SITUATION-BASED FUZZY MULTI-OBJECTIVE OPTIMIZATION TECHNIQUES BY THE UTILITY OF DIVERSE FUZZY NUMBERS

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By

Pinki

Registration Number: 12109444

Supervised By
Dr. Rakesh Kumar (19437)
Department of Mathematics (Associate Professor)
Lovely Professional University

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With gratitude to God for strength, to my family for unconditional love, to my teachers for their wisdom, and to my friends for their constant encouragement - I dedicate this work to you all.

<u>**DECLARATION**</u>

I, hereby declared that the presented work in the thesis entitled "**Situation-Based Fuzzy Multi-Objective Optimization Techniques By The Utility Of Diverse Fuzzy Numbers**" in fulfilment of degree of **Doctor of Philosophy (Ph. D.)** is outcome of research work carried out by me under the supervision of Dr. Rakesh Kumar, working as Associate professor, in the Department of Mathematics of Lovely Professional University, Punjab, India. In keeping with general practice of reporting scientific observations, due acknowledgements have been made whenever work described here has been based on findings of other investigator. This work has not been submitted in part or full to any other University or Institute for the award of any degree.

(Signature of Scholar)

Name of the scholar: Pinki

Registration No.: 12109444

Department of Mathematics

School of Chemical Engineering and Physical Sciences

Lovely Professional University,

Punjab, India

CERTIFICATE

This is to certify that the work reported in the Ph. D. thesis entitled "**Situation-Based Fuzzy Multi-Objective Optimization Techniques By The Utility Of Diverse Fuzzy Numbers**" submitted in fulfillment of the requirement for the award of degree of **Doctor of Philosophy (Ph.D.)** in the Department of Mathematics, is a research work carried out by Pinki, 12109444, is bonafide record of her original work carried out under my supervision and that no part of thesis has been submitted for any other degree, diploma or equivalent course.

(Signature of Supervisor)

Dr. Rakesh Kumar

Associate Professor

Department of Mathematics

School of Chemical Engineering and Physical Sciences

Lovely Professional University,

Punjab, India

Abstract

Multi-objective linear programming problems (MOLPP) are a decision-making tool in many real-world disciplines, such as distribution, production, economics, and ecology. Uncertainty in data makes optimization problems more challenging to resolve since they become fuzzy optimization problems. In this thesis, we look at how well different approaches work in different situations when it comes to addressing fuzzy multi-objective linear programming problems (FMOLPPs).

The ultimate objective of the research is to discover a solution to FMOLPP. The main goal is to make the fuzzy problem simpler to comprehend. Several researchers have employed defuzzification as a method to deal with them. Complex fuzzy numbers are notoriously challenging to decipher, in contrast to basic fuzzy numbers for which defuzzification studies are straightforward. First, to apply defuzzification techniques to any inherent data, a universal classification of all fuzzy numbers is necessary. To finish this job, we classify fuzzy integers according to their components and then give three separate defuzzification methods for each class. After implementing these defuzzification strategies, both basic and complex fuzzy data can be handled. The results from the real-life manufacturing case study show that the two defuzzification methods, α-cut and centroid of area, can effectively handle any balanced intrinsic data.

When dealing with MOLPP concerns that arise from converting fuzzy problems into them, there are various solutions available, both fuzzy and non-fuzzy. The fuzziness that comes with having multiple objectives is frequently too much for traditional, crisp optimization approaches to manage. Modeling complex systems using fuzzy logic, which includes degrees of membership instead of binary judgments, is more realistic. The study conducts a critical analysis of traditional fuzzy methods through their application to real-world case studies, highlighting their shortcomings and practical applications. By looking at these examples, the study explores how intuitionistic and dual-hesitant fuzzy numbers, along with other advanced fuzzy set types, affect optimization results, and how association functions influence both non-linear and linear forms. We use this thorough examination to refine and choose suitable fuzzy strategies for complex, real-world decision-making.

Functions are unfairly treated by existing approaches because of the large variety of functional values. Because of this, the current effort is focused on finding a better normalized distance parameter for membership functions. To validate its usefulness in a controlled situation, we first incorporate it within the basic fuzzy technique, which provides a fundamental framework. After we've proven its worth, we apply it to intuitionistic and dual hesitant fuzzy models, two advanced extensions of fuzzy sets, to see how it does in more complicated decision-making settings. The theoretical basis is fortified, and the normalized distance function's adaptability and robustness across different levels of uncertainty are demonstrated by this gradual integration. To ensure scientific rigor, the investigation begins by analyzing efficacy using hypothetical cases.

After knowing the importance of normalized distance functions and the extension of fuzzy sets in improving the effectiveness of fuzzy multi-objective optimization, the next study focuses on demonstrating their practical implementation across a variety of real-world domains. The study begins by applying the enhanced fuzzy approach with triangular fuzzy numbers to biomimetic systems and transportation problems in smart cities, gradually transitioning from linear to non-linear membership functions to evaluate their impact on optimization outcomes. This progression continues with the application of the enhanced approach to material science, particularly in modeling the composition of titanium alloys. The methodology is further expanded through the integration of triangular intuitionistic fuzzy models within a manufacturing context, showcasing the improvements brought by the normalized distance concept. Lastly, the dual hesitant fuzzy optimization method is employed to address complex production problems, capturing deeper uncertainties through combined hesitant and intuitionistic behaviors. This structured sequence of applications facilitates a comprehensive comparative analysis and effectively demonstrates the robustness, adaptability, and practicality of the proposed methods across diverse, uncertain decision-making environments.

The main contribution of this thesis is the comprehensive and innovative approach it offers for dealing with fuzzy multi-objective optimization issues. It begins with fuzzy logic basics and moves on to practical modeling, comparative analysis, and real-world case applications; it introduces new methodologies and advances in optimization techniques, defuzzification, and fuzzy set extensions. In a variety of complicated real-world situations, the resulting solutions provide decision-makers with strong tools to deal with uncertainty, increase efficiency, and make more informed choices.

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I am eternally grateful to "Bholenath," whose heavenly gifts have sustained me on this path. Nothing would have been achievable if it weren't for the mercy of God.

Dr. Rakesh Kumar, my supervisor, has been an incredible source of encouragement, insightful criticism, and steadfast support during this entire research project. I am thankful to him. Their support and confidence in my work were crucial in getting my thesis done, and their knowledge and hard work were priceless.

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Chapter 1

Introduction

Prior to diving into advanced approaches, it is crucial to have a solid grasp of basic ideas. To construct the theoretical framework that is utilized throughout the thesis, this chapter offers the essential terminology and terminologies pertaining to fuzzy set theory and optimization problems. We hope that by covering these ground rules, the following chapters will provide fuzzy-based optimization strategies with clarity and consistency in their formulation, analysis, and implementation.

Multi-objective optimization (MOO) is an essential tool in the toolbox of decision science and operational study for optimization problems with several competing goals [1]. In MOO, rather than optimizing just one factor as in single-objective optimization, the goal is to discover a group of solutions for multiple objectives. The resulting solution maximizes Pareto efficiency, which means that improving one objective would inevitably lead to the degradation of another objective [2]. Because of this, MOO is especially useful in everyday life, where balancing competing goals is unavoidable. As MOO is a vast area for research, we discuss here only the multi-objective linear programming problem (MOLPP). MOLPP deals only with linear constraints and objective functions.

Fuzzy multi-objective linear programming problem (FMOLPP) builds on conventional MOLPP by utilizing fuzzy set theory. Fuzzy set was first proposed by L. Zadeh in 1965 [3] to deal with the uncertainties that are intrinsic to numerous real-world issues. Fuzzy logic provides a sophisticated and adaptable method to deal with imprecision and uncertainty in decision-making. True or false is not always the best way to evaluate a statement; this is the essential premise of fuzzy logic. Fuzzy sets and linguistic variables are essential parts of fuzzy logic. Sets can be defined with more subtle ambiguity when elements have degrees of membership defined by fuzzy membership functions. One component of fuzzy logic is linguistic variables, which connect the two extremes of mathematical expression precision and human language's inherent imprecision [4]. It provides a way to make quantitative decisions that also take qualitative factors into account. One example of using linguistic variables is to convey concepts like "high," "medium," or "low" in a form that people can understand, making it easier to explain and analyse complicated systems. In many different fields, fuzzy logic, fuzzy sets, and language variables have proven to be practical. By applying fuzzy logic to control systems, we can create intelligent and adaptive controllers capable of handling uncertain input data and changing situations. Due to its ability to represent human reasoning, fuzzy logic is well-suited for use in decision support systems in industries including healthcare, banking, and transportation [5]. According to Zimmermann [6], the concept of fuzzy sets offers a solid foundation for dealing with data inaccuracies and unpredictability. In FMOLPP, we express parameters or variables for objectives and constraints as fuzzy collections, while the functions of membership describe the extent to which a component contributes to a fuzzy set. Fuzzy sets transform into fuzzy numbers once they meet certain characteristics. Depending on the number of components, there are several varieties of fuzzy numbers that alter the behaviour of the membership function. Here are some examples of these fuzzy numbers:

a) *Triangular Fuzzy Numbers*: Defined by three parameters to convey ambiguity straightforwardly and effectively.
b) *Trapezoidal Fuzzy Numbers*: Defined by four parameters for a more versatile form.
c) *Icosikaitetragonal Fuzzy Numbers*: Defined a very complex picture of uncertainty and involves 24 parameters.

Before we can use fuzzy numbers in optimization as parameters, we have to defuzzify the problem. Commonly used methods include the min-max, centroid, the mean of α-cut, and bounded area, etc. These methods allow for the processing of fuzzy objectives and constraints that can be used by standard optimization methods after converting them into crisp values that are equivalent. After the conversion of FMOLPP into MOLPP, the issue is to solve it.

Over the last few decades, classical MOLPP methods have undergone a remarkable evolution. One of the first approaches is the weighted sum method [7], which uses weighted parameters to merge many goals into one. Another is the ε-constraint method, which improves a single goal while transforming the others into limitations with bounds [8], [9]. Similarly, other non-fuzzy approaches are also defined with some advantages and limitations. Although these strategies offer a basic framework for MOLPP, they frequently fail to address the intricacy and unpredictability that come with real-life issues.

To deal with uncertainty, Zimmermann uses fuzzy set theory in his fuzzy approach to solve optimization issues involving many objectives [10]. This technique uses association functions to represent fuzzy goals for each target. It uses the max-min operator, which maximizes the minimal level of fulfilment across every target, to transform many goals into a clear single-objective issue by identifying ideal and nadir points of objectives. This transition enables the application of fuzzy optimization techniques to achieve a fair and robust solution. Zimmermann's method offers a versatile framework for maximizing competing goals while accurately modelling uncertain data.

Intuitionistic fuzzy set (IFS) is introduced by Atanassov in 1986, which adds some uncertainty to fuzzy set theory and makes it even more refined. By combining a membership function, a non-membership function, and a hesitation margin, an intuitionistic fuzzy set can more accurately portray uncertainty [11]-[13]. This extra layer of knowledge allows for more delicate decision-making, particularly when the level of certainty is unclear. The intuitionistic fuzzy approach for MOLPP considers the function of membership and non-membership, as well as the hesitation margin, while optimizing for goals with restrictions [14]. This method provides a more thorough evaluation of uncertainty than simple fuzzy sets, which in turn produces more reliable optimization results. Several applications, like engineering design and resource allocation, have demonstrated promising results when using intuitionistic fuzzy sets in MOLPP to help decision-makers balance competing goals in the face of ambiguity.

To provide for several participation values for a single element, Torra expanded the idea of fuzzy sets in 2010 with the introduction of the hesitant fuzzy set (HFS) [15]. This represents the fact that decision-makers may feel hesitant or unsure while deciding on membership levels. When there are multiple perspectives or insufficiently accurate data to identify a single participation number, hesitant fuzzy sets come in handy. Hesitant fuzzy sets represent the goals in a hesitant fuzzy approach for MOLPP, which deals with multiple membership degrees for a single element. Although numerous membership values increase complexity, they offer a more robust framework than simple fuzzy sets for dealing with uncertainty, which the optimization process must consider

[16]. To optimize the process, researchers use techniques like hesitant fuzzy weighted averaging and hesitant fuzzy aggregation operators to merge the many attributes of membership into a single corresponding value.

Dual hesitant fuzzy set (DHFS), an extension of hesitant fuzzy collections, offers a dual perspective on uncertainty by integrating both hesitant participation and hesitant non-participation values [17]. The sets capture situations where decision-makers are uncertain about the level of membership and non-membership. DHFS provides a comprehensive method for simulating complex MOLPP uncertainty. By using a triangular dual-hesitant fuzzy number to describe goals, dual-hesitant fuzzy MOLPP can be applied. The approach achieves a more complex and well-rounded optimization result by considering both hesitant membership and hesitant non-membership values. Researchers have used methods such as distance measurements and dual hesitant fuzzy aggregation operators [18] to assess and integrate the two hesitant parameters for efficient decision-making.

1.1 Fuzzy Set

According to classical set theory, there are only two possible partnerships of components of a set: either the item in question is a member of the set, or it is not. To indicate membership, a fuzzy set uses an assessment level ranging from 0 to 1. An element's role in a fuzzy set depends on how well it matches its attributes. Let the membership function be denoted as $\acute{m}_{\tilde{F}}$ for each element in a given fuzzy set $\tilde{F}$, presuming that the provided space is X. Consequently, the mathematical representation of a fuzzy set looks like this:

$$\tilde{F} = \{(x, \acute{m}_{\tilde{F}}(x)) : \forall\, x \in X\} \quad (1.1)$$

1.1.1 Membership Function

Assigning a relationship value to an element within the set of all things allows us to determine the likelihood of its inclusion in the fuzzy group. We refer to the function describing this relationship as the membership function. We provide a mathematical illustration of this function over $\tilde{F}$, which can be defined as follows:

$$\acute{m}_{\tilde{F}} : X \to [0,1] \quad (1.2)$$

In the present study, we have used various linear and non-linear membership functions for fuzzy approaches that have used left triangular fuzzy numbers. All membership functions are described mathematically and geometrically in Figure 1.1 - 1.5 to describe almost every scenario of real-life situations associated with these functions for the goal of maximization.

Linear: The simplest form of membership function is linear, according to which the marginal rate of the membership function, concerning the value of x, is constant, which means the value of the membership function increases/decreases from 0 to 1 linearly for the value of x from minimum to maximum in the case of a maximization problem. Linear membership functions are easy to compute and are often used when the relationship between variables is straightforward. These can be concluded from Figure 1.1, and the mathematical expression of it is given below:

$$\acute{m}_{\tilde{F}}(x) = \frac{x - min\ (x)}{max\ (x) - min\ (x)} \tag{1.3}$$

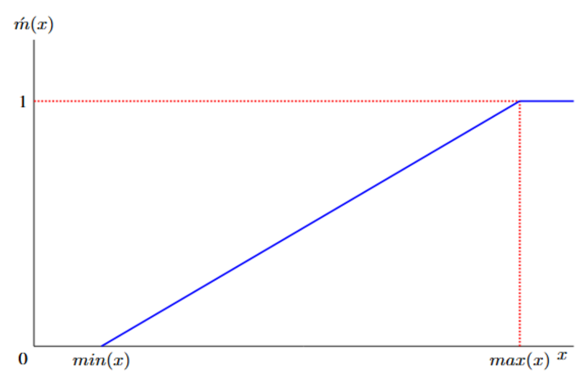


Figure 1.1: Linear membership function

Parabolic: A parabolic membership function is a quadratic function that describes a parabolic curve. It is typically symmetric and represents a gradual increment of slope [19]-[21]. In this case, the rate of the membership function is relatively lower when the value of x is in a weaker position, as given below mathematically and geometrically by eq (1.4) and Figure 1.2, respectively:

$$\acute{m}_{\tilde{F}}(x) = \left(\frac{x - min(x)}{max(x) - min(x)}\right)^2 \tag{1.4}$$

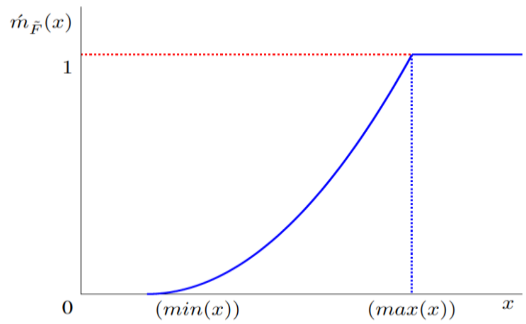


Figure 1.2: Parabolic membership function

Hyperbolic: The hyperbolic similarity scale is characterised by a convex factor for certain values of x and a concave factor for the remaining values [19]-[21]. When the decision maker (DM) improves their place regarding an aim, the marginal rate of satisfaction tends to increase up to a specific value. A convex shape can be used to illustrate this quality in terms of membership degree. After that specific value, when the DM improves their place regarding an aim, the marginal rate of satisfaction tends to drop. This type of action is represented by the concave part of the membership function as shown in Figure 1.3. Here is the entire idiom:

$$\acute{m}_{\tilde{F}}(x) = \frac{1}{2}\, tanh\left[\left(x - \frac{max(x) + min(x)}{2}\right)\alpha\right] + \frac{1}{2} \tag{1.5}$$

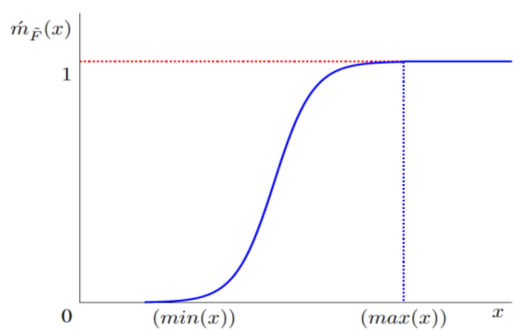


Figure 1.3: Hyperbolic membership function

Exponential: The exponential curve is typically symmetric and represents a gradual increase in slope [22]. The marginal rate of the membership function is relatively higher when x is making progress toward a goal. It can be concluded mathematically and graphically by eq (1.6) and Figure 1.4, respectively:

$$\acute{m}_{\tilde{F}}(x) = \eta_i \left[1 - exp\left\{-\rho_i \frac{x - min(x)}{max(x) - min(x)}\right\}\right] \tag{1.6}$$

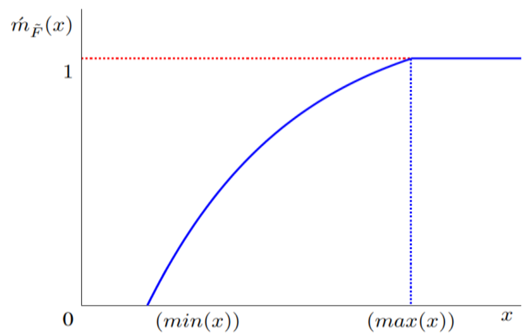


Figure 1.4: Exponential membership function

Sigmoidal: Sigmoidal membership functions are S-shaped curves that represent a smooth, gradual transition from one state to another [22]. The function starts slowly, accelerates in the middle, and then slows down again. This type of membership function is defined using the sigmoid function, which is mathematically represented as a logistic function. The S-shape is ideal for modelling gradual and smooth transitions between states. They look like hyperbolic functions, but their slopes differ from each other. The mathematical and geometrical formations of it are given below via eq (1.7) and Figure 1.5, respectively:

$$\acute{m}_{\tilde{F}}(x) = 1 - \left[\frac{1}{1 + Be^{a\left(\frac{x - \min(x)}{(\max(x) - \min(x))}\right)}}\right] \tag{1.7}$$

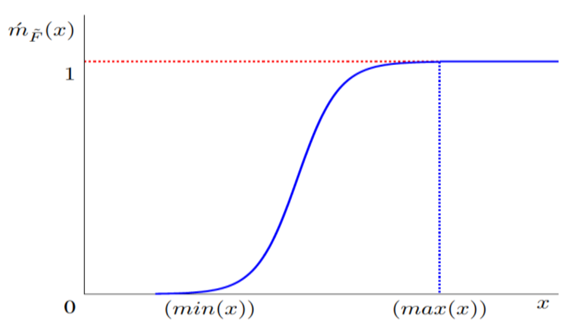


Figure 1.5: Sigmoidal membership function

Each of these membership functions offers unique advantages depending on the nature of the fuzzy system and the relationship between the input variables and their corresponding membership values. By selecting the appropriate type of membership function, the fuzzy system can be better tailored to reflect the real-world phenomena it is intended to model.

1.1.2 α-cut

The α-cut (or the strong α -cut) of a fuzzy set $\tilde{F}$ is the crisp set $\tilde{F}^{\alpha}$ (or the crisp set $\tilde{F}^{\alpha+}$) that contains all the elements of the universal set X whose membership grades in $\tilde{F}$ are greater than or equal to (or strictly greater than) the specified value of α.

$$\tilde{F}^{\alpha} = \{x | \acute{m}_{\tilde{F}}(x) \geq \alpha\} \tag{1.8}$$

$$\tilde{F}^{\alpha+} = \{x | \acute{m}_{\tilde{F}}(x) > \alpha\} \tag{1.9}$$

1.1.3 Support

The support of $\tilde{F}$ is the same as the strong α-cut of $\tilde{F}$ for $\alpha = 0$, which means it contains all the elements of the universal set X whose membership grades in $\tilde{F}$ are greater than 0.

$$S(\tilde{F}) = \{x | \acute{m}_{\tilde{F}}(x) > 0\} \tag{1.10}$$

1.1.4 Convex fuzzy set

A fuzzy set $\tilde{F}$ on X is convex iff its membership function follows the following inequality:

$$\acute{m}_{\tilde{F}}(\lambda x_1 + (1 - \lambda) x_2) \geq min\, \{\acute{m}_{\tilde{F}}(x_1), \acute{m}_{\tilde{F}}(x_2)\} \tag{1.11}$$

for all $x_1, x_2 \in X$ and all $\lambda \in [0, 1]$, where min denotes the minimum operator.

1.1.5 Height of fuzzy set

The height, $h(\tilde{F})$, of a fuzzy set $\tilde{F}$ is the largest membership grade that is obtained by any element in that set. Formally, it is represented as:

$$h(\tilde{F}) = \sup_{x \in X} \acute{m}_{\tilde{F}}(x) \tag{1.12}$$

1.1.6 Normal fuzzy set

A fuzzy set $\tilde{F}$ is called normal when the height of the fuzzy set is equal to 1, which can be mathematically represented by the following:

$$h(\tilde{F}) = 1 \tag{1.13}$$

Which means $\acute{m}_{\tilde{F}}(x) = 1$ for at least one $x \epsilon X$, or mathematically, which can be represented as:

$$\exists\, x \,\epsilon X : \acute{m}_{\tilde{F}}(x) = 1 \tag{1.14}$$

1.1.7 Fuzzy Number

The term "fuzzy number" refers to a fuzzy set that meets the following three criteria:

- $\acute{m}_{\tilde{F}}(\mathrm{x}) = 1$ for at least one $x \epsilon X$.
- The crisp set, which contains all the elements of the universal set X whose membership grades in $\tilde{F}$ are greater than or equal to any value in [0,1], should be a closed interval.
- The crisp set defined above containing the strict inequality should be bounded.

1.1.8 Triangular Fuzzy Number (TFN)

A fuzzy number $\tilde{F}_T$ is a triangular fuzzy number when it can be represented in the form of eq (1.15), according to [23]:

$$\tilde{F}_T = \{([t_1, t_2, t_3], \acute{m}_{\tilde{F}_{TI}}(x));\ \forall\, x \in X\} \tag{1.15}$$

and the membership function of it is represented by:

$$\acute{m}_{\tilde{F}_T}(x) = \begin{cases} 0 & ;\ if\ x \leq t_1 \\ \dfrac{x - t_1}{t_2 - t_1} & ;\ if\ t_1 \leq x \leq t_2 \\ \dfrac{t_3 - x}{t_3 - t_2} & ;\ if\ t_2 \leq x \leq t_3 \\ 0 & ;\ if\ x \geq t_3 \end{cases} \tag{1.16}$$

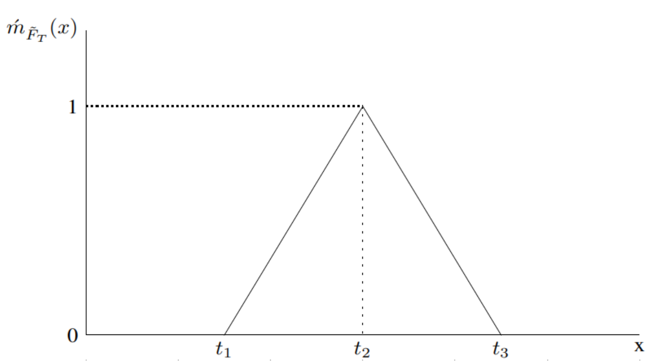


Figure 1.6: Symmetric triangular fuzzy number

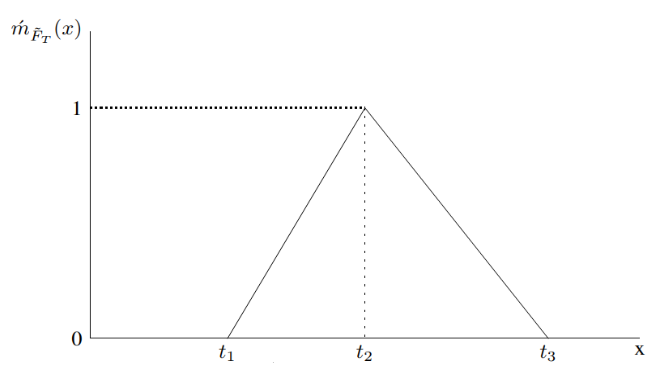


Figure 1.7: Asymmetric triangular fuzzy number

For the traditional fuzzy set, the non-membership degree is always 1 minus the membership degree. Two types of triangular fuzzy numbers exist, which are shown above in Figure 1.6 and Figure 1.7.

- Symmetrical: When $t_2 - t_1 = t_3 - t_2$
- Asymmetrical: When $t_2 - t_1 \neq t_3 - t_2$.

1.2 Extension of Fuzzy Set

This subsection delves into the distinctive mathematical frameworks and features of advanced topics in fuzzy set analysis, including intuitionistic fuzzy sets, hesitant fuzzy sets, and dual hesitant fuzzy sets. We conclude our discussion of their applications by examining their combination with triangular fuzzy numbers. By merging these advanced fuzzy sets, we demonstrate their expanded power to reflect uncertainty and hesitation in decision-making processes in the next chapters. We provide exact mathematical expressions and geometric interpretations to demonstrate the application of these extensions to real optimization problems.

1.2.1 Intuitionistic fuzzy set

Intuitionistic fuzzy set theory expands upon both the traditional set and the collection of fuzzy items. According to this theory, a pair of parameters with real values in the unitary range [0,1] can be used to assess components: one for participation and one for absence of participation. You can find the mathematical formula for this below:

$$\widetilde{F_I} = \{(x, \acute{m}_{\tilde{F}}(x), \acute{n}_{\tilde{F}}(x));\ \forall\, x \in X\} \qquad (1.17)$$

where; $0 \leq \acute{m}_{\widetilde{F_I}}(x) + \acute{n}_{\widetilde{F_I}}(x) \leq 1$.

Fuzzy accumulations can be generated from intuitionistic fuzzy sets when the condition $\acute{m}_{\widetilde{F_I}}(x) + \acute{n}_{\widetilde{F_I}}(x) = 1$ is satisfied, since the connected values of these sets are variants on involvement and failure to participate functions.

1.2.1.1 Intuitionistic triangular fuzzy number (ITFN)

When triangular fuzzy numbers represent the properties of the intuitionistic fuzzy set, then it can be defined as $\tilde{F}_{TI} = \left\{\left([t_1, t_2, t_3], \acute{m}_{\tilde{F}_{TI}}(x)\right), \left([t_1', t_2, t_3'], \acute{n}_{\tilde{F}_{TI}}(x)\right);\ \forall\, x \in X\right\}$, whose functions, membership, and non-membership are $m_{\tilde{F}_{TI}}(x)$ and $n_{\tilde{F}_{TI}}(x)$, respectively:

$$\acute{m}_{\tilde{F}_{TI}}(x) = \begin{cases} 0 \ ; \ if\ x \leq t_1 \\ \dfrac{x - t_1}{t_2 - t_1} \ ; \ if\ t_1 \leq x \leq t_2 \\ \dfrac{t_3 - x}{t_3 - t_2} \ ; \ if\ t_2 \leq x \leq t_3 \\ 0 \ ; \ if\ x \geq t_3 \end{cases} \tag{1.18}$$

$$\acute{n}_{\tilde{F}_{TI}}(x) = \begin{cases} 1 \ ; \ if\ x \leq t_1' \\ \dfrac{t_2 - x}{t_2 - t_1'} \ ; \ if\ t_1' \leq x \leq t_2 \\ \dfrac{x - t_2}{t_3' - t_2} \ ; \ if\ t_2 \leq x \leq t_3' \\ 1 \ ; \ if\ x \geq t_3' \end{cases} \tag{1.19}$$

Here, $\acute{m}_{\tilde{F}_{TI}}(x) + \acute{n}_{\tilde{F}_{TI}}(x) \neq 1$.

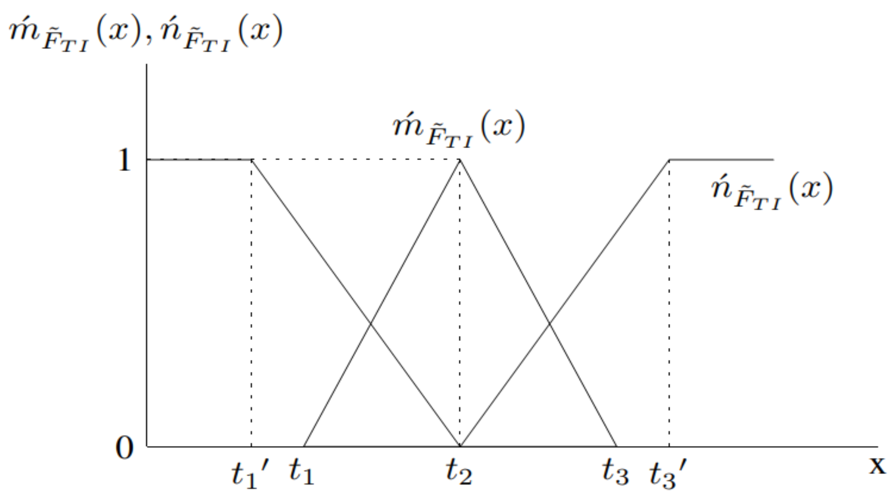


Figure 1.8: Intuitionistic fuzzy number

For a geometrical representation of these types of fuzzy numbers, please refer to Figure 1.8.

1.2.2 Hesitant fuzzy set

A hesitant fuzzy set is defined by the function from the universal set X to [0,1] and is denoted by:

$$\tilde{F}_H = \{(x, h_{\tilde{F}_H}(x)) : \forall\, x \in X\} \tag{1.20}$$

Here, $h_{\tilde{F}_H}$(x) is the hesitant fuzzy element, which is a set of discrete values in [0,1] as the membership degree.

1.2.2.1 Hesitant Triangular Fuzzy Number (HTFN)

When triangular fuzzy numbers represent the properties of the dual hesitant fuzzy sets then it can be defined as $\tilde{F}_{HT} = \{[t_1, t_2, t_3];\ h_{\tilde{F}_{HT}}(x)\}$, whose membership degree for element x according to e^{th} expert is $\acute{m}^{e}_{\tilde{F}_{DHT}}(x)$. Figure 1.9 displays the geometric representation, while eq. (1.16) provides the mathematical representation:

$$\acute{m}^{e}_{\tilde{F}_{HT}}(x) = \begin{cases} 0 \ ;\ if\ x \le t_1 \\ \omega_e\left(\dfrac{x - t_1}{t_2 - t_1}\right) \ ;\ if\ t_1 \le x \le t_2 \\ \omega_e\left(\dfrac{t_3 - x}{t_3 - t_2}\right) \ ;\ if\ t_2 \le x \le t_3 \\ 0 \ ;\ if\ x \ge t_3 \end{cases} \tag{1.21}$$

Where $h_{\tilde{F}_{HT}}(x)$ is a set of discrete values in [0,1] and $0 \le \omega_e \le 1$.

