STUDY OF STABILITY AND BIFURCATION ANALYSIS OF AQUATIC FOOD-WEB MODELS USING DELAY DIFFERENTIAL EQUATIONS

Thesis Submitted for the Award of the Degree of

DOCTOR OF PHILOSOPHY

in

MATHEMATICS

By

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DECLARATION

I, hereby declared that the presented work in the thesis entitled "Study of Stability and Bifurcation Analysis of Aquatic Food-Web Models using Delay Differential Equations" in fulfilment of degree of **Doctor of Philosophy (Ph. D.)** is outcome of research work carried out by me under the supervision Dr. Pankaj Kumar, working as Professor, in the Department of Mathematics, School of Chemical Engineering and Physical Sciences of Lovely Professional University, Punjab, India. In keeping with general practice of reporting scientific observations, due acknowledgements have been made whenever work described here has been based on findings of another investigator. This work has not been submitted in part or full to any other University or Institute for the award of any degree.

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CERTIFICATE

This is to certify that the work reported in the Ph. D. thesis entitled “Study of Stability and Bifurcation Analysis of Aquatic Food-Web Models using Delay Differential Equations” submitted in fulfillment of the requirement for the reward of degree of **Doctor of Philosophy (Ph.D.)** in the Mathematics, is a research work carried out by Ms. Gazal Sharma, 41600108, is bonafide record of her original work carried out under my supervision and that no part of thesis has been submitted for any other degree, diploma or equivalent course.

(Signature of Supervisor)

Name of supervisor: Dr. Pankaj Kumar

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Department: Department of Mathematics

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TO MY FAMILY

I would like to express my deepest gratitude to my family for their unwavering support and encouragement throughout the process of writing this thesis.

To my parents Er. R.N. Sharma and Ms. Madhu Sharma, your unconditional love, patience, and belief in my abilities have been my greatest source of strength. Your guidance and support have been invaluable, and I am forever grateful for the countless sacrifices you have made to help me reach this milestone.

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PAPER PUBLISHED AND COMMUNICATED THROUGH THIS THESIS:

1. Role of Delay in toxin producing Phytoplankton and Zooplankton dynamics presented in International Conference on “Recent Advances in Fundamental and Applied Sciences (RAFAS),2021”.
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3. Dynamics of One-Phytoplankton, Two-Zooplankton System with Square Root Functional Response and Time Lag published in “European Chemical Bulletin”.
4. Phytoplankton-Zooplankton Interaction Harvesting Model: Dynamical Analysis published in “Palestine Journal of Mathematics”.

ABSTRACT

The thesis “**Study of Stability and Bifurcation Analysis of Aquatic Food-Web Models using Delay Differential Equations**” investigates the stability and bifurcation phenomena within aquatic food-web dynamics. The research synthesizes insights from different pivotal research papers, each addressing distinct facets of the complex interactions among phytoplankton, zooplankton, and their environment.

The methodology used for the research is primarily mathematical modelling, specifically using delay differential equations (DDEs). In mathematical ecology, the focus lies not on the direct representation of natural objects, but rather on utilizing mathematical constructs and methodologies as representations of natural phenomena and processes. It serves as a powerful tool for assimilating and consolidating environmental and ecological data. Additionally, the research involves analytical analysis and numerical simulations using MATLAB to explore stability, bifurcation phenomena, and the consequences of time lag on the changing aspects of species within ecosystems. The utilization of MATLAB enhances the reliability and applicability of the findings. Sensitivity analysis of model parameters and validation through mathematical simulations are also conducted. By comprehending the underlying principles guiding our natural theories, mathematical modelling helps prevent the presentation of persuasive arguments that may lack truth or only hold validity under specific conditions.

In natural environments, a species’ pace of population expansion frequently necessitates some time for it to adjust to changes in its own numbers or those of other interacting species. The population age structure, which affects birth and death rates,

varying maturation and gestation periods, such as those of mice (20 days) and elephants (645 days), feeding duration, reaction time, food preservation, resource renewal cycles, and the hunger coefficient in interactions between zooplankton and phytoplankton are some of the factors that contribute to this time delay within the model.

Numerous studies have attempted to ascertain the significance of diverse hydrological characteristics in the progression of plankton blooms, and subsequently explore an appropriate functional result to elucidate the decline in zooplankton inhabitants resulting from toxin-producing phytoplankton (TPP). However, they fall short in explaining a number of important aspects, including the influence of allelopathic interactions on the coexistence and perseverance of phytoplankton species and their direct impact on predators, the regulation of destructive algal blooms or alternations that recur, and the consequence of the time delay necessary for the proclamation of toxic substances. The thesis primarily outlines a structure for planktonic bloom in which the release of toxic constituents or the consequence of toxic phytoplankton is a process that is arbitrated by time delay rather than occurring instantly. This structure can be useful in reducing population oscillations and sustain a stable cohabitation of the species.

The aforementioned mathematical modelling was first applied to study opposing toxin creating phytoplankton on a zooplankton inhabitant. In an environment where harmful phytoplankton and zooplankton interact, this study attempts to examine the consequences of time delay on the constancy and incidence of bifurcation. It examines the dynamics of species within an ecosystem and their interactions with their surroundings. As such, this study integrates sustainability concerns to explore environmental difficulties. Furthermore, this work advances the understanding of delay differential equations conceptually.

Next, the same model was used to investigate the collaborative impacts of phytoplankton on zooplankton. The state variables phytoplankton H_1 and zooplankton C, as well as a companion phytoplankton species H_2 , are incorporated into the model. It is assumed that both phytoplankton defend the inhabitants of zooplankton by cooperating with one another. Hopf bifurcation occurs at critical delay values when delay perturbs the system's equilibrium. Here, MATLAB supports in corroborating analytical findings through numerical simulation.

A different mathematical model was used to investigate the interactions between phytoplankton and two rival zooplankton species. Phytoplankton H, Zooplankton C_1, and Zooplankton C_2 are the related state variables. It is assumed that the inhabitants of Phytoplankton will live along with the inhabitants of Zooplankton that will incline to extinction if the ratio of the mortality rates of Zooplankton to the fundamental of the carrying capacity of inhabitants of Phytoplankton is greater than latter's adaptation competence and rates of predation. When delay is introduced, the system's interior, axial, and boundary equilibria are all disrupted, leading to Hopf bifurcation at critical delay parameters. Sensitivity analysis of model parameters was also conducted, supported by MATLAB simulations.

Additionally, a model of phytoplankton-zooplankton interaction was developed to facilitate commercial harvesting of certain species for food. This model establishes stability criteria across different levels and explores optimal harvesting policies. However, continuous species spreading limits harvesting opportunities and maintaining population equilibrium. Through further research, an optimal balance between biological and bioeconomic equilibrium is sought to support sustainable commercial harvesting practices. Analytical results from the model design are validated through mathematical simulations.

Overall, our study contributes to the advancement of mathematical ecology by elucidating the intricate dynamics of planktonic ecosystems and offering insights into sustainable resource management practices.

Contents

Chapter 1

General Introduction

1.1 Introduction

The scientific subject of ecology studies the number and distribution of living organisms in an ecosystem are influenced by interactions between organisms and their surroundings. Numerous food webs typically come together to form ecosystems. There are several types of food webs that exist, such as those found in microbes, humans, agriculture, detrital, marine, aquatic, soil, and arctic (or polar) food webs. A biological community within an ecosystem is exclusively made up of living things. The need for food that is derived from the sun through plant life links the organisms in a biological community. Herbivores referred to as primary consumers eat the plants. Rabbits, mice, deer, seed-eating birds, cows, horses, and squirrels are a few examples of these.

Secondary consumers that are either carnivores or omnivores eat these creatures. Perhaps, even third- or tertiary-level customers exist. These creatures devour secondary consumers. Like, among the secondary consumers such as owls and snakes are consumed by hawks and mountain lions.

A food cycle is a series of related food chains displaying the movements of energy and matter through an ecosystem. The grazing web, which starts with autotrophs, and the detrital web, which starts with organic debris, are the two main divisions of the food web. There are three categories of creatures that may be distinguished in the food web:

producers, which are plants; consumers, which are both primary and secondary including herbivores, carnivores, or omnivores; lastly decomposers, such as detritivores and decomposers. Food continues to go up the food chain as it always has, with producers and consumers being the components of a bigger structure known as the grazing food web. Food is returned to the soil through the decomposing food web, which is composed of detritivores and decomposers.

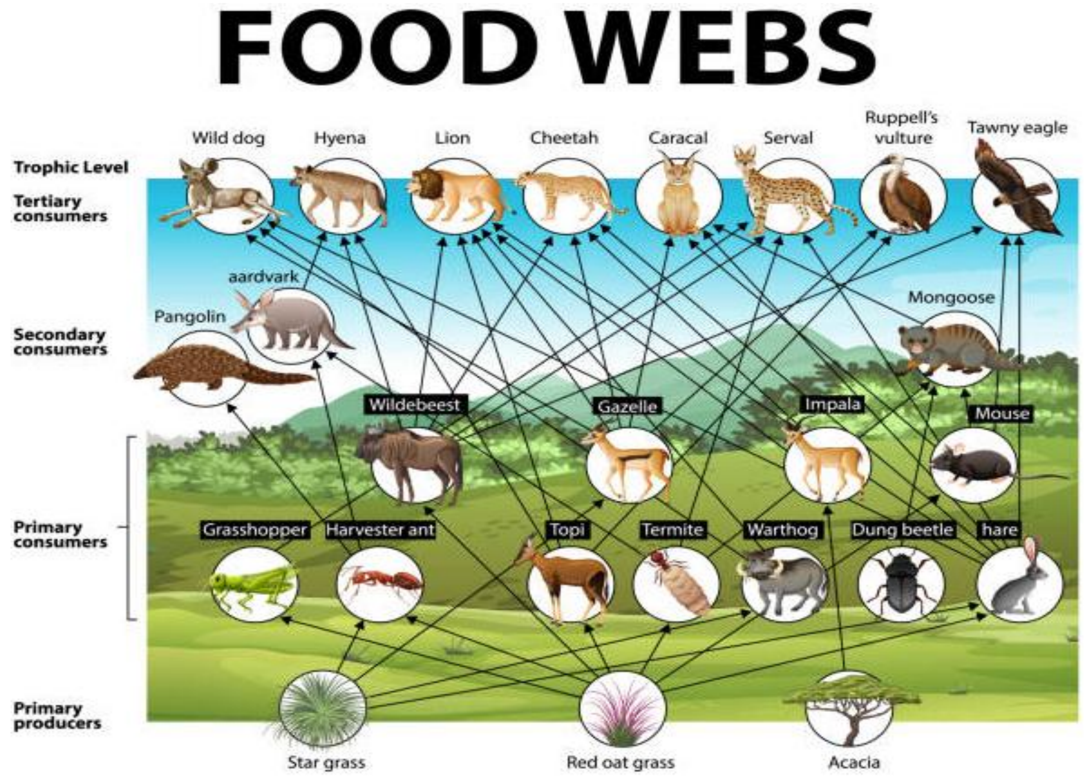

Figure 1.1: Food web diagram

Food webs-interlocking pattern of organisms: - Under natural circumstances, food chains never function as separate sequences; instead, they are linked to one another to form a pattern that interlocks and is known as a food web. The linear arrangement of food chains is rare in nature, and they in fact remain linked to one another through organisms of different kinds at various trophic levels. Hundreds of species are typically connected by their consuming patterns in real food cycle. Food webs are basic components of ecosystem ecology. Some food web in a different ecosystem, with different connected food chains is shown in Figure 1.1.

The stability of an ecosystem in nature is greatly influenced by the food webs.

For instance, a decline in the population of rabbits would inevitably lead to a decline in the population of carnivorous consumers who preferably eat rabbit. Therefore, alternatives (substitutes) serve to maintain the ecosystem's steadiness. Furthermore, survival of all living things depends on a balanced ecosystem. For example, the producers would have died from opposition and overcrowding if the primary consumers—herbivores—had not existed in the natural world. In a similar vein, secondary consumers (carnivores) and primary consumers' survival are interdependent, and so forth. In order for an ecosystem to remain stable, each species is thus kept under some sort of natural control. Any food web's complexity is influenced by the variety of organisms present. As a result, it would rely on two key elements:

1. The length of the food cycle is determined by the diversity of the organisms and their dietary preferences. The food chain would be longer if the organisms' eating habits were more diverse.

2. Alternatives at various consumer chain nodes: The more alternatives, the more the pattern will interlock. The food webs are much more complex in the depth of oceans, seas, and other areas where we discover various species.

In the real world, a species' population growth rate frequently takes time to adjust to changes in either its own population or the population of an interacting species. Age building of the population (affecting rates of births and deaths), period of maturation and gestation which varies for mice (20 days) and elephants (645 days), feeding time, reaction time, food preservation, resource regeneration period, and hunger coefficient in zooplankton phytoplankton interaction are some factors that introduce time delay in the model.

An effective mechanism for integrating and synthesizing environmental and ecological information is mathematical modelling.

Natural objects are not directly addressed in mathematical ecology. It deals with mathematical constructs and procedures that serve as analogues to natural phenomena and processes. The mathematical models only include the most important information that is relevant to the problem at hand, not all of the information about nature that we may be aware of. By understanding the logic behind our theories about nature, mathematical modelling enables us to avoid making convincing arguments that may not be true or are only true in certain circumstances.

Considering the foregoing, the following issues have been researched using mathematical models in this thesis:

1. Functional response of Holling type-II with time-delay is used to model the effects of competing phytoplankton that produce toxins on a zooplankton population.

2. The stability of a system with interacting phytoplankton-toxic phytoplankton-zooplankton species is affected by time delay.
3. An age-based predation phytoplankton-zooplankton model's stability and bifurcation analysis: function of time lag.
4. The impact of time-delay on a multi-team phytoplankton- zooplankton system's dynamics.
5. Dynamic study of a phytoplankton-zooplankton system with mutualistic species opposed to phytoplankton species examined the impacts of time lag and other food sources.
6. Time delay and functional response squared in the dynamics of a single phytoplankton and two zooplankton system.
7. A phytoplankton-zooplankton and host commensal to the zooplankton delay model with host population harvesting.
8. The Asiatic lion, leopard, and ungulates of Gir National Park, India, were studied using a one phytoplankton-two zooplankton delay model.

1.2 Terms used in the Thesis

Commensalism: - It is a class of relationship between two species where one species benefits without affecting the other.

Competition: -It is a class of relationship between two organisms in which one species is harmed due to the presence of the other and vice versa.

Predation: - It depicts a biological relationship in which zooplankton is fed by hunting its phytoplankton. The act of predation causes phytoplankton to die, even though zooplanktons may or may not kill their phytoplankton before eating on them and eventually absorb the phytoplankton's tissue through ingestion.

Gestation period: - Gestation period is the period of time when a fetus develops in mammals, starting with fertilization and ending with birth. The length of this particular phase differs depending on the species. For instance, the gestation period of an elephant is 645 days, but that of a cat is about 58–65 days.

Food webs and chains: - In a wildlife community, feeding process is the primary mechanism that connects the lives of diverse animals. The food and energy stored in an organism's tissues are transferred through a chain when plants and animals grow and are consumed by other creatures. A food web is the resultant network created when food chains connect to one another. In a naturally balanced society, there are an equal number of diverse animals and plants. There might be an impact on the entire web if the balance is off.

1.3 Review of Literature

In a theoretical study on a food cycle model, Klebanoff and Hastings (1994), obtained the normal form at a codimension two point and showed that, for tiny perturbations of the parameters, this normal form can suggest chaos. Later, this idea of a food cycle was examined by Kuznetsov and Rinaldi (1996). They used normal forms and numerical analysis to create a detailed two-parameter bifurcation diagram, taking into account that the zooplankton and super zooplankton death rates can be strongly influenced by variables outside of the system. Differential delay models of population relation with practical persistence was discussed by Cao Y. and Gard T.C. (1997). The practical durability was explored by computing such estimates, using various Liapunov functions, for some basic instances of competition and zooplankton-phytoplankton type, which may involve time delays in the net per capita growth rates. The persistence and overall stability of a delayed non-autonomous zooplankton-phytoplankton system without dominant instantaneous negative feedback were studied by Xu R. and Chen L. (2001). The zeros of transcendental functions were discussed by Ruan S. and Wei J. (2003) with reference to the consistency of delay differential equations using two lags. Braza Peter A.(2003), provides the Holling-Tanner model of bifurcation mechanism for zooplankton-phytoplankton relation using two timing. In this chapter, it has been noted that the Hopf-bifurcation points are separated further, and one limit cycle becomes unstable as the ratio of linear growth rate changes. A population outbreak may cause this situation to change. There was discussion of a zooplankton-phytoplankton model with a stage hierarchy and a continuous maturation time lag by Gourley S.A. and Kuang Y. (2004) (through stage time delay). Through this model, they demonstrated that, for both minor and high values of maturation time lag, if the juvenile mortality rate (through death rate stage) is non-zero, then a globally appealing steady state is the most basic kind of population dynamics. If the resource is dynamic in nature, then the linear stability shows that there is a window in the maturation time lag boundary that produces sustained dynamics of oscillations. Bifurcations of a ratio-reliant zooplankton-phytoplankton mechanism that harvests continuously were studied by Xiao D. and Jennings L.S. (2005). They looked at the ratio-reliant zooplankton phytoplankton model's dynamical characteristics with non-zero constant rate harvesting. As the values of the model's parameters change, it is demonstrated that the concept can undergo many diverse types of bifurcation, including saddle-node, subcritical, and supercritical hopf, Bogdanov Takens, homoclinic, and heteroclinic. The impact of age-based vaccination policies on the dynamics of delay epidemic models was examined by Misra O. P. et al. (2006). A phytoplankton-zooplankton system with dual delays' constancy and analysis of Hopf bifurcation were discussed by Li K. and Wei J. (2009). Wan A. and Wei J. (2010), investigated the delayed Hopf bifurcation analysis of a population with food scarcity. In a delayed zooplankton phytoplankton system, Yuan S. and Zhang F. (2010), investigated consistency and universal bifurcation. Three interacting species were analysed qualitatively in two food-web models based on resources by Kesh D. et al. (1997). The constancy and complication of environmental systems was examined by researchers Upadhyay R. K. et al. (2000). According to their argument, dynamical complexity can exist without structural complexity. The intricate dynamics of a ratio-dependent single phytoplankton, two-zooplankton model have been studied by Hsu S. et al. (2001). El-Owaidy et al. (2001) discussion of a food-web model's mathematical exploration. In a harvested one-zooplankton, two-phytoplankton model, Kumar S. et al. (2002) studied the constancy analysis and Hopf-bifurcation. The Hopf-bifurcation is explained in this chapter using an intriguing one-zooplankton, two phytoplankton model with constant zooplankton harvesting rate. In this case, the harvesting rate is a controlling factor. They discovered that when the harvesting rate exceeds a particular threshold, periodic solutions emerge from stable stationary states. In a food chain model, Upadhyay R. K. (2003) discussed various attractors and the crisis path to chaos. Order and chaos in the food cycle that is dependent on the zooplankton-phytoplankton ratio were discussed by Gakkhar S. and Naji R. K. (2003). They investigated the chaotic dynamics and realistic food cycle model of three species, taking into account type II functional response and zooplankton to phytoplankton ratio dependence for the interaction. Gakkhar S. and Naji R. K. (2003) studied a two phytoplankton, one zooplankton model that included non-linear functional response. When long-term behaviour in this model was examined, they discovered that for a variety of parametric values, the system appeared to exhibit chaos. Bockleman et al. (2004), investigated disorderly cohabitation in a top-zooplankton-mediated competitive exclusive web. Models of the zooplankton-phytoplankton and food web's reactivity and transient dynamics were examined by Neubart et al. (2004). Single -zooplankton and two-phytoplankton model with functional response of Holling type II and no interspecific competition was taken into consideration by Kvrivan and Eisner (2006). They hypothesized that the zooplankton's ideal foraging habits might facilitate the coexistence of all species.

