A STUDY OF CATEGORY AND FUNCTORS ASSOCIATED WITH IF MODULES

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in

Mathematics

By

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DECLARATION

I, Chandni, hereby declared that the presented work in the thesis entitled “A Study Of Category And Functors Associated With IF Modules” in fulfilment of degree of Doctor of Philosophy (Ph. D.) is outcome of research work carried out by me under the supervision of Dr. Nitin Bhardwaj, working as Professor and Deputy Dean, in the, School of Physical Sciences and Chemical Engineering of Lovely Professional University, Punjab, India. In keeping with general practice of reporting scientific observations, due acknowledgements have been made whenever work described here has been based on findings of other investigator. This work has not been submitted in part or full to any other University or Institute for the award of any degree.

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CERTIFICATE

This is to certify that the work reported in the Ph. D. thesis entitled “A Study Of Category And Functors Associated With IF Modules” submitted in fulfillment of the requirement for the award of degree of Doctor of Philosophy (Ph.D.) in the School of Physical Sciences and Chemical Engineering, is a research work carried out by “Chandni”, (41500093), is bonafide record of his/her original work carried out under my supervision and that no part of thesis has been submitted for any other degree, diploma or equivalent course.

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ABSTRACT

In this thesis, we study the category of intuitionistic fuzzy modules by exploring its interaction with various mathematical structures. This research investigates how intuitionistic fuzzy modules relate to concepts in category theory, such as functors, natural transformations, and universal properties. Our focus is on establishing a categorical framework for intuitionistic fuzzy modules. In this framework, the objects represent intuitionistic fuzzy modules over a given commutative ring R, and morphisms preserve the intuitionistic fuzzy structure. We explore the properties of this category, including the existence of products, coproducts, equalizers, coequalizers, pullbacks, images, and inverse images. Furthermore, we investigate the relationships between intuitionistic fuzzy modules and other categorical structures, such as Hom functors, and tensor product functors. The outcomes of this investigation contribute to a more comprehensive understanding of intuitionistic fuzzy modules and their interconnected roles in various mathematical frameworks.

The introduction of fuzzy sets by Zadeh in 1965 marked a significant advancement due to their ability to handle uncertainty and vagueness, which classical crisp sets could not address. Atanassov proposed a generalization of fuzzy sets as intuitionistic fuzzy sets (IFS) in the 1980s, incorporating the degree of non-membership. This extension has found meaningful applications in various fields such as logic programming and medical diagnosis. Biswas was the first to apply the criterion of intuitionistic fuzzy sets in Algebra which led to the introduction of an intuitionistic fuzzy subgroup of a group in [10]. Later on, Hur and others in [19] and [20] brought the perception of intuitionistic fuzzy subring and ideals. B. Davaaz and others in [11] delivered the perception of intuitionistic fuzzy submodule of a module.

Later on, many mathematicians contributed much to the study of intuitionistic fuzzy submodules see [8, 11, 22, 42, 43, 45]. Golan [15] pioneered the research on fuzzy modules, while Lopez-Permouth and Malik [33] dealt with the category of fuzzy modules. Category theory is a general theory of mathematical structures and their relations that was introduced by Samuel Eilenberg and Saunders Mac Lane [13] in the middle of the 20th century in their foundational work on algebraic topology. The category theory is concerned with the mathematical entities and the relationship between them. Categories also emerge as a unifying concept in many fields of mathematics, particularly in all other areas of computer technology and mathematical physics. This thesis explores the concept of universal construction within category theory, a foundational branch of abstract mathematics. Universal constructions provide a powerful framework for understanding mathematical structures and relationships in a broad range of contexts. By investigating the general principles and methodologies underlying universal constructions, this study aims to elucidate their significance and applicability across diverse mathematical landscapes. A systematic examination of key concepts such as initial and terminal objects, equalizers, coequalizers, pullback and intersection, establishes a foundation for comprehending the universal nature of these constructions. The insights derived from this exploration not only deepen our understanding of category theory but also pave the way for insightful applications in various branches of mathematics. Hom-functors of intuitionistic fuzzy modules, which extend the classical notion of homomorphisms to the fuzzy setting, where uncertainties play a crucial role. The abstract investigation navigates through the categorical structure, emphasizing how Hom-functors facilitate the study of relationships, transformations, and compositions among intuitionistic fuzzy modules. This exploration deepens our understanding of fuzzy algebraic structures, offering insights into their versatile applications and theoretical implications. The tensor product in the context of intuitionistic fuzzy modules involves a mathematical operation that combines elements from two such modules to produce a new module. This construction extends the notion of tensor products in classical algebra to the fuzzy domain, accommodating uncertainties inherent in intuitionistic fuzzy domains. The abstract framework of tensor products allows for exploring module interactions, providing a versatile tool for mathematical analysis and modelling in fuzzy algebraic structures. Forgetful functors are mathematical mappings between categories that "forget" some of the structure of objects in one category while preserving others. In the context of algebraic structures, a forgetful functor typically maps objects and morphisms from a more algebraic category to a less algebraic one. In the case of intuitionistic fuzzy modules, a forgetful functor might map these modules to a category of more general mathematical structures, discarding certain specific features of intuitionistic fuzzy modules while retaining others. This process simplifies the analysis or comparison of these modules within a broader mathematical context.

Our present study focuses on intuitionistic fuzzy modules over a commutative ring R with an identity element. During the study, we attempted to develop a parallel theory of category by applying intuitionistic fuzzy techniques. This thesis aims to provide a comprehensive investigation into the category of intuitionistic fuzzy modules, with a focus on advancing both theoretical understanding and practical applications. Through rigorous mathematical analysis, the research aims,

1. To enrich the knowledge of intuitionistic fuzzy sets on algebraic structures like rings and modules.

2. To extend the concepts of module theory to the category theory associated with intu-

itionistic fuzzy theory.

3. To introduce the notion of Kernels and Cokernels in the Category of intuitionistic fuzzy modules.
4. To define new concepts in modules in intuitionistic fuzzy environment.
5. To define zero object associated with the IF module.
6. To define new concepts of coproduct, product, covariant and contravariant functor associated with the IF module.
7. To determine the free, injective and projective modules in the Category of intuitionistic fuzzy modules.

Through these endeavours, the thesis aims to deepen the scholarly understanding of intuitionistic fuzzy modules and broaden their impact on both theoretical mathematics and applied domains.

The research conducted by Hur, Kang, and Song on intuitionistic fuzzy subgroups and subrings expands the understanding of algebraic structures by incorporating the principles of intuitionistic fuzzy logic. By investigating the properties and characteristics of intuitionistic fuzzy subgroups and subrings, their work sheds light on how uncertainty and imprecision can be represented and manipulated within the framework of algebraic systems. This not only deepens our comprehension of intuitionistic fuzzy sets but also broadens the application of fuzzy algebraic structures to various mathematical domains. Consequently, their findings contribute significantly to enriching the knowledge of intuitionistic fuzzy sets on algebraic structures like rings and modules, paving the way for further exploration and advancement in this area of study.

Throughout this thesis, R represents a commutative ring with unity, denoted by 1, where $1 \neq 0$. M is a unitary R-module and θ is a zero element of M and I represents the unit interval $[0, 1]$.

In Chapter 2, we extend the concepts of module theory to the category theory associated with intuitionistic fuzzy theory by defining a category ($\mathbf{C}_{\textbf{R-IFM}}$) of intuitionistic fuzzy modules where the classes of all intuitionistic fuzzy modules and intuitionistic fuzzy R-homomorphisms constitute objects and morphisms. The composition of morphisms is the ordinary composition of functions. Also, we reveal that $Hom(A, B)$ is an abelian group under the ordinary addition of R-homomorphisms, where A and B are any intuitionistic fuzzy submodules. In the context of the additive composition, this structure appears to have a distributive influence on the left and even on the right. We are implementing an important technological tool to "optimally intuitionistic fuzzify" the R-homomorphism families. This capability to intuitionistic fuzzify provides $\mathbf{C}_{\textbf{R-IFM}}$ with the top category structure over $\mathbf{C}_{\textbf{R-M}}$. We prove that the category of intuitionistic fuzzy modules has kernels, Cokernels and define the zero object associated with IF module. Further, we show that $\mathbf{C}_{\textbf{R-IFM}}$ seems to be an additive category, even though it is not an abelian category. Finally, we have shown that the category of fuzzy modules $\mathbf{C}_{\textbf{R-FM}}$ is a subcategory of the category of intuitionistic fuzzy modules $\mathbf{C}_{\textbf{R-IFM}}$ and we established a contravariant functor from the category $\mathbf{C}_{\textbf{R-IFM}}$ to the category $\mathbf{C}_{\textbf{Lat}}$ (= union of all $\mathbf{C}_{\textbf{Lat(R-IFM)}}$, corresponding to each object in $\mathbf{C}_{\textbf{R-M}}$).

In Chapter 3, we extend the notion of intuitionistic fuzzy modules and intuitionistic fuzzy R-homomorphism to intuitionistic fuzzy coretracts (retracts) and intuitionistic fuzzy coretraction

(retraction), and various properties are being investigated. We study free, projective and injective objects in $\mathbf{C_{R\text{-}IFM}}$ and establish their relation with IF-retraction and IF-coretraction in $\mathbf{C_{R\text{-}IFM}}$.

In Category theory, there are many interesting universal objects such as products, coproducts, equalizers, coequalizers, pullbacks and intersections. In Chapter 4, we have introduced the concept of Intuitionistic fuzzy products, Intuitionistic fuzzy coproducts, Intuitionistic fuzzy equalizers, Intuitionistic fuzzy coequalizers, Intuitionistic fuzzy pullbacks and Intuitionistic fuzzy intersections and has tried to get the result about universal objects. We even characterize zero objects, kernels, Cokernels in $\mathbf{C_{R\text{-}IFM}}$.

In Chapter 5, we have explored the concept of the tensor product, Hom-functors, and exact sequences of intuitionistic fuzzy modules which sets the stage for exploring advanced algebraic structures within the framework of fuzzy mathematics. These concepts provide powerful tools for understanding relationships, transformations, and algebraic connections between intuitionistic fuzzy modules. We examine the properties of two Hom functors in the category $\mathbf{C_{R\text{-}IFM}}$. We investigate the relationship between intuitionistic fuzzy projective modules and Hom functors.

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plexities of my research. I am truly grateful for the constructive feedback, unwavering support, and the dedication with which Dr. Sharma contributed to the development and refinement of this dissertation. This academic journey has been immensely rewarding under his mentorship, and I am privileged to have benefited from the expertise and encouragement provided by such an outstanding Co-Supervisor. The countless hours spent in discussions, the constructive critiques, and the encouragement to explore novel avenues of inquiry have left an indelible mark on my academic experience. I am profoundly thankful for the opulent mentorship of Professor Sharma, whose passion for research and dedication to his students have been a constant source of inspiration. This dissertation is a testament to his influence, and I am privileged to have had such a distinguished and supportive guide throughout this transformative journey.

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Contents

Chapter 1

Preliminaries

1.1 Introduction

Categories, functors, and intuitionistic fuzzy modules indeed form the backbone of many mathematical frameworks, particularly in areas like abstract algebra, category theory, and fuzzy logic. This Chapter emphasizes fundamental definitions and outcomes concerning categories, functors, and intuitionistic fuzzy modules, all of which are pivotal to the thesis's development. The work of Eilenberg[13], Mitchell[29], Schubert[40], and others is the framework for this Chapter. The fundamental definition of categories and pertinent findings as outlined by Tom[25], MacLane[27], Rotman[39], and others are covered in section 1.2. The definition of functors and several instances are covered in section 1.3. Functors are defined as structure-preserving maps across categories. In section 1.4, the discussion revolves around the findings regarding the category of R-modules, while in the final section 1.5, the focus shifts to presenting the concept of intuitionistic fuzzy modules.

1.2 Category

Mathematical entities and their relationships are the core topic of category theory. Categories also develop as unifying concepts in numerous other fields of mathematics, especially in mathematical physics and computer science. "A category comprises three essential elements: a composition rule, a collection of morphisms, and a set of objects." As a result, category theory offers a framework for methodically investigating those characteristics and constructs that are only expressible in terms of maps. Category is modeled by the characteristics of the collection of all objects of a certain kind (sets, rings, spaces, modules, graphs) together with the collection of all structure-preserving maps(functions, ring homomorphisms, continuous maps, R-homomorphisms) between them. In 1945, Harvard algebraist Saunders Mac Lane and topologist Samuel Eilenberg published a paper [13] titled "General Theory of Natural Equivalences," which served as a significant milestone in bringing category theory to the attention of the other researcher. To define natural transformations, functors were developed, and to define functors, categories were developed. For conceptual concepts about Category theory and related areas, we follow Awodey [7], Tom Leinster [25], Mitchell [29], Riehl [37], Wyler [55], and others [39, 40, 54].

Definition 1.2.1. [29] A category C is a quadruple (Ob, Hom, id, o) consisting of:

(Cl) Ob, an object class;

(C2) a set of morphisms $Hom_C(X, Y)$ is associated with each ordered object pair (X, Y);

(C3) a morphism $id_X \in Hom_C(X, X)$, with every object X;

(C4) a composition law holds i.e., if $f \in Hom_C(X, Y)$ and $g \in Hom_C(Y, Z)$, $gof \in$

$Hom_C(X, Z)$;

such that it satisfies the following axioms:

(M1) $ho(gof) = (hog)of, \forall f \in Hom_C(X, Y), g \in Hom_C(Y, Z)$ and $h \in Hom_C(Z, W)$;

(M2) $id_Y of = fo\, id_X = f, \forall f \in Hom_C(X, Y)$;

(M3) a set of $Hom_C(X, Y)$ morphisms are pairwise disjoint.

Example 1.2.2. (1) **Set**, the category with sets to be objects, functions to be morphisms, and the usual compositions of functions for compositions.

(2) **Grp**, the category with groups to be objects, group homomorphisms to be morphisms, and their compositions as compositions.

(3) **Ab**, the category with abelian groups to be objects, group homomorphisms to be morphisms, and their compositions as compositions.

Definition 1.2.3. [29] The opposite category C^{op} of the specified category C is constructed when reversing the arrows, i.e., for each ordered object pair (X, Y)

$$Hom_{C^{op}}(Y, X) = Hom_C(X, Y)$$

Definition 1.2.4. [29] Category D is said to be a subcategory of the category C when $ob(D) \subseteq Ob(C)$, $Hom_D(X, Y) \subseteq Hom_C(X, Y)$ $\forall$ ordered object pair (X, Y) and composition of morphisms, and the identity of D should be the same as that of C.

Definition 1.2.5. [29] For the ordered object pair (X, Y) of D, a full subcategory of a category C is a category D if $ob(D) \subseteq Ob(C)$ and $Hom_D(X, Y) = Hom_C(X, Y)$.

Definition 1.2.6. [29] Let $f \in Hom_C(X, Y)$ be a morphism in C. Then f is said to be a coretraction in C if $g \circ f = I_X$ for a unique morphism $g \in Hom_C(Y, X)$. In this case, X is said to be a retract of Y. Dually, a morphism f is said to be a retraction if $f \circ h = I_Y$ for a unique morphism $h \in Hom_C(Y, X)$.

Proposition 1.2.7. *[29] Composition of two coretraction(retraction) is coretraction(retraction).*

Definition 1.2.8. [29] A morphism $f \in Hom_C(X, Y)$ is said to be an isomorphism in a category C when f is both a retraction and a coretraction.

Definition 1.2.9. [29] Let $f \in Hom_C(X, Y)$ be morphism in C. Then f is said to be a monomorphism if $f \circ g = f \circ h$ implies that $g = h$; $\forall g, h \in Hom_C(Z, X)$. Similarly, f is said to be an epimorphism if $g \circ f = h \circ f$ implies that $g = h$, $\forall g, h \in Hom_C(Y, Z)$.

Definition 1.2.10. [29] A category C is called abelian if

1. C does have a zero object.
2. There is a product and a co-product for any pair of objects of C.
3. Each morphism in C does have a kernel and a cokernel.
4. Each monomorphism in C seems to be the kernel of its cokernel.
5. Any epimorphism in C seems to be the cokernel of its kernel.

Example 1.2.11. The category **Ab** is an example of an abelian category.

Definition 1.2.12. [29] A category is said to be balanced if every morphism which is both a monomorphism and an epimorphism is also an isomorphism.

Example 1.2.13. The category **Set** is an example of a balanced category.

Proposition 1.2.14. *If $f \in Hom_C(X, Y)$ is a coretraction (retraction) and is also an epimorphism (respectively monomorphism) then it is an isomorphism.*

Emily Riehl in the book "Category Theory in Context" [37] introduced the concept of a complete and cocomplete category. The study of complete and cocomplete categories enriches our understanding of mathematical structures and their interconnections.

Definition 1.2.15. [37] A category C is said to be

(i) complete if it has products and equalizers.

(ii) cocomplete if it has coproducts and coequalizers.

(iii) bicomplete category if it is both complete and cocomplete.

1.3 Functor

Functors are used in all branches of modern mathematics to relate various categories. Consequently, functors are important in all areas of mathematics that make use of category theory. A structure-preserving mapping across categories is known as a functor in category theory. It maps morphisms of one category to morphisms of the other, and objects of one category to objects of the another, while maintaining the identity and composition of morphisms.

Definition 1.3.1. [13] Let $C = (Ob(C), Hom(C), id, o)$ and $D = (Ob(D), Hom(D), id, o)$ be two categories and let $F_1 : Ob(C) \to Ob(D)$ and $F_2 : Hom(C) \to Hom(D)$ be maps. Then the quadruple $F = (C, D, F_1, F_2)$ is a functor provided:

(i) $X \in Ob(C)$ implies $F_1(X) \in Ob(D)$;

(ii) $f \in Hom(X, Y)$ implies $F_2(f) \in Hom(F_1(X), F_1(Y)), \forall X, Y \in Ob(C)$;

(iii) F_2 preserves composition, i.e., $F_2(gof) = F_2(g)oF_2(f)$, $\forall f \in Hom(X, Y)$ and $g \in Hom(Y, Z)$;

(iv) F preserves identities, i.e., $F_2(e_X) = e_{F_1(X)}, \forall X \in Ob(C)$.

Remark 1.3.2. (i) The notation $F(X)$ is used instead of $F_1(X)$.

(ii) The notation $F(f)$ is used instead of $F_2(f)$.

(iii) Functors from C to D are denoted by the notation $F : C \to D$.

(iv) A functor defined above is called a covariant functor that preserves:

- The domains, the co-domains, and identities.
- The composition of arrows, it especially retains the path of the arrows.

(v) A contravariant functor F is similar to the covariant functor in addition to the other side of the arrow, $F(f) : F(Y) \to F(X)$ and $F(gof) = F(f)oF(g), \forall f \in Hom(X, Y), g \in Hom(Y, Z)$.

Definition 1.3.3. [13] The category C^S formed from a given category C is called a top category over C, if corresponding to every object A in C, the collection $s(A)$ of elements of C with the ordered relation defined on it, form a complete lattice, and the inverse image map $s(f), s(B) \to s(A)$, form a contravariant functor.

Definition 1.3.4. [14] For a category C, assigning every object X to X and every morphism f to the same morphism f in C, we can define a functor $I_C : C \to C$ such that $I_C(X) = X$ and $I_C(f) = f$. This functor is known as identity functor.

Definition 1.3.5. [14] Let $C^{'}$ be a subcategory of C. Define a covariant functor $I : C^{'} \to C$ as $I(X) = X; \forall X \in Ob(C^{'})$ and $I(f) = f$ for all morphisms $f \in C^{'}$. This functor is known as inclusion functor.

1.4 Category of R-modules

Modules generalize the notion of vector spaces and extend the concept of group actions, offering a versatile tool for studying algebraic objects. This section delves into the foundational aspects of the category of modules, exploring its structure, morphisms, and interconnections with other algebraic categories. By examining key concepts such as homomorphisms, direct sums, and submodules, this study aims to establish a comprehensive understanding of the categorical properties of modules. The insights gained not only contribute to the broader field of abstract algebra but also have far-reaching implications in areas such as linear algebra, ring theory, and representation theory. A significant cornerstone of abstract algebra is the "category of R-modules", which provides a unifying framework for understanding diverse algebraic structures. For conceptual concepts about module theory and related areas, we follow [1, 4, 39, 53, 54, 55].

Proposition 1.4.1. *[46] The collection of all R-modules and R-homomorphisms form a category. This category is denoted by $\mathbf{C_{R\text{-}M}}$.*

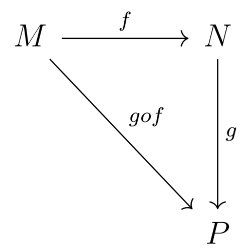
Proof. A category of R-modules $\mathbf{C_{R\text{-}M}}$ consisting of:

(Cl) a class of objects = $Ob(\mathbf{C_{R\text{-}M}})$ = all R-modules;

(C2) a set of $Hom(M, N)_{\mathbf{C_{R\text{-}M}}}$ morphisms for each ordered object pair (M, N) = all R-homomorphisms;

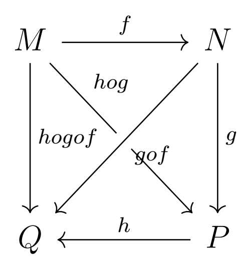
(C3) for each object M an identity R-homomorphism $id_M \in Hom_{\mathbf{C_{R\text{-}M}}}(M, M)$;

(C4) Composition law: For $f \in Hom_{\mathbf{C_{R\text{-}M}}}(M, N)$ and $g \in Hom_{\mathbf{C_{R\text{-}M}}}(N, P)$, there exists a R-homomorphism $g\, o\, f \in Hom(M, P)$ in order for the subsequent diagram commutes



(M1) *h o (g o f) = (h o g) o f*, $\forall f \in Hom_{\mathbf{C_{R\text{-}M}}}(M, N)$, $g \in Hom_{\mathbf{C_{R\text{-}M}}}(N, P)$ and $h \in Hom_{\mathbf{C_{R\text{-}M}}}(P, Q)$;

in order for the subsequent diagram commutes



Therefore associativity of the composition holds.

(M2) $id_N o f = f o\, id_M = f, \forall\, f \in Hom_{\mathbf{C_{R\text{-}M}}}(M, N)$;

(M3) all sets $Hom_{\mathbf{C_{R\text{-}M}}}(M, N)$ are pairwise disjoint.

Thus, a category of R-modules $\mathbf{C_{R\text{-}M}}$ = $(Ob(\mathbf{C_{R\text{-}M}}), Hom(\mathbf{C_{R\text{-}M}}), o)$ consisting of two classes:

(i) a class of objects = $Ob(\mathbf{C_{R\text{-}M}})$ = all R-modules;

(ii) a class of morphisms $\textbf{Hom}(\textbf{C}_{\textbf{R-M}}) = \bigcup\{\textbf{Hom}_{\textbf{C}_{\textbf{R-M}}}(M, N) : M, N \in Ob(\textbf{C}_{\textbf{R-M}})\}$

where $\textbf{Hom}_{\textbf{C}_{\textbf{R-M}}}(M, N)$ are pairwise disjoint sets for each ordered object pair (M, N). □

Example 1.4.2. Consider **C** = $\{M, N, P\}$ as a category with three R-modules $M = \{\bar{0}\}$, $N = \{\bar{0}, \bar{2}\}$, and $P = Z_4$. The Hom-sets are defined as follows: $\textbf{Hom}_{\textbf{C}_{\textbf{R-M}}}(M, M) = \{i_M\}$, $\textbf{Hom}_{\textbf{C}_{\textbf{R-M}}}(N, N) = \{i_N\}$, $\textbf{Hom}_{\textbf{C}_{\textbf{R-M}}}(P, P) = \{i_P\}$, $\textbf{Hom}_{\textbf{C}_{\textbf{R-M}}}(M, N) = \{f\}$, $\textbf{Hom}_{\textbf{C}_{\textbf{R-M}}}(N, P) = \{g\}$, $\textbf{Hom}_{\textbf{C}_{\textbf{R-M}}}(P, M) = \{h\}$ where f, g, h are inclusion R-homomorphisms and i denotes the identity R-homomorphism.

It is evident that $\textbf{C}_{\textbf{R-M}}$ constitutes a category of R-modules.

Proposition 1.4.3. *[39] $\boldsymbol{C_{R\text{-}M}}$ is equipped with products and coproducts.*

Proposition 1.4.4. *[4] In $\boldsymbol{C_{R\text{-}M}}$,*

(i) every monomorphism is the kernel of its cokernels.

(ii)every epimorphism is the cokernels of its kernel.

Proposition 1.4.5. *[4] $\boldsymbol{C_{R\text{-}M}}$ is an additive category.*

Proposition 1.4.6. *[4] $\boldsymbol{C_{R\text{-}M}}$ is an abelian category.*

Lemma 1.4.7. *[35] For a fixed $M \in Ob(\boldsymbol{C_{R\text{-}M}})$, the R-homomorphism $\phi : N \to P$ induces*

a) an R-homomorphism $\phi_* : Hom_{\boldsymbol{C_{R\text{-}M}}}(M, N) \to Hom_{\boldsymbol{C_{R\text{-}M}}}(M, P)$ defined as $\phi_*(\alpha) = \phi \circ \alpha$, $\forall \alpha \in Hom_{\boldsymbol{C_{R\text{-}M}}}(M, N)$.

b) an R-homomorphism $\phi^* : Hom_{\boldsymbol{C_{R\text{-}M}}}(P, M) \to Hom_{\boldsymbol{C_{R\text{-}M}}}(N, M)$ defined by $\phi^*(\beta) = \beta \circ \phi$, $\forall \beta \in Hom_{\boldsymbol{C_{R\text{-}M}}}(P, M)$.

Lemma 1.4.8. *[35] Let $M, N, P \in Ob(\boldsymbol{C_{R\text{-}M}})$ and $\alpha : M \to N$ and $\beta : N \to P$ be R-homomorphisms. Then for any R-module Q*

a) $(\beta\circ\alpha)_* : Hom_{\mathbf{C_{R\text{-}M}}}(Q, M) \to Hom_{\mathbf{C_{R\text{-}M}}}(Q, P)$ *is an* R*-homomorphism such that* $(\beta\circ\alpha)_* = \beta_* \circ \alpha_*$*;*

b) $(\beta\circ\alpha)^* : Hom_{\mathbf{C_{R\text{-}M}}}(P, Q) \to Hom_{\mathbf{C_{R\text{-}M}}}(M, Q)$ *is an* R*-homomorphism such that* $(\beta\circ\alpha)^* = \alpha^* \circ \beta^*$.

Definition 1.4.9. [4] Let S is subset of an R-module M. Then the smallest submodule of M that contains S is L(S), which is the set of all finite linear combinations of the elements of S.

Definition 1.4.10. [4] Let M, N and $P \in Ob(\mathbf{C_{R\text{-}M}})$. An R-homomorphism $\psi : M \times N \to P$ is said to be an R-biadditive provided that for all $a, a_1, a_2 \in M$, $b, b_1, b_2 \in N$ and $r \in R$,

(i) $\psi(a_1 + a_2, y) = f(a_1, y) + f(a_2, b)$;

(ii) $\psi(a, b_1 + b_2) = f(a, y_1) + f(a, b_2)$;

(iii) $\psi(ar, b) = f(x, ry) = rf(a, y)$.

Definition 1.4.11. [4] A tensor product of M and N over R is denoted by $M \otimes N$ and defined as

$$M \otimes N = M \times N/L(S)$$

Being the quotient module of R-module by its submodule, the tensor product $M \otimes N$ is also an R-module. Then there exists an R-homomorphism $\tau : M \times N \to M \otimes N$ such that $\tau(a, b) = (a, b) + L(S)$, for all $a \in M, b \in N$. We will denote $\tau(a, b)$ by $a \otimes b$.

Definition 1.4.12. [1, 4] A tensor product of N and K over R is an R-module $N \otimes K$ which is equipped with an R-biadditive

$$\tau : N \times K \to N \otimes K$$

such that for each R-module P and each R-biadditive $\theta : N \times K \to P$, there is a unique R-homomorphism $\phi : N \otimes K \to P$ such that $\phi \circ \tau = \theta$.

$$\begin{array}{ccc} N \times K & \xrightarrow{\ \tau\ } & N \otimes K \\ & \overset{\theta}{\searrow} & \downarrow{\scriptstyle \phi} \\ & & P \end{array}$$

Theorem 1.4.13. *[1, 4] The tensor product of two R-modules in* ***C*** $_{\textbf{\textit{R-M}}}$ *exists and it is unique upto isomorphism.*

1.5 Intuitionistic fuzzy modules

K.T. Atanassov [5, 6] suggested the interpretation of intuitionistic fuzzy sets that could be a generalized form of fuzzy sets. The exploration towards intuitionistic fuzzy characteristics within module theory has seen significant development. Despite this progress, there remain ample opportunities for additional research in extending these algebraic structures into the domain of intuitionistic fuzzification. Davaaz, in reference [12], expanded upon the idea of an intuitionistic fuzzy set to include H-v-modules. This extension led to the introduction of the theoretical framework for intuitionistic fuzzy H-v-submodules within H-v-modules, prompting the investigation of various associated properties. Later on, numerous mathematicians in [11, 16, 22, 41, 42, 43, 45] made significant contributions to the study of intuitionistic fuzzy submodules.

Definition 1.5.1. [5, 6] A mapping $A = (\mu_A, \nu_A) : X \to I \times I$ is called an intuitionistic fuzzy set(IFS) on X where the mappings $\mu_A : X \to I$ and $\nu_A : X \to I$ denotes the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$ to A, respectively with the condition that $\mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$. An intuitionistic fuzzy set A in X can be represented as an object of the form

$$A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\}.$$

Remark 1.5.2.

(i) When $\mu_A(x) + \nu_A(x) = 1$, i.e., $\nu_A(x) = 1 - \mu_A(x)$. Then A is called a fuzzy set.

(ii) We denote the IFS $A = \{< x, \mu_A(x), \nu_A(x) >; \forall x \in X\}$ by $A = (\mu_A, \nu_A)$.

Definition 1.5.3. [22, 44] An IFS A of an R-module M is called an intuitionistic fuzzy submodule (IFSM) of M, if for every $x, y \in M$ and $r \in R$ the following conditions are satisfied:

(i) $\mu_A(x + y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(x + y) \leq \nu_A(x) \vee \nu_A(y)$;

(ii) $\mu_A(rx) \geq \mu_A(x)$ and $\nu_A(rx) \leq \nu_A(x)$;

(iii) $\mu_A(\theta) = 1$ and $\nu_A(\theta) = 0$, where θ is a zero element of M.

Condition (i) and (ii) can be combined to a single condition $\mu_A(rx + sy) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(rx + sy) \leq \nu_A(x) \vee \nu_A(y)$, where $r, s \in R$ and $x, y \in M$.

Remark 1.5.4.

(i) The set of intuitionistic fuzzy submodules of R-module M is denoted by IFSM(M).

(ii) We denote the IFSM A of an R-module M by $(\mu_A, \nu_A)_M$.

Example 1.5.5. Let $M = \mathbb{R}^2$. Then IFS $A = (\mu_A, \nu_A)_M$ defined as

$$\mu_A(c, d) = \begin{cases} 0.65, & \text{if } (c, d) \neq (0, 0) \\ 1, & \text{if } (c, d) = (0, 0) \end{cases} ; \quad \nu_A(c, d) = \begin{cases} 0.25, & \text{if } (c, d) \neq (0, 0) \\ 0, & \text{if } (c, d) = (0, 0). \end{cases}$$

is an IFSM of M.

Definition 1.5.6. [46] Let K as a submodule of an R-module M. The intuitionistic fuzzy characteristic function of K is defined by χ_K, described by $\chi_K(a) = (\mu_{\chi_K}(a), \nu_{\chi_K}(a))$, where

$$\mu_{\chi_K}(a) = \begin{cases} 1, & \text{if } a \in K \\ \\ 0, & \text{if } a \notin K \end{cases} ; \quad \nu_{\chi_K}(a) = \begin{cases} 0, & \text{if } a \in K \\ \\ 1, & \text{if } a \notin K. \end{cases}$$

Clearly, χ_K is an IFSM of M. The IFSMs $\chi_{\{\theta\}}, \chi_M$ are called trivial IFSMs of module M. Any IFSM of the module M apart from this is called proper IFSM."

Definition 1.5.7. [46] "Let $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B)$ are IFSM of R-modules M and N respectively. Then the map $f : A \to B$ is called an intuitionistic fuzzy R-homomorphism (or IF R-homomorphism) from A to B if

(i) $f : M \to N$ is R-homomorphism and

(ii) $\mu_B(f(a)) \geq \mu_A(a)$ and $\nu_B(f(a)) \leq \nu_A(a), \forall a \in M$.

To avoid confusion between an R-homomorphism $f : M \to N$ and an intuitionistic fuzzy R-homomorphism $f : A \to B$. We denote the latter by $\bar{f} : A \to B$. So, given an IF R-homomorphism $\bar{f} : A \to B$, $f : M \to N$ is the underlying R-homomorphism of $\bar{f}$.

Example 1.5.8. Let $M = (\{0, 1, 2, 3, 4\}, +_4)$ and $N = (\{0, 1\}, +_2)$ be two Z-modules. Define

IFSs $A = (\mu_A, \nu_A)_M$ and $B = (\mu_B, \nu_B)_N$ as

$$\mu_A(a) = \begin{cases} 0.85, & \text{if } a = 0 \\ 0.65, & \text{if } a = 2 \\ 0.45, & \text{if } a = 1, 3 \end{cases} \quad ; \quad \nu_A(a) = \begin{cases} 0, & \text{if } a = 0 \\ 0.35, & \text{if } a = 2 \\ 0.5, & \text{if } a = 1, 3 \end{cases}$$

$$\mu_B(b) = \begin{cases} 0.95, & \text{if } b = 0 \\ 0.35, & \text{if } b = 1 \end{cases} \quad ; \quad \nu_B(b) = \begin{cases} 0, & \text{if } b = 0 \\ 0.5, & \text{if } b = 1. \end{cases}$$

Then A and B are IFSMs of M and N, respectively.

Define the R-homomorphism $f : M \to N$ as $f(a) = 0, \forall a \in M$. Consider $\mu_B(f(0)) = \mu_B(0) = 0.9 \geq 0.8 = \mu_A(0)$, $\mu_B(f(1)) = \mu_B(0) = 0.9 \geq 0.4 = \mu_A(1)$, $\mu_B(f(2)) = \mu_B(0) = 0.9 \geq 0.6 = \mu_A(2)$, $\mu_B(f(3)) = \mu_B(0) = 0.9 \geq 0.4 = \mu_A(3)$. Also,$\nu_B(f(0)) = \mu_B(0) = 0 = 0 = \nu_A(0)$, $\nu_B(f(1)) = \mu_B(0) = 0 \leq 0.5 = \nu_A(1)$, $\nu_B(f(2)) = \mu_B(0) = 0 \leq 0.3 = \nu_A(2)$, $\nu_B(f(3)) = \mu_B(0) = 0 \leq 0.5 = \nu_A(3)$. Thus, $\mu_B(f(a)) \geq \mu_A(a)$ and $\nu_B(f(a)) \leq \nu_A(a), \forall a \in M$.

Hence, $\bar{f} : A \to B$ is an IF R-homomorphism.

Definition 1.5.9. [46] Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ are IFSMs of R-modules M and N respectively and $f : M \to N$ is R-homomorphism. With the help of A and f, we can provide an IF module structure on N by

$$\mu_{f(A)}(b) = \sup\{\mu_A(a) : f(a) = b\} \text{ and } \nu_{f(A)}(b) = \inf\{\nu(a) : f(a) = b\}.$$

It is clear that $f(A) = (\mu_{f(A)}, \nu_{f(A)})$ is an IFSM of and $\bar{f} : A \to f(A)$ is an IF R-homomorphism.

With the help of B and f, we can provide an IF module structure on M by

$$\mu_{f^{-1}(B)}(a) = \mu_B(f(a)) \text{ and } \nu_{f^{-1}(B)}(a) = \nu_B(f(a)).$$

Hence, $f^{-1}(B) = (\mu_{f^{-1}(B)}, \nu_{f^{-1}(B)})$ is an IFSM of M and $\bar{f} : f^{-1}(B) \to B$ is an IF R-homomorphism.

Lemma 1.5.10. *[47] Let M and N are R-modules and $f : M \to N$ be R-homomorphism. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ are IFSM of R-modules M and N respectively and $\bar{f} : A \to B$ is an IF R-homomorphism. Then*

(i) $A \subseteq f^{-1}(f(A))$.

(ii) $A = f^{-1}(f(A))$ if and only if both f and $\bar{f}$ are one-one functions.

(iii)$f(f^{-1}(B)) \subseteq B$.

(iv)$f(f^{-1}(B)) = B$ if and only if both f and $\bar{f}$ are onto functions.

If $f \in Hom(M, N)$ and $\bar{f} \in Hom(A, B)$, define

$$Ker\bar{f} = \{a \in M : \mu_B(f(a)) = 1; \nu_B(f(a)) = 0\}$$

and

$$Im\bar{f} = \{\bar{f}(a) : a \in M\}$$

As $Kerf$ is the pre-image of $\{\theta\}$ under f, we have $Kerf \subseteq Ker\bar{f}$. Especially, if $B = \chi_N$, then we have $Ker\bar{f} = A$, for all $\bar{f} \in Hom(A, B)$.

Proposition 1.5.11. *Let A and B are IFSM of R-modules M and N, respectively, and $\bar{f} : A \to B$ is IF R-homomorphism, then:*

(i) $Ker\bar{f}$ is a submodule of M;

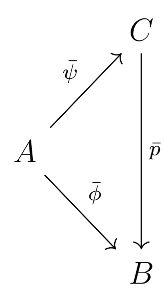
(ii) *The restriction of A to $Ker\bar{f}$ i.e., $A|_{Ker\bar{f}}$ is an IFSM of A.*

Proof. (i) Given that $\bar{f} : A \to B$ is IF R-homomorphism. For zero element θ of M, $\theta \in Ker\bar{f}$. If $r \in R$ and $x \in Ker\bar{f}$, so $\mu_A(\bar{f}(rx)) = \mu_A(r\bar{f}(x)) \geq \mu_A(\bar{f}(x)) = 1$ and $\nu_A(\bar{f}(rx)) = \nu_A(r\bar{f}(x)) \leq \nu_A(\bar{f}(x)) = 0$ implies that $rx \in Ker\bar{f}$. It follows, $-x \in Ker\bar{f}$. Further, if $x, y \in Ker\bar{f}$, Conveniently, we can predict $x + y \in Ker\bar{f}$, which proves the result.