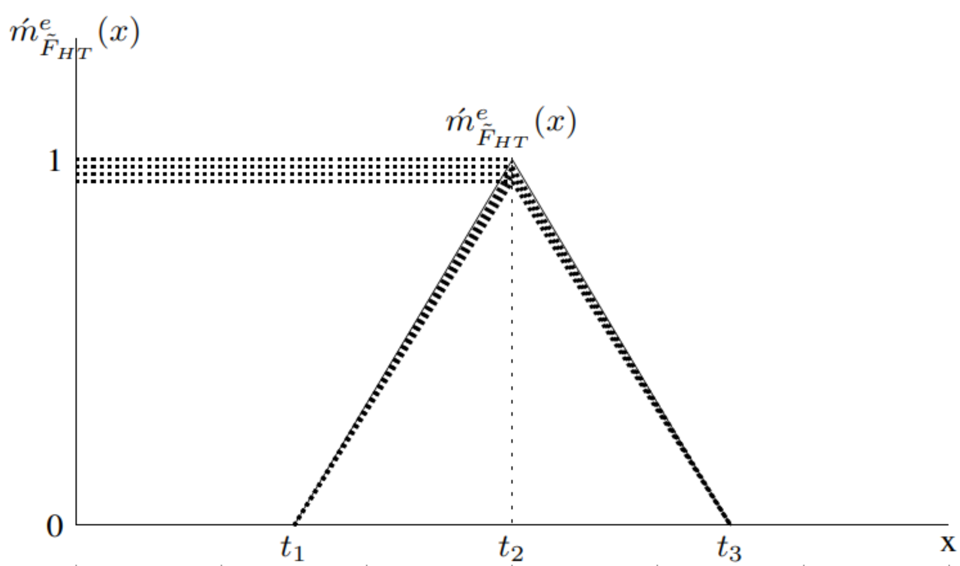


Figure 1.9: Hesitant triangular fuzzy number

1.2.3 Dual hesitant fuzzy set

The fuzzy set is formed by the combination of the properties of the hesitant and intuitionistic fuzzy set, it is called the dual hesitant fuzzy set. A dual hesitant fuzzy set has a set of non-membership degrees for a single element in addition to a membership degree. It is denoted by:

$$\tilde{F}_{DH} \;=\{(x, h_{\tilde{F}_{DH}}(x), g_{\tilde{F}_{DH}}(x)) \colon \forall\, x \in X\} \tag{1.22}$$

1.2.3.1 Dual Hesitant Triangular Fuzzy Number (DHTFN)

When triangular fuzzy numbers represent the properties of the dual hesitant fuzzy sets, then it can be defined as $\tilde{F}_{DHT} \;= \{([t_1, t_2, t_3\,];\; h_{\tilde{F}_{DHT}}(x)), ([t_1', t_2, t_3']; g_{\tilde{F}_{DHT}}(x))\}$ whose membership and non-membership degrees for element x according to the e^{th} expert are $\acute{m}^{e}_{\tilde{F}_{DHT}}(x)$ and $\acute{n}^{e}_{\tilde{F}_{DHT}}(x)$, respectively. Figure 1.10 illustrates the dual hesitant triangular fuzzy number in a geometric manner.

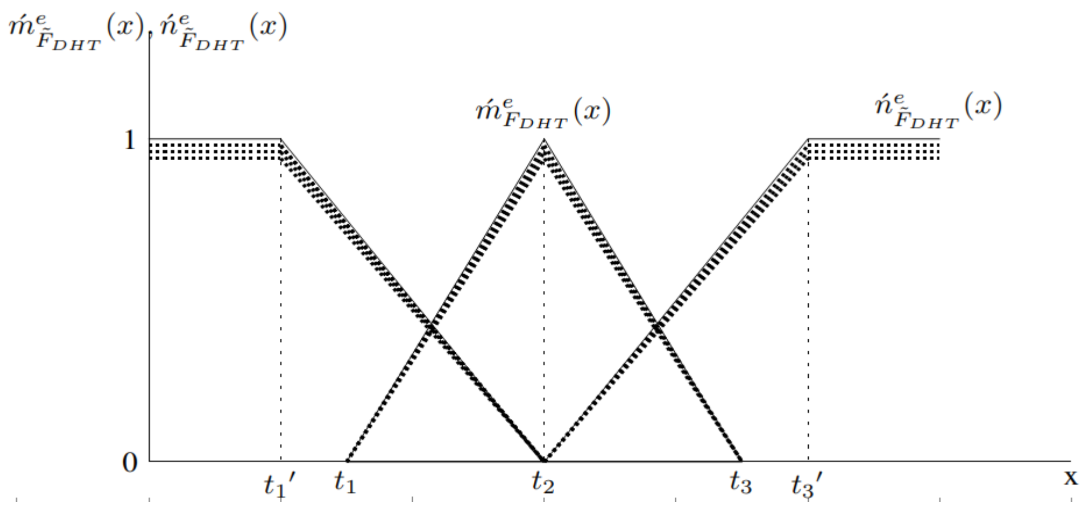


Figure 1.10: Dual hesitant triangular fuzzy number

$$\acute{m}^{e}_{\tilde{F}_{DHT}}(x) = \begin{cases} 0 \;\; ; \; if\; x \le t_1 \\ \omega_e\left(\dfrac{x-t_1}{t_2-t_1}\right) \;\; ; \; if\; t_1 \le x \le t_2 \\ \omega_e\left(\dfrac{t_3-x}{t_3-t_2}\right) \;\; ; \; if\; t_2 \le x \le t_3 \\ 0 \; ; \; if \;\; x \ge \; t_3 \end{cases} \tag{1.23}$$

$$ń^{e}_{\tilde{F}_{DHT}}(x) = \begin{cases} 1 \; ; \; if \; x \leq t_1^{'} \\ ɯ_e\left(\dfrac{t_2 - x}{t_2 - t_1^{'}}\right) \; ; \; if \; t_1^{'} \leq x \leq t_2 \\ ɯ_e\left(\dfrac{x - t_2}{t_3^{'} - t_2}\right) \; ; \; if \; t_2 \leq x \leq t_3^{'} \\ 1 \; ; \; if \; x \geq t_3^{'} \end{cases} \qquad (1.24)$$

1.3 Fuzzy Operations

In theory, novel commands such as union, intersection, and complement can improve knowledge about the fuzzy sets. The level of fuzziness for every component in each set determines the course of action. The construction of uncertain union, intersection, and complement [24] between two fuzzy sets, $\tilde{F}_1$ and $\tilde{F}_2$, are represented below:

1.3.1 Fuzzy union ($\widetilde{\boldsymbol{F}}_{\mathbf{1}} \cup \widetilde{\boldsymbol{F}}_{\mathbf{2}}$)

In mathematical terms, fuzzy union is defined by using the greatest possible value of the membership function.

$$ḿ_{(\tilde{F}_1 \cup \tilde{F}_2)}(x) = max\,\{ḿ_{\tilde{F}_1}(x), ḿ_{\tilde{F}_2}(x)\} \qquad (1.25)$$

1.3.2 Fuzzy intersection ($\widetilde{\boldsymbol{F}}_{\mathbf{1}} \cap \widetilde{\boldsymbol{F}}_{\mathbf{2}}$)

In mathematical terms, fuzzy intersection is defined by using the lowest possible value of the membership function.

$$ḿ_{(\tilde{F}_1 \cap \tilde{F}_2)}(x) = min\,\{ḿ_{\widetilde{F_1}}(x), ḿ_{\widetilde{F_2}}(x)\} \qquad (1.26)$$

1.3.3 Fuzzy complement ($\widetilde{\boldsymbol{F}}^{\boldsymbol{c}}$)

For a fuzzy set, the membership degree of any element of its complement is one minus the degree of that element in that set.

$$ḿ_{(\widetilde{\boldsymbol{F}}^{\boldsymbol{c}})}(x) = 1 - ḿ_{(\widetilde{\boldsymbol{F}})}(x) \qquad (1.27)$$

1.4 Linear Programming Problem (LPP) in Crisp Environment

In this section, we present the exact model of the linear programming problem (LPP) for both one and more objective functions and their representation for standard transportation issues.

1.4.1 Single-objective linear programming problem

On the assumption that each of the goal functions belongs to the maximization category, the following is the usual representation of a single-objective linear optimization problem:

$$max\, f = \sum_{k=1}^{n} p_k x_k \tag{1.28}$$

For the goal, which is represented by f, for every choice variable x_k with parameters p_k and c_k^j as parameters for the objective and each j^{th} constraint, respectively, there is a limit d^j. For all $k = 1,2,..p$ and $j = 1,2,...q$. Subjected to constraints such as:

$$c_k^j x_k \leq d^j \tag{1.29}$$

$$x_k \geq 0 \tag{1.30}$$

If minimization functions are given, then they can be converted into maximization types by multiplying them by -1. The purpose of the LPP is to maximize (or minimize) the objective function by determining the values of the choice variables $x_1, x_2, ..x_p$ that are consistent with all the restrictions. The objective function and restrictions in LPP are both linear. Production planning, resource allocation, transportation logistics, financial management, and an extensive number of other optimization purposes all make use of linear programming.

1.4.2 Integer linear programming problem

In most real-life applications, we need several units of products as decision variables to find out the best solution to optimize the target value. We cannot interpret the number of product units as a decimal or negative number; instead, they must be of the non-negative integer variety. We refer to this issue as an integer-linear programming problem. We provide mathematical representation of these problems such as:

$$max\ f = \sum_{k=1}^{n} p_k x_k \tag{1.31}$$

For all $k = 1,2,..p$ and $j = 1,2, ... q$. Subjected to constraints as:

$$c_k^j x_k \leq d^j \tag{1.32}$$

$$x_k \geq 0 \tag{1.33}$$

$$x_k \in Z^+ \cup 0 \tag{1.34}$$

1.4.3 Multi-objective linear programming problem

When the number of goals in a linear programming problem becomes more than one, it will be a multi-objective linear programming problem. The following is the usual representation of a multi-objective linear optimization problem:

$$max\ f_i = \sum_{k=1}^{n} p_k^i x_k \tag{1.35}$$

For each goal, it is represented by f_i; for every choice variable x_k with parameters p_k^i and c_k^j for each i^{th} objective function and j^{th} restriction, respectively, there is a limit d^j. Let, $F = (f_1, f_2, \dots ..f_l)$ be a vector-valued function. For all $i = 1,2,3,..l$; $k = 1,2,..r$, and $j = 1,2, ... q$ constraints are defined as given in eq (1.29-1.30).

1.4.4 Transportation problem

Let us take p number of origins and q number of targets. x_{rs} are the number of units transported from the r^{th} origin to the s^{th} target. Let f_i be the i^{th} goal from l number of goals that we have to optimize.

$$max/min\ f_i = \sum_{r=1}^{p} \sum_{s=1}^{q} C_{rs}^i\, x_{rs} \tag{1.36}$$

For all $i = 1,2,3,..l$, subjected to:

Supply

$$\sum_{r=1}^{p} x_{rs} = S_s;\ \forall\ s = 1,2,..q \tag{1.37}$$

Demand $$\sum_{s=1}^{q} x_{rs} = D_r;\ \forall\ r = 1,2,..p \quad (1.38)$$

Non-negativity $$x_{rs} \geq 0 \quad (1.39)$$

Balancing condition $$\sum_{r=1}^{p} S_r = \sum_{s=1}^{q} D_s \quad (1.40)$$

If the balanced conditions on the linear transportation problem have an acceptable solution, we will refer to the state of balance criterion as an if requirement. There are precisely pq variables and $p + q$ constraints in a transportation problem.

1.5 Organisation of Thesis

Seven chapters make up the entire thesis, and here is a summary of them:

Chapter 1 includes a brief introduction, and basic definitions of keywords used in the thesis, which contain fuzzy sets, extensions of fuzzy sets, and related operations. The thesis delves into the fundamental framework of linear optimization issues and expands these problems to encompass integer, fuzzy, and multi-objective optimization problems. Here, the basic structure of the standard transportation problem is defined.

Chapter 2 covers research on fuzzy multi-objective optimization, methods for simplifying fuzzy numbers, fuzzy techniques for tackling multi-objective optimization, and advanced fuzzy methods such as intuitionistic and dual hesitant fuzzy approaches. It also outlines the research gaps that previous studies have identified, along with the research goals aimed at addressing them. To achieve the goals, it summarizes the study's chapters.

Chapter 3 includes a classification of various fuzzy numbers based on the number of components on which the functional behaviour changes from the previous one. The chapter defines three types of fuzzy numbers. It provides a complete explanation of the icosikaitetragonal fuzzy number, a complex fuzzy number with 24 components. We also delve into the mathematical and geometric representations of all fuzzy numbers, regardless of their symmetry, linearity, or non-linearity. We discuss a variety of defuzzification techniques for all fuzzy numbers and define a generalized approach. A real-life case study of a manufacturing problem provides a comparative analysis of all approaches, resulting in a conclusion.

Chapter 4 delves into the various real-life areas that fuzzy theory can enhance. As a result, light-period automation, which is under multi-criteria decision-making issues, can make use of fuzzy logic. When it comes to sustainability-focused multi-objective optimization problems in manufacturing, we can apply a preexisting fuzzy technique. It includes intuitionistic fuzzy methods, which improve upon fuzzy methods by incorporating non-membership functions. The results of the comparison study cover every conceivable combination of different membership and non-membership activities. Additionally, it covers the fuzzy methods that have been created using dual hesitant fuzzy sets.

Chapter 5 includes fuzzy methods for optimizing several objectives, one of which is using a normalized distance function as the linear membership function. We expand this approach by adding non-linear membership functions that have curved shapes, like parabolic, hyperbolic, exponential, and sigmoidal. By improving an existing method and adding a non-membership function, we produce a novel intuitionistic fuzzy method. We define both linear and non-linear (best combination) functions using the intuitionistic fuzzy technique. The dual hesitant method, which combines the intuitionistic method with the hesitant fuzzy technique, produces superior results. It covers methods involving the optimal combination of non-linear functions resulting from previous discussion.

Chapter 6 outlines various factors that are used to compare all the functions in the enhanced fuzzy approach with triangular fuzzy numbers. We achieve this by examining a real-life case study of transportation in a smart city. Real-life case studies in biomimetic and material science employ the extended approach. We then implement an enhanced intuitionistic approach with a linear and non-linear nature, using intuitionistic triangular fuzzy numbers, on a real-life manufacturing problem and compare it with existing techniques. The chapter provides a real-world case study of a manufacturing problem and compares the enhanced dual-hesitant fuzzy approach with previous ones. It also provides a comparative study to showcase the uniqueness of our work.

Chapter 7 concludes this thesis. It outlines the goal of the work, and the methods used to achieve it. It also provides directions for new researchers to further study in this area.

Chapter 2

Literature Review

For multi-objective linear programming problems (MOLPP) to represent real-life decision-making situations effectively, it is necessary to include uncertainty and vagueness. Beginning with real-world examples of fuzzy multi-objective optimization problems, this chapter shows how uncertainty is relevant in many different fields. The work continues by discussing defuzzification methods, which reduce complex problems to straightforward multi-objective optimization issues. These methods are essential for turning fuzzy, imprecise data into clear, actionable numbers. Simple, intuitionistic, and dual hesitant fuzzy techniques—which offer more flexible representations of uncertainty to manage diversity of objectives—are included in the fuzzy-based approaches reviewed and categorized in the chapter. We formulate particular objectives for the current study based on the identification of research gaps that have been uncovered by this extensive evaluation. An organized summary is provided at the end of the chapter to help the reader understand the concepts clearly and keep track of them. The research presented in the following chapters builds on the groundwork laid out in Figure 2.1, which explains the systematic flow and aids in comprehending the necessity of sophisticated fuzzy optimization approaches.

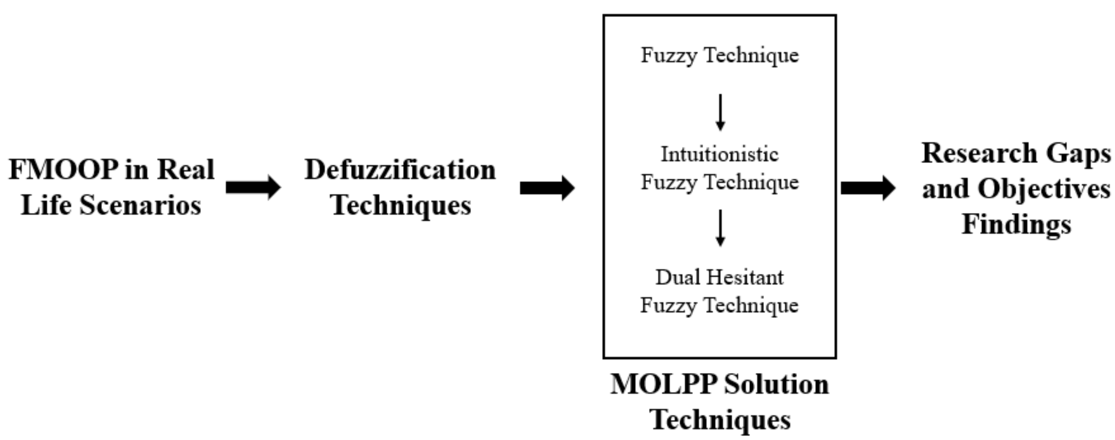


Figure 2.1: Conceptual flow and organization of the chapter

2.1 Review of Real Life Fuzzy Multi-Objective Optimization Problems

Maximizing or minimizing a linear target function within the limitations of a set of linear restrictions is known as a linear programming problem (LPP), a basic technique in the field of operations research. A major step forward in effectively addressing LPP was the simplex approach, which was developed by George Dantzig, according to [25]. It entails navigating through the permissible region's points until the optimal solution emerges. John von Neumann's works in duality theory and game theory established the conceptual foundations of LPP [26]. Many other areas have found uses for LPP throughout the years, including transportation, manufacturing, and inventory problems [27]-[29]. It is worth mentioning that in 1984, Karmarkar [30] devised an algorithm to improve the computational speed by solving LPP in time with polynomials. According to article [31], the researchers Kuhn and Tucker initially addressed the vector-maximization issue, which is the origin of multi-objective optimization involving more than one objective.

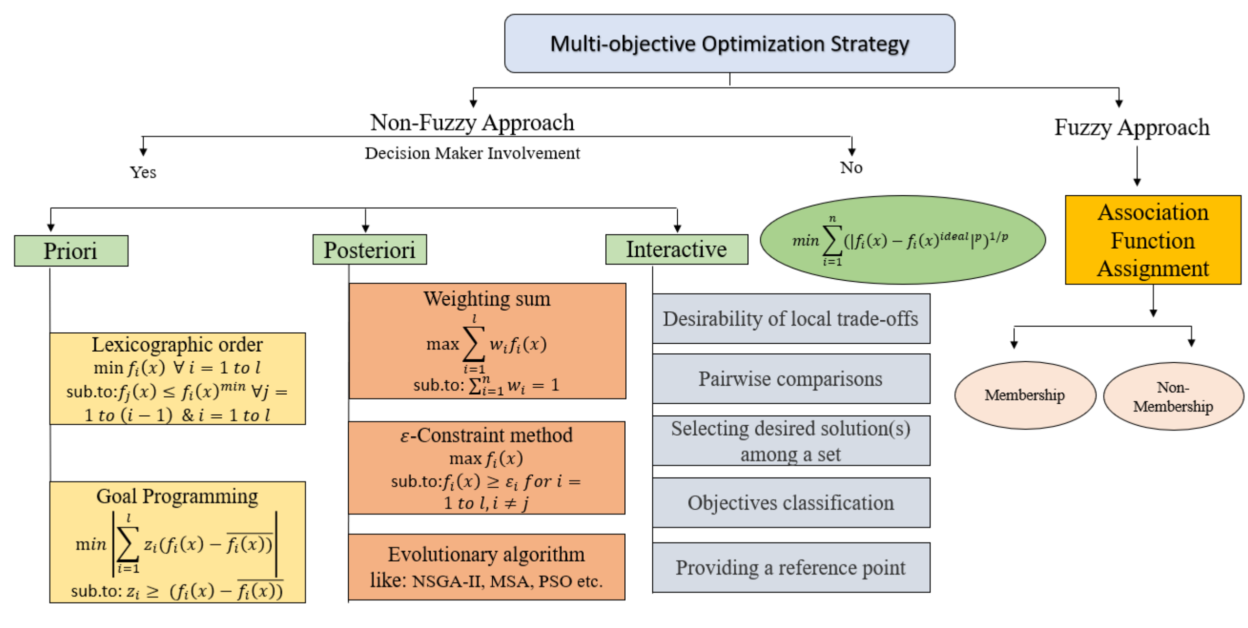


Figure 2.2: Classification of multi-objective optimization approaches

Multiple academics and professionals in operations research, mathematics, and engineering have contributed to the creation of MOO [32]. According to [33], [34], many issues in economics and engineering, such as the compromise between value and efficiency, involve multiple goals that are not binary. Recently, chemical engineers and manufacturers have increasingly used multi-objective optimization. Some of which include [35] chemical recovery, [36] heat treatment, [37] incomplete oxidation, etc. Solving an MOO issue is more complicated than solving a traditional single-objective optimization problem because there is more than one optimal solution. Finding an optimal solution to a problem with many objectives has, thus, been described differently by various scholars. Using various philosophies, we can discover the best outcomes. Different non-fuzzy approaches are classified into two categories, as shown in Figure 2.2, based on DM's involvement in solution selection [38], [39].

One consists of methods that do not include the decision-maker's preference [40]. The second method incorporates preference, further subdividing it into three types: priori, posteriori, and interactive. Priori considers the preferences of the DM before initiating the process, utilizing various approaches such as goal programming [41], [42], lexicographic order [43], to find solutions. Posteriori, which includes methods such as the scalarization method [44]-[47], ε-constraint, and evolutionary algorithms (EAs) [48]-[50], considers preference after the process has been completed. Lastly, the interactive method takes preference into account at each iteration level, with various methods incorporating different preference parameters [51], [52]. The weighted-sum approach uses a weighted vector to aggregate all issues into a single problem. Usually, we set the sum of weights to 1. Despite the weighted-sum method's user-friendliness and simplicity, it comes with two disadvantages. To begin, choosing weights for issues of varying sizes is a challenge [53], [54]. Finding a compromise solution will thus be biased. Additionally, if the optimized problem is not convex, an issue can arise. The ε-constraint method is employed to conquer challenges in multiple non-convex issues. Only one issue is optimized using the ε-constraint approach, while the remaining issues are turned into limits. For every problem, the Îµ vector is found and the limit is applied. This method's optimization of all issues leads to the optimal solution for specific Îµ vectors. By adjusting Îµ, we can get multiple optimal solutions. One drawback of this approach is that it does not work for specific Îµ vectors [8], [9]. Algorithms that employ evolution follow the lead of evolution and natural selection to find the best possible answers. One drawback of EAs is that the algorithm's parameters, such as population size, variation and crossover rates, selection process, and termination criteria, can have a significant impact on how well they work. Finding the optimal values for these parameters to achieve high performance in a variety of problem areas is not an easy or quick task. Goal programming streamlines the optimization problem by emphasizing the need to minimize each objective's deviation from current levels. Its limitation is that it might not always discover Pareto-optimal responses, depending on the target values chosen. Similarly, all approaches have some limitations with their benefits, taking into consideration [55], [56]. One of these approaches' most preferred limitations is that they do not consider uncertainty, which is the most essential parameter for real-life studies. Fuzzy approaches remove the limitations. Table 2.1 discusses some real-life case studies that MOO helps to resolve using a variety of fuzzy and non-fuzzy methods.

Table 2.1: Various fuzzy and non-fuzzy approaches for MOO problems in real-life case studies

Ref	Intuitionistic fuzzy approach	Non-fuzzy approach	Linear membership/non-membership function	Non-Linear membership/non-membership function	Case study area
[57]	×	NSGA-II	×	✓	Industrial
[58]	×	Goal Programming	✓	×	Agriculture
[59]	×	GRA	✓	×	Transportation
[60]	×	Weighted sum	×	✓	Energy storage
[61]	✓	×	✓	×	Three bar truss
[62]	✓	GA	✓	✓	Reliability
[63]	✓	×	✓	×	Irrigation
[64]	×	×	✓	✓	Transportation
[65]	✓	×	×	✓	Three bar truss
[66], [67]	✓	×	×	✓	Designing
[67]	✓	×	✓	✓	Production planning
[68]	×	×	✓	✓	Production planning
[69]	✓	×	✓	×	Portfolio selection
[70]	✓	×	✓	×	Transportation problem

It is common, though, for individuals making decisions to be unaware of the exact value of a parameter. Furthermore, numerous optimization problems [71]-[74] have shown the presence of parameters with uncertain values, known as fuzzy programming problems. Mathematical modelling is one of several fields that make heavy use of linear programming problems with uncertain choice variables or factors.

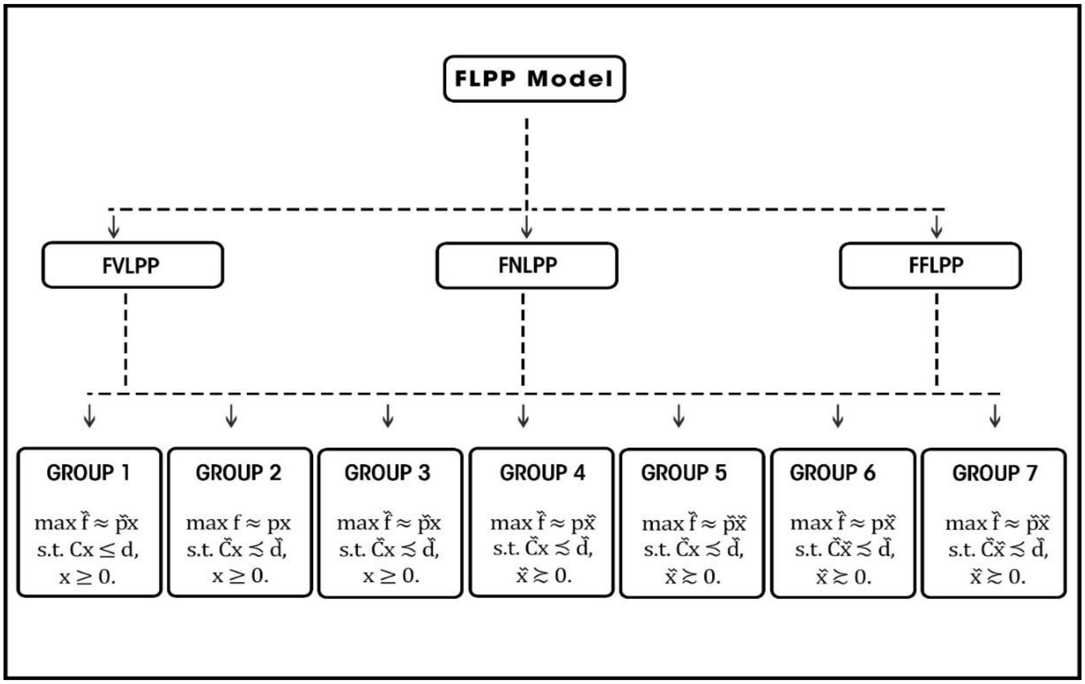


Figure 2.3: Various types and groupings of fuzzy linear programming problems

According to [75], one should interpret the variables and parameters in a fuzzy linear programming problem as fuzzy numbers. The literature has published seven main groups of fuzzy linear programming models, as shown in Figure 2.3. These are variables treated as fuzzy numbers (FVLPP), parameters treated as fuzzy numbers (FNLPP), and both fully fuzzy linear programming problems (FFLPP). They are all subclasses of the larger fuzzy linear programming problem (FLPP).

2.2 Review of Defuzzification Techniques

When certain membership procedure requirements are considered, fuzzy sets transform into fuzzy numbers. The factors' representations might take several forms, reflecting the variety of real-world scenarios. Research publications [76], [77] provide a variety of pictures of fuzzy numbers with triangular, trapezoidal, and pentagonal shapes. It becomes increasingly challenging to handle the situations when the uncertainty of data increases and data ranges expand [78]. The uncertainty concept inspires Icosikaitetragonal fuzzy numbers, which express difficulties with twenty-four elements due to incomplete knowledge and ambiguity [79]. Many studies exist to address the uncertainty associated with the objective function’s coefficients. Some of these studies convert a single objective into multiple objectives, utilizing strategies such as pushing the critical points in the right direction [80], establishing laxicographic order relations [81], [82] using rank correlation are defined for simple fuzzy numbers. Other techniques involve using defuzzification techniques to convert these fuzzy numbers into crisp ones. Similarly, there are various methods to manage the uncertainty associated with constraints. One approach involves separating the components from the inequalities, while another involves applying defuzzification techniques. "Defuzzification" refers to the process of converting fuzzy deduction results into more precise numerical values. It has numerous applications in operational research and is, therefore, an essential component of fuzzy logic. You can use the centroid approach [83], the mean of maxima, the bisector of area, the smallest of maxima, graded mean integral values [84], and a variety of other methods [85] to eliminate fuzzy environment. Studies [78], [86], [87] discuss various defuzzification techniques for both simple and complex fuzzy numbers. These algorithms cannot guarantee defuzzified value accuracy in all cases. We chose the centroid method, the mean of the α-cut, and the bounded area approach for this study because they are the most accurate for both linear and non-linear generalized fuzzy numbers with uniformity according to [88]. Some studies require comparisons of different fuzzy numbers, like game theory. To address these situations, researchers introduce various ranking functions for different fuzzy numbers. Table 2.2 discusses some of these ranking functions or defuzzification techniques.

Table 2.2: Studies with various defuzzification and ranking functions for different fuzzy numbers

Work	Fuzzy number	Ranking function/ Defuzzification technique
[89]	Triangular, Trapezoidal	Magnitude
[90]	Ordered fuzzy number	Centre of circles
[91]	Triangular	Signed distance
[92]	Triangular, Trapezoidal	Graded Mean Integration Value
[93]	Triangular, Trapezoidal, Pentagonal	Signed distance, Graded Mean
[94]	Triangular, Trapezoidal	max-membership, centroid, weighted-average and mean-max
[95]	Hexagonal	Centroid, α-cut
[88]	Hexadecagonal	Centroid, α-cut, average of removal of bounded area, bounded area

2.3 Review of Fuzzy Techniques for MOLPP

Fuzzy methods for multi-objective optimization are a big step forward in applying fuzzy set theory to complicated decision-making situations with a lot of competing goals. With fuzzy optimization, decision-makers can express and implement personal beliefs and uncertainties in the optimization procedure, providing a natural and simple solution to handle the choices and uncertainties associated with multi-objective choices [96]. For each target, there is a proposed linear membership function to specify fuzzy goals. The linear role of membership has many real-world uses due to its simplicity and ease of implementation. To provide a structure to deal with ambiguity in multifaceted decision-making, Zimmermann's method reduced the fuzzy MOLPP issue to a clear single-objective problem by employing the max-min operator. The incorporation of fuzzy set theory into multi-objective optimization frameworks has been the subject of a significant amount of research, with many papers offering novel algorithms and approaches that successfully address a wide range of practical problems. These methods attempt to deal with the complexity and unpredictability of decision-making contexts, including many objectives, by utilizing fuzzy association operators and fuzzy logic. Newer studies have recognized the importance of both linear and non-linear participation functions. Linear functions are used to explain how goals and solutions interact. These formulas are not suited to the problem landscape because they do not capture the complex and nonlinear nature of goals. Many studies have investigated fuzzy multi-objective optimization schemes that use non-linear relationship functions [97], [98] to improve the accuracy of the solutions and the robustness of the algorithms. For more studies, please refer to Table 2.3. Some relevant studies [99]-[101] offer computer programs to fix multi-objective optimization problems that come up because of these complex fuzzy methods.