In their article, Gakkhar S. and Singh B. (2005) discussed the intricate food chain dynamics made up of double phytoplankton’s and a zooplankton. Naji R. K. and Balasim A. T. (2007) conducted research on the dynamic behaviour of the food network model with three species. Gakkhar and Singh B. (2007), investigated the subtleties of a food chain made up of double phytoplanktons and single zooplankton that is actively harvesting. Elettreby M. F. (2009) examined the two-phytoplankton, one-zooplankton model in their paper. Wei Hsiu-Chuan (2010), conducted research pertaining to bifurcation analysis of a four-species food network. In a given a food cycle mechanism including n species and time lags, Liu Q. and Zhou H. (2003), discussed the presence and universal attractivity of intervallic resolutions. The bifurcation and constancy in a harvested single-zooplankton-double-phytoplankton delayed model were studied by Liu Z. and Yuan R. (2006). Algal blooms in two detrimental phytoplankton-zooplankton systems can be controlled by a time delay, according to Sarkar et al. (2007); Zhao J. and Wei J. (2009), studied the bifurcation and stability of two detrimental phytoplankton-zooplankton systems. The dynamic characteristics of a delayed zooplankton phytoplankton model with harvesting was discussed by Kar T. K. and Ghorai Abhijit (2011). The modification of the Leslie-Gower and scheme of Holling-type II can be seen in the manner defined here. Bairagi N.and Jana D. (2011) conducted research on the Hopf bifurcation and constancy caused by delay of zooplankton-phytoplankton mechanism with complex environment. The Hopf bifurcation of a zooplankton-phytoplankton system featuring harvesting and a stage structure was discussed by Ge Zhihao and Yan Jingjing (2011). Zabalo Joaquin (2012), investigates persistence in an intraguild predation model with phytoplankton switching. Misra O. P. et al. (2012), examined the impact of time lag on the constancy of a phytoplankton-toxic phytoplankton-zooplankton interacting species system. A three-part model with two competing, detrimental phytoplankton and a zooplankton with a discrete time lag and functional response of Holling type-II was discussed by Misra O. P. et al. in (2012).

Gupta R. P. and Chandra Peeyush (2013), conducted a bifurcation investigation of the

Leslie-Gower zooplankton-phytoplankton prototype with Michaelis-Menten phytoplankton harvesting. Priyadarshi A. and Gakkhar S. (2013) investigated the dynamics of generalist zooplankton of the Leslie-Gower type in a tritrophic food network system. A phytoplankton-zooplankton model with age-based predation was studied for stability and bifurcation analysis by Misra et O. P. al. in (2013).

1.4 Mathematical Preliminaries

1.4.1 Existence of Unique, Bounded and Positive Solution of Delay differential equation

A differential equation in which the present time derivative is dependent upon the solution and derivatives of previous times is known as a delay differential equation. Here, it is necessary to establish a primary history function as opposed to an initial condition. A delayed state variable can be used to demonstrate the past dependence of a differential equation. The derivative of the state variable is not required in this case. The corresponding delay differential equation with a single delay $\tau > 0$ is given by (Smith, (2010))

$$\dot{x}(t) = f\big(t, x(t), x(t-\tau)\big) \tag{1.1}$$

Assume that $f(t, x, y)$ and $f_x(t, x, y)$ are continuous on R^3. Let $s \in R$ and$\emptyset: [s - \tau, s] \to R$ be constant. We look for a solution $x(t)$ of equation (1.1) satisfying

$$x(t) = \emptyset(t). t \in [s - \tau, s], x(0) = x_0 \tag{1.2}$$

And satisfying equation (1.2) on $t \in [s,\ s + \sigma]$ for some $\sigma > 0$.

Theorem 1 (Existence of unique solution). Let $f(t,x,y)$ and $f_x(t,x,y)$ are continuous on R^3. Let $s \in R$ and $\emptyset: [s-\tau, s] \to R$ be continuous. Next there is, $\sigma > s$ and a distinctive solution of the primary-value problem (1.1) - (1.2) on $[s-\tau, \sigma]$.

Theorem 2 (Boundedness of solution). Let f satisfy the hypothesis of theorem1 and let $x: [s-\tau, \sigma) \to R$ be the non-continuous solution for the primary value problem (1.1)- (1.2). If $\sigma < \infty$ then $\lim_{t\to\sigma-} |x(t)| = \infty$.

Remark. Theorems 1 and 2 extend instantly to the case that $x \in R^n$ and $f: R \times R^n \times R^n \to R^n$, it encompasses several distinct delays as well. $\tau_0 < \tau_1 < \cdots < \tau_m$ where $f = f(t, y(t), y(t-\tau_0), y(t-\tau_1), \dots, y(t-\tau_m))$.

Theorem 3 (Positivity of solution). Suppose that $f: R \times R_+^n \times R_+^n \to R^n$satisfies the hypothesis of theorem 1 for all i, t and for all $x, y \in R_+^n$:

$$x_i = 0 \Rightarrow f_i(t,x,y) \geq 0$$

If the initial data $\emptyset$ in equation (1.2) satisfy $\emptyset \geq 0$, then the corresponding solution $x(t)$ of equation (1.1) satisfy $x(t) \geq 0$ for all $t \geq s$ where it is defined.

1.4.2 Stability by Variational matrix method

Let an autonomous system of equations be

$$\frac{dy}{dt} = f(y) \tag{1.3}$$

Where y is an n-tuple vector i.e. $y = (y_1, y_2, ---y_n)$. Let $\phi(t)$ be the result of system defined above. The linear part of the expansion of the system (1.3) about $\phi(t)$ is given by the variational equation of the system (1.3) with respect to $\phi(t)$, written as

$$\frac{dx}{dt} = f_y(\emptyset(t))x \tag{1.4}$$

Where $f_y(\phi(t)) = \frac{df_i}{(dy_j)_{n\times n}}$ at $\phi(t)$. Since the constancy of the variational structure depicts the constancy of any solution of a non- linear system governed by it, so stability of $x = 0$ determines the stability of $y = \phi(t)$. Particularly, when $\phi(t) = \phi_0$, a constant, the system (1.4) becomes

$$\frac{dx}{dt} = Ax \tag{1.5}$$

Where $A = f_y(\phi_0)$. Since a small perturbation of the system (1.3) is represented by system (1.4), so the stability of $y = \phi_0$ of (1.5) actually gives the stability of the solution of $x = 0$ of (1.4) .The description of constancy of every solution of $x' = Ax$ is given by following theorems. (S. Ahmad & Rao,2014).

Theorem 4 Asymptotically stable solutions exist for all solutions of the system $x' = Ax$,where $A = (a_{ij})$ is a constant matrix, if all of the distinctive roots of A have negative real portions.

Theorem 5 All of the solutions to the system $x' = Ax$ are bounded and hence stable if all of *A's* characteristic roots with multiplicity higher than one have negative real parts and all of its roots with multiplicity of one have non-positive real parts.

Following theorem (S.Ahmad & Rao,2014) to determine the sign of real components of the characteristic equation's roots.

Theorem 6 Hurwitz's Theorem. A prerequisite that is both essential and sufficient for the actual part to be negative of all the roots of the polynomial $L(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + ----+a_n$,with the Hurwitz matrix's primary diagonals of all the minors are positive and have real coefficients.

$$H_n = \begin{bmatrix} a_1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & 1 & 0 & 0 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Theorem 7. Let $\varsigma_{1,}\varsigma_2 \ldots . \varsigma_m$ are all non-negative and $\zeta_i^{j}(j = 0,1,2, \ldots m\colon i = 1,2, \ldots n)$ are constants. As $(\varsigma_1, \varsigma_2, \ldots, \varsigma_m)$ differ, the total of the orders of the zeros of exponential polynomial $P(\chi, e^{-\chi\varsigma_1}, \ldots ., e^{-\chi\varsigma_m})$ only be different when the right half plane is open in the event that the imaginary axis has a zero or crosses it, where,

$$P(\chi, e^{-\chi\varsigma_1}, \ldots ., e^{-\chi\varsigma_m}) = \chi^n + \zeta_1{}^0\chi^{n-1} + \cdots + \zeta_{n-1}{}^0\chi^n + \zeta_n{}^0 + [\zeta_1{}^1\chi^{n-1} + \cdots + \zeta_{n-1}{}^1\chi^n + \zeta_n{}^1]e^{-\chi\varsigma_1} + \cdots + [\zeta_1{}^m\chi^{n-1} + \cdots + \zeta_{n-1}{}^m\chi^n + \zeta_n{}^m]e^{-\chi\varsigma_m}$$

Ruan & Wei (2001),S Ruan & Wei (2001), Shigui Ruan & Wei(2003) proved this theorem using Rouches theorem (Dieudonne, 1960).

1.4.3 Hopf-Bifurcation

Hopf made a significant contribution when expanded on two-to-higher dimensions. Sometimes Hopf bifurcation is also called as “Poincaré-Andronov-Hopf bifurcation”. (Marsden et al. 1978) According to the Hopf-bifurcation theorem, that topological characteristic of flow changes when one or more parameters are changed. The essential observation about flows is that if the stationary point is hyperbolic, that is, if all of the eigenvalues of the linearized flow at the stationary point have non-zero real pars, then the local behaviour of the flow is totally determined by the linearized flow. Therefore, only at parameter values for which a stationary point is non-hyperbolic may bifurcations of stationary points occur. More, precisely, a bifurcation value of a parameter is a value at which the qualitative nature of the flow changes.

The Hopf bifurcation is several orders of magnitude where a two-dimensional centre manifold and a non-hyperbolic stationary point with linearized eigenvalues $\mp i\omega$ make it more difficult to study, and bifurcating results are periodic as different to stationary.

Theorem 8 Hopf-Bifurcation Theorem

Let us consider a family of delay equations with one parameter:

Where $F: C \times R \rightarrow R^n$ is, with according to its reasons, a twice continuously differentiable and represents a stable state at $x = 0$ for every value of μ: $F(0, \mu) \equiv 0$.

Here, we may linearize F about $\emptyset = 0$ as follows:

$$F(\emptyset, \mu) = L(\mu)\emptyset + f(\emptyset, \mu)$$

In this case, $L(\mu): C \rightarrow R^n$ is a linear operator with bounds and f is greater order:

$$\lim_{\emptyset \rightarrow 0} \frac{|f(\emptyset, \mu)|}{||\emptyset||} = 0$$

Following is the characteristic equation about L:

$$|\lambda I - A(\mu, \lambda)| = 0,$$

$$A_{ij}(\mu) = L(\mu)_i(e_\lambda e_j)$$

The roots of this equation constitute the main assumption(H). The characteristic equation will have two simple roots $\mp i\omega$ with $\omega_0 \neq 0$ and no other root can be a multiple of an integer of $i\omega_0$ for $\mu = 0$

Here a root of order one means (Pandey et al. 2016)a simple root. Assuming that the typical equation is expressed as $h(\mu, \lambda) = 0$, then **(H)** implies $h_\lambda(0, i\omega_0) \neq 0$. So, according to the theorem of implied function, there exists a continuously differentiable family of roots $\lambda = \lambda(\mu) = \alpha(\mu) + i\omega(\mu)$ for small-scale μ sustaining $\lambda(0) = i\omega_0$. In specific, $\alpha(0) = 0$ and $\omega(0) = \omega_0$. Further, pre-assumption is that as μ increases through zero, the line of imaginary axis is crossed transversally by these roots. Actually, the assumption is:

$$\alpha'(0) > 0 \tag{1.6}$$

In case $\alpha'(0) < 0,$ we always ensure that equation (1.6) holds by changing the sign of the parameter i.e. we take factor $v = -\mu$. Thus, The positive symbol essentially serves as a normalization that guarantees that if $\mu < 0$, then, a negative real component is present in the pair of roots and if $\mu > 0$, then the real part is positive.

Theorem 9. Let **(H)** and equation (1.6) hold. Subsequently $\varepsilon_0 > 0$ occur, followed by actual value even function $\mu(\varepsilon)$ and $T(\varepsilon) > 0$ fulfilling $\mu(0) = 0$ and $T(\varepsilon) = {}^{2\pi}/_{\omega_0}$, and a non-constant $T(\varepsilon)$- periodic function $p(t, \varepsilon)$ where all functions being continuously differentiable in ε for $|\varepsilon| < \varepsilon_0$, so that $p(t, \varepsilon)$ is a solution of the mentioned equation and $p(t, \varepsilon) = \varepsilon\, q(t, \varepsilon)$ where $q(t, 0)$ is a ${}^{2\pi}/_{\omega_0}$-periodic solution of $q' = L(0)q$.

Additionally, there exist $\mu_0, \beta_0, \delta > 0$ so that it has a non-constant regular solution $x(t)$ of period P for some μ fulfilling $|\mu| < \mu_0$ with $max_t|x_t| < \beta_0$ and $\left|P - {}^{2\pi}/_{\omega_0}\right| < \delta$, then $\mu = \mu(\varepsilon)$ and $x(t) = p(t + \theta, \varepsilon)$ for some $|\varepsilon| < \varepsilon_0$ and some θ.

If F is differentiable five times in a row, then:

$$\mu(\varepsilon) = \mu_1\varepsilon^2 + O(\varepsilon^4)$$

$$T(\varepsilon) = \frac{2\pi}{\omega_0}[1 + \tau_1\varepsilon^2 + O(\varepsilon^4)]$$

Asymptotically, $p(t, \varepsilon)$ is stable if $\mu_1 > 0$ and unstable if $\mu_1 < 0$ if all other characteristic roots for $\mu = 0$ have solidly negative real components, with the exception of $\mp i\omega$.

1.4.4 Sensitivity Analysis of State Variables with respect to Model Parameters

Systematic evaluation of the effects of model parameters on system solutions is called sensitivity analysis. There are number of methods to do sensitivity analysis of systems without delay, but there are only a few methods for sensitivity analysis of systems involving delays. The knowledge of how a small change in model parameter can bring change in the state variable, can be a great help in modelling process. It helps in elimination of ineffective and irrelevant parameters. It gives a complete insight into the overall behaviour of the proposed model.

If all the parameters in the given system (1.1) - (1.2) are considered to be constants, then analysis includes just the calculation of partial derivatives of solution with respect to each parameter (Rihan, 2003). The matrix of sensitivity functions is of the form:

$$S(t) \equiv S(t, \alpha) = \left[\frac{\partial}{\partial \alpha}\right]^T x(t, \alpha) \qquad (1.7)$$

Its jth column is: $S_j(t, \alpha) = \left[\frac{\partial x_j(t,\alpha)}{\partial \alpha_1}, \frac{\partial x_j(t,\alpha)}{\partial \alpha_2}, \dots, \frac{\partial x_j(t,\alpha)}{\partial \alpha_n}\right]^T$

This column vector gives sensitivity of the solution $x_j(t,\alpha)$ for small change in parameter $\alpha_i, i = 1,2,3,\dots,n$.

Theorem 10. $S(t)$ satisfies the delay differential equation:

$$S'(t) = J(t)S(t) + J_\tau(t)S(t-\tau) + B(t), t \geq 0$$

Where $J(t) = \frac{\partial}{\partial x} f(t,x,x_\tau),\ J_\tau(t) = \frac{\partial}{\partial x_\tau} f(t,x,x_\tau),\ B(t) = \frac{\partial}{\partial \alpha} f(t,x,x_\tau)$

1.5 Summary

The present thesis consists of five chapters. In Chapter- 1, a general introduction of the work carried in light of the existing literature is given along with some basic concepts and terminology used in the study and mathematical techniques applied in the analysis of the mathematical models.

In Chapter 2, considered a structure comprising the function of delay and the two opposing toxic phytoplankton and zooplankton. We have examined the system's stability behaviour in the vicinity of the accurate steady states. Our theoretical and numerical findings demonstrate that the system immediately has asymptotic consistency around the internal equilibrium that is positive, which stands for the coexistence each of the three species, at a given threshold of the system characteristics. Based on both qualitative and quantitative analysis, we determine that the predation rates (ρ_1 and ρ_2) of the two harmful phytoplankton are bifurcating parameters. The interior equilibrium point for these parameters displays a stable bifurcating solution when ρ_1 exceeds the edge value ρ_1^*, and when ρ_2^* falls below the edge value ρ_2^* respectively. Likewise, it is discovered that half saturation constants

(h_1 and h_2) are bifurcating parameters, exhibiting a stable bifurcating solution at the interior equilibrium point when h_1 is less than the number set as the threshold h_1^* and h_2^* and beyond the threshold value h_2^* respectively. Interestingly, though, the bifurcating solution's dynamic character is dependent on the values of ρ_1, ρ_2, h_1 and h_2. For a given critical value of the delay parameter, it is seen that the stable interior equilibrium points once more displays Hopf-bifurcation when the time-delay is taken into account within the system.

In the Chapter-3, the impact of time lag on a multi-team phytoplankton-zooplankton system comprising of two phytoplankton and one zooplankton is studied considering that the two phytoplankton-populations help each other when they are subjected to predation. It is demonstrated that the addition of time delay destabilizes the stable balance point of the original system. It is also shown that the Hopf-bifurcation can arise in the system as the period of delay parameter crosses the significant values. Furthermore, utilizing the central manifold reduction theorem and normal form theory, a particular method is devised that defines the route and constancy of the Hopf-bifurcating solution.

The Chapter-4, has taken into consideration a single phytoplankton-two zooplankton system using a functional response of a square root and a phytoplankton growth rate delay. In contrast to the phytoplankton-zooplankton model investigated by Valerio Ajraldi et al. (2011), the inclusion of another competing zooplankton makes the system more stable and prevents limit cycles from occurring in the system spontaneously without delay. Limit cycles naturally emerge under certain situations involving model parameters in the phytoplankton-zooplankton model explored by Valerio Ajraldi et al.

(2011). The conclusion drawn via analysis of stability of the axial balance point (X, 0, 0) is that if the proportion of the zooplankton populations' mortality rates expressed as a square root of the phytoplankton population's carrying capacity is greater than the product of the corresponding conversion effectiveness and rates of predation, the phytoplankton community will survive, and the zooplankton populations will tend to extinct. According to the analysis of stability of the boundary stability point ($X, P_1, 0$), the density dependent mortality rate of the zooplankton population (P_2) reaches extinction if the proportion of the zooplankton's mortality rate (P_1) is less than the ratio of the product of its conversion efficiency and its rate of predation of the zooplankton's mortality rate (P_2) in relation to the product of its rate of predation and conversion effectiveness. The phytoplankton community *(X)* and one of the zooplankton populations (P_1) survive. According to the constancy analysis of the boundary equilibrium point ($X, 0, P_2$), if the constancy level of the phytoplankton inhabitants is lower than the proportion of the zooplankton (P_1) inhabitants' death rate to the sum of its rate of predation, adaptability, and the square root of the phytoplankton population's carrying capacity, then the phytoplankton population *(X)* and the zooplankton population (P_2) would survive and the other zooplankton population (P_1) would go extinct. If the ratio of the zooplankton mortality rate (P_1) to the product of its rate of predation and conversion effectiveness is more than the one-third's square root of the phytoplankton population's carrying capacity, then coexistence exists within the system. Also, when time lag exceeds a significant threshold, maximum cycles occur at each equilibrium point, accounting for the effects of time delay within the system.

In Chapter-5, we have suggested a phytoplankton- zooplankton system with host-commensal to the zooplankton in which time lag is considered in logistic growth of the host species which is undergoing harvesting. In the absence of host species, the system has stable co-existence and after the introduction of host species, the equilibrium level of predator increases preserving the stability behaviour up to a certain value of commensal rate.