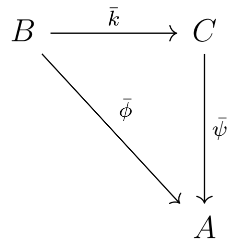
(ii) Let $A|_{Ker\bar{f}} = C = (\mu_C, \nu_C)$, where $\mu_C(x) = \mu_A(x)$ and $\nu_C(x) = \nu_A(x), \forall x \in Ker\bar{f}$. Now it is simple to prove that C is an IFSM of M and $C \subseteq A$.

□

Definition 1.5.12. [44] Let A, B and C are IFSMs of R-modules M, N and P respectively. Then an IFSM A is called an intuitionistic fuzzy projective module (IF-projective), if for every IF R-homomorphism $\bar{\phi} : A \to B$ and IF-epimorphism $\bar{p} : C \to B$, there exists an IF R-homomorphism $\bar{\psi} : A \to C$ such that $\bar{p} \circ \bar{\psi} = \bar{\phi}$, i.e., the subsequent diagram commutes



Definition 1.5.13. [44] Let A, B and C are IFSMs of R-modules M, N and P respectively. Then an IFSM A is called an intuitionistic fuzzy injective module (IF-injective), if for every IF R-homomorphism $\bar{\phi} : B \to A$ and IF-monomorphism $\bar{k} : B \to C$, there exists an IF R-homomorphism $\bar{\psi} : C \to A$ such that $\bar{\psi} \circ \bar{k} = \bar{\phi}$, i.e., the subsequent diagram commutes



Theorem 1.5.14. *[44] Every intuitionistic fuzzy free submodule of an R-module is IF-projective.*

Throughout our thesis, we will denote intuitionistic fuzzy modules using the symbols A, B, C, D, E, F and G, corresponding to R-modules M, N, K, P, Q, S and T, respectively. Notably, $\bar{f} : A \to B$ represents IF R-homomorphism.

Chapter 2

Intuitionistic fuzzy modules: categorical perspectives

2.1 Introduction

Fuzzy algebra, a structure deeply ingrained in various fields including computer science, information technology, theoretical physics, and control engineering, stands as a cornerstone in mathematics. Many ideas of abstract algebra within the framework of fuzzy sets have been extensively studied by researchers since the introduction of fuzzy sets in 1965 [56] . One such researcher is Rosenfeld [38], who became the first person to define the notion of fuzzy subgroups in 1971. Since then, other extensions of this concept have been put forth, particularly in the last few decades. Fuzzy set theory was introduced to the concept of modules in 1975 by Relescu and Nagoita [30]. Atanassov [5, 6] introduced intuitive fuzzy sets in 1986, which are based on the degree of membership and non-membership, adhering to the constraint that their total should not exceed unity. Using the conceptual framework of intuitionistic fuzzy sets, Biswas [10] investigated the intuitionistic fuzzy subsets of a group in 1989 and applied group theory to it.

In recent years, extensive research has focused on fuzzy and intuitionistic fuzzy modules and main submodules, as well as fuzzy and intuitionistic fuzzy prime modules. Hur K. et al. [19, 20] introduced the idea of intuitionistic fuzzy subgroups and ideals, expanding the scope of fuzzy algebraic structures. Mashinchi and Zahedi [28] explored concepts of fuzzy prime and fuzzy primary submodules, contributing to the evolving landscape of fuzzy module theory.

Golan [15] and Lopez-Permouth and Malik [33] made significant contributions to the study of fuzzy modules, examining categories and exact sequences in fuzzy complexes. Categories develop as a unifying concept in many domains of mathematics, particularly in computer technology and mathematical physics. Several other researchers [3, 15, 26, 31, 32, 34, 35, 41, 52, 57, 58] have established and explored theories of fuzzy modules, fuzzy exact sequences of fuzzy complexes, and fuzzy homology of fuzzy chain complexes. Moreover, several mathematicians researched intuitionistic fuzzy submodules and their properties [11, 16, 22, 42, 43, 44, 45]. In this Chapter, we

1. form the category of intuitionistic fuzzy modules ($\mathbf{C_{R\text{-}IFM}}$) in the section 2.2.
2. explore the relationship between the category of R-modules ($\mathbf{C_{R\text{-}M}}$) and the category of intuitionistic fuzzy modules ($\mathbf{C_{R\text{-}IFM}}$) in the section 2.3.
3. examine Optimal intuitionistic fuzzification and investigate that $\mathbf{C_{R\text{-}IFM}}$ is not an abelian category in the section 2.4.
4. develop some categories of IFMs in the section 2.5.

2.2 Category of intuitionistic fuzzy modules $C_{\text{R-IFM}}$

Proposition 2.2.1. *[46] The set $Hom(A, B)$ of all IF R-homomorphisms from A to B forms an additive abelian group. Furthermore, it constitutes a unitary R-module when R is a commutative ring with unity.*

Theorem 2.2.2. *Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ are two IF modules of R-modules M and N respectively. The function $\beta : Hom(A, B) \to I \times I$ on R-module $Hom(A, B)$ is then defined by*

$$\beta(\bar{f}) = (\mu_{\beta(\bar{f})}, \nu_{\beta(\bar{f})})$$

where $\mu_{\beta(\bar{f})} = \wedge\{\mu_B(\bar{f}(a)) : z \in M\}$ and $\nu_{\beta(\bar{f})} = \vee\{\nu_B(\bar{f}(z)) : z \in M\}$ is an IFSM of $Hom(A, B)$.

Proof. As shown in Proposition 2.2.1, $Hom(A, B)$ is an R-module and the scalar multiplication on $Hom(A, B)$ is defined by $(r.\bar{f})(z) = r\ \bar{f}(z), \forall z \in M$. For $r \in R$ and $\bar{f} \in Hom(A, B)$, Consider

$$\begin{aligned}
\mu_{\beta(r.\bar{f})} &= \wedge\{\mu_B((r.\bar{f})(z)) : z \in M\} \\
&= \wedge\{\mu_B((r.\bar{f}(z)) : z \in M\} \\
&\geq \wedge\{\mu_B(\bar{f}(z)) : z \in M\} \\
&= \mu_{\beta(\bar{f})}.
\end{aligned}$$

Thus $\mu_{\beta(r.\bar{f})} \geq \mu_{\beta(\bar{f})}$. Likewise, we are able to exhibit that $\nu_{\beta(r.\bar{f})} \leq \nu_{\beta(\bar{f})}$.

Further, let $\bar{\xi}_1, \bar{\xi}_2 \in Hom(A, B)$ and $z \in M$. Consider

$$\begin{aligned}
\mu_{\beta(\bar{\xi}_1+\bar{\xi}_2)} &= \wedge\{\mu_B((\bar{\xi}_1+\bar{\xi}_2)(z)) : z \in M\} \\
&= \wedge\{\mu_B(\bar{\xi}_1(z)+\bar{\xi}_2(z)) : z \in M\} \\
&\geq \wedge\{\{\mu_B(\bar{\xi}_1(z) \wedge \bar{\xi}_2(z))\} : z \in M\} \\
&= \{\wedge\{\mu_B(\bar{\xi}_1(z)) : z \in M\}\} \wedge \{\wedge\{\mu_B(\bar{\xi}_2(z)) : z \in M\}\} \\
&= \mu_{\beta(\bar{\xi}_1)} \wedge \mu_{\beta(\bar{\xi}_2)}.
\end{aligned}$$

Thus, $\mu_{\beta(\bar{\xi}_1+\bar{\xi}_2)} \geq \mu_{\beta(\bar{\xi}_1)} \wedge \mu_{\beta(\bar{\xi}_2)}$. Likewise, we are able to exhibit that $\nu_{\beta(\bar{\xi}_1+\bar{\xi}_2)} \leq \nu_{\beta(\bar{\xi}_1)} \vee \nu_{\beta(\bar{\xi}_2)}$.

Also, $\mu_{\beta(\bar{0})} = \wedge\{\mu_B(\bar{0}(z)) : z \in M\} = \wedge\{\mu_B(0) : z \in M\} = 1$.

Likewise, we can demonstrate that $\nu_{\beta(\bar{0})} = 0$.

Hence, β is IFSM of R-module $Hom(A, B)$. □

Definition 2.2.3. The category $\mathbf{C_{R\text{-}M}} = (Ob(\mathbf{C_{R\text{-}M}}), \mathbf{Hom}(\mathbf{C_{R\text{-}M}}), o)$ has objects as R-modules and morphisms R-homomorphisms, with composition of morphisms defined as the composition of mappings. An IF-module category $\mathbf{C_{R\text{-}IFM}}$ over the base category $\mathbf{C_{R\text{-}M}}$ is completely described by two mappings:

$$\alpha : Ob(\mathbf{C_{R\text{-}M}}) \to I \times I;$$

$$\beta : Hom(\mathbf{C_{R\text{-}M}}) \to I \times I$$

IF-module category $\mathbf{C_{R\text{-}IFM}}$ consists of

(C1) Ob($\mathbf{C_{R\text{-}IFM}}$) the set of objects as IFSMs on Ob($\mathbf{C_{R\text{-}M}}$), that is, the objects will be $\alpha : Ob(\mathbf{C_{R\text{-}M}}) \to I \times I$;

(C2) $\mathbf{Hom}(\mathbf{C_{R\text{-}IFM}})$ the set of IF R-homomorphisms corresponding to underlying R-homomorphisms from $\mathbf{Hom}(\mathbf{C_{R\text{-}M}})$, i.e., IF R-homomorphisms of the form $\beta : Hom(\mathbf{C_{R\text{-}M}}) \to I \times I$, so that for $f \in \mathbf{Hom}_{\mathbf{C_{R\text{-}M}}}(M, N)$,

$$\beta(\bar{f}) = (\mu_{\beta(\bar{f})}, \nu_{\beta(\bar{f})})$$

as defined in Theorem 2.2.2, a composition law associating to each $f \in \mathbf{Hom}_{\mathbf{C_{R\text{-}M}}}(M, N)$ and $g \in \mathbf{Hom}_{\mathbf{C_{R\text{-}M}}}(N, K)$, an R-homomorphism $gof \in \mathbf{Hom}_{\mathbf{C_{R\text{-}M}}}(M, K)$ exists, so that each of the ensuing axioms holds:

(M1) Associativity: $ho(gof) = (hog)of$, $\forall f \in \mathbf{Hom}_{\mathbf{C_{R\text{-}M}}}(M, N)$, $g \in \mathbf{Hom}_{\mathbf{C_{R\text{-}M}}}(N, K)$ and $h \in \mathbf{Hom}_{\mathbf{C_{R\text{-}M}}}(K, P)$;

(M2) Preservation of morphisms: $\beta(g \circ f) = \beta(g)o\beta(f)$;

(M3) Existence of identity: $\forall M \in Ob(\mathbf{C_{R\text{-}M}})$, identity $i_M \in \mathbf{Hom}_{\mathbf{C_{R\text{-}M}}}(M, M)$ exists satisfying $\beta(i_M) = \alpha(M)$. Thus, the category of IF R-modules can be constructed as

$$\mathbf{C_{R\text{-}IFM}} = (Ob(\mathbf{C_{R\text{-}IFM}}), \mathbf{Hom}(\mathbf{C_{R\text{-}IFM}}), o)$$

Remark 2.2.4. Throughout this thesis, we use the notation $\mathbf{C_{R\text{-}IFM}}$ to represent the category of intuitionistic fuzzy R-modules along with intuitionistic fuzzy R-homomorphisms and the set of all IF R-homomorphism from A to B is denoted by $\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A, B)$.

2.3 Mapping between $\mathbf{C}_{\textbf{R-M}}$ and $\mathbf{C}_{\textbf{R-IFM}}$

In this section, we analyze the relationship between category of R-modules $\mathbf{C}_{\textbf{R-M}}$ with category of IF R-modules $\mathbf{C}_{\textbf{R-IFM}}$ and the existence of the covariant functor between these two categories.

Proposition 2.3.1. *$\mathbf{C}_{\textbf{R-M}}$ is a subcategory of $\mathbf{C}_{\textbf{R-IFM}}$.*

Proof. This can be deduced from Definition 1.2.4, Proposition 2.2.1 and Theorem 2.2.2. □

Proposition 2.3.2. *There exists a covariant functor from $\mathbf{C}_{\textbf{R-M}}$ to $\mathbf{C}_{\textbf{R-IFM}}$.*

Proof. Define $\beta = (\mu_\beta, \nu_\beta) : \mathbf{C}_{\textbf{R-M}} \to \mathbf{C}_{\textbf{R-IFM}}$ by $\beta(M) = (\mu_\beta(M), \nu_\beta(M))$, if for every $a \in M$, $\mu_\beta(a) + \nu_\beta(a) \leq 1$.

Let $f \in \mathbf{Hom}_{\mathbf{C}_{\textbf{R-M}}}(M, N)$. Thus $\beta(f) \in \mathbf{Hom}(\mathbf{C}_{\textbf{R-IFM}})$, where $\beta(f) : \beta(M) \to \beta(N)$ described by

$$\beta(f)(\mu_\beta, \nu_\beta) = (\mu_\beta \circ f^{-1}, \nu_\beta \circ f^{-1}) \text{ ; where}$$

(i) $\mu_\beta(a + b) \geq \mu_\beta(a) \wedge \mu_\beta(b)$

(ii) $\nu_\beta(a + b) \leq \nu_\beta(a) \vee \nu_\beta(b)$

(iii) $\mu_\beta(-a) = \mu_\beta(a)$

(iv) $\nu_\beta(-a) = \nu_\beta(a)$

(v) $\mu_\beta(ra) = \mu_\beta(a)$

(vi) $\nu_\beta(ra) = \nu_\beta(a)$

(vii) $\mu_\beta(0) = 1$

(viii) $\nu_\beta(0) = 0, \forall a, b \in M, r \in R$.

Our aim is to prove that β preserves object, composition, domain, and codomain identity. Let (μ_β, ν_β), $(\mu_{\beta_1}, \nu_{\beta_1}) \in Ob(\mathbf{C_{R\text{-}IFM}})$ such that $(\mu_\beta \circ f^{-1}, \nu_\beta \circ f^{-1}) = (\mu_{\beta_1} \circ f^{-1}, \nu_{\beta_1} \circ f^{-1})$

$\Rightarrow \mu_\beta \circ f^{-1} = \mu_{\beta_1} \circ f^{-1}$ and $\nu_\beta \circ f^{-1} = \nu_{\beta_1} \circ f^{-1}$

$\Rightarrow \mu_\beta = \mu_{\beta_1}$ and $\nu_\beta = \nu_{\beta_1} \Rightarrow (\mu_\beta, \nu_\beta) = (\mu_{\beta_1}, \nu_{\beta_1})$

$\Rightarrow \beta$ is well defined.

Let $f \in \mathbf{Hom_{C_{R\text{-}M}}}(M, N), g \in \mathbf{Hom_{C_{R\text{-}M}}}(N, K)$ then $gof \in \mathbf{Hom_{C_{R\text{-}M}}}(M, K)$. Then, $\beta(f) \in \mathbf{Hom_{C_{R\text{-}IFM}}}(\beta(M), \beta(N))$, $\beta(g) \in \mathbf{Hom_{C_{R\text{-}IFM}}}(\beta(N), \beta(K))$ and $\beta(gof) \in \mathbf{Hom_{C_{R\text{-}IFM}}}(\beta(M), \beta(K))$. For any $(\mu_\beta, \nu_\beta) \in \beta(M)$, we have

$$\begin{aligned}
\beta(g \circ f)(\mu_\beta, \nu_\beta) &= (\mu_\beta \circ (g \circ f)^{-1}, \nu_\beta \circ (g \circ f)^{-1}) \\
&= (\mu_\beta \circ (f^{-1} \circ g^{-1}), \nu_\beta \circ (f^{-1} \circ g^{-1})) \\
&= ((\mu_\beta \circ f^{-1}) \circ g^{-1}, (\nu_\beta \circ f^{-1}) \circ g^{-1}) \\
&= \beta(g)(\mu_\beta \circ f^{-1}, \nu_\beta \circ f^{-1}) \\
&= \beta(g)\beta(f)(\mu_\beta, \nu_\beta).
\end{aligned}$$

Therefore, $\beta(g \circ f) = \beta(g) \circ \beta(f)$.

Moreover, $\beta(i_M)(\mu_\beta, \nu_\beta) = (\mu_\beta \circ i_M^{-1}, \nu_\beta \circ i_M^{-1}) = (\mu_\beta, \nu_\beta)$ implies that $\beta(i_M)$ is the identity element in $\mathbf{Hom}(\mathbf{C_{R\text{-}IFM}})$. Hence, $\beta : \mathbf{C_{R\text{-}M}} \to \mathbf{C_{R\text{-}IFM}}$ is a covariant functor. $\square$

2.4 Optimal intuitionistic fuzzification

In this section, we show that the category $\mathbf{C}_{\textbf{R-IFM}}$ forms a top category over the category $\mathbf{C}_{\textbf{R-M}}$. To prove this, we first construct a category $\mathbf{C}_{\textbf{Lat(R-IFM)}}$ of complete lattices corresponding to every object in $\mathbf{C}_{\textbf{R-M}}$ and then show that corresponding to each morphism in $\mathbf{C}_{\textbf{R-M}}$, there exists a contravariant functor from $\mathbf{C}_{\textbf{R-IFM}}$ to the category $\mathbf{C}_{\textbf{Lat}}$ (=union of all $\mathbf{C}_{\textbf{Lat(R-IFM)}}$, corresponding to each object in $\mathbf{C}_{\textbf{R-M}}$) that preserve infima. Finally, we define the notion of kernel and cokernel for the category $\mathbf{C}_{\textbf{R-IFM}}$ and show that $\mathbf{C}_{\textbf{R-IFM}}$ is not an abelian category.

Lemma 2.4.1. *Let M and N are R-modules and $f : M \to N$ be R-homomorphism.*

(i) If $A = (\mu_A, \nu_A)$ is an IFSM of M, then there exists an IFSM $f(A) = (\mu_{f(A)}, \nu_{f(A)})$ of N such that for any IFSM (μ_B, ν_B) of N, the map $\bar{f} : A \to B$ is an IF R-homomorphism if and only if $f(A) \subseteq B$.

(ii) If $B = (\mu_B, \nu_B)$ is an IFSM of N, then there exists an IFSM $f^{-1}(B) = (\mu_{f^{-1}(B)}, \nu_{f^{-1}(B)})$ of M such that for any IFSM A of M, the map $\bar{f} : A \to B$ is an IF R-homomorphism if and only if $A \subseteq f^{-1}(B)$.

Proof. (i) Now, $\bar{f} : A \to B$ is an IF R-homomorphism if and only if for each $z \in M$, $\mu_B(f(z)) \geq \mu_A(z)$ and $\nu_B(f(z)) \leq \nu_A(z)$. Let $t \in N$ be any element, then $\mu_{f(A)}(t) = \vee\{\mu_A(z) : f(z) = t\} \leq \mu_A(z) \leq \mu_B(f(z))$. Likewise, we are able to exhibit that $\nu_{f(A)}(t) \geq \nu_B(f(z))$ i.e., $f(A) \subseteq B$.

(ii)Now, $\bar{f} : A \to B$ is an IF R-homomorphism if and only if for each $z \in M$, $\mu_B(f(z)) \geq \mu_A(z)$ and $\nu_B(f(z)) \leq \nu_A(z)$. Now, $\mu_{f^{-1}(B)}(z) = \mu_B(f(z)) \geq \mu_A(z)$ and $\nu_{f^{-1}(B)}(z) = \nu_B(f(z)) \leq \nu_A(z)$ implies that $A \subseteq f^{-1}(B)$. □

Observe that if $f \in Hom(M, N)$, now for each IFSM A $[B]$ on M $[N]$ one will have IFSMs $f(A)$ $[f^{-1}(B)]$, we conclude that f is trivially intuitionistic fuzzified relative to A $[B]$. In particular, we will say that for each IFSM A $[B]$ of $M[N]$, we have obtained IF R-homomorphism $\bar{f} : A \to \chi_N$ $[\bar{f} : \chi_M \to B]$.

Lemma 2.4.2. *The set $s(M) = \{(\mu, \nu) : M \to I \times I : (\mu, \nu)$ is IF module of R-module M $\}$ form a complete lattice associated with the order relation $(\mu_1, \nu_1) \leq (\mu_2, \nu_2)$ if $\mu_1(a) \leq \mu_2(a)$ and $\nu_1(a) \geq \nu_2(a)$, $\forall a \in M$.*

Proof. Let $\{(\mu_i, \nu_i) : i \in J\}$ be a collection of elements of $s(M)$. Then infimum and supremum on $s(M)$ are explicitly specified as:

$\wedge_{i\in J}(\mu_i, \nu_i)(a) = (Inf_{i\in J}\{\mu_i(a)\}, Sup_{i\in J}\{\nu_i(a)\})$

and

$\vee_{i\in J}(\mu_i, \nu_i)(a) = (Inf_{i\in J}\{\mu(a) : (\mu_i, \nu_i) \in s(M)$ and $\mu_i \leq \mu, \forall i \in J\}, Sup_{i\in J}\{\nu(a) : (\mu_i, \nu_i) \in s(M)$ and $\nu_i \geq \nu, \forall i \in J\})$.

Then $s(M)$ form a complete lattice. □

Remark 2.4.3.

(i) The least element of $s(M)$ is $\bar{0}$ and the greatest element of $s(M)$ is $\bar{1}$.

(ii) $s(M)$ under the order relation defined above form a category where

$Ob(s(M))$ = all IFSMs of M and $Hom(s(M))$ = order relation defined above.

(iii) Supremum can also be defined as $\vee_{i\in J}(\mu_i, \nu_i)(a) = (Sup_{i\in J}\{\mu_i(a)\}, Inf_{i\in J}\{\nu_i(a)\})$, which only holds for IFSs but does not hold for IFSMs including when J is finite.

For e.g., let $M = Z$-module Z and IFSMs (μ_1, ν_1) and (μ_2, ν_2) of M described as:

$$(\mu_1, \nu_1)(t) = \begin{cases} (1, 0), & \text{if } t \text{ is even} \\ (0, 1), & \text{if } t \text{ is odd} \end{cases} ; \quad (\mu_2, \nu_2)(t) = \begin{cases} (1, 0), & \text{if } 3|t \\ (0, 1), & \text{if } 3 \nmid t. \end{cases}$$

Take $(\mu_1, \nu_1) \vee (\mu_2, \nu_2) = (\mu_3, \nu_3)$, where $\mu_3(t) = max\{\mu_1(t), \mu_2(t)\}$ and $\nu_3(t) = min\{\nu_1(t), \nu_2(t)\}$. Here we can check that $(\mu_3, \nu_3))$ is not an IFSM of M, for $0 = \mu_3(1) = \mu_3(3-2) \ngeq \mu_3(3) \wedge \mu_3(2) = 1$ and $1 = \nu_3(1) = \nu_3(3-2) \nleq \nu_3(3) \vee \nu_3(2) = 0$.

Lemma 2.4.4. *The set $t(M) = \{(\mu, \nu) : M \to I \times I : (\mu, \nu)$ is IF module of R-module M $\}$ form a complete lattice associated with the order relation $(\mu_1, \nu_1) \leq (\mu_2, \nu_2)$ if $\mu_1(a) \geq \mu_2(a)$ and $\nu_1(a) \leq \nu_2(a)$ $\forall a \in M$.*

Proof. Let $\{(\mu_i, \nu_i) : i \in J\}$ be a collection of elements of $t(M)$. Then infimum and supremum on $t(M)$ are explicitly specified as :

$\wedge_{i\in J}(\mu_i, \nu_i)(a) = (Sup_{i\in J}\{\mu_i(a)\}, Inf_{i\in J}\{\nu_i(a)\})$

and

$\vee_{i\in J}(\mu_i, \nu_i)(a) = (Inf_{i\in J}\{\mu(a) : (\mu_i, \nu_i) \in t(M)$ and $\mu_i \leq \mu, \forall i \in J\}, Sup_{i\in J}\{\nu(a) : (\mu_i, \nu_i) \in t(M)$ and $\nu_i \geq \nu, \forall i \in J\})$.

Then $t(M)$ forms a complete lattice. □

Remark 2.4.5. $t(M)$ under the order relation defined above form a category where $Ob(t(M))$ = all IFSMs of M and $Hom(t(M))$ = order relation as defined above.

Theorem 2.4.6. *$\boldsymbol{C_{R\text{-}IFM}}$ is a top category over $\mathbf{C_{R\text{-}M}}$.*

Proof. This becomes sufficient to prove that, with every $M \in Ob(\mathbf{C_{R\text{-}M}})$, the corresponding complete lattice $s(M)$ specified in Lemma 2.4.2. For each $f \in \mathbf{Hom}_{\mathbf{C_{R\text{-}M}}}(M, N)$, $s(f) : s(N) \to s(M)$ defined as $s(f)(\mu_B, \nu_B) = (\mu_{f^{-1}(B)}, \nu_{f^{-1}(B)}), \forall (\mu_B, \nu_B) \in s(N)$ determine a contravariant functor $s : \mathbf{C_{R\text{-}IFM}} \to \mathbf{C_{Lat}}$. Thus, we are trying to prove that

(i) $s(f)$ preserve infima, for all $f \in \mathbf{Hom}_{\mathbf{C_{R\text{-}M}}}(M, N)$;

(ii) $s(g \circ f) = s(f) \circ s(g)$, for each $f \in \mathbf{Hom}_{\mathbf{C_{R\text{-}M}}}(M, N)$ and $g \in \mathbf{Hom}_{\mathbf{C_{R\text{-}M}}}(N, K)$, and

(iii) the identity function $s(i_M) : s(M) \to s(M)$ exists for every identity R-homomorphism $i_M : M \to M$.

Consider $\{(\mu_{B_i}, \nu_{B_i}) : i \in J\} \subset s(N)$ is a non-empty subfamily of $s(N)$, and let $a \in M$. Then,

$$\begin{aligned}
s(f)[\wedge(\mu_{B_i}, \nu_{B_i})](a) &= (Inf\{\mu_{f^{-1}(B_i)}\}, Sup\{\nu_{f^{-1}(B_i)}\})(a) \\
&= (Inf\{\mu_{f^{-1}(B_i)}(a)\}, Sup\{\nu_{f^{-1}(B_i)}(a)\}) \\
&= (Inf\{\mu_{B_i}(f(a))\}, Sup\{\nu_{B_i}(f(a))\}) \\
&= (Inf\{\mu_{B_i}\}, Sup\{\nu_{B_i}\})(f(a)) \\
&= \wedge(\mu_{B_i}, \nu_{B_i})(f(a)) \\
&= \wedge(\mu_{B_i}(f(a)), \nu_{B_i}(f(a))) \\
&= \wedge(\mu_{f^{-1}(B_i)}(a), \nu_{f^{-1}(B_i)}(a)) \\
&= \wedge(\mu_{f^{-1}(B_i)}, \nu_{f^{-1}(B_i)})(a) \\
&= \wedge[s(f)(\mu_{B_i}, \nu_{B_i})](a).
\end{aligned}$$

Thus, $s(f)$ preserves infima. Let $f : M \to N, g : N \to K$ is homomorphism, and let

$(\mu_C, \nu_C) \in s(K)$ and $k \in K$, then

$$\begin{aligned} s(gof)(\mu_C, \nu_C)(k) &= (\mu_{(gof)^{-1}(C)}, \nu_{(gof)^{-1}(C)})(k) \\ &= (\mu_{(f^{-1}og^{-1})(C)}(k), \nu_{(f^{-1}og^{-1})(C)}(k)) \\ &= (\mu_{(f^{-1}(g^{-1}(C)))}(k), \nu_{(f^{-1}(g^{-1}(C)))}(k)) \\ &= s(f)(\mu_{g^{-1}(C)}(k), \nu_{g^{-1}(C)}(k)) \\ &= s(f)(s(g)(\mu_C(k), \nu_C(k))) \\ &= s(f)s(g)(\mu_C, \nu_C)(k). \end{aligned}$$

Thus, $s(gof) = s(f) \circ s(g)$.

Further, $i_M : M \to M$ is the identity R-homomorphism, such that $i_M(a) = a, \forall a \in M$. Then $s(i_M)$ be the identity element in $\mathbf{Hom}(\mathbf{C_{R\text{-}IFM}})$, for if $(\mu_A, \nu_A) \in s(M)$ be any element, then $s(i_M)(\mu_A, \nu_A)(a) = (\mu_{i_M^{-1}(A)}(a), \nu_{i_M^{-1}(A)}(a)) = (\mu_{i_M(A)}(a), \nu_{i_M(A)}(a)) = (\mu_A(a), \nu_A(a)) = (\mu_A, \nu_A)(a)$. Hence proved. □

Remark 2.4.7. There exists a covariant functor $t : \mathbf{C_{R\text{-}IFM}} \to \mathbf{C_{Lat}}$ so $t(f) : t(M) \to t(N)$ preserves suprema and is defined by $t(f)(\mu_A, \nu_A) = (\mu_{f(A)}, \nu_{f(A)}), \forall(\mu_A, \nu_A) \in t(M)$ so that $t(g \circ f) = t(g) \circ t(f), \forall f : M \to N, g : N \to K$.

Proof. It is very simple to find that $t(f)$ preserves suprema and $t(i_M)$ is an identity element in $\mathbf{Hom}(\mathbf{C_{R\text{-}IFM}})$. Furthermore, we have

$$\begin{aligned} t(gof)(\mu_A, \nu_A)(a) &= (\mu_{(gof)(A)}(a), \nu_{(gof)(A)}(a)) \\ &= (\mu_{g(f(A))}(a), \nu_{g(f(A))}(a)) \\ &= t(g)(\mu_{f(A)}(a), \mu_{f(A)}(a)) \\ &= t(g)(t(f)(\mu_A(a), \nu_A(a))) \\ &= t(g)t(f)(\mu_A(a), \nu_A(a)) \\ &= t(g)t(f)(\mu_A, \nu_A)(a) \end{aligned}$$

Thus $t(gof) = t(g) \circ t(f)$. Hence, the result is proved. □

Lemma 2.4.8. *(i) Let $\{M_i : i \in J\}$, N are R-modules and $\mathfrak{A} = \{f_i : M_i \to N : i \in J\}$ be a collection of R-homomorphisms. If $\{A_i : i \in J\}$ is a collection of IFSMs of M_i, then there exists a smallest IFSM $B = (\mu_B, \nu_B)$ of N so that $\bar{f}_i : A_i \to B$ is an IF R-homomorphism, $\forall i \in J$, where $(\mu_B, \nu_B) = (\mu, \nu)^{\mathfrak{A}} = (\mu^{\mathfrak{A}}, \nu^{\mathfrak{A}})$, here $\mu_B = \mu^{\mathfrak{A}} = \vee\{\mu_{f_i(A_i)} : i \in J\}$ and $\nu_B = \nu^{\mathfrak{A}} = \wedge\{\nu_{f_i(A_i)} : i \in J\}$.*

(ii) Let M and $\{N_i : i \in J\}$ are R-modules and $\mathfrak{B} = \{g_i : M \to N_i : i \in J\}$ be a collection of R-homomorphisms. If $\{B_i : i \in J\}$ are IFSMs of N_i, then there exists a largest IFSM $A = (\mu_A, \nu_A)$ of M so that $\bar{g}_i : A \to B_i$ is an IF R-homomorphism, $\forall i \in J$, where $(\mu_A, \nu_A) = (\mu, \nu)_{\mathfrak{B}} = (\mu_{\mathfrak{B}}, \nu_{\mathfrak{B}})$, here $\mu_A = \mu_{\mathfrak{B}} = \wedge\{\mu_{g_i^{-1}(B_i)} : i \in J\}$ and $\nu_A = \nu_{\mathfrak{B}} = \vee\{\nu_{g_i^{-1}(B_i)} : i \in J\}$.

Proof. (i) Using Lemma 2.4.1(i), for each $i \in J$, A_i is IFSM of M_i, there exists IFSM $f_i(A_i)$ on N so that for every IFSM $B = (\mu_B, \nu_B)$ of N, $\bar{f}_i : A_i \to B$ is an IF R-homomorphism if and only if $f_i(A_i) \subseteq B$, i.e., $\mu_B \geq \mu_{f_i(A_i)}$ and $\nu_B \leq \nu_{f_i(A_i)}$. Let $\mu^{\mathfrak{A}} = \vee\{\mu_{f_i(A_i)} : i \in J\}$ and $\nu^{\mathfrak{A}} = \wedge\{\nu_{f_i(A_i)} : i \in J\}$. Subsequently, the consequence follows.

(ii) Using Lemma 2.4.1(ii), for each $i \in J$, B_i is IFSM of N, then there exists an IFSM $g_i^{-1}(B_i)$ of M, such that for any IFSM $A = (\mu_A, \nu_A)$ of M, $\bar{g}_i : A \to B_i$ is an IF R-homomorphism if and only if $A \subseteq g_i^{-1}(B_i)$, i.e., $\mu_A \leq \mu_{g_i^{-1}(B_i)}$ and $\nu_A \geq \nu_{g_i^{-1}(B_i)}$. Let $\mu_{\mathfrak{B}} = \wedge\{(\mu_{g_i^{-1}(B_i)} : i \in J\}$ and $\nu_{\mathfrak{B}} = \vee\{(\nu_{g_i^{-1}(B_i)} : i \in J\}$. Subsequently, the consequence follows. □

Lemma 2.4.9. *(i) Let $\{A_i : i \in J\}$ are IFSMs of $M_i, i \in J$ and $\mathfrak{A} = \{f_i : M_i \to N : i \in J\}$ be a family of R-homomorphisms and R-homomorphism $g : N \to K$ then*

$$(\mu, \nu)^{\mathfrak{A}_1} = t(g)(\mu, \nu)^{\mathfrak{A}}, \textit{ where } \mathfrak{A}_1 = \{gof_i : M_i \to K : i \in J\}.$$

(ii) Let $\{B_i : i \in J\}$ are IFSMs of $N_i, \forall i \in J$ and $\mathfrak{B} = \{g_i : M \to N_i : i \in J\}$ be a family of R-homomorphisms and $h : K \to M$ an R-homomorphism then

$$(\mu, \nu)_{\mathfrak{B}_1} = s(h)(\mu, \nu)_{\mathfrak{B}}, \textit{ where } \mathfrak{B}_1 = \{g_i oh : K \to N_i : i \in J\}.$$

Proof.

(i) Let $\mathfrak{A}_1 = \{g_i = gof_i : N_i \to K : i \in J\}$ be the collection of R-homomorphisms. Then, by Lemma 2.4.8(i), there exists IFSM $C = (\mu_C, \nu_C)$ of K such that $g_i : A_i \to C$ is IF R-homomorphism, $\forall i \in J$, where $(\mu_C, \nu_C) = (\mu, \nu)^{\mathfrak{A}_1} = (\mu^{\mathfrak{A}_1}, \nu^{\mathfrak{A}_1})$, here $\mu^{\mathfrak{A}_1} = \vee\{\mu_{g_i(A_i)} : i \in J\}$ and $\nu^{\mathfrak{A}_1} = \wedge\{\nu_{g_i(A_i)} : i \in J\}$. Consider

$$\begin{aligned}
(\mu,\nu)^{\mathfrak{A}_1} &= \vee\{(\mu_{g_i(A_i)},\nu_{g_i(A_i)}) : i \in J\} \\
&= \vee\{(\mu_{(gof_i)(A_i)},\nu_{(gof_i)(A_i)}) : i \in J\} \\
&= \vee\{(\mu_{(g(f_i(A_i)))},\nu_{(g(f_i(A_i)))}) : i \in J\} \\
&= \vee\{t(g)(\mu_{f_i(A_i)},\nu_{f_i(A_i)}) : i \in J\} \\
&= t(g) \vee \{(\mu_{f_i(A_i)},\nu_{f_i(A_i)}) : i \in J\} \\
&= t(g)(\mu,\nu)^{\mathfrak{A}}.
\end{aligned}$$

(ii) Let $\mathfrak{B}_1 = \{h_i = g_i o h : K \to N_i : i \in J\}$ be the collection of R-homomorphisms. Then by Lemma 2.4.8(ii), there exists IFSM $A = (\mu_A, \nu_A)$ of K such that $h_i : A \to C_i$ is IF R-homomorphism, $\forall i \in J$, where $(\mu_A,\nu_A) = (\mu,\nu)_{\mathfrak{B}_1} = (\mu_{\mathfrak{B}_1},\nu_{\mathfrak{B}_1})$, here $\mu_{\mathfrak{B}_1} = \wedge\{\mu_{h_i^{-1}(C_i)} : i \in J\}$ and $\nu_{\mathfrak{B}_1} = \vee\{\nu_{h_i^{-1}(C_i)} : i \in J\}$. Now, we have

$$\begin{aligned}
(\mu,\nu)_{\mathfrak{B}_1} &= \wedge\{(\mu_{h_i^{-1}(C_i)},\nu_{h_i^{-1}(C_i)}) : i \in J\} \\
&= \wedge\{(\mu_{(g_i oh)^{-1}(C_i)},\nu_{(g_i oh)^{-1}(C_i)}) : i \in J\} \\
&= \wedge\{(\mu_{(h^{-1}og_i^{-1})(C_i)},\nu_{(h^{-1}og_i^{-1})(C_i)}) : i \in J\} \\
&= \wedge\{(\mu_{h^{-1}(g_i^{-1}(C_i))},\nu_{h^{-1}(g_i^{-1}(C_i))}) : i \in J\} \\
&= \wedge\{s(h)(\mu_{g_i^{-1}(C_i)},\nu_{g_i^{-1}(C_i)}) : i \in J\} \\
&= s(h) \wedge \{(\mu_{g_i^{-1}(C_i)},\nu_{g_i^{-1}(C_i)}) : i \in J\} \\
&= s(h)(\mu,\nu)_{\mathfrak{B}}.
\end{aligned}$$

Thus, $(\mu, \nu)_{\mathfrak{B}_1} = s(h)(\mu, \nu)_{\mathfrak{B}}$. □

Remark 2.4.10. From Lemma 2.4.8 and Lemma 2.4.9, we are able to optimally intuitionistically fuzzify f_i [g_i], in respect to the family of IFSMs $\{A_i : i \in J\}$ [$\{B_i : i \in J\}$].