Table 2.3: Different nature of membership functions for fuzzy approach

Study	Nature of membership function	Shape of membership function
[102]	Linear	Linear
[103]	Non-linear	Hyperbolic
[104]	Non-linear	Inverse hyperbolic
[105]	Non-linear	Exponential
[106]	Non-linear	Sigmoid
[107]	Linear	Piecewise

2.4 Review of Intuitionistic Fuzzy Techniques for MOLPP

The advantage of intuitionistic fuzzy sets is that the level of dissatisfaction is considered in addition to the degree of adoption, as it also takes decision-makers' inability into account. Angelov [108], extends the concept of fuzzy optimization to intuitionistic fuzzy optimization, focusing on maximizing adoption and minimizing rejection. Because intuitionistic fuzzy optimization has so many applications in a wide range of issues related to optimization, researchers have been interested in the theory and practice of this optimization technique.

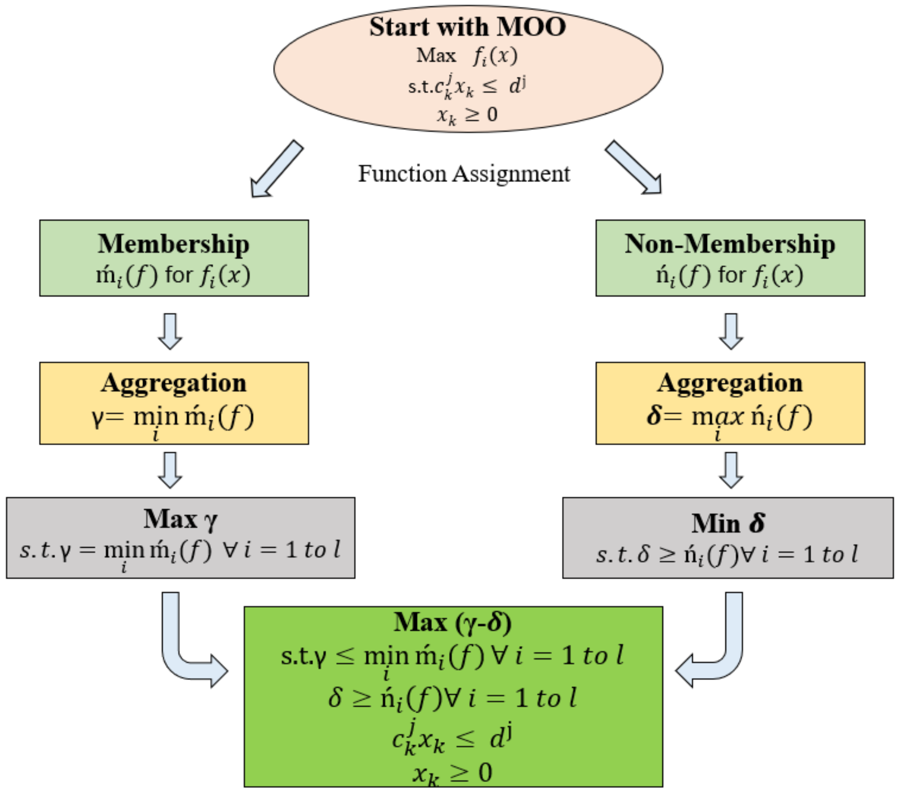


Figure 2.4: Generalized steps for the intuitionistic fuzzy approach

In situations where traditional fuzzy sets are insufficient to describe an uncertain set, one alternative approach is to utilize the concept of an IFS. IFS theory is just fuzzy set theory with a wider scope. Given the radically deficient and unpredictable nature of human judgment and comprehension, it is reasonable to assume that IFS could mimic these processes. In 2016, Sarkar and Roy [109] introduced an Intuitionistic Fuzzy Optimization (IFO) method, shown in Figure 2.4. It takes into account the functions that are not members and are not linear. In this study, researchers examined two target functions as constraints: bridge weight and displacement of the loaded joint, with stress in the truss components serving as a limiting factor. Li (2008) [110] created a way to solve problems with more than one attribute in an intuitive fuzzy setting by using linear programming techniques for multidimensional analysis of preference (LINMAP). Through numerical examples, the author demonstrates the use of LINMAP's ability to address multiple-attribute problems in both fuzzy and crisp environments, including Atanassov's intuitionistic fuzzy environment. Hernandez and Uddameri (2010) used a multi-criteria selection methodology based on Atanassov's Intuitionistic Fuzzy Sets (A-IFS) concept to determine agricultural optimal management methods [111]. They demonstrate A-IFS through a case analysis in the South Texas, USA, region. In the end, the solution ranks the options and determines that "brush control" and "irrigation scheduling" were the best and worst choices, respectively. Table 2.4 provides an estimate of the approach's application areas.

Table 2.4: Various intuitionistic fuzzy approaches in different real-life sectors

Study	Nature of membership function	Nature of non-membership function	Application area
[112]	Linear	Linear	Irrigation system
[113]	Linear & Non-linear	Linear & Non-linear	Production planning
[114]	Linear	Linear	Transportation planning
[115]	Linear	Linear	Structural modelling
[116]	Linear	Linear	Agriculture production
[117]	Linear	Linear	Portfolio selection
[118]	Linear	Linear	Vendor selection

2.5 Review of Dual Hesitant Fuzzy Techniques for MOLPP

Contrary to popular belief, specialists do not agree on the optimal values for optimizing problem's factors; as a result, a single degree of participation is insufficient to solve the optimization issue effectively; instead, an assortment of parameter grades is required. In this case, hesitant fuzzy sets, not intuitionistic or fuzzy sets, are crucial [119]. When it comes to decision-making, both intuitionistic and hesitant fuzzy sets work well with parallel relationship functions [120]. Using dual hesitant fuzzy sets, which consider the fact that the two category values don't behave in a straight line, adds more detail to the model for ambiguity and is a significant improvement [121]. This technique provides a more robust structure to express and resolve optimization issues with multiple goals, particularly in cases where the uncertainty is complex and multifaceted.

2.6 Research Gaps from Existing Studies

While several studies have provided defuzzification techniques for various fuzzy numbers and various fuzzy approaches using simple and extended fuzzy sets to solve fuzzy multi-objective optimization problems, there are still several research gaps that need to be filled to fully understand fuzzy set theory in a multi-objective optimization environment. Some of them are given below:

- Conventional defuzzification strategies, including the centroid area, mean of α-cut, and bounded area, have been around for a while, but they haven't been tested on complex imprecise numbers like icosikaitetragonal fuzzy numbers.
- Due to the substantial variation in the values of the functions, the simple distance function from the ideal solution for association functions is not effective.
- Non-linear association functions can handle the complexity present in real-world, multi-objective optimization problems. As a result, it is necessary to associate these functions with modified approaches.
- No study has ever taken an intuitionistic fuzzy framework that depicts all real-world scenarios and methodically looked at every possible combination of various linear and non-linear functions for membership and non-membership functions.

2.7 Objectives of the Research

The purpose of this thesis is to conduct a thorough review of all existing studies, identify any gaps, and develop new, enhanced fuzzy techniques for solving multi-objective optimization problems using various membership functions and fuzzy numbers. Then, we have to apply these new techniques to various real-life applications, conducting comparative studies between the results obtained from these techniques and those from existing technologies. The following list outlines the primary goals of the thesis:

- Study and analyze the existing optimization techniques in a fuzzy, multi-objective optimization environment.
- Develop a range of optimization strategies for multi-objective optimization problems employing a variety of membership functions and fuzzy numbers to improve ideal outcomes.
- Apply effective methods to real-world circumstances and compare them to the methods now in use, as well as come up with situation-based strategies for addressing multi-objective optimization difficulties in a fuzzy or uncertain environment.

Chapter 3

Classification of Various Fuzzy Numbers with Defuzzification Techniques

Defuzzification techniques are known to be the initial step in solving fuzzy multi-objective optimization issues by reducing them to simpler optimization problems. Numerous defuzzification strategies exist for basic fuzzy numbers, as discussed in the previous chapter. However, as real-world issues have become more complex and uncertain, new forms of advanced fuzzy numbers have emerged. To accurately characterize and solve these complicated fuzzy numbers, broad and flexible defuzzification algorithms are required.

Consequently, this chapter starts with describing different fuzzy number representations and sorts them according to the number of components they have. Then we discuss here three of the most preferred defuzzification techniques for all these fuzzy numbers. After that, an icosikaitetragonal complex fuzzy number, just found, is appropriately described. We use an icosikaitetragonal fuzzy number in a real-life case study of the manufacturing department to validate the effectiveness of all discussed defuzzification techniques. This chapter concludes based on these results. From fuzzy number categorization and representation to defuzzification technique application and case study evaluation, Figure 3.1 lays out the chapter's structure to help with understanding and organizing. This image serves as a visual roadmap to guide readers through the theoretical advancements and practical implementations discussed in this chapter.

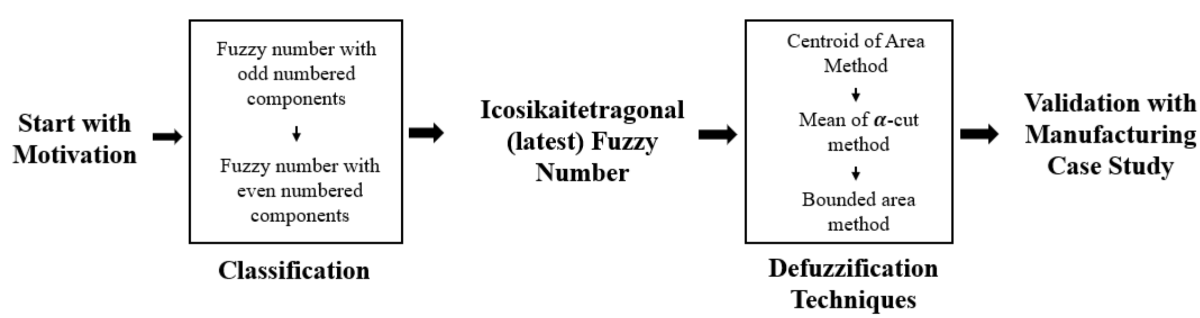


Figure 3.1: Conceptual flow and organization of the chapter

3.1 Introduction

Once some properties are added, a fuzzy set can be transformed into a fuzzy number. We can categorize the fuzzy numbers into different kinds based on their distinct characteristics. Here, we categorize fuzzy numbers based on the number of components that change their behaviour. We categorize fuzzy numbers into two types: one with odd-numbered components such as triangular, pentagonal, heptagonal, etc., and another with even-numbered components. We further sub-categorize even-numbered fuzzy numbers into two types: those with multiples of 4, such as tetragonal, octagonal, etc., and those without multiples of 4, such as hexagonal, decagonal, etc. Depending on the situation, uncertainty can be present as fuzzy numbers from any of these categories. Researchers have conducted defuzzification studies for simple fuzzy numbers, or those with fewer components, to address this uncertainty. However, complex fuzzy numbers have not been the focus of these studies. So, a generalized study is required to handle any type of situation.

Although fuzzy numbers can exhibit both symmetric and asymmetric behaviour, here we will discuss only symmetric fuzzy numbers. With the help of existing approaches, we have developed mathematical formulations and representations for each fuzzy number, explaining various defuzzification techniques for them. We use a real-life case study of manufacturing to demonstrate the novelty of our work and conduct a comparative analysis among various techniques based on a specific situation. The introduction of a new fuzzy number with 24 constituent elements occurred recently. Rare studies exist to provide information about this fuzzy number. Therefore, we have conducted a thorough analysis of its representation and the methods that have been used to defuzzify it. We conduct the case study in the context of this fuzzy number.

3.2 Fuzzy Number with Odd Numbered Components

The membership function for such types of fuzzy numbers can be mathematically represented by the function defined in eq (3.1). The graphical representation of the membership function is defined by $(n-1)$ intervals. Its behavior is increasing up to $\left(\frac{n+1}{2}\right)^{th}$ component, then decreasing, but the slope of the increment can be different for various intervals. If the fuzzy number is non-linear as shown in Figure 3.2, then the change rate in each interval will not be constant.

$$\acute{m}(x) = \begin{cases} 0 & if\ x < t_1 \\ \left(\frac{2}{n-1}\right)\left(\frac{x-t_1}{t_2-t_1}\right)^p & if\ t_1 \leq x \leq t_2 \\ \left(\frac{2}{n-1}\right)\left(\frac{x-t_2}{t_3-t_2}\right)^p & if\ t_2 \leq x \leq t_3 \\ \cdots & \cdots \\ \left(\frac{2}{n-1}\right)\left(\frac{x-t_{\frac{n-1}{2}}}{t_{\frac{n+1}{2}}-t_{\frac{n-1}{2}}}\right)^p & if\ t_{\frac{n-1}{2}} \leq x \leq t_{\frac{n+1}{2}} \\ \left(\frac{2}{n-1}\right)\left(\frac{t_{\frac{n+3}{2}}-x}{t_{\frac{n+3}{2}}-t_{\frac{n+1}{2}}}\right)^p & if\ t_{\frac{n-1}{2}} \leq x \leq t_{\frac{n+1}{2}} \\ \cdots & \cdots \\ \left(\frac{2}{n-1}\right)\left(\frac{t_{n-2}-x}{t_{n-1}-t_{n-2}}\right)^p & if\ t_{n-2} \leq x \leq t_{n-1} \\ \left(\frac{2}{n-1}\right)\left(\frac{t_{n-1}-x}{t_n-t_{n-1}}\right)^p & if\ t_{n-1} \leq x \leq t_n \\ 0 & if\ x \geq t_n \end{cases} \quad (3.1)$$

When $p = 1$ in eq (3.1), the fuzzy number behaves linearly as shown in Figure 3.3; otherwise, it behaves non-linearly.

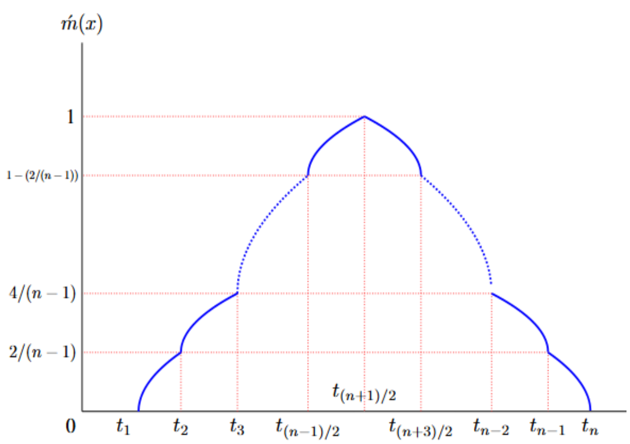


Figure 3.2: Non-linear symmetric fuzzy number with an odd number of components

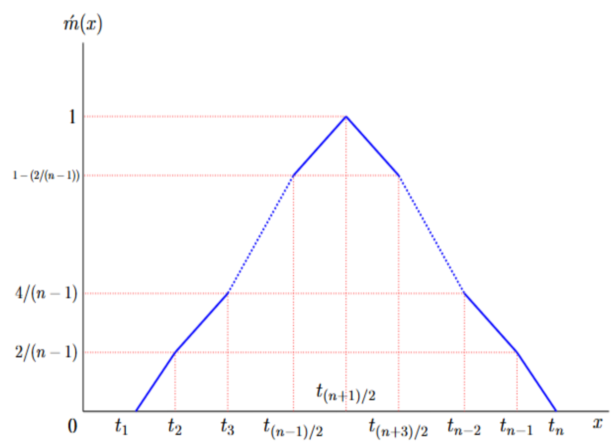


Figure 3.3: Linear symmetric fuzzy number with an odd number of components

3.3 Fuzzy Number with Even Numbered Components (Not multiple of 4)

Out of $(n-1)$ intervals of the function, the function's behavior is increasing up upto $\left(\frac{n}{2}-1\right)^{th}$ intervals, then constant for one interval. The function decreases from $\left(\frac{n}{2}+1\right)^{th}$ interval to $(n-1)^{th}$ interval. In the case of a non-linear fuzzy number, the rate of change in every interval is not constant. The mathematical representation is given through eq (3.2) with $p = 1$, then the fuzzy number shows linear behavior as in Figure 3.5; otherwise, it is non-linear as in Figure 3.4.

$$\acute{m}(x) = \begin{cases} 0 & if\ x < t_1 \\ \left(\frac{2}{n-1}\right)\left(\frac{x-t_1}{t_2-t_1}\right)^p & if\ t_1 \le x \le t_2 \\ \left(\frac{2}{n-1}\right)\left(\frac{x-t_2}{t_3-t_2}\right)^p & if\ t_2 \le x \le t_3 \\ \dots & \dots \\ \left(\frac{2}{n-1}\right)\left(\frac{x-t_{\frac{n-2}{2}}}{t_{\frac{n}{2}}-t_{\frac{n-2}{2}}}\right)^p & if\ t_{\frac{n-2}{2}} \le x \le t_{\frac{n}{2}} \\ 1 & if\ t_{\frac{n}{2}} \le x \le t_{\frac{n+2}{2}} \\ \left(\frac{2}{n-1}\right)\left(\frac{t_{\frac{n+4}{2}}-x}{t_{\frac{n+4}{2}}-t_{\frac{n+2}{2}}}\right)^p & if\ t_{\frac{n+2}{2}} \le x \le t_{\frac{n+4}{2}} \\ \dots & \dots \\ \left(\frac{2}{n-1}\right)\left(\frac{t_{n-2}-x}{t_{n-1}-t_{n-2}}\right)^p & if\ t_{n-2} \le x \le t_{n-1} \\ \left(\frac{2}{n-1}\right)\left(\frac{t_{n-1}-x}{t_n-t_{n-1}}\right)^p & if\ t_{n-1} \le x \le t_n \\ 0 & if\ x \ge t_n \end{cases} \quad (3.2)$$

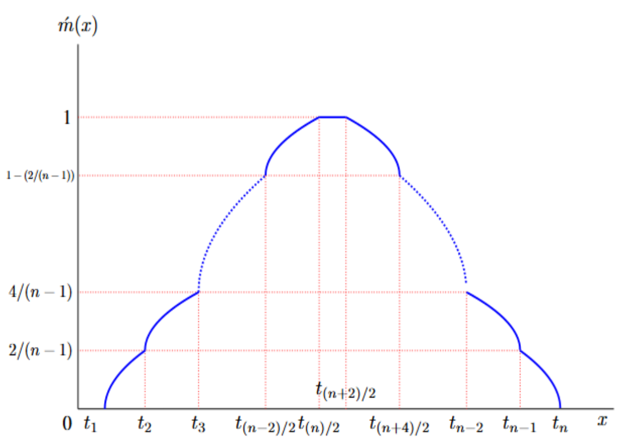


Figure 3.4: Non-linear symmetric fuzzy number with an even number (not multiple of 4) of components

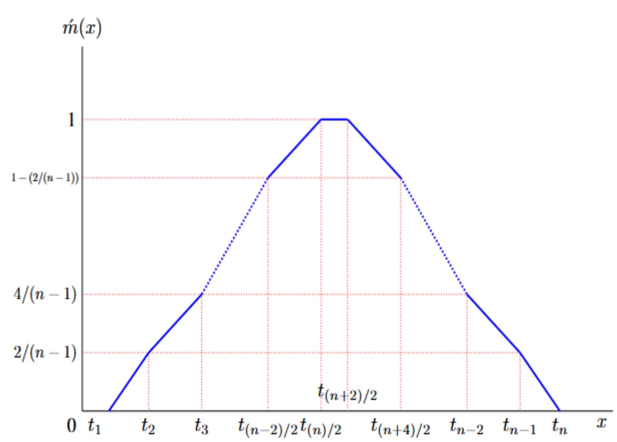


Figure 3.5: Linear symmetric fuzzy number with an even number (not multiple of 4) of components

3.4 Fuzzy Number with Even Numbered Components (multiple of 4)

From the first interval of $(n-1)$ intervals, the membership function increases, becomes constant in the next interval, and then the same pattern increases and decreases. This pattern continues until the $\left(\frac{n+2}{2}\right)^{th}$ component, at which point the function exhibits decreasing behavior instead of increasing; otherwise, it remains unchanged.

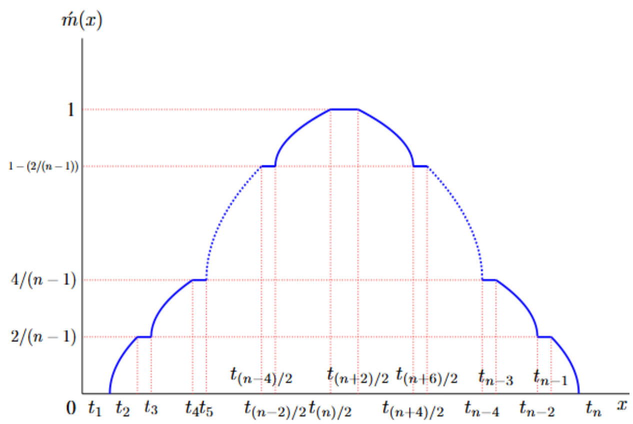


Figure 3.6: Non-linear symmetric fuzzy number with an even number (multiple of 4) of components

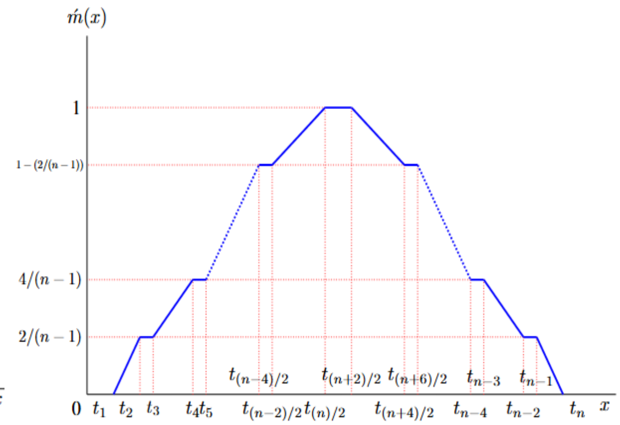


Figure 3.7: Linear symmetric fuzzy number with an even number (multiple of 4) of components

The mathematical formulation of the membership function is given below with eq (3.3):

$$\acute{m}(x) = \begin{cases} 0 & if\ x < t_1 \\ \left(\frac{2}{n-1}\right)\left(\frac{x-t_1}{t_2-t_1}\right)^p & if\ t_1 \le x \le t_2 \\ 1 & if\ x \le t_3 \\ \left(\frac{2}{n-1}\right)\left(\frac{x-t_2}{t_3-t_2}\right)^p & if\ t_3 \le x \le t_4 \\ 1 & if\ t_4 \le x \le t_5 \\ \dots & \dots \\ \left(\frac{2}{n-1}\right)\left(\frac{x-t_{\frac{n-2}{2}}}{t_{\frac{n}{2}}-t_{\frac{n-2}{2}}}\right)^p & if\ t_{\frac{n-2}{2}} \le x \le t_{\frac{n}{2}} \\ 1 & if\ t_{\frac{n}{2}} \le x \le t_{\frac{n+2}{2}} \\ \left(\frac{2}{n-1}\right)\left(\frac{t_{\frac{n+4}{2}}-x}{t_{\frac{n+4}{2}}-t_{\frac{n+2}{2}}}\right)^p & if\ t_{\frac{n+2}{2}} \le x \le t_{\frac{n+4}{2}} \\ \dots & \dots \\ 1 & if\ t_{n-4} \le x \le t_{n-3} \\ \left(\frac{2}{n-1}\right)\left(\frac{t_{n-2}-x}{t_{n-1}-t_{n-2}}\right)^p & if\ t_{n-3} \le x \le t_{n-2} \\ 1 & if\ t_{n-2} \le x \le t_{n-1} \\ \left(\frac{2}{n-1}\right)\left(\frac{t_{n-1}-x}{t_n-t_{n-1}}\right)^p & if\ t_{n-1} \le x \le t_n \\ 0 & if\ x \ge t_n \end{cases} \quad (3.3)$$

In the case of a non-linear fuzzy number as in Figure 3.6, the rate of change in every interval of a non-linear fuzzy number is not constant. If in eq (3.3), $p = 1$; the rate of change becomes constant as shown in Figure 3.7.

3.5 Icosikaitetragonal Fuzzy Number

The 24 defining components of the icosikaitetragonal fuzzy number make them unique among all existing fuzzy numbers. For modelling complicated real-world situations where accuracy is paramount, this number provides a fine-grained and thorough depiction of uncertainty.

$$\acute{m}(x)= \begin{cases} 0 & if\ x < t_1 \\ \left(\frac{1}{6}\right)\left(\frac{x-t_1}{t_2-t_1}\right)^p & if\ t_1 \le x \le t_2 \\ 1/6 & if\ t_2 \le x \le t_3 \\ \frac{1}{6}+\left(\frac{1}{6}\right)\left(\frac{x-t_3}{t_4-t_3}\right)^p & if\ t_3 \le x \le t_4 \\ 2/6 & if\ t_4 \le x \le t_5 \\ \frac{2}{6}+\left(\frac{1}{6}\right)\left(\frac{x-t_5}{t_6-t_5}\right)^p & if\ t_5 \le x \le t_6 \\ 3/6 & if\ t_6 \le x \le t_7 \\ \frac{3}{6}+\left(\frac{1}{6}\right)\left(\frac{x-t_7}{t_8-t_7}\right)^p & if\ t_7 \le x \le t_8 \\ 4/6 & if\ t_8 \le x \le t_9 \\ \frac{4}{6}+\left(\frac{1}{6}\right)\left(\frac{x-t_9}{t_{10}-t_9}\right)^p & if\ t_9 \le x \le t_{10} \\ 5/6 & if\ t_{10} \le x \le t_{11} \\ \frac{5}{6}+\left(\frac{1}{6}\right)\left(\frac{x-t_{11}}{t_{12}-t_{11}}\right)^p & if\ t_{11} \le x \le t_{12} \\ 1 & if\ t_{12} \le x \le t_{13} \\ \frac{5}{6}+\left(\frac{1}{6}\right)\left(\frac{t_{14}-x}{t_{14}-t_{13}}\right)^p & if\ t_{13} \le x \le t_{14} \\ 5/6 & if\ t_{14} \le x \ge t_{15} \\ \frac{4}{6}+\left(\frac{1}{6}\right)\left(\frac{t_{16}-x}{t_{16}-t_{15}}\right)^p & if\ t_{15} \le x \le t_{16} \\ 4/6 & if\ t_{16} \le x \le t_{17} \\ \frac{3}{6}+\left(\frac{1}{6}\right)\left(\frac{t_{18}-x}{t_{18}-t_{17}}\right)^p & if\ t_{17} \le x \le t_{18} \\ 3/6 & if\ t_{18} \le x \le t_{19} \\ \frac{2}{6}+\left(\frac{1}{6}\right)\left(\frac{t_{20}-x}{t_{20}-t_{19}}\right)^p & if\ t_{19} \le x \le t_{20} \\ 2/6 & if\ t_{20} \le x \le t_{21} \\ \frac{1}{6}+\left(\frac{1}{6}\right)\left(\frac{t_{22}-x}{t_{22}-t_{21}}\right)^p & if\ t_{21} \le x \le t_{22} \\ 1/6 & if\ t_{22} \le x \le t_{23} \\ \left(\frac{1}{6}\right)\left(\frac{t_{24}-x}{t_{24}-t_{23}}\right)^p & if\ t_{23} \le x \le t_{24} \\ 0 & if\ x \ge t_{24} \end{cases} \quad (3.4)$$

24 is a multiple of 4, so it falls under the second category with the second subcategory. Consequently, we can derive their non-linear membership function from eq (3.3). If they are non-linear, their geometric expression can be represented through Figure 3.8, and their mathematical representation via eq (3.4). The transformation into linear expressions, as depicted in Figure 3.9, takes place when we apply p=1 into eq (3.4).

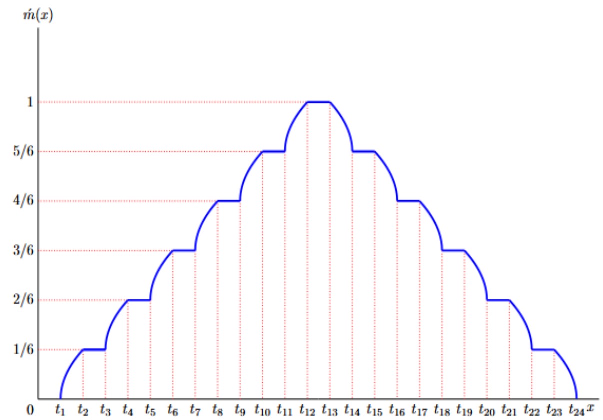


Figure 3.8: Non-linear representation of a symmetric icosikaitetragonal fuzzy number

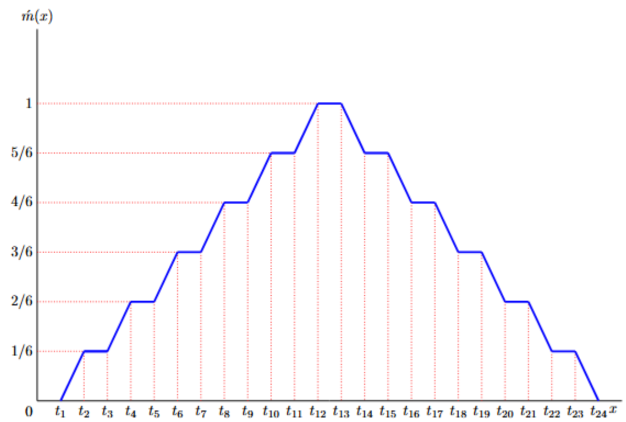


Figure 3.9: Linear representation of a symmetric icosikaitetragonal fuzzy number

3.6 Defuzzification

The bounded area method, centroid method, and mean of α-cut are the three defuzzification strategies that we have used in this research. We chose these methods due to their superior accuracy [88]. Techniques for defuzzification are defined for all of the categories of fuzzy numbers with linear nature discussed in sections 3.2, 3.3, and 3.4, considering cases 1, 2, and 3, respectively.