Further it is noted that if commensal rate increases then limit cycles occurs and the system loses its stability. It is shown numerically that the phytoplankton-zooplankton system shows oscillatory behaviour in the absence of host species when intrinsic growth rate is less than or equal to harvesting rate of host species ($r_2 \leq H$). If $r_2 > H$ then the host species will survive and the phytoplankton-zooplankton system will exhibit similar oscillatory behaviour with larger amplitude. From the analysis it is observed that the role of harvesting rate is opposite to that of commensal rate with regard to the underlying system's dynamic behaviour. Following the consideration of the effect of time-delay within the system, limit cycles arise when the time lag exceeds certain significant levels for interior equilibrium points that were otherwise stable without delay. It is also shown analytically as well as numerically that the length of delay increases as harvesting rate increases and specific growth rate of host species decreases.

Chapter 2

Role of Delay in toxin producing Phytoplankton and Zooplankton dynamics

Abstract

The effects of zooplankton exposed to toxic phytoplankton are studied using a mathematical model proposed in this chapter. Phytoplankton H_1, Phytoplankton H_2, and Zooplankton C are the state variables linked to them. The assumption is that the population of zooplankton is negatively impacted by the toxicity released by two phytoplanktons. The system's internal equilibrium was disturbed by the delay addition, leading to Hopf bifurcation at the critical delay parameter value. MATLAB is used to provide numerical simulation to complement analytical findings.

Keywords: Phytoplankton, Zooplankton, Plant biomass, Hopf bifurcation, Toxic material.

2.1 Introduction

Over the past three decades, harmful plankton blooms have increased worldwide as studied by Anderson (1989); Hallegraeff (1993) & Smayda (1990). The mechanisms underlying hazardous algal blooms (HAB) and how to regulate them have recently drawn more attention to researchers like Blaxter et.al (1998); Huang et.al (2006); Sarkar et.al (2007) & Zhao and Wei (2009).

It is a kind of ecological model that has extensively been studied and established by Li and Chen (2009) far along with other different plankton models. Also, Chattopadhyay et.al (2002) & Khare et. al (2010) in their work attempt to a certain the role that various hydrological conditions have in the development of plankton blooms and to explore the appropriate form of functional reaction to explain the reason behind the declination in zooplankton population by toxin-producing phytoplankton (TPPs). One of the most important factors is the drop in zooplankton pressure dure to grazing is caused by phytoplankton's production of toxins as explained by Keating (1976) & Kirk and Gilbert (1992). The exclusion principle explains the phenomenon where zooplankton steers clear of regions abundant in certain phytoplankton species, such as Rhizosolenia, Coscinodiscus, and Phaeocystis, etc. either because of dense phytoplankton concentration or because they produce some toxic and unpleasant elements Odum (1971).

Buskey and Stockwell (1993) research in the southern coast of Texas explains that while Aureococcus anophagefferens, a chrysophyte, bloom, there is declination of population of micro and meso zooplankton. The research findings demonstrated that harmful phytoplankton contributes to an increase in zooplankton populations, which in turn has a major impact on interactions between both. But, it cannot be denied that not all the above-mentioned research were able to explain various features like allelopathic interaction on phytoplankton’s persistence and coexistence, its immediate effect on predators, the regulation of oscillations or dangerous algal blooms, the impact of delay needed to discharge toxic substances. At this point, it is important to highlight that numerous predational functional response arrangements and toxin releases provide a variety of intriguing dynamics of the system Sarkar and Malchow (2005).

The study presents planktonic bloom mechanism where the release of noxious elements and the consequences of hazardous phytoplankton is not an instantaneous but rather are arbitrated by certain amount of time and which can be effective in reducing population oscillations and in turn, which is very crucial in maintaining a co-existence stability between the organisms. Several research data indicate that the harmful phytoplankton bloom eventually causes death of zooplankton and therefore the time delay biological study is being considered extremely important. The number of Paracalanus (zooplankton) populations is found to be declining after a period of bloom of toxic phytoplankton Noctiluca scintillans as explained in mathematical and empirical explanations Chattopadhyay et.al (2002). This fact allows some substantial liberty in the model construction considering the delay factor. According to Sarkar et al. (2005), hazardous compounds are categorized into Holling type II functions and the predational functional response are linear responses. The research was based on plankton that produces toxin and their impacts on phytoplankton-zooplankton system. The authors also proposed that the roles of delay and environmental variations in above mentioned adverse dynamics of phytoplankton and zooplankton may yield interesting outcomes and should be considered for further investigation.

2.2 Statistical Model

The current research was inspired by interactions discovered in theory and experiment between different kinds of phytoplankton-zooplankton interactions. The differential equation system that governs the dynamics are as follows. The three state variables are:

$H_1(t)$: ratio of the first type of dangerous phytoplankton at time t

$H_2(t)$: ratio of the second type of dangerous phytoplankton at time t

$C(t)$: concentrations of the Zooplankton $at\ time\ t$

$$\frac{dH_1}{dt} = \gamma_1 H_1 \left(1 - \frac{H_1}{s}\right) - k_1 H_1 H_2 - \beta_1 H_1 C \tag{2.1}$$

$$\frac{dH_2}{dt} = \gamma_2 H_2 \left(1 - \frac{H_2}{s}\right) - k_2 H_1 H_2 - \beta_2 H_2 C \tag{2.2}$$

$$\frac{dC}{dt} = (\alpha_1 H_1 + \alpha_2 H_2)C - dC - \emptyset_1 H_1(t-\tau)C - \emptyset_2 H_2(t-\tau)C \tag{2.3}$$

The additional primary conditions mentioned below are included in the system of equations (1-3);

a. $H_1(0) \geq 0$

b. $H_2(0) \geq 0$

c. $C(0) \geq 0$

2.2.1 Delineation of the variables

Parameters	Description
γ_1	The rate of harmful phytoplankton's expansion H_1
γ_2	The rate of harmful phytoplankton's expansion H_2
s	Carrying Capacity of both phytoplankton species
β_1	The predation of phytoplankton’s rate H_1 by zooplankton C
β_2	The predation of phytoplankton’s rate H_2 by zooplankton C
α_1	The conversion rates of Zooplankton C on phytoplankton H_1

α_2	The conversion rates of Zooplankton C on phytoplankton H_2
d	Zooplankton’s Natural death rate C
$\emptyset_1$	Proportion of toxic release by harmful phytoplankton H_1 on zooplankton C
$\emptyset_2$	Proportion of toxic release by harmful phytoplankton H_2 on zooplankton C
k_1	Inhibitory detrimental phytoplankton effect H_1 on the other phtoplankton H_2
k_2	Inhibitory detrimental phytoplankton effect H_2 on the other phtoplankton H_1
τ	Delay parameter

2.3 Model Without Delay

$$\frac{dH_1}{dt} = \gamma_1 H_1\left(1-\frac{H_1}{s}\right) - k_1 H_1 H_2 - \beta_1 H_1 C \tag{2.4}$$

$$\frac{dH_2}{dt} = \gamma_2 H_2\left(1-\frac{H_2}{s}\right) - k_2 H_1 H_2 - \beta_2 H_2 C \tag{2.5}$$

$$\frac{dC}{dt} = \big((\alpha_1-\emptyset_1)H_1 + (\alpha_2-\emptyset_2)H_2\big)C - dC \tag{2.6}$$

2.4 Interior equilibrium of Model

Solving $\frac{dH_1{}^*}{dt} = 0\,, \frac{dH_2{}^*}{dt} = 0 \; and \, \frac{dC}{dt}$ simultaneously,

$$H_1{}^* = \frac{(\beta_1\gamma_1 - \beta_1\gamma_2)(\alpha_2-\emptyset_2) - d(k_1\beta_2 - \frac{\gamma_2\beta_1}{s})}{\left[\left(\frac{\gamma_1\beta_2}{s} - k_2\beta_1\right)(\alpha_2-\emptyset_2) - (\alpha_1-\emptyset_1)(k_1\beta_2 - \frac{\gamma_2\beta_1}{s})\right]}$$

$$H_2{}^* = \frac{(\beta_2\gamma_1 - \beta_1\gamma_2)(\alpha_1-\emptyset_1) - d(\frac{\gamma_1\beta_2}{s} - k_2\beta_1)}{\left[\left(k_1\beta_2 - \frac{\gamma_2\beta_1}{s}\right)(\alpha_1-\emptyset_1) - (\alpha_2-\emptyset_2)(\frac{\gamma_1\beta_2}{s} - k_2\beta_1)\right]}$$

$$C^* = \frac{1}{\beta_1}\left(\gamma_1 - \frac{\gamma_1}{s}\left(\frac{(\beta_1\gamma_1 - \beta_1\gamma_2)(\alpha_2 - \emptyset_2) - d\left(k_1\beta_2 - \frac{\gamma_2\beta_1}{s}\right)}{\left[\left(\frac{\gamma_1\beta_2}{s} - k_2\beta_1\right)(\alpha_2 - \emptyset_2) - (\alpha_1 - \emptyset_1)\left(k_1\beta_2 - \frac{\gamma_2\beta_1}{s}\right)\right]}\right)\right.$$
$$\left. - k_1\left(\frac{(\beta_2\gamma_1 - \beta_1\gamma_2)(\alpha_1 - \emptyset_1) - d(\frac{\gamma_1\beta_2}{s} - k_2\beta_1)}{\left[\left(k_1\beta_2 - \frac{\gamma_2\beta_1}{s}\right)(\alpha_1 - \emptyset_1) - (\alpha_2 - \emptyset_2)(\frac{\gamma_1\beta_2}{s} - k_2\beta_1)\right]}\right)\right)$$

2.5 Study of Interior Equilibrium and Local Hopf-Bifurcation

We will only look at the interior equilibrium's dynamic behaviour in this section. It is evident that there is only one positive equilibrium in the system of equations (2.1), (2.2), and (2.3).

$$\begin{vmatrix} \mu - m_1 & -m_2 & -m_3 \\ -m_4 & \mu - m_5 & -m_6 \\ -m_7 & -m_8 & \mu - m_9 \end{vmatrix} = 0$$

The following is the exponential characteristic equation for equilibrium E^* :

$$\mu^3 + P_1\mu^2 + P_2\mu + P_3 + e^{-\mu\tau}(Q_1\mu + Q_2) \qquad (2.7)$$

Where $P_1 = (\delta_1 + \delta_2 + \delta_3 + \gamma N^*), P_2 = (\delta_1\delta_2 + \delta_2\delta_3 + \delta_3\delta_1 + \delta_1\gamma Q^* + S_{NM}\gamma P^*),$

$$P_3 = (\delta_1\delta_2\delta_3 + \delta_2 S_{NM}\gamma P^*), Q_1 = \alpha Z^*(\delta_1 + \delta_3 + \gamma Q^*), Q_2 = \delta_2\delta_3\alpha Z^*.$$

Clearly P_1, P_2, P_3, Q_1, Q_2 will be positive.

Now, $\mu = i\omega$ satisfies (2.7) if and only if

$$(i\omega)^3 + P_1(i\omega)^2 + P_2(i\omega) + P_3 + (Q_1(i\omega) + Q_2)e^{-i\omega\tau} = 0 \qquad (2.8)$$

Splitting Real and imaginary parts:

$$P_3 - P_1\omega^2 + Q_2 cos\ \omega\tau + Q_1\omega \sin\omega\tau = 0 \qquad (2.9)$$

$$P_2\omega - \omega^3 + Q_1\omega \, cos \, \omega\tau - Q_2 \sin \omega\tau = 0 \qquad (2.10)$$

Which gives:

$$\omega^6 + \left({P_1}^2 - {Q_1}^2 - 2P_2\right)\omega^4 + \left({P_2}^2 - {Q_2}^2 + 2Q_1Q_2 - 2P_1P_3\right)\omega^2 +$$

$$\left({P_3}^2 - {Q_2}^2\right) = 0 \qquad (2.11)$$

Let $a = \left({P_1}^2 - {Q_1}^2 - 2P_2\right), b = \left({P_2}^2 - {Q_2}^2 + 2Q_1Q_2 - 2P_1P_3\right), c = \left({P_3}^2 - {Q_2}^2\right).$

If $\omega^2 = t$, then equation (2.11) becomes:

$$t^3 + at^2 + bt + c = 0 \qquad (2.12)$$

Lemma 1. Conditions applied on (2.12) yields following results,

1) If $c < 0$, then (2.12) has at least one positive root.
2) If $c \geq 0$ and $(a^2 - 3b) \leq 0$, then (2.12) has no root that is positive.
3) If $c \geq 0$ and $(a^2 - 3b) > 0$, then (2.12) roots will be positive roots if and only

$$if \ \upsilon = \frac{-a \pm (a^2 - 3b)}{3} > 0 \ and \ h(\upsilon) \leq 0$$

Lemma 2. Assume $h(t_0) = \ (3t^2 + 2at_0 + b_0)$ and the conditions of Routh-Hurwitz are fulfilled. When (j=0,1,2...), $t\eta(\tau) = t(\tau) + i\omega(\tau)$ fulfilling $t(\tau_j) = 0, \omega(\tau_j) = \omega_0$, where

$$\tau_j = \frac{1}{\omega} \, a\gamma c \cos\left(\frac{-\left(Q_1\omega^2(P_2 - \omega^2) + (P_3 - P_1\omega^2)(Q_2)\right)}{(Q_2)^2 + (Q_1\omega_0)^2} + 2j\pi\right)$$

then $\pm i\omega_0$ are simple roots. If $t^j(\tau_j) = \left.\frac{R_e\eta(\tau)}{d\tau}\right|_{\eta = i \, t_0} \quad f = 0$

Using Lemma 2, we can quickly determine the equations (2.1), (2.2), and (2.3) system's stability and bifurcation.

Theorem: If Routh-Hurwitz criterion is satisfied, then the Cciterion examines the polynomial equation's coefficients to see if the system has any roots in the complex plane's right-hand half. If there are no origin in the partial right of the complex plane, the system is constant.

Results have been proved by “Ahmad, S. & Rao, (1999)”.

2.6 Numerical Results

Representing (2.1) to (2.3) by utilising the set of values mentioned below:

$\gamma_1 = 2.5,\ \gamma_2 = 2.55,\ s = 20, k_1 = .01,\ \beta_1 = .66,\ k_2 = 0.02,\ \beta_2 = 0.55,\ \alpha_1 = 0.43,\ \alpha_2 = 0.21,\ d = 0.1$

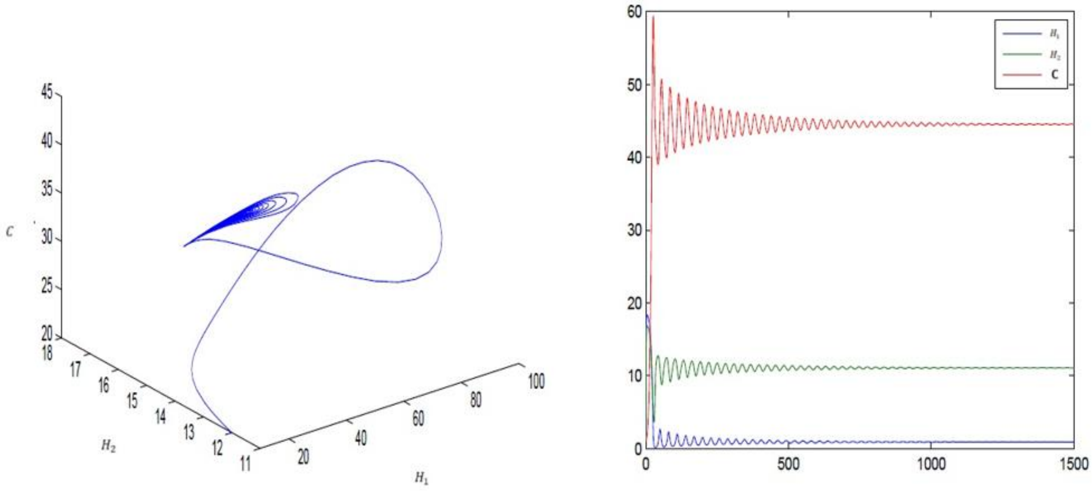

Figure 2.1: The system's positive internal equilibrium point E* is stable instantly when $\beta_1 = 0.66 < \beta_1^* = 0.683535$.

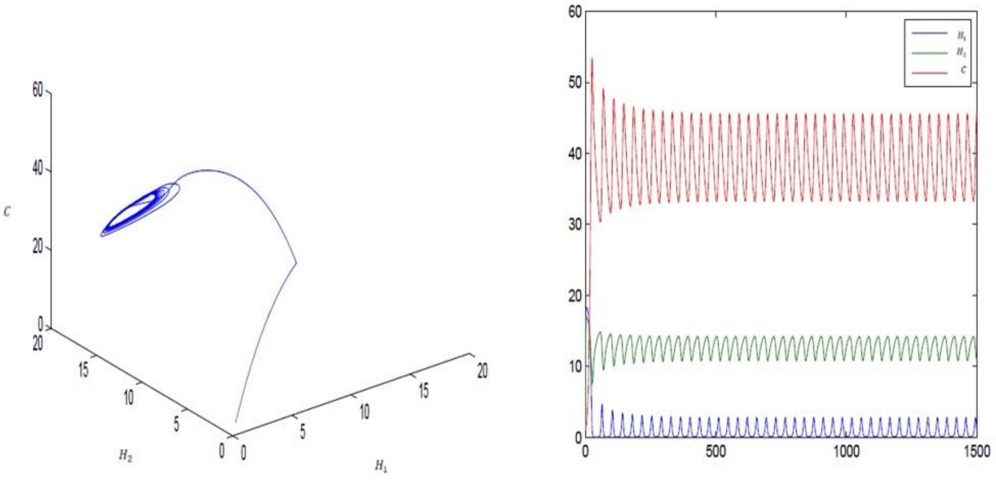

Figure 2.2: When $\beta_1 = 0.75 > \beta_1^* = 0.683535$, A Hopf-bifurcation appears as a effect of the structure's positive internal equilibrium point E^* losing stability quickly.

2.7 Conclusion

Researchers have previously proven that phytoplankton that produces toxin can be considered as an agent that controls the end of planktonic blooms using experimental results and mathematical models. However, such studies do not take into account the presence of two hazardous phytoplankton. Furthermore, the impact of time delay can’t be overlooked in this circumstance. This work examined a three-component model with two competing substances i.e. toxic phytoplankton and zooplankton. The theoretical and numerical data obtained reveals that the system is stable for a specific threshold of system parameters, reflecting the coexistence of three species, around the positive interior equilibrium.

Chapter 3

Effect of Time-Delay on the teamed-up Phytoplankton-Zooplankton Dynamics

Abstract

The effects of teamed up phytoplankton on zooplankton are studied using a mathematical model proposed in this chapter. The associated state variables are Phytoplankton H_1, Phytoplankton H_2 and Zooplankton C. The assumption is that both Phytoplankton help each other against Zooplankton which adversely effects the Zooplankton population. The addition of delay disrupted the system's interior equilibrium, causing Hopf bifurcation at the delay parameter's crucial value. MATLAB is used to assist analytical findings with numerical simulation.

3.1 Introduction

The Universal existence and importance of Zooplankton-Phytoplankton dynamics has piqued the curiosity of both applied mathematicians and ecologists. Many models have two or more interacting species systems that have been developed, considering the effects of age, structure, functional response, time lag, crowding, switching, and other factors.

Many creatures can be found in nature that form teams and they have moved in teams from one place to another. There are two major benefits: the first is an increase in foraging efficiency and the second is the reduction of predation risk. The biological motivation of our study is related to the system in which the two types of Phytoplankton are living in close quarters, and which are of one type which are being assaulted by Zooplankton.

The issue of multi-team games is a relatively new one. It's being looked into by Elettreby M. (2009); Liu Y. and Simaan, M.A. (2003) gave a model to investigate a novel multi-team Phytoplankton-Zooplankton system where Phytoplankton teams assist one another in the presence of Zooplankton. Equilibrium solutions of the model are derived, and their stability analysis is performed. The persistence analysis in the model without considering the team behaviour is also carried out in this chapter.