Theorem 2.4.11. *The category of IF modules* $\boldsymbol{C_{R\text{-}IFM}}$ *has kernels and cokernels.*

Proof. "Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IFSM of R-modules M and N, respectively. Let $\bar{f} : A \to B$ be an IF R-homomorphism corresponding to the R-homomorphism $f : M \to N$." For ***Ker f***, there exists an inclusion map $g : \ker f \to M$ in order for the subsequent diagram commutes

$$\begin{array}{ccc} Kerf & \xrightarrow{\;g\;} & M \\ & \searrow^{fog=0} & \downarrow f \\ & & N \end{array}$$

$$\begin{array}{ccccc} Kerf & \xrightarrow{\qquad g \qquad} & M & \xrightarrow{\qquad f \qquad} & N \\ & \searrow^{(\mu_{g^{-1}(A)}, \nu_{g^{-1}(A)})} & \downarrow (\mu_A, \nu_A) & \swarrow_{(\mu_B, \nu_B)} & \\ & & I \times I & & \end{array}$$

For ***Ker*** $\bar{f}$, there exists an inclusion map $\bar{g} : g^{-1}(A) \to A$ in order for the subsequent diagram commutes

$$\begin{array}{ccc} g^{-1}(A) & \xrightarrow{\;\bar{g}\;} & A \\ & \searrow^{\bar{f}o\bar{g}=\bar{0}} & \downarrow \bar{f} \\ & & B \end{array}$$

Therefore, the kernel of $\bar{f}$ is defined as $g^{-1}(A)$ with the inclusion map $\bar{g} : g^{-1}(A) \to A$.

Thus, the kernel of $\bar{f}$ is given as $((\ker f, g^{-1}(A)), \bar{g})$, where the inclusion map is $g : \ker f \to$

M.

Similarly, the cokernel of $\bar{f}$ is defined as $((N/Imf, \pi(B)), \bar{\pi})$, where the projection map $\pi : N \to N/Imf$ and $\bar{\pi} : B \to B_{N/Imf}$. □

Remark 2.4.12. Although the category of IF modules $\mathbf{C_{R\text{-}IFM}}$ has kernels and cokernels even then it is not an abelian category. By definition of the abelian category, every monomorphism should be normal, i.e, every monomorphism is a kernel of some morphism. An IF R-homomorphism $\bar{h} : C \to A$ of IFSM C of M on being normal (i.e., being a kernel) C should be identical to $g^{-1}(A)$. Consequently, for $M \neq \{\theta\}$, the IF R-homomorphism $\bar{1} : \chi_{\{\theta\}} \to \chi_M$ is a sub-object of χ_M, which is not a kernel. Thus, $\mathbf{C_{R\text{-}IFM}}$ is not an abelian category.

Theorem 2.4.13. *$\mathbf{C_{R\text{-}IFM}}$ has zero object.*

Proof. Define the R-module Z comprising only the identity element θ and an IFS $A_0 : Z \to I \times I$ as

$$\mu_{A_0}(\theta) = 1 \text{ and } \nu_{A_0}(\theta) = 0$$

Since there is exactly one IF R-homomorphism $\phi_0 : A_0 \to A$ satisfying $\phi_0(\theta) = \theta$, $\mu_{A_0}(\theta) = \mu_A(\phi_0(\theta))$ and $\nu_{A_0}(\theta) = \nu_A(\phi_0(\theta))$, $\mathbf{C_{R\text{-}IFM}}$ has initial object A_0. Also, $\mathbf{C_{R\text{-}IFM}}$ has terminal object A_0 as there is exactly one IF R-homomorphism $\bar{\psi_0} : A \to A_0$ such that $\psi_0(x) = \theta$, $\mu_{A_0}(\psi_0(x)) = \mu_A(x) = 1$ and $\nu_{A_0}(\psi_0(x)) = \nu_A(x) = 0$. Thus, A_0 is both initial as well as terminal object. Hence, A_0 is zero object in the category $\mathbf{C_{R\text{-}IFM}}$. □

2.5 Exploring distinct categories of IFMs

Within this section, we exploring distinct categories of intuitionistic fuzzy modules, aiming to elucidate its unique properties and theoretical implications. By delving into its distinct characteristics, we deepen our understanding of IFM theory. We also discuss the relationship between these categories.

Theorem 2.5.1. *The collection of IFSMs together with IF R-homomorphisms and their composition form a category. It is denoted by* $\boldsymbol{C_{R\text{-}IFM}}$.

Proof. The proof is a consequence of definition 2.2.3. □

Definition 2.5.2. [42] Let M and N be two R-modules and let A, B be two IFSMs of M and N respectively. Let $f : M \to N$ be a R-homomorphism. Then $\bar{f}$ is called a weak intuitionistic fuzzy R-homomorphism(WIF R-homomorphism) of A onto B if $f(A) \subseteq B$. The R-homomorphism f is called an intuitionistic fuzzy R-homomorphism(IF R-homomorphism) of A onto B if $f(A) = B$. We say that A is an intuitionistic fuzzy homomorphic to B and we write it as $A \approx B$. If $f : M \to N$ be an R-isomorphism, then f is called a weak intuitionistic fuzzy R-isomorphism(WIF R-isomorphism) from A onto B if $f(A) \subseteq B$ and f is called an intuitionistic fuzzy R-isomorphism(IF R-homomorphism) if $f(A) = B$ and we write it as $A \cong B$.

Remark 2.5.3. Every IF R-homomorphism is a WIF R-homomorphism; but converse does not hold

Example 2.5.4. Let $M = (Z_{18}, +_{18})$ be Z-module and $N = (< 3 >, +_{18})$ be submodule of

M. Define IFSs A and B on M and N respectively as:

$$\mu_A(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0.8, & \text{if } x = 3, 15 \\ 0.6, & \text{if } x = 6, 12 \\ 0.5, & \text{if } x = 9 \\ 0.3, & \text{if } x = 2, 4, 8, 10, 14, 16 \\ 0, & \text{if } x = 1, 5, 7, 11, 13, 17 \end{cases} ; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x = 0 \\ 0.1, & \text{if } x = 3, 15 \\ 0.2, & \text{if } x = 6, 12 \\ 0.4, & \text{if } x = 9 \\ 0.6, & \text{if } x = 2, 4, 8, 10, 14, 16 \\ 0.8, & \text{if } x = 1, 5, 7, 11, 13, 17. \end{cases}$$

$$\mu_B(y) = \begin{cases} 1, & \text{if } y = 0 \\ 0.3, & \text{if } y = 3, 15 \\ 0.6, & \text{if } y = 6, 12 \\ 0.4, & \text{if } y = 9 \end{cases} ; \quad \nu_B(y) = \begin{cases} 0, & \text{if } y = 0 \\ 0.6, & \text{if } y = 3, 15 \\ 0.3, & \text{if } y = 6, 12 \\ 0.5, & \text{if } y = 9 \end{cases}$$

$$\phi(y) = \begin{cases} 0, & \text{if } x = 0 \\ 3, & \text{if } x = 3 \\ 6, & \text{if } x = 6 \\ 9, & \text{if } x = 9 \\ 12, & \text{if } x = 12 \\ 15, & \text{if } x = 15. \end{cases}$$

We can verify that A and B are IFSMs of M and N respectively and the mapping $\phi : N \to M$ defined above is an R-homomorphism. We can verify that this defines a WIF R-homomorphism from B into A, which is not an IF R-homomorphism.

Proposition 2.5.5. *Composite of two WIF R-homomorphisms is also WIF R-homomorphism.*

Proof. Let $A \in IFSM(M), B \in IFSM(N), andC \in IFSM(K)$ and let $\bar{f} : A \to B$ and $\bar{\sigma} : B \to C$ be WIF R-homomorphisms respectively. Therefore, $f(A) \subseteq B$ and $\sigma(B) \subseteq C$. We seek to establish that $(\bar{\sigma} \circ \bar{f})(A) \subseteq C$.

Let $k \in K$. Then $\mu_{\sigma(B)}(k) \leq \mu_C(k)$ and $\nu_{\sigma(B)}(k) \geq \nu_C(k)$. Also, corresponding to this $k \in K$, we have $\sigma^{-1}(k) \in N$ which implies $\mu_{f(A)}(\sigma^{-1}(k)) \leq \mu_B(\sigma^{-1}(k))$ and $\nu_{f(A)}(\sigma^{-1}(k)) \geq \nu_B(\sigma^{-1}(k))$. Thus, $\mu_{\sigma(f(A))}(k) \leq \mu_{\sigma(B)}(k)$ and $\nu_{\sigma(f(A))}(k) \geq \nu_{\sigma(B)}(k)$. From this, we conclude that $\mu_{\sigma f(A)}(k) \leq \mu_{\sigma(B)}(k) \leq \mu_C(k)$ and $\nu_{\sigma f(A)}(k) \geq \nu_{\sigma(B)}(k) \geq \nu_C(k)$. Thus, $\sigma \circ f(A) \subseteq C$, concluding that $\bar{\sigma} \circ \bar{f}$ is WIF R-homomorphism from A onto C. □

Theorem 2.5.6. *The collection of intuitionistic fuzzy modules together with weak intuitionistic fuzzy R-homomorphisms and their composition form a category.*

Proof. **Associativity**: Let $A \in IFSM(M), B \in IFSM(N), C \in IFSM(K)$ and $D \in IFSM(P)$. Also, let $\bar{f} : A \to B, \bar{\sigma} : B \to C$ and $\bar{h} : C \to D$ be WIF R-homomorphisms corresponding to the R-homomorphisms $f : M \to N, \sigma : N \to K$ and $h : K \to P$ respectively. Then $f \circ (\sigma \circ h) = (f \circ \sigma) \circ h$ as R-homomorphisms from M into P. For $z \in M$

$$\mu_{(f\circ(\sigma\circ h))A}(z) = \mu_{f\circ(\sigma(h(A)))}(z) = \mu_{\sigma(h(A))}(f^{-1}(z)) = \mu_{h(A)}(\sigma^{-1}(f^{-1}(z)))$$

$$= \mu_{h(A)}((f \circ \sigma)^{-1}(z)) = \mu_A(h^{-1}((f \circ \sigma)^{-1}(z))) = \mu_A(((f \circ \sigma) \circ h)^{-1}(z)) = \mu_{((f\circ\sigma)\circ h)A}(z).$$

Likewise, we are able to exhibit that $\nu_{(f\circ(\sigma\circ h))A}(z) = \nu_{((f\circ\sigma)\circ h)A}(z)$. Hence, $\bar{f} \circ (\bar{\sigma} \circ \bar{h})$ and $(\bar{f} \circ \bar{\sigma}) \circ \bar{h}$ are equal as WIF R-homomorphisms from A into D.

Identity: Let $A \in IFSM(M)$ and $B \in IFSM(N)$ be IFSMs and $I_M : M \to M$ be the identity R-isomorphism on M. Then, $I_A : A \to A$ is the corresponding IF R-isomorphism. For any $z \in M$, $\mu_{I_A(A)}(z) = \mu_A(I_A^{-1}(z)) = \mu_A(z)$.

Likewise, we are able to exhibit that $\nu_{I_A(A)}(z) = \nu_A(z)$. Further, let $\bar{f} : A \to B, \bar{\sigma} : B \to C$ be WIF R-homomorphism corresponding the R-homomorphism $f : M \to N, \sigma : N \to K$ respectively. Then $\mu_{(f \circ I_A)A}(z) = \mu_A((f \circ I_A)^{-1}(z)) = \mu_A((I_A)^{-1} \circ (f^{-1}(z))) = \mu_A(f^{-1}(z)) = \mu_{f(A)}(z)$. Likewise, we are able to exhibit that $\nu_{(f \circ I_A)(A)}(z) = \nu_{f(A)}(z)$. Therefore, we have $\bar{f} \circ I_A = \bar{f}$. Likewise, we are able to exhibit that $I_A \circ \bar{\sigma} = \bar{\sigma}$. Hence, I_A is the identity IF R-isomorphism of A. □

Remark 2.5.7. We shall denote the above category by $\mathbf{C_{WR\text{-}IFM}}$.

Theorem 2.5.8. *The collection of intuitionistic fuzzy modules together with weak intuitionistic fuzzy R-isomorphisms and their composition form a category.*

Remark 2.5.9. We shall denote the above category by $\mathbf{C_{WRI\text{-}IFM}}$.

Theorem 2.5.10. *The collection of intuitionistic fuzzy modules together with intuitionistic fuzzy R-isomorphisms and their composition form a category.*

Remark 2.5.11. We shall denote the above category by $\mathbf{C_{RI\text{-}IFM}}$.

We have formed four categories of intuitionistic fuzzy modules, viz., $\mathbf{C_{WR\text{-}IFM}}$, $\mathbf{C_{R\text{-}IFM}}$, $\mathbf{C_{WRI\text{-}IFM}}$ and $\mathbf{C_{RI\text{-}IFM}}$. Of these, $\mathbf{C_{RI\text{-}IFM}}$ is a subcategory of both $\mathbf{C_{WRI\text{-}IFM}}$ and $\mathbf{C_{R\text{-}IFM}}$. Both $\mathbf{C_{WRI\text{-}IFM}}$ and $\mathbf{C_{R\text{-}IFM}}$ are subcategories of $\mathbf{C_{WR\text{-}IFM}}$. None is a full subcategory. $\mathbf{C_{WRI\text{-}IFM}}$ is not a subcategory of $\mathbf{C_{R\text{-}IFM}}$, and $\mathbf{C_{R\text{-}IFM}}$ is not a subcategory of $\mathbf{C_{WRI\text{-}IFM}}$.

Chapter 3

Some special morphisms in the category $\mathbf{C_{R\text{-}IFM}}$

Within the category $\mathbf{C_{R\text{-}IFM}}$, the study of special morphisms emerges as a focal point, providing a nuanced understanding of the relationships between these mathematical structures. Special morphisms play a pivotal role in capturing the unique characteristics and transformations within intuitionistic fuzzy modules, offering insights into their behavior and interactions. This Chapter investigates various types of special morphisms, including coretractions, retractions, monomorphisms, epimorphisms, and isomorphisms, within the context of intuitionistic fuzzy modules. Through a systematic exploration of their properties and significance, the research aims to shed light on the categorical structure of intuitionistic fuzzy modules and enhance our ability to discern and characterize their distinctive features.

3.1 Introduction

Agnes [2] explored the special morphisms in the category of fuzzy sets and delineated the necessary conditions for a morphism to be deemed a coretraction or retraction. In this Chapter, we extend the notion of intuitionistic fuzzy modules and intuitionistic fuzzy R- homomorphism to intuitionistic fuzzy coretracts (retracts) and intuitionistic fuzzy coretraction (retraction), and various properties are being investigated.

This Chapter turns its attention to the investigation of specific special types of morphisms. In this Chapter, we

1. introduce two special type of morphisms, namely Retraction and Coretraction in the category ($\mathbf{C_{R\text{-}IFM}}$) of intuitionistic fuzzy modules.

2. obtain the condition under which an intuitionistic fuzzy R-homomorphism in $\mathbf{C_{R\text{-}IFM}}$ to be a retraction or a coretraction.

3. acquire some equivalent statements for these two morphisms.

4. study free, projective and injective objects in $\mathbf{C_{R\text{-}IFM}}$ and establish their relation with retraction and coretraction in $\mathbf{C_{R\text{-}IFM}}$.

3.2 Some special morphisms

In this section, we study and define some special morphisms like coretraction, retraction, monomorphism, epimorphism, isomorphism etc. in the category $\mathbf{C_{R\text{-}IFM}}$.

Definition 3.2.1. An IF R-homomorphism $\bar{f} : A \to B$ is said to be an intuitionistic fuzzy coretraction(IF-coretraction) if there exists an IF R-homomorphism $\bar{\sigma} : B \to A$ such that

$$\bar{\sigma} \circ \bar{f} = I_A.$$

In other words, $\bar{f}$ is an IF-coretraction, if it is left invertible. In this case , the IF R-homomorphism $\bar{\sigma}$ is called a left inverse of $\bar{f}$.

Lemma 3.2.2. *Composite of two IF-coretractions is also an IF-coretraction in* $\mathbf{C_{R\text{-}IFM}}$.

Proof. Let $\bar{f} : A \to B$ and $\bar{\sigma} : B \to C$ be two IF-coretractions in $\mathbf{C_{R\text{-}IFM}}$. So, IF R-homomorphisms $\bar{u} : B \to A$ and $\bar{v} : C \to B$ exists such that

$$\bar{u} \circ \bar{f} = I_A \text{ and } \bar{v} \circ \bar{\sigma} = I_B.$$

$$\begin{aligned} Now, (\bar{u} \circ \bar{v}) \circ (\bar{\sigma} \circ \bar{f}) &= \bar{u} \circ (\bar{v} \circ \bar{\sigma}) \circ \bar{f} \text{ [Using associativity of composition]} \\ &= \bar{u} \circ I_B \circ \bar{f} \\ &= \bar{u} \circ \bar{f} \\ &= I_A. \end{aligned}$$

Thus $\bar{u} \circ \bar{v} : A \to C$ is left inverse of $\bar{\sigma} \circ \bar{f}$. Hence, $\bar{\sigma} \circ \bar{f}$ is an IF-coretraction in $\mathbf{C_{R\text{-}IFM}}$. □

Proposition 3.2.3. *Let A and B are IFSMs of R-modules M and N respectively and $f : M \to N$ be a R-homomorphism. If an IF R-homomorphism $\bar{f} : A \to B$ is an IF-coretraction in* $\mathbf{C_{R\text{-}IFM}}$*, then both f and $\bar{f}$ are one-one functions.*

Proof. Since $\bar{f} : A \to B$ is an IF-coretraction in $\mathbf{C_{R\text{-}IFM}}$. Therefore, an IF R-homomorphism $\bar{\sigma} : B \to A$ exists such that $\bar{\sigma} \circ \bar{f} = I_A$. By lemma 1.5.10 , both f and $\bar{f}$ are one-one functions. □

The converse of Proposition 3.2.3 does not hold. Refer to the example provided below for elucidation:

Example 3.2.4. Assume $M = \mathbb{Z}_2$ and $N = \mathbb{Z}_4$. Clearly, M, N are Z-modules. Consider $A = \chi_M$ and $B = \chi_N$. Then A and B are IFSMs of Z-modules M and N respectively. Define the mapping $f : M \to N$ by $f(0) = 0, f(1) = 2$. Clearly, f is one one Z-homomorphism. Also, $\mu_B(f(z)) \geq \mu_A(z)$ and $\nu_B(f(z)) \leq \nu_A(z), \forall z \in M$. Note that $\bar{f}$ is one one IF Z-homomorphism. However, there exists no IF Z-homomorphism $\bar{\sigma} : B \to A$ such that $\bar{\sigma} \circ \bar{f} = I_A$. That is, $\bar{f} : A \to B$ is not an IF-coretraction.

Definition 3.2.5. An IF R-homomorphism $\bar{f} : A \to B$ is said to be an intuitionistic fuzzy retraction (IF-retraction), if an IF R-homomorphism $\bar{\sigma} : B \to A$ exists that satisfies

$$\bar{f} \circ \bar{\sigma} = I_B.$$

An IFSM B is said to be retract of an IFSM A. In other words, an IF R-homomorphism $\bar{f}$ is an intuitionistic fuzzy retraction if it is right invertible. An IF R-homomorphism $\bar{\sigma}$ in the above definition is called a right inverse of $\bar{f}$.

Lemma 3.2.6. *Composite of two intuitionistic fuzzy retractions is also an intuitionistic fuzzy retraction in* $\boldsymbol{C_{R\text{-}IFM}}$.

Proof. Similar to the proof of Lemma 3.2.2, this can also be demonstrated. □

Proposition 3.2.7. *Let A and B are IFSM of R-modules M and N respectively and $f : M \to$*

N is R-homomorphism. If an IF R-homomorphism $\bar{f} : A \to B$ is an IF-retraction in $\mathbf{C_{R\text{-}IFM}}$, then both f and $\bar{f}$ are onto functions.

Proof. It can be easily prove by using Lemma 1.5.10. □

The converse of Proposition 3.2.7 does not hold. Refer to the example provided below for elucidation:

Example 3.2.8. Assume $M = \mathbb{Z}_2$ and $N = \mathbb{Z}_2$. Clearly, M, N are Z-modules. Define IFS A and B on M and N respectively as

$$\mu_A(z) = \begin{cases} 1, & \text{if } z = 0 \\ 0.5, & \text{if } z = 1 \end{cases}, \quad \nu_A(z) = \begin{cases} 0, & \text{if } z = 0 \\ 0.4, & \text{if } z = 1 \end{cases},$$

and $\mu_B(t) = 1, \nu_B(t) = 0, \forall t \in N$. Clearly, A and B are IFSM of M and N respectively. Define $f : M \to N$ as $f(0) = 0, f(1) = 1$. Clearly, f is an onto Z-homomorphism. Also, $\mu_B(f(z)) \geq \mu_A(z)$ and $\nu_B(f(z)) \leq \nu_A(z), \forall z \in M$. Note that $\bar{f}$ is onto IF Z-homomorphism. However there exists no IF Z-homomorphism $\bar{\sigma} : B \to A$ such that $\bar{f} \circ \bar{\sigma} = I_B$. Thus, $\bar{f} : A \to B$ is not an IF-retraction.

Definition 3.2.9. An IF-homomorphism $\bar{f} \in Hom_{C_{R-IFM}}(A, B)$ is said to be an intuitionistic fuzzy monomorphism (IF-monomorphism) if $\bar{f} \circ \bar{\sigma} = \bar{f} \circ \bar{h}$ implies that $\bar{\sigma} = \bar{h}$ for all $\bar{\sigma}, \bar{h} \in Hom_{\mathbf{C_{R\text{-}IFM}}}(C, A)$; i.e. left cancellation holds in $\mathbf{C_{R\text{-}IFM}}$.

Definition 3.2.10. An IF-homomorphism $\bar{f} \in Hom_{\mathbf{C_{R\text{-}IFM}}}(A, B)$ is said to be an intuitionistic fuzzy epimorphism (IF-epimorphism) if $\bar{\sigma} \circ \bar{f} = \bar{h} \circ \bar{f}$ implies that $\bar{\sigma} = \bar{h}$ for all $\bar{\sigma}, \bar{h} \in Hom_{\mathbf{C_{R\text{-}IFM}}}(B, C)$; i.e. right cancellation holds in $\mathbf{C_{R\text{-}IFM}}$.

Lemma 3.2.11. *Composite of two IF-monomorphisms is also an IF-monomorphism in* $\mathbf{C_{R\text{-}IFM}}$.

Lemma 3.2.12. *Composite of two IF-epimorphisms is also an IF-epimorphism in* $\mathbf{C_{R\text{-}IFM}}$.

Remark 3.2.13. (i) Since every IF-coretraction has a left inverse, it implies that left cancellation holds in $\mathbf{C_{R\text{-}IFM}}$. Then, it follows that every IF-coretraction is an IF-monomorphism.

(ii) Since every IF-retraction has a right inverse, it implies that right cancellation holds in $\mathbf{C_{R\text{-}IFM}}$. Then, it follows that every IF-retraction is an IF-epimorphism.

(iii) An IF R-homomorphism is an IF R-isomorphism if and only if it is both IF-coretraction and IF-retraction.

Lemma 3.2.14. *In* $\mathbf{C_{R\text{-}IFM}}$,

(i) underlying maps of epimorphisms are surjective, and

(ii) underlying maps of monomorphisms are injective.

Proof. (i) Suppose $\bar{f} : A \to B$ be IF-epimorphism in $\mathbf{C_{R\text{-}IFM}}$, and let $g, h : N \to K$ in $\mathbf{C_{R\text{-}M}}$ such that $g \circ f = h \circ f$. The IF R-homomorphisms $\bar{g}, \bar{h} : B \to \bar{1}_K$ derived by straightforwardly applying intuitionistic fuzzification to g and h with respect to B, following that, $\bar{g} \circ \bar{f} = \bar{h} \circ \bar{f}$. Consequently, $\bar{g} = \bar{h}$, leading to $g = h$. Hence, f is an epimorphism in $\mathbf{C_{R\text{-}M}}$. Since epimorphisms are surjective in abelian categories and $\mathbf{C_{R\text{-}M}}$ is an abelian category, f is surjective.

(ii) The proof is similar. □

Let's recall the definition of balanced category

Definition 3.2.15. "A category C is said to be balanced if every morphism is an isomorphism."

Theorem 3.2.16. *The category* $\mathbf{C_{R\text{-}IFM}}$ *is not balanced category.*

Proof. In example 3.2.8, we prove that $\bar{f}$ is not an IF-retraction even though $\bar{f}$ is an IF R-homomorphism. Thus, every IF R-homomorphism is not an IF R-isomorphism. Hence, the category $\mathbf{C_{R\text{-}IFM}}$ is not balanced category. □

3.3 Exploring special morphisms in the context of intuitionistic fuzzy projective and injective modules

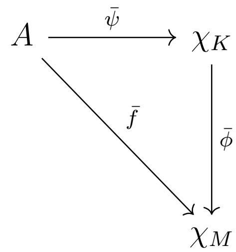
We will study free, projective and injective objects in $\mathbf{C_{R\text{-}IFM}}$ and establish their relation with morphism in $\mathbf{C_{R\text{-}IFM}}$ and retraction (coretraction) in this section.

Theorem 3.3.1. *$A \in Ob(\boldsymbol{C_{R\text{-}IFM}})$ is IF-projective if and only if, $M \in Ob(\boldsymbol{C_{R\text{-}M}})$ is projective and $A = \bar{0}_M$.*

Proof. Firstly, let A be a projective object in $\mathbf{C_{R\text{-}IFM}}$. Let N and K be two R-modules and $f : M \to N$ be R-homomorphism and $\phi : K \to N$ be epimorphism.

Take $B = \chi_N$ and $C = \chi_K$ such that B, C becomes IFSMs of R-modules N and K respectively. Thus, $\bar{f} : A \to B$ becomes an IF R-homomorphism and $\bar{\phi} : C \to B$ becomes an IF-epimorphism obtained by trivially intuitionistic fuzzifying f and ϕ in $\mathbf{C_{R\text{-}IFM}}$. As A is a projective object in $\mathbf{C_{R\text{-}IFM}}$, an IF R-homomorphism $\bar{\psi} : A \to C$ exists that satisfies $\bar{\phi} \circ \bar{\psi} = \bar{f}$. This implies the existence of an R-homomorphism $\psi : M \to K$ that satisfies $\phi \circ \psi = f$. Hence, M is projective object in $\mathbf{C_{R\text{-}M}}$.

If $A \neq \bar{0}_M$, then there exist no IF R-homomorphism $\bar{\psi} : A \to \chi_K$ i.e., A will no longer be a projective object in $\mathbf{C_{R\text{-}IFM}}$ as the following diagram fails to commute



□

Theorem 3.3.2. *Retraction of projective objects in $\mathbf{C_{R\text{-}IFM}}$ are projective.*

Proof. Suppose that A be a projective object in $\mathbf{C_{R\text{-}IFM}}$ and let B be retract of A.

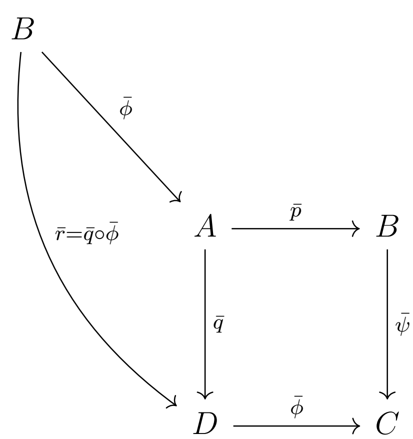
For an IF R-homomorphism $\bar{f} : A \to B$, an another IF R-homomorphism $\bar{\phi} : B \to A$ exists so that $\bar{f} \circ \bar{\phi} = 1_B$. We claim that B is an IF-projective in $\mathbf{C_{R\text{-}IFM}}$.

Consider an arbitrary IF R-homomorphism $\bar{\psi} : B \to C$ and an IF-epimorphism $\bar{p} : D \to C$ in $\mathbf{C_{R\text{-}IFM}}$.

Now, $\mu_C \circ (\bar{\psi} \circ \bar{f}) = (\mu_C \circ \bar{\psi}) \circ \bar{f} = \mu_B \circ \bar{f} = \mu_A$. Similarly, we can prove $\nu_C \circ (\bar{\psi} \circ \bar{f}) = \nu_A$.

As a result, $\bar{\psi} \circ \bar{f} \in \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A, C)$ becomes an IF R-homomorphism. As A is an IF-projective, an IF R-homomorphism $\bar{q} : A \to D$ exists that satisfies $\bar{p} \circ \bar{q} = \bar{\psi} \circ \bar{f}$.

Then, $(\bar{p} \circ \bar{q}) \circ \bar{\phi} = (\bar{\psi} \circ \bar{f}) \circ \bar{\phi} = \bar{\psi} \circ (\bar{f} \circ \bar{\phi}) = \bar{\psi} \circ I_B = \bar{\psi}$. Consequently, $\bar{p} \circ \bar{r} = \bar{\psi}, where\ \bar{r} = \bar{q} \circ \bar{\phi}$.



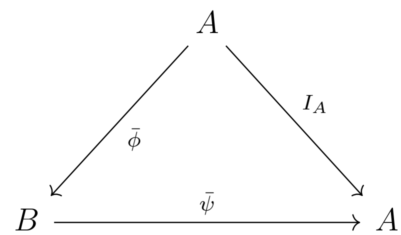
Consider $\mu_D \circ \bar{r} = \mu_D \circ (\bar{q} \circ \bar{\phi}) = (\mu_D \circ \bar{q}) \circ \bar{\phi} = \mu_A \circ \bar{\phi} = \mu_B$.

Thus, $\bar{r} : B \to D$ becomes an IF R-homomorphism satisfying $\bar{p} \circ \bar{r} = \bar{\psi}$.

Hence, B is an IF-projective in $\mathbf{C_{R\text{-}IFM}}$. □

Theorem 3.3.3. *If A is a projective object in $\mathbf{C_{R\text{-}IFM}}$, then every IF-epimorphism $\bar{\psi} : B \to A$ is an intuitionistic fuzzy retraction, where $B \in \mathbf{C_{R\text{-}IFM}}$.*

Proof. Since A is a projective object in $\mathbf{C_{R\text{-}IFM}}$ and $\bar{\psi} : B \to A$ be an IF-epimorphism, it follows that the depicted diagram commutes



Consequently, an IF R-homomorphism $\bar{\phi} : A \to B$ exists that satisfies

$$\bar{\psi} \circ \bar{\phi} = I_A$$

Hence, $\bar{\psi}$ is an IF-retraction. □

Theorem 3.3.4. *$A \in Ob(\mathbf{C_{R\text{-}IFM}})$ is free if and only if, $M \in Ob(\mathbf{C_{R\text{-}M}})$ is free and $A = \bar{0}_M$.*

Proof. Let A be a free object in $\mathbf{C_{R\text{-}IFM}}$. Then A is free IFSM of an R-module M. As every IF free submodule of a module is IF projective in $\mathbf{C_{R\text{-}IFM}}$, A is an IF-projective. Further, according to Theorem [3.3.1], M is a projective module and $A = \bar{0}_M$. So, it suffices to demonstrate that M is a free R-module.

For $P \neq \emptyset$, let $i : P \to M$ be a R-homomorphism. For any R-module N, let $f : P \to N$ be a R-homomorphism. Let $B = \bar{0}_N$ be an IFSM of N and $D = \bar{0}_P$ be an IFSM of P such that D

is a basis of an IFSM A. So, $\bar{i} : D \to A$ and $\bar{f} : D \to B$ are IF R-homomorphisms obtained by trivially intuitionistic fuzzifying i and f.

As A is a free IFSM, then $\bar{\phi} \circ \bar{i} = \bar{f}$ for a unique IF R-homomorphism $\bar{\phi} : A \to B$.

$$\begin{array}{ccc} D & \xrightarrow{\ \bar{i}\ } & A \\ & \searrow^{\bar{f}} & \downarrow{\scriptstyle \bar{\phi}} \\ & & B \end{array}$$

Consequently, $\phi \circ i = f$ for each R-homomorphism $\phi : M \to N$.

Thus, M is a free R-module. □

Theorem 3.3.5. *$(\mu_0, \nu_0)_M$ is an IF-projective if and only if, $(\mu_0, \nu_0)_M$ is a direct summand of a free object in* $\boldsymbol{C_{R\text{-}IFM}}$.

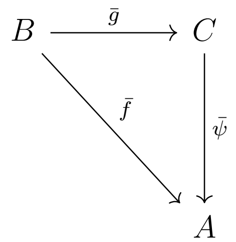
Proof. Firstly, let $(\mu_0, \nu_0)_M$ be an IF-projective object in $\mathbf{C_{R\text{-}IFM}}$. Then, according to Theorem [3.3.1], M is a projective module in $\mathbf{C_{R\text{-}M}}$. Since projective object is a direct summand of free module in $\mathbf{C_{R\text{-}M}}$, a free R-module F and an R-module K exists satisfying $F = K \oplus M$. Then $(\mu_0, \nu_0)_F = (\mu_0, \nu_0)_K \oplus (\mu_0, \nu_0)_M$.

Conversely, if $(\mu_0, \nu_0)_F = (\mu', \nu')_K \oplus (\mu'', \nu'')_M$ with the inclusion maps $i_K : K \to F$ and $i_M : M \to F$, then $\mu_0(i_K(z)) \geq \mu'(z)$, $\nu_0(i_K(z)) \leq \nu'(z)$ hence $\mu' = \mu_0$ and $\nu' = \nu_0$ and similarly, we can have $\mu'' = \mu_0$ and $\nu'' = \nu_0$. Thus, $(\mu_0, \nu_0)_M$ is an IF-projective. □

Theorem 3.3.6. *$A \in Ob(\boldsymbol{C_{R\text{-}IFM}})$ is an IF-injective if and only, if $M \in Ob(\boldsymbol{C_{R\text{-}M}})$ is an injective and $A = \bar{1}$.*

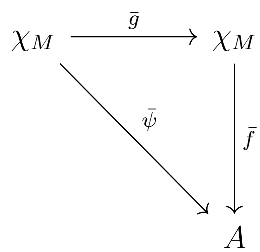
Proof. Let A be an injective object in $\mathbf{C_{R\text{-}IFM}}$. Let N and K be two R-modules and an R-homomorphism $f : M \to N$ and a monomorphism $g : N \to K$ in $\mathbf{C_{R\text{-}M}}$. Take $B = \chi_N$

and $C = \chi_K$ such that B and C becomes IFSMs of N and K respectively, $\bar{f} : B \to A$ becomes IF R-homomorphism and $\bar{g} : B \to C$ becomes IF-monomorphism obtained by trivially intuitionistic fuzzifying f and g in $\mathbf{C_{R\text{-}IFM}}$. As A is an injective object in $\mathbf{C_{R\text{-}IFM}}$, an IF R-homomorphism $\bar{\psi} : C \to A$ exists that satisfying $\bar{\psi} \circ \bar{g} = \bar{f}$.



Thus, $\psi \circ g = f$ for a R-homomorphism $\psi : K \to M$. Hence, M is injective object.

If $A \neq \bar{1}$, then there is no existence of an IF R-homomorphism $\bar{\psi} : \chi_M \to A$ i.e., A will no longer be an injective object $\mathbf{C_{R\text{-}IFM}}$ as the following diagram fails to commute

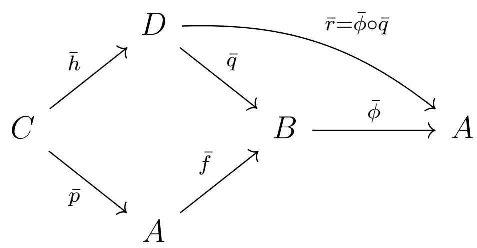


□

Theorem 3.3.7. *Let $\bar{f} : A \to B$ be a IF-coretraction. If B is an IF-injective, then so is A.*

Proof. Since $\bar{f} : A \to B$ is an IF-coretraction. Therefore, $\bar{g} \circ \bar{f} = 1_A$ for a unique IF R-homomorphism $\bar{\phi} : B \to A$. Now, we will show that A is an IF-injective.

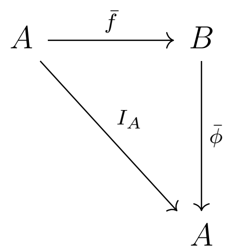
Let $\bar{h} : C \to D$ be a IF-monomorphism and $\bar{p} : C \to A$ be any IF R-homomorphism in $\mathbf{C_{R\text{-}IFM}}$. Then $\bar{f} \circ \bar{p} : C \to B$ is an IF R-homomorphism. For an IF-injective module B, an IF R-homomorphism $\bar{q} : D \to B$ exists such that $\bar{q} \circ \bar{h} = \bar{f} \circ \bar{p}$ which implies that $\bar{\phi} \circ \bar{q} \circ \bar{h} = \bar{\phi} \circ \bar{f} \circ \bar{p} = \bar{p}$. This gives us $\bar{r} \circ \bar{h} = \bar{p}$, where $\bar{r} = \bar{\phi} \circ \bar{q}$.



Hence, A is an IF-injective. $\square$

Theorem 3.3.8. *If* $A \in Ob(\mathbf{C_{R\text{-}IFM}})$ *is IF-injective, then every IF-injective* $\bar{f} : A \to B$ *is an intuitionistic fuzzy coretraction, where* $B \in Ob(\mathbf{C_{R\text{-}IFM}})$.