3.6.1 Centroid of area method

This technique focuses on the membership function's centre of gravity to get a discrete value. Many smaller areas make up the overall dispersion of the membership function, which represents the entire control operation. To extract the defuzzified number from a continuous, ambiguous set, one must first determine the area and centre of each sub-region and then add all these parts together. In the context of continuous membership functions, the COA-defuzzified value x^* is characterized as:

$$x^* = \frac{\sum_{k=1}^{n} t_k \acute{m}(x)_k}{\sum_{k=1}^{n} \acute{m}(x)_k} \tag{3.5}$$

Here t_k indicates the sample element, $\acute{m}(x)_k$ is the membership function, and n represents the number of elements in the sample. For continuous membership function, x^* is defined as:

$$x^* = \frac{\int x\, \acute{m}(x) dx}{\int \acute{m}(x) dx} \tag{3.6}$$

- ***Case 1***

$$\int x\, \acute{m}(x) dx = \frac{\left(t_n^2 + 2t_{n-1}^2 + \cdots + 2t_{\frac{n+3}{2}}^2 + t_{\frac{n+1}{2}}^2 + t_{\frac{n+1}{2}} t_{\frac{n+3}{2}} + t_{\frac{n+3}{2}} t_{\frac{n+5}{2}} + .. + t_{n-1} t_n\right)}{3(n-1)} - \frac{\left(t_1^2 + 2t_2^2 + \cdots + 2t_{\frac{n-1}{2}}^2 + t_{\frac{n+1}{2}}^2 + t_{\frac{n-1}{2}} t_{\frac{n+1}{2}} + t_{\frac{n-1}{2}} t_{\frac{n-3}{2}} + .. + t_1 t_2\right)}{3(n-1)} \tag{3.7}$$

$$\int \acute{m}(x) dx = \frac{\left(t_n + 2t_{n-1} + \cdots + 2t_{\frac{n+3}{2}} + 2t_{\frac{n-1}{2}} + .. + 2t_2 + t_1\right)}{(n-1)} \tag{3.8}$$

- ***Case 2***

$$\int x\, \acute{m}(x) dx = \frac{\left(t_n^2 + 2t_{n-1}^2 + \cdots + 2t_{\frac{n}{2}+2}^2 + t_{\frac{n}{2}+1}^2 + t_{\frac{n}{2}+2} t_{\frac{n}{2}+1} + t_{\frac{n}{2}+3} t_{\frac{n}{2}+2} + .. + t_{n-1} t_n\right)}{3(n-2)} - \frac{\left(t_1^2 + 2t_2^2 + \cdots + 2t_{\frac{n}{2}-1}^2 + t_{\frac{n}{2}}^2 + t_{\frac{n}{2}} t_{\frac{n}{2}-1} + t_{\frac{n}{2}-1} t_{\frac{n}{2}-2} + .. + t_1 t_2\right)}{3(n-2)} \tag{3.9}$$

$$\int \acute{m}(x) dx = \frac{\left(t_n + 2t_{n-1} + \cdots 2t_{\frac{n}{2}+2} + t_{\frac{n}{2}+1} + t_{\frac{n}{2}} + 2t_{\frac{n}{2}-1} + 2t_{\frac{n}{2}-2} + .. + 2t_2 + t_1\right).}{(n-2)} \tag{3.10}$$

- ***Case 3***

$$\int x\,\acute{m}(x)dx$$
$$= \frac{\left(t_n^2 + t_{n-1}^2 + \cdots + t_{\frac{n}{2}+2}^2 + t_{\frac{n}{2}+1}^2 + t_{\frac{n}{2}+1}t_{\frac{n}{2}+2} + t_{\frac{n}{2}+3}t_{\frac{n}{2}+4} + .. + t_{n-1}t_n\right)}{\frac{6n}{4}} \tag{3.11}$$
$$- \frac{\left(t_1^2 + t_2^2 + \cdots + t_{\frac{n}{2}-1}^2 + t_{\frac{n}{2}}^2 + t_{\frac{n}{2}}t_{\frac{n}{2}-1} + t_{\frac{n}{2}-2}t_{\frac{n}{2}-3} + .. + t_1t_2\right)}{\frac{6n}{4}}$$

$$\int \acute{m}(x)dx = \frac{\left(t_n + t_{n-1} + \cdots + t_{\frac{n}{2}} + t_{\frac{n}{2}-1} + .. + t_2 + t_1\right).}{\frac{n}{2}} \tag{3.12}$$

3.6.2 Mean of $\boldsymbol{\alpha}$-cut method

The α-cut of a fuzzy number is the collection of all the elements of the fuzzy number whose membership degree is greater than or equal to α. α −cut is categorized as left α −cut collected from left of the center of the domain and right α −cut collected from the right of the center of domain. **Appendix A** provides a mathematical definition for the left α -cut and the right α -cut for all categories of fuzzy numbers. We gather all the collections and then calculate the average of these collections for all $\alpha \epsilon [0,1]$, presenting the results mathematically:

$$x^* = \int_{\alpha=0}^{1} \frac{\left(L^{-1}(\alpha) + R^{-1}(\alpha)\right)}{2} d\alpha \tag{3.13}$$

- ***Case 1***

$$x^* = \frac{\left(t_n + 2t_{n-1} + \cdots + 2t_{\frac{n+1}{2}} + 2t_{\frac{n-1}{2}} + .. + 2t_2 + t_1\right)}{2(n-1)} \tag{3.14}$$

- ***Case 2***

$$x^* = \frac{\left(t_n + 2t_{n-1} + \cdots 2t_{\frac{n}{2}+2} + t_{\frac{n}{2}+1} + t_{\frac{n}{2}} + 2t_{\frac{n}{2}-1} .. + 2t_2 + t_1\right)}{2(n-2)} \tag{3.15}$$

- ***Case 3***

$$x^* = \frac{\left(t_n + t_{n-1} + \cdots t_{\frac{n}{2}+2} + t_{\frac{n}{2}+1} + t_{\frac{n}{2}} + t_{\frac{n}{2}-1} .. + t_2 + t_1\right)}{n} \tag{3.16}$$

3.6.3 Bounded area method

The defuzzification process in this case involves calculating the mean of the areas. We first calculate the area for each i^{th} trapezium. Next, we calculate the mean of all these areas. The mathematical formulation for mean value is given below:

$$x^* = \frac{\sum A_i}{|i|} \tag{3.17}$$

- ***Case 1***

$$x^* = \frac{\left(t_n + 2t_{n-1} + \cdots - 2t_{\frac{n-1}{2}} + 2t_{\frac{n+3}{2}} - .. - 2t_2 - t_1\right)}{(n-1)} \tag{3.18}$$

- ***Case 2***

$$x^* = \frac{\left(t_n + 2t_{n-1} + \cdots 2t_{\frac{n}{2}+2} + t_{\frac{n}{2}+1} - t_{\frac{n}{2}} - 2t_{\frac{n}{2}-1} .. - 2t_2 - t_1\right)}{(n-1)} \tag{3.19}$$

- ***Case 3***

$$x^* = \frac{\left(t_n + t_{n-1} + \cdots t_{\frac{n}{2}+2} + t_{\frac{n}{2}+1} - t_{\frac{n}{2}} - t_{\frac{n}{2}-1} .. - t_2 - t_1\right).}{\frac{n}{2}} \tag{3.20}$$

3.6.4 Icosikaitetragonal fuzzy number

The defuzzification techniques for the number that we have discussed in sections 3.6.1-3.6.3 are given below:

Centroid of Area Method:

$$x^* = \frac{\sum_{i=13}^{24} t_i^2 + \sum_{i=7}^{12} t_{2i}t_{2i-1} - \sum_{i=1}^{12} t_i^2 - \sum_{i=1}^{6} t_{2i}t_{2i-1}}{3\sum_{i=1}^{24} t_i} \tag{3.21}$$

Mean of $\boldsymbol{\alpha}$*-cut method:*

$$x^* = \frac{\sum_{i=1}^{24} t_i}{24} \tag{3.22}$$

Bounded area method:

$$x^* = \frac{\sum_{i=13}^{24} t_i}{12} - \frac{\sum_{i=1}^{12} t_i}{12} \tag{3.23}$$

3.7 Manufacturing Problem in Fuzzy Environment

In a manufacturing optimization challenge, for instance, it is not necessary for every item to be of high quality and fully marketable at a given price. The products may have flaws that prevent them from selling at the set price. Due to unforeseen circumstances, the market price of the final product and the raw materials that have been used to make it can fluctuate. Therefore, prices and/or productions are not completely predictable, but rather, they are often imprecise or non-deterministic. Consequently, optimization issues involving these variables necessitate the use of non-classical approaches. To clarify a real-life scenario, we applied the symmetric icosikaitetragonal fuzzy number to the data provided in [122]. The study delves into a park of six machine types that will be used to manufacture three distinct items. There is a current capacity portfolio available, with prices based on machine type and machine hours per week. Table 3.1 shows all the data about the issue. Here all the parameters are given according to eq (2.24) - (2.26) from chapter 2.

Table 3.1: Data for the manufacturing problem in the icosikaitetragonal fuzzy numbered environment

Element of array	Fuzzy number
p_1^1	(6,10,14,18,22,26,30,34,38,42,46,50,54,58,62,66,70,74,78,82,86,90,94,98)
p_2^1	(12,20,28,36,44,52,60,68,76,84,92,100,108,116,124,132,140,148,156,164,172,180,188,196)
p_3^1	(6.5,7.5,8.5,9.5,10.5,11.5,12.5,13.5,14.5,15.5,16.5,17.5,18.5,19.5,20.5,21.5,22.5,23.5,24.5,25.5,26.5,27.5,28.5,29.5)
p_1^2	(26,32,38,44,50,56,62,68,74,80,86,92,98,104,110,116,122,128,134,140,146,152,158,164)
p_2^2	(20,25,30,35,40,45,50,55,60,65,70,75,80,85,90,95,100,105,110,115,120,125,130,135)

p_3^2	(6,10,14,18,22,26,30,34,38,42,46,50,54,58,62,66,70,74,78,82,86,90,94,98)
p_1^3	(3,5,7,9,11,13,15,17,19,21,23,25,27,29,31,33,35,37,39,41,43,45,47,49)
p_2^3	(12,20,28,36,44,52,60,68,76,84,92,100,108,116,124,132,140,148,156,164,172,180,188,196)
p_3^3	(20,25,30,35,40,45,50,55,60,65,70,75,80,85,90,95,100,105,110,115,120,125,130,135)
c_1^1	(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24)
c_2^1	(6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29)
c_1^2	(0.8,1,1.2,1.4,1.6,1.8,2,2.2,2.4,2.6,2.8,3,3.2,3.4,3.6,3.8,4,4.2,4.4,4.6,4.8,5,5.2,5.4)
c_2^2	(3.5,4,4.5,5,5.5,6,6.5,7,7.5,8,8.5,9,9.5,10,10.5,11,11.5,12,12.5,13,13.5,14,14.5,15)
c_3^2	(2.5,3, 3.5,4,4.5,5,5.5,6,6.5,7,7.5,8,8.5,9,9.5,10,10.5,11,11.5,12,12.5,13,13.5,14)
c_1^3	(1.2,2,2.8,3.6,4.4,5.2,6,6.8,7.6,8.4,9.2,10,10.8,11.6,12.4,13.2,14,14.8,15.6,16.4,17.2,18,18.8,19.6)
c_2^3	(2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25)
c_3^3	(4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27)
c_1^4	(1.8,2,2.4,2.8,3.2,3.6,4,4.4,4.8,5.2,5.6,6,6.4,6.8,7.2,7.6,8,8.4,8.8,9.2,9.6,10,10.4,10.8)
c_3^4	(5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28)
c_2^5	(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24)
c_3^5	(1.5,2,2.5,3, 3.5,4,4.5,5,5.5,6,6.5,7,7.5,8,8.5,9,9.5,10,10.5,11,11.5,12,12.5,13)
c_1^6	(4,4.5,5,5.5,6,6.5,7,7.5,8,8.5,9,9.5,10,10.5,11,11.5,12,12.5,13,13.5,14,14.5,15,15.5)
c_2^6	(4,4.5,5,5.5,6,6.5,7,7.5,8,8.5,9,9.5,10,10.5,11,11.5,12,12.5,13,13.5,14,14.5,15,15.5)
c_3^6	(1.8,2,2.2,2.4,2.6,2.8,3,3.2,3.4,3.6,3.8,4,4.2,4.4,4.6,4.8,5,5.2,5.4,5.6,5.8,6,6.2,6.2,6.4)
d^1	(1180,1200,1220,1240,1260,1280,1300,1320,1340,1360,1380,1400,1420,1440,1460,1480,1500,1520,1540,1560,1580, 1600,1620,1640)
d^2	(835,850,865,880,895,910,925,940,955,970,985,1000,1015,1030,1045,1060,1075,1090,1105,1120,1135,1150,1165,118 0)
d^3	(1475,1500,1525,1550,1575,1600,1625,1650,1675,1700,1725,1750,1775,1800,1825,1850,1875,1900,1925,1950,1975, 2000,2025,2050)
d^4	(1105,1125,1145,1165,1185,1205,1225,1245,1265,1285,1305,1325,1345,1365,1385,1405,1425,1445,1465,1485,1505,1 525,1545,1565)
d^5	(735,750,765,780,795,810,825,840,855,870,885,900,915,930,945,960,975,990,1005,1020,1035,1050,1065,1080)
d^6	(910,925,940,955,970,985,1000,1015,1030,1045,1060,1075,1090,1105,1120,1135,1150,1165,1180,1195,1210,1225,11 40,1155)

3.7.1 Defuzzification with proposed techniques

Here, the suggested methods are used to defuzzify the fuzzy parameters in the manufacturing case study. The defined fuzzy numbers are sequentially subjected to the Centroid of Area Method, Alpha-Cut Method, and Bounded Area Method, with the corresponding formulations described in eq (3.21), (3.22), and (3.23). To facilitate additional analysis and comparison, these procedures are used to transform the fuzzy data into crisp values as shown in Table 3.2.

Table 3.2: Defuzzified values of manufacturing data

Element of array	**Centroid method**	**α-cut**	**Bounded area**
p_1^1	52	52	48
p_2^1	104	104	96
p_3^1	18	18	12
p_1^2	95	95	72
p_2^2	77.5	77.5	60
p_3^2	52	52	48
p_1^3	26	26	
p_2^3	104	104	96
p_3^3	77.5	77.5	60
c_1^1	12.5	12.5	12
c_2^1	17.5	17.5	12
c_1^2	3.1	3.1	2.4
c_2^2	9.25	9.25	6
c_3^2	8.25	8.25	6
c_1^3	10.4	10.4	9.6
c_2^3	13.5	13.5	12
c_3^3	15.5	15.5	12
c_1^4	6.2	6.2	2.4
c_3^4	16.5	16.5	12
c_2^5	12.5	12.5	12
c_3^5	7.25	7.25	6
c_1^6	9.75	9.75	6
c_2^6	9.75	9.75	6
c_3^6	4.1	4.1	2.4
d^1	1410	1410	240
d^2	1007.5	100.75	96
d^3	1762.5	1762.5	300
d^4	1335	1335	240

d^5	907.5	907.5	180
d^6	1082.5	1082.5	180

3.7.2 Results

- We can see in Table 3.2 that the Centroid of Area technique and the Mean of the α-Cut method produce the same results when the intervals between the components of symmetric fuzzy numbers are equal. This proves that they remain stable when subjected to conditions of uniform distribution.
- There is a noticeable difference using the Bounded Area technique. The Bounded Area technique does not capture the true diversity in fuzziness because it consistently yields the same result for all fuzzy integers with the same component intervals.

3.8 Conclusion

The chapter's study allows for the geometrical representation of any type of fuzzy number, facilitating a more accurate understanding of uncertainty situations. In situations where the interval between any two components is identical for symmetric fuzzy numbers, the results obtained using the centroid area method and the mean of the α-cut method are identical, as shown in Table 3.1. However, the results can vary when using the bounded area method. In crisp conditions, we can see that the two approaches produce identical results. However, when we apply the bounded area method to fuzzy numbers displaying the same interval difference between components, its shortcoming becomes apparent. The outcomes are the same for different numbers. So, for the sake of this production model, we shall restrict ourselves to using just the two processes in which centroid area and α-cut are taken into consideration.

3.8.1 Major Findings

- We can capture more information and complexity when depicting ambiguity with arbitrary fuzzy numbers, enhancing flexibility in modelling uncertainty.
- Since the Bounded Area method is not sensitive enough to be used for the current production model, further optimization will be carried out utilizing the centroid and α-cut techniques exclusively.

Chapter 4

Conventional Fuzzy-Decision Making and Multi-Objective Optimization Techniques Utilizing Various Fuzzy Numbers

Prior to solving fuzzy multi-objective optimization problems, it is crucial to simplify them using defuzzification techniques, which were covered extensively in the preceding chapter. The following chapters expand on this idea by constructing fuzzy-based methods for solving these reduced optimization issues. In this chapter, we start by looking at different real-life case studies to assess existing fuzzy techniques. Because of this, we can better comprehend the effects of non-linear and linear association functions on optimization results in real-world settings and analyze the effects of extended fuzzy sets.

The chapter's organizational flow is shown in Figure 4.1. It starts with the use of triangular fuzzy numbers in traffic light management to show how fuzzy set theory can be used to decision-making. Next, it moves on to different traditional fuzzy optimization approaches, such as the min-max technique and more complex ones using intuitionistic and dual hesitant fuzzy sets.

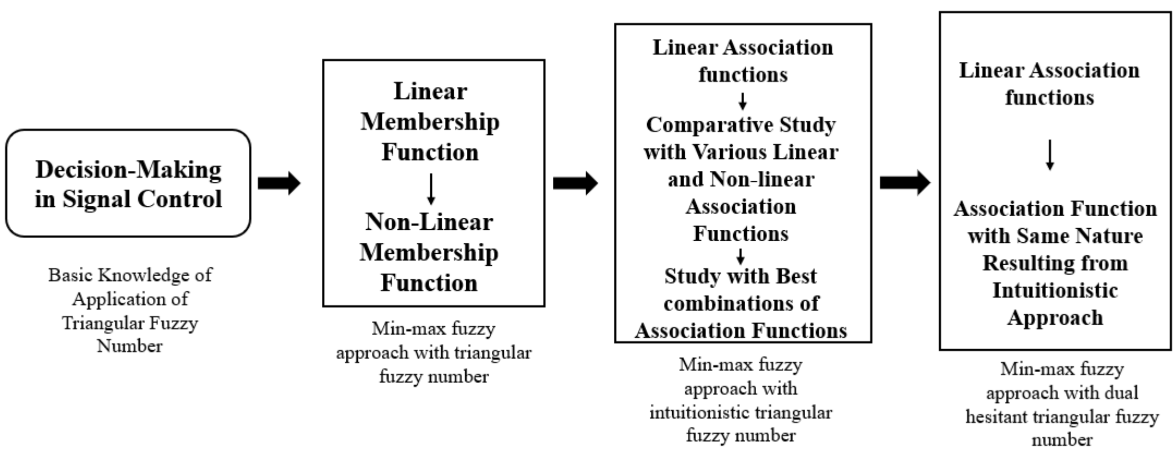


Figure 4.1: Conceptual flow and organization of the chapter

In each part, we compare the two types of association functions, linear and non-linear, and see how well they handle MOLPP. By following this structured procedure, we will be able to fully grasp the latest developments in fuzzy models as they pertain to real-world case studies. Then, a method using triangular, dual-hesitant fuzzy numbers was described that is valuable for the scenario of having multiple experts define association functions.

4.1 Introduction

The use of fuzzy logic and its inference rules can enhance the results of automation systems with multi-criteria decision-making problems. Here we have taken a problem of this type in which green light duration for traffic symbols is examined through them. When it comes to the ever-changing traffic patterns of growing cities, the old-fashioned fixed durations for green lights at junctions just aren't worth it. This has led to an increased awareness of the need to use novel strategies, most notably a system to automatically choose the green light period [123]. This chapter aims to dynamically increase green light durations using fuzzy logic inference algorithms to tackle the complex relationship between traffic congestion and urbanization. The concept of traffic management encompasses a holistic approach to address urban congestion and improve transportation efficiency. Due to these issues, numerous studies on intelligent transportation systems are currently underway. This encompasses a wide range of research topics, including traffic control with automated traffic signals [124], fuzzy logic [125],[126], swarm intelligence [127], genetic algorithms [128], and multiagent-oriented networks [129]. Researchers of [128],[130] have utilized a wireless sensor network for real-time traffic tracking, integrating it with fuzzy logic to determine the length of the green light and dynamically manage traffic at intersections. Systems based on fuzzy logic can make instantaneous adjustments to signal timing in response to incoming data. Because it can handle vague or incomplete data, fuzzy logic works well. Numerous studies [131], [132] have shown that fuzzy logic-based traffic signal control improves traffic flow and reduces congestion. Fuzzy logic's applications go far beyond tweaking signal timing. It is becoming more commonplace in real-time traffic management systems that account for factors like congestion, road conditions, and even the weather. We demonstrated the malleability of fuzzy logic to handle intricate decision-making by introducing a real-time traffic control system based on it. Several criteria oversee managing traffic lights, but a growing number of criteria will make it difficult to compute inference rules. A process to determine the relative weight of each criterion is required to select the more appropriate criteria for calculation. To provide insight into the relative relevance of the choice criteria under consideration, multi-criteria decision-making (MCDM) approaches make use of criterion weights [133]. Researchers have developed various models to assess the relative importance of these characteristics. Methods such as Best Worst (BWM), Full Consistency (FUCOM), Level-Based Weight Assessment (LBWA), and AHP are well-known. Several types of investigations have made use of the SWARA approach because of its straightforward and minimally invasive procedure [134]. But its biggest flaw is that it can't verify results using consistency levels [135]. The ability to identify the measurement of consistency has led to increased use of FUCOM, BWM, LBWA, and AHP recently. Among the methods, the FUCOM algorithm handles the fewest pairwise contrasts [136]. BWM becomes exceedingly complex when dealing with many pairwise comparisons [137]. The LBWA model [137] is similar to the FUCOM approach, and it enables weight computation using a minimum number of pairwise comparisons. This strategy offers the benefit of adding criteria without complicating the algorithm. Besides these advantages, the LBWA model should highlight the decision-maker's ability to further adjust the weight coefficients using the elasticity coefficient [138]. Scientific article [139] extensively references the AHP as a framework for multiple-criteria decision-making. According to [140], the AHP method's hierarchical structure allows for more efficient and transparent targeting of each criterion. But when dealing with subjective human assessments, the ambiguity and vagueness make the AHP approach ineffective. To deal with this limitation, the strategy is enhanced by [141] to effectively manage variability and ambiguity. By fusing AHP with fuzzy set theory, this modification, also known as fuzzy AHP, combines AHP with fuzzy set theory, resulting in more plausible and precise illustrations of the decision-making procedure. Fuzzy integers and linguistic variables in fuzzy AHP can express the relative importance of each set of criteria. For this research, we used fuzzy AHP. The AHP technique's ease of use and the fact that users can input judgment data easily without needing complicated mathematical expertise are the two major benefits.

From the above study, we can be aware of using membership functions for linguistic variables and their inference rules, which provide the direction for the use of membership functions in MOO. The first fuzzy approach for MOO is developed by Zimmerman. The use of this approach is discussed for sustainability issues in this study. The urgency of taking action to reduce greenhouse gas emissions is increasing all the time [142]. Due to the significant contribution of emissions from storage and manufacturing to global warming, business leaders must devise a sustainable green supply chain [143] in the manufacturing sector. Existing studies [144],[145] do not include the improved inventory model, which considers shortages to reduce costs and environmental concerns, in their sustainability requirements. This chapter's work will determine the necessary quantity for manufacturing and backordering, aiming to optimize profits while minimizing the ecological costs associated with industrial emissions and waste management. The inventory cost function encompasses the costs associated with initial setup, inventory, and any shortages. It has also considered several goals, including the production system's per-cycle waste and the overall penalty cost. The model also considers defects in the manufacturing process and shortages, thereby enhancing its realism. This case induces a multi-objective optimization problem due to the simultaneous involvement of multiple goals. Several non-fuzzy techniques exist to solve these problems, but Zimmerman's fuzzy approach, as discussed in Chapter 1, is more appropriate. As a result, the fuzzy technique transforms the multi-objective crisp problem into a single-objective optimization problem by considering the linear degrees of the membership function. The model is used to determine the quantity of products per production cycle to maintain sustainability (economic as well as environmental). The membership function's non-linearity influences our discussions in Chapter 2. To determine the impact of various non-linear functions, we conduct a financial portfolio analysis. Studies on intuitionistic fuzzy sets and MOO have previously focused on linear membership and non-membership functions. Fuzzy approaches with non-linear membership functions outperformed linear ones in accurately capturing uncertainty's intricacies, according to the results of the case study about financial portfolios. Despite these advancements, a significant gap remains in the literature: no research has ever approached a real-world intuitionistic fuzzy framework that systematically examined every conceivable combination of membership and non-membership functions. Since most prior research has focused on singular cases, little is known about how these combinations compare in various real-world contexts. Therefore, this research uses a case study from the agricultural sector to investigate the matter.

Most of the time, experts disagree on the optimal degrees' assignment for the elements in the optimization problem, necessitating multiple levels of involvement and a variety of parameter grades to find a solution. This goes against popular opinion. In this context, hesitant fuzzy sets, not intuitionistic or fuzzy sets, are crucial [119]. In the context of parallel relationship functions, both intuitionistic and hesitant fuzzy sets are equally effective for decision-making [120] Dual-hesitant fuzzy sets are an innovative tool for modelling ambiguity, which consider the non-linear behaviour of the two category values and give an extra-detailed model [121]. However, the same restriction remains when applying the method to the class of non-linear functions. Our study, based on the comparison results with the intuitionistic approach, provides the formulation of a strategy with the optimal combination of association functions.

4.2 Fuzzy Decision-Making in Signal Control with Triangular Fuzzy Number

The Analytic Hierarchy Process (AHP), which streamlines the selection process, allows for the assignment of weights to various criteria. This weight assignment allows criteria to be prioritized based on their perceived relevance, aligning with end-user preferences and priorities. A strategic approach reduces computational complexities and ensures that the decision-making criteria accurately reflect the concerns and expectations of all parties involved. So, customer preference analysis and green light period management are the two primary components of the proposed method. The complete process encompasses two distinct phases for the entire process, as illustrated in Figure 4.2:

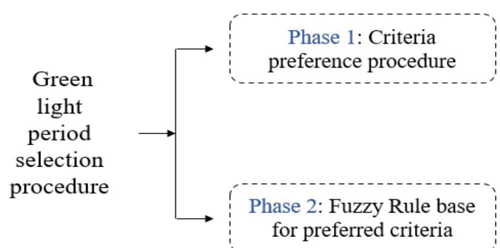


Figure 4.2: Different phases of automatic green light signal control process

Phase 1: Examining the preferences of the criteria: To consider customer preferences in the decision-making process complicates criteria selection. Moreover, figuring out what people really want is not a simple task. Drivers can input data into our automatic traffic signal period selection system through a designated mobile app. Having this information at their disposal can help decision-makers to understand consumers' views on service and, consequently, choose the right criteria to satisfy customers' expectations. The process involves several steps, which are shown in Figure 4.3.

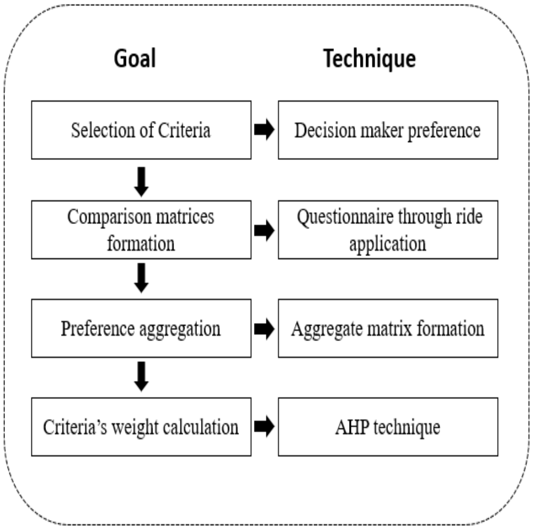


Figure 4.3: Research flow diagram for criteria preference procedure

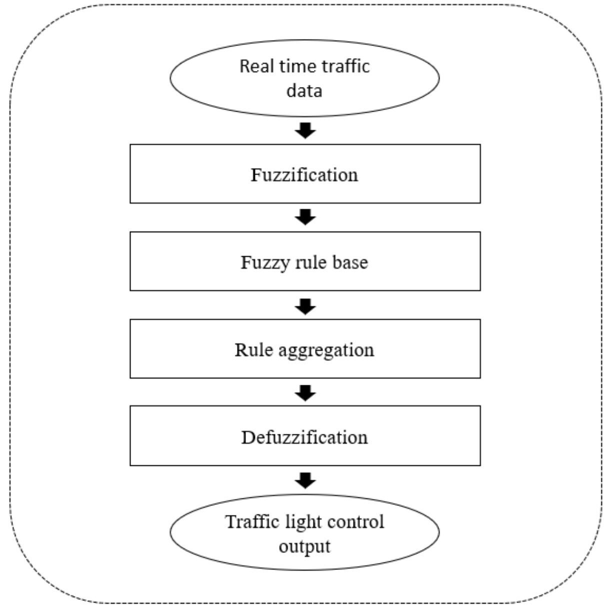


Figure 4.4: Research flow diagram of fuzzy rules for traffic light period control

- ***Step 1: Selection of criteria***

Several criteria can be taken into consideration for the selection process of vehicles based on the requirements of various decision-makers. Here we are discussing some of them. An explanation of each desired criterion considered is as follows:

i. *Vehicle Density:* The density of vehicles measures the number of automobiles there are per unit area. The duration of green lights should be adjusted based on the number of vehicles on the road to avoid congestion and keep traffic moving smoothly.

ii. *Intersection Type and Geometry:* Vehicles' travel times are affected by the geometry and intricacy of the intersection. We can think about things like turn lanes and different approaches, which effect the vehicles crossing speed at signal points.

iii. *Pedestrian and Cyclist Activity:* A more inclusive and safe traffic signal design would include enough green time for pedestrian crossings and room for bicycles.

iv. *Waiting time:* Improving traffic flow requires reducing wait times at crossings. Vehicles are expected to spend less time waiting at red lights if green light lengths are adjusted.

v. *Weather condition*: Inconvenient weather conditions, including rain, can affect the flow of transportation. To compensate for decreased visibility and changed driving conditions caused by heavy rain, the length of time that green lights remain on might be changed.

vi. *Safety concern:* The first priority is to make sure that vehicles are operated safely. The durations of green lights should be sufficient to provide safe acceleration and deceleration, therefore decreasing the likelihood of accidents.

- ***Step 2: Prioritization of green light period selection criteria***

At this point, we have to calculate the relative strengths of each criterion from the perspective of a single traveler. We can utilize fuzzy linguistic variables, transformed into fuzzy integers, to denote the relative significance of each set of criteria. The representation of the fuzzy evaluation matrix, which is based on a pairwise comparison of fuzzy numbers, is as follows:

$$\widetilde{M} = \begin{bmatrix} \widetilde{m_{11}} & \widetilde{m_{12}} & \dots & \widetilde{m_{1k}} \\ \widetilde{m_{21}} & \widetilde{m_{22}} & \dots & \widetilde{m_{2k}} \\ \dots & \dots & \dots & \dots \\ \widetilde{m_{l1}} & \widetilde{m_{l2}} & \dots & \widetilde{m_{lk}} \end{bmatrix} \tag{4.1}$$

In which $\widetilde{m_{lk}}$ denotes the relative weight of criterion l in respect of k, with $l = k = \{1,2\ldots, n\}$. The number of matrices obtained is proportional to the number of passengers who consented to answer the survey using the on-demand method.

- ***Step 3: Construct aggregated fuzzy decision matrix.***

A group judgment and an approximation of the collective choices can be obtained by fusing these individual traveler opinions once the prioritization of the green light period evaluation criteria has been completed. To do this, we employ the definitions offered by [146] to build an aggregated fuzzy decision matrix. Let the responses be in the simplest form of fuzzy numbers, which is a triangular fuzzy number. That is represented by $\widetilde{m}_{lk}$= (a_{lk}, b_{lk}, c_{lk}), and $\widetilde{M'}$ is the aggregated matrix. Then:

$$a'_{lk} = \min_{m\epsilon\{1,2,..p\}} a_{lkm} \tag{4.2}$$

$$b'_{lk} = \left(\prod_{m=1}^{p} b_{lkm}\right)^{1/p} \tag{4.3}$$

$$c'_{lk} = \max_{m\epsilon\{1,2,..p\}} c_{lkm} \tag{4.4}$$

- ***Step 4: Weighing the significance of criteria***

At this stage, we can compute the weight vector $W = (w1, w2, \dots, wn)$ of the selected criteria, which signifies the weight wj of each criterion j. We apply the steps given in the study [147] to determine the precedence variable of the fuzzy matrix $\widetilde{M'}$ using the extent analysis technique.

$$\widetilde{M'} = \begin{bmatrix} \widetilde{m'_{11}} & \widetilde{m'_{12}} & \dots & \widetilde{m'_{1k}} \\ \widetilde{m'_{21}} & \widetilde{m'_{22}} & \dots & \widetilde{m'_{2k}} \\ \dots & \dots & \dots & \dots \\ \widetilde{m'_{l1}} & \widetilde{m'_{l2}} & \dots & \widetilde{m'_{lk}} \end{bmatrix} \tag{4.5}$$

Here; $\widetilde{m'}_{lk}$= ($a'_{lk}, b'_{lk}, c'_{lk}$).

