)}\gamma_2 p_{(\tau,\omega)} \subseteq \sigma^{-1}(\breve{G})$. Then $\sigma(x_{(\eta,\theta)})\gamma_1\sigma(y_{(\beta,\vartheta)})\gamma_2\sigma(z_{(\tau,\omega)}) \subseteq \breve{G}$, i.e., $(\sigma(h))_{(\eta,\theta)}\gamma_1(\sigma(k))_{(\beta,\vartheta)}\gamma_2(\sigma(p))_{(\tau,\omega)} \subseteq \breve{G}$, which imply that $(\sigma(h))_{(\eta,\theta)}\gamma_1(\sigma(k))_{(\beta,\vartheta)} \subseteq \breve{G}$ or $(\sigma(h))_{(\eta,\theta)}\gamma_2(\sigma(p))_{(\tau,\omega)} \subseteq f(\breve{G})$ or $(\sigma(k))_{(\beta,\vartheta)}\gamma_2(\sigma(p))_{(\tau,\omega)} \subseteq f(\breve{G})$. Since f is global, it follows that $h_{(\eta,\theta)}\gamma_1 k_{(\beta,\vartheta)} \subseteq \sigma^{-1}(\breve{G})$ or $h_{(\eta,\theta)}k_{(\beta,\vartheta)}\gamma_2 p_{(\tau,\omega)} \subseteq \sigma^{-1}\left(f(\breve{G})\right) = f\left(\sigma^{-1}(\breve{G})\right)$ or $k_{(\beta,\vartheta)}\gamma_2 p_{(\tau,\omega)} \subseteq \sigma^{-1}\left(f(\breve{G})\right) = f\left(\sigma^{-1}(\breve{G})\right)$. Hence $\sigma^{-1}(\breve{G})$ is an IF2 $-$Af $-$PrI of H_1.

It can be easily verified that if σ is a Γ-$Hom_{H_1}^{H_2}$, then $\sigma^{-1}(\sigma(G)) = G$ for every $G \in \mathcal{G}(H)$ that contains $Ker(\sigma)$.

Theorem 7.3.8. *Let σ is surjective Γ-$Hom_{H_1}^{H_2}$ of Γ-Rings and let G be an IFI of H_1 that contains $Ker(\sigma)$. Then G is an IF2 $-Af$ $-$PrI of H_1 iff $\sigma(G)$ is an IF2 $-Af$ $-$PrI of H_2, where f is a global IFI expansion.*

Proof. If $\sigma(G)$ is an IF2 $-$Af $-$PrI of H_2, then G is an IF2 $-$Af $-$PrI of H_1, by Theorem (7.3.7) and $G = \sigma^{-1}(\sigma(G))$. Suppose that G is an IF2 $-$Af $-$PrI of H_1. Let $h_{(\eta,\theta)}, k_{(\beta,\vartheta)}, p_{(\tau,\omega)} \in IFP(H_2)$ and $\gamma_1, \gamma_2 \in \Gamma$ s.t. $h_{(\eta,\theta)}\gamma_1 k_{(\beta,\vartheta)}\gamma_2 p_{(\tau,\omega)} \subseteq \sigma(G)$. Since σ is surjective we have $\sigma(a) = h, \sigma(b) = k, \sigma(c) = p$, for some $a, b, c \in H_1$.Then $\sigma(a_{(\eta,\theta)}\gamma_1 b_{(\beta,\vartheta)}\gamma_2 c_{(\tau,\omega)}) = (\sigma(a))_{(\eta,\theta)}\gamma_1(\sigma(b))_{(\beta,\vartheta)}\gamma_2(\sigma(c))_{(\tau,\omega)} =$ $h_{(\eta,\theta)}\gamma_1 k_{(\beta,\vartheta)}\gamma_2 p_{(\tau,\omega)} \subseteq \sigma(G)$, which imply that $a_{(\eta,\theta)}\gamma_1 b_{(\beta,\vartheta)}\gamma_2 c_{(\tau,\omega)} \subseteq \sigma^{-1}(\sigma(G)) = G$. Since G is an IF2 $-$Af $-$PrI of H_1, it follows that $a_{(\eta,\theta)}\gamma_1 b_{(\beta,\vartheta)} \subseteq G$ or $a_{(\eta,\theta)}\gamma_2 c_{(\tau,\omega)} \subseteq f(G)$ or $b_{(\beta,\vartheta)}\gamma_2 c_{(\tau,\omega)} \subseteq f(G)$, i.e., $h_{(\eta,\theta)}\gamma_1 k_{(\beta,\vartheta)} \subseteq \sigma(G)$ or $h_{(\eta,\theta)}\gamma_2 p_{(\tau,\omega)} \subseteq \sigma(f(G))$ or $k_{(\beta,\vartheta)}\gamma_2 p_{(\tau,\omega)} \subseteq \sigma(f(G))$. As f is global, we have

$$f(G) = f\left(\sigma^{-1}(\sigma(G))\right) = \sigma^{-1}(f(\sigma(G))$$

and so $\sigma(f(G)) = \sigma\left(\sigma^{-1}\left(f(\sigma(G))\right)\right) = f(\sigma(G))$. Since σ is surjective. Therefore $\sigma(G)$ is an IF2 $-$Af $-$PrI of H_2. This completes the proof.