Proof. Since A is an injective object in $\mathbf{C_{R\text{-}IFM}}$ and $\bar{f} : A \to B$ be an IF-injective. Thus, we obtain



As the above diagram is commutative, an IF R-homomorphism $\bar{\phi} : B \to A$ exists that satisfying $\bar{\phi} \circ \bar{f} = I_A$. Hence, $\bar{f}$ is an intuitionistic fuzzy coretraction. $\square$

Chapter 4

Construction of some universal objects in $C_{\textbf{R-IFM}}$

4.1 Introduction

The concept of intuitionistic fuzzy modules has since become a focal point in category theory, offering a framework for addressing universal contractions. We will start with some basic definitions as presented in [48]. For more sources on category theory, readers can consult on of [1, 27]. Maclane introduced various universal objects such as product, equalizer, pullback and dual, namely coproduct, coequalizer, and pushout in general topology. Behera [9] introduced fuzzy equalizers, fuzzy coequalizers, fuzzy pullbacks, and fuzzy pushouts for fuzzy topological spaces. Results related to these universal objects were also studied. Rashmanlou, Hamouda, and others [17, 18, 36] respected researchers in the field of category theory made a significant contribution by introducing the concept of intuitionistic fuzzy topological spaces. This study aims to extend the foundational concepts of category theory, such as pullback, intersections, images, and inverse-images, into the domain of intuitionistic fuzzy modules. These constructions, are essential for elucidating relationships between mathematical entities, and find a natural adaptation within the context of intuitionistic fuzzy modules, encapsulates the uncertainty and vagueness inherent in many real-world phenomena. Our exploration delves into the universal construction of these categorical concepts, unveiling their significance and applicability within the framework of intuitionistic fuzzy modules. The construction of universal objects in the $\mathbf{C_{R\text{-}IFM}}$ category represents a burgeoning area of research with promising applications across diverse domains.

4.2 Equalizers and coequalizers

Theorem 4.2.1. *For a given family of IFSMs $\{A_i | i \in J\}$ of R-modules $\{M_i | i \in J\}$ respectively, the following properties hold:*

(i) An IFSM A exists on $\prod_{i\in J} M_i$ with a family of IF R-homomorphisms $\{\bar{p}_i \in \boldsymbol{Hom}_{\boldsymbol{C_{R\text{-}IFM}}}(A, A_i) | i \in J\}$;

(ii) For any IFSM B of R-module N, equipped with a family of IF R-homomorphisms $\{\bar{\phi}_i \in \boldsymbol{Hom}_{\boldsymbol{C_{R\text{-}IFM}}}(B, A_i) | i \in J\}$, then $\forall i \in J$; $\bar{p}_i \circ \bar{\theta} = \bar{\phi}_i$ for a unique IF R-homomorphism $\bar{\theta} : B \to A$.

Proof. (i) Let $p_i : M \to M_i$ be the canonical projection mapping, where $M = \prod_{i\in J} M_i$. An IFS A on M is defined as:

$$\mu_A(z) = \wedge\{\mu_{A_i}(p_i(z)) | i \in J\} \text{ and } \nu_A(z) = \vee\{\nu_{A_i}(p_i(z)) | i \in J\}, \forall z = \textstyle\prod_{i\in J} z_i \in M.$$

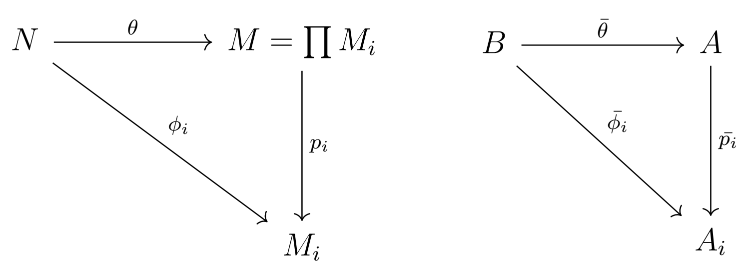
It can be verify that A is an IFSM on M. Moreover, since $\{p_i : M \to M_i | i \in J\}$ are projection mappings and $\mu_A(z) = \wedge\{\mu_{A_i}(p_i(z))|i \in J\} \leq \mu_{A_i}(p_i(z))$; $\nu_A(z) = \vee\{\nu_{A_i}(p_i(z))|i \in J\} \geq \nu_{A_i}(p_i(z)), \forall i \in J$, implies that $\bar{p_i} : A \to A_i$ is an IF R-homomorphism.

(ii) Consider the family of IF R-homomorphisms $\{\bar{\phi_i} : B \to A_i | i \in J\}$ in $\mathbf{C_{R\text{-}IFM}}$. Then, the corresponding R-homomorphisms $\{\phi_i : N \to M_i | i \in J\}$ are in the $\mathbf{C_{R\text{-}M}}$. By the universal property of product in $\mathbf{C_{R\text{-}M}}$, for each $i \in J$, $p_i \circ \theta = \phi_i$ for a unique R-homomorphism $\theta : N \to M$.

For each $y \in N$, let $\theta(y) \in M$ with $p_i(\theta(y)) = \phi_i(y)$, yielding $\bar{p_i} \circ \bar{\theta} = \bar{\phi_i}, \forall i \in J$. Now,

$$\begin{aligned} \mu_B(y) &\leq \mu_{A_i}(\phi_i(y)) \\ &= \mu_{A_i}(p_i(\theta(y))) \\ &= (\mu_{A_i} \circ p_i)(\theta(y)) \\ &= \mu_A(\theta(y)). \end{aligned}$$

i.e., $\mu_B(y) \leq \mu_A(\theta(y))$. Additionally, $\nu_B(y) \geq \nu_A(\theta(y))$, implying that $\bar{\theta} : B \to A$ is an IF R-homomorphism.



For Uniqueness, suppose $\bar{\xi} : B \to A$ is another IF R-homomorphism satisfying $\bar{p_i} \circ \bar{\xi} = \bar{\phi_i}$. So, we have $\bar{p_i} \circ \bar{\xi} = \bar{p_i} \circ \bar{\theta}$. Since each $p_i : M \to M_i, i \in J$ is a projection mapping, it follows $\bar{\xi} = \bar{\theta}$.

Hence, $\bar{\theta} : B \to A$ is a unique IF R-homomorphism with $\bar{p_i} \circ \bar{\theta} = \bar{\phi_i}, \forall i \in J$.

Therefore, the category $\mathbf{C_{R\text{-}IFM}}$ has product $\prod_{i\in J} A_i$. □

Remark 4.2.2. Above theorem highlights that the IFSM A on $M = \prod_{i\in J} M_i$ can be viewed as the direct product of the IFSMs A_i. This is denoted as $A = \prod_{i\in J} A_i$. Consequently, it implies that the category of intuitionistic fuzzy modules indeed possesses a product.

Theorem 4.2.3. *For a given family of IFSMs $\{A_i | i \in J\}$ of R-modules $\{M_i | i \in J\}$ respectively, the following properties hold:*

(i) An IFSM A exists on $\coprod_{i\in J} M_i$, along with a set of IF R-homomorphisms $\{\bar{q}_i : \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A_i, A) | i \in J\}$;

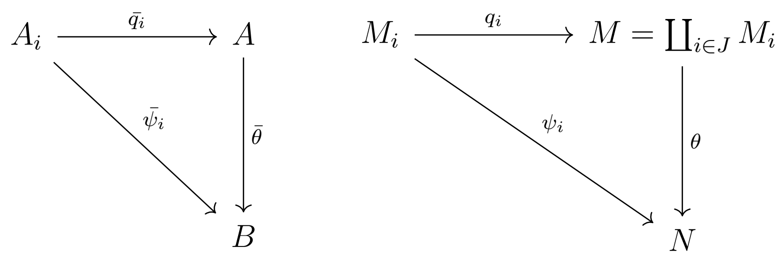
(ii) For any IFSM B of R-module N, equipped with a family of IF R-homomorphisms $\{\bar{\psi}_i : A_i \to B | i \in J\}$, for each $i \in J$, then $\theta \circ \bar{q}_i = \bar{\psi}_i$ for a unique IF R-homomorphism $\bar{\theta} : A \to B$.

Proof. (i) Let $M = \coprod_{i\in J} M_i$ be the coproduct of disjoint union of R-modules $\{M_i | i \in J\}$ and let $q_i : M_i \to M$ be the canonical injection mapping such that $q_i(x_i) = x_i; \forall x_i \in M_i, i \in J$.
An IFS A on M is defined as:

$$\mu_A(x_i) = \wedge\{\mu_{A_i}(x_i) | i \in J\} \text{ and } \nu_A(x_i) = \vee\{\nu_{A_i}(x_i) | i \in J\}, \forall x_i \in M_i, i \in J.$$

Then, A is an IFSM on M. Moreover, for each $i \in J$ and $\forall x_i \in M_i$, we have
$\mu_A(x_i) \leq \mu_{A_i}(x_i) = \mu_{A_i}(q_i(x_i))$ and $\nu_A(x_i) \geq \nu_{A_i}(x_i) = \nu_{A_i}(q_i(x_i))$ indicating that $\bar{q}_i : A_i \to A$ is an IF R-homomorphism. Therefore, $\bar{q}_i \in \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A_i, A), \forall i \in J$.

(ii) Let B is an IFSM of an R-module N with a family of IF R-homomorphisms $\{\bar{\psi}_i : A_i \to B | i \in J\}$ in $\mathbf{C_{R\text{-}IFM}}$. Define $\psi_i : M_i \to N$ as an R-homomorphism by $\psi_i(x_i) = y$, where $y \in N$.



Define an R-homomorphism $\theta : M \to N$ as $\theta(x_i) = y, \forall y \in N$. This implies $\theta \circ q_i = \psi_i$ and

$$\begin{aligned} \mu_B(\theta(x_i)) &= \mu_B(y) \\ &= \mu_B(\psi_i(x_i)) \\ &\geq \mu_{A_i}(x_i). \end{aligned}$$

Further, $\mu_B(\theta(x_i)) \geq \mu_{A_i}(x_i)$ and $\nu_B(\theta(x_i)) \leq \nu_{A_i}(x_i)$. Thus, $\bar{\theta} : A \to B$ is an IF R-homomorphism satisfying $\bar{\theta} \circ \bar{q_i} = \bar{\psi_i}$; $\forall i \in J$. Hence, the category $\mathbf{C_{R\text{-}IFM}}$ has coproduct $\coprod_{i \in J} A_i$. □

Remark 4.2.4. The IFSM A on $M = \coprod_{i \in J} M_i$, serves as the coproduct of the IFSMs of A_i. This is denoted s $A = \coprod_{i \in J} A_i$. Therefore, it confirms that the category of intuitionistic fuzzy modules indeed possess a coproduct.

Definition 4.2.5. Let $\bar{f}, \bar{g} : A \to B$ are IF R-homomorphisms. An intuitionistic fuzzy equalizer (IF-equalizer) is defined as a pair $(E, \bar{e})$, where $E = (\mu_E, \nu_E)_Q$ is an IFSM of R-module Q and $\bar{e} \in \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(E, A)$, the following properties hold:

i) $\bar{f} \circ \bar{e} = \bar{g} \circ \bar{e}$, and

ii) For any IFSM $E_1 = (\mu_{E_1}, \nu_{E_1})_{Q_1}$ of an R-module Q_1 and $\bar{e_1} \in \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(E_1, A)$, if $(E_1, \bar{e_1})$ is another pair satisfying $\bar{f} \circ \bar{e_1} = \bar{g} \circ \bar{e_1}$, then $\bar{e} \circ \bar{p} = \bar{e_1}$ for a unique IF R-homomorphism $\bar{p} : E_1 \to E$.

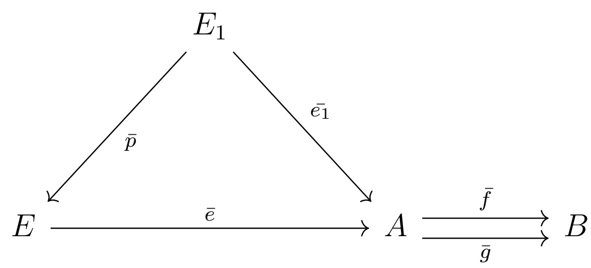


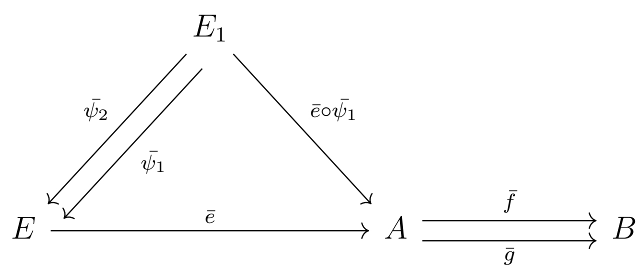
Figure 4.1: Equalizer

Remark 4.2.6. In particular, for an IFSM E of an R-module Q to be considered an IF-equalizer, it must possess an element $\bar{e} \in \mathbf{Hom}_{\mathbf{C}_{\textbf{R-IFM}}}(E, A)$ that satisfies the aforementioned condition. Additionally, E must satisfy the conditions $\mu_E(z) = \mu_A(\bar{e}(z))$; $\nu_E(z) = \nu_A(\bar{e}(z))$, $\forall z \in Q$.

Proposition 4.2.7. *Let $\bar{e} : E \to A$ be an IF-equalizer for IF R-homomorphisms $\bar{f}, \bar{g} : A \to B$. Then, $\bar{e}$ is an IF-monomorphism. Additionally, any two IF-equalizers for $\bar{f}$ and $\bar{g}$ are isomorphic as IFSMs.*

Proof. Given that $\bar{e} : E \to A$ is an IF-equalizer for $\bar{f}, \bar{g} : A \to B$, $\bar{f} \circ \bar{e} = \bar{g} \circ \bar{e}$.

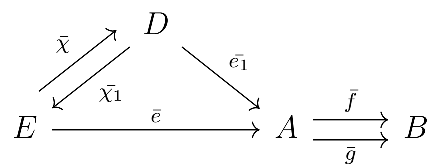
Now, suppose $\bar{\psi_1}, \bar{\psi_2} : E_1 \to E$ are IF R-homomorphisms such that $\bar{e} \circ \bar{\psi_1} = \bar{e} \circ \bar{\psi_2}$.



Considering $\bar{f} \circ (\bar{e} \circ \bar{\psi_1}) = \bar{f} \circ (\bar{e} \circ \bar{\psi_2}) = (\bar{f} \circ \bar{e}) \circ \bar{\psi_2} = (\bar{g} \circ \bar{e}) \circ \bar{\psi_2} = \bar{g} \circ (\bar{e} \circ \bar{\psi_2}) = \bar{g} \circ (\bar{e} \circ \bar{\psi_1})$.

By the uniqueness of an IF-equalizer, $\bar{\psi_1} = \bar{\psi_2}$, thus $\bar{e}$ is an IF-monomorphism.

Let $\bar{e_1} : D \to A$ be another IF-equalizer for $\bar{f}, \bar{g} : A \to B$, then there exist IF R-homomorphisms $\bar{\chi_1} : D \to E$ and $\bar{\chi} : E \to D$ to maintain the commutativity of the diagram



$\bar{e} \circ \bar{\chi_1} = \bar{e_1}$ and $\bar{e_1} \circ \bar{\chi} = \bar{e}$. Therefore, we have

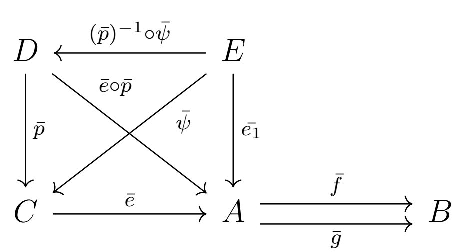
$$\bar{e} \circ (\bar{\chi_1} \circ \bar{\chi}) = (\bar{e} \circ \bar{\chi_1}) \circ \bar{\chi} = \bar{e_1} \circ \bar{\chi} = \bar{e} = \bar{e} \circ I_E.$$

Thus, $\bar{\chi_1} \circ \bar{\chi} = I_E$. Similarly, we can show that $\bar{\chi} \circ \bar{\chi_1} = I_D$. Hence, $\bar{\chi_1}$ and $\bar{\chi}$ are IF R-isomorphisms. Consequently, E and D are isomorphic as IFSMs. □

Remark 4.2.8. In Proposition 4.2.7, the status of $\bar{e}$ or $\bar{e_1}$ as a strong monomorphism is undetermined. Nevertheless, if one of them qualifies as a strong monomorphism, then the other must also be a strong monomorphism. Consequently, E and D would be isomorphic as IFSMs.

Proposition 4.2.9. *Let $\bar{e} : C \to A$ be an IF-equalizer for $\bar{f}, \bar{g} : A \to B$ and let D be an IFSM isomorphic to an IFSM C. Then, D is also an IF-equalizer for $\bar{f}$ and $\bar{g}$.*

Proof. Let $\bar{p} : D \to C$ be an IF R-isomorphism. Firstly, we claim that $\bar{e} \circ \bar{p} : D \to A$ forms an IF-equalizer for $\bar{f}$ and $\bar{g}$. Since $\bar{e}$ is an IF-equalizer for $\bar{f}$ and $\bar{g}$, $\bar{g} \circ \bar{e} = \bar{f} \circ \bar{e}$.
Condition (i) for an IF-equalizer is satisfied as $\bar{f} \circ (\bar{e} \circ \bar{p}) = (\bar{f} \circ \bar{e}) \circ \bar{p} = (\bar{g} \circ \bar{e}) \circ \bar{p} = \bar{g} \circ (\bar{e} \circ \bar{p})$.
Suppose that there exists $\bar{e_1} : E \to A$ satisfying $\bar{f} \circ \bar{e_1} = \bar{g} \circ \bar{e_1}$. Since $\bar{e}$ is an IF-equalizer, $\bar{e} \circ \bar{\psi} = \bar{e_1}$ for a unique IF R-homomorphism $\bar{\psi} : E \to C$.



By the commutativity of above diagram and $\bar{p}$ is an IF R-isomorphism, then $(\bar{p})^{-1} \circ \bar{\psi} : E \to D$ becomes an IF R-homomorphism. Consider $(\bar{e} \circ \bar{p})((\bar{p})^{-1} \circ \bar{\psi}) = \bar{e} \circ (\bar{p} \circ (\bar{p})^{-1}) \circ \bar{\psi} = \bar{e} \circ I_D \circ \bar{\psi} =$

$\bar{e} \circ \bar{\psi} = \bar{e_1}$.

For uniqueness, let $\bar{t} : E \to D$ such that $\bar{e} \circ \bar{p} \circ \bar{t} = \bar{e_1}$, which implies $\bar{e} \circ \bar{p} \circ \bar{t} = \bar{e} \circ \bar{\psi}$.

Since $\bar{e}$ is an IF monomorphism, we have $\bar{p} \circ \bar{t} = \bar{\psi}$, implying $\bar{t} = (\bar{p})^{-1} \circ \bar{\psi}$.

This proves the uniqueness.

Also, for $z \in P$, $\mu_D(z) = \mu_C(\bar{p}(z)) = \mu_A(\bar{e}(\bar{p}(z))) = \mu_A((\bar{e} \circ \bar{p})(z))$.

Similarly, we can show $\nu_D(z) = \nu_A((\bar{e} \circ \bar{p})(z))$. Hence, D is also an IF-equalizer of $\bar{f}$ and $\bar{g}$. □

Proposition 4.2.10. *I_A is an IF-equalizer for $\bar{f}, \bar{g} : A \to B$ if and only if $\bar{f} = \bar{g}$.*

Proof. Firstly, let $\bar{f} = \bar{g}$. This implies, $\bar{f} \circ I_A = \bar{g} \circ I_A$.

Let $\bar{h} : C \to A$ be an IF R-homomorphism satisfying $\bar{f} \circ \bar{h} = \bar{g} \circ \bar{h}$, ensuring commutativity of the figure 4.2 The commutativity implies $I_A \circ \bar{h} = \bar{h}$ demonstrating the uniqueness. Thus,

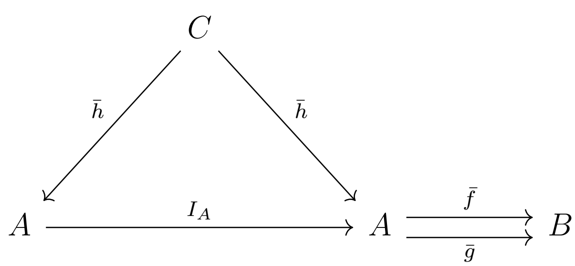


Figure 4.2: Equalizer:I_A

I_A is an IF-equalizer.

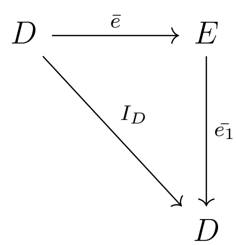
Conversely, let I_A be an IF-equalizer, then $\bar{f} \circ I_A = \bar{g} \circ I_A$, implying $\bar{f} = \bar{g}$. □

Proposition 4.2.11. *Let $\bar{e} : C \to A$ is an IF-equalizer of $\bar{f}, \bar{g} : A \to B$. If $\bar{e}$ is an IF-epimorphism, then $\bar{e}$ is an IF R-isomorphism.*

Proof. Since $\bar{e} : C \to A$ is an IF-equalizer for $\bar{f}$ and $\bar{g}$, we have $\bar{f} \circ \bar{e} = \bar{g} \circ \bar{e}$. Additionally as, $\bar{e}$ is an IF-epimorphism, therefore it is right cancellable. Therefore, $\bar{f} = \bar{g}$. By Proposition [4.2.10], I_A is an IF-equalizer for $\bar{f}$ and $\bar{g}$, implying that $\bar{e} \circ \bar{k} = I_A$ for a unique IF R-homomorphism $\bar{k} : A \to C$. Hence, $\bar{e} = (\bar{k})^{-1}$, demonstrating that $\bar{e}$ is an IF R-isomorphism. □

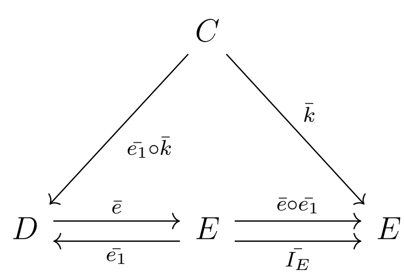
Proposition 4.2.12. *Every IF-coretraction is an IF-equalizer in* $\boldsymbol{C}_{\textbf{\textit{R-IFM}}}$.

Proof. Let $D = (\mu_D, \nu_D)_P$ and $E = (\mu_E, \nu_E)_Q$ are IFSMs of R-modules P and Q respectively, with $\bar{e} : D \to E$ being an IF-coretraction, an IF R-homomorphism $\bar{e_1} : E \to D$ exists that satisfying $\bar{e_1} \circ \bar{e} = I_D$, implying the next diagram commutes:



To prove that $\bar{e}$ is an IF-equalizer for I_E and $\bar{e} \circ \bar{e_1}$, firstly we will show that $I_E \circ \bar{e}$ = $(\bar{e} \circ \bar{e_1}) \circ \bar{e}$. Now $(\bar{e} \circ \bar{e_1}) \circ \bar{e} = \bar{e} \circ (\bar{e_1} \circ \bar{e}) = \bar{e} \circ I_D = \bar{e} = I_E \circ \bar{e}$. Therefore, the first condition of IF-equalizer is satisfied.

Let $\bar{k} : C \to E$ be an IF R-homomorphism such that $I_E \circ \bar{k}$ = $(\bar{e} \circ \bar{e_1}) \circ \bar{k}$. This implies that $\bar{k}$ = $\bar{e} \circ (\bar{e_1} \circ \bar{k})$. As a result, subsequent diagram commutes



, confirming the uniqueness. Hence, $\bar{e}$ is an IF-equalizer for I_E and $\bar{e} \circ \bar{e_1}$. □

Example 4.2.13. Let $P = Q = \mathbf{Z}_2$ be two Z-modules. Define IFSs $D = (\mu_D, \nu_D)_P$ and $E = (\mu_E, \nu_E)_Q$ on P and Q respectively as

$$\mu_D(d) = \begin{cases} 1, & \text{if } d = 0 \\ 0.5, & \text{if } d = 1 \end{cases}; \quad \nu_D(d) = \begin{cases} 0, & \text{if } d = 0 \\ 0.4, & \text{if } d = 1 \end{cases}$$

$\mu_E(b) = 1, \nu_E(b) = 0$, for every $b \in \mathbf{Z}_2$. Then it's straightforward to confirm that D and E are IFSMs of P and Q respectively. Define $h, h_1 : P \to Q$ as $h(0) = 0, h(1) = 1, h_1(0) = 0, h_1(1) = 0$. Clearly h, h_1 are R-homomorphism, for

$\mu_E(h(0)) = \mu_E(0) = 1 \geq 1 = \mu_D(0), \nu_E(h(0)) = \nu_E(0) = 0 \leq 0 = \nu_D(0),$

$\mu_E(h(1)) = \mu_E(1) = 1 \geq 0.5 = \mu_D(1), \nu_E(h(1)) = \nu_E(1) = 0 \leq 0.4 = \nu_D(1).$

Thus, $\mu_E(h(d)) \geq \mu_D(d)$ and $\nu_E(h(d)) \leq \nu_D(d), \forall d \in P$.

Also, $\mu_E(g(0)) = \mu_E(0) = 1 \geq 1 = \mu_D(0), \nu_E(h_1(0)) = \nu_E(0) = 0 \leq 0 = \nu_D(0),$

$\mu_E(h_1(1)) = \mu_E(0) = 1 \geq 0.5 = \mu_D(1), \nu_E(h_1(1)) = \nu_E(0) = 0. \leq 0.4 = \nu_D(1).$

Thus, $\mu_E(h_1(d)) \geq \mu_D(d)$ and $\nu_E(h_1(d)) \leq \nu_D(d), \forall d \in P$.

Hence, $\bar{h}, \bar{h_1} : D \to E$ are IF R-homomorphisms.

Let $K = \{z \in P : h(z) = h_1(z)\}$ be a submodule of P. Then $K = \{0\}$. Define an R-homomorphism $e : K \to P$ as $e(z) = z, \forall z \in K$. It is easy to verify that $\bar{e}$ is an IF R-homomorphism satisfying $\bar{h} \circ \bar{e} = \bar{h_1} \circ \bar{e}$. Let E is an IFSM on K defined as $\mu_E(0) = 1, \nu_e(0) = 0$. Thus we have $\mu_D(0) = \mu_E(e(0))$ and $\nu_D(0) = \nu_E(e(0))$. Hence, the pair $(E, \bar{e})$ forms an IF-equalizer.

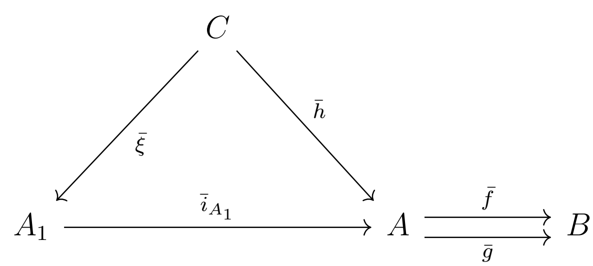
Proposition 4.2.14. *$C_{R\text{-}IFM}$ has equalizers.*

Proof. Let $\bar{f}, \bar{g} : A \to B$ be an IF R-homomorphisms. Define $M_1 = \{c \in M : f(c) = g(c)\}$ which is clearly a submodule of M. Let A_1 be the restriction of A to M_1, defined as $\mu_{A_1}(c) = \mu_A(c)$ and $\nu_{A_1}(c) = \nu_A(c)$; $\forall c \in M_1$, implying A_1 becomes an IFSM of M_1 and inclusion mapping $i : M_1 \to M$ yields a strong IF R-homomorphism $\bar{i}_{A_1} : A_1 \to A$, defined as $\mu_A(\bar{i}_{A_1}(c)) = \mu_{A_1}(c) = \mu_A(c)$ and $\nu_A(\bar{i}_{A_1}(c)) = \nu_{A_1}(c) = \nu_A(c)$.

Now, it's claimed that $\bar{i}_{A_1} : A_1 \to A$ is an IF-equalizer for $\bar{f}$ and $\bar{g}$.

For $d \in M$, $(\bar{f} \circ \bar{i}_{A_1})(d) = \bar{f}(\bar{i}_{A_1}(d)) = \bar{f}(d) = \bar{g}(d) = \bar{g}(\bar{i}_{A_1}(d)) = (\bar{g} \circ \bar{i}_{A_1})(d)$ implying

$$\bar{f} \circ \bar{i}_{A_1} = \bar{g} \circ \bar{i}_{A_1}.$$



Let $\bar{h} \in \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(C, A)$ satisfying $\bar{f} \circ \bar{h} = \bar{g} \circ \bar{h}$. Define $\bar{\xi} : C \to A_1$ as $\bar{\xi}(k) = \bar{h}(k)$, $\forall k \in K$. For $k \in K$, $(\bar{f} \circ \bar{h})(k) = (\bar{g} \circ \bar{h})(k)$, implying $\bar{f}(\bar{h}(k)) = \bar{g}(\bar{h}(k))$. Thus, $\bar{\xi}$ is well-defined. Furthermore, $(\bar{i}_{A_1} \circ \bar{\xi})(k) = \bar{i}_{A_1}(\bar{\xi}(k)) = \bar{\xi}(k) = \bar{h}(k)$ shows that $\bar{i}_{A_1} \circ \bar{\xi} = \bar{h}$. Since $\bar{i}_{A_1}$ is a strong IF R-homomorphism, $\bar{\xi}$ is unique. Finally, to demonstrate that $\bar{\xi}$ is an IF R-homomorphism, consider $\mu_C(k) \leq \mu_A(\bar{h}(k)) = \mu_A(\bar{\xi}(k)) = \mu_{A_1}(\bar{\xi}(k))$, showing $\mu_{A_1}(\bar{\xi}(k)) \geq \mu_C(k)$. Similarly, it can be shown that $\nu_{A_1}(\bar{\xi}(k)) \leq \nu_C(k)$. Hence, $\bar{\xi}$ is an IF R-homomorphism, concluding that $\mathbf{C_{R\text{-}IFM}}$ possesses IF-equalizers. □

Remark 4.2.15. In $\mathbf{C_{R\text{-}IFM}}$, every strong IF R-homomorphism is an IF-equalizer.

Definition 4.2.16. [21] Let ρ be an intuitionistic fuzzy equivalence relation on R-module M. For each $a \in M$, the IFS $\rho_a = (\mu_{\rho_a}, \nu_{\rho_a}) : M \to I \times I$ on M defined as

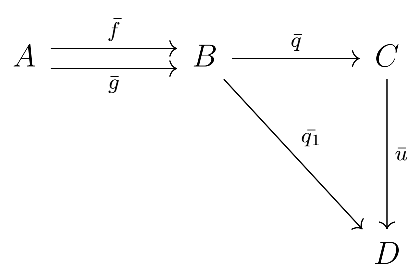
$$\mu_{\rho_a}(x) = \mu_\rho(a, x) \text{ and } \nu_{\rho_a}(x) = \nu_\rho(a, x), \forall x \in M.$$

is called an intuitionistic fuzzy equivalence class of ρ containing a. The set $\{\rho_a : a \in M\}$ is called the IF quotient set of M by ρ and is denoted by M/ρ. We can also write $M/\rho = \{[a] : a \in M\}$. In fact M/ρ form an R-module with respect to the operations defined by $\rho_a + \rho_b = \rho_{a+b}$ and $r\rho_a = \rho_{ra}, \forall a, b \in M, r \in R$ called the quotient submodule induced by ρ.

Definition 4.2.17. Let $\bar{f}, \bar{g} : A \to B$ are IF R-homomorphisms. An intuitionistic fuzzy coequalizer (IF-coequalizer) is defined as a pair $(C, \bar{q})$, where $C = (\mu_C, \nu_C)_K$ is an IFSM of an R-module K and $\bar{q} \in \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(B, C)$, if the following conditions hold:

(i) $\bar{q} \circ \bar{f} = \bar{q} \circ \bar{g}$ and

(ii) For any IFSM $E = (\mu_E, \nu_E)_Q$ and $\bar{q_1} \in \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(B, E)$, if $(E, \bar{q_1})$ is another pair satisfying $\bar{q_1} \circ \bar{f} = \bar{q_1} \circ \bar{g}$, then $\bar{u} \circ \bar{q} = \bar{q_1}$ for a unique IF R-homomorphism $\bar{u} : C \to E$.



Proposition 4.2.18. *Every IF-retraction is an IF-coequalizer in $\mathbf{C_{R\text{-}IFM}}$.*

Proposition 4.2.19. *$\mathbf{C_{R\text{-}IFM}}$ have coequalizers.*

Proof. Let $A, B \in Ob(\mathbf{C_{R\text{-}IFM}})$ and $\bar{f}, \bar{g} : A \to B$ be IF R-homomorphisms. Assume that ρ is the smallest IF-equivalence relation on N satisfying $\bar{f}(c) \sim \bar{g}(c), \forall c \in M$.

Let $K = N/\rho$ =$\{[y] : y \in N\}$, $\phi : N \to K$ is the canonical mapping.

Now, construct an IFS C on K as

$$\mu_C([k]) = \vee\{\mu_B(z) : z \in [k]\} \text{ and } \nu_C([k]) = \wedge\{\nu_B(z) : z \in [k]\}$$

It is easy to confirm that C is an IFSM on K. Define $\bar{\phi} : B \to C$ as

$$\bar{\phi}(y) = [y], \text{ for each } y \in N.$$

Clearly, $\bar{\phi}$ is IF R-homomorphism as $\mu_B(y) \leq \vee\{\mu_B(z) : z \in [y]\} = \mu_C([y]) = \mu_C(\bar{\phi}(y))$ and $\nu_B(y) \geq \wedge\{\nu_B(z) : z \in [y]\} = \nu_C([y]) = \nu_C(\bar{\phi}(y))$.

To show that $\bar{\phi}$ is IF-coequalizer of $\bar{f}$ and $\bar{g}$, observe that for each $d \in M$, $\bar{f}(d) \sim \bar{g}(d)$ thus $\bar{\phi}(\bar{f}(d)) = [\bar{f}(d)] = [\bar{g}(d)] = \bar{\phi}(\bar{g}(d))$, i.e. $\bar{\phi} \circ \bar{f}$ = $\bar{\phi} \circ \bar{g}$.

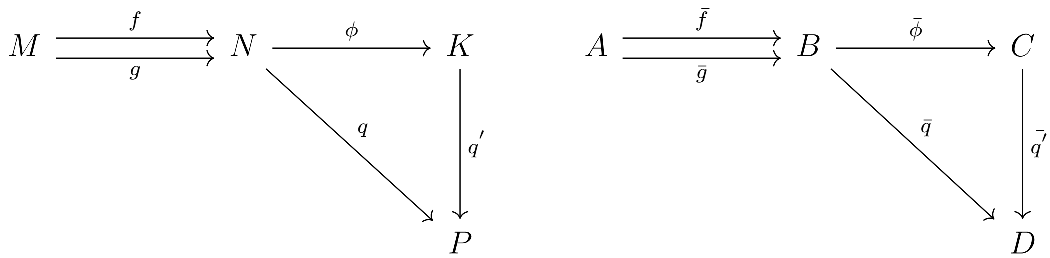
Let D be an IFSM on P and $\bar{q} : B \to D$ be an IF R-homomorphism satisfying $\bar{q} \circ \bar{f}$ = $\bar{q} \circ \bar{g}$. Define $\rho_1 = \{(y, y_1) \in N \times N : q(y) = q(y_1)\}$ as an IF equivalence relation on N . As $q \circ f = q \circ g$ implies $(f(d), g(d)) \in \rho_1$ for $d \in M$, concluding that $\rho \subseteq \rho_1$. Therefore, ρ is the smallest IF equivalence relation contain $\{(f(d), g(d)) : d \in M\}$. Define $q^{'} : K \to P$ by $q^{'}([y]) = q(y), \forall [y] \in K$.

Let $y, y_1 \in N$ such that $[y] = [y_1]$. If $(y, y_1) \in \rho$, then $(y, y_1) \in \rho_1$, we have $q(y) = q(y_1)$ implies that $q^{'}([y]) = q^{'}([y_1])$. Thus, $q^{'}$ is well-defined. For $y \in N$, $(\bar{q}^{'} \circ \bar{\phi})(y) = \bar{q}^{'}(\bar{\phi}(y)) = \bar{q}^{'}([y]) = \bar{q}(y)$ implying $\bar{q}^{'} \circ \bar{\phi} = \bar{q}$. There is only need to prove that $\bar{q}^{'} : C \to D$ an IF R-homomorphism.

As $\bar{q}$ is an IF R-homomorphism, for $[y] \in K$, we have

$\mu_C([y]) = \vee\{\mu_B(z) : z \in [y]\} \leq \vee\{\mu_D(\bar{q}(z)) : z \in [y]\} = \mu_D(\bar{q}^{'}([y]))$, indicating that $\mu_D(q^{'}([y])) \geq \mu_C([y])$. Similarly, $\nu_D(q^{'}([y])) \leq \nu_C([y])$.

Consequently, $\bar{q}^{'}$ is a unique IF R-homomorphism from C to D, concluding that $\bar{\phi}$ is IF-coequalizer of $\bar{f}$ and $\bar{g}$.



This completes the proof. □

Theorem 4.2.20. *$\boldsymbol{C_{R\text{-}IFM}}$ is complete and cocomplete.*

Proof. The completeness and cocompleteness of the category $\mathbf{C_{R\text{-}IFM}}$ are demonstrated by Theorem 4.2.1, Theorem 4.2.3, Proposition 4.2.14, and Proposition 4.2.19. □

Remark 4.2.21. According to Theorem 4.2.20, it can be inferred that the category of intuitionistic fuzzy modules, $\mathbf{C_{R\text{-}IFM}}$, is indeed bicomplete.