I. First, use fuzzy arithmetic procedures to determine the total number of elements in every row of the fuzzy matrix $\widetilde{M'}$.

$$RS_l = \sum_{k=1}^{n} \widetilde{m'}_{lk} \tag{4.6}$$

II. Next, standardize the sums of the rows by doing the following:

$$\widetilde{N_l} = RS_l / \sum_{k=1}^{n} RS_k \tag{4.7}$$

III. Thirdly, determine the level of certainty that $\widetilde{N_l} \geq \widetilde{N_k}$, that is described as:

$$D(\widetilde{N_l} \geq \widetilde{N_k}) = \begin{cases} 1 & if \;\; b_l > b_k \\ \dfrac{c_l - a_k}{(c_l - b_l) + (b_k - a_k)} & if \;\; a_k < c_l \\ 0 & otherwise \end{cases} \tag{4.8}$$

IV. Lastly, the following equation is used to calculate the degree of perspective with respect to the other $(n-1)$ fuzzy integers:

$$D\left(\widetilde{N_l} \geq \widetilde{N_k} \middle| k = 1,2 \dots n; l \neq k\right) = \min_{k\epsilon\{1,2,..n;l\neq k\}} D(\widetilde{N_l} \geq \widetilde{N_k}) \quad (4.9)$$

V. It is concluded by defining the weight vectors of the fuzzy matrix $\widetilde{M'}$ as $W = (w1, w2, w3, .. wl)$, where:

$$wl = \frac{D\left(\widetilde{N_l} \geq \widetilde{N_k} \middle| k = 1,2 \dots n; l \neq k\right)}{\sum_{i=1}^{n} D\left(\widetilde{N_l} \geq \widetilde{N_k} \middle| k = 1,2 \dots n; l \neq k\right)} \quad (4.10)$$

Phase 2: Traffic Light Period Control: A fuzzy rule base is a critical component of a fuzzy logic-based traffic light control system. A set of IF-THEN rules governs the control of traffic lights in real-time traffic conditions. We design these rules to capture the complex and often imprecise relationships between different variables that affect traffic flow. We explain the fuzzy rule base in a traffic light control system below, based on the flow diagram provided in Figure 4.4.

- ***Step 1: Real-time data collection***

This step includes the procedure to collect the real-time data, meaning the values of selected criteria at a specific time, which can be in the form of a quantitative or qualitative pattern.

- ***Step 2: Fuzzification***

Before creating fuzzy rules, we need to define linguistic variables that represent aspects of the traffic situation, such as "vehicle density," "traffic flow," "waiting time," and "road occupancy" that we have discussed earlier. Typically, different membership functions divide these variables into categories like "low," "medium," and "high." For more information on how this scale is created, see Rao [148], [149].

- ***Step 3: Fuzzy Rules***

The linguistic variables guide the development of fuzzy rules. Each rule consists of an IF part (antecedent) and a THEN part (consequent). The IF part specifies the conditions or input variables, while the THEN part specifies the control action or output variable. For example, a fuzzy rule might be: If vehicle density is high and waiting time is long, then increase the green time for that direction, means the green light period should be high.

- ***Step 4: Rule Aggregation***

Concurrently applying multiple rules requires aggregation of their outputs. Typically, we do this by considering the "firing strength" of each rule, which relies on how well the input variables meet the conditions in the IF part of the rule.

- ***Step 5: Defuzzification***

We need to convert the aggregated result back into a crisp, non-fuzzy value to control the traffic lights. This process is called defuzzification.

4.2.1 Numerical experiment

We provide hypothetical data here to demonstrate the impact of the proposed approach. For simplification, only two respondents are considered for phase 1. Given that the solution will take two phases, we have divided the problem into the two parts listed below.

Phase 1: Examining the preferences of the criteria: Here we have discussed the criteria that are selected for consideration and their preference weights associated with them.

- ***Step 1***

Here we have a decision-making situation where six criteria, C1, C2, C3, C4, C5, and C6, are being considered. The criteria are the same as we have discussed in the earlier section.

- ***Step 2***

Two reaction matrices, M_1 and M_2, are produced when two decision-makers separately offer their respective comparisons of the criteria provided through Table 4.1 and Table 4.2, respectively. The matrix elements m_{ij}^1 and m_{ij}^2 show the preference scores for the relative comparison of criteria Ci and Cj, as given by the first and second decision-makers, respectively. Use the given matrices to determine the aggregated weight preferences for each criterion. The estimation values in this case come from the [0,10] interval.

Table 4.1: Response of first rider towards different criteria

M_1	C1	C2	C3	C4	C5	C6
C1	(1,1,1)	(8,9,10)	(8,9,10)	(6,7,8)	(2,3,4)	(2,3,4)
C2	$(8,9,10)^{-1}$	(1,1,1)	(6,7,8)	(4,5,6)	(2,3,4)	(2,3,4)
C3	$(8,9,10)^{-1}$	$(6,7,8)^{-1}$	(1,1,1)	(4,5,6)	(2,3,4)	(2,3,4)
C4	$(6,7,8)^{-1}$	$(4,5,6)^{-1}$	$(4,5,6)^{-1}$	(1,1,1)	(4,5,6)	(4,5,6)
C5	$(2,3,4)^{-1}$	$(2,3,4)^{-1}$	$(2,3,4)^{-1}$	$(4,5,6)^{-1}$	(1,1,1)	(4,5,6)
C6	$(2,3,4)^{-1}$	$(2,3,4)^{-1}$	$(2,3,4)^{-1}$	$(4,5,6)^{-1}$	$(4,5,6)^{-1}$	(1,1,1)

Table 4.2: Response of second rider towards different criteria

M_2	C1	C2	C3	C4	C5	C6
C1	(1,1,1)	(8,9,10)	(4,5,6)	(8,9,10)	(8,9,10)	(2,3,4)
C2	$(8,9,10)^{-1}$	(1,1,1)	(2,3,4)	(4,5,6)	(4,5,6)	(8,9,10)
C3	$(4,5,6)^{-1}$	$(2,3,4)^{-1}$	(1,1,1)	(8,9,10)	(6,7,8)	(8,9,10)
C4	$(8,9,10)^{-1}$	$(4,5,6)^{-1}$	$(8,9,10)^{-1}$	(1,1,1)	(2,3,4)	(6,7,8)
C5	$(8,9,10)^{-1}$	$(4,5,6)^{-1}$	$(6,7,8)^{-1}$	$(2,3,4)^{-1}$	(1,1,1)	(6,7,8)
C6	$(2,3,4)^{-1}$	$(8,9,10)^{-1}$	$(8,9,10)^{-1}$	$(6,7,8)^{-1}$	$(6,7,8)^{-1}$	(1,1,1)

- ***Step 3***

The aggregated weight for criterion Ci results from combining the preferences in both matrices. The final weight vector ought to show an all-encompassing perspective that considers the feedback from both decision-makers. To determine appropriate and consistent weights for each criterion, the calculation makes use of a technique like the Analytic Hierarchy Process (AHP), which uses pairwise comparison matrices.

After aggregation of the above two matrices, the resulting matrix $\widetilde{M'}$ is provided in Table 4.3:

Table 4.3: Aggregation of matrices given in table 4.1 and 4.2

$\widetilde{M'}$	C1	C2	C3	C4	C5	C6
C1	(1,1,1)	(8,9,10)	(4,6.7,10)	(6,7.9,10)	(2,5.2,10)	(2,3,4)
C2	(0,1,2)	(1,1,1)	(2,4.6,8)	(4,5,6)	(2,3.9,6)	(2,5.2,10)
C3	(0,2.2,6)	(2,4.6,8)	(1,1,1)	(4,6.7,10)	(2,4.6,8)	(2,5.2,10)
C4	(0,1.7,4)	(4,5,6)	(0,2.2,6)	(1,1,1)	(2,3.9,6)	(4,5.9,8)
C5	(0,2.6,8)	(4,5.9,8)	(2,4.6,8)	(4,5.9,8)	(1,1,1)	(4,5.9,8)

C6	(6,7,8)	(0,2.6,8)	(0,2.6,8)	(2,3.9,6)	(2,3.9,6)	(1,1,1)

- ***Step 4***

Specifically, we want to find the weight vector W in a mathematical sense so that:

An array of weights, $W = (w1, w2, w3, w4, w5, w6)$, exists. By using formula eq (4.6) - (4.10):

$$W = (0.196, 0.152, 0.171, 0.15, 0.173, 0.156) \tag{4.11}$$

Now any number of criteria can be selected based on these weight factors.

Phase 2: Green light period calculation: Let the three criteria that are chosen according to weight factors are weather factors, vehicle density, and waiting time in our fuzzy logic-based traffic light control. Weather conditions can significantly affect traffic flow, and integrating them into the control system can further enhance its adaptability.

- ***Step 1***

We continue to collect real-time data from sensors at the intersection, including vehicle density (V) and waiting time (W). Additionally, we now collect weather-related data, such as rain intensity (R) in millimeters per hour. For this example, let's assume that vehicle density is 40 vehicles/minute, waiting time is 60 seconds, and rain intensity is 5 mm/hour. We have the following data:

- ***Step 2***

We define fuzzy numbers for linguistic variables for vehicle density (V), waiting time (W), and rain intensity (R) through Table 4.4:

Table 4.4: Data used for analysis according to DM

Criteria	Fuzzy number	Linguistic variable
Vehicle Density (vehicles/minute)	0,15,30 20,35,50 40,55	Low(L) Medium(M) High(H)
Waiting time (second)	0,15,30 20,40,60 50,70	Short(S) Medium(M) Long(L)
Rain Intensity(mm/hr)	0,2.5,5 4,7,10 9,12	Light(L) Moderate(M) Heavy(H)
	0,15,30	Small

Green light period(seconds)	25,40,55 50,65	Medium High

Figure 4.4 and Table 4.5, respectively, display the geometrical representation and mathematical formulation of the given criteria:

Table 4.5: Linear membership functions' formulation for all criteria

Criteria	Level	Membership Function
Vehicle density(vehicle/minute)	Low(L)	$\acute{m}_{VL}(x) = \begin{cases} 0 & if\ x \geq 30 \\ \frac{(30-x)}{15} & if\ 15 < x < 30 \\ 1 & if\ x \leq 15 \end{cases}$
	Medium(M)	$\acute{m}_{VM}(x) = \begin{cases} \frac{(x-20)}{15} & if\ 20 < x \leq 35 \\ \frac{(50-x)}{15} & if\ 35 < x \leq 50 \\ 0 & otherwise \end{cases}$
	High(H)	$\acute{m}_{VH}(x) = \begin{cases} 0 & if\ x \leq 40 \\ \frac{(x-40)}{15} & if\ 40 < x < 55 \\ 1 & if\ x \geq 55 \end{cases}$
Waiting time (second)	Short(S)	$\acute{m}_{WS}(x) = \begin{cases} 0 & if\ x \geq 30 \\ \frac{(30-x)}{15} & if\ 15 < x < 30 \\ 1 & if\ x \leq 15 \end{cases}$
	Medium(M)	$\acute{m}_{WM}(x) = \begin{cases} \frac{(x-20)}{20} & if\ 20 < x \leq 40 \\ \frac{(60-x)}{20} & if\ 40 < x \leq 60 \\ 0 & otherwise \end{cases}$
	Long(L)	$\acute{m}_{WL}(x) = \begin{cases} 0 & if\ x \leq 50 \\ \frac{(x-50)}{20} & if\ 50 < x < 70 \\ 1 & if\ x \geq 70 \end{cases}$
Rain intensity(mm/hour)	Light(L)	$\acute{m}_{RL}(x) = \begin{cases} 0 & if\ x \geq 5 \\ \frac{(5-x)}{2.5} & if\ 2.5 < x < 5 \\ 1 & if\ x \leq 2.5 \end{cases}$
	Moderate(M)	$\acute{m}_{RM}(x) = \begin{cases} \frac{(x-4)}{3} & if\ 4 < x \leq 7 \\ \frac{(10-x)}{3} & if\ 7 < x \leq 10 \\ 0 & otherwise \end{cases}$
	Heavy(H)	$\acute{m}_{RLO}(x) = \begin{cases} 0 & if\ x \leq 9 \\ \frac{(x-9)}{3} & if\ 9 < x < 12 \\ 1 & if\ x \geq 12 \end{cases}$

Green light period(G)	Small(S) Medium(M) High(H)	$m_{GS}(x) = \begin{cases} 0 & if\ x \geq 30 \\ \frac{(30-x)}{15} & if\ 15 < x < 30 \\ 1 & if\ x \leq 15 \end{cases}$ $\acute{m}_{GM}(x) = \begin{cases} \frac{(x-25)}{15} & if\ 25 < x \leq 40 \\ \frac{(55-x)}{15} & if\ 40 < x \leq 55 \\ 0 & otherwise \end{cases}$ $\acute{m}_{GH}(x) = \begin{cases} 0 & if\ x \leq 50 \\ \frac{(x-50)}{15} & if\ 50 < x < 65 \\ 1 & if\ x \geq 65 \end{cases}$

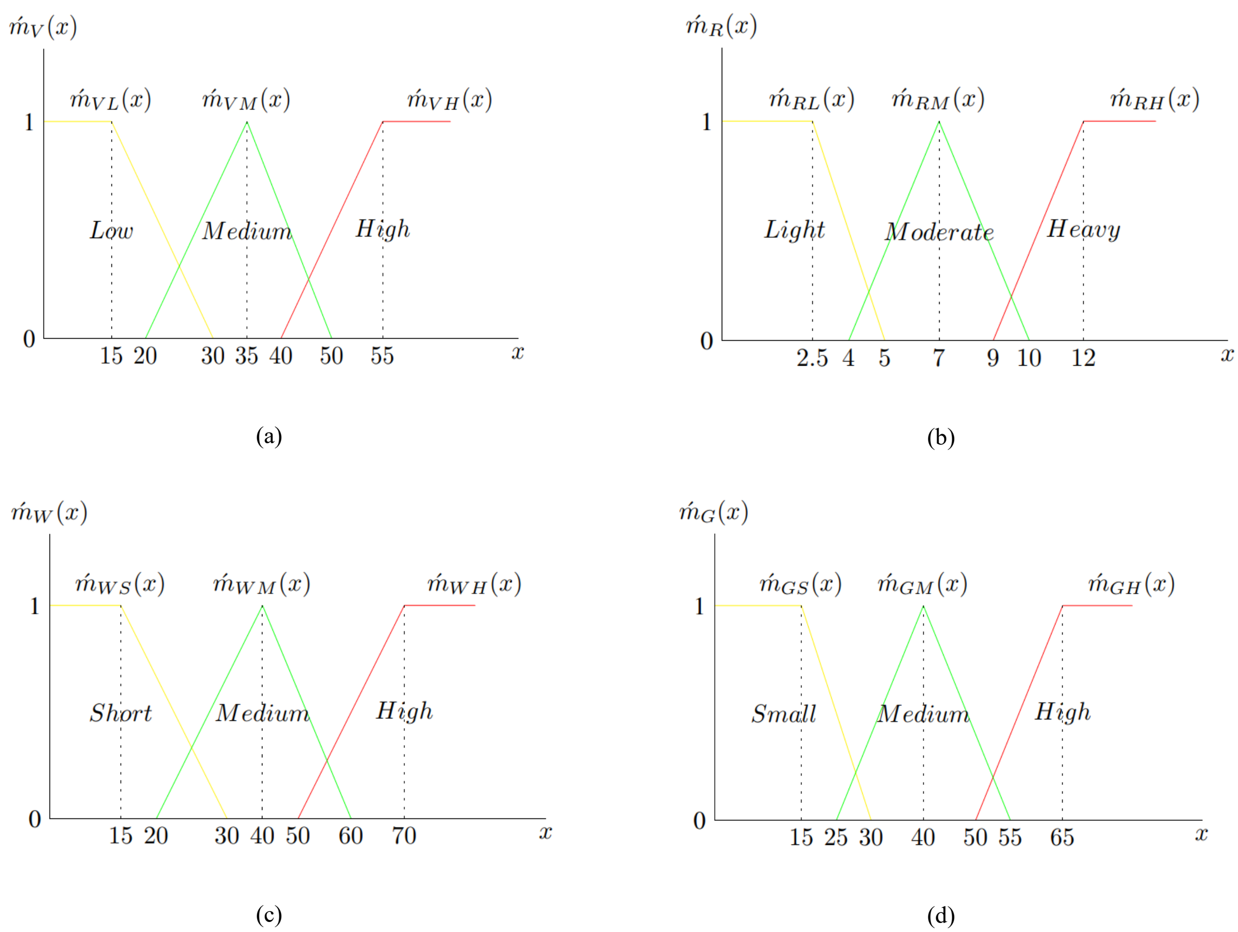


Figure 4.5: Membership assignment for various functions (a) Vehicle density (b) Rain intensity (c) Waiting time (d) Green light period

- ***Step 3***

The decision maker can combine the three given criteria to create 27 fuzzy rules, as shown in Table 4.6 below:

Table 4.6: Fuzzy rules decided by decision maker

Green light period	**Cases (Vehicle density, waiting time, rain intensity)**
High	LLL, LLM, MLL, MLM, HSL, HSM, HML, HMM, HLL, HLM, HLH
Small	LSM, LSH, LMH, MSH, MMH,
Medium	LSL, LML, LMM, LLH, MSL, MSM, MML, MMM, MLH, HSH, HSM, HMH

- ***Step 4***

Here the data of all criteria are collected and aggregated according to their membership values to define the condition according to the fuzzy rules defined in step 3.

- ***Step 5***

We need to convert the aggregated result back into a crisp, non-fuzzy value to control the traffic lights. This process is called defuzzification. Common methods include the centroid area, the mean of α -cut, and the bounded area method, all of which we have discussed in Chapter 3. In our study, we will focus on the mean of α -cut.

4.2.2 Solution

The solution to the above issue will be broken into two problems. In the first problem, we must choose the required criteria, and in the second, we must calculate the green light period based on the selected criteria.

Phase 1: Let's consider an example where we have considered all six criteria, but we need to select only three criteria to simplify the calculation. Please identify these three criteria.

The criterion's weight preferences are listed as follows:

$C1 > C5 > C3 > C6 > C2 > C4$

Based on this order, we can choose any number of criteria according to our suitability. We prioritize the criteria with the highest weight parameters. So, the preferred three criteria are:

$C1, C5, C3$.

Phase 2: After the selection of criteria, the second phase starts for the selection of timing for the green light period.

- ***Step 1***

Let us take an example of a data set for which, at a given time, vehicle density is 22, waiting time is 20 seconds, and rain intensity is 5 mm/hr. Next, we must determine the time of the green light signal.

- ***Step 2***

The procedure to define the fuzzy rules is included. Here, we have considered the rules described in Table 4.6.

- ***Step 3***

At this stage, the value of membership functions associated with the given quantities of criteria is calculated in Table 4.7 according to the defined membership functions in Table 4.5.

Table 4.7: Membership functions defined for different criterias

Criteria	Membership function's value
Membership functions satisfied by vehicle density (22)	$ḿ_{VL}(22) = \begin{cases} 0 & if\ x \geq 30 \\ \frac{(30-x)}{15} & if\ 15 < x < 30 \\ 1 & if\ x \leq 15 \end{cases} = 8/15$ $ḿ_{VM}(22) = \begin{cases} \frac{(x-20)}{15} & if\ 20 < x \leq 35 \\ \frac{(50-x)}{15} & if\ 35 < x \leq 50 \\ 0 & otherwise \end{cases} =2/15$
Membership functions satisfied by waiting time (20)	$ḿ_{WS}(20) = \begin{cases} 0 & if\ x \geq 30 \\ \frac{(30-x)}{15} & if\ 15 < x < 30 \\ 1 & if\ x \leq 15 \end{cases} = 2/3$ $ḿ_{WM}(20) = \begin{cases} \frac{(x-20)}{20} & if\ 20 < x \leq 40 \\ \frac{(60-x)}{20} & if\ 40 < x \leq 60 \\ 0 & otherwise \end{cases} = 0$
Membership functions satisfied by rain intensity (5)	$ḿ_{RL}(5) = \begin{cases} 0 & if\ x \geq 5 \\ \frac{(5-x)}{2.5} & if\ 2.5 < x < 5 \\ 1 & if\ x \leq 2.5 \end{cases} = 0$ $ḿ_{RM}(5) = \begin{cases} \frac{(x-4)}{3} & if\ 4 < x \leq 7 \\ \frac{(10-x)}{3} & if\ 7 < x \leq 10 \\ 0 & if\ x \geq 10 \end{cases} = 1/3$

- ***Step 4***

Total resulted combinations: LSL, LSM, LML, LMM, MSL, MSM, MML, MMM.

- ***Step 5***

By applying the mean of the α-cut method of defuzzification for all the eight combinations:

$$\left.\begin{aligned}
LSL&: \left(\frac{8}{15},\frac{2}{3},0\right) = 0.466\\
LSM&: \left(\frac{8}{15},\frac{2}{3},\frac{1}{3}\right) = 0.55\\
LML&: \left(\frac{8}{15},0,0\right) = 0.133\\
LMM&: \left(\frac{8}{15},0,\frac{1}{3}\right) = 0.2167\\
MSL&: \left(\frac{2}{15},\frac{2}{3},0\right) = 0.2\\
MSM&: \left(\frac{2}{15},\frac{2}{3},\frac{1}{3}\right) = 0.283\\
MML&: \left(\frac{2}{15},0,0\right) = 0.033\\
MMM&: \left(\frac{2}{15},0,\frac{1}{3}\right) = 0.1167
\end{aligned}\right\} \tag{4.12}$$

4.2.3 Results

Now, the maximum of all these values is 0.55, which aligns with the LSM condition and, based on fuzzy rules, indicates a brief period of green light.

After applying the membership functions for a small portion of the green light period along the membership degree 0.55, the crisp values are as follows: $\frac{(30-x)}{15} = 0.55 => x = 21.75$

So, the time for the green light signal will be 21.75 seconds.

4.3 Conventional Fuzzy Min-Max Approach for Multi-Objective Linear Programming Problems

The optimal solution x' for the standard MOLPP, as determined by equations (1.25) - (1.27), exists only for any other value of x:

$$f_i(x') \geq f_i(x), \ \forall\, i = 1, \dots l \ or \quad f_i(x') > f_i(x), \ \text{ with a minimum of } one\ i\ \epsilon\{1, \dots l\} \tag{4.13}$$

According to this approach, once we have determined the extremes for each objective activity, we can use a distance minimization method based on two hyperplanes. To develop participation measures, we can approximate these distances using fuzzy parameters. This rule states that increasing the value of the fuzzy parameter will result in a decrease in the perpendicular distances.

As the values for the objective parameters converge on the optimal value, we find a locally optimal solution.

In this context, we assume the set of constraints for the original MOLPP to be a convex closed set. Therefore, the distance function (measured in units of x, separation from the hyperplane, denoted by $(\overline{f}_i - f_i(x))$) is defined as follows:

$$Ð_i(x) = |\overline{f}_i - f_i(x)| \tag{4.14}$$

If we are given some constraints, we may determine the maximum value of a single objective using the notation $\overline{f}_i$ = maximum value of $f_i(x)$. The formula for the optimal possible $f_i(x)$ can be found in terms of the distance parameter:

$$\overline{f}_i - f_i(x) = 0 \tag{4.15}$$

The functional value that characterizes the maximum separation of any two hyperplanes can be expressed as follows:

$$Max\ Ð_i(x) = \overline{f}_i - f_i^n,\ \forall\ i = 1,2,\ldots l \tag{4.16}$$

Here f_i^n is the nadir point of the n^{th} function. Let $\bar{Ð}_i = \{(Max\ Ð_i(x))\ \forall\ i = 1,2,\ldots l\}$. The next step is to figure out the best way to set up a distance membership function:

$$ḿ(Ð_i(x)) = \begin{cases} 0 & if\ Ð_i(x) \geq \bar{Ð}_i \\ \dfrac{(\bar{Ð}_i - Ð_i(x))}{\bar{Ð}_i} & if\ 0 < Ð_i(x) < \bar{Ð}_i \\ 1 & if\ Ð_i(x) \leq 0 \end{cases} \tag{4.17}$$

Our participation functions' connection to one another is represented by a new kind of characteristic. For inclusion of all membership functions, the minimum operator is defined as follows:

$$\gamma \leq \frac{(\bar{Ð}_i - Ð_i(x))}{\bar{Ð}_i} \tag{4.18}$$

Now our aim is to maximize the association function's value to minimize the distance. So, the situation can be boiled down to a single objective.

$$
\left.\begin{array}{c}
\text{Max } \gamma \\
\text{Subjected to: } -f_i(x) + Ð_i\gamma \leq Ð_i - \overline{f}_i \\
c_k^j x_k \leq d^j \\
x_k \geq 0
\end{array}\right\} \tag{4.19}
$$

That is amenable to the usual optimization tools and methods for solving.

4.3.1 Sustainability in manufacturing

This section provides a specific aspect of the model formulation for sustainable manufacturing and inventory planning [150]. First, the basic ideas behind the framework's creation and the mathematical models used are explained. Next, we break down the decision variables. Next to this, the sustainability goals that need to be improved are listed, and lastly, the necessary limits are laid out.

a) Assumptions:

- There is just one cycle time for the product.
- After clearing the backlog, the demand is fully satisfied.
- Price changes do not affect demand.
- Stock is routinely replenished.
- It's possible to have shortages approved.
- The horizon (in terms of time) is infinite.
- The cost of stocking supplies and their acquisition price are both known and stable in theory.
- There are no quantity discounts offered.
- We only keep a single supply of each item in stock.
- Prevention of pollution and safe disposal are essential tenets of the waste management industry.
- We are reducing harmful emissions and adjusting inventories and output accordingly.
- We measure the potential cost of pollution as a percentage of production.
- Acceptance of a broken manufacturing element.

Symbol	Meaning
D_R	Average yearly demand
P_S	Selling price per item

α	Factor of incorrect product production
t_i	Inventory time
t_s	Shortage time
Q	Output Quantity per cycle
Q_S	Shortage quantity per cycle
C_P	Cost of manufacturing a single unit
C_S	Cost of setup on a per-item basis
C_i	Cost of maintaining inventory per quantity held and per cycle
C_{short}	Shortage cost per unit quantity per cycle
Ψ	Unit cost of waste disposed to the environment per item
C_{ie}	Industrial emission cost associated per item per cycle
C_{ec}	Electricity consumption cost associated per item per cycle
C_{ve}	Vehicle emission cost associated per item per cycle
C_{Ψ}	Total waste penalty cost available per production
C_{pc}	Total pollution control cost available per production
r	Space required per item
R	Total space available per cycle
B_T	Total inventory budget available per cycle
T_t	Total time of one cycle

b) Decision variables:

To optimize our objective of maintaining a healthy ecosystem while meeting the needs of our customers and cutting down on waste, we must calculate the current output and backlog of our operations. In this case, the deciding factors are treated as whole numbers.

c) Objective Functions:

This part outlines three equations of objectives for optimization. The initial goal is to maximize the inventory system's profit for each unit of time.

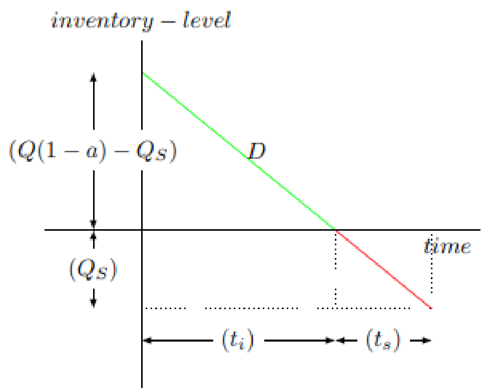


Figure 4.6: Inventory level model for one cycle

The inventory system's second goal is to minimize waste generation throughout each cycle, while the third goal aims to minimize potential financial losses due to penalties during each cycle. The inventory model, which includes destruction and shortage factors, is provided in Figure 4.6. Here are the corresponding mathematical equations for each goal:

Profit
$$Max\ P = \frac{P_S Q(1-\alpha)}{T_t} - \frac{C_P Q}{T_t} - \frac{C_S D_R}{(1-\alpha)Q} - \frac{C_i(Q(1-\alpha)-Q_S)^2}{2Q(1-\alpha)} - \frac{C_{short} {Q_S}^2}{2Q(1-\alpha)} \quad (4.20)$$

Waste
$$Min\ W = Q\Psi\alpha \quad (4.21)$$

Penalty
$$Min\ E = Q(C_{ie} + C_{ec}) + C_{ve}Q(1-\alpha) \quad (4.22)$$

Restrictions
$$\frac{Q(1-\alpha)-Q_S}{t_i} = \frac{Q_S}{t_s} = \frac{Q(1-\alpha)}{(t_i+t_s)} = D_R \quad (4.23)$$

d) Constraints associated:

This part outlines the numerous constraints or limitations required to accomplish multi-objective optimization. There are a total of five equations there. Regardless of outcomes, eq (4.24) sets the initial investment as the limit for the right-hand side of the equation. As shown in eq (4.25), the storage facility has a limited capacity and can only store a specific number of units at any given time. Eq (4.26) determines the maximum allowable level of environmental waste disposal in the inventory system. To keep the cost of pollution control as low as possible throughout inventory manufacturing, eq (4.27) makes some conservative assumptions. Alternatively, the model prohibits negative quantities and ordering costs, citing eq (4.28). The following mathematical form illustrates the limitations:

Total budget $$C_P Q - \frac{C_S D}{(1-\alpha)Q} - \frac{C_i(Q(1-\alpha)-Q_S)^2}{2Q(1-\alpha)} - \frac{C_{short}{Q_S}^2}{2Q(1-\alpha)} \leq B_T \quad (4.24)$$

Space capacity $$rQ(1-\alpha) \leq R \quad (4.25)$$

Waste restriction $$Q\Psi\alpha \leq C_\Psi \quad (4.26)$$

Pollution cost $$Q(C_{ie} + C_{ec}) + C_{ve}Q(1-\alpha) \leq C_{pc} \quad (4.27)$$

Non-negativity $$Q, Q_S, P, W, E \geq 0 \quad (4.28)$$

4.3.1.1 Numerical experiment

Here, we have used industrial data provided in Table 4.8 to simulate realistic conditions, which is taken from a secondary source [151]. The experiment will show all the procedures, from manufacturing to selling, followed by any firm to maximize its profit while minimizing penalty costs.

Table 4.8: Values of parameters according to case study

Parameter		**Value**
Demand rate per unit time (year)	$\tilde{D}_R$	496.66
Unit selling price	$\tilde{P}_S$	2408.33
Production factor of defective product per cycle	$\tilde{\alpha}$	0.15
Production cost per unit item	$\tilde{C}_P$	1091.66
Setup cost per cycle	$\tilde{C}_S$	38.83
Inventory holding cost per unit quantity per cycle	$\tilde{C}_i$	26.33
Shortage cost per unit quantity per cycle	$\tilde{C}_{short}$	32.16
Unit cost of waste disposed to the environment per item	$\tilde{\Psi}$	445
Unit cost of industrial emissions associated per cycle	$\tilde{C}_{ie}$	129.33
Unit cost of electricity consumption associated per cycle	$\tilde{C}_{ec}$	109.16
Unit cost of vehicle emissions associated per cycle	$\tilde{C}_{ve}$	124.16
Total waste penalty cost available per production	$\tilde{C}_\Psi$	5450
Total pollution control cost available per production	$\tilde{C}_{pc}$	42166.66
Space required per item	r	2
Total space available per cycle	R	42
Total inventory budget available per unit time per cycle	$\tilde{B}_T$	1000000

4.3.1.3 Formulation of MOLPP:

With the aid of the aforementioned objective functions and constraints, we formulate the MOLPP problem as follows:

$$\left.\begin{array}{c} Max\ P = 558257.82 - \dfrac{22688.59}{Q} - \dfrac{15.49\left(0.85Q - Q_S\right)^2}{Q} - \dfrac{18.92{Q_S}^2}{Q} \\ Min\ \ W = 66.75Q \\ Min\ E = 343.696Q \\ Sub.to: 637863.36 + \dfrac{22688.59}{Q} + \dfrac{15.49\left(0.85Q - Q_S\right)^2}{Q} + \dfrac{18.92{Q_S}^2}{Q} \leq 1000000 \\ 1.7Q \leq 42 \\ 66.75Q \leq 5450 \\ 343.696Q \leq 42166.66 \\ \dfrac{Q(1-\alpha)}{\tilde{T}_t} = D_R \\ Q, Q_S, \tilde{T}_t, P, W, E \geq 0 \end{array}\right\} \quad (4.29)$$

4.3.1.4 Solution by proposed approach:

LINGO 18.0 × 64 software calculates the best possible objective function values given the constraints you provide.