7.4 Conclusion

This chapter, introduces the concept of IFf-PrIs (2-absorbing f-primary ideals), which serves as a unification of the notions of IFPIs (2-absorbing ideals) and IFPrIs (2-APrIs) within a Γ-Ring. The exploration of these concepts signifies a new direction towards establishing the foundation for studying the decomposition property for IFf-PrI (2-absorbing f-primary ideal).

Chapter 8

Extensions Of Intuitionistic Fuzzy Ideal Of Γ-Rings

8.1 Introduction

The concept of extensions of fuzzy ideal with respect to an element in the Γ-semiring was introduced by Venkateshwarlu, Rao, and Narayana in [67]. By using this concept, they characterized FPI and FSPI. In this chapter, notion of extension of IFI with respect to an element of Γ-Ring is investigated and characterization of IFPIs and IFSPIs has been innovated.

8.2 Extensions Of Intuitionistic Fuzzy Ideal Of Γ-Rings

The concept of extensions of IFI of Γ-Rings has been coined and characterization of IFPI and IFSPI has been done in this section.

Definition 8.2.1. Suppose H is a Γ-Ring. Take any IFS G of H and $h \in H$. The IFS $< h, G >$ defined by $\mu_{<h,G>}(k) = Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 r\gamma_2 k)\}$ and $\nu_{<h,G>}(k) = Sup_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_G(h\gamma_1 r\gamma_2 k)\}$ is said to be extension of G by h, where $k \in H$.

Proposition 8.2.2. *Let H be a commutative Γ-Ring. Take G is an IFI of H and $h \in H$, then the extension $< h, G >$ of G by h is an IFI of H.*

Proof. Clearly $< h, G >$ is an IFS of H. Let $r_1, r_2 \in H$, $\gamma \in \Gamma$, we have

$$\begin{aligned}\mu_{<h,G>}(r_1-r_2) &= Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 r\gamma_2(r_1-r_2))\}\\ &= Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 r\gamma_2 r_1 - h\gamma_1 r\gamma_2 r_2))\}\\ &\geq Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 r\gamma_2 r_1)\wedge\mu_G(h\gamma_1 r\gamma_2 \mathrm{r}_2)\}\\ &= \{Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}(\mu_G(h\gamma_1 r\gamma_2 r_1))\}\wedge\{Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}(\mu_G(h\gamma_1 r\gamma_2 r_1))\}\\ &= \mu_{<h,G>}(r_1)\wedge\mu_{<h,G>}(r_2).\end{aligned}$$

Thus $\mu_{<h,G>}(r_1-r_2)\geq\mu_{<h,G>}(r_1)\wedge\mu_{<h,G>}(r_2)$. In the same manner this can seen that $\nu_{<h,G>}(r_1-r_2)\leq\nu_{<h,G>}(r_1)\vee\nu_{<h,G>}(r_2)$. Also,

$$\begin{aligned}\mu_{<h,G>}(r_1\gamma r_2) &= Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 r\gamma_2(r_1\gamma r_2))\}\\ &= Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G((h\gamma_1 r\gamma_2 r_1)\gamma r_2)\}\\ &\geq Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 r\gamma_2 r_1)\}\}\\ &= \mu_{<h,G>}(r_1).\end{aligned}$$

Since H is a comm. Γ-Ring $r_1\gamma r_2 = r_2\gamma r_1$, for all $r_1, r_2\in H, \gamma\in\Gamma$.

$$\begin{aligned}\mu_{<h,G>}(r_1\gamma r_2) &= \mu_{<h,G>}(r_2\gamma r_1) = Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 r\gamma_2(r_2\gamma r_1))\}\\ &= Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G((h\gamma_1 r\gamma_2 r_2)\gamma r_1)\}\\ &\geq Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 r\gamma_2 r_2)\}\\ &= \mu_{<h,G>}(r_2)\end{aligned}$$

Thus $\mu_{<h,G>}(r_1\gamma r_2)\geq\mu_{<h,G>}(r_1)\vee\mu_{<h,G>}(r_2)$. Similarly, we can show $\nu_{<h,G>}(r_1\gamma r_2)\leq\mu_{<h,G>}(r_1)\wedge\nu_{<h,G>}(r_2)$. Hence $< h, G >$ is an IFI of H.

Example 8.2.3. Consider $H=\Gamma=Z_9=\{0,1,2,3,\ldots.,8\}$ under the operations addition modulo 9 and multiplication modulo 9. Then H is a Γ-Ring. Define an IFS G of H as

$$\mu_G(h)=\begin{cases}1, & \text{if } h=0\\ 0.4, & \text{if } h\in\{3,6\}\\ 0.7, & \text{otherwise}\end{cases};\quad \nu_G(h)=\begin{cases}0, & \text{if } h=0\\ 0.5, & \text{if } h\in\{3,6\}\\ 0.2, & \text{otherwise.}\end{cases}$$

It is easy to verify that G is not an IFI of H, for $\mu_G(4-1)=\mu_G(3)=0.4\ngeq 0.7=\mu_G(4)\wedge\mu_G(1)$. However, the extension of G by 3, i.e., the IFS $< 3+G >$ is defined as

$$\mu_{<3+G>}(h)=\begin{cases}1, & \text{if } h\in\{0,3,6\}\\ 0.4, & \text{otherwise}\end{cases};\quad \nu_{<3+G>}(h)=\begin{cases}0, & \text{if } h\in\{0,3,6\}\\ 0.5, & \text{otherwise.}\end{cases}$$

is an IFI of H.