4.3 Intersections and pullbacks

Definition 4.3.1. Let $\bar{\sigma} : A \to C$ and $\bar{\tau} : B \to C$ are IF R-homomorphisms. Then, for these IF R-homomorphisms, the intuitionistic fuzzy pullback(IF-pullback) is a triplet $(D, \bar{\sigma}_1, \bar{\tau}_1)$ with $D = (\mu_D, \nu_D)_P$, $\bar{\sigma}_1 \in \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(D, A)$, and $\bar{\tau}_1 \in \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(D, B)$ if Figure-4.3 commutes and the following properties hold:

(i) $\bar{\sigma} \circ \bar{\sigma}_1 = \bar{\tau} \circ \bar{\tau}_1$ and

ii) Universal Property: if another triplet $(E, \bar{\phi}, \bar{\psi})$ with $E = (\mu_E, \nu_E)_P$, $\bar{\phi} \in \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(E, A)$ and $\bar{\psi} \in \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(E, B)$ satisfying $\bar{\sigma} \circ \bar{\phi} = \bar{\tau} \circ \bar{\psi}$, then $\bar{\phi} = \bar{\sigma}_1 \circ \bar{\theta}$ and $\bar{\psi} = \bar{\tau}_1 \circ \bar{\theta}$ for a unique IF R-homomorphism $\bar{\theta} : E \to D$.

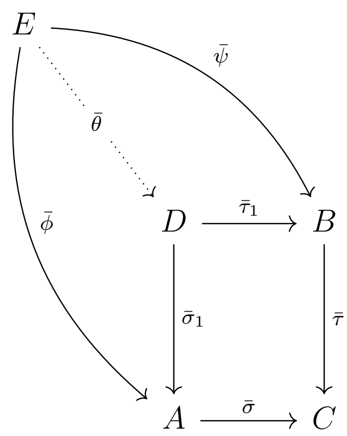


Figure 4.3: IF-Pullback

Proposition 4.3.2. ***$C_{R\text{-}IFM}$*** *have pullbacks.*

Proof. Let $\bar{\sigma} \in \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A, C)$ and $\bar{\tau} \in \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(B, C)$ in $\mathbf{C_{R\text{-}IFM}}$.

As a subset of $M \times N$, consider $P = \{(d_1, d_2) \in M \times N | \sigma(d_1) = \tau(d_2)\}$. Define IFSM $D = (\mu_D, \nu_D)$ on R-module P as $\mu_D = (\mu_A \wedge \mu_B)|_P$ and $\nu_D = (\nu_A \vee \nu_B)|_P$ such that

$\mu_D(d_1, d_2) = \{\mu_A(d_1) \wedge \mu_B(d_2) | \sigma(d_1) = \tau(d_2)\}$ and $\nu_{\mathrm{D}}(\mathrm{d}_1, \mathrm{d}_2) = \{\nu_{\mathrm{A}}(\mathrm{d}_1) \vee \nu_{\mathrm{B}}(\mathrm{d}_2) | \sigma(\mathrm{d}_1) = \tau(\mathrm{d}_2)\}$

Define the projection maps $\sigma_1(d_1, d_2) = d_1$ and $\tau_1(d_1, d_2) = d_2$; $\forall (d_1, d_2) \in P$.

We want to claim that the triplet $(D, \bar{\sigma}_1, \bar{\tau}_1)$ is IF-pullback.

Consider $\sigma(\sigma_1(d_1, d_2)) = \sigma(d_1) = \tau(d_2) = \tau(\tau_1(d_1, d_2))$.

By intuitionistic fuzzification, $\bar{\sigma} \circ \bar{\sigma}_1 = \bar{\tau} \circ \bar{\tau}_1$.

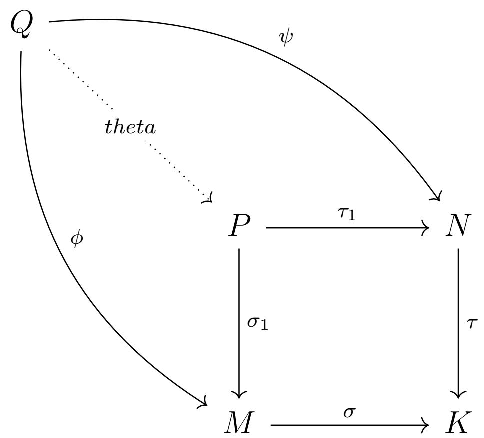
We now need to prove the universal property.

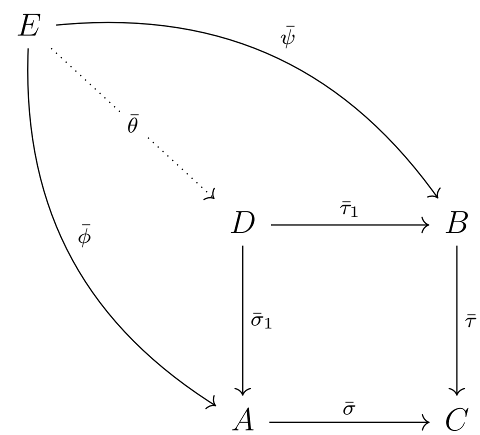
Consider the another triplet $(E, \bar{\phi}, \bar{\psi})$ with $E = (\mu_E, \nu_E)_Q$, $\bar{\phi} \in \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(E, A)$ and $\bar{\psi} \in \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(E, B)$ satisfying $\bar{\sigma} \circ \bar{\phi} = \bar{\tau} \circ \bar{\psi}$. Define R-homomorphism $\theta : Q \to P$ as

$$\theta(c) = (\phi(c), \psi(c))$$

For $(\phi(c), \psi(c)) \in P$, $\sigma(\phi(c)) = \tau(\psi(c))$. Therefore, θ is well-defined.

Since the category $\mathbf{C_{R\text{-}M}}$ has pullbacks. Then θ is unique R-homomorphism which satisfies $\phi = \sigma_1 \circ \theta$ and $\psi = \tau_1 \circ \theta$. Subsequently, it is sufficient to show that $\bar{\theta} \in \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(E, D)$ is an IF R-homomorphism.





Consider $\mu_{\mathrm{D}}(\theta(\mathrm{z})) = \mu_{\mathrm{D}}(\phi(\mathrm{z}), \psi(\mathrm{z})) = \mu_{\mathrm{A}}(\phi(\mathrm{z})) \wedge \mu_{\mathrm{B}}(\psi(\mathrm{z})) \geq \mu_{\mathrm{E}}(\mathrm{z}) \wedge \mu_{\mathrm{E}}(\mathrm{z}) = \mu_{\mathrm{E}}(\mathrm{z})$

which implies $\mu_D(\theta(z)) \geq \mu_E(z)$. Likewise, we are able to show that $\nu_D(\theta(z)) \leq \nu_E(z)$.

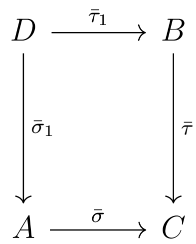
Thus, $\bar{\theta} \in \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(E, D)$ is an IF R-homomorphism that satisfies $\bar{\phi} = \bar{\sigma}_1 \circ \bar{\theta}$.

Hence, pullbacks exists in the $\mathbf{C_{R\text{-}IFM}}$. □

Remark 4.3.3. If $\bar{\sigma} : A \to C$ and $\bar{\tau} : B \to C$ are IF R-homomorphisms then IF-pullback can be constructed by defining IFSM $D = (\mu_D, \nu_D)$ on R-module P as

$$\mu_D = (\mu_A \wedge \mu_B)|_P \text{ and } \nu_D = (\nu_A \vee \nu_B)|_P$$

where $P = \{(d_1, d_2) \in M \times N | \sigma(d_1) = \tau(d_2)\}$ with the projection maps $\sigma_1(d_1, d_2) = d_1$ and $\tau_1(d_1, d_2) = d_2$, $\forall (d_1, d_2) \in P$. This is characteristic of IF-pullback with IF-product and D is sub-object of $A \times B$ in $\mathbf{C_{R\text{-}IFM}}$.



Definition 4.3.4. For a given IFSM A of R-module M equipped with a family of IF R-homomorphisms $\{\bar{\sigma}_i : A_i \to A : i \in J\}$. Then, a strong IF R-homomorphism $\bar{\phi} : B \to A$ is said to be an intuitionistic fuzzy intersection(IF-intersection) of the family if figure-4.4 commutes and the following properties are satisfied:

(i) for each $i \in J$, there exist IF R-homomorphisms $\bar{\tau}_i : B \to A_i$ such that $\bar{\phi} = \bar{\sigma}_i \circ \bar{\tau}_i$ and

(ii) Universal Property: For any IFSM $C = (\mu_C, \nu_C)_K$ and IF R-homomorphism $\bar{h} \in \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(C, A)$ satisfying $\bar{h} = \bar{\sigma}_i \circ \bar{\gamma}_i$ for IF R-homomorphisms $\bar{\gamma}_i : C \to A_i$ for $i \in J$ then $\bar{h} = \bar{\phi} \circ \bar{\theta}$ for a unique IF R-homomorphism $\bar{\theta} : C \to B$.

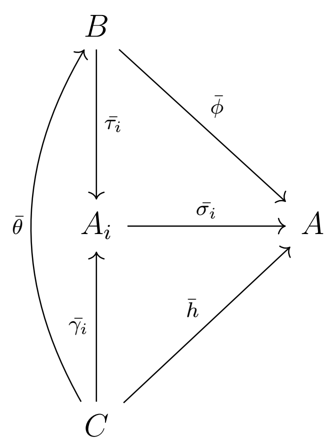


Figure 4.4: IF-Intersection

Proposition 4.3.5. ***$C_{R\text{-}IFM}$*** *have intersections.*

Proof. We aim to demonstrate the existence of an IF-intersection for any family of subobjects of the IFSM A, which is an R-module M in $\mathbf{C_{R\text{-}IFM}}$.

Consider the family of IFSMs $\{A_i = (\mu_{A_i}, \nu_{A_i}) | i \in J\}$ of R-modules $\{M_i | i \in J\}$, along with a family of IF R-homomorphism $\{\bar{\sigma}_i : A_i \to A | i \in J\}$ representing sub-objects of A. Each $\bar{\sigma}_i$ is IF-monomorphism, implying $\sigma_i : M_i \to M$ is injective mapping in $\mathbf{C_{R\text{-}M}}$ for each $i \in J$. Consequently, $\sigma_i(M_i)$ forms a submodule of M, isomorphic to M_i.

Let $N = \cap_{i \in J} \sigma_i(M_i) \subseteq M$.

Assume $N = \emptyset$. It follows that there is a unique R-homomorphism from N to any other R-module, and is evident as $\emptyset : \emptyset \to A$, serving as the IF-intersection.

Let us assume that $N = \cap_{i \in J} \sigma_i(M_i) \neq \emptyset$.

Define an IFSM $B = (\mu_B, \nu_B)$ on R module N, where

$$\mu_B(y) = \mu_A(y) \text{ and } \nu_B(y) = \nu_A(y), \forall y \in N.$$

Consequently, $\bar{i}_B : B \to A$ is strong IF R-homomorphism in $\mathbf{C_{R\text{-}IFM}}$.

To establish $(B, \bar{i_B})$ as the IF-intersection of the family $\{\bar{\sigma}_i : A_i \to A\}_{i \in J}$, observe that if $y \in N = \cap_{i \in J} \sigma_i(M_i)$ then a unique $x_i \in M_i$ exists so that $y = \sigma_i(x_i)\ \forall i \in J$.

Let $\sigma_i|^{M_i}$ is the corestriction of σ_i on $\sigma_i(M_i)$, with $(\sigma_i|^{M_i})^{-1}$ denoting its inverse mapping. Let $\tau_i = (\sigma_i|^{M_i})^{-1}|_N : N \to M_i$ as the restriction of $(\sigma_i|^{M_i})^{-1}$ on N for each $i \in J$.

Thus, we obtain a well-defined function $\tau_i : N \to M_i$ defined as

$$\tau_i(y) = x_i \text{ if } y = \sigma_i(x_i); \forall i \in J.$$

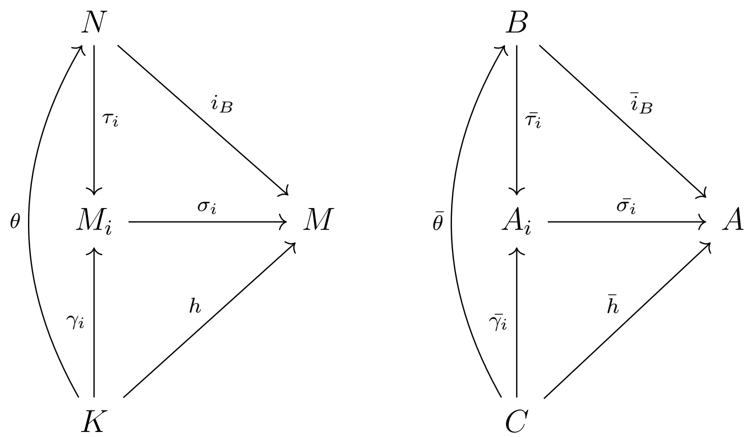
Consider $\mu_{A_i}(\tau_i(y)) = \mu_{A_i}(x_i) = \mu_{A_i}(\sigma_i(x_i)) = \mu_A(y) = \mu_B(y)$.

Similarly, $\nu_{A_i}(\tau_i(y)) = \nu_B(y)$.

Therefore, $\bar{\tau}_i \in \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(B, A_i)$ and $\bar{i}_B = \bar{\sigma}_i \circ \bar{\tau}_i$ for each $i \in J$.

For the universal property verification, consider an IFSM $C = (\mu_C, \nu_C)_K$ and $\bar{h} \in \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(C, A)$

such that $\bar{h} = \bar{\sigma_i} \circ \bar{\gamma_i}$ for IF R-homomorphisms $\bar{\gamma_i} : C \to A_i$.



Define $\bar{\theta} : C \to B$ such that for $z \in K$, $h(z) = \sigma_i \circ \gamma_i(z) = \sigma_i(\gamma_i(z)) \in \sigma_i(M_i)$ for all $i \in J$.

Hence, $h(z) \in \cap_{i\in J}\sigma_i(M_i) = N$.

Take θ as a restriction mapping of h on N. Therefore, $\theta : K \to N$ is defined as $\theta(z) = h(z)$ $\forall z \in K$.

Since the category $\mathbf{C_{R\text{-}M}}$ has intersection, R-homomorphism $\theta : K \to N$ is unique, satisfying $i_N \circ \theta = h$. We only need to demonstrate that $\bar{\theta}$ is IF R-homomorphism.

For $z \in K$, $\mu_B(\theta(z)) = \mu_B(h(z)) = \mu_A(h(z)) \geq \mu_C(z)$.

Likewise, we are able to show that $\nu_B(\theta(z)) \leq \nu_C(z)$.

Thus, $\bar{\theta}$ is IF R-homomorphism satisfying $\bar{i_B} \circ \bar{\theta} = \bar{h}$.

Therefore, $(B, \bar{i}_B)$ is the IF-intersection of the family $\{\bar{\sigma_i} : A_i \to A\}_{i\in J}$. □

4.4 Images and inverse- images

Definition 4.4.1. Let $\bar{f} : A \to B$ be IF R-homomorphism and $\bar{\sigma} : I \to B$ be IFSM of B. Then, I is said to be an IF-image of $\bar{f}$ if figure-4.5 commutes, and the following conditions hold:

(i) $\bar{f} = \bar{\sigma} \circ \bar{f}_1$ for some IF R-homomorphism $\bar{f}_1 : A \to I$;

(ii) Universal Property: for any IFSM $\bar{\tau} : J \to B$ of B satisfying $\bar{f} = \bar{\tau} \circ \bar{f}_2$ for some IF R-homomorphism $\bar{f}_2 : A \to J$ then $\bar{\sigma} = \bar{\tau} \circ \bar{\theta}$ for a unique IF R-homomorphism $\bar{\theta} : I \to J$.

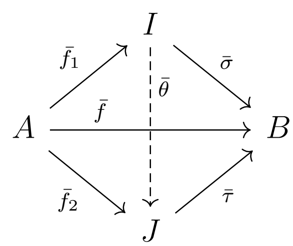


Figure 4.5: IF-image

Lemma 4.4.2. *$C_{R\text{-}IFM}$ have images.*

Proof. Let $\bar{f} : A \to B$ be any given IF R-homomorphism. We define $f(M) = \{f(x) | x \in M\}$.

Define $f_1 : M \to f(M)$ as $f_1(x) = f(x); \forall x \in M$.

The IFSM $\bar{f}(A)$ of $f(M)$ is defined as $\mu_{\bar{f}(A)}(f(x)) = \mu_B(f(x))$ and $\nu_{\bar{f}(A)}(f(x)) = \nu_B(f(x))$.

Let $i_{f(M)} : f(M) \to N$ and $\bar{i}_{\bar{f}(A)} : \bar{f}(A) \to B$ be respective inclusion mappings. Thus, $\bar{f}(A)$ is IFSM of B. Indeed, $\bar{i}_{\bar{f}(A)}$ is a strong IF-monomorphism.

Now, we aim to prove that $\bar{i}_{\bar{f}(A)} : \bar{f}(A) \to B$ is an IF-image of $\bar{f}$.

(i) For all $x \in M$, $i_{f(M)}(f_1(x)) = i_{f(M)}(f(x)) = f(x)$, so $i_{f(M)} \circ f_1 = f$. By intuitionistic fuzzification, $\bar{i}_{\bar{f}(A)} \circ \bar{f}_1 = \bar{f}$.

(ii) Next, we need to verify the universal property.

Suppose there exists an IF R-homomorphism $\bar{f}_2 : A \to C$ and a strong IF-monomorphism

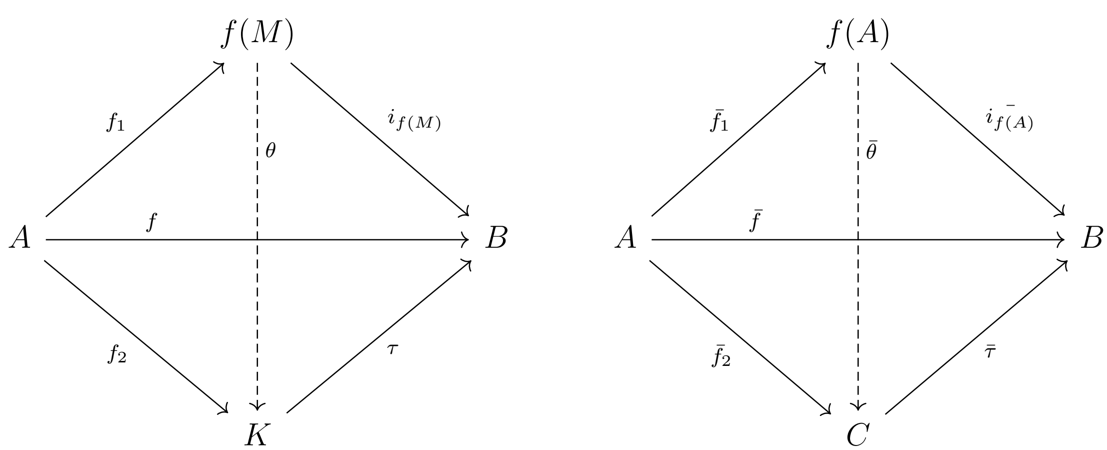
$\bar{\tau} : C \to B$ such that $\bar{f} = \bar{\tau} \circ \bar{f}_2$. Define $\theta : f(M) \to K$ by $\theta(f(x)) = f_2(x)$. It is well-defined as

$$\begin{aligned} f(x_1) &= f(x_2) \\ (\tau \circ f_2)(x_1) &= (\tau \circ f_2)(x_2) \\ \Rightarrow f_2(x_1) &= f_2(x_2) \text{ (Since } \tau \text{ is injective)} \\ \Rightarrow \theta(f(x_1)) &= \theta(f(x_2)). \end{aligned}$$

Since $\mathbf{C_{R\text{-}M}}$ has images, then θ is unique R-homomorphism such that $\tau \circ \theta = i_{f(M)}$. Hence, it is sufficient to show that $\bar{\theta} \in \mathbf{Hom_{C_{R\text{-}IFM}}}(f(A), C)$ is an IF R-homomorphism. Consider

$$\begin{aligned} \mu_{f(A)}(f(x)) &\leq \mu_B(i_{f(x)} f(x)) \\ &= \mu_B(\tau \circ \theta)(f(x)) \\ &= \mu_C(\theta(f(x)) \\ Thus, \mu_C(\theta(f(x)) &\geq \mu_{\bar{f}(A)}(f(x)). \end{aligned}$$

Likewise, we are able to show that $\nu_C(\theta(f(x)) \leq \nu_{\bar{f}(A)}(f(x))$.



Thus, $\bar{\theta} : f(A) \to C$ is an IF R-homomorphism which satisfies $\bar{\tau} \circ \bar{\theta} = \bar{i}_{f(A)}$.

Hence, $\bar{i}_{\bar{f}(A)} : \bar{f}(A) \to B$ is an IF-image of $\bar{f}$. $\square$

Definition 4.4.3. Let $\bar{f} : A \to B$ be given IF R-homomorphism and $\bar{\sigma} : C \to B$ be IFSM of B. An object $I \in \mathbf{C_{R\text{-}IFM}}$ is said to be an IF-inverse image of C by $\bar{f}$ if there exists IF R-homomorphisms $\bar{\tau}_1 : I \to C$ and $\bar{\tau}_2 : I \to A$ such that figure-6 commutes and the following conditions hold:

(i) $\bar{\sigma} \circ \bar{\tau}_1 = \bar{f} \circ \bar{\tau}_2$ and

(ii) Universal Mapping: for any IF R-homomorphisms $\bar{\delta}_1 : J \to C$ and $\bar{\delta}_2 : J \to A$ such that $\bar{\sigma} \circ \bar{\delta}_1 = \bar{f} \circ \bar{\delta}_2$ then $\bar{\tau}_1 \circ \bar{\theta} = \bar{\delta}_1$ and $\bar{\tau}_2 \circ \bar{\theta} = \bar{\delta}_2$ for a unique IF R-homomorphism $\bar{\theta} : J \to I$.

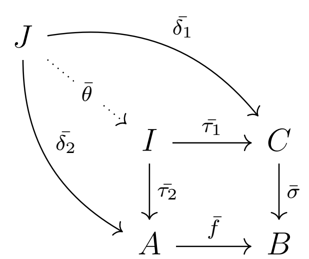


Figure 4.6: IF-inverse image

Lemma 4.4.4. *$\mathbf{C_{R\text{-}IFM}}$ have inverse images.*

Proof. Let $\bar{f} : A \to B$ be any IF R-homomorphism and $\bar{\sigma} : C \to B$ be IFSM of B.

Then $\bar{\sigma}$ is strong IF R-homomorphism. Let $P = \{x \in M | f(x) \in \sigma(K)\}$.

Let $x_1, x_2 \in P$ such that $f(x_1) = \sigma(k_1)$ and $f(x_2) = \sigma(k_2)$ for $k_1, k_2 \in K$.

Consider $f(ax_1 + bx_2) = af(x_1) + bf(x_2) = a\sigma(k_1) + b\sigma(k_2) = \sigma(ak_1 + bk_2) \in \sigma(K)$ which implies that $ax_1 + bx_2 \in P$. Consequently, P is submodule of M.

Define IFS I of an R-module P as $\mu_I(x) = \mu_A(x)$ and $\nu_I(x) = \nu_A(x), \forall x \in M$.

From this, we have I is an IFSM of A. Let $\bar{i_I} : I \to A$ be an inclusion mapping. For all $x \in P$,

$\mu_A(i_I(x)) = \mu_A(x) = \mu_I(x)$ and $\nu_A(i_I(x)) = \nu_A(x) = \nu_I(x)$.

Hence, $\bar{i_I} : I \to A$ is a strong IF R-homomorphism. Therefore, $\bar{i_I} : I \to A$ is an IFSM of A.

Now, we want to prove that I is an IF-inverse image of A.

Define an IF R-homomorphism $\bar{\tau_1} : I \to C$ as follows:

Since σ is injective , $x \in P$ implies that there is a unique $k \in K$ such that $f(x) = \sigma(k)$.

Define $\tau_1 : P \to K$ as

$$\tau_1(x) = k \text{ if } f(x) = \sigma(k)$$

Consider

$$\begin{aligned} \sigma(\tau_1(x)) &= \sigma(k), iff(x) = \sigma(k) \\ &= f(x) \\ &= f(i_p(x)). \end{aligned}$$

Thus, $\sigma \circ \tau_1 = f \circ i_P$.

For $x \in M$, $\mu_I(x) = \mu_A(x) \leq \mu_B(\sigma(k)) = \mu_C(k) = \mu_C(\tau_1(x))$.

Consequently, $\mu_C(\tau_1(x)) \geq \mu_I(x)$. Similarly, $\nu_C(\tau_1(x)) \leq \mu_I(x)$. Hence, $\bar{\tau_1} : I \to C$ is an IF R-homomorphism satisfying $\bar{\sigma} \circ \bar{\tau_1} = \bar{f} \circ \bar{i_P}$.

Next, we have to verify the universal property.

Suppose there exists IF R-homomorphisms $\bar{\delta_1} : J \to C$ and $\bar{\delta_2} : J \to A$ such that $\bar{\sigma} \circ \bar{\delta_1} = \bar{f} \circ \bar{\delta_2}$, where J is an IFSM of R-module Q.

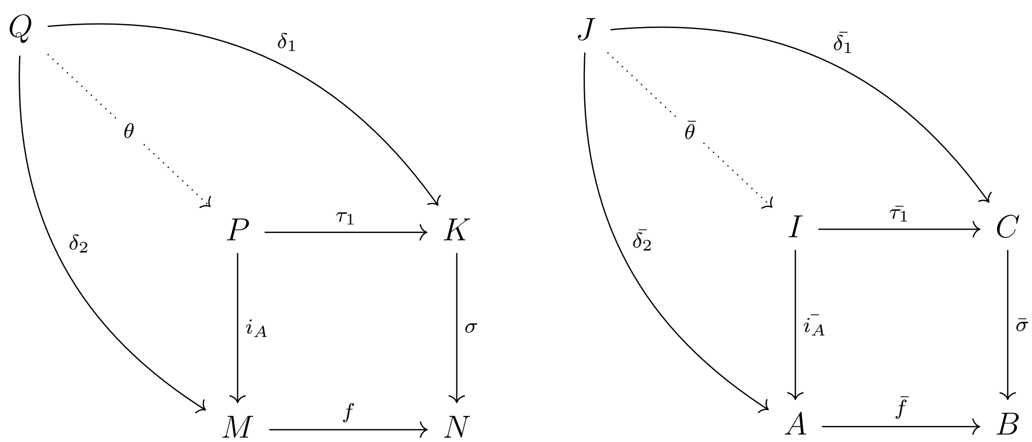
Now, we define $\bar{\theta} : J \to I$ as follows:

Let $t \in Q$. Then $\delta_2(t) \in M$ and $\delta_1(t) \in K$ such that $f(\delta_2(t)) = \sigma(\delta_1(t)) \in \sigma(K)$ which implies $\delta_2(t) \in P$ and $\tau_1(\delta_2(t)) = \delta_1(t)$. Define $\theta : Q \to P$ as $\theta(t) = \delta_2(t)$. Since, category $\mathbf{C}_{\textbf{R-M}}$ has inverse image. As a result, $\theta : Q \to P$ is unique R-homomorphism which satisfies $\tau_1 \circ \theta = \delta_1$ and $\tau_1 \circ \theta = \delta_1$.

We only need to show that $\bar{\theta}$ is an IF R-homomorphism. For $t \in Q$

$$
\begin{aligned}
\mu_J(t) &\leq \mu_A(\delta_2 t) \\
&= \mu_I(i_I(\delta_2 t)) \text{ (Since } i_I \text{ is an IF } R\text{-homomorphism)} \\
&= \mu_I(\delta_2(t)) \\
&= \mu_I(\theta(t)).
\end{aligned}
$$

Hence, $\mu_I(\theta(t)) \geq \mu_J(t)$. Likewise, we are able to show that $\nu_I(\theta(t)) \leq \nu_J(t)$.



Then $\bar{\theta} : J \to I$ is a unique IF R-homomorphism that satisfies $\bar{\tau_1} \circ \bar{\theta} = \bar{\delta}_1$ and $\bar{\tau}_1 \circ \bar{\theta} = \bar{\delta}_1$.

Hence, $\bar{i}_I : I \to A$ is an IF-inverse image of A. □

Remark 4.4.5.

(i) In any category, any two images/inverse-images are isomorphic.

(ii) IF-inverse image of IFSM $\bar{i}_C : C \to B$ by $\bar{f} : A \to B$ can be taken as $f^{-1}(C)$ whose membership and non-membership functions are inclusion mappings.

4.5 Some implications

We now show how IF-equalizer, IF-intersection, IF-monomorphism, and IF-inverse image relate to IF-pullback.

Proposition 4.5.1. *Consider the following square-1*

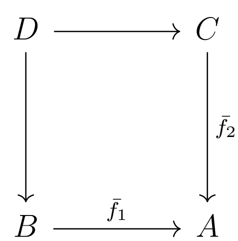


Figure 4.7: Square-1

where $\bar{f}_2$ is strong IF-monomorphism then the above square is IF-pullback if and only if $D = (\bar{f}_1)^{-1}(C)$.

Proof. From the definitions of IF-inverse image and IF-pullbacks, the result follows straightforwardly. □

Example 4.5.2. Let $M = (Z_6, +_6)$ be Z-module, $N = (Z_4, +_4)$ be Z-module and $K =$

$(\{0,3\},+_6)$ be submodule of M. Define IFSs A, B and C on M, N and K respectively as:

$$\mu_A(x)=\begin{cases}1, & \text{if } x=0\\ 0.3, & \text{if } x=2,4\\ 0.2, & \text{if } x=1,3,5\end{cases}\quad;\quad \nu_A(x)=\begin{cases}0, & \text{if } x=0\\ 0.5, & \text{if } x=1,3,5\\ 0.6, & \text{if } x=2,4.\end{cases}$$

$$\mu_B(y)=\begin{cases}1, & \text{if } y=0\\ 0.6, & \text{if } y=2\\ 0.3, & \text{if } y=1,3\end{cases}\quad;\quad \nu_B(y)=\begin{cases}0, & \text{if } y=0\\ 0.3, & \text{if } y=2\\ 0.6, & \text{if } y=1,3.\end{cases}$$

and $\mu_C=\mu_A|_K$ and $\nu_C=\nu_A|_K$. It is easy to verify that A, B and C are IFSMs.

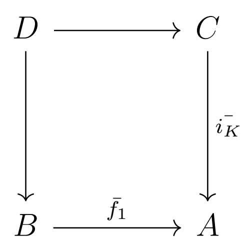


Figure 4.8: Square-2

Define $f_1 : N \to M$ as $f_1(0)=f_1(2)=0$ and $f_1(1)=f_1(3)=1$ and $i_K : K \to M$ as inclusion mapping.

Therefore, $\bar{i}_K$ is a strong IF-monomorphism.

Suppose that $P=(f_1)^{-1}(K)=N$. The IF-inverse image of $\bar{f}_1$ is then $D=(\bar{f}_1)^{-1}(C)$ and hence it represents the IF-pullback of $\bar{f}_1$ and $\bar{i}_K$.

Theorem 4.5.3. *In* **$C_{R\text{-}IFM}$**, *IF-pullbacks exists if and only if IF-equalizers exists.*

Proof. Firstly, suppose that IF-pullbacks exists in **$C_{R\text{-}IFM}$**.

Let $\bar{f}, \bar{g} : A \to B$ be IF R-homomorphisms. Define

$$\bar{\delta_1} : M \to N \times N \text{ as } \delta_1(x) = (f(x), g(x))$$

and

$$\bar{\delta_2} : N \to N \times N \text{ as } \delta_2(y) = (y, y).$$

Our aim to claim that $\bar{\delta_1}$ and $\bar{\delta_2}$ are IF R-homomorphisms.

$\mu_{B\times B}(\delta_1(x)) = \mu_{B\times B}(f(x), g(x)) \geq (\mu_B(f(x)) \wedge \mu_B(g(x))) \geq (\mu_A(x) \wedge \mu_A(x)) = \mu_A(x)$.

Thus, $\mu_{B\times B}(\delta_1(x)) \geq \mu_A(x)$. Likewise, we are able to show that $\nu_{B\times B}(\delta_1(x)) \leq \nu_A(x)$.

Consequently, $\bar{\delta_1}$ is an IF R-homomorphism. In similar argument, $\bar{\delta_2}$ is an IF R-homomorphism.

Let the triplet $(D, \bar{\sigma}_1, \bar{\tau}_1)$ be IF-pullback with $D = (\mu_D, \nu_D)_P$, $\bar{\sigma}_1 \in$ $\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(D, A)$ and $\bar{\tau}_1 \in$ $\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(D, B)$ such that the following square-3 commutes and satisfies

$$\bar{\delta_1} \circ \bar{\sigma}_1 = \quad \bar{\delta_2} \circ \bar{\tau}_1 \tag{4.5.1}$$

We now want to prove that D is an IF-equalizer. Consider the following figure-10:

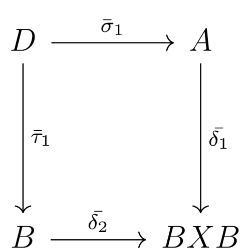


Figure 4.9: Square-3

First, we will prove that $\bar{f} \circ \bar{\sigma}_1 = \bar{g} \circ \bar{\sigma}_1$. Let $\{\bar{\rho}_i : B \times B \to B; i = 1, 2\}$ are the IF-projective mappings. From the square-4, we have

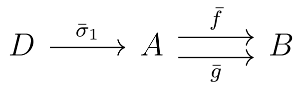


Figure 4.10: D as IF-equalizer

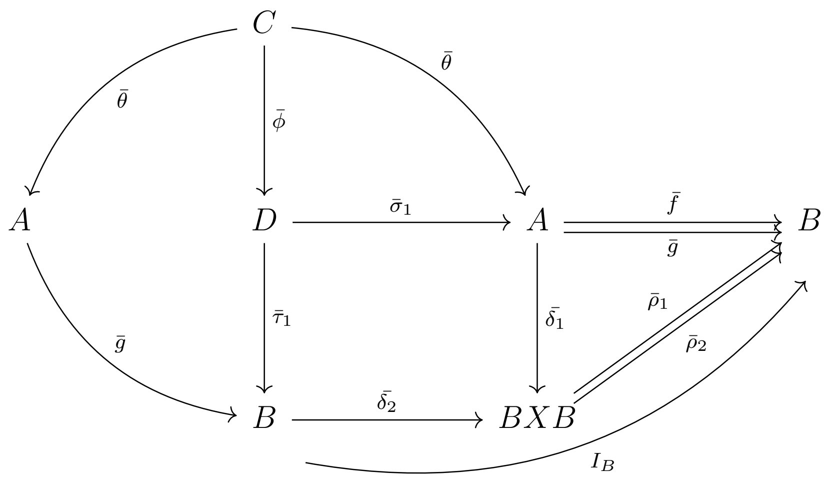


Figure 4.11: Square-4

$$\left.\begin{aligned} \bar{\rho}_1 \circ \bar{\delta}_1 &= \bar{f} \\ \bar{\rho}_2 \circ \bar{\delta}_1 &= \bar{g} \\ \bar{\rho}_1 \circ \bar{\delta}_2 &= I_B \\ \bar{\rho}_2 \circ \bar{\delta}_2 &= I_B \end{aligned}\right\} \tag{4.5.2}$$

By the associativity property of $\mathbf{C_{R\text{-}IFM}}$ and equation (4.1) and (4.2) holds , we have

$\bar{f} \circ \bar{\sigma}_1 = \bar{\rho}_1 \circ \bar{\delta}_1 \circ \bar{\sigma}_1 = \bar{\rho}_1 \circ \bar{\delta}_2 \circ \bar{\tau}_1 = I_B \circ \bar{\tau}_1 = \bar{\tau}_1$ and

$\bar{g} \circ \bar{\sigma}_1 = \bar{\rho}_2 \circ \bar{\sigma} \circ \bar{\sigma}_1 = \bar{\rho}_2 \circ \bar{\tau} \circ \bar{\tau}_1 = I_B \circ \bar{\tau}_1 = \bar{\tau}_1$.

Consequently, we have $\bar{f} \circ \bar{\sigma}_1 = \bar{g} \circ \bar{\sigma}_1$.

Next, we have to verify the universal property.

Let $C = (\mu_C, \nu_C)_K$ be an IFSM and $\bar{\theta} : C \rightarrow A$ be an IF R-homomorphism such that

$$\bar{f} \circ \bar{\theta} = \bar{g} \circ \bar{\theta}.$$

Consider $\bar{\rho}_1 \circ \bar{\delta}_1 \circ \bar{\theta} = \bar{f} \circ \bar{\theta} = \bar{g} \circ \bar{\theta} = I_B \circ \bar{g} \circ \bar{\theta} = \bar{\rho}_1 \circ \bar{\delta}_2 \circ \bar{g} \circ \bar{\theta}$ and $\bar{\rho}_2 \circ \bar{\delta}_1 \circ \bar{\theta} = \bar{g} \circ \bar{\theta} = I_B \circ \bar{g} \circ \bar{\theta} = \bar{\rho}_2 \circ \bar{\delta}_2 \circ \bar{g} \circ \bar{\theta}$. As $\{\bar{\rho}_i | i = 1, 2\}$ are projection mappings. Thus, $\bar{\delta}_1 \circ \bar{\theta} = \bar{\delta}_2 \circ \bar{g} \circ \bar{\theta}$. Since D is the IF-pullback of $\bar{\sigma}_1$ and $\bar{\tau}_1$, there exists a unique IF R-homomorphism $\bar{\phi} : C \to D$ such that $\bar{\theta} = \bar{\sigma}_1 \circ \bar{\phi}$ and which further establishes the universal property of IF-equalizer. Hence, IF-equalizers exists in $\mathbf{C_{R\text{-}IFM}}$.

Conversely, suppose that IF-equalizers exist in $\mathbf{C_{R\text{-}IFM}}$. Let $\bar{f} : A \to C$ and $\bar{g} : B \to C$ are IF R-homomorphisms and let $\bar{t}_1 : A \times B \to A$ and $\bar{t}_2 : A \times B \to B$ are the IF-projective mappings. Consider an IF R-homomorphism $\bar{e} : E \to A \times B$ with an IF-equalizer $E = (\mu_E, \nu_E)_P$ for IF R-homomorphisms $\bar{f} \circ \bar{t}_1, \bar{g} \circ \bar{t}_2 : A \times B \to C$ that satisfies $(\bar{f} \circ \bar{t}_1) \circ \bar{e} = (\bar{g} \circ \bar{t}_2) \circ \bar{e}$ (refer to figure-12).