The introduction of time delay in Phytoplankton-Zooplankton system is because of the factors such as age-structure, maturation period, gestation period (ranging from about 20 days to about 645 days), feeding time, reaction time and resource regeneration time. The system with delay received more attention because of its complex dynamical behaviour such as the emergence of periodic solution and bifurcation phenomena. We'd want to make a point about here that there are a number of excellent publications on bifurcation and stability for ecological models without delay or with delay, we're talking about Abdusalam H. and Fahmy, E. (2003); Freedman, H. et al. (1989); Gakkhar S. and Kamel N.R. (2003); Gakkhar, S. et al.(2009); Hui J. and Chen L.(2006); Jing Z. and Yang J.(2006); Liu Z. and Yuan R. May R.(1973); Meng X. and Wei J.(2004); Song Y. and Yuan S.(2007); Zhao, J., Wei, J.(2009) and the reference therein. Time-integrated models delay in various biological systems are extensively researched Beretta E. and Kuang Y. (1998); Gopalsamy K. (1992); Kuang Y. (1993); MacDonald, N. (1976). Richard Shine discovered that no specimen showed signs of zooplankton poisoning and that the majority of zooplanktons consumed phytoplanktons before discarding them. Both Zooplanktons immediately learnt to stay away from Phytoplankton after just one or two encounters. Zooplankton rejected phytoplanktons for the duration of the sample (22 days), indicating that phytoplanktons are detrimental when considering long-term retention Webb, J. K. et al. (2011).

According to Christina G. Halpin, zooplankton increase their consumption by artificially elevating their nutritional levels, and they lower it by ceasing protein enrichment. This demonstrates that phytoplankton can perceive and use this information to determine the nutritional composition of harmful phytoplankton, opening up new possibilities for phytoplankton defence evolution.

It is evident that individual variations in nutrition may result in hitherto unheard-of levels of toxicity in phytoplankton, which may help to explain why certain species have ontogenetic defence mechanisms Halpin C. et al. (2014). New methods against Zooplanktons are vitally needed for the survival of endangered species, according to Read J. et al.(2016). According to Ramos R.S. et al.(2017), flubendiamide tended to remain on the surface for a longer period of time.The population pallescens are also maintained by these substances, and they make more sustainable sophisticated pest control strategies possible.

Though coral gobies could be prey, Gratzer B. et al. (2015) pointed out that E. Gobiodon was clearly preferred by the non-toxic monitoring fish of Fasciatus. While pursuing a goby, the predator did not favour any particular animal over the others. Since these gobies have a skin toxin, residing on the ground is highly risky, as shown by their unusual post-capture avoidance. Recent research by Banerjee M. & Venturino E. (2011) examined a phytoplankton-toxic zooplankton-interacting device without taking into consideration the temporal lag brought on by the gestation period. But the report's authors neglected to account for the temporal gap throughout the gestation stage. The approach put out by them considers the practical form of the predator displaying avoidance behaviour in the face of large concentrations of toxic phytoplankton, and the predator replicates instantaneously after predating the prey. In the natural world, however, it makes more sense to accommodate the temporal lag of the gestation period. In light of this, the work of Banerjee M. & Venturino E. (2011) is extended by incorporating time delay into the dynamical equation of zooplankton. The link between phytoplankton and zooplankton under certain external conditions was emphasised by Rodgers Makwinja (2021). In this work, Fanny Chenillat et al. (2021) demonstrate how zooplankton feeding patterns are portrayed in these models. The approach provided by Elettreby, M. (2009) assumes that Zooplankton reproduces instantaneously after Phytoplankton is devoured; in reality, though, it is more likely to take the time lag of the gestation period into account. In this study, the work of Elettreby, M. (2009) is expanded by adding time delay to the model, taking into account time-lag for gestation period.

3.2 Mathematical Model

Consider a Phytoplankton-Zooplankton system that is delayed in which both Phytoplankton help each other against Zooplankton. In this model, the assumption is that the Zooplankton takes time τ for the gestation period . Accordingly, the model can be expressed as

$$\frac{dH_1(t)}{dt} = mH_1(t)\big(1 - H_1(t)\big) - H_1(t)C(t) + H_1(t)H_2(t)C(t) \qquad 3.1$$

$$\frac{dH_2(t)}{dt} = nH_2(t)\big(1 - H_2(t)\big) - H_2(t)C(t) + H_1(t)H_2(t)C(t) \qquad 3.2$$

$$\frac{dC(t)}{dt} = -oC^2(t) + pH_1(t-\tau)C(t) + qH_2(t-\tau)C(t) \qquad 3.3$$

If the desired time delay is $\tau > 0$ for the Zooplankton's gestation time, $H_1(t)$ and $H_2(t)$ are densities of two teams of Phytoplankton, $C(t)$ is the density of Zooplankton. All the parameter takes positive values, i.e., $m, n, o, p, q > 0$. The system comes with the following starting functionality (3.1-3.3):

$$(H_1(\gamma), H_2(\gamma), C(\gamma)) \in K_+ = \mathrm{K}\left((-\tau, 0), R^3\right), H_1(0), H_2(0), C(0) > 0.$$

3.3 Point of interior equilibrium and stability analysis

The system of equations mentioned in (3.1-3.3) has eight equilibria with certain conditions for non-negativity. Since the remaining seven equilibria do not show any impact of delay in the internal stability of outcomes, we shall only confine our analysis to the stability and local Hopf-bifurcation of interior equilibrium.Let $E^*({H_1}^*, {H_2}^*, C^*)$ denote the interior equilibrium where,

${H_1}^* = \frac{no+q(1-\sqrt{\frac{n}{m}})}{q+p\sqrt{n/m}}, {H_2}^* = \frac{o\sqrt{mn}-p(1-\sqrt{\frac{n}{m}})}{q+p\sqrt{n/m}}, C^* = \sqrt{mn}\ and\ E^*$ exist under the following conditions

$$o\sqrt{mn} \le p + q, om + p > p\sqrt{\frac{n}{m}}, on + q > q\sqrt{m/n}$$

Now we'll look at the equilibrium's $E^*({H_1}^*, {H_2}^*, C^*)$ local stability. Let $\alpha_1 = H_1 - {H_1}^*, \alpha_2 = H_2 - {H_2}^*$ and $\alpha_3 = C - C^*$.

After that, the system (3.1-3.3) can be written as follows:

$$\frac{d\alpha_1}{dt} = -m{H_1}^*\alpha_1 - {H_1}^*\alpha_3 + {H_1}^*{H_2}^*\alpha_3 + {H_1}^*C^*\alpha_2 - m{\alpha_1}^2 - (1 - {H_2}^*)\alpha_1\alpha_3 + {H_1}^*\alpha_2\alpha_3 + C^*\alpha_1\alpha_2 + \alpha_1\alpha_2\alpha_3 \qquad 3.4$$

$$\frac{d\alpha_2}{dt} = -n{H_2}^*\alpha_2 - {H_2}^*\alpha_3 + {H_1}^*{H_2}^*\alpha_3 + {H_2}^*C^*\alpha_1 - n{\alpha_2}^2 - (1 - {H_1}^*)\alpha_2\alpha_3 + {H_2}^*\alpha_1\alpha_3 + C^*\alpha_1\alpha_2 + \alpha_1\alpha_2\alpha_3 \qquad 3.5$$

$$\frac{d\alpha_3}{dt} = -oC^*\alpha_3 + pC^*\alpha_1(t-\tau) + qC^*\alpha_2(t-\tau) - o{\alpha_3}^2 + p\alpha_1(t-\tau)\alpha_3 + q\alpha_2(t-\tau)\alpha_3 \qquad 3.6$$

To study the stability of the equilibrium $E^*({H_1}^*, {H_2}^*, C^*)$, it's enough to examine the constancy of the origin for the structure (3.4-3.6). As the linearized system (3.4-3.6) is at (0,0,0), its characteristic equation is,

$$\lambda^3 + X\lambda^2 + Y_1\lambda + e^{-\lambda\tau}(Y_2\lambda + Z_2) = 0 \qquad (3.7)$$

$$X = m{H_1}^* + n{H_2}^* + oC^*$$

$$Y_1 = no{H_2}^*C^* + m{H_1}^*(n{H_2}^* + oC^*) - {H_1}^*{H_2}^*C^{*2},$$

$$Y_2 = q{H_2}^*C^*(1 - {H_1}^*) + p{H_1}^*C^*(1 - {H_2}^*)$$

$$Z_2 = mqH_1{}^*H_2{}^*C^*(1 - H_1{}^*) + pH_1{}^*H_2{}^*C^{*2}(1 - H_1{}^*) + qH_1{}^*H_2{}^*C^{*2}(1 - H_2{}^*)$$
$$+ npH_1{}^*H_2{}^*C^*(1 - H_2{}^*)$$

When $\tau = 0$, equation (3.7) becomes,

$$\lambda^3 + X\lambda^2 + (Y_1 + Y_2)\,\lambda + Z_2 = 0 \tag{3.8}$$

According to the Routh-Hurwitz, the balance E^* is stable in the local asymptote when $\tau = 0$, criterion, if

$$(H_1)\ X > 0,\ (Y_1 + Y_2) > 0,\ Z_2 > 0,\ X\,(Y_1 + Y_2) > Z_2$$

hold. As seen in the following, the time delay can affect the stability of equilibrium $E*$ and cause Hopf bifurcation when it surpasses certain critical levels.

Lemma I Suppose the conditions in (H_1) are satisfied. Then equation (3.7) with $\tau = \tau_j (j = 0,1,-----)$ possesses a basic pair of totally imaginary conjugate roots $\pm i\omega_0$, where

$$\tau_j = \frac{1}{\omega_0}[arcos\frac{\omega_0{}^2(\omega_0{}^2Y_2 + Z_2X - Y_1Y_2)}{Y_2{}^2\omega_0{}^2 + Z_2{}^2} + 2j\prod]$$

Further, we have the following:

1. If $\tau \in [0, \tau_0)$, the equation of all roots (3.7) has undesirable factual parts.
2. If $\tau = \tau_0$, equation (3.7) has a duo of conjugate virtuously complex roots $\pm i\omega_0$, and the real components of all the other roots are negative.

Lemma II Suppose $h(x_0) = (3x_0^2 + 2px_0 + q) \neq 0$ and the conditions in (H_1) are satisfied. For (j=0,1...), denote $(\tau) =$ Ա$(\tau) + i\omega(\tau)$ function as the equation's root (3.7) fulfilling Ա $(\tau_J) = 0$, $\omega\ (\tau_j) = \omega_0$, where

$$\tau_j = \frac{1}{\omega_0}[\arccos\frac{{\omega_0}^2({\omega_0}^2 Y_2 + Z_2 X - Y_1 Y_2)}{{Y_2}^2{\omega_0}^2 + {Z_2}^2} + 2j\prod]$$

Then $\pm i\omega_0$ are simple roots. If the transversality condition (H_1),Ꮭ'$(\tau_j) =$

$$\frac{Re\lambda(\tau)}{d\tau}\Big|_{\lambda = i\omega_0 \neq 0}$$

holds, a Hopf bifurcation occurs for (3.4-3.6) at Ꮭ $= 0$ and $\tau = \tau_j$.

Theorem Suppose the conditions in (H_1) are satisfied. If $\tau \in [0, \tau_0)$, subsequently (3.4-3.6)'s zero solution is asymptotically stable.

The proof of Lemmas and theorem have been conveyed in the publication given by Ruan S. and Wei J. (2001).

3.4 Direction and Stability of the Hopf-Bifurcating Solution

It is also known to be of interest to determine the period, stability, and direction of these bifurcating periodic solutions. In this part, we will create the explicit equations specifying the characteristics of the Hopf-bifurcation at the critical value τ_j using normal form theory and centre manifold reduction due to Hassard B.D. et al. (1981).

Applying time scaling, $t \to \frac{t}{\tau}$ to normalize the delay τ, system (3.4-3.6) is

$$\frac{d\alpha_1}{dt} = -mH_1{}^*\alpha_1 - H_1{}^*\alpha_3 + H_1{}^*H_2{}^*\alpha_3 + H_1{}^*H_2{}^*\alpha_2 - m\alpha_1{}^2 - (1 - H_2{}^*)\alpha_1\alpha_3 + H_1{}^*\alpha_2\alpha_3 + C^*\alpha_1\alpha_2 + \alpha_1\alpha_2\alpha_3 \qquad 3.9$$

$$\frac{d\alpha_2}{dt} = -nH_2{}^*\alpha_2 - H_2{}^*\alpha_3 + H_1{}^*H_2{}^*\alpha_3 + H_2{}^*C^*\alpha_1 - n\alpha_2{}^2 - (1 - H_1{}^*)\alpha_2\alpha_3$$

$$+ H_2{}^*\alpha_1\alpha_3 \ + C^*\alpha_1\alpha_2 + \alpha_1\alpha_2\alpha_3 \qquad 3.10$$

$$\frac{d\alpha_3}{dt} = -oC^*\alpha_3 + pC^*\alpha_1(t-1) + qC^*\alpha_2(t-1) - o\alpha_3{}^2 + p\alpha_1(t-1)\alpha_3$$

$$+ \alpha_2(t-1)\alpha_3 \qquad 3.11$$

As a result, we'll be able to work in the phase, K=K ((-1,0),R_+^3). Denote the critical value $\tau_j \ by \, \tau_0$ without sacrificing generality . Let $\tau = \tau_0 + \varphi$,then $\varphi = 0$ is a Hopf-bifurcation value of the system (3.9-3.11). To make notations simpler, we rewrite (3.9-3.11) as,

$$\alpha{'}(t) = M_\varphi(\alpha_t) + G(\varphi, \alpha_t) \qquad (3.12)$$

Where $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))^U \in R^3, \alpha_t(\gamma) \in K$ is defined by $\alpha_t(\gamma) = \alpha(t + \gamma), and \ M_\varphi = K \to S, G: SXK \to S$ and further

$$M_{\varphi\varepsilon} = (\tau_0 + \varphi)\begin{bmatrix} -mH_1{}^* & H_1{}^*C^* & -H_1{}^* + H_1{}^*H_2{}^* \\ H_2{}^*C & -nH_2{}^* & -H_2{}^* + H_1{}^*H_2{}^* \\ 0 & 0 & -oC^* \end{bmatrix}\begin{bmatrix} \varepsilon_1(0) \\ \varepsilon_2(0) \\ \varepsilon_3(0) \end{bmatrix} + (\tau_0$$
$$+ \varphi)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ pC^* & qC^* & 0 \end{bmatrix}\begin{bmatrix} \varepsilon_1(-1) \\ \varepsilon_2(-1) \\ \varepsilon_3(-1) \end{bmatrix}$$

and

$$G(\varphi, \varepsilon) = (\tau_0 + \varphi)\begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix}$$

respectively, where,

$$G_1 = -m\varepsilon_1{}^2(0) - (1 - H_2{}^*)\varepsilon_1(0)\varepsilon_3(0) + H_1{}^*\varepsilon_2(0)\varepsilon_3(0) + C^*\varepsilon_1(0)\varepsilon_2(0)$$
$$+ \varepsilon_1(0)\varepsilon_2(0)\varepsilon_3(0)$$

$$\begin{aligned} G_2 = -n\varepsilon_2{}^2(0) - (1 - H_1{}^*)\varepsilon_2(0)\varepsilon_3(0) + H_2{}^*\varepsilon_1(0)\varepsilon_3(0) + C^*\varepsilon_1(0)\varepsilon_2(0) \\ + \varepsilon_1(0)\varepsilon_2(0)\varepsilon_3(0) \end{aligned}$$

$$G_3 = -o\varepsilon_3{}^2(0) + p\varepsilon_1(-1)\varepsilon_3(0) + q\varepsilon_2(-1)\varepsilon_3(0),$$

$$\varepsilon(\gamma) = (\varepsilon_1(\gamma)\varepsilon_2(\gamma)\varepsilon_3(\gamma))^U \in K((-1,0), S)$$

According to Reisz illustration theorem, there occurs a function $\eta(\gamma, \varphi)$ of limited dissimilarity for $\gamma \in [-1,0]$, in such a way that, $M_\varphi \varepsilon = \int_0^1 d\eta(\gamma, 0)\varepsilon(\gamma)\ for\ \varepsilon \in K.$

As a matter of fact, we can select

$$\begin{aligned} \eta(\gamma, \varphi) = (\tau_0 + \varphi)\begin{bmatrix} -mH_1{}^* & H_1{}^*C^* & -H_1{}^* + H_1{}^*H_2{}^* \\ H_2{}^*C & -nH_2{}^* & -H_2{}^* + H_1{}^*H_2{}^* \\ 0 & 0 & -oC^* \end{bmatrix}\begin{bmatrix} \varepsilon_1(0) \\ \varepsilon_2(0) \\ \varepsilon_3(0) \end{bmatrix}\delta(\gamma) + (\tau_0 \\ + \varphi)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ pC^* & qC^* & 0 \end{bmatrix}\delta(\gamma + 1) \end{aligned}$$

where δ is the Dirac delta function.For $\varepsilon \in K([-1,0], R_+^3)$,we establish

$$X(\varphi)\varepsilon = \begin{cases} \dfrac{d\varepsilon\gamma}{d\gamma}, & \gamma \in [-1,0) \\ \displaystyle\int_{-1}^{0} d\eta(\gamma, 0)\varepsilon(\gamma), & \gamma = 0 \end{cases}$$

and

$$S(\varphi)\varepsilon = \begin{cases} 0, \gamma \in [-1,0), \\ G(\varphi, \varepsilon)\ \gamma = 0. \end{cases}$$

Hence, (3.12) is equal to

$$\alpha_t{}' = X(\varphi)\alpha_t + S(\varphi)\alpha_t \qquad (3.13)$$

For $\Psi \in K^1([-1,0], R_+^3), define$

$$X^*\Psi(s) = \begin{cases} -\dfrac{d\Psi(s)}{d\gamma} & ,s \in [-1,0) \\ \displaystyle\int_{-1}^{0} d\eta^U(-t,0)\Psi(-t), & s = 0. \end{cases}$$

And bilinear inner product

$$<\Psi(s), \varepsilon(\gamma) \geq \bar{\Psi}(0)\varepsilon(0) - \int_{-1}^{0}\int_{\xi=\theta}^{0} \bar{\Psi}(\xi-\gamma)d\eta(\gamma)\varepsilon(\xi)d\xi \tag{3.14}$$

We know that X^* and $X = X(0)$ are adjoint operators. We are aware that $i\omega_0$ are eigen values of X (0). Thus, they are eigen values of X^*. Assume that $r(\gamma) = r(0)e^{i\omega_0\gamma}$ is an eigenvector of X (0) related to the eigenvalue $i\omega_0$. So, $X(0) = i\omega_0 r(\gamma)$. When $\gamma = 0$, we acquire

$$[i\omega_0 I - \int_{-1}^{0} d\eta(\gamma)e^{i\omega_0\gamma}]r(0) = 0,$$

which generates $r(0) = (1, \sigma_1, \rho_1)^U, where$

$$\sigma_1 = \frac{(H_1{}^* - H_1{}^*H_2{}^*)H_2{}^*C^* + (H_2{}^* - H_1{}^*H_2{}^*)(i\omega_0 + mH_1{}^*)}{H_1{}^*C^*(H_2{}^* - H_1{}^*H_2{}^*) - (H_1{}^* - H_1{}^*H_2{}^*)(i\omega_0 + nH_2{}^*)}$$

$$\rho_1 = \frac{H_1{}^*H_2{}^*C^{*2} - (i\omega_0 + mH_1{}^*)(i\omega_0 + nH_2{}^*)}{H_1{}^*C^*(H_2{}^* - H_1{}^*H_2{}^*) - (H_1{}^* - H_1{}^*H_2{}^*)(i\omega_0 + nH_2{}^*)}$$

In the same way, it may be confirmed that $r^*(s) = D(1, \sigma_2, \rho_2)e^{i\omega_0\tau_0 s}$ is the eigen vector of X^* related to $-i\omega_0$,where

$$\sigma_2 = \frac{(H_1{}^* - H_1{}^*H_2{}^*)H_2{}^*C^* + (H_2{}^* - H_1{}^*H_2{}^*)(mH_1{}^* - i\omega_0)}{H_1{}^*C^*(H_2{}^* - H_1{}^*H_2{}^*) - (H_1{}^* - H_1{}^*H_2{}^*)(nH_2{}^* - i\omega_0)}$$

$$\rho_2 = \frac{H_1{}^*H_2{}^*C^{*2} - (i\omega_0 + mH_1{}^*)(i\omega_0 + nH_2{}^*)}{H_1{}^*C^*(H_2{}^* - H_1{}^*H_2{}^*) - (H_1{}^* - H_1{}^*H_2{}^*)(nH_2{}^* - i\omega_0)}$$

From (3.14) we have $<r^*(s), r(\gamma) >=0$

Following algorithms in Hassard, B.D. et al. (1981) and the coordinates are calculated describing the centre manifold K_0 using the same notations as at $\varphi = 0$.Let α_t be a solution of equation (3.12) with $\varphi = 0$.