Proposition 8.2.4. *Suppose H is a commutative Γ-Ring. If G is an IFI of H and $h \in H$. Then these axioms are true*

1. $G \subseteq < h, G >$
2. $< (h\gamma)^{n-1}h, G > \subseteq < (h\gamma)^n h, G >$, *where* $\gamma \in \Gamma$
3. *If* $h \in Supp(G)$, *then* $Supp(< h, G >) = H$, *where* $Supp(G)$ *is eloborated as* $Supp(G) = \{h \in H: \mu_G(h) > 0, \nu_G(h) < 1\}$.

Proof. **(1)** Let $k \in H$. Since G is an IFI of H, so $\mu_{<h,G>}(k) = Inf_{r \in H, \gamma_1, \gamma_2 \in \Gamma}\{\mu_G(h\gamma_1 r\gamma_2 k)\} \geq \mu_G(k)$ and $\nu_{<h,G>}(k) = Sup_{r \in H, \gamma_1, \gamma_2 \in \Gamma}\{\nu_G(h\gamma_1 r\gamma_2 k)\} \leq \nu_G(k), \forall k \in H$

Thus $G \subseteq < h, G >$.

(2) Let $n \in \mathbf{N}$, $k \in H$. Since G is an IFI of H, we have

$$\begin{aligned} \mu_{<(h\gamma)^n h, G>}(k) &= Inf_{r \in H, \gamma_1, \gamma_2 \in \Gamma}\{\mu_G((h\gamma)^n h\gamma_1 r\gamma_2 k)\} \\ &= Inf_{r \in H, \gamma_1, \gamma_2 \in \Gamma}\{\mu_G((h\gamma(h\gamma)^{n-1} h\gamma_1 r\gamma_2 k)\} \\ &\geq Inf_{r \in H, \gamma_1, \gamma_2 \in \Gamma}\{\mu_G((h\gamma)^{n-1} h\gamma_1 r\gamma_2 k)\} \\ &= \mu_{<(h\gamma)^{n-1} h, G>}(k). \end{aligned}$$

Thus $\mu_{<(h\gamma)^n h, G>}(k) \geq \mu_{<(h\gamma)^{n-1} h, G>}(k)$. In the same manner, it can be shown that $\nu_{<(h\gamma)^n h, G>}(k) \leq \nu_{<(h\gamma)^{n-1} h, G>}(k)$, for all $k \in H$. Thus $< (h\gamma)^{n-1}h, G > \subseteq < (h\gamma)^n h, G >$.

(3) Since $< h, G >$ is an IFI of H, so for $k \in H$

$\mu_{<h,G>}(k) = Inf_{r \in H, \gamma_1, \gamma_2 \in \Gamma}\{\mu_G(h\gamma_1 r\gamma_2 k)\} \geq \mu_G(h) > 0$ and

$\nu_{<h,G>}(k) = Sup_{r \in H, \gamma_1, \gamma_2 \in \Gamma}\{\nu_G(h\gamma_1 r\gamma_2 k)\} \leq \nu_G(h) < 1$. This implies $k \in Supp(< h, G >)$. So $H \subseteq Supp(< h, G >)$. But $Supp(< h, G >) \subseteq H$ always implies that $Supp(< h, G >) = H$.

Theorem 8.2.5. *Suppose H is a Γ-Ring and G is an IFPI of H. Then for all $h, k \in H$ $Inf_{r \in H, \gamma_1, \gamma_2 \in \Gamma}\{\mu_G(h\gamma_1 r\gamma_2 k)\} = \mu_G(h) \vee \mu_G(k)$ and $Sup_{r \in H, \gamma_1, \gamma_2 \in \Gamma}\{\nu_G(h\gamma_1 r\gamma_2 k)\} = \nu_G(h) \wedge \nu_G(k)$. Conversely, suppose G is an IFI of a Γ-Ring H s.t. $Img(G) = \{(1,0), (\lambda, \zeta)\}$, where $\lambda, \zeta \in [0,1)$ s.t. $\lambda + \zeta \leq 1$ and $Inf_{r \in H, \gamma_1, \gamma_2 \in \Gamma}\{\mu_G(h\gamma_1 r\gamma_2 k)\} =$*

$\mu_G(h) \vee \mu_G(k)$ *and* $Sup_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_G(h\gamma_1 r\gamma_2 k)\} = \nu_G(h) \wedge \nu_G(k)$ *holds* $\forall h,k \in H$, *then* G *is an IFPI of H.*

Proof. Let G be an IFPI of H. Then (i) $G(0_H) = (1,0)$ (ii) G_* is a PI of H (iii) $Img(G) = \{(1,0),(\lambda,\zeta)\}$, where $\lambda,\zeta \in [0,1)$ s.t. $\lambda + \zeta \leq 1$.

Clearly $Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 r\gamma_2 k)\} = 1$ or λ and $Sup_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_G(h\gamma_1 r\gamma_2 k)\} = 0$ or ζ.