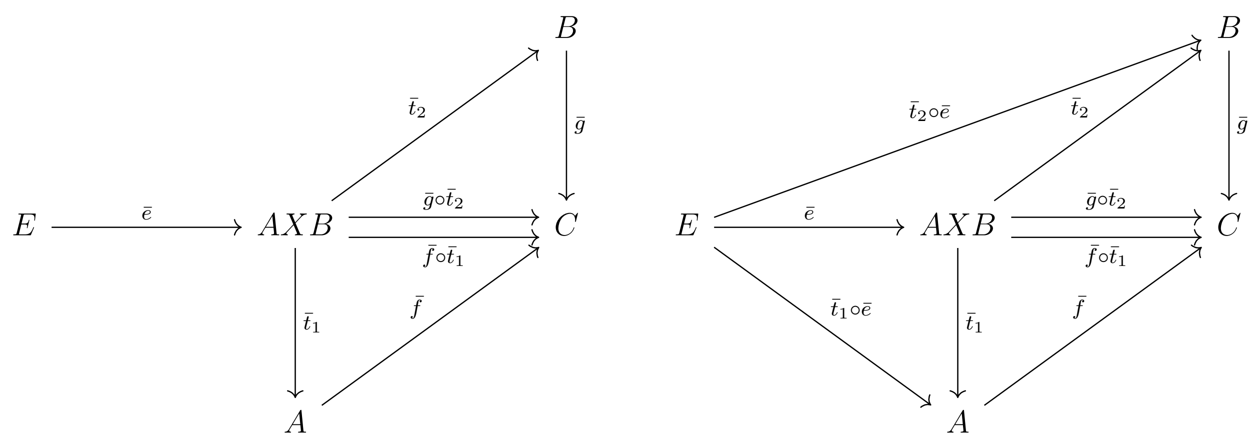


Figure 4.12: IF-equalizer-E

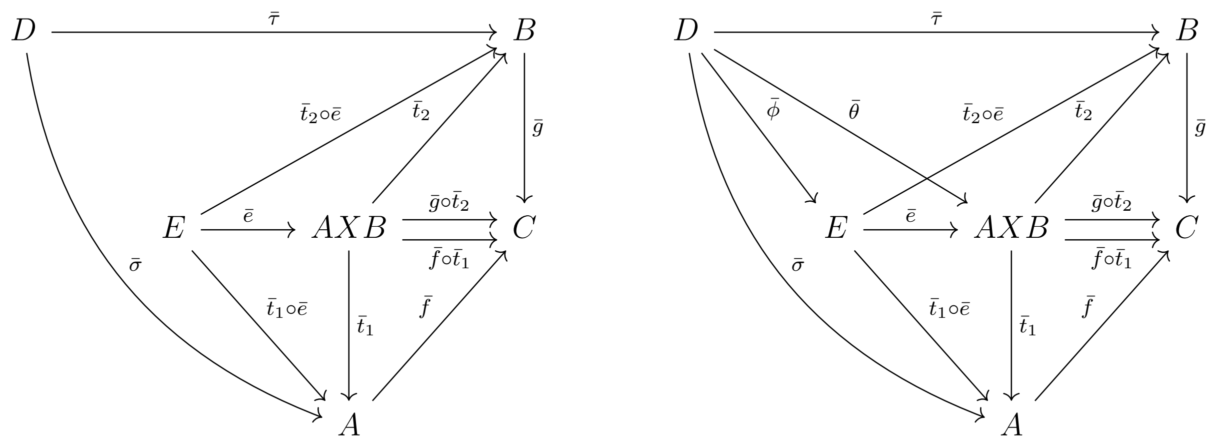
We want to claim that $(E, \bar{t}_1 \circ \bar{e}, \bar{t}_2 \circ \bar{e})$ is an IF-pullback for the IF R-homomorphisms $\bar{f}$ and $\bar{g}$. By associativity, we have

$$\bar{f} \circ (\bar{t}_1 \circ \bar{e}) = \bar{g} \circ (\bar{t}_2 \circ \bar{e}) \tag{4.5.3}$$

Then, first condition of IF-pullback satisfied.

<u>For Universal mapping</u>

Let $D = (\mu_D, \nu_D)_Q \in Ob(\mathbf{C_{R\text{-}IFM}})$, $\bar{\sigma} \in \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(D, A)$ and $\bar{\tau} \in \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(D, B)$ satisfying $\bar{f} \circ \bar{\sigma} = \bar{g} \circ \bar{\tau}$, i.e. the following square-5 commutes. By uniqueness property of projection



(a) Square-5 (b) Square-6

maps and theorem (4.2.1), a unique IF R-homomorphism $\bar{\xi} : D \to A \times B$ sexists that satisfies $\bar{t}_1 \circ \bar{\xi} = \bar{\sigma}$ and $\bar{t}_2 \circ \bar{\xi} = \bar{\tau}$.

Consider $(\bar{f} \circ \bar{t}_1) \circ \bar{\xi} = \bar{f} \circ (\bar{t}_1 \circ \bar{\xi}) = \bar{f} \circ \bar{\sigma} = \bar{g} \circ \bar{\tau} = \bar{g} \circ (\bar{t}_2 \circ \bar{\xi}) = (\bar{g} \circ \bar{t}_2) \circ \bar{\xi}$. By uniqueness of IF-equalizer, a unique IF R-homomorphism $\bar{\phi} : D \to E$ exists that satisfies $\bar{e} \circ \bar{\phi} = \bar{\xi}$. From this, we can conclude that $(\bar{t}_1 \circ \bar{e}) \circ \bar{\phi} = \bar{\sigma}$ and $(\bar{t}_2 \circ \bar{e}) \circ \bar{\phi} = \bar{\tau}$. Hence, E is IF-pullback. □

Proposition 4.5.4. *Consider the following square-7*

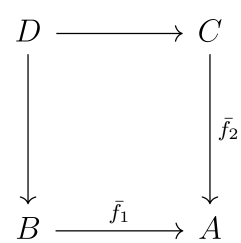


Figure 4.14: Square-7

where $\bar{f}_1$ and $\bar{f}_2$ are strong IF-monomorphisms then the above square is IF-pullback if and only if D is the IF-intersection of B and C.

Proof. Since $\bar{f}_1$ and $\bar{f}_2$ are strong IF-monomorphism then B and C are IFSMs of A. From the definitions of IF-intersection and IF-pullback, it conclude that D is an IF-pullback if and only if D is the IF-intersection of B and C. □

Example 4.5.5. Let $M = (Z_6, +_6)$ be Z-module. Take $N = M$, $K = (\{0, 3\}, +_6)$ and $P = (\{0\}, +_6)$ as submodules of M. Define IFSs A, B, C and D on M, N, K and P respectively as:

$$\mu_A(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0.3, & \text{if } x = 2, 4 \\ 0.2, & \text{if } x = 1, 3, 5 \end{cases} \quad ; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x = 0 \\ 0.5, & \text{if } x = 1, 3, 5 \\ 0.6, & \text{if } x = 2, 4. \end{cases}$$

$\mu_B = \mu_A|_N$ and $\nu_B = \nu_A|_N$, $\mu_C = \mu_A|_K$ and $\nu_C = \nu_A|_K$ and $\mu_D = \mu_A|_P$ and $\nu_D = \nu_A|_P$.

It can be easily verify that B, C and D are IFSMs of M. Then the mappings $\bar{f}_N : B \to A$,

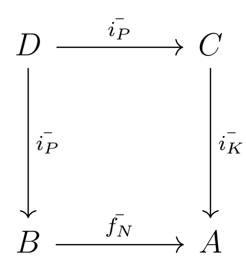


Figure 4.15: Square-8

$\bar{i}_K : C \to A$ are strong IF-monomorphism. Then D is the IF-intersection of B and C and hence the IF-pullback of $\bar{f}_N$ and $\bar{i}_K$.

Proposition 4.5.6. *Consider the IF-pullback diagram*

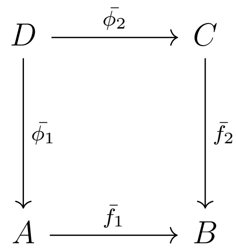


Figure 4.16: Square-9

(i) If $\bar{f}_1$ is an IF monomorphism then $\bar{\phi}_2$ is also an IF monomorphism.

(ii) If $\bar{f}_1$ is an IF-retraction then $\bar{\phi}_2$ is also an IF-retraction.

(iii) If $\bar{f}_1$ is an IF R-isomorphism then $\bar{\phi}_2$ is also an IF R-isomorphism.

(iv) If $\bar{f}_1$ is an IF-equalizer then $\bar{\phi}_2$ is also an IF-equalizer.

Proof. (i) Let $\bar{f}_1 : A \to B$ is an IF-monomorphism. Since the square-9 is an IF-pullback in $\mathbf{C_{R\text{-}IFM}}$, we have

$$\bar{f}_1 \circ \bar{\phi}_1 = \bar{f}_2 \circ \bar{\phi}_2 \tag{4.5.4}$$

Suppose there exists IF R-homomorphisms $\bar{\xi}_1, \bar{\xi}_2 : E \to D$ such that the square-10 is commutative and

$$\bar{\phi}_2 \circ \bar{\xi}_1 = \bar{\phi}_2 \circ \bar{\xi}_2 \tag{4.5.5}$$

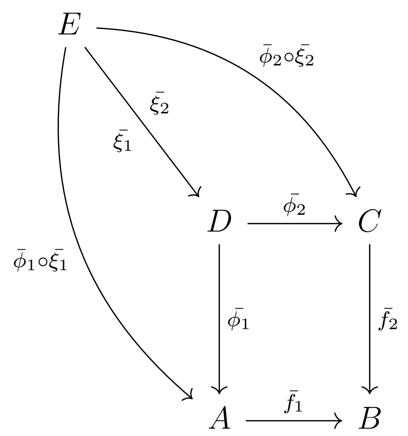


Figure 4.17: Square-10

Then

$$\begin{aligned} \bar{f}_1 \circ (\bar{\phi}_1 \circ \bar{\xi}_1) &= (\bar{f}_1 \circ \bar{\phi}_1) \circ \bar{\xi}_1 \\ &= (\bar{f}_2 \circ \bar{\phi}_2) \circ \bar{\xi}_1 \\ &= \bar{f}_2 \circ (\bar{\phi}_2 \circ \bar{\xi}_1). \end{aligned}$$

using (4.5.5) implies

$$\bar{f}_1 \circ (\bar{\phi}_1 \circ \bar{\xi}_1) = \bar{f}_2 \circ (\bar{\phi}_2 \circ \bar{\xi}_2) \quad (4.5.6)$$

using (4.5.4) implies

$$\bar{f}_1 \circ (\bar{\phi}_1 \circ \bar{\xi}_1) = \bar{f}_1 \circ (\bar{\phi}_1 \circ \bar{\xi}_2) \quad (4.5.7)$$

Since $\bar{f}_1 : A \to B$ is an IF-monomorphism, then

$$\bar{\phi}_1 \circ \bar{\xi}_1 = \bar{\phi}_1 \circ \bar{\xi}_2 \quad (4.5.8)$$

Since Square-10 is IF-pullback and equation(4.5.6) holds in $\mathbf{C_{R\text{-}IFM}}$, a unique IF R-homomorphism

$\bar{h} : E \to D$ exists that satisfies

$$\left.\begin{aligned} \bar{\phi}_1 \circ \bar{h} = & \quad \bar{\phi}_1 \circ \bar{\xi}_1 \\ \bar{\phi}_2 \circ \bar{h} = & \quad \bar{\phi}_2 \circ \bar{\xi}_2 \end{aligned}\right\} \tag{4.5.9}$$

From equations (4.5.5) and (4.5.8), concludes that $\bar{h} = \bar{\xi}_1$ and $\bar{h} = \bar{\xi}_2$ both satisfying (4.9).

By uniqueness of IF-pullback in $\mathbf{C_{R\text{-}IFM}}$, $\bar{\xi}_1 = \bar{\xi}_2$. Hence, $\bar{\phi}_2$ is IF-monomorphism.

(ii) Let $\bar{f}_1 : A \to B$ is an IF-retraction. So, an IF R-homomorphism $\bar{g}_1 : B \to A$ exists that satisfy

$$\bar{f}_1 \circ \bar{g}_1 = \quad I_A. \tag{4.5.10}$$

For $\bar{g}_1 \circ \bar{f}_2 : C \to A$,

$$\begin{aligned} \bar{f}_1 \circ (\bar{g}_1 \circ \bar{f}_2) &= (\bar{f}_1 \circ \bar{g}_1) \circ \bar{f}_2 \\ &= I_A \circ \bar{f}_2 \\ &= \bar{f}_2 \\ &= \bar{f}_2 \circ I_C. \end{aligned}$$

Since the Square-11 is IF-pullback, a unique IF R-homomorphism $\bar{h} : C \to D$ exists which satisfy

$$\left.\begin{aligned} \bar{\phi}_1 \circ \bar{h} = & \quad \bar{g}_1 \circ \bar{f}_2 \\ \bar{\phi}_2 \circ \bar{h} = & \quad I_C \end{aligned}\right\} \tag{4.5.11}$$

Thus, $\bar{\phi}_2$ is IF-retraction in $\mathbf{C_{R\text{-}IFM}}$.

(iii) Let $\bar{f}_1 : A \to B$ is an IF R-isomorphism, which indicates both IF-coretraction and

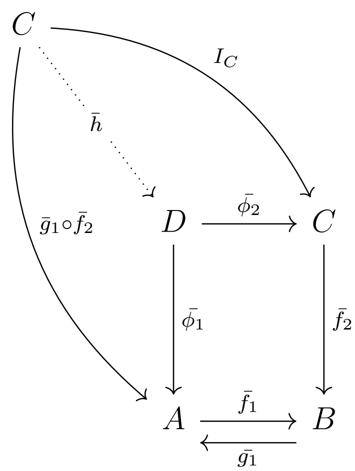


Figure 4.18: Square-11

IF-retraction. Since every IF-coretraction is IF-monomorphism, $\bar{f}_1$ is IF-monomorphism and IF-retraction. $\bar{\phi}_2$ is both IF-monomorphism and IF-retraction by (i) and (ii). $\bar{\phi}_2$ is both IF-monomorphism and IF-epimorphism since every IF-retraction is IF-epimorphism. Hence, $\bar{\phi}_2$ is IF R-isomorphism in $\mathbf{C_{R\text{-}IFM}}$.

(iv) Let $\bar{f}_1 : A \to B$ is an IF-equalizer of $\bar{\xi}_1, \bar{\xi}_2 : B \to E$.

$$\bar{\xi}_1 \circ \bar{f}_1 = \quad \bar{\xi}_2 \circ \bar{f}_1 \tag{4.5.12}$$

We want to claim that $\bar{\phi}_2$ is IF-equalizer for $\bar{\xi}_1 \circ \bar{f}_2$ and $\bar{\xi}_2 \circ \bar{f}_2$.

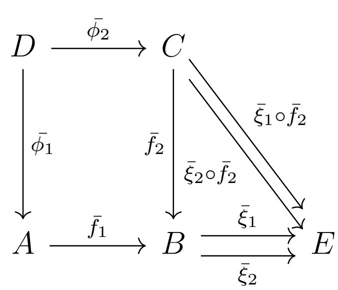


Figure 4.19: Square-12

Consider

$$\begin{aligned}(\bar{\xi_1} \circ \bar{f_2}) \circ \bar{\phi_2} &= \bar{\xi_1} \circ (\bar{f_2} \circ \bar{\phi_2}) \\ &= \bar{\xi_1} \circ (\bar{f_1} \circ \bar{\phi_1}) \\ &= (\bar{\xi_1} \circ \bar{f_1}) \circ \bar{\phi_1} \\ &= (\bar{\xi_2} \circ \bar{f_1}) \circ \bar{\phi_1} \\ &= \bar{\xi_2} \circ (\bar{f_1} \circ \bar{\phi_1}) \\ &= \bar{\xi_2} \circ (\bar{f_2} \circ \bar{\phi_2}).\end{aligned}$$

Thus

$$(\bar{\xi_1} \circ \bar{f_2}) \circ \bar{\phi_2} = (\bar{\xi_2} \circ \bar{f_2}) \circ \bar{\phi_2}. \tag{4.5.13}$$

which proves the first condition of IF-equalizer.

For Uniqueness, suppose $\bar{s} : F \to C$ such that

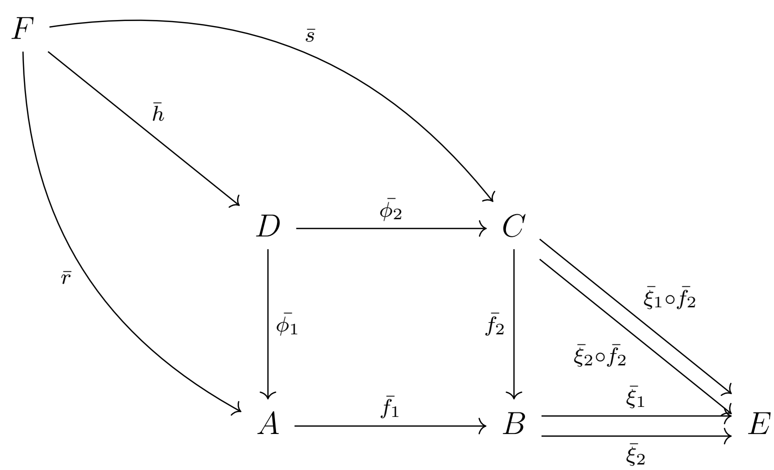


Figure 4.20: Square-13

$$(\bar{\xi}_1 \circ \bar{f}_2) \circ \bar{s} = (\bar{\xi}_2 \circ \bar{f}_2) \circ \bar{s} \tag{4.5.14}$$

$$\bar{\xi}_1 \circ (\bar{f}_2 \circ \bar{s}) = \bar{\xi}_2 \circ (\bar{f}_2 \circ \bar{s}) \tag{4.5.15}$$

By the universal mapping property of IF-equalizer, $\bar{r} : F \to A$ a unique IF R-homomorphism exists such that

$$\bar{f}_1 \circ \bar{r} = \bar{f}_2 \circ \bar{s} \tag{4.5.16}$$

Since square-13 is IF-pullback and equation(4.14) holds, a unique IF R-homomorphism $\bar{h}$: $F \to D$ exists such that

$$\bar{\phi}_2 \circ \bar{h} = \bar{s} \tag{4.5.17}$$

which proves the uniqueness. Hence, $\bar{\phi}_2$ is an IF-equalizer for $\bar{\xi}_1 \circ \bar{f}_2$ and $\bar{\xi}_2 \circ \bar{f}_2$. □

Proposition 4.5.7. *Consider the square-14*

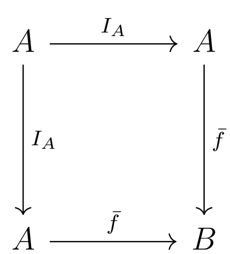


Figure 4.21: Square-14

An IF R-homomorphism $\bar{f} : A \to B$ is an IF-monomorphism if and only if the square-14 is a IF-pullback.

Proof. Firstly, let $\bar{f} : A \to B$ is an IF-monomorphism. Clearly, $\bar{f} \circ I_A = \bar{f} \circ I_A$. Thus, square-14 is commutative which implies that the first condition of IF-pullback holds.

For Universal mapping

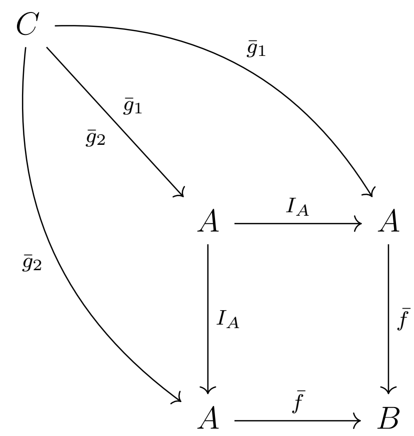


Figure 4.22: Square-15

Let $(G, \bar{g}_1, \bar{g}_2)$ is a triplet with $C = (\mu_C, \nu_C)_K, \bar{g}_1, \bar{g}_2 \in \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(C, A)$ satisfying

$$\bar{f} \circ \bar{g}_1 = \bar{f} \circ \bar{g}_2.$$

Since $\bar{f}$ is an IF-monomorphism. Thus

$$\bar{g}_1 = \bar{g}_2 \tag{4.5.18}$$

It is simple to verify from Square-15 that

$$I_A \circ \bar{g}_1 = \bar{g}_1 \text{ and } I_A \circ \bar{g}_2 = \bar{g}_2$$

where I_A is the identity IF R-homomorphism from A to A. From (4.5.18), it proves that $\bar{g}_1$ is a unique IF R-homomorphism from C to A. Thus the square-14 is IF-pullback.

Conversely, let the given square-14 be IF-pullback. Suppose there exists IF R-homomorphism $\bar{g}_1, \bar{g}_2 : C \to A$ that satisfies

$$\bar{f} \circ \bar{g}_1 = \bar{f} \circ \bar{g}_2.$$

Then, a unique IF R-homomorphism $\bar{h} : C \to A$ that satisfies

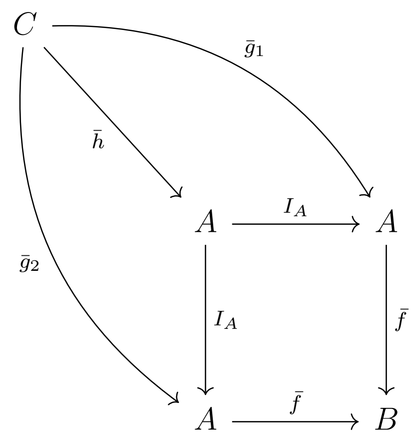


Figure 4.23: Square-16

$$\begin{aligned} I_A \circ \bar{h} &= \bar{g}_1 \\ \text{and } I_A \circ \bar{h} &= \bar{g}_2 \\ \Rightarrow \bar{g}_1 &= \bar{g}_2. \end{aligned}$$

Hence, $\bar{f}$ is an IF-monomorphism. □

Proposition 4.5.8. *Let the right square-II in the following commutative diagram be IF-pullback. Then the outer rectangle is IF-pullback if and only if the left square-I is IF-pullback.*

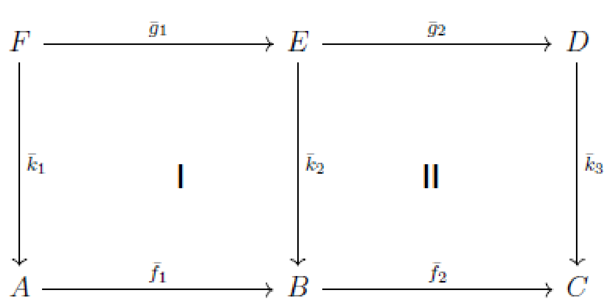


Figure 4.24: rectangle

Proof. Firstly, let the left square-I is IF-pullback. We want to prove that outer rectangle is IF-pullback. The commutativity of Figure-4.24 makes it possible to obtain

$$\bar{k}_3 \circ \bar{g}_2 \circ \bar{g}_1 \quad = \quad \bar{f}_2 \circ \bar{f}_1 \circ \bar{k}_1.$$

Consequently, in the outer rectangle, the first IF-pullback condition is satisfied.

For Universal mapping

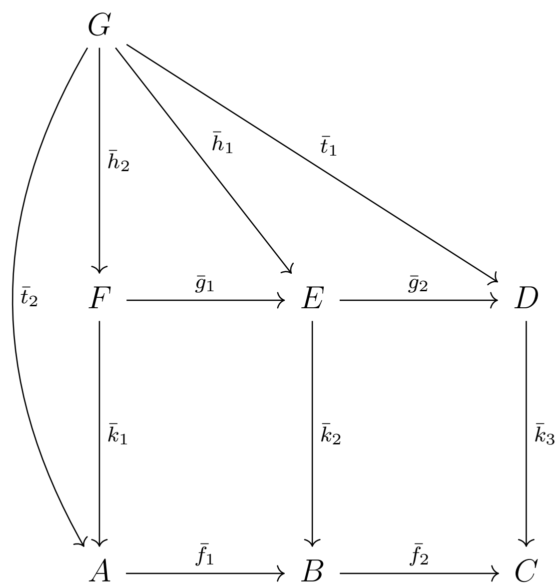


Figure 4.25: Square-17

Let $(G, \bar{t}_1, \bar{t}_2)$ is a triplet with $G = (\mu_G, \nu_G)_T$, $\bar{t}_1 \in \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(G, D)$ and $\bar{t}_2 \in \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(G, A)$ satisfying

$$\bar{k}_3 \circ \bar{t}_1 \quad = \quad (\bar{f}_2 \circ \bar{f}_1) \circ \bar{t}_2$$
$$\Rightarrow \bar{k}_3 \circ \bar{t}_1 \quad = \quad \bar{f}_2 \circ (\bar{f}_1 \circ \bar{t}_2)$$

As the right square-II is IF-pullback, then $\bar{h}_1 : G \to E$ is a unique IF R-homomorphism such that

$$\left.\begin{aligned} \bar{g}_2 \circ \bar{h}_1 = &\quad \bar{t}_1 \\ \bar{k}_2 \circ \bar{h}_1 = &\quad \bar{f}_1 \circ \bar{t}_2 \end{aligned}\right\} \tag{4.5.19}$$

Since the right square-I is IF-pullback, then $\bar{h}_2 : G \to F$ is a unique IF R-homomorphism such that

$$\left.\begin{aligned} \bar{g}_1 \circ \bar{h}_2 = &\quad \bar{h}_1 \\ \bar{k}_1 \circ \bar{h}_2 = &\quad \bar{t}_2 \end{aligned}\right\} \tag{4.5.20}$$

We can determine that

$$\begin{aligned} \bar{g}_2 \circ \bar{g}_1 \circ \bar{h}_2 &= \bar{g}_2 \circ \bar{h}_1 \text{(by (4.20))} \\ &= \bar{t}_1 \text{(by (4.5.19))} \end{aligned}$$

As the consequence, $\bar{h}_2 : G \to F$ is a unique IF R-homomorphism satisfying

$$\bar{k}_1 \circ \bar{h}_2 = \bar{t}_2 \text{ and } \bar{g}_2 \circ \bar{g}_1 \circ \bar{h}_2 = \bar{t}_1.$$

Thus, the outer rectangle is IF-pullback.

Conversely, let us consider that the outer rectangle is IF-pullback. Moreover, the right square-II is assumed to be IF-pullback.

We now want to prove that the left square-I is IF-pullback. The commutativity of Figure-4.24 makes it possible to derive

$$\bar{f}_1 \circ \bar{k}_1 = \bar{k}_2 \circ \bar{g}_1 \tag{4.5.21}$$

which shows that the first condition of IF-pullback in left square-I.

<u>For Universal mapping</u>

Let $(H, \bar{s}_1, \bar{s}_2)$ be a triplet with $H = (\mu_H, \nu_H)_T$, $\bar{s}_1 \in \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(H, E)$ and $\bar{s}_2 \in \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(H, A)$ satisfying

$$\bar{k}_2 \circ \bar{s}_1 = \bar{f}_1 \circ \bar{s}_2 \tag{4.5.22}$$

Consider

$$\begin{aligned}(\bar{k}_3 \circ \bar{g}_2) \circ \bar{s}_1 &= (\bar{f}_2 \circ \bar{k}_2) \circ \bar{s}_1 \\ &= \bar{f}_2 \circ (\bar{k}_2 \circ \bar{s}_1) \\ &= \bar{f}_2 \circ (\bar{f}_1 \circ \bar{s}_2) \\ &= (\bar{f}_2 \circ \bar{f}_1) \circ \bar{s}_2\end{aligned}$$

Consequently, we obtain

$$\bar{k}_3 \circ (\bar{g}_2 \circ \bar{s}_1) = (\bar{f}_2 \circ \bar{f}_1) \circ \bar{s}_2 \tag{4.5.23}$$

The outer rectangle being IF-pullback implies that a unique IF R-homomorphism $\bar{h} : H \to F$ exists satisfying

$$(\bar{g}_2 \circ \bar{g}_1) \circ \bar{h} = \bar{g}_2 \circ \bar{s}_1 \tag{4.5.24}$$

$$\bar{k}_1 \circ \bar{h} = \bar{s}_2 \tag{4.5.25}$$

Since both $\bar{g}_1 \circ \bar{h}, \bar{s}_1 : H \to E$ satisfy $\bar{g}_2 \circ (\bar{g}_1 \circ \bar{h}) = \bar{g}_2 \circ \bar{s}_1$ and given that the right square-II forms an IF-pullback, according to uniqueness, we obtain

$$\bar{g}_1 \circ \bar{h} = \bar{s}_1. \tag{4.5.26}$$

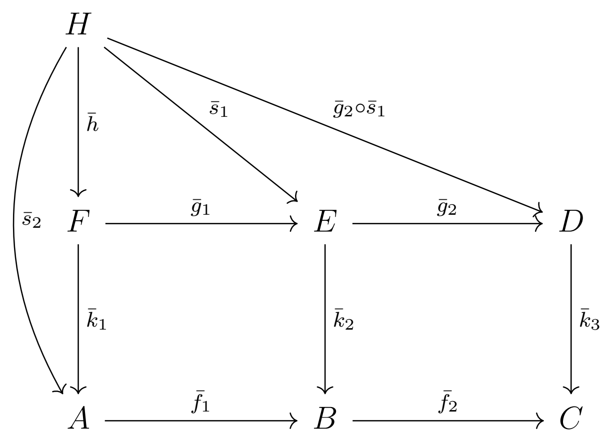


Figure 4.26: Square-18

As a result of (4.5.25) and (4.5.26), we are able to determine that $\bar{h} \in$ $\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(H, F)$ is a unique satisfying

$$\bar{k}_1 \circ \bar{h} = \bar{s}_2 \text{ and } \bar{g}_1 \circ \bar{h} = \bar{s}_1.$$

Hence, the left square-I is an IF-pullback. □

Chapter 5

Some functors in the category $\mathbf{C_{R\text{-}IFM}}$

5.1 Introduction

Hom-functors for intuitionistic fuzzy modules extend traditional module theory to account for uncertainty. They capture morphisms between modules, considering both membership and non-membership degrees. This abstraction facilitates a formalized understanding of structure-preserving transformations in scenarios of ambiguity. Pan [31, 32] gave fuzzy module $Hom(\mu_A, \nu_B)$ and examined the functors $Hom(\mu_A, -)$ and $Hom(-, \nu_A)$. Properties of the two functors $Hom(\mu_A, -)$, $Hom(-, \nu_A)$ in the fuzzy module category explored by Liu in [26]. Additionally researcher studied the connection between fuzzy projective module and Hom-functor and also between the Hom-functors and tensor product functors. Rana [35] studied the functors associated with fuzzy modules. For fuzzy module categories, Permouth [33] examined Morita theory and provided a definition for tensor products. In this Chapter, we investigate some functors in the category $\mathbf{C_{R\text{-}IFM}}$. For a commutative ring R, we present the concept of Hom functors- $\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A, -)$ and $\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(-, A)$ in the category $C_{\mathbf{R\text{-}IFM}}$ and investigate their properties. We elucidate completely the characterization of intuitionistic fuzzy projective modules through Hom functor and show that an IFSM A is projective if and only if the functor $\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A,-)$ preserves the short exact sequence $\bar{0} \rightarrow A \overset{\bar{f}}{\rightarrow} B \overset{\bar{g}}{\rightarrow} C \rightarrow \bar{0}$ in $C_{\mathbf{R\text{-}IFM}}$. Furthermore, we investigate the functor $\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(\bar{O}_{Re},-)$ by defining an intuitionistic fuzzy R-homomorphism $\Gamma_A : \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(\bar{O}_{Re},A) \rightarrow eA$, where e is an idempotent element of the semi-perfect commutative ring R. Also, we analyse the existence of the tensor product of two IFSMs. Finally, we investigate the association between Hom-functor and tensor functor in the category $C_{\mathbf{R\text{-}IFM}}$.

5.2 Hom-functors in $\mathbf{C_{R\text{-}IFM}}$

In this section, we study Hom functors- $\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A,-)$ and $\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(-,A)$ associated with the category $\mathbf{C_{R\text{-}IFM}}$ of intuitionistic fuzzy modules.

Lemma 5.2.1. *For a fixed IFSM A, an IF R-homomorphism $\bar{g} : B \rightarrow C$ induces*

a) an IF R-homomorphism $\bar{g}_* : \boldsymbol{Hom}_{C_{R\text{-}IFM}}(A,B) \rightarrow \boldsymbol{Hom}_{C_{R\text{-}IFM}}(A,C)$ defined by

$$\bar{g}_*(\bar{f}) = \bar{g} \circ \bar{f} \ \ \forall \bar{f} \in \boldsymbol{Hom}_{C_{R\text{-}IFM}}(\mathrm{A},\mathrm{B}).$$

b) an IF R-homomorphism $\bar{g}^* : \boldsymbol{Hom}_{C_{R\text{-}IFM}}(C,A) \rightarrow \boldsymbol{Hom}_{C_{R\text{-}IFM}}(B,A)$ defined as

$$\bar{g}^*(\bar{\phi}) = \bar{\phi} \circ \bar{g} \ \ \forall \ \bar{\phi} \in \boldsymbol{Hom}_{C_{R\text{-}IFM}}(\mathrm{C},\mathrm{A}).$$

Proof. Let A and B are IFSM of R-modules M and N respectively and $\bar{f} \in \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A,B)$. According to Theorem(2.2.2), $\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A,B)$ is IFSM of R-module, which is defined by the function $\beta : Hom(A, B) \to I \times I$ as

$$\beta(\bar{f}) = (\mu_{\beta(\bar{f})}, \nu_{\beta(\bar{f})})$$

where $\mu_{\beta(\bar{f})} = \wedge\{\mu_B(\bar{f}(a)) : a \in M\}$ and $\nu_{\beta(\bar{f})} = \vee\{\nu_B(\bar{f}(a)) : a \in M\}$.

(a)

$$\begin{aligned}
\mu_{\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A,C)}(\bar{g}_*(\bar{f})) &= \mu_{\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A,C)}(\bar{g} \circ \bar{f}) \\
&= \wedge\{\mu_C((\bar{g} \circ \bar{f})(a)) : a \in M\} \\
&= \wedge\{\mu_C(\bar{g}(\bar{f}(a))) : a \in M\} \\
&\geq \wedge\{\mu_B(\bar{f}(a)) : a \in M\} \\
&= \mu_{\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A,B)}(\bar{f}) \\
\Rightarrow \mu_{\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A,C)}(\bar{g}_*(\bar{f})) &\geq \mu_{\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A,B)}(\bar{f}).
\end{aligned}$$

Likewise, we can exhibit that

$$\nu_{\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A,C)}(\bar{g}_*(\bar{f})) \leq \nu_{\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A,B)}(\bar{f}).$$

Thus, $\bar{g}_*$ is an IF R-homomorphism.

(b)

$$\mu_{\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(B,A)}(\bar{g}^*(\bar{\phi})) = \mu_{\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(B,A)}(\bar{\phi} \circ \bar{g})$$

$$\begin{aligned}
\therefore \mu_{\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(B,A)}(\bar{g}^*(\bar{\phi})) &= \wedge\{\mu_A((\bar{\phi}\circ\bar{g}))(b)) : b \in N\} \\
&= \wedge\{\mu_A(\bar{\phi}(\bar{g}(b))) : b \in N\} \\
&\geq \wedge\{\mu_A(\bar{\phi}(c)) : c = g^{-1}(b) \in K\} \\
&= \mu_{\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(C,A)}(\bar{\phi}) \\
\Rightarrow \mu_{\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(B,A)}(\bar{g}^*(\bar{\phi})) &\geq \mu_{\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(C,A)}(\bar{\phi}).
\end{aligned}$$

Likewise, we can exhibit that

$$\nu_{\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(B,A)}(\bar{g}^*(\bar{\phi})) \leq \nu_{\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(C,A)}(\bar{\phi}).$$

Hence, $\bar{g}^*$ is an IF R-homomorphism. □

Proposition 5.2.2. *For any $A \in Ob(\boldsymbol{C_{R\text{-}IFM}})$, $\bar{\alpha} \in \boldsymbol{Hom_{C_{R\text{-}IFM}}}(B, C)$ and $\bar{\beta} \in \boldsymbol{Hom_{C_{R\text{-}IFM}}}(C, D)$, there exists an IF R-homomorphism*

a) $(\bar{\beta}\circ\bar{\alpha})_* : \boldsymbol{Hom_{C_{R\text{-}IFM}}}(A, B) \to \boldsymbol{Hom_{C_{R\text{-}IFM}}}(A, D)$ such that

$$(\bar{\beta}\circ\bar{\alpha})_* = \bar{\beta}_* \circ \bar{\alpha}_*$$

b) $(\bar{\beta}\circ\bar{\alpha})^* : \boldsymbol{Hom_{C_{R\text{-}IFM}}}(D, A) \to \boldsymbol{Hom_{C_{R\text{-}IFM}}}(B, A)$ such that

$$(\bar{\beta}\circ\bar{\alpha})^* = \bar{\alpha}^* \circ \bar{\beta}^*$$

Definition 5.2.3. (Covariant Functor $Hom^{\mathbf{A}}$)

For a fixed IFSM A, Let $Hom^{\mathbf{A}} = \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A, -)$ is set of all IF R-homomorphisms from

IFSM A to any other IFSM. Additionally, for each IF R-homomorphism $\bar{g} : B \to C$, let $Hom^{\mathbf{A}}(\bar{g}) = \bar{g}_* : \mathbf{Hom}_{\mathbf{C}_{\textbf{R-IFM}}}(A, B) \to \mathbf{Hom}_{\mathbf{C}_{\textbf{R-IFM}}}(A, C)$ be an IF R-homomorphism defined as

$$\bar{g}_*(\bar{f}) = \bar{g} \circ \bar{f} \;\; \forall \bar{f} \in \mathbf{Hom}_{\mathbf{C}_{\textbf{R-IFM}}}(\mathrm{A}, \mathrm{B}).$$

Lemma [5.2.1] and Proposition [5.2.2] thus make it straightforward to prove that $Hom^{\mathbf{A}} = \mathbf{Hom}_{\mathbf{C}_{\textbf{R-IFM}}}(A, -)$ is a covariant functor.