Max $P = \bar{P} = 557164.7$ at point (24,9)

Min $W = \bar{W} = 66.75$ at point (1,1)

Min $E = \bar{E} = 343.696$ at point (1,1)

So, the nadir points for all objectives are:

$P^n = 535549.96$ at $(1,1)$, $W^n = 1602$ at $(24,9)$, $E^n = 8248.704$ at $(24,9)$

Now, the model-predicted distance functions are as follows:

$$Đ_1 = |\bar{P} - P| = 557164.7 - \left(558257.82 - \frac{22688.59}{Q} - \frac{15.49(0.85Q - Q_S)^2}{Q} - \frac{18.92{Q_S}^2}{Q}\right) \quad (4.30)$$

$$Đ_2 = |\bar{W} - \mathrm{W}| = 66.75 - 66.75Q \quad (4.31)$$

$$Đ_3 = |\bar{E} - \mathrm{E}| = 343.696 - 343.696Q \quad (4.32)$$

Maximum separations for two objectives can be expressed as:

$$Max\ Đ_1 = \quad |\bar{P} - \quad P^n| = \quad 21614.74 \tag{4.33}$$

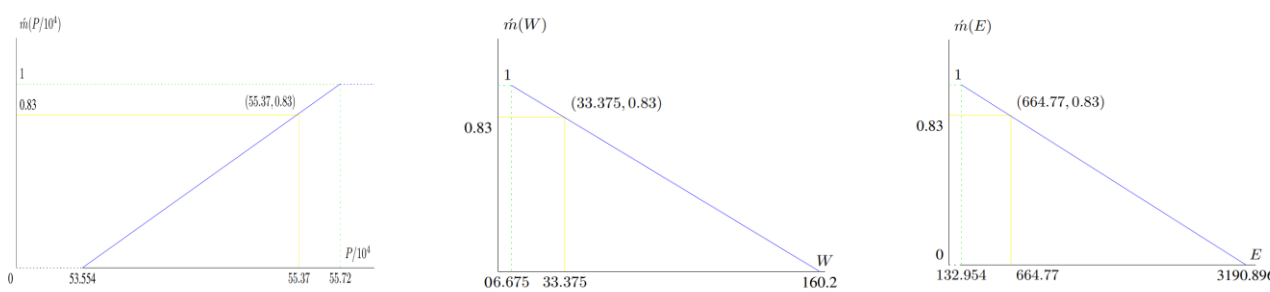
$$Max\ Đ_2 = \quad |\bar{W} - \quad W^n| = \quad 1535.25 \tag{4.34}$$

$$Max\ Đ_3 = \quad |\bar{E} - \quad E^n| = \quad 7905.008 \tag{4.35}$$

The resulting single objective optimization problem will be:

$$\left.\begin{array}{c} Max\ \gamma \\ Sub.\,to\!: 21614.74\gamma \leq 20521.62 - \dfrac{22688.59}{Q} - \dfrac{15.49\left(0.85Q - Q_S\right)^2}{Q} - \dfrac{18.92{Q_S}^2}{Q} \\ -1535.25\gamma \leq -1602 + 66.75Q \\ -7905.008\gamma \leq -7905.008 + 343.696Q \\ 1.7Q \leq 42 \\ 66.75Q \leq 5450 \\ 343.696Q \leq 42166.66 \\ Q, Q_S \geq 0 \\ 0 \leq \gamma \leq 1 \end{array}\right\} \tag{4.36}$$

The membership functions of the objective show linear behaviour from their nadir point to the maximum point as given in Figure 4.7. The following graph of objective function values and satisfaction levels of membership grade:



(a) (b) (c)

Figure 4.7: Membership grade's graph for all functions (a) Profit function (b) Waste function (c) Penalty function

4.3.1.5 Results

We use LINGO 18.0 × 64 software to achieve the outcomes. The value of the satisfaction level is 0.8260850. By plugging in values for the choice variables and using the provided equality constraints, the model determines the optimal solutions provided in Table 4.9.

Table 4.9: Optimum values for resulted variables by proposed model

Decision variable	Optimum value
Q	5
Q_S	1
T_t	0.01007
t_i	0.00805655
t_s	0.002013
γ	0.8260850
P	553683.59
W	333.75
E	1718.48

4.4 Conventional Fuzzy Approach with Non-linear Membership Functions

The approach will be the same as described in Section 4.3, besides the nature of membership functions. Chapter 1 describes the non-linear type of the membership function [152]. For the distance parameter, these membership functions act like the one given below:

Hyperbolic Membership function: Eq (1.5) assists to define the hyperbolic membership function associated with the function in terms of the distance parameter. In particular, the hyperbolic level of membership can be characterised as follows:

$$\acute{m}(Đ_i(x)) = \begin{cases} 1 & if\ Đ_i(x) \leq 0 \\ (\frac{1}{2}(tanh\ ((-Đ_i(x) + \frac{\overline{Đ_i(x)}}{2})\delta) + \frac{1}{2}) & if 0 < Đ_i(x) < \bar{Đ}_i \\ 0 & if\ Đ_i(x) \geq \bar{Đ}_i \end{cases} \quad (4.37)$$

Here, $\delta = |\frac{6}{\bar{f}_i(x)+f_i^n(x)}|$.

Parabolic membership function: Eq (1.4) assists in defining the parabolic membership function associated with the function in terms of the distance parameter. In particular, the parabolic level of membership can be characterised as follows:

$$ḿ(Đ_i(x)) = \begin{cases} 1 & if\, Đ_i(x) \le 0 \\ \frac{(\bar{Đ}_i - Đ_i(x))^2}{(\bar{Đ}_i)^2} & if\ 0 < Đ_i(x) < \bar{Đ}_i \\ 0 & if\ Đ_i(x) \ge \bar{Đ}_i \end{cases} \tag{4.38}$$

Exponential membership function: Eq (1.6) assists in defining the exponential membership function associated with the function in terms of the distance function. Each objective goal has been associated with the following exponential membership function [22]:

$$ḿ(Đ_i(x)) = \begin{cases} 1 & if\, Đ_i(x) \le 0 \\ \rho_i[1 - exp\left\{-\eta_i \frac{|\bar{Đ}_i - Đ_i(x)|}{(\bar{Đ}_i)}\right\}] & if\ 0 < Đ_i(x) < \bar{Đ}_i \\ 1 & if\ Đ_i(x) \ge \bar{Đ}_i \end{cases} \tag{4.39}$$

In this case, ρ and η must be positive or they must be negative.

Sigmoidal membership function: Eq (1.7) assists in defining the sigmoidal membership function associated with the function in terms of the distance parameter. In this chapter, we formally define the S-curve membership function [153]:

$$ḿ(Đ_i(x)) = \begin{cases} 1 & if\, Đ_i(x) \le 0 \\ 1 - \left(\frac{1}{1 + Be^{\alpha\left(\frac{\bar{Đ}_i - Đ_i(x)}{(\bar{Đ}_i)}\right)}}\right) & if\ 0 < Đ_i(x) < \bar{Đ}_i \\ 0 & if\ Đ_i(x) \ge \bar{Đ}_i \end{cases} \tag{4.40}$$

In eq (4.34), the symbol $ḿ(Đ_i(x))$ denotes the participation value forms for which $\alpha > 0$. When $\alpha \to \propto$, ambiguity grows, but when $\alpha = 0$, precision is shown. Experts using a combination of hypotheses and trials must determine the value of the parameter. One advantage of the revised S-curve is that its membership function can be tailored to the data at hand.To simplify, we'll set α = 13.813, B = 0.001001001 for this study on the basis of the results of existing studies.

4.4.1 Finantial Situation

The data is taken from the secondary source [80]. A company is considering participating in three projects aimed at enhancing the quality of life in their city, with the intention of creating investment plans spanning a period of four years. Right now, the company has $4 million to invest in projects.

Table 4.11 illustrates the projected return on financing, expressed in millions of dollars, for the company's financing over a four-year period. Table 4.11 also displays cash flow streams for these three projects, assuming full participation. As time goes on, the accuracy of the earnings from earlier investments may decrease. Interest earnings from investment accounts, for instance, fluctuate with economic conditions. Therefore, these fluctuations take the form of fuzzy triangular numbers. The uncertain range has a triangular probabilistic model (4, 4.5, and 5.7) with the most likely result set at 4.5, the foremost defeatist at 4, and the highest idealistic at 5.7. The company will take over the management of some older, middle-class housing as part of the second project, with the stipulation that it will be burned down after four years unless certain initial improvements are made. The business has the option to obtain one-year loans with variable interest rates ranging from (5,6,6.7), (5,6,7), (4.5,5.8,6.5), (5.5,6.2,7.2) percent for each of the first four years. At any given time, the maximum loan amount is $2,000,000, and the total principal due cannot exceed $4,000,000. You can invest excess cash at various annual rates ranging from (3,3.7,4), (3.5,4.5,5.2), (4,4.8,5.5), and (4.5,5,6) percent annually. Consider the question of how to maximize the firm's net worth after four years. Ignore taxes and assume that the firm's less-than-full ownership of a project will have a proportional impact on all of the project's cash outflows. Next, we can model this issue as a linear program. To accomplish this, Table 4.10 talks about the assumed symbols.

Table 4.10: Symbols used in mathematical formulation of financial situation

A	Part of contribution in 1st project
B	Part of contribution in 2nd project
C	Part of contribution in 3rd project
D_k	borrowing costs for time periods k = 1,2,3,4
L_i	money lend in year i= 1,2,3,4
W	total wealth after four phases

a) *Objective function:*

To maximize the total wealth:

Wealth
$$Max\ W \approx (4.5,5.2,6.7)A + (-2,-1.8,-1.2)B + (5.7,6.1,6.5)C - (1.055,1.062,1.072)D_4 + (1.045,1.05,1.06)L_4 \tag{4.41}$$

c) Constraints associated:

- First investment can never be greater than \$4 million, and interest will not be considered with borrowed money.

- Investment for every period cannot be greater than its income stream.
- Company can only take out a loan and invest in a surplus fund of up to \$2,000,000 at any given time.

Sub. to.
$$\left.\begin{array}{c} 2.5A + 2.5B + 2C - D_1 + L_1 \leq 4 \\ 0.8A - 0.6B + 1.9C + (1.05,1.06,1.067)D_1 - (1.03,1.037,1.04)L_1 - D_2 + L_2 \lesssim (0.34, 0.39, 0.42) \\ -A - 0.5B - C + (1.05,1.06,1.07)D_2 - (1.035,1.045,1.052)L_2 - D_3 + L_3 \lesssim (0.3, 0.36, 0.47) \\ -1.4A - 1.8B - 2C + (1.045,1.058,1.065)D_3 - (1.04,1.048,1.055)L_3 - D_4 + L_4 \lesssim (0.32, 0.35, 0.4) \\ A, B, C \in [0,1], D_k, L_i \in [0,2] \end{array}\right\} \quad (4.42)$$

4.4.1.1 Numerical Data:

The numerical data for the cash flow in three years is given here in Table 4.11.

Table 4.11: The Financing Situation: Managing Projects Over Time (in million dollars)

Period (year)	**0**	**1**	**2**	**3**	**4**
Income stream	4	(0.34, 0.39, 0.42)	(0.3, 0.36, 0.47)	(0.32, 0.35, 0.4)	(0.35, 0.37, 0.39)
Project 1	-2.5	-0.8	1	1.4	(4.5,5.2,6.7)
Project 2	-2.5	0.6	0.5	1.8	(-2,-1.8,-1.2)
Project 3	-2.0	-1.9	1	2	(5.7,6.1,6.5)

4.4.1.2 Formulation of MOLPP:

Using the defuzzification method (lexicographic order relation) [81], [82] for the fuzzy objective function and using the ranking method [41],[154] for the defuzzification of constraints, the above problem given by eq (4.42) is converted into the following classical multi-objective optimization problem:

$$\left.\begin{array}{c} max\ \ (5.2A\ -1.8B\ +6.1C\ -1.062\,D_4+\ 1.05L_4) \\ max\ -(2.2A\ +\ 0.8B\ +0.8C\ +0.017D_4\ +\ 0.015L_4) \\ max\,(11.2A\ -3.2B\ +\ 12.2C\ +\ 2.127D_4\ +\ 2.105L_4) \\ 2.5A\ +2.5B\ +2C\ -D_1\ +\ L_1\ \leq\ 4 \\ 0.8A\ -0.6B\ +1.9C\ +1.06D_1\ -1.037L_1\ -D_2\ +L_2\ \leq\ 0.39 \\ 0.8A\ -0.6B\ +1.9C\ +1.055D_1\ -1.0335L_1\ -D_2\ +L_2\ \leq\ 0.365 \\ 0.8A\ -0.6B\ +1.9C\ +1.0635D_1\ -1.0385L_1\ -D_2\ +L_2\ \leq\ 0.405 \\ -1A\ -0.5B\ -\ C\ +\ 1.06\,D_2\ -\ 1.045L_2\ -D_3\ +L_3\ \leq\ 0.36 \\ -1A\ -0.5B\ -\ C\ +\ 1.055\,D_2\ -\ 1.04L_2\ -D_3\ +L_3\ \leq\ 0.33 \\ -1A\ -0.5B\ -\ C\ +\ 1.065\,D_2\ -\ 1.0485L_2\ -D_3\ +L_3\ \leq\ 0.415 \\ -1.4A\ -\ 1.8B\ -2C\ +\ 1.058D_3\ -1.048L_3\ -D_4\ +L_4\ \leq\ 0.35 \\ -1.4A\ -\ 1.8B\ -2C\ +\ 1.0515D_3\ -1.044L_3\ -D_4\ +L_4\ \leq\ 0.335 \\ -1.4A\ -\ 1.8B\ -2C\ +\ 1.0615D_3\ -1.0515L_3\ -D_4\ +L_4\ \leq\ 0.375 \\ A,B,C\,\epsilon\,[0,1]\,,D_k,L_i\epsilon\,[0,2] \end{array}\right\} \quad (4.43)$$

The optimal value of the objective functions subject to the specified restrictions is determined by the simplex method.

Max $f_1 = \overline{f}_1 = 12.0911$ and nadir point of $f_1 = f_1^n = -3.924$.

Max $f_2 = \overline{f}_2 = 3.2854$ and nadir point of $f_2 = f_2^n = 0$.

Max $f_3 = \overline{f}_3 = 24.9859$ and nadir point of $f_3 = f_3^n = -7.454$.

4.4.1.3 Solution by proposed approach

The membership functions with different natures for all functions move from their nadir point to the maximum point. The single-objective optimization problems after using the above method with different membership functions are provided below and shown in Figure 4.8 – Figure 4.12:

a) Linear Membership Function:

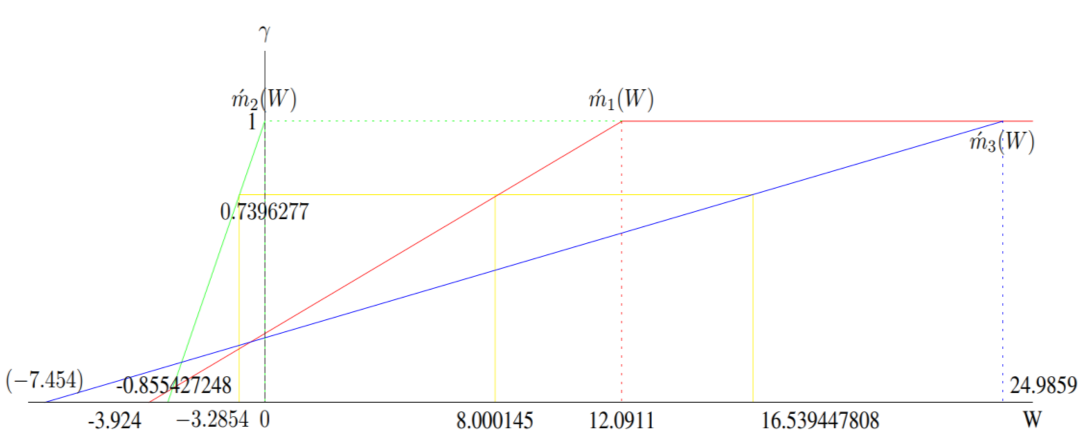


Figure 4.8: Linear membership degrees associated with wealth functions

$$
\left.\begin{array}{c}
max\ \gamma \\
sub.to: 16.0151\gamma - (5.2A - 1.8B + 6.1C - 1.062 D_4 + 1.05L_4) \leq 3.924 \\
3.2854\gamma + (2.2A + 0.8B + 0.8C + 0.017D_4 + 0.015L_4) \leq 3.2854 \\
32.4399\gamma - (11.2A - 3.2B + 12.2C + 2.127D_4 + 2.105L_4) \leq 7.454 \\
2.5A + 2.5B + 2C - D_1 + L_1 \leq 4 \\
0.8A - 0.6B + 1.9C + 1.06D_1 - 1.037L_1 - D_2 + L_2 \leq 0.39 \\
0.8A - 0.6B + 1.9C + 1.055D_1 - 1.0335L_1 - D_2 + L_2 \leq 0.365 \\
0.8A - 0.6B + 1.9C + 1.0635D_1 - 1.0385L_1 - D_2 + L_2 \leq 0.405 \\
-1A - 0.5B - C + 1.06 D_2 - 1.045L_2 - D_3 + L_3 \leq 0.36 \\
-1A - 0.5B - C + 1.055 D_2 - 1.04L_2 - D_3 + L_3 \leq 0.33 \\
-1A - 0.5B - C + 1.065 D_2 - 1.0485L_2 - D_3 + L_3 \leq 0.415 \\
-1.4A - 1.8B - 2C + 1.058D_3 - 1.048L_3 - D_4 + L_4 \leq 0.35 \\
-1.4A - 1.8B - 2C + 1.0515D_3 - 1.044L_3 - D_4 + L_4 \leq 0.335 \\
-1.4A - 1.8B - 2C + 1.0615D_3 - 1.0515L_3 - D_4 + L_4 \leq 0.375 \\
A, B, C \in [0,1], D_k, L_i \in [0,2], 0 \leq \gamma \leq 1
\end{array}\right\} \quad (4.44)
$$

b) Hyperbolic membership function:

$$
\left.\begin{array}{c}
max\ \gamma \\
Sub.to: \gamma \leq \left(\frac{1}{2}\right)\left(tanh\left(W - \frac{8.1671}{2}\right)\left(\frac{6}{8.1671}\right)\right) + \frac{1}{2} \\
\gamma \leq \left(\frac{1}{2}\right)\left(tanh\left(W - \frac{(-3.284)}{2}\right)\left(\frac{6}{-3.284}\right)\right) + \frac{1}{2} \\
\gamma \leq \left(\frac{1}{2}\right)\left(tanh\left(W - \frac{17.5319}{2}\right)\left(\frac{6}{17.5319}\right)\right) + \frac{1}{2} \\
2.5A + 2.5B + 2C - D_1 + L_1 \leq 4 \\
0.8A - 0.6B + 1.9C + 1.06D_1 - 1.037L_1 - D_2 + L_2 \leq 0.39 \\
0.8A - 0.6B + 1.9C + 1.055D_1 - 1.0335L_1 - D_2 + L_2 \leq 0.365 \\
0.8A - 0.6B + 1.9C + 1.0635D_1 - 1.0385L_1 - D_2 + L_2 \leq 0.405 \\
-1A - 0.5B - C + 1.06 D_2 - 1.045L_2 - D_3 + L_3 \leq 0.36 \\
-1A - 0.5B - C + 1.055 D_2 - 1.04L_2 - D_3 + L_3 \leq 0.33 \\
-1A - 0.5B - C + 1.065 D_2 - 1.0485L_2 - D_3 + L_3 \leq 0.415 \\
-1.4A - 1.8B - 2C + 1.058D_3 - 1.048L_3 - D_4 + L_4 \leq 0.35 \\
-1.4A - 1.8B - 2C + 1.0515D_3 - 1.044L_3 - D_4 + L_4 \leq 0.335 \\
-1.4A - 1.8B - 2C + 1.0615D_3 - 1.0515L_3 - D_4 + L_4 \leq 0.375 \\
A, B, C \in [0,1], D_k, L_i \in [0,2], 0 \leq \gamma \leq 1
\end{array}\right\} \quad (4.45)
$$

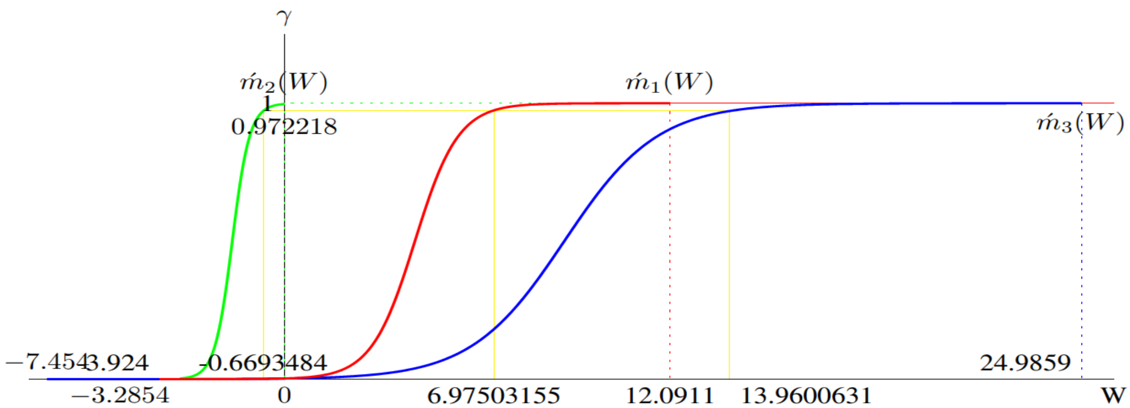


Figure 4.9: Hyperbolic membership degrees associated with wealth functions

c) Parabolic membership function:

$$\left.\begin{array}{c}
max\ \gamma \\
Sub.to: 256.48342801\gamma \leq (5.2A - 1.8B + 6.1C - 1.062D_4 + 1.05L_4 + 3.924)^2 \\
10.79385316\gamma \leq ((2.2A + 0.8B + 0.8C + 0.017D_4 + 0.015L_4) - 3.2854)^2 \\
1052.34711201\gamma \leq (11.2A - 3.2B + 12.2C - 2.127D_4 + 2.105L_4 + 7.454)^2 \\
2.5A + 2.5B + 2C - D_1 + L_1 \leq 4 \\
0.8A - 0.6B + 1.9C + 1.06D_1 - 1.037L_1 - D_2 + L_2 \leq 0.39 \\
0.8A - 0.6B + 1.9C + 1.055D_1 - 1.0335L_1 - D_2 + L_2 \leq 0.365 \\
0.8A - 0.6B + 1.9C + 1.0635D_1 - 1.0385L_1 - D_2 + L_2 \leq 0.405 \\
-1A - 0.5B - C + 1.06D_2 - 1.045L_2 - D_3 + L_3 \leq 0.36 \\
-1A - 0.5B - C + 1.055D_2 - 1.04L_2 - D_3 + L_3 \leq 0.33 \\
-1A - 0.5B - C + 1.065D_2 - 1.0485L_2 - D_3 + L_3 \leq 0.415 \\
-1.4A - 1.8B - 2C + 1.058D_3 - 1.048L_3 - D_4 + L_4 \leq 0.35 \\
-1.4A - 1.8B - 2C + 1.0515D_3 - 1.044L_3 - D_4 + L_4 \leq 0.335 \\
-1.4A - 1.8B - 2C + 1.0615D_3 - 1.0515L_3 - D_4 + L_4 \leq 0.375 \\
A, B, C \in [0,1], D_k, L_i \in [0,2], 0 \leq \gamma \leq 1
\end{array}\right\} \quad (4.46)$$

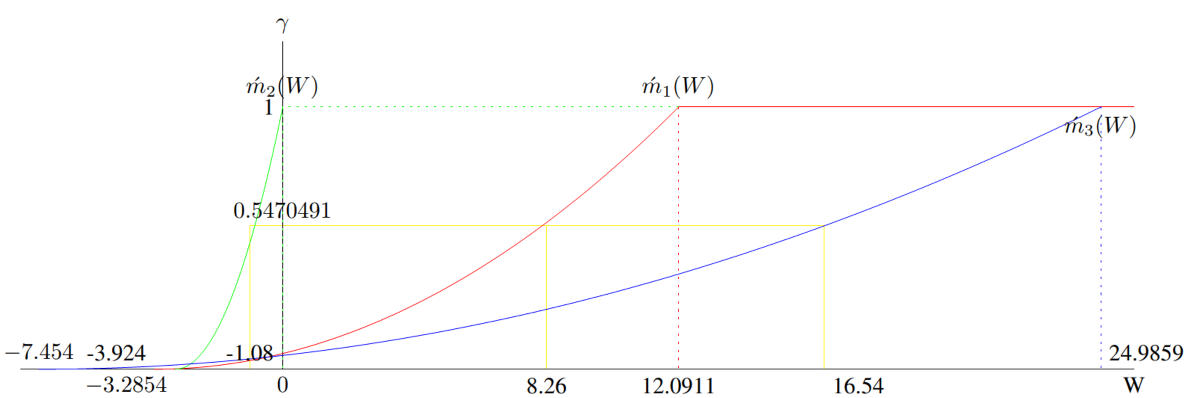


Figure 4.10: Parabolic membership degrees associated with wealth functions

d) *Exponential membership function:* (taking $\eta_q = 1.2, \rho_q = -ln\left(\frac{1}{6}\right)$)

$$
\left.
\begin{array}{c}
max\ \gamma \\
Sub.to: \gamma \leq (1.2)\left(1 - exp\frac{\left(ln\left(\frac{1}{6}\right)\left(-5.2A + 1.8B - 6.1C + 1.062D_4 - 1.05L_4 - 3.924\right)\right)}{16.0151}\right) \\
\gamma \leq (1.2)\left(1 - exp\frac{\left(ln\left(\frac{1}{6}\right)\left(2.2A + 0.8B + 0.8C + 0.017D_4 + 0.015L_4 - 3.2854\right)\right)}{3.2854}\right) \\
\gamma \leq (1.2)\left(1 - exp\frac{\left(-ln\left(\frac{1}{6}\right)\left(11.2A - 3.2B + 12.2C - 2.127D_4 + 2.105L_4 + 7.454\right)\right)}{32.4399}\right) \\
2.5A + 2.5B + 2C - D_1 + L_1 \leq 4 \\
0.8A - 0.6B + 1.9C + 1.06D_1 - 1.037L_1 - D_2 + L_2 \leq 0.39 \\
0.8A - 0.6B + 1.9C + 1.055D_1 - 1.0335L_1 - D_2 + L_2 \leq 0.365 \\
0.8A - 0.6B + 1.9C + 1.0635D_1 - 1.0385L_1 - D_2 + L_2 \leq 0.405 \\
-1A - 0.5B - C + 1.06D_2 - 1.045L_2 - D_3 + L_3 \leq 0.36 \\
-1A - 0.5B - C + 1.055D_2 - 1.04L_2 - D_3 + L_3 \leq 0.33 \\
-1A - 0.5B - C + 1.065D_2 - 1.0485L_2 - D_3 + L_3 \leq 0.415 \\
-1.4A - 1.8B - 2C + 1.058D_3 - 1.048L_3 - D_4 + L_4 \leq 0.35 \\
-1.4A - 1.8B - 2C + 1.0515D_3 - 1.044L_3 - D_4 + L_4 \leq 0.335 \\
-1.4A - 1.8B - 2C + 1.0615D_3 - 1.0515L_3 - D_4 + L_4 \leq 0.375 \\
A, B, C\ \epsilon\ [0,1], D_k, L_i \epsilon\ [0,2], 0 \leq \gamma \leq 1
\end{array}
\right\} \quad (4.47)
$$

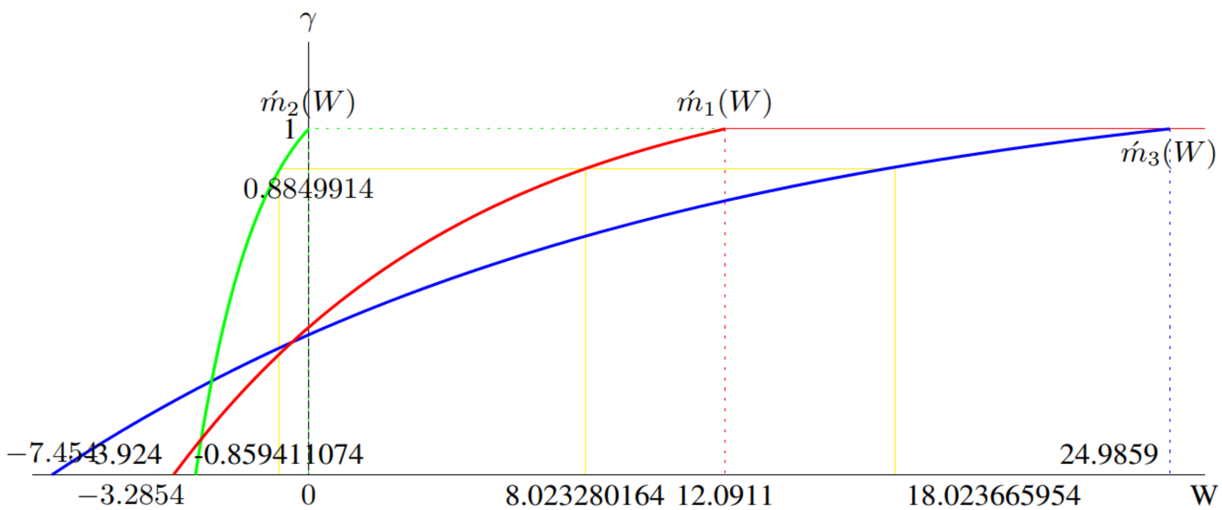


Figure 4.11: Exponential membership degrees associated with wealth functions

e) Sigmoidal membership function:

$$\left.\begin{array}{c} \max \gamma \\ Sub.to{:}\ \gamma \le 1 - \left(\frac{1}{1+(0.001001001)e^{0.843766196(5.2A-1.8B+6.1C-1.062D_4+1.05L_4+3.924)}}\right) \\ \gamma \le 1 - \left(\frac{1}{1+(0.001001001)e^{(-4.2043587)\left(\left((2.2A+0.8B+0.8C+0.017D_4+0.015L_4)-3.2854\right)\right)}}\right) \\ \gamma \le 1 - \left(\frac{1}{1+(0.001001001)e^{(0.425803)(11.2A-3.2B+12.2C-2.127D_4+2.105L_4+7.454)}}\right) \\ 2.5A + 2.5B + 2C - D_1 + L_1 \le 4 \\ 0.8A - 0.6B + 1.9C + 1.06D_1 - 1.037L_1 - D_2 + L_2 \le 0.39 \\ 0.8A - 0.6B + 1.9C + 1.055D_1 - 1.0335L_1 - D_2 + L_2 \le 0.365 \\ 0.8A - 0.6B + 1.9C + 1.0635D_1 - 1.0385L_1 - D_2 + L_2 \le 0.405 \\ -1A - 0.5B - C + 1.06D_2 - 1.045L_2 - D_3 + L_3 \le 0.36 \\ -1A - 0.5B - C + 1.055D_2 - 1.04L_2 - D_3 + L_3 \le 0.33 \\ -1A - 0.5B - C + 1.065D_2 - 1.0485L_2 - D_3 + L_3 \le 0.415 \\ -1.4A - 1.8B - 2C + 1.058D_3 - 1.048L_3 - D_4 + L_4 \le 0.35 \\ -1.4A - 1.8B - 2C + 1.0515D_3 - 1.044L_3 - D_4 + L_4 \le 0.335 \\ -1.4A - 1.8B - 2C + 1.0615D_3 - 1.0515L_3 - D_4 + L_4 \le 0.375 \\ A, B, C \in [0,1], D_k, L_i \in [0,2], 0 \le \gamma \le 1 \end{array}\right\} \quad (4.48)$$

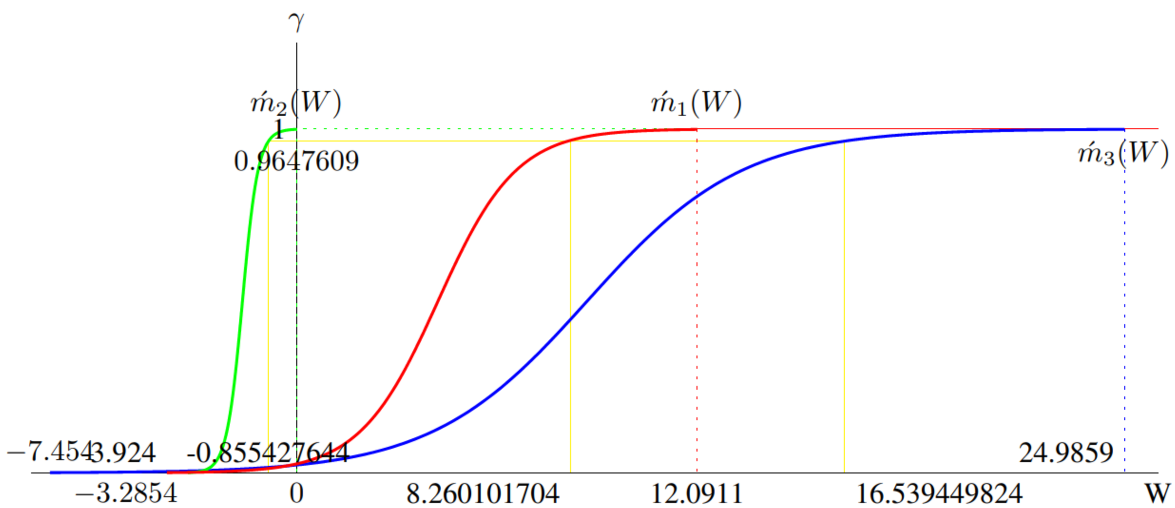


Figure 4.12: Sigmoidal membership degrees associated with wealth functions

4.4.1.4 Results

From the above study, we can see that the different membership functions give different satisfaction levels. Table 4.12 below presents the numerical satisfaction level results from various membership functions:

Table 4.12: Numerical results of satisfaction level

Membership Function	Optimal Value
Linear	0.739628
Hyperbolic	0.972218
Parabolic	0.547049
Exponential	0.884991
Sigmoidal	0.964761

Based on the data in Table 4.12, we can conclude that the hyperbolic membership function provides superior solutions to the given numerical problem. Therefore, we can rank the models based on their ability to ensure either DM pleasure or success. Thus, you could base model effectiveness on DM approval or performance: hyperbolic > s-curve > exponential > linear > parabola.