Case(i) Let $\mu_G(h) \vee \mu_G(k) = 1$. Suppose $\mu_G(h) = 1$, then $\nu_G(h) = 0$. This implies that $h \in G_*$. Since G_* is an ideal of H so $h\gamma_1 r\gamma_2 k \in G_*$, for all $\gamma_1,\gamma_2 \in \Gamma$ and for all $r,k \in H$. Therefore $\mu_G(h\gamma_1 r\gamma_2 k) = 1$ and $\nu_G(h\gamma_1 r\gamma_2 k) = 0$, for all $\gamma_1,\gamma_2 \in \Gamma$, $r,k \in H$. Hence $Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 r\gamma_2 k)\} = 1 = \mu_G(h) \vee \mu_G(k)$ and $Sup_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_G(h\gamma_1 r\gamma_2 k)\} = 0 = \nu_G(h) \wedge \nu_G(k)$.

Case(ii) Let $\mu_G(h) \vee \mu_G(k) = \lambda$. Then atleast one of $\mu_G(h)$ or $\mu_G(k)$ is λ. Suppose $\mu_G(h) = \lambda$ and so $\nu_G(h) = \zeta$. This implies $h \notin G_*$. Hence $h\Gamma H\Gamma k \nsubseteq G_*$. Thus $\exists' s$, $\gamma_1,\gamma_2 \in \Gamma$ and $r \in H$ such that $h\gamma_1 r\gamma_2 k \notin G_*$. Hence $\mu_G(h\gamma_1 r\gamma_2 k) \neq 1$ and $\nu_G(h\gamma_1 r\gamma_2 k) \neq 0$. As $Img(G) = \{(1,0),(\lambda,\zeta)\}$, so we have $\mu_G(h\gamma_1 r\gamma_2 k) = \lambda$ and $\nu_G(h\gamma_1 r\gamma_2 k) = \zeta$. Thus $Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 r\gamma_2 k)\} = \lambda = \mu_G(h) \vee \mu_G(k)$ and $Sup_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_G(h\gamma_1 r\gamma_2 k)\} = \zeta = \nu_G(h) \wedge \nu_G(k)$.

Conversely, to prove the converse it is sufficient to show that G_* is a PI of H. Suppose $h,k \in H$ s.t. $h\Gamma H\Gamma k \subseteq G_*$. Therefore for all $\gamma_1,\gamma_2 \in \Gamma$, $r \in H$, $h\gamma_1 r\gamma_2 k \in G_*$. So $\mu_G(h\gamma_1 r\gamma_2 k) = 1$ and $\nu_G(h\gamma_1 r\gamma_2 k) = 0$, for all $\gamma_1,\gamma_2 \in \Gamma$ and $r \in H$.

Hence $Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 r\gamma_2 k)\} = 1$ and $Sup_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_G(h\gamma_1 r\gamma_2 k)\} = 0$. Therefore

$\mu_G(h) \vee \mu_G(k) = 1$ and $\nu_G(h) \wedge \nu_G(k) = 0$. This indicates that $\mu_G(h) = 1, \nu_G(h) = 0$ or $\mu_G(k) = 1, \nu_G(k) = 0$, i.e., $h \in G_*$ or $k \in G_*$. Thus G_* is a PI of H. Hence G is an IFPI of H.

Proposition 8.2.6. *Suppose H be a Γ-Ring and G is an IFPI of H and $h \in H$, then* $\mu_{<h,G>}(k) = Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_{<h\gamma_1 r\gamma_2 h,G>}(k)\}$ *and* $\nu_{<h,G>}(k) = Sup_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_{<h\gamma_1 r\gamma_2 h,G>}(y)\}, \forall k \in H$.

Proof. Now

$$
\begin{aligned}
Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\left(\mu_{<h\gamma_1 r\gamma_2 h,G>}(k)\right) &= Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{Inf_{r\in H}(h'\gamma_1' r\gamma_2' k)\}, \text{ where } h' = h\gamma_1 r\gamma_2 h \\
&= Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h') \vee \mu_G(k)\} \text{ as } G \text{ is an IFPI} \\
&= Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 r\gamma_2 h) \vee \mu_G(k)\}\} \\
&= Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 r\gamma_2 h)\} \vee Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(k)\} \\
&= \mu_G(h) \vee \mu_G(h) \vee \mu_G(k) = \mu_G(h) \vee \mu_G(k) \\
&= Inf_{r\in H,\gamma_3,\gamma_4\in\Gamma}\{\mu_G(h\gamma_3 r\gamma_4 k)\} \text{ as } G \text{ is an IFPI} \\
&= \mu_{<h,G>}(k).
\end{aligned}
$$

The same argument can be used to prove other results.

Definition 8.2.7. Suppose H be a Γ-Ring and $N \subseteq H$ and $h \in H$, we define

$$< h, N >= \{k \in H | h\Gamma H\Gamma k \subseteq N\}$$

Proposition 8.2.8. *Suppose H is a Γ-Ring and $\emptyset \neq N \subseteq H$. Then $< h, \chi_N >= \chi_{<h,N>}$ for every $h \in H$.*

Proof. Suppose $k \in H$. Now $\mu_{<h,\chi_N>}(k) = Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_{\chi_N}(h\gamma_1 r\gamma_2 k)\} = 1$ or 0 and $\nu_{<h,\chi_N>}(k) = Sup_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_{\chi_N}(h\gamma_1 r\gamma_2 k)\} = 0$ or 1.