Proposition 5.2.4. *Let* $\bar{g} \in \boldsymbol{Hom}_{\boldsymbol{C}_{\textbf{\textit{R-IFM}}}}(B, C)$*. Then* $\bar{g}$ *is IF-monomorphism in* $\boldsymbol{C}_{\textbf{\textit{R-IFM}}}$ *if and only if* $Hom^{\boldsymbol{A}}(\bar{g}) = \bar{g}_* : \boldsymbol{Hom}_{\boldsymbol{C}_{\textbf{\textit{R-IFM}}}}(A, B) \to \boldsymbol{Hom}_{\boldsymbol{C}_{\textbf{\textit{R-IFM}}}}(A, C)$ *is IF-monomorphism for each* $A \in ob(\boldsymbol{C}_{\textbf{\textit{R-IFM}}})$*.*

Proof. Let $\bar{g}$ is IF-monomorphism in $\mathbf{C}_{\textbf{R-IFM}}$.

Suppose $\bar{\alpha_1}, \bar{\alpha_2} \in \mathbf{Hom}_{\mathbf{C}_{\textbf{R-IFM}}}(A, B)$ such that $\bar{g}_*(\bar{\alpha_1}) = \bar{g}_*(\bar{\alpha_2})$. From the definition of covariant functor $Hom^{\mathbf{A}}$, we have

$$\begin{aligned} \bar{g} \circ \bar{\alpha_1} &= \bar{g} \circ \bar{\alpha_2} \\ \bar{\alpha_1} &= \bar{\alpha_2} \;\; \text{; Since } \bar{g} \text{ is IF-monomorphism.} \end{aligned}$$

Thus, $Hom^{\mathbf{A}}(\bar{g})$ is IF-monomorphism.

Conversely, suppose that $Hom^{\mathbf{A}}(\bar{g})$ is IF-monomorphism for each $A \in Ob(\mathbf{C}_{\textbf{R-IFM}})$.

Let $\bar{\alpha_1}, \bar{\alpha_2} \in \mathbf{Hom}_{\mathbf{C}_{\textbf{R-IFM}}}(A, B)$ such that $\bar{g} \circ \bar{\alpha_1} = \bar{g} \circ \bar{\alpha_2}$. Then $\bar{g}_*(\bar{\alpha_1}) = \bar{g}_*(\bar{\alpha_2})$. By assumption, $\bar{\alpha_1} = \bar{\alpha_2}$. Thus, $\bar{g}$ is IF-monomorphism in $\mathbf{C}_{\textbf{R-IFM}}$. □

Corollary 5.2.5. $Hom^{\boldsymbol{A}}$ *is a monofunctor for each* $A \in Ob(\boldsymbol{C}_{\textbf{\textit{R-IFM}}})$*.*

Proposition 5.2.6. *Let $\bar{g} \in \mathbf{Hom}_{\mathbf{C}_{\text{R-IFM}}}(B, C)$ be an IF-coretraction in $\mathbf{C}_{\text{R-IFM}}$. Then $Hom^{\mathbf{A}}(\bar{g})$ is IF-coretraction in $\mathbf{C}_{\text{R-IFM}}$ for each $A \in Ob(\mathbf{C}_{\text{R-IFM}})$.*

Proof. Let $\bar{g} : B \to C$ be an IF-coretraction in $\mathbf{C}_{\text{R-IFM}}$. Therefore, an IF R-homomorphism $\bar{\phi} : C \to B$ exists in $\mathbf{C}_{\text{R-IFM}}$ such that $\bar{\phi} \circ \bar{g} = I_B$. Since every IF-coretraction is IF-monomorphism in $\mathbf{C}_{\text{R-IFM}}$, $\bar{g}$ is an IF-monomorphism in $\mathbf{C}_{\text{R-IFM}}$. By proposition [5.2.4], $Hom^{\mathbf{A}}(\bar{g})$ is IF-monomorphism in $\mathbf{C}_{\text{R-IFM}}$. Consider

$$
\begin{aligned}
((\bar{\phi} \circ \bar{g})_*)(\bar{f}) &= (\bar{\phi} \circ \bar{g}) \circ \bar{f} \\
&= (I_B) \circ \bar{f} \\
&= I_B(\bar{f}) \\
Thus, ((\bar{\phi} \circ \bar{g})_*)(\bar{f}) &= I_B(\bar{f})
\end{aligned}
$$

which implies that $(\bar{\phi} \circ \bar{g})_* = I_B$. By proposition 5.2.2(a), $\bar{\phi}_* \circ \bar{g}_* = I_B$. Hence, $Hom^{\mathbf{A}}(\bar{g}) : \mathbf{Hom}_{\mathbf{C}_{\text{R-IFM}}}(A, B) \to \mathbf{Hom}_{\mathbf{C}_{\text{R-IFM}}}(A, C)$ is IF-coretraction for each $A \in Ob(\mathbf{C}_{\text{R-IFM}})$. □

Corollary 5.2.7. *The functor $Hom^{\mathbf{A}}$ preserves coretraction.*

Definition 5.2.8. (Contravariant Functor $\mathbf{Hom}_{\mathbf{A}}$)

For a fixed IFSM A, Let $\mathbf{Hom}_{\mathbf{A}} = \mathbf{Hom}_{\mathbf{C}_{\text{R-IFM}}}(-, A)$ is set of all IF R-homomorphisms from any other IFSM to IFSM A. Additionally, for each IF R-homomorphism $\bar{g} : B \to C$, define an IF R-homomorphism $\mathbf{Hom}_{\mathbf{A}}(\bar{g}) = \bar{g}^* : \mathbf{Hom}_{\mathbf{C}_{\text{R-IFM}}}(C, A) \to \mathbf{Hom}_{\mathbf{C}_{\text{R-IFM}}}(B, A)$ as

$$\bar{g}^*(\bar{\phi}) = \bar{\phi} \circ \bar{g} \;\; \forall \, \bar{\phi} \in \mathbf{Hom}_{\mathbf{C}_{\text{R-IFM}}}(\mathrm{C}, \mathrm{A}).$$

Lemma [5.2.1] and Proposition [5.2.2] thus make it straightforward to prove that $\mathbf{Hom_A}$=$\mathbf{Hom}_{\mathbf{C}_{\textbf{R-IFM}}}(-, A)$ is a contravariant functor.

Proposition 5.2.9. *Let* $\bar{g} \in \boldsymbol{Hom}_{\boldsymbol{C}_{\textbf{\textit{R-IFM}}}}(B, C)$ *be an IF* R*-homomorphism. Then* $\bar{g}$ *is IF-epimorphism in* $\boldsymbol{C}_{\textbf{\textit{R-IFM}}}$ *if and only if* $\boldsymbol{Hom_A}(\bar{g}) = \bar{g}^* : \boldsymbol{Hom}_{\boldsymbol{C}_{\textbf{\textit{R-IFM}}}}(C, A) \to \boldsymbol{Hom}_{\boldsymbol{C}_{\textbf{\textit{R-IFM}}}}(B, A)$ *is IF-epimorphism for each* $A \in Ob(\boldsymbol{C}_{\textbf{\textit{R-IFM}}})$.

Proof. Let $\bar{g}$ is IF-monomorphism in $\mathbf{C}_{\textbf{R-IFM}}$.

Suppose $\bar{\phi_1}, \bar{\phi_2} \in \mathbf{Hom}_{\mathbf{C}_{\textbf{R-IFM}}}(C, A)$ such that $\bar{g}^*(\bar{\phi_1}) = \bar{g}^*(\bar{\phi_2})$. Then $\bar{\phi_1} \circ \bar{g} = \bar{\phi_2} \circ \bar{g}$. Since $\bar{g}$ is IF-epimorphism, $\bar{\phi_1} = \bar{\phi_2}$. Thus, $Hom^{\mathbf{A}}(\bar{g})$ is IF-epimorphism.

Conversely, $\mathbf{Hom_A}(\bar{g})$ is IF-epimorphism for all $A \in Ob(\mathbf{C}_{\textbf{R-IFM}})$. Suppose $\bar{\phi_1}, \bar{\phi_2} \in \mathbf{Hom}_{\mathbf{C}_{\textbf{R-IFM}}}(C, A)$ such that $\bar{\phi_1} \circ \bar{g} = \bar{\phi_2} \circ \bar{g}$. Then $\bar{g}^*(\bar{\phi_1}) = \bar{g}^*(\bar{\phi_2})$ implies $\bar{\phi_1} = \bar{\phi_2}$ by assumption. Thus, $\bar{g}$ is IF-epimorphism in $\mathbf{C}_{\textbf{R-IFM}}$. □

Corollary 5.2.10. $\boldsymbol{Hom_A}$ *is a epifunctor for each* $A \in Ob(\boldsymbol{C}_{\textbf{\textit{R-IFM}}})$.

Proposition 5.2.11. *Let* $\bar{g} : B \to C$ *be an IF-retraction in* $\boldsymbol{C}_{\textbf{\textit{R-IFM}}}$. *Then* $\boldsymbol{Hom_A}(\bar{g}) : \boldsymbol{Hom}_{\boldsymbol{C}_{\textbf{\textit{R-IFM}}}}(C, A) \to \boldsymbol{Hom}_{\boldsymbol{C}_{\textbf{\textit{R-IFM}}}}(B, A)$ *is IF-retraction for all* $A \in Ob(\boldsymbol{C}_{\textbf{\textit{R-IFM}}})$.

Proof. Let $\bar{g} : B \to C$ be an IF-retraction in $\mathbf{C}_{\textbf{R-IFM}}$. Therefore, an IF R-homomorphism $\bar{h} : C \to B$ exists in $\mathbf{C}_{\textbf{R-IFM}}$ such that $\bar{g} \circ \bar{h} = I_A$. Since every IF-retraction is IF-epimorphism in $\mathbf{C}_{\textbf{R-IFM}}$, $\bar{g}$ is an IF-epimorphism in $\mathbf{C}_{\textbf{R-IFM}}$. By Proposition [5.2.9], $\mathbf{Hom_A}(\bar{g})$ is IF-epimorphism in $\mathbf{C_{R\text{-}IFM}}$. Consider

$$\begin{aligned} ((\bar{g}\circ\bar{h})^{*})(\bar{\phi}) &= \bar{\phi}\circ(\bar{g}\circ\bar{h}) \\ &= \bar{\phi}\circ(I_A) \\ &= (\bar{\phi})(I_A) \\ Thus, ((\bar{g}\circ\bar{h})^{*})(\bar{\phi}) &= (\bar{\phi})(I_B) \end{aligned}$$

which implies that $(\bar{g}\circ\bar{h})^{*} = I_A$. By Proposition [5.2.2], $\bar{g}^{*}\circ\bar{h}^{*} = I_A$. Hence, $\mathbf{Hom_A}(\bar{g}) = \bar{g}^{*} : \mathbf{Hom_{C_{R\text{-}IFM}}}(C, A) \to \mathbf{Hom_{C_{R\text{-}IFM}}}(B, A)$ is IF-retraction for all $A \in Ob(\mathbf{C_{R\text{-}IFM}})$. □

Corollary 5.2.12. *The functor* $\boldsymbol{Hom_A}$ *preserves retraction.*

Proposition 5.2.13. $Hom^{\mathbf{A}}$ *preserves isomorphism in* $\mathbf{C_{R\text{-}IFM}}$.

Proof. Let $\bar{g} : B \to C$ be an IF R-isomorphism in $\mathbf{C_{R\text{-}IFM}}$. We have to prove that $Hom^{\mathbf{A}}(\bar{g})$ is an IF R-isomorphism in $\mathbf{C_{R\text{-}IFM}}$. By assumption, there exists $\bar{h} : C \to B$ such that $\bar{h}\circ\bar{g} = I_B$ and $\bar{g}\circ\bar{h} = I_C$, implying that $Hom^{\mathbf{A}}[\bar{h}\circ\bar{g}] = Hom^{\mathbf{A}}(I_B)$ and $Hom^{\mathbf{A}}[\bar{g}\circ\bar{h}] = Hom^{\mathbf{A}}(I_C)$. This implies that $Hom^{\mathbf{A}}(\bar{h})\circ Hom^{\mathbf{A}}(\bar{g}) = I_{Hom^{\mathbf{A}}(B)}$ and $Hom^{\mathbf{A}}(\bar{g})\circ Hom^{\mathbf{A}}(\bar{h}) = I_{Hom^{\mathbf{A}}(C)}$. Hence, $Hom^{\mathbf{A}}$ preserves isomorphism in $\mathbf{C_{R\text{-}IFM}}$. □

Remark 5.2.14. From above results, we can conclude that

(i) $\mathbf{Hom_{C_{R\text{-}IFM}}}(A, -)$ is a covariant functor.

(ii) $\mathbf{Hom_{C_{R\text{-}IFM}}}(-, A)$ is a contravariant functor.

(iii) $\mathbf{Hom_{C_{R\text{-}IFM}}} : \mathbf{C_{R\text{-}IFM}} \to \mathbf{C_{R\text{-}IFM}}$ is an invariant functor, exhibiting properties of both covariant and contravariant functors.

5.3 Forgetful functors

Explore connections between the category of intuitionistic fuzzy modules and other mathematical structures, providing insights into how these relationships can enhance our understanding of both intuitionistic fuzzy modules and the broader mathematical landscape. "This section explores three forgetful functors originating from the $\mathbf{C_{R\text{-}IFM}}$ category and analyzes their preservation properties."

Proposition 5.3.1. *There exists a forgetful functor from the category of intuitionistic fuzzy modules to the category of intuitionistic fuzzy sets.*

Proof. Let $\mathbf{C_{IFS}}$ denote the category of intuitionistic fuzzy sets. We define a forgetful functor $F_1 : \mathbf{C_{R\text{-}IFM}} \to \mathbf{C_{IFS}}$ such that

(i) for each object $A \in Ob(\mathbf{C_{R\text{-}IFM}})$, F_1 associates it with an object $F_1(A) \in Ob(\mathbf{C_{IFS}})$ by retaining the underlying set structure along with its degree of membership as well as its degree of non-membership, while discarding the module structure.

(ii) F_1 preserves morphisms between objects, meaning that if IF R-homomorphism $\bar{f} : A \to B$ exists in $\mathbf{C_{R\text{-}IFM}}$, then the corresponding morphism between their underlying intuitionistic fuzzy sets A and B is preserved under F_1.

Therefore, a forgetful functor F_1 from $\mathbf{C_{R\text{-}IFM}}$ to $\mathbf{C_{IFS}}$, mapping structures from $\mathbf{C_{R\text{-}IFM}}$ to $\mathbf{C_{IFS}}$ by retaining the essential set properties while discarding the module structure. □

Proposition 5.3.2. *There exists a forgetful functor from the category of intuitionistic fuzzy modules to the category of sets.*

Proof. Let $\mathbf{C_S}$ denote the category of sets. We define a forgetful functor $F_2 : \mathbf{C_{R\text{-}IFM}} \to \mathbf{C_S}$ such that

(i) for each object $A \in Ob(\mathbf{C_{R\text{-}IFM}})$, F_2 associates it with a set $F_2(A) = X \in Ob(\mathbf{C_S})$ by discarding the module structure while retaining only the underlying set.

(ii) F_2 preserves morphisms between objects, meaning that if IF R-homomorphism $\bar{f} : A \to B$ exists in $\mathbf{C_{R\text{-}IFM}}$, then the corresponding morphism between their underlying sets X and Y is preserved under this functor F_2.

Therefore, a forgetful functor F_2 from $\mathbf{C_{R\text{-}IFM}}$ to $\mathbf{C_S}$, discarding the module structure of IFSMs and retaining only its underlying set structure. □

Proposition 5.3.3. *There exists a forgetful functor from* $\mathbf{C_{R\text{-}IFM}}$ *to* $\mathbf{C_{R\text{-}M}}$.

Proof. Define a forgetful functor $F_3 : \mathbf{C_{R\text{-}IFM}} \to \mathbf{C_{R\text{-}M}}$ such that

(i) for each object $A \in Ob(\mathbf{C_{R\text{-}IFM}})$, F_3 associates it with a module $F_3(A) = M \in Ob(\mathbf{C_{R\text{-}M}})$ by discarding the intuitionistic fuzzy structure while retaining only the underlying module structure.

(ii)F_3 preserves morphisms between objects, meaning that if IF R-homomorphism $\bar{f} : A \to B$ exists in $\mathbf{C_{R\text{-}IFM}}$, then the corresponding R-homomorphism between their underlying R-modules M and N is preserved under this functor F_3.

Therefore, a forgetful functor F_3 from $\mathbf{C_{R\text{-}IFM}}$ to $\mathbf{C_{R\text{-}M}}$, discarding the intuitionistic fuzzy characteristics and retaining only its underlying module structure. □

Definition 5.3.4. The forgetful functor

(a) $F_1 : \mathbf{C_{R\text{-}IFM}} \to \mathbf{C_{IFS}}$ that assigns every IFSM A to the underlying IFS A and every IF R-homomorphism $\bar{f} : A \to B$ to the corresponding IFS morphism.

(b) $F_2 : \mathbf{C_{R\text{-}IFM}} \to \mathbf{C_S}$ that assigns every IFSM A to the underlying set M and every IF R-homomorphism $\bar{f} : A \to B$ to the corresponding set map $f : M \to N$.

(c) $F_3 : \mathbf{C_{R\text{-}IFM}} \to \mathbf{C_{R\text{-}M}}$ that assigns every IFSM A to the underlying R-module M and every IF R-homomorphism $\bar{f} : A \to B$ to the corresponding underlying R-homomorphism $f : M \to N$.

The forgetful functor preserves various important properties of morphisms from the $\mathbf{C_{R\text{-}IFM}}$ category to the other categories.

Proposition 5.3.5. *The forgetful functor* $F_1 : \boldsymbol{C_{R\text{-}IFM}} \to \boldsymbol{C_{IFS}}$ *preserves*

(i) coretraction,

(ii) monomorphisms,

(iii) retractions,

(iv) epimorphisms.

Proposition 5.3.6. *The forgetful functor* $F_2 : \boldsymbol{C_{R\text{-}IFM}} \to \boldsymbol{C_S}$ *preserves*

(i) coretraction,

(ii) monomorphisms,

(iii) retractions,

(iv) epimorphisms.

Proposition 5.3.7. *The forgetful functor* $F_3 : \boldsymbol{C_{R\text{-}IFM}} \to \boldsymbol{C_{R\text{-}M}}$ *preserves* *(i) coretraction,*

(ii) monomorphisms,

(iii) retractions,

(iv) epimorphisms.

5.4 Intuitionistic fuzzy exact sequences in $\mathbf{C_{R\text{-}IFM}}$

Throughout the Chapter, we will take $A = (\mu_A, \nu_A)_M$, $B = (\mu_B, \nu_B)_N$, $C = (\mu_C, \nu_C)_K$, $D = (\mu_D, \nu_D)_P$, $E = (\mu_E, \nu_E)_Q$ and $F = (\mu_F, \nu_F)_S$ be IFSMs of R-modules M, N, K, P, Q and S respectively.

In the theory of R-modules, a sequence of the form

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0$$

is said to be short exact sequence in $\mathbf{C_{R\text{-}M}}$ when f is a monomorphism, g is an epimorphism and $Im(f) = ker(g)$. In this section, we extend this notion to intuitionistic fuzzy modules and establish several results.

Definition 5.4.1. An intuitionistic fuzzy short exact sequence in $\mathbf{C_{R\text{-}IFM}}$ is a sequence of the form

$$\bar{0} \longrightarrow A \xrightarrow{\bar{f}} B \xrightarrow{\bar{g}} C \longrightarrow \bar{0}$$

when $Im(\bar{f}) = ker(\bar{g})$, $\bar{g}$ is an IF-epimorphism and $\bar{f}$ is a IF-monomorphism. We abbreviate an intuitionistic fuzzy short exact sequence as IFSE sequence.

Example 5.4.2. Let $A = \chi_Z$, $B = \chi_{nZ}$ and $C = \chi_{\frac{Z}{nZ}}$ then

$$\bar{0} \longrightarrow A \xrightarrow{\bar{i}} B \xrightarrow{\bar{\pi}} C \longrightarrow \bar{0}$$

is a IFSE sequence where $\bar{i}$ and $\bar{\pi}$ are IF-inclusion map and natural IF-epimorphism respectively.

Remark 5.4.3. $\mathbf{Hom_{C_{R\text{-}IFM}}}(A, -)$ is not left exact in $\mathbf{C_{R\text{-}IFM}}$; i.e., we want to claim that the IFSE sequence

$$\bar{0} \longrightarrow \mathbf{Hom_{C_{R\text{-}IFM}}}(A, \bar{0}) \xrightarrow{F\bar{f}=\bar{f}_*} \mathbf{Hom_{C_{R\text{-}IFM}}}(A, A) \xrightarrow{F\bar{g}=\bar{g}_*} \mathbf{Hom_{C_{R\text{-}IFM}}}(A, B)$$

is not exact, where $F = \mathbf{Hom_{C_{R\text{-}IFM}}}(A, -)$. However the sequence given in above example is exact. Define $\bar{\rho_1} : Z \to Z$ as $\bar{\rho_1}(n) = 6n$ and $\bar{\rho_2} : Z \to Z$ as $\bar{\rho_2}(n) = 12n$. Clearly, $\bar{\rho_1}$ and $\bar{\rho_2}$ are in $KerF\bar{g}$. Thus, $|KerF\bar{g}| \geq 2$. Since $\mathbf{Hom_{C_{R\text{-}IFM}}}(A, 0_M)$ contains only zero IF R-homomorphism, so we obtain $ImF\bar{f} \neq KerF\bar{g}$. Hence, $\mathbf{Hom_{C_{R\text{-}IFM}}}(A, -)$ is not left exact.

Definition 5.4.4. Let $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B)$ are IFSM of R-modules M and N respectively and $\mathbf{Hom_{C_{R\text{-}IFM}}}(A, B)$ is the set of IF R-homomorphisms from A to B. An IF-R homomorphism $\bar{f} \in \mathbf{Hom_{C_{R\text{-}IFM}}}(A, B)$ is said to be an Intuitionistic fuzzy split(IF-split), if there is an IF R-homomorphism $\bar{g} \in \mathbf{Hom_{C_{R\text{-}IFM}}}(B, A)$ such that $\bar{g} \circ \bar{f} = I_A$.

Theorem 5.4.5. *For A be any IFSM, let* $\bar{0} \longrightarrow B \xrightarrow{\bar{f}} C \xrightarrow{\bar{g}} D \longrightarrow \bar{0}$ *be IFSE sequence. Then*

$$\bar{0} \longrightarrow \boldsymbol{Hom_{C_{R\text{-}IFM}}}(A, B) \xrightarrow{\bar{f}_*} \boldsymbol{Hom_{C_{R\text{-}IFM}}}(A, C) \xrightarrow{\bar{g}_*} \boldsymbol{Hom_{C_{R\text{-}IFM}}}(A, D)$$

is IFSE sequence if and only if for any $\phi \in Hom(M, N)$ which $f_*(\phi) = \psi$, where $\bar{\psi} \in ker\bar{g}_*$, $\phi^{-1}(\mu_B) \geqslant \mu_A$ and $\phi^{-1}(\nu_B) \leqslant \nu_A$.

Proof. Firstly, let

$$\bar{0} \longrightarrow \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A,B) \xrightarrow{\bar{f}_*} \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A,C) \xrightarrow{\bar{g}_*} \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A,D)$$

is IFSE sequence. Thus, $Im\bar{f}_* = Ker\bar{g}_*$. For $\bar{\phi} \in \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A,B)$, we have

$$\phi^{-1}(\mu_B) \geqslant \mu_A, \phi^{-1}(\nu_B) \leqslant \nu_A$$

and also $\bar{f}_*(\phi) = \bar{\psi}$, where $\bar{\psi} \in ker\bar{g}_*$ which is lifted to R-homomorphism $\phi \in \mathbf{Hom}_{\mathbf{C_{R\text{-}M}}}(M,N)$ with the result that $f_*(\phi) = \psi$.

Conversely, let $\phi \in \mathbf{Hom}_{\mathbf{C_{R\text{-}M}}}(M,N)$ with $f_*(\phi) = \psi$, where $\bar{\psi} \in ker\bar{g}_*$ and $\phi^{-1}(\mu_B) \geqslant \mu_A$ and $\phi^{-1}(\nu_B) \leqslant \nu_A$. This shows that $\bar{\phi} \in \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A,B)$. Since $f_*(\phi) = \psi$ which gives $f \circ \phi = \psi$. By intuitionistic fuzzification, we have $\bar{f} \circ \phi = \bar{\psi}$, stating that $\bar{f}_*(\phi) = \bar{\psi}$. Thus, $Im\bar{f}_* = Ker\bar{g}_*$. □

Theorem 5.4.6. *Let A, B, C and D be IFSMs of R-modules M,N,K and P respectively. Let*

$$\bar{0} \longrightarrow B \xrightarrow{\bar{g}} C \xrightarrow{\bar{h}} D$$

is IFSE sequence in $\boldsymbol{C_{R\text{-}IFM}}$, where $\bar{g}$ is IF split. Then $\boldsymbol{Hom_{C_{R\text{-}IFM}}}(A,-)$ preserves the sequence.

Proof. Let $F = \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A,-)$. We will show that the sequence

$$\bar{0} \longrightarrow \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A,B) \xrightarrow{F\bar{g}=\bar{g}_*} \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A,C) \xrightarrow{F\bar{h}=\bar{h}_*} \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A,D)$$

is exact. Clearly $F\bar{g} = \bar{g}_*$ is monic as $\bar{g}$ is IF split. We claim that $Im\bar{g}_* = Ker\bar{h}_*$,i.e. $Im\bar{g}_* \subseteq Ker\bar{h}_*$ and $Im\bar{g}_* \supseteq Ker\bar{h}_*$. Let $\bar{\phi} \in Im\bar{g}_*$ such that $\bar{\phi} \doteq \bar{g}_*(\bar{\psi}) = \bar{g} \circ \bar{\psi}$; where

$\bar{\psi} \in \mathbf{Hom_{C_{R\text{-}IFM}}}(A, B)$.

$$\begin{aligned} \mu_{Hom(A,D)}(\bar{h} \circ \bar{\phi}) &= \mu_{Hom(A,D)}(\bar{h} \circ \bar{g} \circ \bar{\psi}) \\ &= \wedge\{\mu_D((\bar{h} \circ \bar{g} \circ \bar{\psi})(a)) : a \in M\} \\ &= \wedge\{\mu_D(\bar{h} \circ \bar{g})(\bar{\psi}(a)) : a \in M\} \\ &= \wedge\{\mu_D(\bar{h} \circ \bar{g})(b) : \bar{\psi}(a) = b \in N\} \\ &= \{1\}[\because Im\bar{g} = Ker\bar{h}] \\ &= 1. \end{aligned}$$

So, $Im\bar{g}_* \subseteq Ker\bar{h}_*$. Now, we will prove that $Im\bar{g}_* \supseteq Ker\bar{h}_*$.

Let $\bar{\phi} \in Ker\bar{h}_*$. Then we have

(i) $\mu_{Hom(A,D)}(\bar{h}_*(\bar{\phi})) = 1$ which implies that $\wedge\{\mu_D((\bar{h} \circ \bar{\phi})(a)) : a \in M\} = 1$. Thus,

$$\mu_D(\bar{h}(\bar{\phi}(a))) = 1, \forall a \in M \tag{5.4.1}$$

(ii) $\nu_{Hom(A,D)}(\bar{h}_*(\bar{\phi})) = 0$ which implies that $\vee\{\nu_D((\bar{h} \circ \bar{\phi})(a)) : a \in M\} = 0$. Thus,

$$\nu_D(\bar{h}(\bar{\phi}(a))) = 0, \forall a \in M \tag{5.4.2}$$

From equation (5.4.1) and (5.4.2), $Im\bar{\phi} \in Ker\bar{h} = Im\bar{g}$.

As $\bar{g}$ is monic, $\bar{g}(b) = \bar{\phi}(a)$ for a unique $b \in N$. Define $\bar{k} : A \to B$ as $\bar{k}(a) = b$. Now, we will prove that $\bar{k}$ is an IF R-homomorphism. As $\bar{g}$ is IF split, $\bar{\rho} \circ \bar{g} = I_B$ for a unique IF R-homomorphism $\bar{\rho} : C \to B$. For every $a \in M$, we have

$\mu_B(\bar{k}(a)) = \mu_B(b) = \mu_B(\bar{\rho} \circ \bar{g}(b)) \geq \mu_C(\bar{g}(b)) = \mu_C(\bar{\phi}(a)) \geq \mu_A(a)$.

Consider $\bar{g}_*(\bar{k}) = \bar{g} \circ \bar{k} = \bar{\phi}$. Then $\bar{\phi} \in Im\bar{g}_*$. Thus $Im\bar{g}_* \supseteq Ker\bar{h}_*$. That concludes the proof. □

Theorem 5.4.7. *Let*

$$B \xrightarrow{\bar{g}} C \xrightarrow{\bar{h}} D \longrightarrow \bar{0}$$

be a IFSE sequence in $\boldsymbol{C_{R\text{-}IFM}}$*, where* $\bar{g}$ *is IF-split. Let* $G = \boldsymbol{Hom_{C_{R\text{-}IFM}}}(-, A)$*. Then the induced sequence*

$$\bar{0} \longrightarrow \boldsymbol{Hom_{C_{R\text{-}IFM}}}(D, A) \xrightarrow{G\bar{h}=\bar{h}^*} \boldsymbol{Hom_{C_{R\text{-}IFM}}}(C, A) \xrightarrow{G\bar{g}=\bar{g}^*} \boldsymbol{Hom_{C_{R\text{-}IFM}}}(B, A)$$

is IFSE sequence.

Proof. The demonstration exhibits a resemblance to Theorem [5.4.6]. □

Andersen and Fuller [4] explored the connection between projective modules and semi-perfect rings. Theorem 27.11 [4] states that if P is a projective R-module, there exists a set of idempotent elements (e_i) for $i \in J$ in a commutative semi-perfect ring R, such that $P \cong \coprod_{i \in J} Re_i$, where each e_i belongs to $E(R)$, the set of idempotent elements of the ring R. By theorem 3.3.1, every IF projective module is a zero IFSM, we can derive the following result:

Proposition 5.4.8. *Let* R *be a commutative semi-perfect ring. For a IF-projective module* $A = \bar{0}_M$*, there exists a set of idempotent elements* e_i *in* R *such that* $\bar{0}_M \cong \coprod_{i \in J} \bar{O}_{Re_i}$.

Proposition 5.4.9. *[22] Let* $A = (\mu_A, \nu_A)_M$ *be IFSM of the* R*-module* M*, and let* $e \in E(R)$*. Then, we define* eA *as an IFSM of* eM*, denoted by* $eA = (\mu_{eA}, \nu_{eA})_M$*, where:* $\mu_{eA}(z) = \vee\{\mu_A(y) : z = ey, y \in M\}$ *and* $\nu_{eA}(z) = \wedge\{\nu_A(y) : x = ey, z \in M\}$ *for every z in M.*

Let $E(R)$ denote the set of idempotent elements in the commutative semi-perfect ring R. Now, we will study the functor $\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(\bar{O}_{Re}, -)$ where $e \in E(R)$.

Define a map

$$\Gamma_A : Hom(\bar{O}_{Re}, A) \to eA$$

$$\bar{\phi} \mapsto \bar{\phi}|_{eA}.$$

Lemma 5.4.10. *Γ_A is an IF R-isomorphism, where $A \in Ob(\boldsymbol{C_{R\text{-}IFM}})$.*

Proof. Suppose $ex \in eM$. Define mapping $\bar{\phi} : \bar{O}_{Re} \to A$ as $\bar{\phi}(re) = rex$; where $r \in R$ and $x \in M$. For $\bar{\phi} \in Hom(\bar{O}_{Re}, A)$, $\Gamma_A(\bar{\phi}) = ex$ demonstrating that Γ_A is surjective. If $\bar{\phi} \in Hom(\bar{O}_{Re}, A)$, we observe that $\bar{\phi}$ is determined by $\bar{\phi}|_{eA}$, establishing Γ_A is an injective mapping. Also, for $\bar{\phi} \in Hom(\bar{O}_{Re}, A)$,

$$\beta_1(\bar{\phi}) = \mu_A(\bar{\phi}(e)), \beta_2(\bar{\phi}) = \nu_A(\bar{\phi}(e))$$

where $\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(\bar{O}_{Re}, A) = (\beta_1, \beta_2)$.

Consider $\mu_{eA}(\Gamma_A(f))$ = $\mu_{eA}(ex)$ = $\vee\{\mu_A(ey) : x = ey, y \in M\}$ = $\{\mu_A(\bar{\phi}(e))\}$ = $\beta_1(\bar{\phi})$ and $\nu_{eA}(\Gamma_A(f))$ = $\nu_{eA}(ex)$ = $\wedge\{\nu_A(ey) : x = ey, y \in M\}$ = $\{\nu_A(\bar{\phi}(e))\}$ = $\beta_2(\bar{\phi})$. Thus, we conclude that

$$\mu_{eA}(\Gamma_A(f)) = \beta_1(\bar{\phi}), \nu_{eA}(\Gamma_A(f)) = \beta_2(\bar{\phi}).$$

Hence, Γ_A is an intuitionistic fuzzy R-isomorphism. □

Proposition 5.4.11. *Consider the following commutative figure [5.1] for $\bar{\alpha}$, $\bar{\gamma}$ are IF-isomorphisms and $\bar{\beta}$ is IF quasi-isomorphisms :*

$$\begin{array}{ccccccccc} \bar{0} & \longrightarrow & D & \xrightarrow{\bar{\phi}} & E & \xrightarrow{\bar{\psi}} & F & \longrightarrow & \bar{0} \\ & & \downarrow{\scriptstyle \bar{\alpha}} & & \downarrow{\scriptstyle \bar{\beta}} & & \downarrow{\scriptstyle \bar{\gamma}} & & \\ \bar{0} & \longrightarrow & A & \xrightarrow{\bar{f}} & B & \xrightarrow{\bar{g}} & C & \longrightarrow & \bar{0} \end{array}$$

Figure 5.1: IFSE

Then bottom row forms an IFSE sequence in $\boldsymbol{C_{R\text{-}IFM}}$ *if and only if the top row does.*

Proof. Firstly, Let the bottom row be an IFSE sequence. Then $Im\bar{f} = Ker\bar{g}$, $\bar{g}$ is IF-epimorphism and $\bar{f}$ is IF-monomorphism. We want to show that

(i) $\bar{\phi}$ is IF-monomorphism;

(ii) $\bar{\psi}$ is IF-epimorphism;

(iii) $Im(\bar{\phi}) = Ker(\bar{\psi})$.

Let $p_1, p_2 \in P$ such that $\bar{\phi}(p_1) = \bar{\phi}(p_2)$. As the figure 5.1 is commutative, we have

$$\begin{aligned} \bar{f}(\bar{\alpha}(p_1)) &= \bar{\beta}(\bar{\phi}(p_1)) \\ &= \bar{\beta}(\bar{\phi}(p_2)) \text{ [By Assumption]} \\ &= \bar{f}(\bar{\alpha}(p_2)) \text{ [By Commutativity]} \\ \Rightarrow p_1 &= p_2 \text{ [As } \bar{\mathrm{f}}\bar{\alpha} \text{ is IF} - \text{monomorphism].} \end{aligned}$$

Thus, $\bar{\phi}$ is IF-monomorphism.

Let $s \in S$ and $\bar{\gamma}(s) = k$, where $k \in K$. Since $\bar{g}$ is IF-epimorphism, $\bar{g}(n) = k$ for $n \in N$. As $\bar{\beta}$ is an IF quasi-isomorphisms, $n = \bar{\beta}(q)$ for some $q \in Q$.

Consider $k = \bar{g}(n) = \bar{g}(\bar{\beta}(q)) = \bar{\gamma}(\bar{\psi}(q))$, implying $\bar{\gamma}(s) = \bar{\gamma}(\bar{\psi}(q))$. Since $\bar{\gamma}$ is IF-monomorphism, $\bar{\psi}(q) = s$, concluding $\bar{\psi}$ is an IF-epimorphism.

Let $p_1 \in P$. Now, we aim to show that $\bar{\phi}(p_1) \in Ker(\bar{\psi})$. Since $\bar{f}(\bar{\alpha}(p_1)) \in Im(\bar{f}) = Ker(\bar{g})$, $\mu_C(\bar{g}(\bar{f}(\bar{\alpha}(p_1)))) = 1$ and $\nu_C(\bar{g}(\bar{f}(\bar{\alpha}(p_1)))) = 0$. So, $\mu_C(\bar{\gamma}(\bar{\psi}(\bar{\phi}(p_1)))) = 1$ and $\nu_C(\bar{\gamma}(\bar{\psi}(\bar{\phi}(p_1)))) = 0$. Since $\bar{\gamma}$ is IF-isomorphism, $\mu_F(\bar{\psi}(\bar{\phi}(p_1))) = \mu_C(\bar{\gamma}(\bar{\psi}(\bar{\phi}(p_1)))) = 1$ and $\nu_F(\bar{\psi}(\bar{\phi}(p_1))) = \nu_C(\bar{\gamma}(\bar{\psi}(\bar{\phi}(p_1))) = 0$, implying $\bar{\phi}(p_1) \in Ker(\bar{\psi})$. Thus, $Im(\bar{\phi}) \subseteq Ker(\bar{\psi})$.