4.5 Conventional Intuitionistic Fuzzy Multi-Objective Optimization Approach

The definitions of membership functions use the simple distance functions we discussed in eq (4.14), but the range of functions for membership and non-membership will differ in an intuitive fuzzy approach. In the case of an intuitionistic fuzzy approach, the minimum value of the function will be less than that considered for the membership function. We can attribute this phenomenon to the existence of hesitation. Therefore, the distance operator's maximum value will vary for the two association functions listed below:

f) For membership function:

$$Max\ Đ_i(x) = \overline{f}_i - f_i^n,\ \forall\ i = 1,2,\dots l \tag{4.49}$$

g) For non-membership function:

$$Max\ Đ'_i(x) = \overline{f}_i - f_i^w,\ \forall\ i = 1,2,\dots l \tag{4.50}$$

Where f_i^n, f_i^w are the nadir and worst values of functions whose separation defines the hesitancy level. Let $\overline{Đ}_i = \{\{Max\ \ Đ_i(x)\}, \overline{Đ'}_i = \{\{Max\ \ Đ'_i(x)\}; \forall\ i =\ 1,2,\dots l\}$. The next step is to figure out the best way to set up a distance relationship function:

$$\acute{m}(Đ_i(x)) = \begin{cases} 0 & if\ Đ_i(x) \geq \bar{Đ}_i \\ \frac{(\bar{Đ}_i - Đ_i(x))}{\bar{Đ}_i} & if\ 0 < Đ_i(x) < \bar{Đ}_i \\ 1 & if\ Đ_i(x) \leq 0 \end{cases} \tag{4.51}$$

$$\acute{n}(Đ_i(x)) = \begin{cases} 1 & if\ Đ_i(x) \geq \bar{Đ'}_i \\ 1 - \frac{(\bar{Đ'}_i - Đ_i(x))}{\bar{Đ'}_i} & if\ 0 < Đ_i(x) < \bar{Đ'}_i \\ 0 & if\ Đ_i(x) \leq 0 \end{cases} \tag{4.52}$$

New features represent the relationship between our participation and non-participation functions. We aim to enhance the value of membership and diminish the value of non-membership functions, so we have determined two distinct parameters for each function.

$$\gamma \leq \frac{(\bar{Đ}_i - Đ_i(x))}{\bar{Đ}_i} \tag{4.53}$$

$$\delta \geq 1 - \frac{(\bar{Đ'}_i - Đ_i(x))}{\bar{Đ'}_i} \tag{4.54}$$

This statement reduces the situation to a single goal.

$$\left.\begin{array}{c} Max\ \gamma - \delta \\ Subjected\ to\colon \gamma \leq \frac{(\bar{Đ}_i - Đ_i(x))}{\bar{Đ}_i} \\ \delta \geq 1 - \frac{(\bar{Đ'}_i - Đ_i(x))}{\bar{Đ'}_i} \\ c_k^j x_k \leq d^j \\ x_k \geq 0 \\ 0 \leq \gamma, \delta \leq 1 \\ \gamma + \delta \leq 1 \\ 0 \leq \gamma - \delta \leq 1 \end{array}\right\} \tag{4.55}$$

For minimization and maximization, the nature of the functions shows different natures of membership functions and non-membership functions as we move from the minimum to the maximum value of functions. The intuitionistic approach to maximization says that the minimum value of the objective function goes down for non-membership degrees because the sum of membership and non-membership degrees should be less than or equal to 1. Figure 4.13 shows this. However, in the case of minimization, the maximum value will exceed the expectation, as shown in Figure 4.14.

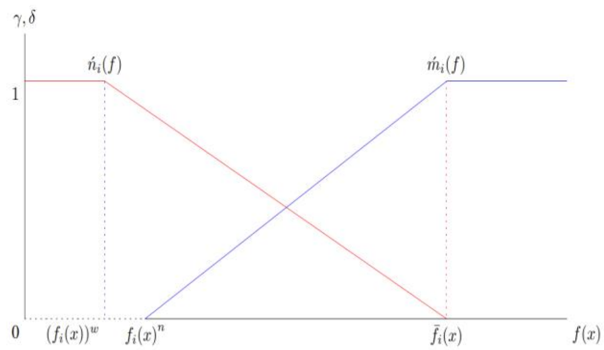


Figure 4.13: Membership and non-membership degree for maximization according to the intuitionistic approach

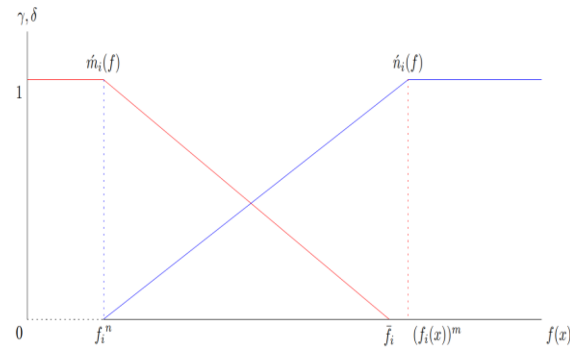


Figure 4.14: Membership and non-membership degree for minimization according to the intuitionistic approach

4.5.1 Transportation Problem

Let there be two production sites, O_1 and O_2, for the company, as well as three warehouses: T_1, T_2, and T_3, and transportation cost and time per unit quantity of commodity from origins O_1 and O_2 to destinations T_1, T_2, and T_3, represented by the notation C_{rs},T_{rs} from the r^{th} site to the s^{th} target.

4.5.1.1 Numerical data:

The cost and time values for source r to destination s are given by the corresponding elements of matrices C_{rs} and T_{rs}, respectively.

$$C_{rs} = \begin{bmatrix} 5 & 7 & 2 \\ 5 & 3 & 3 \end{bmatrix} \tag{4.56}$$

$$T_{rs} = \begin{bmatrix} 5 & 4 & 5 \\ 3 & 7 & 4 \end{bmatrix} \tag{4.57}$$

The second source has 90 units available, and the first one has 40 units available. The desired quantity for each item is 35, 30, and 65 units for the first, second, and third warehouses, respectively. The company aims to minimize the overall transportation cost and time by determining the optimal amount of product to ship from each starting point to each ending point.

4.5.1.2 Formulation of MOLPP:

The problem can be stated in standard form, which is given in Chapter 2, as follows:

Cost $$Min\ f_1 = 5x_{11} + 7x_{12} + 2x_{13} + 5x_{21} + 3x_{22} + 3x_{23} \tag{4.58}$$

Time	$Min\ f_2 = 5x_{11} + 4x_{12} + 5x_{13} + 3x_{21} + 7x_{22} + 4x_{23}$	(4.59)
Demand	$\left.\begin{array}{r} x_{11} + x_{21} = 35 \\ x_{12} + x_{22} = 30 \\ x_{13} + x_{23} = 65 \end{array}\right\}$	(4.60)
Supply	$\left.\begin{array}{r} x_{11} + x_{12} + x_{13} = 40 \\ x_{21} + x_{22} + x_{23} = 90 \end{array}\right\}$	(4.61)
Non-negativity	$x_{rs} \geq 0$	(4.62)

4.5.1.3 Solution:

The membership functions are defined from the nadir point to the maximum point of functions and non-membership from the worst to the maximum points with a linear nature. We reduce the given problem to a single objective LPP after applying the proposed approach:

$$\left.\begin{array}{c} Max\ (\lambda - \delta) \\ sub.to. 50\lambda \leq 570 - (5x_{11} + 7x_{12} + 2x_{13} + 5x_{21} + 3x_{22} + 3x_{23}) \\ 120\lambda \leq 615 - (5x_{11} + 4x_{12} + 5x_{13} + 3x_{21} + 7x_{22} + 4x_{23}) \\ 160\mu \geq -420 + (5x_{11} + 7x_{12} + 2x_{13} + 5x_{21} + 3x_{22} + 3x_{23}) \\ 155\mu \geq -495 + (5x_{11} + 4x_{12} + 5x_{13} + 3x_{21} + 7x_{22} + 4x_{23}) \\ x_{11} + x_{21} = 35 \\ x_{12} + x_{22} = 30 \\ x_{13} + x_{23} = 65 \\ x_{11} + x_{12} + x_{13} = 40 \\ x_{21} + x_{22} + x_{23} = 90 \\ x_{rs} \geq 0 \end{array}\right\} \quad (4.63)$$

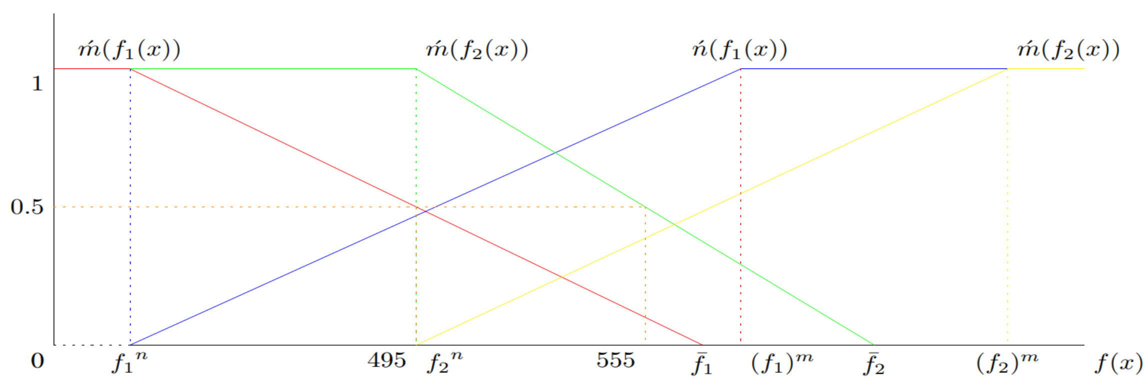


Figure 4.15: Graphical representation of satisfaction and dissatisfaction level

By LINGO 18.0 × 64 software solution, the issue is given as: λ = 0.5 and **δ**= 0.46875 at point (0,15,25,35,15,40) which can be seen from Figure 4.15. At this point, functional values are given by $f_1 = 495$ and $f_2 = 555$.

4.5.1.4 Results

The study aims to increase the satisfaction level and decrease the dissatisfaction level through an intuitionistic approach. Therefore, according to Table 4.13, the intuitionistic fuzzy approach represents a novel departure from the traditional fuzzy approach.

Table 4.13: Satisfaction and dissatisfaction values for fuzzy and intuitionistic approaches

Parameter	Fuzzy Approach	Intuitionistic Fuzzy Approach
$\boldsymbol{\lambda}$	0.3170732	0.5
$\boldsymbol{\delta}$	0.6829268	0.4687500

4.6 Conventional Intuitionistic Approach with Combination of Linear and Non-linear Association Functions

In chapter 1, we covered five different types of association functions. We generate a total of 25 scenarios for the global approximation of association functions, utilizing these five functions for both membership and non-membership parameters. We explain the linear and non-linear membership functions in equations (4.17), (4.31) - (4.34). So, we show only the mathematical formulation for the non-membership functions here:

4.6.1 Non-membership function

Non-membership functions are typically defined by their opponents. When satisfaction rises from 0 to 1, dissatisfaction falls. For the fuzzy technique, the non-membership function is therefore 1 minus the membership function. The intuitionistic fuzzy approach, on the other hand, requires an objective function minimum that is lower than the fuzzy approach's minimum. In this study, we used the lowest level of the objective function as the minimum level, even though it represents the membership function's lowest point. The function for the objective of the non-participation (ṅ) function having $\overline{f(x)}_i$ and $f(x)^w{}_i$ as positive and negative ideal values, respectively, is presented for maximization objectives from $1 \leq i \leq l$:

a) Linear non-membership function:

$$
ń(Đ_i(x)) = \begin{cases} 1 & if\ Đ_i(x) \geq \bar{Đ}'_i \\ 1 - \frac{(\bar{Đ}'_i - Đ_i(x))}{\bar{Đ}'_i} & if\ 0 < Đ_i(x) < \bar{Đ}'_i \\ 0 & if\ Đ_i(x) \leq 0 \end{cases} \tag{4.64}
$$

b) Hyperbolic non- membership function:

$$
ń(Đ_i(x)) = \begin{cases} 1 & if\ Đ_i(x) \geq \bar{Đ}'_i \\ \frac{1}{2}\left(tanh\left(\left(-Đ_i(x) + \frac{\bar{Đ}'_i}{2}\right)\delta\right)\right) - \frac{1}{2} & if\ 0 < Đ_i(x) < \bar{Đ}'_i \\ 0 & if\ Đ_i(x) \leq 0 \end{cases} \tag{4.65}
$$

c) Parabolic non- membership function:

$$
ń(Đ_i(x)) = \begin{cases} 1 & if\ Đ_i(x) \geq \bar{Đ}'_i \\ 1 - \frac{\left(\bar{Đ}'_i - Đ_i(x)\right)^2}{\bar{Đ}'^2_i} & if\ 0 < Đ_i(x) < \bar{Đ}'_i \\ 0 & if\ Đ_i(x) \leq 0 \end{cases} \tag{4.66}
$$

d) Exponential non-membership function:

$$
ń(Đ_i(x)) = \begin{cases} 1 & if\ Đ_i(x) \geq \bar{Đ}'_i \\ 1 - \eta\left[1 - exp\left\{-\rho\frac{|\bar{Đ}'_i - Đ_i(x)|}{(\bar{Đ}'_i)}\right\}\right] & if\ 0 < Đ_i(x) < \bar{Đ}'_i \\ 0 & if\ Đ_i(x) \leq 0 \end{cases} \tag{4.67}
$$

e) Sigmoidal non-membership function:

$$
ń(Đ_i(x)) = \begin{cases} 1 & if\ Đ_i(x) \geq \bar{Đ}'_i \\ \left(\frac{1}{1 + Be^{\alpha\left(\frac{\bar{Đ}'_i - Đ_i(x)}{(\bar{Đ}'_i)}\right)}}\right) & if\ 0 < Đ_i(x) < \bar{Đ}'_i \\ 1 & if\ Đ_i(x) \leq 0 \end{cases} \tag{4.68}
$$

We can choose a membership function in five different ways, as well as a non-membership function in five different ways. By combining different aspects of the two roles, we can realize 25 distinct hybrid scenarios.

4.6.2 Agriculture case study

Here we discuss crop production planning for a district, Baksa, in Assam. The data includes details about different crops in different seasons, along with the availability of land, fertilizers, and labour schedules. It also defines the production of every crop and the profit per unit land area. We must maximize total production while maximizing profit. We describe a total of 12 types of crops here. To maximize total profit and production with limited resources, we must maximize the area for each crop. Table B of [155] provides the data.

4.6.2.1 Numerical data:

We worked with secondary data on the agriculture programming problem in [155]. Table B of [155] provides the data.

4.6.2.2 Formulation of MOLPP:

Different parts of problems are defined below:

a) *Decision variable:*

The decision variables, listed below, determine the values of the objective function, necessitating our optimization of the functions:

x_{11}	Winter Rice
x_{12}	Rape and Master
x_{13}	Jute
x_{14}	Gram
x_{21}	Summer Rice
x_{22}	Lentil
x_{23}	Ginger
x_{24}	Turmeric
x_{25}	Garlic
x_{26}	Potato
x_{31}	Autumn
x_{32}	Maize

b) *Objective function: (Maximization)*

The objective functions, profit and production, are what we need to optimize in this scenario:

Production	$P_1 = 3849x_{11} + 1880x_{12} + 2290x_{13} + 870x_{14} + 4460x_{21} + 1122x_{22} + 17212x_{23} + 27910x_{24} + 5410x_{25} + 25015x_{26} + 3155x_{31} + 3079x_{32}$	(4.69)
Profit	$P_2 = 75265x_{11} + 13322x_{12} + 18009x_{13} + 40777x_{14} + 95992x_{21} + 41064x_{22} + 695142x_{23} + 1504078x_{24} + 132746x_{25} + 118245x_{26} + 49955x_{31} + 37842x_{32}$	(4.70)

c) Constraints:

The restrictions listed below determine the availability or demand for resources and products, respectively:

Kharif Season

$$\begin{cases} 150x_{11} + 80x_{12} + 170x_{13} + 80x_{14} \leq 310 \\ x_{11} + x_{12} + x_{13} + x_{14} \leq 137955 \\ 60x_{11} + 40x_{12} + 20x_{13} + 15x_{14} \leq 464300 \\ 20x_{11} + 35x_{12} + 20x_{13} + 35x_{14} \leq 530500 \\ 40x_{11} + 15x_{12} + 20x_{13} + 0x_{14} \leq 169400 \end{cases} \tag{4.71}$$

Rabi Season

$$\begin{cases} 150x_{21} + 80x_{22} + 188x_{23} + 300x_{24} + 120x_{25} + 120x_{26} \leq 310 \\ x_{21} + x_{22} + x_{23} + x_{24} + x_{25} + x_{26} \leq 137955 \\ 40x_{21} + 15x_{22} + 20x_{23} + 30x_{24} + 100x_{25} + 60x_{26} \leq 99200 \\ 20x_{21} + 35x_{22} + 60x_{23} + 50x_{24} + 80x_{25} + 50x_{26} \leq 140700 \\ 20x_{21} + 0x_{22} + 20x_{23} + 60x_{24} + 60x_{25} + 50x_{26} \leq 92200 \end{cases} \tag{4.72}$$

Summer Season

$$\begin{cases} 150x_{31} + 100x_{32} \leq 310 \\ x_{31} + x_{32} \leq 137955 \\ 40x_{31} + 60x_{32} \leq 271000 \\ 20x_{31} + 40x_{32} \leq 302200 \\ 20x_{31} + 40x_{32} \leq 88600 \end{cases} \tag{4.73}$$

Production limit

$$\begin{cases} 3849x_{11} + 4460x_{21} + 3155x_{31} \leq 650 \\ 1122x_{22} \geq 40 \\ 25015x_{32} \geq 450 \end{cases} \tag{4.74}$$

Non-neg.

$$x_{ij} \geq 0 \tag{4.75}$$

4.6.2.3 Solution:

After solving both objectives separately with constraints by LINGO 18.0 × 64:

Max $P_1 = \overline{P_1} = 80952.17$ at point (0.1688750, 3.558359, 0, 0, 0, 0.03565062, 0, 0, 0, 2.559566, 0, 3.1).

Max $P_2 = \overline{P_2} = 1808004$ at point (0, 0, 0, 3.875, 0, 0.03565062, 0, 1.016631, 0, 0.01798921, 0, 3.1).

Nadir point of $P_1 = {P_1}^n$= 41780.32 at point (0, 0, 0, 3.875, 0, 0.03565062, 0, 1.016631, 0, 0.01798921, 0, 3.1).

Nadir point of $P_2 = {P_2}^n$= 481544.874 at point (0.1688750, 3.558359, 0, 0, 0, 0, 0, 0.03565062, 0, 0, 0, 2.559566, 0, 3.1).

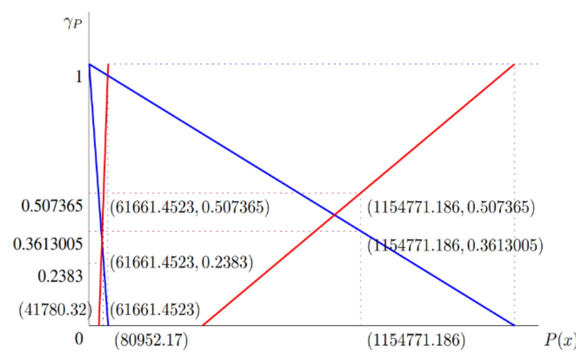
Worst point of $P_1 = {P_1}^w$= 0, worst point of $P_2 = {P_2}^w$= 0.

We assign the membership and non-membership functions to both functions and then use the provided methodology to transform the multiple objectives problem into a single objective problem, utilizing a hybrid approach to the functions' natures. To avoid duplication, the representation of a single-objective optimization problem does not specify the problem's initial constraints, which are the same as those included in the resulting new problem. Only new constraints are shown here.

Association functions lead to additional constraints: The approach transforms membership functions into restrictions for a single objective's problem. Here, a total of 25 cases are given, so for every case the restrictions generated are graphically defined in Figure 4.16.

1) Linear-Linear (LL)

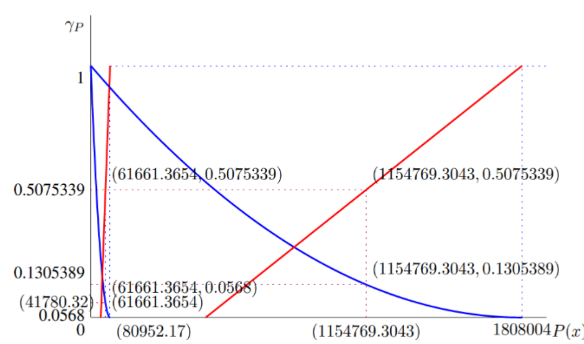
$$\left.\begin{array}{r} \gamma \le \dfrac{p_1 - 41780.32}{80952.17 - 41780.32} \\ \gamma \le \dfrac{p_2 - 481544.874}{1808004 - 481544.874} \\ \delta \ge \dfrac{80952.172 - p_1}{80952.172} \\ \delta \ge \dfrac{1808004 - p_2}{1808004} \\ 1 \ge (\gamma + \boldsymbol{\delta}) \ge \gamma \ge \boldsymbol{\delta} \ge \mathbf{0} \end{array}\right\}$$



(a)

2) Linear-Parabola (LP)

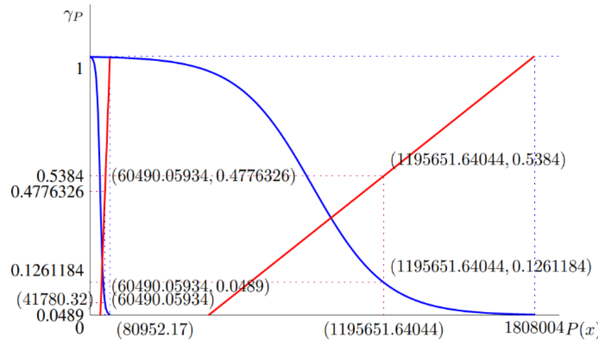
$$\left.\begin{array}{r} \gamma \le \dfrac{p_1 - 41780.32}{80952.17 - 41780.32} \\ \gamma \le \dfrac{p_2 - 481544.874}{1808004 - 481544.874} \\ \delta \ge \left(\dfrac{80952.172 - p_1}{80952.172}\right)^2 \\ \delta \ge \left(\dfrac{1808004 - p_2}{1808004}\right)^2 \\ 1 \ge (\gamma + \boldsymbol{\delta}) \ge \gamma \ge \boldsymbol{\delta} \ge \mathbf{0} \end{array}\right\}$$



(b)

3) Linear-Hyperbolic (LH)

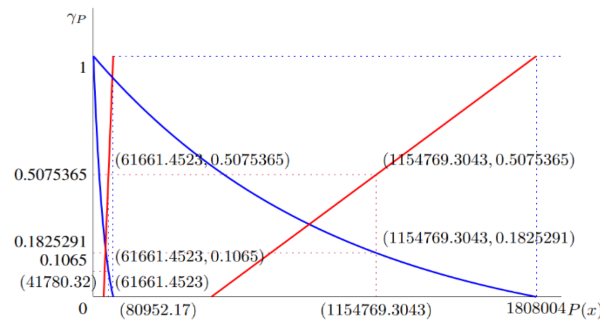
$$\left.\begin{aligned}
\gamma &\le \frac{p_1-41780.32}{80952.17-41780.32}\\
\gamma &\le \frac{p_2-481544.874}{1808004-481544.874}\\
\delta &\ge \frac{1}{2}-\frac{1}{2}tanh\left(\left(P_1-\frac{80952.17}{2}\right)\left(\frac{6}{80952.17}\right)\right)\\
\delta &\ge \frac{1}{2}-\frac{1}{2}tanh\left(\left(P_2-\frac{1808004}{2}\right)\left(\frac{6}{1808004}\right)\right)\\
1 \ge (\gamma+\boldsymbol{\delta}) &\ge \gamma \ge \boldsymbol{\delta} \ge \mathbf{0}
\end{aligned}\right\}$$



(c)

4) Linear-Exponential (LE)

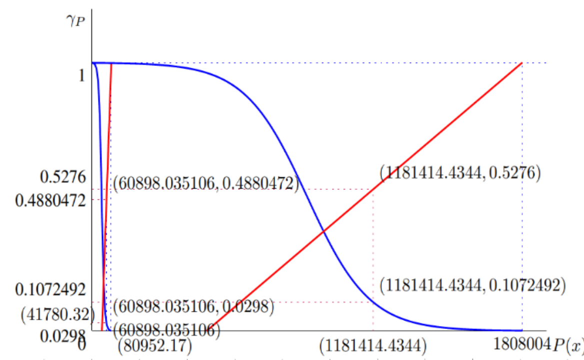
$$\left.\begin{aligned}
\gamma &\le \frac{p_1-41780.32}{80952.17-41780.32}\\
\gamma &\le \frac{p_2-481544.874}{1808004-481544.874}\\
\delta &\ge 1-1.2\left[1-exp\left\{-1.79\frac{p_1}{(80952.17)}\right\}\right]\\
\delta &\ge 1-1.2\left[1-exp\left\{-1.79\frac{p_2}{(1808004)}\right\}\right]\\
1 \ge (\gamma+\boldsymbol{\delta}) &\ge \gamma \ge \boldsymbol{\delta} \ge \mathbf{0}
\end{aligned}\right\}$$



(d)

5) Linear-sigmoidal (LS)

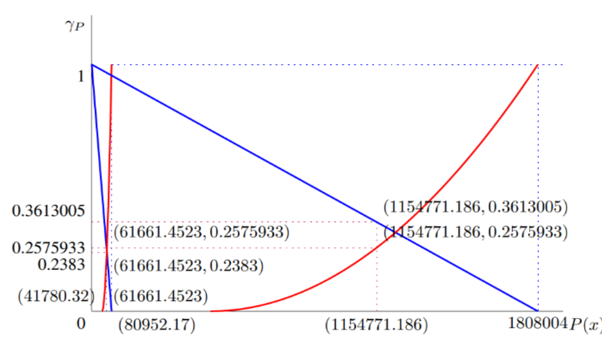
$$\left.\begin{aligned}
\gamma &\le \frac{p_1-41780.32}{80952.17-41780.32}\\
\gamma &\le \frac{p_2-481544.874}{1808004-481544.874}\\
\delta &\ge \frac{1}{1+0.001001001\exp\left(13.813\times\frac{p_1}{80952.17}\right)}\\
\delta &\ge \frac{1}{1+0.001001001\exp\left(13.813\times\frac{p_2}{1808004}\right)}\\
1 \ge (\gamma+\boldsymbol{\delta}) &\ge \gamma \ge \boldsymbol{\delta} \ge \mathbf{0}
\end{aligned}\right\}$$



(e)

6) Parabolic-Linear (PL)

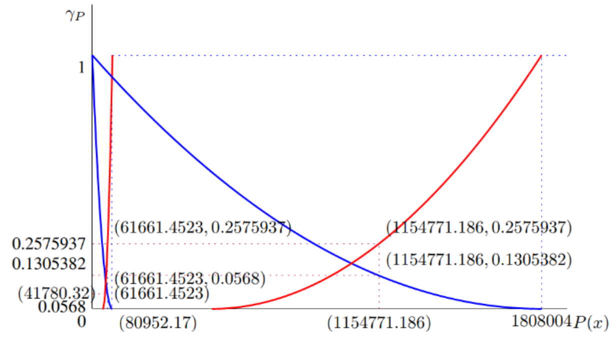
$$\left.\begin{aligned}
\gamma &\le \left(\frac{p_1-41780.32}{80952.172-41780.32}\right)^2\\
\gamma &\le \left(\frac{p_2-481544.874}{1808004-481544.874}\right)^2\\
\delta &\ge \frac{80952.172-p_1}{80952.172}\\
\delta &\ge \frac{1808004-p_2}{1808004}\\
1 \ge (\gamma+\boldsymbol{\delta}) &\ge \gamma \ge \boldsymbol{\delta} \ge \mathbf{0}
\end{aligned}\right\}$$



(f)

7) Parabolic-Parabolic (PP)