Case(i) If $\mu_{<h,\chi_N>}(k) = 1$ and so $\nu_{<h,\chi_N>}(k) = 0$ and therefore, $Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_{\chi_N}(h\gamma_1 r\gamma_2 k)\} = 1$ and $Sup_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_{\chi_N}(h\gamma_1 r\gamma_2 k)\} = 0$. This implies $h\gamma_1 r\gamma_2 k \in N$, for all $\gamma_1, \gamma_2 \in \Gamma, r \in H$ and so $k \in < h, N >$. Hence $\mu_{\chi_{<h,N>}}(k) = 1, \nu_{\chi_{<h,N>}}(k) = 0$. So here in case (i), this is true that $< h, \chi_N >= \chi_{<h,N>}$.

Case(ii) If $\mu_{<h,\chi_N>}(k) = 0$ and so $\nu_{<h,\chi_N>}(k) = 1$ and therefore, $Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_{\chi_N}(h\gamma_1 r\gamma_2 k)\} = 0$ and $Sup_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_{\chi_N}(h\gamma_1 r\gamma_2 k)\} = 1$. Hence $h\gamma_1 r\gamma_2 k \notin N$, for some $\gamma_1, \gamma_2 \in \Gamma, r \in H$. This implies $k \notin < h, N >$. Hence

$\mu_{\chi_{<h,N>}}(k) = 0, \nu_{\chi_{<x,N>}}(y) = 1$. So here in case (ii) also this is true that $< h, \chi_N >= \chi_{<h,N>}$. Hence the result proved.

Theorem 8.2.9. *Suppose H is a Γ-Ring. If G is an IFPI of H and $h \in H$ be s.t. $h \notin G_*$, then*

$< h, G >= G$. Conversely, let G be an IFI of H s.t. $Img(G) = \{(1,0), (\lambda, \zeta)\}$, where $\lambda, \zeta \in [0,1)$ s.t. $\lambda + \zeta \leq 1$. If $< h, G >= G$, for some $h \in H$ for which $G(h) = (\lambda, \zeta)$, then G is an IFPI of H.

Proof. Let G be an IFPI of H. Then (i) $G(0_H) = (1,0)$ (ii) G_* is a PI of H (iii) $Img(G) = \{(1,0), (\lambda, \zeta)\}$, where $\lambda, \zeta \in [0,1)$ such that $\lambda + \zeta \leq 1$. Let $h \in H$

Case(i) If $h \in G_*$, then $h\gamma_1 r\gamma_2 h \in G_*$ for all $\gamma_1, \gamma_2 \in \Gamma, r, h \in H$. So $Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 r\gamma_2 h)\} = 1 = \mu_G(k)$ and $Sup_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_G(h\gamma_1 r\gamma_2 k)\} = 0 = \nu_G(k)$. That is $\mu_{<h,G>}(k) = \mu_G(k)$ and $\nu_{<h,G>}(k) = \nu_G(k)$, i.e., $< h, G > (k) = G(k)$.

Case(ii) Let $k \notin G_*$. Is G_* is a PI of H, $h\gamma_1 r\gamma_2 k \notin G_*$, for some $\gamma_1, \gamma_2 \in \Gamma, r, h \in H$. So $Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 r\gamma_2 k)\} = \lambda = \mu_G(k)$ and $Sup_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_G(h\gamma_1 r\gamma_2 k)\} = \zeta = \nu_G(k)$, i.e., $\mu_{<h,G>}(k) = \mu_G(k)$ and $\nu_{<h,G>}(k) = \nu_G(k)$, i.e., $< h, G > (k) = G(k)$. So in both the cases we get $< h, G >= G$.

Conversely, let $h, k \in H$.

Case(i) Let $\mu_G(h) = \lambda, \nu_G(h) = \zeta$. Now $\mu_G(k) = \mu_{<h,G>}(k) = Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 r\gamma_2 k)\}$ and $\nu_G(k) = \nu_{<h,G>}(k) = Sup_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_G(h\gamma_1 r\gamma_2 k)\}$. Since $Img(G) = \{(1,0), (\lambda, \zeta)\}$, where $\lambda, \zeta \in [0,1)$ such that $\lambda + \zeta \leq 1$. Now $\mu_G(k) \geq \lambda = \mu_G(h)$ and $\nu_G(k) \leq \zeta = \nu_G(h)$. So $\mu_G(h) \vee \mu_G(k) = \mu_G(k)$ and $\nu_G(h) \wedge \nu_G(k) = \nu_G(k)$. Therefore we have

$Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 r\gamma_2 k)\} = \mu_G(h) \vee \mu_G(k)$ and $Sup_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_G(h\gamma_1 r\gamma_2 k)\} = \nu_G(h) \wedge \nu_G(k)$.

Case(ii) Let $\mu_G(h) = 1, \nu_G(h) = 0$, then $h \in G_*$. As G is an IFI of H, G_* is an ideal of H. Hence $h\gamma_1 r\gamma_2 k \in G_*$, $\forall \gamma_1, \gamma_2 \in \Gamma, r, k \in H$. So $Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 r\gamma_2 k)\} = 1 = \mu_G(h) \vee \mu_G(k)$ and $Sup_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_G(h\gamma_1 r\gamma_2 k)\} = 0 = \nu_G(h) \wedge \nu_G(k)$, $\forall h, k \in H$. Hence using the converse of Theorem (8.2.5) G is an IFPI of H.

Theorem 8.2.10. *Suppose H is a Γ-Ring. If G is an IFPI of H & $h \in H$ be s.t. $h \in G_*$, then*

$< h, G >= \chi_H$.