Define $z(t) =< r^*(s), \alpha_t(\gamma)>$,

$$Y(t, \gamma) = \alpha_t(\gamma) - 2Rez(t)r(\gamma) \tag{3.15}$$

Upon the central manifold K_0 we possess

$$Y(t, \gamma) = Y(z(t)\overline{z}(t), \gamma),$$

Where

$$Y(z, \overline{z}, \gamma) = Y_{20}(\gamma)\frac{z^2}{2} + Y_{11}(\gamma)z\overline{z} + Y_{02}(\gamma)\frac{\overline{z}^2}{2} + \cdots ...$$

z and $\overline{z}$ are centre manifold, K_0 local coordinated pointing in the direction of r^* and $\overline{r^*}$. Point to remember that Y is real if α_t is real. Only viable solutions are taken into account. For result $\alpha_t \in K_0$ of (3.12), as $\varphi = 0$,

$$z'(t) = i\omega_0\tau_0 z + < \overline{r^*}(\gamma), G(0, Y(z, \overline{z}, \gamma) + 2Re\{z(t)r(\gamma)\}) >$$

$$= i\omega_0\tau_0 z + \overline{r^*}(0)G(0, Y(z, \overline{z}, 0) + 2Re\{z(t)(\gamma)\})$$

$$= i\omega_0\tau_0 z + \overline{r^*}(0)G(z, \overline{z})$$

We recast the formula as,

$$z'(t) = i\omega_0\tau_0 z(t) + h(z, \overline{z}) \tag{3.16}$$

Where

$$h(z,\overline{z}) = \overline{r}^*(0)G_0(z,\overline{z})$$

$$= h_{20}(\gamma)\frac{z^2}{2} + h_{11}(\gamma)z\overline{z} + h_{02}(\gamma)\frac{\overline{z}^2}{2} + h_{21}\frac{z^2\overline{z}}{2} + \cdots \qquad (3.17)$$

Noticing

$$\alpha_t(\gamma) = (\alpha_{1t}, \alpha_{2t}, \alpha_{3t}) = Y(t,\gamma) + zr(\gamma) + \overline{zr}(\gamma),$$

and $r(0) = (1, \sigma_1, \rho_1)^U e^{iw_0\tau_0\gamma}$, we calculate the values of $\alpha_{1t}(0), \alpha_{2t}(0), \alpha_{3t}(0), \alpha_{1t}(-1), \alpha_{2t}(-1)$ and comparing the co-efficients with (3.17), we observe that $Y_{20}(\gamma)$ *and* $Y_{11}(\gamma)$ are there in h_{21}.

Hence, we need to further compute them from (3.13) and (3.15),

$$Y' = {\alpha_t}' - z'r - \overline{z}'r$$

$$= \begin{cases} XY - 2Re\{\overline{r}^*(0)G_0 r(\gamma)\}, \gamma \in [-1,0), \\ XY - 2Re\{\overline{r}^*(0)G_0 r(0)\} + G_0, \gamma = 0 \end{cases}$$

$$= X + I(z,\overline{z},\gamma), \qquad (3.18)$$

Where

$$I(z,\overline{z},\gamma) = I_{20}(\gamma)\frac{z^2}{2} + I_{11}(\gamma)z\overline{z} + I_{02}(\gamma)\frac{\overline{z}^2}{2} + I_{21}\frac{z^2\overline{z}}{2} + \cdots \ldots \qquad (3.19)$$

Conversely, yet, on K_0 in close proximity to the origin,

$$Y' = Y_z z' + Y_{\overline{z}}\overline{z}'.$$

Comparing the coefficient after expanding the series,

$$[X - 2i\omega_0 J]X_{20}(\gamma) = -I_{20}(\gamma),$$

$$XY_{11}(\gamma) = -I_{11}(\gamma) \tag{3.20}$$

By (3.16), we are aware that for $\gamma \in [-1,0)$,

$$I(z, \overline{z}, \varphi) = -\overline{r}^*(0)\overline{G}_0 r(\gamma) - r^*(0)\overline{G}_0 \overline{r}(\gamma) = -hr(\gamma) - \overline{h}\,\overline{r}(\gamma).$$

Analysing the coefficients in relation to (3.18) we acquire for $\gamma \in [-1,0]$ that

$$I_{20}(\gamma) = -h_{20} r(\gamma) - \overline{h}_{02}\overline{r}(\gamma),$$

$$I_{11}(\gamma) = -h_{11} r(\gamma) - \overline{h}_{11}\overline{r}(\gamma)$$

By using equations (3.18), (3.19) and the definition of X, we can get

$$Y_{20}(\gamma) = 2i\omega_0\tau_0 Y_{20}(\gamma) + h_{20} r(\gamma) + \overline{h}_{02}\overline{h}(\gamma)$$

Solving for $Y_{20}(\gamma)\ and\ Y_{11}(\gamma)$

$$I(z, \overline{z}, \gamma) = -2Re\{\overline{r}^*(0) G_0 r(0)\} + G_0,$$

We have

$$I_{20}(\gamma) = -h_{20} r(\gamma) - \overline{h}_{02}\overline{r}(\gamma) + Gz^2,$$

$$I_{11}(\gamma) = -h_{11} r(\gamma) - \overline{h}_{11}\overline{r}(\gamma) + G_{z\overline{z}},$$

Where

$$G_0 = G_{z^2}\frac{z^2}{2} + Gz\overline{z} + G_{\overline{z}^2}\frac{\overline{z}^2}{2} + \cdots \ldots$$

Hence, combining the definition of X, we get

$$\begin{bmatrix} 2i\omega_0 + mH_1{}^* & -H_1{}^*C^* & H_1{}^* - H_1{}^*H_2{}^* \\ -H_2{}^*C^* & 2i\omega_0 + nH_2{}^* & H_2{}^* - H_1{}^*H_2{}^* \\ -pC^*q^{-2i\omega_0\tau_0} & -qC^*e^{-2i\omega_0\tau_0} & 2i\omega_0 + oC^* \end{bmatrix} E_1$$

$$= -2\begin{bmatrix} m + (1 - H_2{}^*)\rho_1 - \sigma_1(H_1{}^*\rho_1 + C^*) \\ (n\sigma_1{}^2 + (1 - H_1{}^*)\sigma_1\rho_1) - \sigma_1C^* - \rho_1H_2{}^*) \\ \rho_1(o\rho_1 - pe^{-i\omega_0\tau_0} - e\sigma_1e^{-i\omega_0\tau_0}) \end{bmatrix}$$

and

$$\begin{bmatrix} H_1{}^* & -H_1{}^*C^* & H_1{}^* - H_1{}^*H_2{}^* \\ -H_2{}^*C^* & nH_2{}^* & H_2{}^* - H_1{}^*H_2{}^* \\ -pC^* & -qC^* & oC^* \end{bmatrix} E_2$$

$$=-2\begin{bmatrix} m + (1 - H_2{}^*)Re\{\rho_1\} - H_1{}^*Re\{\overline{\rho}_1\sigma_1\} - C^*Re\{\sigma_1\} \\ (\sigma_1\overline{\sigma}_1n + (1 - H_1{}^*)Re\{\sigma_1\overline{\rho}_1\} - H_2{}^*Re\{\overline{\rho}_1\} - C^*Re\{\sigma_1\}) \\ (o\rho_1\overline{\rho}_1 - pRe\{\rho_1\} - qRe\{\rho_1\overline{\sigma}_1\}e^{i\omega_0\tau_0} \end{bmatrix}$$

Then h_{21} can be expressed by the parameters.

Hence, we can see that h_{ij} can be determined by the parameters. Therefore, below mentioned quantities are:

$$\left.\begin{aligned} K_1(0) = \frac{i}{2w_0\tau_0}\left(h_{11}h_{20} - 2|h_{11}|^2 - \frac{|h_{02}|^2}{3}\right) + \frac{h_{21}}{3} \\ \varphi_2 = -\frac{Re\{D_1(0)\}}{Re\{\lambda'(\tau_0)\}}, \\ \beta_2 = 2Re\{D_1(0)\} \\ U_2 = -\frac{Im\{D_1(0)\} + \varphi_2 Im\{\lambda'(\tau_0)\}}{\tau_0\omega_0} \end{aligned}\right\} \qquad (3.21)$$

3.5 Numerical Example

Numerical simulations for system of (3.1-3.3) have been performed using MATLAB. We take two set of parametric values as given in the literature Elettreby M. (2009):

Parameters	**Parametric Values**
M	1.2
N	1.4
O	1
P	1
Q	2
Positive interior equilibrium point	$E^*(0.4025, 0.4468, 1.2961)$
τ_j	$1.7387 + 2j\pi/0.6499$
Result	An equilibrium point that is positive is stable when $\tau < \tau_0$ (Figure 1) and the Hopf-Bifurcation is observed in at τ

Furthermore, the direction and stability of periodic solutions that bifurcate and break away from positive equilibrium at the critical point τ_j can be regulated. e.g., when $\tau = \tau_0 = 1.7387$, $K_1(0) = -1.5156 - 6.2274i$. It follows from (3.21) that $\varphi_2 > 0$ and $\beta_2 < 0$. As a result, when $\tau > \tau_0$, the bifurcation occurs, and the orbits that fit are orbitally asymptotically stable, as illustrated in Figure 2. The amount of the crucial delay τ_0 grows as the intrinsic growth rate of Phytoplankton (H_1) and conversion efficiency q of Zooplankton diminishes.

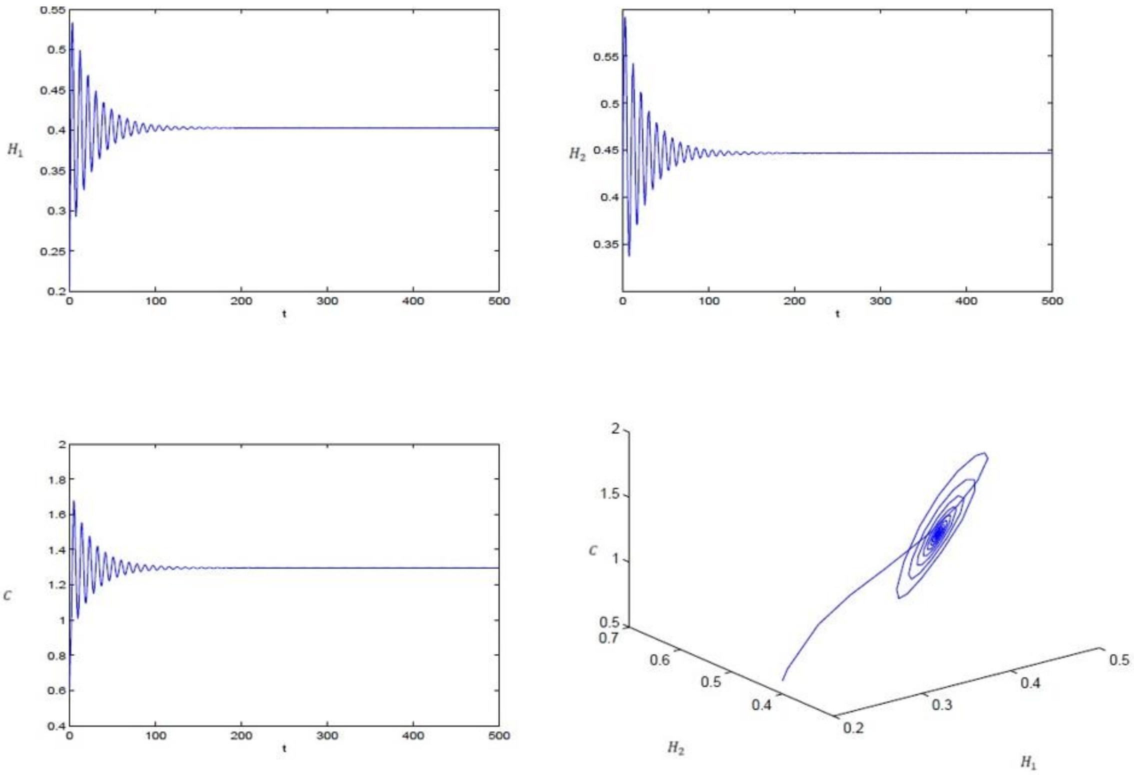

Figure 3.1: The internal balance point E^* is positive for system (3.1-3.3) and has asymptotic stability when $\tau = 1.5 < \tau_0 = 1.7387$. Here the initial value is (0.2 0.4 0.6).

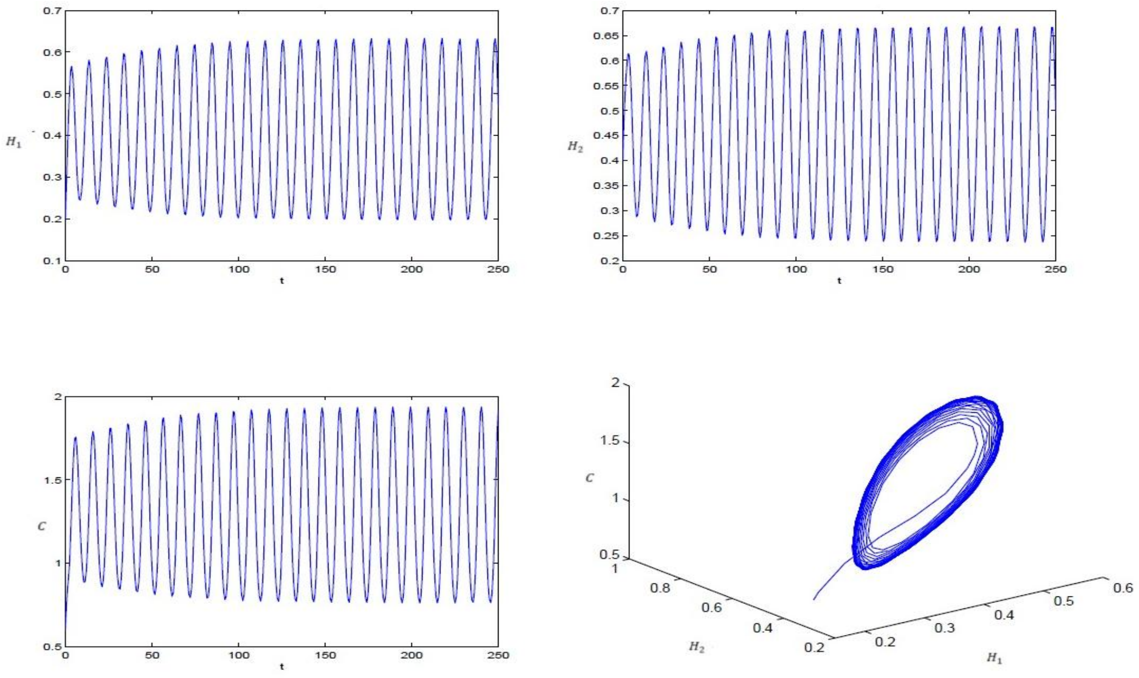

Figure 3.2: When $\tau = 1.85 > \tau_0 = 1.7387$, the point of positive internal equilibrium is E^* of system (3.1-3.3)

a) Becomes unstable

b) Hopf-bifurcation takes place

c) the orbitally and asymptotically stable bifurcating periodic solution

d) In this case, the initial value is (0.2 0.4 0.6).

3.6 Conclusion

In this chapter, the impact of time lag on a multi team phytoplankton-zooplankton system was investigated, where we considered two phytoplankton and one zooplankton. The assumption is that the two phytoplankton-populations assist each other when they are preyed upon. Further it is demonstrated that the system's stable equilibrium point becomes unstable when a time delay is introduced. It is shown that the Hopf-bifurcation can occur when the delay parameter crosses the critical levels. Additionally, utilising the centre manifold reduction theorem and normal form theory, an explicit approach is constructed that establishes the direction and stability of a Hopf- bifurcating solution.

Chapter 4

Dynamics of One-Phytoplankton Two-Zooplankton System with Square Root Functional Response and Time Lag

Abstract: In this chapter, the effects of Phytoplankton and two competing Zooplanktons are studied using a mathematical model. The associated state variables are Phytoplankton H,Zooplankton C_1 and Zooplankton C_2. The assumption is that if the ratio of mortality rates of Zooplankton to the radical of the carrying volume of the Phytoplankton inhabitants is larger than their respective conversion efficiency and predation rates, the Phytoplankton inhabitants will survive as well as Zooplankton inhabitants will tend to extinction. The addition of delay disrupted the system's interior, axial and boundary equilibrium, and at the critical point of the delay parameter, Hopf bifurcation occurs. Sensitivity analysis is performed on the model parameters. MATLAB is used to assist analytical findings with numerical simulation.

Keywords: Phytoplankton, Zooplankton, Toxic Material, Hopf bifurcation, Stability, Interior Equilibrium.

4.1 Introduction

Many animals establish groups in nature, and they also move in groups from one location to another. Zooplanktons benefit from the development of groups (herds) because it improves their effectiveness at foraging and reduces their danger of predation. As a result, the herd's actions prevent the extinction of Zooplankton, which group together to protect themselves from predators. One of the most fascinating population dynamics phenomena is the aggregation of Zooplanktons. Many authors' models have taken into account various functional responses of Phytoplankton-Zooplankton forming groups Cosner et al. (1999) and Venturino (2011). Ajraldi et al. (2011) investigated that employing a two-breed system where the members of first breed live in groups while those belonging to the other breed live alone. They've explained everything from competitiveness to symbiosis to predation in populations. Phytoplankton-Zooplankton interaction is defined, where limit cycles organically arise Braza (2012). Beretta and Kuang, (1998) also looked at the functional response of the square root in the Zooplankton-Phytoplankton model and found in that community behavior more in the area of the emergence other than typical models that don't include herd behavior. Due to the square root word, this makes ecological sense.

Models with delays are more realistic as time lags are present in almost all biological scenarios and cause frequent changes in demography. Numerous authors have carried out in-depth studies on time delay in a range of biological systems. Cushing (1977), Wangersky and Cunningham (1957), MacDonald (1976), Gopalswamy (1992), Kuang (1993), Chakraborty et al. (2011), Ajraldi et al. (2011). In Phytoplankton-Zooplankton models, a wide variety of functional responses have been studied Kar and Matsuda (2007); Khare et al. (2011); Holt and Lawto (1994)

In their natural habitat, zooplankton depends not only on phytoplankton density but also on the abundance and existence of host species as examined by Srinivasu and

Prasad (2010); Srinivasu et al. (2007); Merfield et al. (2004). Several studies have demonstrated the importance of the host-commensal interaction in the survival or extinction of a variety of organisms as specified by Nouhuys and Kraft (2012); Vargas- Leon and Alcaraz (2013); Wang (2013); Zhang (2012); Bhattacharyya and Pal (2013). Real-world examples of harvesting causing the extinction of some species have also negatively impacted other species that depend on these species. Many publications have been published in recent years analyzing Zooplankton-Phytoplankton models with harvesting Ghosh and Kar (2013); Yuan and Pei (2013); Zhang et al. (2013); Khan et al. (2021); Chenilla et al. (2021); Ruan and Wei (2001).