Let $q \in Ker(\bar{\psi})$, so $\mu_F(\bar{\psi}(q)) = 1$ and $\nu_F(\bar{\psi}(q)) = 0$. As a result, $\mu_C(\bar{\gamma}(\bar{\psi}(q))) = 1$ and $\nu_C(\bar{\gamma}(\bar{\psi}(q))) = 0$. According to commutativity, $\mu_C(\bar{g}(\bar{\beta}(g))) = 1$ and $\nu_C(\bar{g}(\bar{\beta}(q))) = 0$, which implies that $\beta(q) \in Ker(\bar{g}) = Im(\bar{f})$ and so, there exists $m \in M$ satisfying $\bar{f}(m) = \beta(q)$, along with $p_1 \in P$ such that $\bar{\alpha}(p_1) = m$.

$$
\begin{aligned}
\bar{\beta}(\bar{\phi}(p_1)) &= \bar{f}(\bar{\alpha}(p_1)) \\
&= \bar{f}(m) \\
&= \bar{\beta}(q) \\
\Rightarrow q &= \bar{\phi}(p_1) \text{ [As } \bar{\beta} \text{ is IF} - \text{isomorphism].}
\end{aligned}
$$

Thus, $Ker(\bar{\psi}) \subseteq Im(\bar{\phi})$. Hence, $Ker(\bar{\psi}) = Im(\bar{\phi})$. □

Lemma 5.4.12. *Let the IFSE sequence*

$$\bar{0} \longrightarrow A \xrightarrow{\bar{f}} B \xrightarrow{\bar{g}} C \longrightarrow \bar{0}$$

within $\boldsymbol{C_{R\text{-}IFM}}$ *and for* $e \in E(R)$, $e\bar{f} = \bar{f}|_{eA}$ *and* $e\bar{g} = \bar{g}|_{eB}$ *be the restriction mappings on* $\bar{f}$ *and* $\bar{g}$ *respectively. Then the sequence*

$$\bar{0} \longrightarrow eA \xrightarrow{e\bar{f}} eB \xrightarrow{e\bar{g}} eC \longrightarrow \bar{0}$$

is IFSE sequence $\boldsymbol{C_{R\text{-}IFM}}$.

Proof. Let $ek \in eK$. As $\bar{g}$ is an IF-epimorphism, it follows that $\bar{g}(b) = k$ for $b \in N$. Consider,

$$\begin{aligned} e\bar{g}(eb) &= e^2\bar{g}(b) \\ &= e\bar{g}(b) \\ &= e(k) \\ &= ek \end{aligned}$$

Thus $e\bar{g}$ is an IF-epimorphism. Since $Im(\bar{f}) \subseteq Ker(\bar{g})$, it is clear that $Im(e\bar{f}) \subseteq Ker(e\bar{g})$. Let $eb \in Ker(e\bar{g})$. Since $Ker(e\bar{g}) \subseteq Ker(\bar{g})$, therefore there exists $x \in M$ such that $\bar{f}(x) = eb$. For $e^2 = e$ and $ex \in eM$, we have $e\bar{f}(ex) = e^2\bar{f}(x) = e\bar{f}(x) = e.eb = eb$. Thus, $Ker(e\bar{g}) \subseteq Im(e\bar{f})$. This concludes the proof. □

Proposition 5.4.13. *The functor* $\textbf{\textit{Hom}}_{\boldsymbol{C}_{\textbf{\textit{R-IFM}}}}(\bar{O}_{Re}, -)$ *preserves the IFSE-sequence*

$$\bar{0} \longrightarrow A \xrightarrow{\bar{f}} B \xrightarrow{\bar{g}} C \longrightarrow \bar{0}$$

in $\boldsymbol{C}_{\textbf{\textit{R-IFM}}}$*, where* $e \in E(R)$.

Proof. Take into account the following commutative diagram within $\mathbf{C}_{\textbf{R-IFM}}$.

$$\begin{array}{ccccccccc} \bar{0} & \longrightarrow & \mathbf{Hom}_{\mathbf{C}_{\textbf{R-IFM}}}(\bar{O}_{Re}, A) & \xrightarrow{\bar{f}_*} & \mathbf{Hom}_{\mathbf{C}_{\textbf{R-IFM}}}(\bar{O}_{Re}, B) & \xrightarrow{\bar{g}_*} & \mathbf{Hom}_{\mathbf{C}_{\textbf{R-IFM}}}(\bar{O}_{Re}, C) & \longrightarrow & \bar{0} \\ & & \downarrow{\scriptstyle \Gamma_A} & & \downarrow{\scriptstyle \Gamma_B} & & \downarrow{\scriptstyle \Gamma_C} & & \\ \bar{0} & \longrightarrow & eA & \xrightarrow{e\bar{f}} & eB & \xrightarrow{e\bar{g}} & eC & \longrightarrow & \bar{0} \end{array}$$

By lemma [5.4.10], Γ_A, Γ_B and Γ_C are IF-isomorphisms. By Proposition 5.4.11, bottom row is IFSE sequence iff top row is IFSE sequence. By lemma [5.4.12], the sequence

$$\bar{0} \longrightarrow eA \xrightarrow{e\bar{f}} eB \xrightarrow{e\bar{g}} eC \longrightarrow \bar{0}$$

is a IFSE sequence. Hence, bottom row is exact. Hence, the functor $\mathbf{Hom}_{\mathbf{C}_{\textbf{R-IFM}}}(\bar{O}_{Re}, -)$ preserves the sequence in $\mathbf{C}_{\textbf{R-IFM}}$. □

Lemma 5.4.14. *Let $A = (\mu_A, \nu_A)_M$ be IFSM of R-module M and $e_i \in E(R)$, where $i \in J$. Then,*

$$\boldsymbol{Hom}_{\boldsymbol{C}_{\textbf{R-IFM}}}(\coprod_{i \in J} \bar{O}_{Re_i}, A) \cong \prod_{i \in J}(e_i A)$$

Proof. Since $\mathbf{Hom}_{\mathbf{C}_{\textbf{R-M}}}(-, M)$ converts coproducts into products. Let

$$\phi : \mathbf{Hom}_{\mathbf{C}_{\textbf{R-M}}}(\coprod_{i \in J} Re_i, M) \to \prod_{i \in J} \mathbf{Hom}_{\mathbf{C}_{\textbf{R-M}}}(Re_i, M)$$

defined by

$$\tau \mapsto (\tau \circ \rho_i = \tau_i)_{i \in J}$$

be an R-isomorphism, where ρ_j is the injection mapping $Re_j \to \coprod_{i \in J} Re_i$.

Let $\mathbf{Hom}_{\mathbf{C}_{\textbf{R-IFM}}}(\bar{O}_{Re_i}, A) = (\beta_{1i}, \beta_{2i})$ and $\mathbf{Hom}_{\mathbf{C}_{\textbf{R-IFM}}}(\coprod_{i \in J} \bar{O}_{Re_i}, A) = (\beta_1, \beta_2)$. Based on the evidence presented in lemma [5.4.10], it follows that

$$\beta_{1i}(\bar{\tau}_i) = \mu_A(\bar{\tau}_i(e_i)), \beta_{2i}(\bar{\tau}_i) = \nu_A(\bar{\tau}_i(e_i)); \forall i \in J.$$

Consider $\mu_A(\sum_{i \in J} \bar{\tau}_i(r_i e_i)) = \wedge_{i \in J} \mu_A(\bar{\tau}_i(r_i e_i)) = \wedge_{i \in J} \mu_A(r_i \bar{\tau}_i(e_i)) \geqslant \wedge_{i \in J} \mu_A(\bar{\tau}_i(e_i))$, where

$r_{i,}e_i \in R, i \in J$. For $\bar{\tau} \in \mathbf{Hom_{C_{R\text{-}IFM}}}(\coprod_{i\in J} \bar{O}_{Re_i}, A)$, we obtain

$$\begin{aligned}
\beta_1(\bar{\tau}) &= \wedge\{\mu_A \circ \bar{\tau}_i(r_ie_i) : (r_ie_i)_{i\in J} \in \coprod_{i\in J} Re_i\} \\
&= \{\mu_A(\sum_{i\in J} \bar{\tau}_i(r_ie_i)) : (r_ie_i)_{i\in J} \in \coprod_{i\in J} Re_i\} \\
&= \wedge\{\mu_A(\bar{\tau}_i(e_i)) : i \in J\} \\
&= \wedge\{\beta_{1i}(\bar{\tau}_i) : i \in J\} \\
&= \prod_{i\in J} \beta_{1i}((\bar{\tau}_i)_{i\in J}) \\
&= \prod_{i\in J} \beta_{1i} \circ \phi(\bar{\tau}).
\end{aligned}$$

So, it implies

$$\mathbf{Hom_{C_{R\text{-}IFM}}}(\coprod_{i\in J} \bar{O}_{Re_i}, A) \cong \prod_{i\in J} \mathbf{Hom_{C_{R\text{-}IFM}}}(\bar{O}_{Re_i}, A) \tag{5.4.3}$$

By lemma [5.4.12],

$$\begin{aligned}
Hom(\bar{O}_{Re_i}, A) &\cong e_iA \\
\Rightarrow \prod_{i\in J} Hom(\bar{O}_{Re_i}, A) &\cong \prod_{i\in J} e_iA
\end{aligned} \tag{5.4.4}$$

From equation (5.4.3) and (5.4.4), we can conclude that

$$\mathbf{Hom_{C_{R\text{-}IFM}}}(\coprod_{i\in J} \bar{O}_{Re_i}, A) \cong \prod_{i\in J} (e_iA) \tag{5.4.5}$$

□

Theorem 5.4.15. *Let A, B, C and D be IFSMs of R-modules M, N, K and P respectively. Let*

$$\bar{0} \longrightarrow B \xrightarrow{\bar{\phi}} C \xrightarrow{\bar{\psi}} D \longrightarrow \bar{0}$$

is IFSE sequence in $\mathbf{C_{R\text{-}IFM}}$*. Then the functor* $\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A,-)$ *preserves the sequence if and only if* A *is an IF-projective module.*

Proof. First, suppose $\bar{\psi}$ is an IF-epimorphism and $\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A,-)$ preserves the IFSE sequence.

Let $C' = C|_K$, where $K = ker\bar{\psi} = \{b \in N : \mu_D(\bar{\psi}(b)) = 1, \nu_D(\bar{\psi}(b)) = 0\}$. Then C' is IF-submodule of C, and we obtain the IFSE sequence

$$\bar{0} \longrightarrow C' \xrightarrow{\bar{i}} C \xrightarrow{\bar{\psi}} D \longrightarrow \bar{0}$$

where $\bar{i}$ is the inclusion map.

Since $\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A,-)$ preserves the IFSE sequence, $\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A,-)$ preserves the IF-epimorphism $\bar{\psi}$, implying that A is IF-projective.

Conversely, assume that A is an IF-projective. With respect to $e_i \in E(R)$ and Proposition [5.4.8], we have

$$A \cong \coprod_{i \in J} \bar{O}_{Re_i}$$

Let the sequence

$$\bar{0} \longrightarrow B \xrightarrow{\bar{\phi}} C \xrightarrow{\bar{\psi}} D \longrightarrow \bar{0}$$

be an IFSE sequence in $\mathbf{C_{R\text{-}IFM}}$. The sequence

$$\bar{0} \longrightarrow \prod_{i\in J} e_i B \longrightarrow \prod_{i\in J} e_i C \longrightarrow \prod_{i\in J} e_i D \longrightarrow \bar{0}$$

is also IFSE sequence based on lemma [5.4.12]. By employing Lemma [5.4.14], we can construct the subsequent commutative diagram:

$$\begin{array}{ccccccccc}
\bar{0} \rightarrow & \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(\coprod_{i\in J}\bar{O}_{Re_i}, B) & \rightarrow & \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(\coprod_{i\in J}\bar{O}_{Re_i}, C) & \rightarrow & \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(\coprod_{i\in J}\bar{O}_{Re_i}, D) & \rightarrow \bar{0} \\
 & \downarrow & & \downarrow & & \downarrow & \\
\bar{0} \longrightarrow & \prod_{i\in J} e_i B & \longrightarrow & \prod_{i\in J} e_i C & \longrightarrow & \prod_{i\in J} e_i D & \longrightarrow \bar{0}
\end{array}$$

Given that the bottom row constitutes an IFSE sequence, it follows that the top row also represents an IFSE sequence. Thus, it is established that $\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A, -)$ preserves the IFSE sequence. □

5.5 Tensor product functor in $\mathbf{C_{R\text{-}IFM}}$

Tensor product structure provides the most innovative approach to connecting two modules. The structure and characteristics of tensor products constructed from two intuitionistic fuzzy modules in the $C_{\mathbf{R\text{-}IFM}}$ category are covered in this section. In addition, we investigate whether there is a connection between Hom functor and tensor product functor in this category.

Definition 5.5.1. Define $D \times E : P \times Q \to I \times I$ as an IFS on $P \times Q$ by $D \times E = \{< (p, q), (\mu_{D\times E})(p, q), (\nu_{D\times E})(p, q) >: (p, q) \in P \times Q\}$ where

$$\mu_{D\times E}(p, q) = (\mu_D \times \mu_E)(p, q) = \vee\{\mu_D(p), \mu_E(q)\}, \nu_{D\times E}(p, q) = (\nu_D \times \nu_E)(p, q) = \wedge\{\nu_D(p), \nu_E(q)\},$$

$$\mu_{D\times E}(\sum(p_i, q_i)) = (\mu_D \times \mu_E)(\sum(p_i, q_i)) = \wedge\{\vee\{\mu_D(p_i), \mu_E(q_i)|i \in J\}\}$$

and

$$\nu_{D\times E}(\sum(p_i, q_i)) = (\nu_D \times \nu_E)(\sum(p_i, q_i)) = \vee\{\wedge\{\nu_D(p_i), \nu_E(q_i)|i \in J\}\}$$

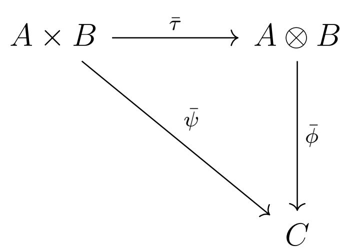
Proposition 5.5.2. *$A \times B$ is an IFSM on $M \times N$.*

Definition 5.5.3. An intuitionistic fuzzy biadditive (IF-biadditive) is a mapping $\bar{\phi} : A \times B \to C$ satisfies

(i) $\phi : M \times N \to K$ is R-biadditive and

(ii) $\mu_C(\bar{\phi}(\sum(x_i, y_i))) \geq (\mu_A \times \mu_B)(\sum(x_i, y_i))$ and $\nu_C(\bar{\phi}(\sum(x_i, y_i))) \leq (\nu_A \times \nu_B)(\sum(x_i, y_i))$; $\forall \sum(x_i, y_i) \in M \times N$.

Definition 5.5.4. An intuitionistic fuzzy tensor product of two IFSMs, A and B, denoted as $A \otimes B$, is endowed with an IF-biadditive mapping $\bar{\tau} : A \times B \to A \otimes B$, which satisfies the property that for any IFSM C over an R-module K, along with every IF-biadditive function $\bar{\psi} : A \times B \to C$, then $\bar{\phi} \circ \bar{\tau} = \bar{\psi}$ for a unique IF R-homomorphism $\bar{\phi} : A \otimes B \to C$ and the subsequent diagram commutes:



and

$$\mu_C(\bar{\phi}(x \otimes y)) \geq (\mu_A \times \mu_B)(x, y)$$

$$\nu_C(\bar{\phi}(x \otimes y)) \leq (\nu_A \times \nu_B)(x, y)$$

Remark 5.5.5. We will denote intuitionistic fuzzy tensor product by IFT-product.

Theorem 5.5.6. *The intuitionistic fuzzy tensor product exists in $\boldsymbol{C_{R\text{-}IFM}}$ and is uniquely determined up to isomorphism.*

Proof. Let $A = (\mu_A, \nu_A)$, $B = (\mu_B, \nu_B)$ and $C = (\mu_C, \nu_C)$ be IFSM's of R-modules M, N and K respectively. Let $\bar{\tau} : A \times B \to A \otimes B$ be the IFT-product of A and B.

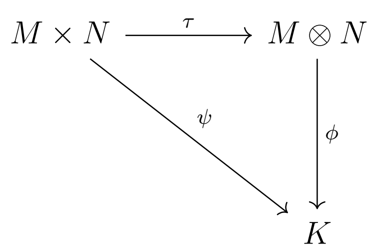
Define the mapping $A \otimes B : M \otimes N \to I \times I$ as

$$(\mu_A \otimes \mu_B)(\sum(x_i \otimes y_i)) = \vee\{(\mu_A \times \mu_B)(\sum(x_i^{'}, y_i^{'}))| \sum(x_i^{'} \otimes y_i^{'}) = \sum(x_i \otimes y_i)\}$$

and

$$(\nu_A \otimes \nu_B)(\sum(x_i \otimes y_i)) = \wedge\{(\nu_A \times \nu_B)(\sum(x_i^{'}, y_i^{'}))| \sum(x_i^{'} \otimes y_i^{'}) = \sum(x_i \otimes y_i)\}$$

From this, it can be easily check that $\bar{\tau}$ is IF-biadditive. Let $\bar{\psi} : A \times B \to C$ be IF-biadditive. Since the tensor product of two R-modules exists and is unique up to isomorphism in $\mathbf{C}_{\textbf{R-M}}$, it follows that for any R-biadditive map $\psi : M \times N \to K$, there exists a unique R-homomorphism $\phi : M \otimes N \to K$ such that $\phi \circ \tau = \psi$.



We only need to show that $\bar{\phi} : A \otimes B \to C$ is an IF R-homomorphism, i.e., we want to claim that, $\forall \sum(x_i \otimes y_i) \in M \times N$

$$\mu_C(\bar{\phi}(\sum(x_i \otimes y_i))) \geq (\mu_A \otimes \mu_B)(\sum(x_i \otimes y_i))$$

$$\nu_C(\bar{\phi}(\sum(x_i \otimes y_i))) \leq (\nu_A \otimes \nu_B)(\sum(x_i \otimes y_i))$$

Let $\sum(x_i^{'} \otimes y_i^{'}) = \sum(x_i \otimes y_i)$. Consider

$$\begin{aligned}
\mu_C(\phi(\sum(x_i^{'} \otimes y_i^{'}))) &= \mu_C(\sum \phi(x_i^{'} \otimes y_i^{'})) \\
&\geq \wedge\{\mu_C(\phi(x_i^{'} \otimes y_i^{'}))\} \\
&= \wedge\{\mu_C(\phi \circ \tau)(x^{'}, y^{'}))\} \\
&= \wedge\{\mu_C(\psi)(x^{'}, y^{'}))\} \\
&\geq \wedge\{(\mu_A \times \mu_B)(x^{'}, y^{'})\} \\
&= (\mu_A \times \mu_B)(\sum(x_i^{'} \otimes y_i^{'})) \\
\Rightarrow \mu_C(\bar{\phi}(\sum(x_i \otimes y_i))) &\geq (\mu_A \otimes \mu_B)(\sum(x_i \otimes y_i)).
\end{aligned}$$

In the similar manner, we have $\nu_C(\bar{\phi}(\sum(x_i \otimes y_i))) \leq (\nu_A \otimes \nu_B)(\sum(x_i \otimes y_i))$.

Hence, the IFT-product exists in $\mathbf{C}_{\textbf{R-IFM}}$ and it is unique upto isomorphism. □

In $\mathbf{C}_{\textbf{R-M}}$, $R \otimes M \cong M$ if $M \in Ob(C_{R-M})$. We can get the following result in $\mathbf{C}_{\textbf{R-IFM}}$ by using this fact:

Proposition 5.5.7. *Let $A \in Ob(\mathbf{C}_{\textbf{R-IFM}})$. Then, we have $\bar{0} \otimes A \cong \bar{0}$.*

The preservation of epimorphisms by tensor product functors is a crucial property in the study of intuitionistic fuzzy modules, as its ensures that certain algebraic structures and properties are maintained under tensor product operations, facilitating their application in various mathematical contexts. For $M \in Ob(C_{R-M})$, $M \otimes -$ is right exact. In the domain of $\mathbf{C}_{\textbf{R-IFM}}$, we have now reached the following finding:

Proposition 5.5.8. *Let $A \in Ob(\mathbf{C}_{\textbf{R-IFM}})$. Then $A \otimes -$ preserves epimorphisms in $\mathbf{C}_{\textbf{R-IFM}}$.*

Proof. Consider the IFSE sequence $B \xrightarrow{\bar{g}} C \longrightarrow \bar{0}$ in $\mathbf{C}_{\textbf{R-IFM}}$, thus establishing $\bar{g}$ is an IF-epimorphism. Since $M \otimes -$ is right exact, we have

$$A \otimes B \xrightarrow{I_A \otimes \bar{g}} A \otimes C \longrightarrow \bar{0}$$

Now, we only need to prove that $I_A \otimes \bar{g}$ is IF R-homomorphism. For any $\sum(x_i \otimes y_i) \in M \times N$, we have

$(\mu_A \otimes \mu_C)((I_A \otimes \bar{g})(\sum(x_i \otimes y_i)))$

= $(\mu_A \otimes \mu_C)(\sum(x_i \otimes \bar{g}(y_i)))$

= $\vee\{(\mu_A \times \mu_C)(\sum(x_i^{'}, k_i^{'}))| \sum(x_i^{'} \otimes k_i^{'}) = \sum(x_i \otimes \bar{g}(y_i))\}$

$\geq \vee\{(\mu_A \times \mu_B)(\sum(x_i^{'}, y_i^{'}))|\bar{g}(y_i^{'}) = k_i^{'}$ and $\sum(\mathrm{x_i^{'}} \otimes \mathrm{k_i^{'}}) = \sum(\mathrm{x_i} \otimes \bar{\mathrm{g}}(\mathrm{y_i}))\}$

$\geq \vee\{(\mu_A \times \mu_B)(\sum(x_i^{''}, y_i^{''}))| \sum(x_i^{''} \otimes y_i^{''}) = \sum(x_i \otimes y_i)\}$

= $(\mu_A \otimes \mu_B)(\sum(x_i \otimes y_i))$.

Thus $(\mu_A \otimes \mu_C)((I_A \otimes \bar{g})(\sum(x_i \otimes y_i))) \geq (\mu_A \otimes \mu_B)(\sum(x_i \otimes y_i))$.

Similarly, we can show that $(\nu_A \otimes \nu_C)((I_A \otimes \bar{g})(\sum(x_i \otimes y_i))) \leq (\nu_A \otimes \nu_B)(\sum(x_i \otimes y_i))$.

We therefore obtain the desired result. □

Proposition 5.5.9. *Let $A \in Ob(\mathbf{C}_{\textbf{R-IFM}})$. Then $- \otimes A$ preserves epimorphisms in $\mathbf{C}_{\textbf{R-IFM}}$.*

We'll now examine how the tensor product functor and the Hom-functor are related in the category $\mathbf{C}_{\textbf{R-IFM}}$. This connection is established through a natural isomorphism known as the Hom-Tensor adjunction. It establishes a relationship between $\mathbf{Hom}_{\mathbf{C}_{\textbf{R-IFM}}}(B \otimes A, C)$ and $\mathbf{Hom}_{\mathbf{C}_{\textbf{R-IFM}}}(A, \mathbf{Hom}_{\mathbf{C}_{\textbf{R-IFM}}}(B, C))$.

Theorem 5.5.10. *(Adjoint Isomorphism)*

In $\boldsymbol{C_{R\text{-}IFM}}$*, there exists IF quasi-isomorphisms*

$$\tau : \boldsymbol{Hom_{C_{R\text{-}IFM}}}(E \otimes D, C) \cong_Q \boldsymbol{Hom_{C_{R\text{-}IFM}}}(D, \boldsymbol{Hom_{C_{R\text{-}IFM}}}(E, C));$$

$$\tau^{'} : \boldsymbol{Hom_{C_{R\text{-}IFM}}}(D \otimes E, C) \cong_Q \boldsymbol{Hom_{C_{R\text{-}IFM}}}(D, \boldsymbol{Hom_{C_{R\text{-}IFM}}}(E, C)).$$

Proof. We demonstrate the existence of the initial quasi-isomorphism. For $D, E, C \in Ob(\mathbf{Hom_{C_{R\text{-}IFM}}})$ and by existence of tensor product in $\mathbf{C_{R\text{-}IFM}}$, a unique IF R-homomorphism $\phi \in \mathbf{Hom_{C_{R\text{-}IFM}}}(E \otimes D, C)$ exists such that

$$\mu_C(\phi(q \otimes p)) \geq (\mu_E \otimes \mu_D)(q \otimes p)$$

$$\nu_C(\phi(q \otimes p)) \leq (\mu_E \otimes \mu_D)(q \otimes p).$$

With due reference to Theorem 2.75 [20], we will define the following IF R-homomorphisms:

For $p \in P$ and $q \in Q$, define $\phi_p : E \to C$ as $\phi_p(q) = \phi(q \otimes p)$,

$\bar{\phi} : D \to \mathbf{Hom_{C_{R\text{-}IFM}}}(E, C)$ as $\bar{\phi}(q) = \phi_p$ and

$\tau : \mathbf{Hom_{C_{R\text{-}IFM}}}(E \otimes D, C) \to \mathbf{Hom_{C_{R\text{-}IFM}}}(D, \mathbf{Hom_{C_{R\text{-}IFM}}}(E, C))$ as $\tau(\phi) = \bar{\phi}$.

It is therefore necessary to prove that ϕ_p, $\bar{\phi}$ are IF R-homomorphisms and τ are IF R-isomorphism.

(i) To begin with, we will show that ϕ_p is IF R-homomorphism. For $q \in Q$, we have

$$\mu_C(\phi_p(q)) = \mu_C(\phi(q \otimes p)) \geq (\mu_E \otimes \mu_D)(q \otimes p) \geq (\mu_E \times \mu_D)(q, p) = \vee\{\mu_E(q), \mu_D(p)\} \geq \mu_E(q).$$

Likewise, we can exhibit that $\nu_C(\phi_p(q)) \leq \nu_E(q)$. Thus, ϕ_p is an IF R-homomorphism.

(ii) Secondly, we will proceed to establish that $\bar{\phi}$ acts as an IF R-homomorphism. For $p \in P$,

$$\begin{aligned}
\mu_{\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(E,C)}(\bar{\phi}(p)) &= \mu_{\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(E,C)}(\phi_p) \\
&= \wedge\{\mu_C(\phi_p(q))|q \in Q\} \\
&= \wedge\{\mu_C(\phi(q \otimes p)|q \in Q\} \\
&\geq \wedge\{(\mu_E \otimes \mu_D)(q \otimes p)|p \in P, q \in Q\} \\
&\geq \wedge\{\vee\{\mu_E(q), \mu_D(p)\}|p \in P, q \in Q\} \\
&\geq \mu_D(p).
\end{aligned}$$

Likewise, we can exhibit that $\nu_{\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(E,C)}(\bar{\phi}(p)) \leq \nu_D(p)$.

(iii) Finally, we will show that τ is an IF R-isomorphism. For $\phi \in \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(E \otimes D, C)$, Consider

$$\begin{aligned}
\mu_{\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(D,\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(E,C))}(\tau(\phi)) &= \mu_{\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(D,\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(E,C))}(\bar{\phi}) \\
&= \wedge\{\mu_{\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(E,C)}(\bar{\phi}(p))|p \in P\} \\
&= \wedge\{\mu_{\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(E,C)}(f_p)|p \in P\} \\
&= \wedge\{\wedge\{\mu_C(\phi_p(q))|q \in Q\}|p \in P\} \\
&= \wedge\{\wedge\{\mu_C(\phi(q \otimes p))|q \in Q\}|p \in P\} \\
&= \wedge\{\mu_C(\phi(q \otimes p))|q \in Q, p \in P\} \\
&= \mu_{\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(E\otimes D,C)}(\phi).
\end{aligned}$$

Similarly,we can prove $\nu_{\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(D,\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(E,C))}(\tau(\phi)) = \nu_{\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(E\otimes D,C)}(\phi)$.

This is called adjoint isomorphism. □

Remark 5.5.11. Natural isomorphism can be obtained by the adjoint isomorphism theorem (5.5.10) as

$$\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(E \otimes D, C) \cong_Q \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(D, \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(E, C))$$

Thus, $E \otimes \Box$ is the right adjoint of $Hom(\Box, E)$.

Fixing any two IF-modules D, E, C, each $\tau = \tau_{D,E,C}$ is a natural isomorphism:

$$\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(E \otimes \Box, C) \cong_Q \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(\Box, \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(E, C));$$

$$\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(E \otimes D, \Box) \cong_Q \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(D, \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(E, \Box));$$

$$\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(\Box \otimes D, C) \cong_Q \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(D, \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(\Box, C))$$

The below figure is commutative for $\theta : D \to D^{'}$

$$\begin{array}{ccc}
\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(E \otimes D^{'}, C) & \xrightarrow{\tau_{D^{'},E,C}} & \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(D^{'}, \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(E, C)) \\
\Big\downarrow (I_E \otimes \theta)^* & & \Big\downarrow \theta* \\
\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(E \otimes D, C) & \xrightarrow{\tau_{D,E,C}} & \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(D, \mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(E, C))
\end{array}$$

Chapter 6

Overall conclusion

This thesis delves into the category of intuitionistic fuzzy modules, investigating their mathematical properties and applications. Through systematically exploring the category of intuitionistic fuzzy modules, the study establishes fundamental definitions and theorems that contribute to the theoretical foundation of this mathematical structure. Furthermore, applications of intuitionistic fuzzy modules in diverse fields exemplify their versatility and relevance. The findings of this research enhance our understanding of the category of intuitionistic fuzzy modules and pave the way for their effective utilization in various mathematical and computational contexts.

Building upon the framework of the category of intuitionistic fuzzy modules, this study extends its focus to intricate aspects, such as special morphisms and the construction of some universal objects within this category. The investigation involves the development of novel concepts and methodologies to characterize the relationships between intuitionistic fuzzy modules, providing deeper insights into their structural properties.

In the Chapter 2, we studied the category of intuitionistic fuzzy modules $\mathbf{C_{R\text{-}IFM}}$ over the category of R-modules $\mathbf{C_{R\text{-}M}}$ by constructing a contravariant functor from the category $\mathbf{C_{R\text{-}IFM}}$ to the category $\mathbf{C_{Lat}}$ (= union of all $\mathbf{C_{Lat(R\text{-}IFM)}}$, corresponding to each object in $\mathbf{C_{R\text{-}M}}$). We showed that $\mathbf{C_{R\text{-}M}}$ is a subcategory of $\mathbf{C_{R\text{-}IFM}}$. Further, we showed that $\mathbf{C_{R\text{-}IFM}}$ is a top category is not an abelian.

Within the $\mathbf{C_{R\text{-}IFM}}$ category, the study of special morphisms emerges as a focal point, providing a nuanced understanding of the relationships between these mathematical structures. Special morphisms are pivotal in capturing the unique characteristics and transformations within intuitionistic fuzzy modules, offering insights into their behaviour and interactions. This work investigated various types of special morphisms, including coretractions, retractions, monomorphisms, epimorphisms, and isomorphisms, within the context of intuitionistic fuzzy modules. Through systematically exploring their properties and significance, the research aims to shed light on the categorical structure of intuitionistic fuzzy modules and enhance our ability to discern and characterize their distinctive features. In Chapter 3, we have proved that the $\mathbf{C_{R\text{-}IFM}}$ is not a balanced category. Further, we proved that if an IF R-homomorphism $\bar{f} : A \to B$ is a coretraction (respectively, retraction), then both f and $\bar{f}$ are one-one (respectively, onto) functions; but the converse does not hold. Exploring special morphisms of intuitionistic fuzzy modules provides valuable insights into the algebraic structures of these modules. By investigating the properties and behaviours of these morphisms, researchers can enhance the theoretical foundations and implications of intuitionistic fuzzy modules, fostering advancements in mathematical and computational fields.

In Chapter 4, we established the existence of IF-products, IF-coproducts, IF-equalizers and IF-coequalizers in the category $\mathbf{C_{R\text{-}IFM}}$. Using these outcomes, we demonstrated the completeness and cocompleteness of the category $\mathbf{C_{R\text{-}IFM}}$. Consequently, we established $\mathbf{C_{R\text{-}IFM}}$ as a bicomplete category which shows the categorical goodness of intuitionistic fuzzy modules. This research contributes a thorough examination of constructing universal objects in the category of intuitionistic fuzzy modules, with a particular emphasis on pullbacks, intersections, images, and inverse images.

Hom functors play a crucial role by capturing morphisms between intuitionistic fuzzy modules, facilitating a deeper understanding of their interplay. Additionally, the study of exact sequences offers a systematic way to analyze the relationships and structures in sequences of intuitionistic fuzzy modules. In Chapter 5, we have established that $\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A,-)$ is a covariant and $\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(-,A)$ is a contravariant functor. Additionally, when researching the intuitionistic fuzzy short exact sequence(IFSE sequence), we have defined an intuitionistic fuzzy R-isomorphism $\Gamma_A : Hom(\bar{O}_{Re}, A) \to A$ determined by $\bar{f} \to \bar{f}(e)$; where $e \in E(R)$. Using this, we have studied the relationship between the Hom-functors and IF-projective modules. The tensor product, a key focus, allows us to extend module operations to accommodate uncertainties, reflecting the nuanced nature of intuitionistic fuzzy environments. Furthermore, we established the existence of the tensor product in the category $\mathbf{C_{R\text{-}IFM}}$. We then investigated the relation between Hom functor and tensor product functor in this category.

This thesis aims to delve into the theoretical foundations and applications of these concepts, offering insights into the intricate algebraic nature of intuitionistic fuzzy modules and their role in addressing uncertainty within mathematical structures. Through this exploration, we aim to provide a comprehensive understanding of these structures and their significance in the broader context of fuzzy mathematics.

One significant direction for future research is the extension of cohomology concepts to intuitionistic fuzzy modules. In the future, investigating special morphisms within intuitionistic fuzzy modules can provide insights into their homological properties. For instance, constructing injective resolutions via these morphisms aids in computing Ext and Tor functors, which is crucial for understanding module structures. Additionally, utilizing special morphisms between projective intuitionistic fuzzy modules enables the construction of projective resolutions, vital for computing Hom and Ext functors, thus enhancing comprehension of algebraic and homological properties. A key future direction is extending cohomology concepts to intuitionistic fuzzy modules, potentially bridging fuzzy logic with established mathematical theories. This extension enriches intuitionistic fuzzy module theory and broadens understanding of cohomological methods, fostering interdisciplinary collaborations and advancing mathematical research.

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List of Abbreviation and Symbols used

IFS	Intuitionistic fuzzy set
$IFS(X)$	Set of all intuitionistic fuzzy sets of X
$\mu_A(x)$	Degree of membership of the element x in the IFS A of X
$\nu_A(x)$	Degree of non-membership of the element x in the IFS A of X
$f(A)$	Image of the IFS A under the map f
$f^{-1}(A)$	Inverse image of the IFS A under the map f
$Kerf$	Kernel of the map f
$Img(A)$	Set of values of the intuitionistic fuzzy set A
$IFSM(M)$	Set of all intuitionistic fuzzy Submodules of M
χ_K	Intuitionistic fuzzy characteristic function of K where K is a submodule of an R-module M
$\mathbf{C_{R\text{-}M}}$	Category of R-module
$\mathbf{C_{R\text{-}IFM}}$	Category of intuitionistic fuzzy module

$\mathbf{C_{WR\text{-}IFM}}$	Category of intuitionistic fuzzy modules with intuitionistic fuzzy weak R-homomorphisms
$\mathbf{C_{WRI\text{-}IFM}}$	Category of intuitionistic fuzzy modules with intuitionistic fuzzy weak R-isomorphisms
$\mathbf{C_{RI\text{-}IFM}}$	Category of intuitionistic fuzzy modules with intuitionistic fuzzy R-isomorphisms
IF R-homomorphism	Intuitionistic fuzzy R-homomorphism
$\mathbf{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A, B)$	Set of all intuitionistic fuzzy R-homomorphisms from A to B
$IFSES$	Intuitionistic fuzzy short exact sequence
IF-coretraction	Intuitionistic fuzzy coretraction
IF-retraction	Intuitionistic fuzzy retraction
IF-projective	Intuitionistic fuzzy projective
IF-injective	Intuitionistic fuzzy injective
IF-equalizer	Intuitionistic fuzzy equalizer
IF co-equalizer	Intuitionistic fuzzy co-equalizer
IF-pullback	Intuitionistic fuzzy pullback
IF-intersection	Intuitionistic fuzzy intersection
IF-image	Intuitionistic fuzzy image
IF-inverse image	Intuitionistic fuzzy inverse image
IFT-product	Intuitionistic fuzzy tensor product

List of Publications

1. Category of Intuitionistic Fuzzy Modules, 2022 , MDPI Mathematics, 10, 399.(SCOPUS Journal)
2. Special Types of Morphisms in the Category C_{R-IFM}, 2023, Notes on Intuitionistic Fuzzy Sets, 29(4), 351-364.(SCOPUS Journal)
3. Tensor Product of Intuitionistic Fuzzy Modules,2023, J. of Ramanujan Society of Mathematics and Mathematical Sciences, 11(1), 63-78.
4. Some Special Objects in the Category of Intuitionistic Fuzzy Modules (C_{R-IFM}), 2024, Advances in Fuzzy Sets and Systems, 29(1), 1-23.
5. Constructions of Some Universal Objects in the Category C_{R-IFM}, 2024, Palestine Journal of Mathematics, (Accepted).
6. Hom-Functors and Exact Sequences of C_{R-IFM} in Neural Network, 2024, Journal of Mathematical Sciences, (SPRINGER),(Communicated).

List of Conferences and Workshops

1. Presented a research paper entitled "Categorical Properties of Intuitionistic Fuzzy Groups" on IFSCOM2021 " 7th IFS and Contemporary Mathematics Conference" in TURKEY, on 25-29 May, 2021.

2. Presented a research paper entitled "Special types of morphisms in the category C_{R-IFM}" in the Two day International Conference on "Recent Development and Techniques in Pure And Applied Mathematics(ICRDTPM-2023)" organised by School of Mathematics, Algappa University, Karaikudi on 23-24 March, 2023.

3. Participated in the Workshop on "Intuitionistic Fuzzy Sets" organised by Slovak Academy of Sciences on 15 December, 2023.

4. Participated and presented a research paper entitled "Constructions of some universal objects in the category C_{R-IFM}" in the 5th International Conference on "Recent Advances in Fundamental and Applied Sciences(RAFAS-2024)" organised by School of Chemical Engineering and Physical Sciences, Lovely Professional University on 19-20 April, 2024.