Proof. Suppose G be an IFPI of H. Then (i) $G(0_H) = (1,0)$ (ii) G_* is a prime ideal of H (iii) $Img(G) = \{(1,0), (\lambda, \zeta)\}$, where $0 \le \lambda, \zeta < 1$ s.t. $\lambda + \zeta \le 1$. Let $y \in H$. As $h \in G_*$, then $h\gamma_1 r\gamma_2 k \in G_*$, for all $\gamma_1, \gamma_2 \in \Gamma, r, k \in H$. So $\mu_{<h,G>}(k) = Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 r\gamma_2 k)\} = 1 = \mu_{\chi_H}(k)$ and $\nu_{<h,G>}(k) = Sup_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 r\gamma_2 k)\} = 0 = \nu_{\chi_H}(k)$, $\forall k \in H$. Hence $< h, G >= \chi_H$.

Corollary 8.2.11. *Suppose M is an ideal of a Γ-Ring H. If M is a PI of H then for $h \in H$,* $< h, \chi_M >= \chi_M$.

Proof. Suppose M be a PI of H. Then χ_M is an IFPI of H Now $h \notin H$ implies $h \notin (\chi_M)_*$, we have by Theorem (8.2.9) $< h, \chi_M >= \chi_M$.

Theorem 8.2.12. *Let H be a commutative Γ-Ring and G be an IFS of H s.t. $< h, G >= G$ for every $h \in H$. Then G is constant.*

Proof. For $h, k \in H$ we have

$$\begin{aligned}\mu_G(h) &= \mu_{<k,G>}(h), \text{ as } < h, G >= G \text{ for every } h \in H\\ &= Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(k\gamma_1 r\gamma_2 h)\} = Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 r\gamma_2 k)\}\\ &= \mu_{<h,G>}(k) = \mu_G(k).\end{aligned}$$

Thus $\mu_G(h) = \mu_G(k)$. Similarly, this can be depicted $\nu_G(h) = \nu_G(k)$, for all $h, k \in H$. Hence G is constant.

Proposition 8.2.13. *Let H be a Γ-Ring and G is an IFPI of H. Then either $< h, G >$ is an IFPI of H or $< h, G >$ is constant.*

Proof. Let G be an IFPI of H and $h \in H$

Case(i) If $h \notin G_*$. By Theorem (8.2.9) $< h, G >= G$. This proves that $< h, G >$ is an IFPI of H.

Case(ii) If $h \in G_*$. Then $h\gamma_1 r\gamma_2 k \in G_*$, $for\ all$ $\gamma_1, \gamma_2 \in \Gamma, r, k \in H$. Hence $\mu_{<h,G>}(k) = Inf_{r\in H, \gamma_1,\gamma_2 \in \Gamma}\{\mu_G(h\gamma_1 r\gamma_2 k)\} = 1; \nu_{<h,G>}(k) =$ $Sup_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 r\gamma_2 k)\} = 0$, for all $k \in H$. This proves $< h, G >$ is a constant.

Proposition 8.2.14. *Suppose H is a commutative Γ-Ring and G is an IFSPI of H iff* $G(h\gamma h) = G(h), \forall h \in H,$ *and* $\forall \gamma \in \Gamma$.

Proof. Let $\mathbb{G}_1$be an IFI of *Γ-Ring H such that* $\mathbb{G}_1(h\gamma h) = \mathbb{G}_1(h)$, $\forall h \in H$ and $\forall \gamma \in \Gamma$. Let $\mathbb{G}_2$ be an IFI of H s.t. $\mathbb{G}_2 \Gamma \mathbb{G}_2 \subseteq \mathbb{G}_1$. Let $\mathbb{G}_2 \nsubseteq \mathbb{G}_1$. Then $\exists k \in H$ s.t. $\mu_{\mathbb{G}_2}(k) > \mu_{\mathbb{G}_1}(k)$ and $\nu_{\mathbb{G}_2}(k) < \nu_{\mathbb{G}_1}(k)$.

Now $\mu_{\mathbb{G}_2\Gamma\mathbb{G}_2}(k\gamma k) \geq \mu_{\mathbb{G}_2}(k) > \mu_{\mathbb{G}_1}(k)$ and $\nu_{\mathbb{G}_2\Gamma\mathbb{G}_2}(k\gamma k) \leq \nu_{\mathbb{G}_2}(k) < \mu_{\mathbb{G}_1}(k)$. Again $\mu_{\mathbb{G}_1}(k) = \mu_{\mathbb{G}_1}(k\gamma k) \geq \mu_{\mathbb{G}_2\Gamma\mathbb{G}_2}(k\gamma k)$ and $\nu_{\mathbb{G}_1}(k) = \nu_{\mathbb{G}_1}(k\gamma k) \leq \nu_{\mathbb{G}_2\Gamma\mathbb{G}_2}(k\gamma k)$. This implies that $\mathbb{G}_2\Gamma\mathbb{G}_2 = \mathbb{G}_1$, which is not true. Hence $\mathbb{G}_2 \subseteq \mathbb{G}_1$. Thus $\mathbb{G}_1$ is an IFSPI of H.

Conversely, let G be an IFSPI of H. Now for any $h \in H$, we have

$$\begin{aligned} \mu_G(h) &= Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 r\gamma_2 h)\} (\text{ from prop. (2.2.11)}) \\ &\geq Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 x\gamma_2 h)\} \\ &\geq \mu_G(h\gamma_i h). \end{aligned}$$

Again $\mu_G(h\gamma_i h) \geq \mu_G(h)$. Thus $\mu_G(h\gamma_i h) = \mu_G(h)$. In the same manner it can be shown that $\nu_G(h\gamma_i h) = \nu_G(h)$. That is $G(h\gamma h) = G(h) \forall h \in H, \gamma \in \Gamma$.