4.2 Mathematical Model

The current work arose from theoretical and experimental findings on the interplay of hazardous algal blooms with various types of phytoplankton-zooplankton interactions. Following system of differential equations governs the dynamics.

$$\frac{dH}{dT} = \gamma H\left(1 - \frac{H(T-\tau)}{K}\right) - \widehat{\beta_1}\sqrt{H}C_1 - \widehat{\beta_2}\sqrt{H}C_2 \tag{4.1}$$

$$\frac{dC_1}{dT} = \widehat{\alpha_1}\widehat{\beta_1}\sqrt{H}C_1 - \widehat{d_1}C_1 \tag{4.2}$$

$$\frac{dC_2}{dT} = \widehat{\alpha_2}\widehat{\beta_2}\sqrt{H}C_2 - \widehat{d_2}C_2 - \hat{\imath}{C_2}^2 \tag{4.3}$$

with initial conditions$H(0) > 0, C_1(0) > 0, C_2(0) > 0.$

We use following for non- dimentionalising the equations:

$\hat{h} = \frac{H}{S}, y_1 = \frac{C_1\widehat{\beta_1}}{\sqrt{S}}, y_2 = \frac{C_2\widehat{\beta_2}}{\gamma\sqrt{S}}, t = \frac{\gamma T}{2}$ and setting out the new variables such as

$$k_1 = \frac{2\widehat{\alpha_1}\widehat{\beta_1}\sqrt{S}}{\gamma}$$

$$, k_2 = \frac{2\widehat{\alpha_2}\widehat{\beta_2}\sqrt{S}}{\gamma}, d_1 = \frac{\widehat{d_1}}{\widehat{\alpha_1}\widehat{\beta_1}\sqrt{S}}, d_2 = \frac{\widehat{d_2}}{\widehat{\alpha_2}\widehat{\beta_2}\sqrt{S}}, d_3 = \frac{\widehat{d_3}}{\widehat{\alpha_1}\widehat{\beta_2}^{2}}.$$

The following equations make up the non-dimensionalized system:

Table 1. Description of parameters.

The indicant $'i'$ may have a rational value of 1 or 2.

Parameter	**Description**
$\hat{h}$	Density of Phytoplankton
y_1	Density of First Zooplankton
y_2	Density of Second Zooplankton
$\widehat{d_\imath}$	Natural mortality rate
$\widehat{\beta_\imath}$	Predatoriness rates
$\hat{k}$	rate of intraspecies competition
γ	Phytoplankton's intrinsic growth rate
S	Carrying capacity
$\widehat{\alpha_\imath}$	Rate of conversion

$$\frac{d\hat{h}}{dt} = \hat{h}\left(1 - \hat{h}(t - \tau)\right) - \sqrt{\hat{h}}y_1 - \sqrt{\hat{h}}y_2, \tag{4.4}$$

$$\frac{dy_1}{dt} = k_1 y_1 \left(\sqrt{\hat{h}} - d_1\right) \tag{4.5}$$

$$\frac{dy_2}{dt} = k_2 y_2 \left(\sqrt{\hat{h}} - d_2 - d_3 y_2\right) \tag{4.6}$$

$$\frac{dh}{dt} = h\big(1 - h^2(t-\tau)\big) - y_1 - y_2 \tag{4.7}$$

$$\frac{dy_1}{dt} = k_1 y_1 (h - d_1) \tag{4.8}$$

$$\frac{dy_2}{dt} = k_2 y_2 (h - d_2 - d_3 y_2) \tag{4.9}$$

with starting parameters h (0) >0, $y_1(0) > 0$, $y_2(0) > 0$.

4.3 Equilibrium of the Model

The points of balance of the system (4.7) - (4.9) are given by:

$$h\big(1 - h^2(t-\tau)\big) - y_1 - y_2 = 0 \tag{4.10}$$

$$k_1 y_1 (h - d_1) = 0 \tag{4.11}$$

$$k_2 y_2 (h - d_2 - d_3 y_2) = 0 \tag{4.12}$$

On solving (4.10) - (4.12), following five points of balance or equilibrium for the system (4.7) - (4.9) are achieved:

1) The point of origin $E_0 = (0,0,0)$.
2) The point of axial equilibrium $E_A = (1,0,0)$.
3) The point of equilibrium at the boundary $E_{B_1} = \big(d_1, d_1\big(1 - {d_1}^2\big), 0\big)$and

$$E_{B_2} = (\tilde{h}, 0, \widetilde{y_2})$$

where $\widetilde{y_2} = \tilde{h}(1 - \tilde{h}^2)$ and $\tilde{h}$ represents the cubic equation's positive root.

$$d_3\tilde{h}^3 + (1 - d_3)\tilde{h} - d_2 = 0. \qquad (4.13)$$

The point of equilibrium at the boundary E_{B_1} occurs if $d_1 < 1$ and boundary equilibrium point E_{B_2} exists if $\tilde{h} < 1$.

4) The point of inner equilibrium $E^* = ({x_1}^*, {y_1}^*, {y_2}^*)$ where ${x_1}^* = d_1, {y_2}^* = \frac{d_1 - d_2}{d_3}\, and {y_1}^* = d_1\left(1 - {d_1}^2\right) - \frac{d_1 - d_2}{d_3}$. The point of inner equilibrium E^*exists if $d_1 > d_2 \; and \; d_1 d_3 + d_2 > d_1(1 + {d_1}^2 d_3)$.

Dynamical actions when $\tau = 0$

We will now investigate the system's dynamical behaviour in relation to each of the five possible equilibria. The system's variational matrix (4.7)-(4.9) is

$$V = \begin{bmatrix} 1 - 3h^2 & -1 & -1 \\ k_1 y_1 & k_1(h - d_1) & 0 \\ k_2 y_2 & 0 & k_2(h - d_2 - 2d_3 y_2) \end{bmatrix}$$

The characteristic V at the equilibrium point equation E_0 is

$$(1 - \lambda)(k_1 d_1 + \lambda)(k_1 d_1 + \lambda)$$

The V-specific characteristic equation at E_A is

$$(2 + \lambda)(k_1(1 - d_1 + \lambda)(k_1 d_1 + \lambda) = 0.$$

The eigenvalues at E_A are -2, $k_1(1 - d_1)$ and $k_2(1 - d_2), d_1 > 1$ and $d_2 > 1$.

If the equilibrium point E_A is fixed then further the points E_{B_1}, E_{B_2} and E^* do not occur.

If E_{B_1}, E_{B_2} and E^* exist then point E_A is not stable.

The point E_{B_1} will have the equation:

$$(k_2(d_1 - d_2) - \lambda)\left(\lambda^2 + \left(3{d_1}^2 - 1\right)\lambda + k_1 d_1\right) = 0 \qquad (4.14)$$

The points at E_{B_1} for (4.7) -(4.9) are locally asymptotically stable if $d_2 > d_1 \; and \; d_1 > \frac{1}{\sqrt{3}}$.

Also, if E_{B_1} is stable then E^* does not exist else it stable.

The point E_{B_2} will have the equation:

$$(k_1(\tilde{h} - d_1) - \lambda)(\lambda^2 + (3\tilde{x}^2 + k_2 d_3 \tilde{y}_2 - 1)\lambda + k_2 \tilde{y}_2 \qquad (4.15)$$

The points at E_{B_2} for (4.7) -(4.9) possess local asymptotic stability if $d_1 > \tilde{h}$ and $\tilde{h} > \frac{1}{\sqrt{3}}$.

The point E^* will have the equation:

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0 \qquad (4.16)$$

$$A_1 = k_2 d_3 {y_2}^* + 3{d_1}^2 - 1$$

$$A_2 = \left(3{d_1}^2 - 1\right)k_2 d_3 {y_2}^* + k_2 {y_2}^* + k_1 {y_1}^*$$

$$A_3 = k_1 k_2 {y_1}^* d_3 {y_2}^*$$

Dynamical actions when $\tau > 0$

The characteristic equation for (4.7)-(4.9) around at any equilibrium point is

$$\lambda^3 + P_1\lambda^2 + P_2\lambda + P_3 + e^{-\lambda\tau}(Q_1\lambda^2 + Q_2\lambda + Q_3) = 0 \qquad (4.17)$$

where

$$P_1 = -1 + k_1d_1 + k_2d_2 - k_1h - k_2h + h^2 + 2k_2d_3y_2, Q_1 = 2h^2,$$

$$\begin{aligned} P_2 = {} & -k_1d_1 - k_2d_2 + k_1k_2d_1d_2 + k_1h + k_2h - k_1k_2d_1h - k_1k_2d_2h + k_1k_2h^2 + k_1d_1h^2 \\ & + k_2d_2h^2 - \ k_1h^3 - k_2h^3 + k_1y_1 + k_2y_2 - 2k_2d_3y_2 + 2k_1k_2d_1d_3y_2 \\ & - 2k_1k_2d_3hy_2 + \ 2k_2d_3h^2y_2 \end{aligned}$$

$$Q_2 = 2k_1d_1h^2 + 2k_2d_2h^2 - 2k_1h^3 - 2k_2h^3 + 4k_2d_3h^2y_2,$$

$$\begin{aligned} P_3 = {} & -k_1k_2d_1d_2 + k_1k_2d_1h + k_1k_2d_2h - k_1k_2h^2 + k_1k_2d_1d_2h^2 - k_1k_2d_1h^3 \\ & - k_1k_2d_2h^3 + k_1k_2h^4 + k_1k_2d_2y_1 - k_1k_2hy_1 + k_1k_2d_1y_2 \\ & - 2k_1k_2d_1d_3y_2 - k_1k_2hy_2 + 2k_1k_2d_3hy_2 + 2k_1k_2d_1d_3h^2y_2 \\ & - 2k_1k_2d_3h^3y_2 + 2k_1k_2d_3y_1y_2. \end{aligned}$$

$$Q_3 = 2k_1k_2d_1d_2h^2 - 2k_1k_2d_1h^3 - 2k_1k_2d_2h^3 + 2k_1k_2h^4 + 4k_1k_2d_1d_3h^2y_2 - 4k_1k_2d_3x^3y_2.$$

In order to ensure stability of the equilibrium point, all the eigenvalues in characteristic equation (4.17) must have a negative real component. It is challenging to determine the circumstances in which all of equation (4.17)'s roots will have a detrimental part. When $\tau = 0$, equation (4.17) changes to

$$\lambda^3 + (P_1 + Q_1)\lambda^2 + (P_2 + Q_2)\lambda + (P_3 + Q_3) = 0 \qquad (4.18)$$

By Routh-Hurwitz criterion,

If $(P_1 + Q_1) > 0, (P_3 + Q_3) > 0, (P_1 + Q_1)(P_2 + Q_2) > (P_3 + Q_3)$, therefore the real portions of all the roots of equation (4.10) will be negative.

Assuming that $\lambda = 0$ is the solution to the equation (4.18), then $(P_3 + Q_3) = 0$. As a result, this condition conflicts with the second one. Therefore $\lambda = 0$ cannot be solution to equation (4.18). Let's suppose that for some $\tau \geq 0, i\omega$ with $\omega > 0$ is a result of (4.18), so

$$-i\omega^3 - P_1\omega^2 + iP_2\omega + P_3 + (cos\omega\tau - isin\omega\tau)(-Q_1\omega + iQ_2\omega + Q_3) = 0(4.19)$$

Putting the fictional and actual components apart,

$$P_3 - P_1\omega^2 + (Q_3 - Q_1\omega^2)cos\omega\tau + Q_2\omega sin\omega\tau = 0 \qquad (4.20)$$

$$P_2\omega - \omega^3 + Q_2\omega cos\omega\tau - (Q_3 - Q_1\omega^2)sin\omega\tau = 0 \qquad (4.21)$$

which leads to $$\omega^6 + m\omega^4 + n\omega^2 + s = 0 \qquad (4.22)$$

where

$$m = {P_1}^2 - {Q_1}^2 - 2P_2, n = {P_2}^2 - {Q_2}^2 + 2Q_1Q_3 - 2P_1P_3, s = {P_3}^2 - {Q_3}^2$$

Let $u = \omega^2$,then equation (22) becomes

$$u^3 + mu^2 + nu + s = 0 \qquad (4.23)$$

Lemma 1: The following outcomes apply to the polynomial equation (4.23):

1) There is at least one positive root in equation (5.15), if $s < 0$.

2) There is no positive root in equation (4.23), if $s \geq 0$ and$(m^2 - 3n) \leq 0$.

3) Equation (4.23) has positive roots if and only if $s \geq 0$ and$(m^2 - 3n) > 0$, and

$$u_1^* = \frac{-m+\sqrt{m^2-3n}}{3} > 0 \text{ and } h(u_1^*) \leq 0.$$

Lemma 2: Suppose $h(u_0) = (3u_0^2 + 2mu_0 + n) \neq 0$ and the conditions in (y_1) are satisfied. For (j=0,1...), denote $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$be the root of equation (4.24) satisfying $\alpha(\tau_j) = 0, \omega(\tau_j) = \omega_0$,where

$$\tau_j = \frac{1}{\omega_0}(\arccos\frac{-(Q_2\omega_0{}^2(P_2 - \omega_0{}^2) + (P_3 - P_1\omega_0{}^2)(Q_3 - Q_1\omega_0{}^2))}{(Q_3 - Q_1\omega_0{}^2)^2 + (Q_2Q_3 - Q_1\omega_0)^2} + 2j\Pi)$$

Then$\pm i\omega_0$are simple roots. In the event that the transversality requirement,

$$(\mathbf{y_2})\ \alpha'(\tau_j) = \frac{\mathrm{Re}\lambda(\tau)}{d\tau}\Big|_{\lambda=i\omega_0} \neq 0$$

holds, the system (4.7)-(4.9) experiences a Hopf bifurcation at any equilibrium point, and $\tau = \tau_j$

The evidence of Lemmas have been conveyed in the publication given by Ruan, S. & Wei; On the Zeros of a Third-Degree Exponential Polynomial with Applications to A Delayed Model for The Control of Testosterone Secretion, IMA J. Math. Appl. Medic. Biol., 18, pp. 41-52.

4.4 Numerical Representation

The use numerical simulations run in Matlab to support all of the prior analytical conclusions. Regarding the group of parameters, $k_1 = 1, k_2 = 1, d_3 = 0.58,$ the parameter-dependent stability areas; d_1 and d_2.Plots showing the system's equilibrium points are shown. Here, the starting point is (41,.11,.11).

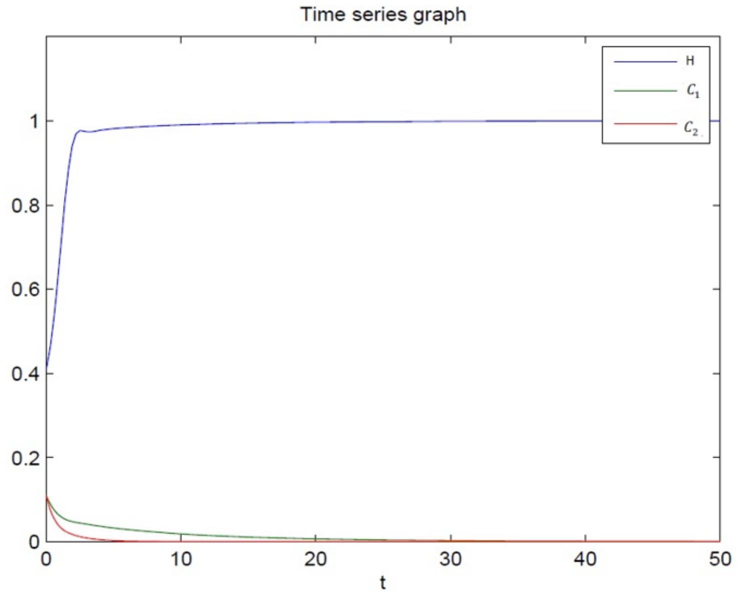

Figure 4.1: For $d_1 = .76, d_2 = 1.6$ and $E^* = (0.7600\ , 0.0452\ ,0.2760)$, the system exhibits asymptotic stability.

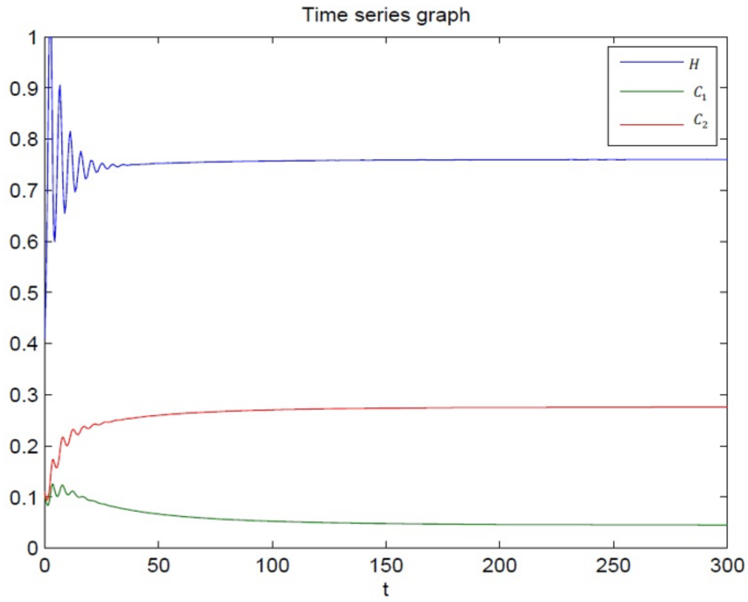

Figure 4.2: For $d_1 = 0.76,\ d_2 = .6$ and $E^* = (0.76000.0452\ 0.2760),\ \tau = 0.95$ and $\tau_0 = 0.920973$. If $\tau < \tau_0$ the system exhibits asymptotic stability.

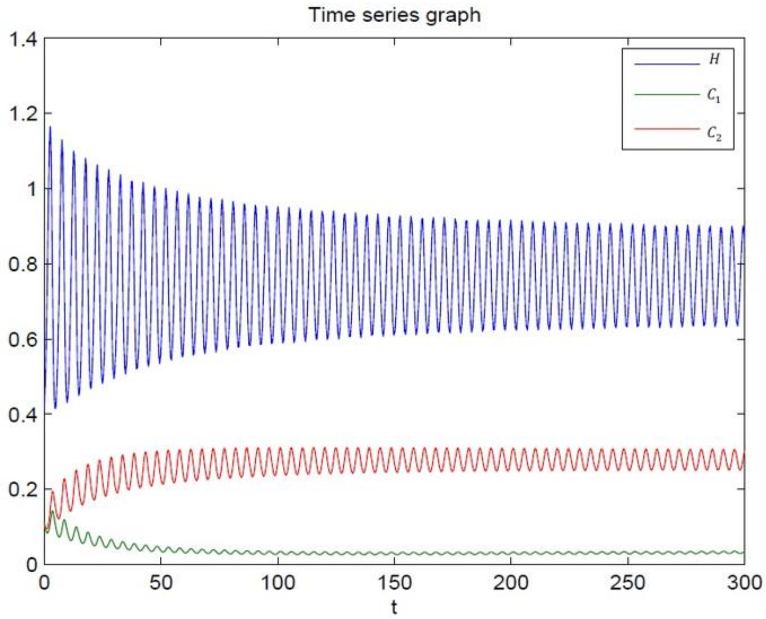

Figure 4.3: For $d_1 = .76, d_2 = 0.6$ *and* $E^* = (0.76000.0452\ 0.2760),\ \tau = .924$ and$\tau_0 = 0.920973.$ If $\tau > \tau_0$, the system becomes unstable, a Hopf-bifurcation takes place.

Similar result have been exhibited using Boundary and Axial Equilibrium. Numerical simulation for a certain set of parameters makes it clear that when only the system's Phytoplankton population survives, the delay is shortest; when all system species survive, it is longest.

4.5 Analysis of the State Variables' Sensitivity to the Model's Parameters

The model in this study has constant values. For approximation of the general coefficients, the "Direct Method" is utilized as conveyed in the publication given by Rihan. Here the All parameters are taken to be constants, and the original framework is used to simultaneously solve sensitivity equations. Then doing partial differentiation of the solution relating to each parameter may be all that is required for sensitivity analysis in this scenario if all of the parameters that appear in the system model (4.7)-(4.9) are assumed to be constants.

The solution (H, C_1, C_2) of partial differentiation with respect to t, for example, result in the following set of sensitivity equations:

$$\frac{\partial S_1}{\partial t} = (K_h - K_h C_2) - \alpha H(t-\tau)C_1 - \delta_1 H$$

$$\frac{\partial S_2}{\partial t} = \beta H(t-\tau)y - \delta_2 C_2$$

$$\frac{\partial S_3}{\partial t} = I - \gamma H C_1 - \delta_3 C_2$$

Where $S_1 = \frac{\partial H}{\partial \widehat{d_1}}, S_2 = \frac{\partial C_1}{\partial \widehat{d_1}}$ and $S_3 = \frac{\partial C_2}{\partial \widehat{d_1}}$.

4.6 Variable Sensitivity to Parameter $\widehat{d_1}$

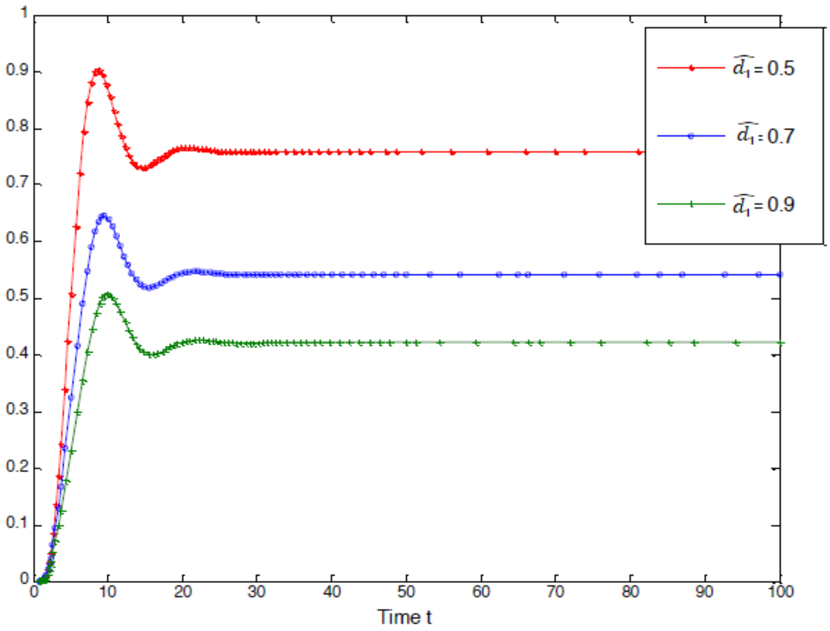

Figure 4.4: An illustration of the time series between minor variations for various values of coefficient $\widehat{\mathrm{d}_1}$

Figures 4 ,demonstrates that as we decrease the value of $\widehat{d_1}$ from 0.9 to 0.5, here is slight variation and shift in the state variables' values concentration of H and C_1, which ultimately remain constant and tend to zero. It predicts that state variables H and C_1 will be less sensitive to the factor $\widehat{d_1}$. However, given the same range of $\widehat{d_1}$ values, the variable for state C_2 experiences a significant change. It displays a decline in the delayed value. It also continues to be stable.

Hence, we conclude that with increase in mortality rate overall population of Phytoplankton and Zooplankton is decreasing.

4.7 Conclusion

According to both our theoretical and numerical findings, the system exhibits asymptotic constancy around the internal equilibrium that is positive and represents the cohabitation of all species, for a particular threshold of the system parameters. Additionally, it is ascertained by the examination of the equilibrium points' stability that when the ratio of a Zooplankton's mortality rate along with the multiplication of its conversion effective performance and spoliation is more than the radical of one-third of the Phytoplankton inhabitants' carrying capacity, the system is said to be coexisting. Limit cycles arise at all symmetry points when the time delay crosses a threshold value after taking the effective time delay on the body. This study also uses the direct method to examine the responsiveness of model solutions to modifications of delay differential system parameters. It is demonstrated that the sensitivity functions allow one to identify particular parameters and enhance one's comprehension of the role that particular model parameters play. The oscillation and value changes that go along with state variable sensitivity to parameter changes indicate the parameter is essential to the model and that the solution is sensitive to changes in the parameter. The state variable nutrient concentration H is the parameter that is least sensitive, according to sensitivity analysis. While rates of plant biomass decrease with a decline in the delayed value of utilization coefficient and become unstable, rates of plant biomass increase with a decline in the delayed value of consumption coefficient and remain stable.

Chapter 5

Phytoplankton-Zooplankton Interaction Harvesting Model: Dynamical Analysis

Abstract

In order to facilitate the commercial harvesting of certain species for food, we have created a model of the interaction between phytoplankton and zooplankton in this work. In order to keep the population at a suitable equilibrium level even when the species is continuously exploited, criteria for local stability, instability, and global stability are developed. Additionally, some threshold harvesting levels are investigated. However, if the species are spreading continuously, harvesting has limited opportunities to explore and keep the population at an appropriate equilibrium position. In light of additional research, this system's biological and bioeconomic equilibrium is attained in such a way that an optimal harvesting policy is also supported commercially. The results obtained using the analytical approach at the end of model design are supported by simulations that were carried out using mathematical methods. As seen from the results, both the analytical strategies and mathematical simulations produced consistent results, thereby validating the model's accuracy and reliability.

Keywords—Phytoplankton-Zooplankton, Boundedness, Harvesting Policy.

5.1 Introduction

The food web in marine habitats relies on the flow of energy, which serves as the base of the food chain and is generated and recycled by organisms within the ecosystem. Plankton, phytoplankton in particular is crucial for both environmental stability and aquaculture because they produce oxygen for living organisms and also remove half of the universe's carbon dioxide. Aquatic ecologists have always been captivated by the disequilibrium dynamics of phytoplankton explosions, or abrupt rise and fall of population. Sarkar et al. (2005) and Chattopadhyay et al. (2002) examined the two major reasons for planktonic blooming that are frequently accompanied by massive cell lysis and fast population collapse.Also, the rapid increase in bacterial population leads to the fast deoxygenation of water which might be dangerous for aquatic life. While zooplankton species like jellyfish, krill, and acetes are exploited for food today, phytoplankton species like nori, kelp, and eucheuma have been a fascinating research area. According to Chakraborty et al. (2008) and Pal et al. (2007), the importance of different functional forms in phytoplankton-zooplankton interactions in the presence of nutrients in the plankton system has been studied using a variety of research models in recent years. A second-order fractional step method for two-dimensional delay parabolic partial differential equations with a small parameter was presented by L. Govindarao and Abhishek Das in 2022. M'Hammed Ziane and Abderrahim El Moussaouy (2023) talked about a new generalisation of Hopfian modules.

It is discovered and reported after following research on the phytoplankton-zooplankton system by Chakraborty et al. (2008) that phytoplankton that produces toxins may be utilized as regulating agents to stop plankton blooms. Subsequently, the impact of harvesting on the plankton system has been rather insignificant. Lv et al. (2010) put forth a model for the harvesting of a phytoplankton-zooplankton system that produces toxins. Edwards and Brindley (1996); Ruan (1993) and Busenberg et al. (1990) studied one phytoplankton model with harvesting with two zooplankton. They found that only zooplankton with a higher biomass

conversion ratio and a lower natural death rate survive in the absence of harvesting, whereas lower biomass conversion ratio and a higher natural death rate may become more common following harvesting. Several oscillation theorems for nonlinear fractional differential equations with impulsive effect were discussed by A. Rahee et al (2022). According to Murray (2002), both population growth and species interactions in this study follow the law of mass effect. New stability and stabilization criteria for continuous systems with time delays were discussed by M. Rathika (2021).

The chapter begins by developing a problem model that outlines the significant parameters in section 2. Next, it examines the positivity and boundedness, and equilibrium of the model in section 3 and 4 respectively. Following this, specific theorems and propositions are applied to obtain results. Subsequently, in section 5, a harvesting policy is implemented, and simulations are conducted, leading to the final conclusions in section 7.

5.2 Formation of Model

In this section, a system of simultaneous differential equations is used to model the interaction between phytoplankton and zooplankton. These equations are represented by:

$$\frac{dH}{dt} = \gamma_1 H - \frac{\gamma_1 H^2(t-\tau)}{s_1} - d_1 H - c_1 E H - \alpha_1 P C \quad (5.1)$$

$$\frac{dC}{dt} = \gamma_2 C - \frac{\gamma_2 C^2(t-\tau)}{s_2} - d_2 C - c_2 E - \alpha_2 H C - \alpha H C \quad (5.2)$$

with the initial conditions $H(0) = H_0 > 0, C(0) = C_0 > 0$.

5.3 Description of Parameters:

Parameters	**Description**
H	Density of Phytoplankton population
C	Density of Zooplankton population
γ_1	The phytoplankton's inherent rate of expansion H
γ_2	The zooplankton’s inherent rate of expansion C
s_1	Carrying Capacity of phytoplankton species
s_2	Carrying Capacity of zooplankton species
α_1	The predation rate of phytoplankton H by zooplankton C
α_2	Biomass conversion rate
α	Ratio of toxic production per unit of phytoplankton biomass
d_1	Mortality rate of phytoplankton H
d_2	Mortality rate of zooplankton C
c_1, c_2	Catchability coefficients
E	Harvesting Effort
U	Harvesting cost per unit effort
B	Hamiltonian

5.4 Positivity and Boundedness of Solution

In this section, the positivity and boundedness of the system is discussed with values of H(t) and C(t) for all $t = 0$ given as: $H(0) = H_0 > 0, C(0) = C_0 > 0.$

$$H(t) = H(0)exp\int_0^t \left(\gamma_1\left(1-\frac{H}{s_1}\right) - (d_1 + c_1E) - \alpha_1 C\right) ds \geq 0 \tag{5.3}$$

and

$$C(t) = C(0)exp\int_0^t \left(\gamma_2\left(1-\frac{C}{s_2}\right) - (d_2 + c_2E) + (\alpha_2 - \alpha)C\right) ds \geq 0 \tag{5.4}$$

Further, $\frac{dC}{dt} \leq \gamma_1 C\left(1-\frac{H}{s_1}\right) - c_1EH$ and $\frac{dH}{dt} \leq \gamma_2 H\left(1-\frac{H}{s_2}\right) - c_2EC$

By using a standard comparison theorem:

$\lim_{t\to+\infty} H(t) \leq \xi_1$ and $\lim_{t\to+\infty} C(t) \leq \xi_2$

Where $\xi_1 = \max\,[H(0), \frac{s_1(\gamma_1 - c_1E)}{\gamma_1}]$ and $\xi_2 = \max\left[C(0), \frac{s_2(\gamma_2 - c_2E)}{\gamma_2}\right]$

Thus, under the specified initial conditions, all solution curves for the system equation (5.1) and equation (5.2) enter the region.

5.5 Equilibrium of the Model

The level of equilibrium of phytoplankton grows as the harvesting rate increases, whereas the host species' equilibrium decreases. It is also noted that when harvesting rates increase, the balance of zooplankton species decreases. This part examines the existence of Hopf bifurcation and observes the coexisting equilibrium's local and global stability. The possible steady states of the system are as follows:

i. $R_0 = (0,0)$

ii. $R_1 = \left(\frac{(\gamma_1 - d_1 - c_1 E)s_1}{\gamma_1}, 0\right)$, exist if $E < \frac{\gamma_1 - d_1}{c_1}$

iii. $R_2 = \left(\frac{(\gamma_2 - d_2 - c_2 E)s_2}{\gamma_2}, 0\right)$, exist if $E < \frac{\gamma_1 - d_1}{c_1}$,and

iv. The interior equilibrium $R_* = (H_*, C_*)$ where $H_* = \frac{\frac{(\gamma_2(\gamma_1 - d_1 - c_1 E)}{\alpha_1 s_2} - (\gamma_2 - d_2 - c_2 E)}{(\frac{\gamma_1 \gamma_2}{s_1 s_2 \alpha_1} + \alpha_2 - \alpha)}$ and $Z_* = \frac{\left(\gamma_1 - d_1 - c_1 E - \frac{\gamma_1 H_*}{s_1}\right)}{\alpha_1}$ exist if $\frac{\gamma_2 - d_2}{c_2} < E < \frac{(\gamma_1 - d_1 - \frac{\gamma_1 H^*}{s})}{c_1}$.

Proposition 1. As given in equation (5.1) and equation (5.2), $R_0 = (0,0)$, $E > max\,(\frac{\gamma_1 d_1}{c_1}, \frac{\gamma_2 d_2}{c_2})$ it is asymptotically stable. Further when $E < \frac{\gamma_{1 - d_1}}{c_1}$ always hold, R_0, R_1 exists and if $E < \frac{\gamma_1(\gamma_2 d_2)s_1(\alpha_2 - \alpha)}{\gamma_1 c_2 - c_1 s_1(\alpha_2 - \alpha)}$, the equilibrium of free zooplankton R_1becomes stable asymptotically.

Proposition 2. As given in equation (5.1) and equation (5.2), if $\frac{\gamma_1(\gamma_2 - \mu_2) - (r_1 - \mu_1)K_1(\rho_2 - \alpha)}{r_1 C_2 - C_1 K_1(\rho_2 - \alpha)} < E < min(\frac{r_1 \mu_1}{C_1} \frac{r_2 \mu_2}{C_2})R_0$, R_1 and R_2 exist, and R_0, R_1 become unstable, R_2 is locally asymptotically stable if

$$E < \frac{r_2(r_1 \mu_1) - (r_2 \mu_2)K_2 \rho_1}{r_2 C_1 - C_2 K_2 \rho_2}$$

Characteristic equation at R^* is, $\lambda^2 - traceJ\,\lambda + detJ = 0$

Where$trace\,J = (\gamma_1 - d_1 - c_1 E) - \frac{2\gamma_1 H^*}{s_1} - \alpha_1 C^*$ (5.5)

Theorem1. For the system equation (5.1), if,

$\frac{\gamma_2 - d_2}{c_2} < E < \frac{\gamma_1 - d_1}{c_1} - \frac{\gamma_1 H_*}{c_1 s_2}$, then R^* exists and is locally asymptotically stable if $trace\,J < 0$ and $det\,J > 0$.

For the global stability of the equilibria, we have the following theorems:

Theorem 2.

The extinction equilibria R_0 is globally asymptotically stable (GAS)if E > max$\left[\frac{\gamma_1 d_1}{C_1}, \frac{\gamma_2 d_2}{C_2}\right]$.

Theorem 3.

The interior equilibrium R^* is globally asymptotically stable (GAS) in the positive quadrant. The proofs of the theorem can be verified from Salle & Lefschetz (1961).

In this section, we shall only study dynamical behaviour of the interior equilibrium. It is obvious that system of equation (5.1) and equation (5.2) has a unique positive equilibrium. The equilibrium exponential characteristic equation is given by:

$$\begin{vmatrix} \lambda - m_1 & -m_2 \\ -m_3 & \lambda - m_4 \end{vmatrix} \tag{5.6}$$

which gives us;

$$\lambda^2 - (m_1 + m_4)\lambda + m_1 m_4 - m_2 m_3 = 0 \text{ where}$$

$$m_1 = (r_1 - \mu_1 - c_1 E - \rho_1 Z) - \frac{r_1 P\, e^{-\lambda\tau}}{K}, m_2 = (\rho_2 - \alpha)Z, m_3 = \rho_1 P\,,$$

$$m_4=(r_2-\mu_2 - c_2E) + (\rho_2 - \alpha)P - \tfrac{r_2}{K_2}Ze^{-\lambda\tau}$$

5.6 Optimal Harvesting Policy

The goal of this section is finding the best harvesting strategy for the highest sustainable yield, assuring that the regulatory body will accomplish its goal. We examine the current value by:

$$R = \int_0^\infty e^{-\delta t}(p_1c_1H(t) + p_2c_2C(t) - U)E(t)dt$$

where the instantaneous annual discount rate is represented by δ. Consequently, our goal is to maximize R^* based on equation (5.1) and equation (5.2) and in relation to the control limits 0≤ E≤ E_{max}, where E_{max} is the maximum amount of harvesting that can be done. The related Hamiltonian function is obtained by applying the Pontryagin Maximum Principle:

$$H = e^{-\delta t}(p_1c_1H + p_2c_2C - U)E + \lambda_1\left(r_1H\left(1 - \tfrac{H}{K_1}\right) - (\mu_1 - c_1E\right)H - p_1HC) +$$

$$\lambda_2(r_2C(1-\tfrac{C}{K_2})_2 - (\mu_2 + c_2E)C + (\rho_2 - \alpha)HC)$$

$$= \sigma(t)E + \lambda_1\left(r_1\, H\left(1-\tfrac{H}{K_1}\right) - \mu_1H - \rho_1HC\right) + \lambda_2\, (r_2C\, (1 - \tfrac{C}{K_2}) - (\mu_2C + \rho_2HC -$$

$$\alpha HC).$$

where $\lambda_{1,}\lambda_2$ are the operators for the adjoint that fulfil the equations and σE represents the switching function:

$$\frac{d\lambda_1}{dt} = \frac{\partial B}{\partial H}, \tag{5.7}$$

$$\frac{d\lambda_2}{dt} = \frac{\partial B}{\partial C}, \tag{5.8}$$

In this instance, the optimization problem will yield only singular control since the Hamilton B is a linear control variable $E(t)$.The following is therefore a prerequisite for maximizing Hamilton B under the singular control variable $E(t)$:

Putting $\frac{\partial H}{\partial E} = 0$ becomes:

$$e^{-\delta t}(p_1 c_1 H + p_2 c_2 C - U) - \lambda_1 c_1 H - \lambda_2 c_2 C) = 0 \; or \; (\lambda_1 c_1 H + \lambda_2 c_2 C) = e^{-\delta t}, \tag{5.9}$$

As a result, equation (5.7) and equation (5.8) becomes:

$$\frac{d\lambda_1}{dt} = -c_1 p_1 E e^{-\delta t} - \lambda_1 \left(r_1 - \frac{2r_1 H}{K_1} - (\mu_1 + c_1 E) - p_1 C \right) - (p_2 - \alpha) C \lambda_2 \tag{5.10}$$

$$\frac{d\lambda_2}{dt} = -c_1 p_1 E e^{-\delta t} + p_1 p \lambda_1 - \lambda_2 \left(r_2 - \frac{2r_2}{K_2} \right) - \mu_2 + c_2 E + (p_2 - \alpha) H) \tag{5.11}$$

The ideal equilibrium in order to find the

$$E = \frac{r_1}{c_1}\left[\left(1 - \frac{H^*}{K_1}\right) - \mu_{1-\rho_1 C^*}\right] = \frac{r_2}{c_2}\left[(1 - \frac{C^*}{K_2}) - \mu_2 + \rho^2 - \alpha) H^*\right] \tag{5.12}$$

In the given equation (5.9) and equation (5.10), it is possible to obtain the simultaneous linear equation system that follows:

$$\frac{d\lambda_1}{dt} = -c_1 p_1 E e^{-\delta t} + \frac{r_1 H^* \lambda_1}{K_1} - (\rho_2 - \alpha) C^* \lambda_2 \tag{5.13}$$

$$\frac{d\lambda_2}{dt} = -c_2 p_2 E e^{-\delta t} + \rho_1 H^* \lambda_1 \, \frac{r_2 C^* \lambda_2}{K_2} \tag{5.14}$$

Eliminating λ_2 from equation (5.13) and equation (5.14), we have:

$$\frac{d\lambda_1}{dt} = -T_1\lambda_1 = -T_2 e^{-\delta t}, \tag{5.15}$$

Where $T_1 = \frac{r_1}{c_1} + \frac{(\rho_{2-\alpha})c_1}{c_2} H^*$ and $T_2 = c_1p_1E + \frac{(\rho_{2-\alpha})}{c_2}(c_1p_1H^* + c_2p_2C^* - U$

On solving equation (5.15), we get, $\lambda_1 = \frac{T_2}{T_1+\delta} - e^{-\delta t}$ and using this value in equation (5.14), we obtain,

$$\frac{d\lambda_1}{dt} - R_1\lambda_1 = -R_2 e^{-\delta t} \tag{5.16}$$

Which results into $\lambda_2 = \frac{R_2}{R_1+\delta} e^{-\delta t}$

Where $R_1 = \frac{r_2 Z^*}{K_2}$ and $R_2 = c_2p_2\mathrm{E} - \rho_1 \frac{T_2}{T_1+\delta} - H^*$

As t→∞, it remains bounded. Putting λ_1 and λ_2 in equation (5.9), It is possible to obtain the simultaneous linear equation system that follows:

$$c_1\left(p_1 - \frac{T_2}{T_1+\delta}\right)H + c_2\left(p_2 - \frac{R_2}{R_1+\delta}\right)Z = U \tag{5.17}$$

Considering the values of T_1, T_2, R_1, R_2 and from equation (5.17), it can now be deduced that:

$$\pi(H,C,E) = c_1p_1H + c_2p_2C - U = \left(\frac{c_1T_2}{T_1+\delta}\right)\mathrm{H} + \frac{c_2N_2}{N_1+\delta}\,\mathrm{C} \to 0 \text{ as } \delta \to \infty \tag{5.18}$$

Consequently, $\pi(H_\infty, C_\infty, E, t) = 0$ is the net economic revenue. This suggests that harvesting will cease to occur in the event of an infinite discount rate, as the net economic revenue falls to zero. Furthermore, equation (5.18) demonstrates that the optimal equilibrium will only see the net economic rent maximized if a zero discount rate is provided. Once more, equation (5.18) demonstrates that $c_1 T_2(R_1+\delta)H + c_2 R_2(T_1+\delta)$ is of $Q(\delta)$ and $(T_1+\delta)(R_1+\delta)C$ of $Q(\delta^2)$.Therefore, maximizing π occurs when δ = 0.