Proposition 8.2.15. *Let H be a commutative Γ-Ring and G be an IFSPI of H. Then* $< h, G >$ *is an IFSPI of H for every* $h \in H$.

Proof. Suppose G is an IFSPI of H and $h \in H$. Now by Proposition (8.2.2) $< h, G >$ is an IFI of H. For every $k \in H, \gamma \in \Gamma$, this is true

$$
\begin{aligned}
\mu_{<h,G>}(k) &= Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 r\gamma_2 k)\}\\
&= Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G\{(h\gamma_1 r\gamma_2 k)\gamma(h\gamma_1 r\gamma_2 k)\}\}(\text{ as } G \text{ is IFSPI })\\
&= Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G\{(h\gamma_1 m\gamma_2 k)\gamma(k\gamma_1 h\gamma_2 r)\}\}\\
&\geq Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 r\gamma_2 k\gamma)\}\\
&= Inf_{r\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G\big(h\gamma_1 r\gamma_2 (k\gamma k)\big)\}\\
&= \mu_{<h,G>}(k\gamma k).
\end{aligned}
$$

Again $\mu_{<h,G>}(k\gamma k) \geq \mu_G(k)$, as $< h, G >$ is IFI of H. Thus $\mu_{<h,G>}(k\gamma k) = \mu_G(k)$. Similarly, we can show $\nu_{<h,G>}(k\gamma k) = \nu_G(k)$ for all $k \in H, \gamma \in \Gamma$, by proposition (8.2.14) $< h, G >$ will be an IFSPI of H.

Corollary 8.2.16. *Suppose H is a commutative Γ-Ring and $\{G_i : i \in J\}$ be a non-empty family of IFSPIs of H. If $\mu_G(h) = Inf_{i\in J}\{\mu_{G_i}(h)\}$ and $\nu_G(h) = Sup_{i\in J}\{\nu_{G_i}(h)\}$. Then take any $h \in H$, $< h, G >$ will be an IFSPI of H.*

Proof. Clearly, G is an IFS of H. Let $r_1, r_2 \in H, \gamma \in \Gamma$, then

$$
\begin{aligned}
\mu_G(r_1 - r_2) &= Inf_{i\in J}\{\mu_{G_i}(r_1 - r_2)\}\\
&\geq Inf_{i\in J}\{\mu_{G_i}(r_1) \wedge \mu_{G_i}(r_1)\}\\
&= \{Inf_{i\in J}\{\mu_{G_i}(r_1)\}\} \wedge \{Inf_{i\in J}\{\mu_{G_i}(\mathrm{r}_2)\}\}\\
&= \mu_G(\mathrm{r}_1) \wedge \mu_G(r_2).
\end{aligned}
$$

Similarly, we can show that $\nu_G(r_1 - r_2) \leq \nu_G(\mathrm{r}_1) \vee \nu_G(\mathrm{r}_2)$. Also

$$
\begin{aligned}
\mu_G(r_1\gamma r_2) &= Inf_{i\in J}\{\mu_{G_i}(r_1\gamma r_2)\}\\
&\geq Inf_{i\in J}\{\mu_{G_i}(\mathrm{r}) \vee \mu_{G_i}(r_1)\}\\
&= \{Inf_{i\in J}\{\mu_{G_i}(r_1)\}\} \vee \{Inf_{i\in J}\{\mu_{G_i}(r_2)\}\}\\
&= \mu_G(r_1) \vee \mu_G(r_2).
\end{aligned}
$$

In the same way, we prove that $\nu_G(r_1\gamma r_2) \geq \nu_G(r_1) \wedge \nu_G(r_2)$. Thus G will be an IFI of H. Let $a \in H$, $\gamma \in \Gamma$, we have $\mu_G(a) = Inf_{i\in J}\{\mu_{\mathbb{G}_i}(a)\} = Inf_{i\in J}\{\mu_{\mathbb{G}_i}(a\gamma a)\} = \mu_G(a\gamma a)$, as each $\mathbb{G}_i$ is IFSPIs of H. In the same way, we prove that $\nu_G(a) = \nu_G(a\gamma a)$, for all $\gamma \in \Gamma$. Then by proposition (2.2.11), $< x, G >$ is an IFSPI of H.

Corollary 8.2.17. *Let H be a comm.* Γ*-Ring and* $\{P_i : i \in J\}$ *is a family of SPI of H with at least one element and* $P = \bigcap_{i \in J} P_i \neq \emptyset$*. Them* $< x, \chi_P >$ *is an IFSPI of H for every* $x \in H$.

Proof. Since $P = \bigcap_{i \in J} P_i$, is IPI of H. Then χ_P will be an IFSPI of H. Thus by proposition (8.2.15) $< x, \chi_P >$ will become an IFSPI of H.

8.3 Conclusion

In the last chapter, the notion of extensions of IFI with respect to an element in the Γ-Ring is investigated and characterization of IFPI and IFSPI has been innovated.