5.7 Numerical Example

(i) Allocating values $r_1 = 6, r_2 = 0.1$, $K_1 = 50$, $K_2 = 30$, $\rho_1 = 0.3$, $\rho_2 = 0.25$, $\alpha = 0.04$, $\mu_1 = 0.05$, $\mu_2 = 0.08$, $c_1 = 0.22$, $c_2 = 0.1$, E = 0.5 It is easy to find $\frac{r_1\mu_1}{c_1}$=27.0455, $\frac{r_2\mu_2}{c_2}$= 0.2, $\frac{r_1\mu_1}{c_1} - \frac{r_1 H^*}{c_1 K_1}$= 26.8005,Trace J = 0.1182 < 0 and Det J = 0.5490 > 0 then R^* = (0.4490, 19.2871) is globally asymptotically stable.

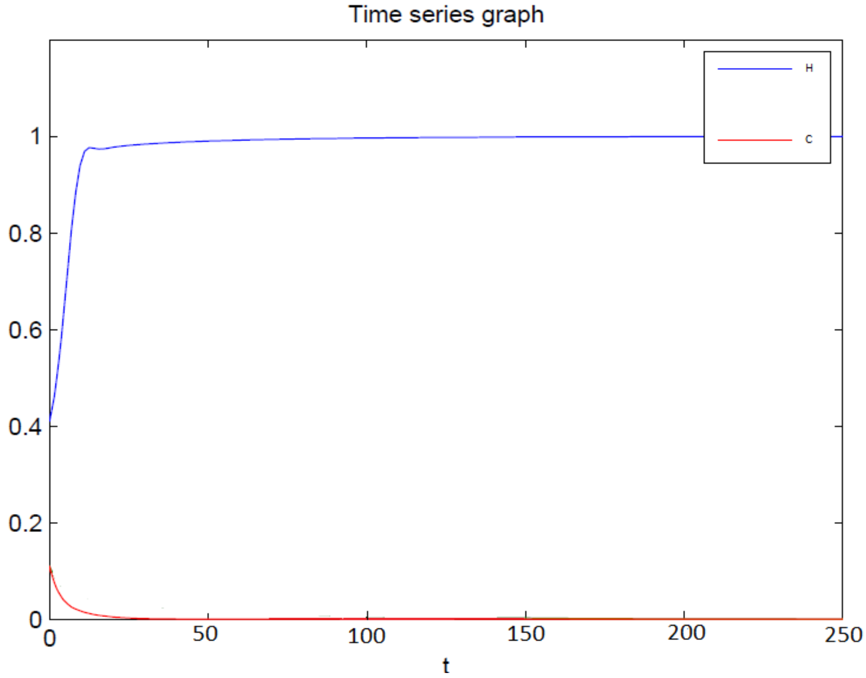

Figure 5.1 The graph shows that the system is globally asymptotically stable under above mentioned parameters.

(ii) Choosing E = 30, it is easy to verify that E > (max$\frac{r_1\mu_1}{c_1}, \frac{r_2\mu_2}{c_2}$) = max (27.0455, 0.2) = 27.0455,p_1 = 2, p_2 = 3, C = 5.9, δ = 0.03 and E = 0.5 or 1, the system loses its stability and Hopf-Bifuracation occurs.

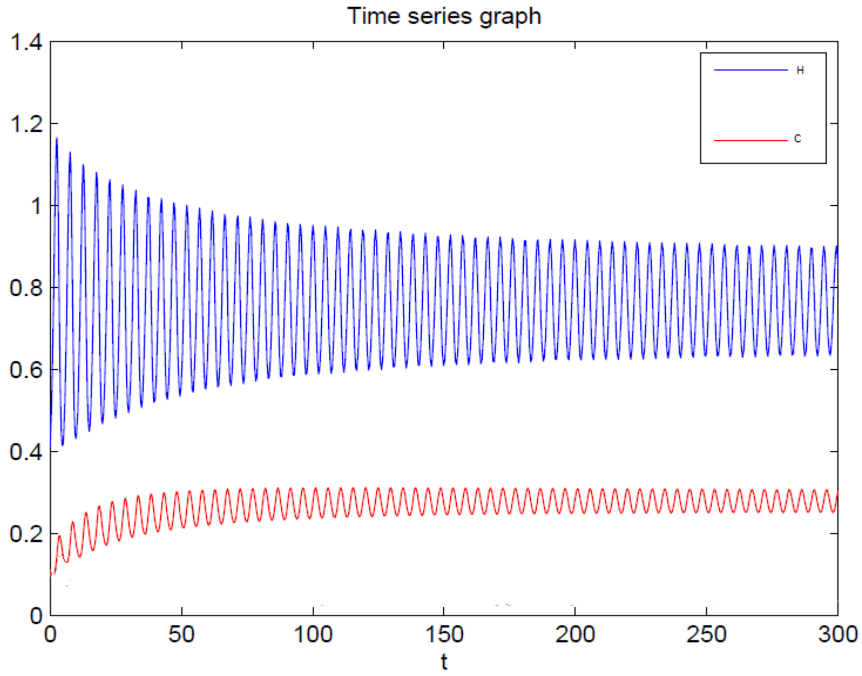

Figure 5.2 The graph shows that the system loses its stability and Hopf- Bifurcation occurs when there is change of parameters.

5.8 Conclusion

The chapter explores a mathematical model elucidating the dynamics of phytoplankton-zooplankton interactions, considering the impact of harvesting on certain species exploited for commercial purposes. The study assumes logistic growth for both populations and incorporates the release of toxic substances by phytoplankton, affecting the grazing pressure of predators on their prey. The analysis reveals that excessive harvesting can lead to irreversible population extinction. The research establishes the existence and global asymptotic stability of an interior equilibrium under specific conditions. Numerical results indicate variations in population levels for phytoplankton and zooplankton in the presence and absence of harvesting. Specifically, when harvesting is absent (E=0), an interior equilibrium is observed at a lower population level, denoted as $R* = (0.0575, 19.8102)$, in contrast to the presence of harvesting, where the equilibrium is at $R* = (0.2815, 19.3555)$. The chapter also investigates optimal harvesting policies, emphasizing the maximization of present value revenues and adherence to transversality conditions for constant shadow prices over time. Both analytical and numerical evidence demonstrates that the duration of the delay grows as the rate of harvesting rises and the specific growth rate of the host species falls. The study concludes that zero discounting optimizes economic revenue, while an infinite discount rate results in complete economic rent dissipation. Future research may entail a collaborative study of two phytoplankton and two zooplankton species to observe the alteration in system dynamics and its long-term effects in response to the evolving climate.

Chapter 6

Conclusion & Future Scope

6.1 Conclusion

The culmination of this thesis offers a thorough analysis of the complex dynamics regulating aquatic environments, with a particular emphasis on the relationships between populations of zooplankton and phytoplankton that produce toxins. This research offers a strong foundation for comprehending these interactions and their consequences for ecosystem sustainability and stability through the accomplishment of three main objectives.

First, the thesis models, together with a thorough stability study, how competing toxin-producing phytoplankton affects a zooplankton population. Scientists have proved that toxin-producing phytoplankton play a role in regulating the death of plankton blooms. Both mathematical modelling and experimental data were used to obtain the result for the same. The prior research, however, did not take into account the simultaneous presence of two different forms of hazardous phytoplankton and also failed to consider the possible effects of systemic delays in time as well. However, in this thesis, a more intricate model was designed to fill up these gaps. The model includes three key components: two types of toxic phytoplankton and one zooplankton. The main aim of this objective with the incorporation of these elements was to create a more realistic representation of the interactions within plankton communities. To comprehend the behaviour of these three-component systems, the study used numerical simulations in addition to theoretical analysis. The outcomes present that, in some circumstances, the system may attain stability. Also, when the system's parameters-like growth rates and interaction strengths-fall inside a specific threshold, the system reaches a stable state.

Furthermore, in this case, the idea of a "positive interior equilibrium" is equally considered. Instead of one or more species going extinct, it alludes to a balanced situation in which the populations of all three species remain positive. When this balance is reached, it indicates that the ecosystem can support all three species over the long term, even in the face of competition and toxins. This implies that all three species-two harmful phytoplankton species and one zooplankton species-can coexist without causing the others to totally disappear. With the first objective thus accomplished, the study offers insight into the interactions and coexistence of multiple harmful species in marine environments and emphasises the significance of taking time delays and multiple competing species into account in ecological models, resulting in more precise predictions and possible management approaches for harmful algal blooms.

The second objective of the study explores the Hopf bifurcation and direction analysis of a hazardous phytoplankton-zooplankton system. By extending the examination into the crucial turning points that may result in cyclical behaviours and maybe oscillatory dynamics within the ecosystem, this objective provides a greater understanding of the circumstances that give rise to these bifurcations. The goal was to investigate how time delays affect a complicated biological system that includes one kind of zooplankton and two types of phytoplankton. The system's primary characteristic is the mutual support between the two phytoplankton populations in the face of zooplankton predation. This mutual aid might take the form of different ecological interactions, as when one species produces a good for the other, or it could take the form of indirect advantages like changing the habitat to lessen the pressure from predators. Moreover, the dynamics of the system are greatly impacted by the addition of a time delay. The time it takes for the impacts of predation to affect phytoplankton development or for phytoplankton populations to react to environmental changes are two examples of real-world processes that may be represented by time delays. Additionally, the study demonstrates that adding a certain time delay causes the system's stable equilibrium point to become unstable. A Hopf bifurcation, which is the turning point from a stable condition to periodic oscillations, is caused by this instability. In other words, beyond a critical threshold of the delay parameter, the system no longer returns to its equilibrium but instead starts to oscillate.

For this, the examination made use of mathematical methods to comprehend the nature of these oscillations. Using centre manifold reduction theorem and normal form theory, the study develops an explicit procedure to ascertain the stability (i.e., whether the oscillations are stable or unstable) and direction (i.e., whether the oscillations expand or diminish) of the solutions originating from the Hopf bifurcation. Consequently, the second objective of the thesis offers a thorough analysis of how time lags can cause an ecological system to become unstable and result in intricate oscillatory behaviour. Understanding these dynamics is crucial for predicting and managing ecological systems, particularly in scenarios where time delays cannot be ignored.

In the third objective, the study models the dynamics of interacting aquatic populations using delay differential equations and sensitivity analysis. The objective considers the natural time lags in biological processes and assesses the impact these time lags have on ecosystem stability and dynamics. All species live in the system, as shown by the theoretical and numerical examination, which show that the system stabilises around a positive internal equilibrium. This stability happens when the square root of one-third of the phytoplankton's carrying capacity is exceeded by the ratio of zooplankton mortality rate, together with its conversion efficiency and resource consumption. In addition, the introduction of time delays may result in limit cycles, which are periodic oscillations. When the time delay exceeds a crucial level, these cycles appear. The thesis examined how modifications to time delays and other system factors impact the model's results using a direct approach. Sensitivity functions can aid in determining which parameters have the most effects on the model. The state variable nutrient concentration H is the parameter that is least sensitive, according to sensitivity analysis. On the other hand, changes in the utilisation and consumption coefficients' delay values cause a significant reaction in the plant biomass rates. To be more precise, when the utilisation coefficient delays less, the system becomes unstable, but when the consumption coefficient delays less, it stays stable.

Further, the model examines the interactions between phytoplankton and zooplankton as well as the impact of harvesting on commercially fished species. It is anticipated that both populations would expand significantly, with toxins released by phytoplankton impacting zooplankton feeding. Overharvesting has the potential to cause an irreversible drop in population. Under specific conditions, the analysis validates the existence and global asymptotic stability of an inner equilibrium. Examining the numerical findings, we find that the system reaches an internal equilibrium at lower population levels without harvesting (E=0): $R\ast=(0.0575,19.81102)$ R∗=(0.0575,19.81102). At higher population levels, however, harvesting causes the equilibrium to shift: $R\ast=(0.2815,19.3555)$ R∗=(0.2815,19.3555). The best harvesting tactics that would maximise present value revenues while maintaining stable shadow prices over time. The findings indicate that longer delays are linked to greater rates of harvesting and lower rates of specific growth of the host species. The greatest amount of economic income is produced by zero discounting, but the entire amount of economic rent is dispersed by an infinite discount rate.

Future research may entail a collaborative study of two phytoplankton and two zooplankton species to observe the alteration in system dynamics and its long-term effects in response to the evolving climate. Together, these objectives form a cohesive study that not only advances theoretical understanding but also provides practical insights for maintaining and managing aquatic ecosystems in the face of environmental challenges.

6.2 Future Scope

The stock market is a complicated system that is impacted by a wide range of factors, such as investor behaviour, economic data, and developments in geopolitics. Economists, mathematicians, and financial analysts have long struggled to comprehend and forecast its dynamics. The application of mathematical models to understand the behaviour of the stock market has gained attention in recent years. One such mathematical model is Delay differential equations (DDEs), which provide a potent framework for capturing time delays and feedback mechanism inherent in financial systems. The use of DDEs to simulate stock market dynamics and its consequences for financial analysis and decision-making is explored in this article:

6.2.1. The Need for Mathematical Modelling in the Stock Market

- **a.** The dynamics of the stock market are complicated and sometimes nonlinear, making it difficult to comprehend using conventional approaches.
- **b.** A systematic way to examine and forecast market behaviour is offered by mathematical models, which are helpful for economists, policymakers, and investors.
- **c.** Conventional models, on the other hand, frequently overlook important elements like feedback loops and time delays, which have a big impact on market dynamics.

Introduction to Delay Differential Equations:

- **d.** A set of differential equations is known as delay differential Equations (DDEs) which accounts for time delays into the dynamics of a system.
- **e.** They are perfect for researching financial systems with memory effects because they are especially well-suited for simulating phenomena where past conditions affect future behaviour.
- **f.** DDEs can record various aspects of the stock market, such as investor behaviour, herd behaviour, and the propagation of information through the market.

6.2.2. Application of Delay Differential Equations (DDEs) in Stock Market Modeling

a. Incorporating time delays into account when modelling the dynamics of stock prices: DDEs may be used to simulate the time lag between the release of information and its effect on stock prices.

b. DDE models have the ability to depict the feedback loop that exists between market prices and investor behaviour, providing insight into the formation of market bubbles and crashes.

c. DDEs can assist in identifying the best trading strategies that take transaction costs and market dynamics into consideration by introducing time delays into trading algorithms.

6.2.3. Challenges and Future Directions:

a. Despite its potential, DDE models have difficulties with model validation and parameter estimates, particularly in strident and unpredictable market environments.

b. Future studies may concentrate on improving DDEs models by including other variables such regulatory changes, market microstructure, and liquidity constraints.

c. To create reliable DDEs models that can guide investment decisions and risk management strategies, collaboration between practitioners, economists, and mathematicians is essential.

In conclusion, Delay Differential Equations (DDEs) present a viable framework for simulating the complex dynamics of the stock market, offering perceptions into the views and behaviour of investors and strategies for trading. We are able to anticipate and reduce financial risks more accurately and get a deeper understanding of the fundamental processes influencing stock prices by including time delays into mathematical models. The way we examine and negotiate the minutiae of the stock market might be completely transformed by more study and cooperation in this field.

There are uses for Delay Differential Equations (DDEs) outside of the stock market. Here are some other domains in which DDE-based mathematical models may be used in the future:

1) **Epidemiology and Public Health:** Since there is frequently a lag between an individual’s infection and the onset of their ability to infect others, DDEs may be utilised to simulate the spread of infectious diseases. By incorporating these delays, epidemiologists may enhance their comprehension of disease dynamics and assess the efficacy of treatments like immunisation drives and social distancing measures.
2) **Neuroscience:** DDEs may be applied to simulate brain systems, where signal propagation delays between neurons are common. Researchers can gain a better understanding of brain processes including information processing, synchronisation, and neuronal oscillations by using these models.
3) **Chemical Engineering:** DDEs may be used to model chemical reaction networks, which frequently have delays because of the time it takes for reactants to diffuse or to occur. Further, these models may be used to develop more effective reactors and optimise chemical processes.
4) **Ecology**: In ecological systems, where there are frequently delays owing to gestation durations, maturation times, and migratory patterns, population

dynamics may be modelled using DDEs. These models may be used to forecast population fluctuations, interactions between species, and the effects of changing environmental conditions on ecosystems.

5) **Control Systems**: DDEs may be used to simulate time-delayed systems, such industrial processes, transportation networks, and communication networks. In order to guarantee stability, performance, and dependability in complex systems, these models may be used to assist create strong control techniques.
6) **Climate Science:** DDEs are useful for modelling climate feedback loops, in which variations in greenhouse gas emissions frequently take time to affect weather patterns and world temperatures. These models can assist in refining climate forecasts and assisting in the decision-making process for climate change mitigation policies.
7) **Economics and Finance:** DDEs may also be used to simulate macroeconomic dynamics, interest rate dynamics, and exchange rate changes and can assist in the better understanding of economic risks and fluctuations by policymakers, central banks, and financial institutions.

Thus, DDEs may be used to describe systems with time delays in various fields and real-world problems since they provide an adaptable framework. We may anticipate seeing increasingly complex and perceptive implementations of DDE models in the future as mathematical methods and computational power continue to evolve.

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