Overall Conclusion

In this thesis, an attempt has been made to study IFIs within the Γ-Ring, with particular emphasis on their structure. The concept of IFCI within a Γ-Ring has been explored, establishing a connection between IFCI and its level cut sets (3.2.7) and (3.2.9). A relationship between Aut(H) and Aut(OR) has been derived (3.3.19), along with a one-to-one mapping between IFCI(H) and IFCI(OR) (3.3.22).

Furthermore, the fundamental concepts of IFPrI and IFPR in Γ-Ring have been investigated, demonstrating that IFPrI of a Γ-Ring forms a two-valued IFS with the base set being a PrI (4.3.17). It has also been shown that the IFPR of an IFPrI is an IFPI (4.3.20). The homeomorphic behavior of IFPrI and IFPR in Γ-Ring was established (4.4.3), (4.4.7), (4.4.8), (4.4.9). The notion of (IF2−APrI) in Γ-Ring has been explored, proving that every IF2−AI of Γ-Ring is an IF2−APrI (4.5.7), but the converse is not true (4.5.8). Additionally, it has been established that the intersection of two IF2-APrIs of a Γ-Ring may not be an IF2−APrIs (4.5.12); however, the intersection of a finite number of IFP−2−APrIs of a Γ-Ring is an IFP−2−APrI (4.5.11).

Furthermore, the IF version of the Lasker-Noether theorem for a commutative Noetherian Γ-Ring has been established, proving that every IFI G in a commutative Noetherian Γ-Ring can be decomposed as the intersection of a finite number of IFPrIs (5.2.8). This decomposition is called an IF primary decomposition. In addition to exploring the IF primary decomposition, it has been demonstrated that, in the case of the minimal IF primary decomposition of an IFI G, the set of all IF associated PIs of G is independent of the particular decomposition (5.3.11).

The structure space on the IFPIs(H) of commutative Γ-Ring with unity (6.2.2) has also been investigated. It has been shown that this structured space is always

T_0 (6.3.1) but not T_2 (6.3.6); however, when H is a Boolean Γ-Ring, then it is a T_2 space (6.3.7). Furthermore, a subspace of the structure space, which is always compact (6.3.8), has been identified. Additionally, a relationship between the two different structure spaces has been established when there is a Γ-Ring homomorphism between two Γ-Rings (6.4.6). Moreover, the structure space is connected if and only if 0 and e are the only idempotent elements in H (6.5.4).

Further, the two notions of IFPIs (2-AIs) and IFPrIs (2A-PrIs) of a Γ-Ring have been unified into IFf-PrI (2-Af-PrIs)), where f is a map from the set of all IFIs (2-AIs) into itself called the ideal expansion map. It has also been shown that the intersection of a finite number of IFf-PrIs (2-Af-PrIs) of a Γ-Ring is again an IFf-PrI (2-Af-PrIs) provided the mapping f is an intersection-preserving (7.2.14) and (7.3.6). Additionally, it has been proven that the image and pre-image of an IF$f-$PrI (2-Af-PrIs) under the Γ-Ring homomorphism between two Γ-rings are IFf-PrIs (2-Af-PrIs), provided the mapping f is both intersection-preserving and globally expansive (7.2.16), (7.2.15), (7.3.7) and (7.3.8).

Finally, the notion of extensions of intuitionistic fuzzy ideals with respect to an element in the Γ-ring has been introduced, and the characterization of intuitionistic fuzzy prime ideals (8.2.13) and intuitionistic fuzzy semi-prime ideals has been undertaken (8.2.15).

Nevertheless, there remain results in crisp set theory related to the topics covered in this thesis that need investigation in the IF setting over Γ-Ring. Many ideas in algebra related to the theory of Γ-Ring, such as the "structure of primitive Γ-Ring" and "higher separation axioms for the structure space on the set of prime ideals", have yet to be defined or explored in the IF analogs.

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List Of Publications

Paper Published In International Journals

- Sharma, P. K., Lata, H. Intuitionistic fuzzy characteristic ideal of a Γ-Ring, South East Asian Journal of Mathematics and Mathematical Sciences, Vol. 18, No. 1, 2022, 49-70. (Scopus Indexed Journal)
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List Of Conferences

Paper Presentations In International Conferences

- Extensions of intuitionistic fuzzy ideal of Γ-Ring, presented in the international conference on Recent Trends in Mathematics, held at H.P. University, Shimla from 6-7th September, 2021.
- A study on intuitionistic fuzzy $2-$absorbing primary ideals in Γ-Ring, presented in the 25th Jubilee Edition of the International Conference on Intuitionistic Fuzzy Sets held at Bulgaria from 9-10th September, 2022 and published in Notes On Intuitionistic Fuzzy Sets, Vol. 28, No. 3, 2022, 280-292.
- Expansion of intuitionistic fuzzy ideals of Γ-Ring, presented in the international conference on Algebra, Analysis and Applications, Organized by Manipal Institute of Higher Education, from Jan, 06-08, 2023, at Manipal Academy of Higher Education

List Of Workshops

- FDP On "Research Methodology" organized by Department of Management Sciences of Balaji Institute of Technology & Science Narsampet ,Warangal , Telangana on 13-07-2020 to 17-07-2020.
- FDP on "LATEX and Its Applications for Researchers" organized by Department of Information Science & Engineering Vidyavardhaka College of Engineering, Mysuru from 20-07-2020 to 24-07-2020.
- Short Term Course on Scientific Writing using Typesetting Software LaTeX organized

by Lovely Professional University, Phagwara, Punjab from 25-04-2022 to 30-04-2022.