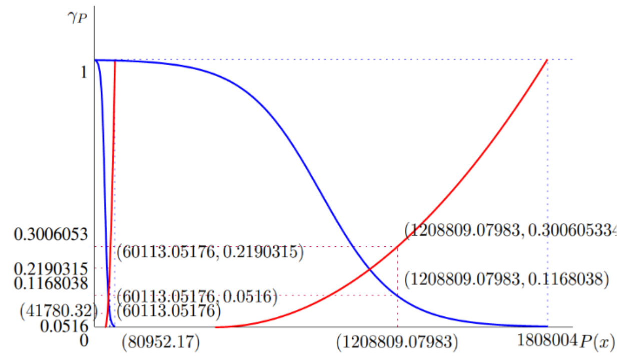
$$\left.\begin{aligned}
\gamma &\le \left(\frac{p_1 - 41780.32}{80952.172 - 41780.32}\right)^2 \\
\gamma &\le \left(\frac{p_2 - 481544.874}{1808004 - 481544.874}\right)^2 \\
\delta &\ge \left(\frac{80952.172 - p_1}{80952.172}\right)^2 \\
\delta &\ge \left(\frac{1808004 - p_2}{1808004}\right)^2 \\
1 \ge (\gamma + \boldsymbol{\delta}) &\ge \gamma \ge \boldsymbol{\delta} \ge \mathbf{0}
\end{aligned}\right\}$$



(g)

8) Parabolic-Hyperbolic (PH)

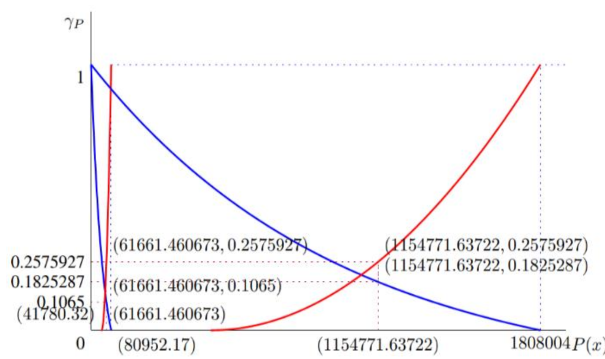
$$\left.\begin{aligned}
\gamma &\le \left(\frac{p_1 - 41780.32}{80952.172 - 41780.32}\right)^2 \\
\gamma &\le \left(\frac{p_2 - 481544.874}{1808004 - 481544.874}\right)^2 \\
\delta &\ge \frac{1}{2} - \frac{1}{2} tanh\left(\left(P_1 - \frac{80952.17}{2}\right)\left(\frac{6}{80952.17}\right)\right) \\
\delta &\ge \frac{1}{2} - \frac{1}{2} tanh\left(\left(P_2 - \frac{1808004}{2}\right)\left(\frac{6}{1808004}\right)\right) \\
1 \ge (\gamma + \boldsymbol{\delta}) &\ge \gamma \ge \boldsymbol{\delta} \ge \mathbf{0}
\end{aligned}\right\}$$



(h)

9) Parabolic-Exponential (PE)

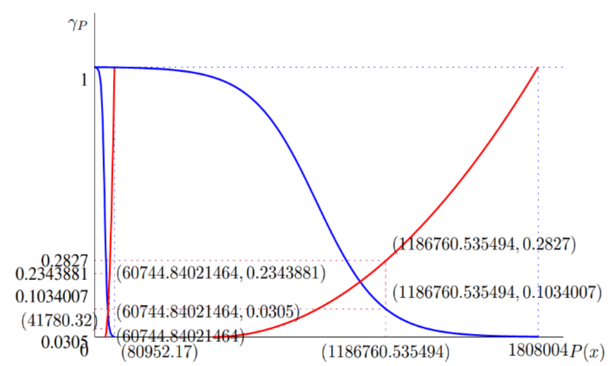
$$\left.\begin{aligned}
\gamma &\le \left(\frac{p_1 - 41780.32}{80952.172 - 41780.32}\right)^2 \\
\gamma &\le \left(\frac{p_2 - 481544.874}{1808004 - 481544.874}\right)^2 \\
\delta &\ge 1 - 1.2\left[1 - exp\left\{-1.79\frac{p_1}{(80952.17)}\right\}\right] \\
\delta &\ge 1 - 1.2\left[1 - exp\left\{-1.79\frac{p_2}{(1808004)}\right\}\right] \\
1 \ge (\gamma + \boldsymbol{\delta}) &\ge \gamma \ge \boldsymbol{\delta} \ge \mathbf{0}
\end{aligned}\right\}$$



(i)

10) Parabolic-Sigmoidal (PS)

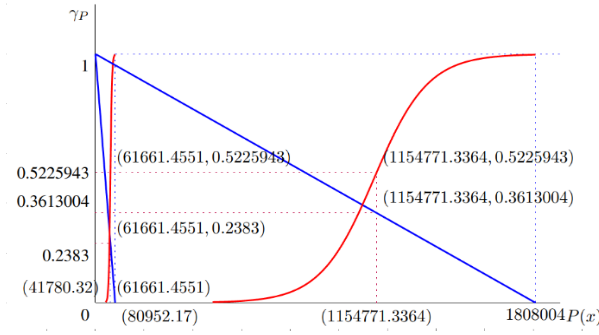
$$\left.\begin{aligned}
\gamma &\le \left(\frac{p_1 - 41780.32}{80952.172 - 41780.32}\right)^2 \\
\gamma &\le \left(\frac{p_2 - 481544.874}{1808004 - 481544.874}\right)^2 \\
\delta &\ge \frac{1}{1 + 0.001001001\, \exp\left(13.813 \times \frac{p_1}{80952.17}\right)} \\
\delta &\ge \frac{1}{1 + 0.001001001\, \exp\left(13.813 \times \frac{p_2}{1808004}\right)} \\
1 \ge (\gamma + \boldsymbol{\delta}) &\ge \gamma \ge \boldsymbol{\delta} \ge \mathbf{0}
\end{aligned}\right\}$$



(j)

11) Hyperbolic-Linear (HL)

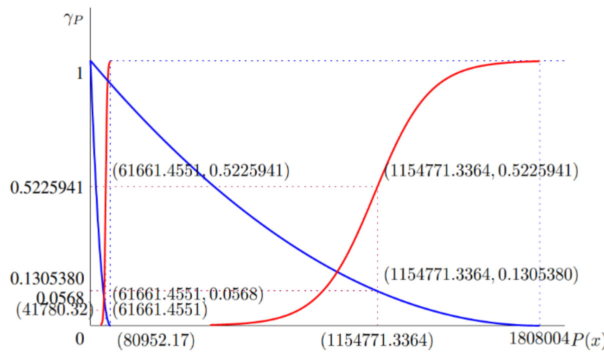
$$\left.\begin{array}{c}
\gamma \le \frac{1}{2}+\frac{1}{2}tanh\left(\left(P_1-\frac{\overline{P_1}+{P_1}^n}{2}\right)\left(\frac{6}{\overline{P_1}+{P_1}^n}\right)\right)\\
\gamma \le \frac{1}{2}+\frac{1}{2}tanh\left(\left(P_2-\frac{\overline{P_2}+{P_2}^n}{2}\right)\left(\frac{6}{\overline{P_2}+{P_2}^n}\right)\right)\\
\delta \ge \frac{80952.172-p_1}{80952.172}\\
\delta \ge \frac{1808004-p_2}{1808004}\\
1 \ge (\gamma+\boldsymbol{\delta}) \ge \gamma \ge \boldsymbol{\delta} \ge \mathbf{0}
\end{array}\right\}$$



(k)

12) Hyperbolic-Parabolic (HP)

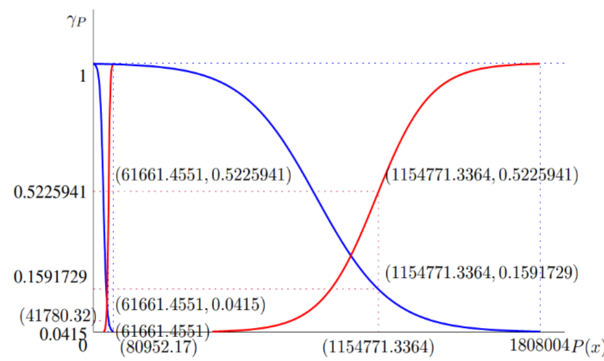
$$\left.\begin{array}{c}
\gamma \le \frac{1}{2}+\frac{1}{2}tanh\left(\left(P_1-\frac{\overline{P_1}+{P_1}^n}{2}\right)\left(\frac{6}{\overline{P_1}+{P_1}^n}\right)\right)\\
\gamma \le \frac{1}{2}+\frac{1}{2}tanh\left(\left(P_2-\frac{\overline{P_2}+{P_2}^n}{2}\right)\left(\frac{6}{\overline{P_2}+{P_2}^n}\right)\right)\\
\delta \ge \left(\frac{80952.172-p_1}{80952.172}\right)^2\\
\delta \ge \left(\frac{1808004-p_2}{1808004}\right)^2\\
1 \ge (\gamma+\boldsymbol{\delta}) \ge \gamma \ge \boldsymbol{\delta} \ge \mathbf{0}
\end{array}\right\}$$



(l)

13) Hyperbolic-Hyperbolic (HH)

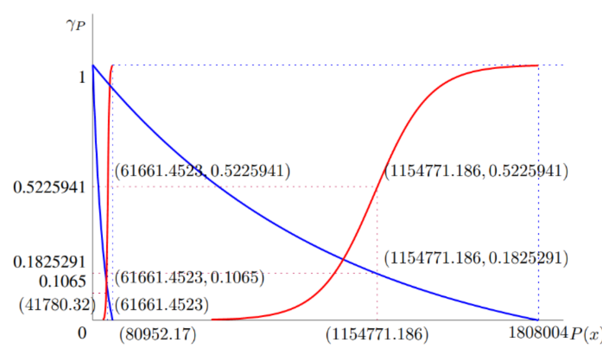
$$\left.\begin{array}{c}
\gamma \le \frac{1}{2}+\frac{1}{2}tanh\left(\left(P_1-\frac{\overline{P_1}+{P_1}^n}{2}\right)\left(\frac{6}{\overline{P_1}+{P_1}^n}\right)\right)\\
\gamma \le \frac{1}{2}+\frac{1}{2}tanh\left(\left(P_2-\frac{\overline{P_2}+{P_2}^n}{2}\right)\left(\frac{6}{\overline{P_2}+{P_2}^n}\right)\right)\\
\delta \ge \frac{1}{2}-\frac{1}{2}tanh\left(\left(P_1-\frac{80952.17}{2}\right)\left(\frac{6}{80952.17}\right)\right)\\
\delta \ge \frac{1}{2}-\frac{1}{2}tanh\left(\left(P_2-\frac{1808004}{2}\right)\left(\frac{6}{1808004}\right)\right)\\
1 \ge (\gamma+\boldsymbol{\delta}) \ge \gamma \ge \boldsymbol{\delta} \ge \mathbf{0}
\end{array}\right\}$$



(m)

14) Hyperbolic-Exponential (HE)

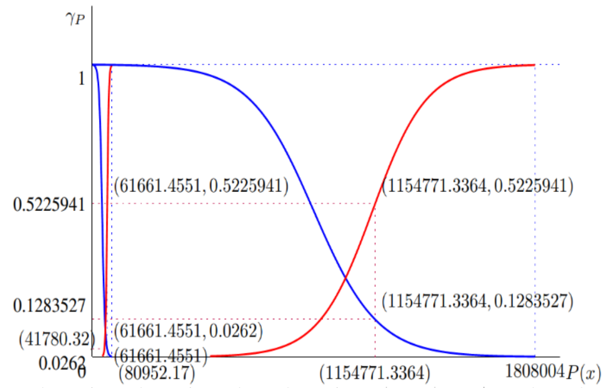
$$\left.\begin{array}{c}
\gamma \le \frac{1}{2}+\frac{1}{2}tanh\left(\left(P_1-\frac{\overline{P_1}+{P_1}^n}{2}\right)\left(\frac{6}{\overline{P_1}+{P_1}^n}\right)\right)\\
\gamma \le \frac{1}{2}+\frac{1}{2}tanh\left(\left(P_2-\frac{\overline{P_2}+{P_2}^n}{2}\right)\left(\frac{6}{\overline{P_2}+{P_2}^n}\right)\right)\\
\delta \ge 1-1.2\left[1-exp\left\{-1.79\frac{p_1}{(80952.17)}\right\}\right]\\
\delta \ge 1-1.2\left[1-exp\left\{-1.79\frac{p_2}{(1808004)}\right\}\right]\\
1 \ge (\gamma+\boldsymbol{\delta}) \ge \gamma \ge \boldsymbol{\delta} \ge \mathbf{0}
\end{array}\right\}$$



(n)

15) Hyperbolic-Sigmoidal (HS)

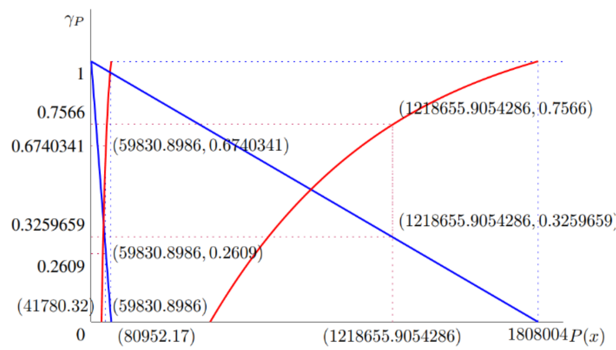
$$\gamma \leq \frac{1}{2}+\frac{1}{2}tanh\left(\left(P_1-\frac{\overline{P_1}+P_1{}^n}{2}\right)\left(\frac{6}{\overline{P_1}+P_1{}^n}\right)\right)$$
$$\gamma \leq \frac{1}{2}+\frac{1}{2}tanh\left(\left(P_2-\frac{\overline{P_2}+P_2{}^n}{2}\right)\left(\frac{6}{\overline{P_2}+P_2{}^n}\right)\right)$$
$$\delta \geq \frac{1}{1+0.001001001\exp\left(13.813\times\frac{p_1}{80952.17}\right)}$$
$$\delta \geq \frac{1}{1+0.001001001\exp\left(13.813\times\frac{p_2}{1808004}\right)}$$
$$1 \geq (\gamma+\boldsymbol{\delta}) \geq \gamma \geq \boldsymbol{\delta} \geq \mathbf{0}$$



(o)

16) Exponential-Linear (EL)

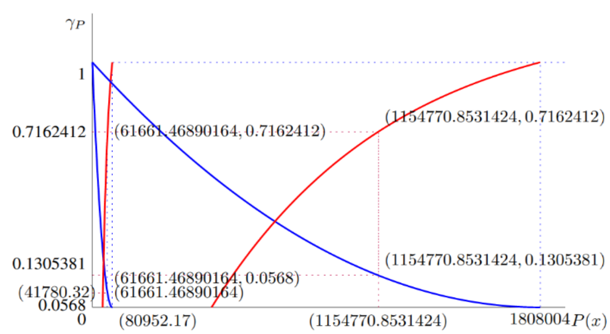
$$\left.\begin{array}{c}\gamma \leq 1.2\left[1-exp\left\{-1.79\frac{p_1-41780.32}{(80952.17-41780.32)}\right\}\right]\\ \gamma \leq 1.2\left[1-exp\left\{-1.79\frac{p_2-481544.874}{(1808004-481544.874)}\right\}\right]\\ \delta \geq \frac{80952.172-p_1}{80952.172}\\ \delta \geq \frac{1808004-p_2}{1808004}\\ 1 \geq (\gamma+\boldsymbol{\delta}) \geq \gamma \geq \boldsymbol{\delta} \geq \mathbf{0}\end{array}\right\}$$



(p)

17) Exponential-Parabolic (EP)

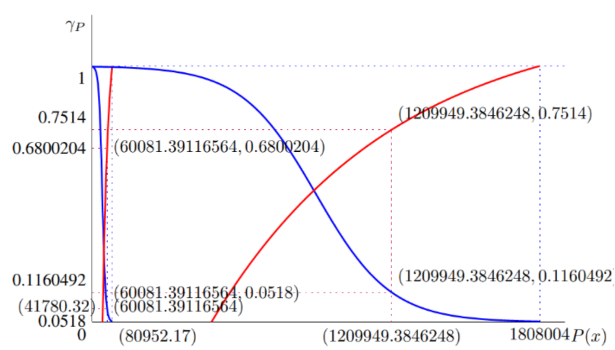
$$\left.\begin{array}{c}\gamma \leq 1.2\left[1-exp\left\{-1.79\frac{p_1-41780.32}{(80952.17-41780.32)}\right\}\right]\\ \gamma \leq 1.2\left[1-exp\left\{-1.79\frac{p_2-481544.874}{(1808004-481544.874)}\right\}\right]\\ \delta \geq \left(\frac{80952.172-p_1}{80952.172}\right)^2\\ \delta \geq \left(\frac{1808004-p_2}{1808004}\right)^2\\ 1 \geq (\gamma+\boldsymbol{\delta}) \geq \gamma \geq \boldsymbol{\delta} \geq \mathbf{0}\end{array}\right\}$$



(q)

18) Exponential-Hyperbolic (EH)

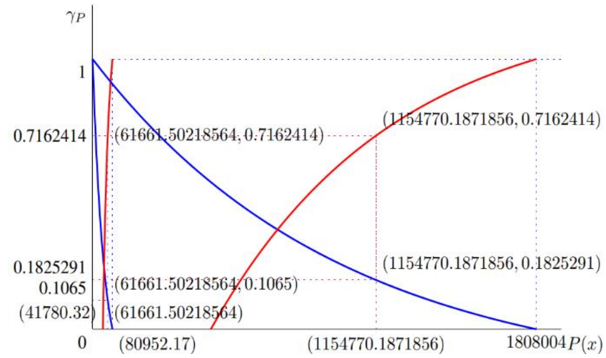
$$\left.\begin{array}{c}\gamma \leq 1.2\left[1-exp\left\{-1.79\frac{p_1-41780.32}{(80952.17-41780.32)}\right\}\right]\\ \gamma \leq 1.2\left[1-exp\left\{-1.79\frac{p_2-481544.874}{(1808004-481544.874)}\right\}\right]\\ \delta \geq \frac{1}{2}-\frac{1}{2}tanh\left(\left(P_1-\frac{80952.17}{2}\right)\left(\frac{6}{80952.17}\right)\right)\\ \delta \geq \frac{1}{2}-\frac{1}{2}tanh\left(\left(P_2-\frac{1808004}{2}\right)\left(\frac{6}{1808004}\right)\right)\\ 1 \geq (\gamma+\boldsymbol{\delta}) \geq \gamma \geq \boldsymbol{\delta} \geq \mathbf{0}\end{array}\right\}$$



(r)

19) Exponential-Exponential (EE)

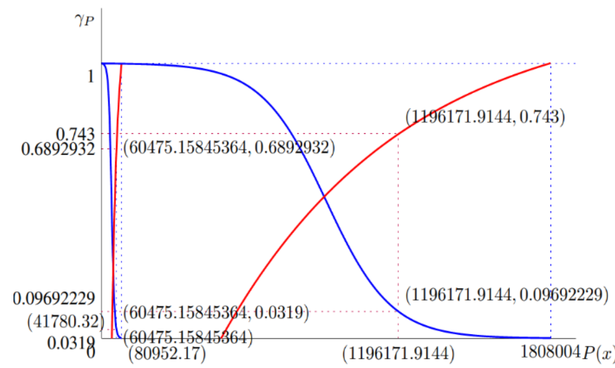
$$\left.\begin{array}{c} \gamma \leq 1.2\left[1 - exp\left\{-1.79\frac{p_1 - 41780.32}{(80952.17 - 41780.32)}\right\}\right] \\ \gamma \leq 1.2\left[1 - exp\left\{-1.79\frac{p_2 - 481544.874}{(1808004 - 481544.874)}\right\}\right] \\ \delta \geq 1 - 1.2\left[1 - exp\left\{-1.79\frac{p_1}{(80952.17)}\right\}\right] \\ \delta \geq 1 - 1.2\left[1 - exp\left\{-1.79\frac{p_2}{(1808004)}\right\}\right] \\ 1 \geq (\gamma + \boldsymbol{\delta}) \geq \gamma \geq \boldsymbol{\delta} \geq \mathbf{0} \end{array}\right\}$$



(s)

20) Exponential-Sigmoid (ES)

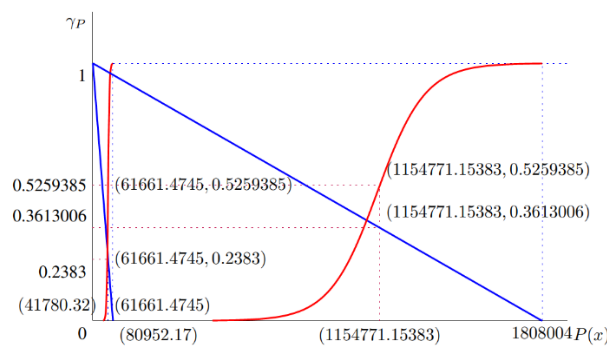
$$\left.\begin{array}{c} \gamma \leq 1.2\left[1 - exp\left\{-1.79\frac{p_1 - 41780.32}{(80952.17 - 41780.32)}\right\}\right] \\ \gamma \leq 1.2\left[1 - exp\left\{-1.79\frac{p_2 - 481544.874}{(1808004 - 481544.874)}\right\}\right] \\ \delta \geq \frac{1}{1 + 0.001001001\,\exp\left(13.813 \times \frac{p_1}{80952.17}\right)} \\ \delta \geq \frac{1}{1 + 0.001001001\,\exp\left(13.813 \times \frac{p_2}{1808004}\right)} \\ 1 \geq (\gamma + \boldsymbol{\delta}) \geq \gamma \geq \boldsymbol{\delta} \geq \mathbf{0} \end{array}\right\}$$



(t)

21) Sigmoidal-Linear (SL)

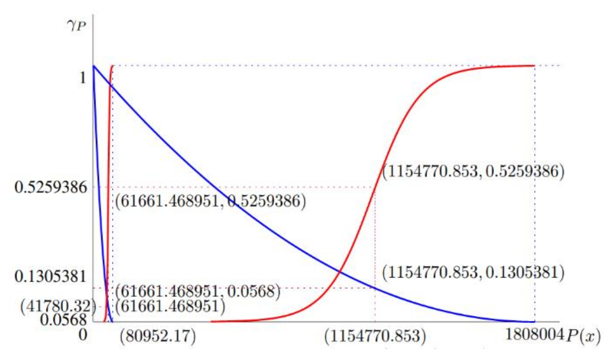
$$\left.\begin{array}{c} \gamma \leq 1 - \frac{1}{1 + 0.001001001\,\exp\left(13.813 \times \frac{p_1 - 41780.32}{(80952.17 - 41780.32)}\right)} \\ \gamma \leq 1 - \frac{1}{1 + 0.001001001\,\exp\left(13.813 \times \frac{p_2 - 481544.874}{(1808004 - 481544.874)}\right)} \\ \delta \geq \frac{80952.172 - p_1}{80952.172} \\ \delta \geq \frac{1808004 - p_2}{1808004} \\ 1 \geq (\gamma + \boldsymbol{\delta}) \geq \gamma \geq \boldsymbol{\delta} \geq \mathbf{0} \end{array}\right\}$$



(u)

22) Sigmoidal-Parabolic (SP)

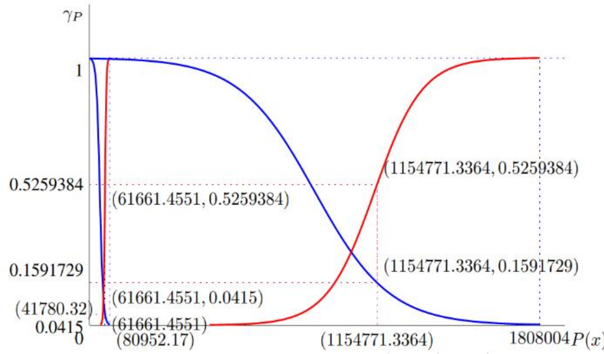
$$\left.\begin{array}{c} \gamma \leq 1 - \frac{1}{1 + 0.001001001\,\exp\left(13.813 \times \frac{p_1 - 41780.32}{(80952.17 - 41780.32)}\right)} \\ \gamma \leq 1 - \frac{1}{1 + 0.001001001\,\exp\left(13.813 \times \frac{p_2 - 481544.874}{(1808004 - 481544.874)}\right)} \\ \delta \geq \left(\frac{80952.172 - p_1}{80952.172}\right)^2 \\ \delta \geq \left(\frac{1808004 - p_2}{1808004}\right)^2 \\ 1 \geq (\gamma + \boldsymbol{\delta}) \geq \gamma \geq \boldsymbol{\delta} \geq \mathbf{0} \end{array}\right\}$$



(v)

23) Sigmoidal-Hyperbolic (SH)

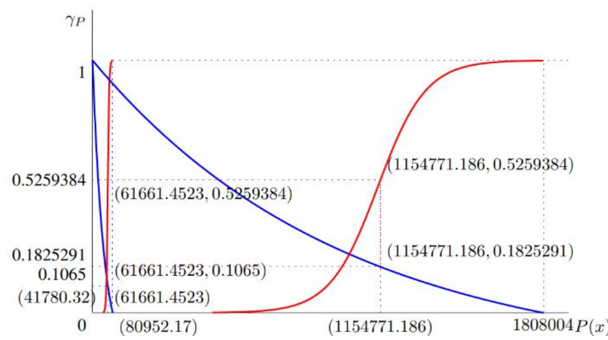
$$\left.\begin{array}{c}
\gamma \leq 1-\frac{1}{1+0.001001001\exp\left(13.813\times\frac{p_1-41780.32}{(80952.17-41780.32)}\right)} \\
\gamma \leq 1-\frac{1}{1+0.001001001\exp\left(13.813\times\frac{p_2-481544.874}{(1808004-481544.874)}\right)} \\
\delta \geq \frac{1}{2}-\frac{1}{2}tanh\left(\left(P_1-\frac{80952.17}{2}\right)\left(\frac{6}{80952.17}\right)\right) \\
\delta \geq \frac{1}{2}-\frac{1}{2}tanh\left(\left(P_2-\frac{1808004}{2}\right)\left(\frac{6}{1808004}\right)\right) \\
1 \geq (\gamma+\boldsymbol{\delta}) \geq \gamma \geq \boldsymbol{\delta} \geq \mathbf{0}
\end{array}\right\}$$



(w)

24) Sigmoidal-Exponential (SE)

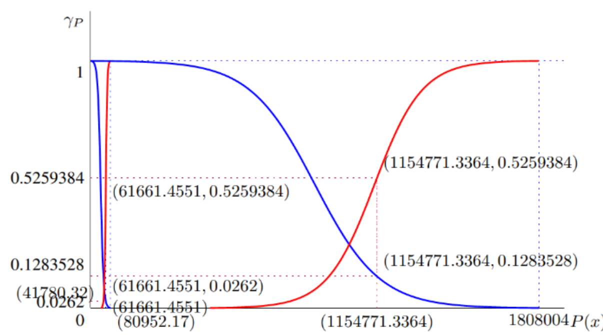
$$\left.\begin{array}{c}
\gamma \leq 1-\frac{1}{1+0.001001001\exp\left(13.813\times\frac{p_1-41780.32}{(80952.17-41780.32)}\right)} \\
\gamma \leq 1-\frac{1}{1+0.001001001\exp\left(13.813\times\frac{p_2-481544.874}{(1808004-481544.874)}\right)} \\
\delta \geq 1-1.2\left[1-exp\left\{-1.79\frac{p_1}{(80952.17)}\right\}\right] \\
\delta \geq 1-1.2\left[1-exp\left\{-1.79\frac{p_2}{(1808004)}\right\}\right] \\
1 \geq (\gamma+\boldsymbol{\delta}) \geq \gamma \geq \boldsymbol{\delta} \geq \mathbf{0}
\end{array}\right\}$$



(x)

25) Sigmoidal-Sigmoidal (SS)

$$\left.\begin{array}{c}
\gamma \leq 1-\frac{1}{1+0.001001001\exp\left(13.813\times\frac{p_1-41780.32}{(80952.17-41780.32)}\right)} \\
\gamma \leq 1-\frac{1}{1+0.001001001\exp\left(13.813\times\frac{p_2-481544.874}{(1808004-481544.874)}\right)} \\
\delta \geq \frac{1}{1+0.001001001\exp\left(13.813\times\frac{p_1}{80952.17}\right)} \\
\delta \geq \frac{1}{1+0.001001001\exp\left(13.813\times\frac{p_2}{1808004}\right)} \\
1 \geq (\gamma+\boldsymbol{\delta}) \geq \gamma \geq \boldsymbol{\delta} \geq \mathbf{0}
\end{array}\right\}$$



(y)

Figure 4.16: Graphical approach for various scenarios under intuitionistic fuzzy environment (a) Linear v/s Linear association functions (b) Linear v/s Parabolic association functions (c) Linear v/s Hyperbolic association functions (d) Linear v/s Exponential association functions (e) Linear v/s Sigmoidal association functions (f) Parabolic v/s Linear association functions (g) Parabolic v/s Parabolic association functions (h) Parabolic v/s Hyperbolic association functions (i) Parabolic v/s Exponential association functions (j) Parabolic v/s Sigmoidal association functions (k) Hyperbolic v/s Linear association functions (l) Hyperbolic v/s Parabolic association functions (m) Hyperbolic v/s Hyperbolic association functions (n) Hyperbolic v/s Exponential association functions (o) Hyperbolic v/s Sigmoidal association functions (p) Exponential v/s Linear association functions (q) Exponential v/s Parabolic association functions (r) Exponential v/s Hyperbolic association functions (s) Exponential v/s Exponential association functions (t) Exponential v/s Sigmoidal association functions (u) Sigmoidal v/s Linear association functions (v) Sigmoidal v/s Parabolic association functions (w) Sigmoidal v/s Hyperbolic association functions (x) Sigmoidal v/s Exponential association functions (y) Sigmoidal v/s Sigmoidal association functions

4.6.2.4 Results

These optimized points, obtained from LINGO 18.0×64, are used to determine the values of numerous parameters. In resulting Tables 4.14-4.16, rows show the behaviours of membership functions, while columns show the behaviours of non-membership functions. We obtain the comparison parameters in the following way:

- Satisfaction level: Minimum of both membership degrees.

$$\gamma = \min_{i=1,2} \acute{m}_i \tag{4.76}$$

- Dissatisfaction level: Maximum of both non-membership degrees.

$$\delta = \max_{i=1,2} \acute{n}_i \tag{4.77}$$

- Difference between satisfaction and dissatisfaction level:

$$\gamma - \delta \tag{4.78}$$

The values for them are given below with the help of Table 4.14:

Table 4.14: Satisfaction values for 25 different scenarios

	Linear	**Parabolic**	**Hyperbolic**	**Exponential**	**Sigmoid**
Linear	0.507365	0.2575933	0.5225943	0.6740341	0.5259385
Parabolic	0.5075339	0.2575937	0.5225941	0.7162412	0.5259386
Hyperbolic	0.4776326	0.2190315	0.5225941	0.6800204	0.5259384
Exponential	0.5075365	0.2575927	0.5225941	0.7162414	0.5259384
Sigmoid	0.4880472	0.2343881	0.5225941	0.6892932	0.5259384

Table 4.15: Dissatisfaction values for 25 different scenarios

	Linear	**Parabolic**	**Hyperbolic**	**Exponential**	**Sigmoid**
Linear	0.3613005	0.3613005	0.3613004	0.3259659	0.3613006
Parabolic	0.1305389	0.1305382	0.1305380	0.1305381	0.1305381
Hyperbolic	0.1261184	0.1168038	0.1591729	0.1160492	0.1591729
Exponential	0.1825291	0.1825287	0.1825291	0.1825291	0.1825291
Sigmoid	0.1072492	0.1034007	0.1283527	0.09692229	0.1283528

Table 4.16: Difference between satisfaction and dissatisfaction values for 25 different scenarios

	Linear	**Parabolic**	**Hyperbolic**	**Exponential**	**Sigmoid**

Linear	0.146236	-0.1037072	0.1612938	0.3480682	0.1646379
Parabolic	0.3769951	0.1270554	0.3920561	0.5857031	0.3954005
Hyperbolic	0.3515142	0.1022277	0.3634213	0.5639711	0.4844681
Exponential	0.3250074	0.07506407	0.3400650	0.5337124	0.3434093
Sigmoid	0.3807980	0.1309874	0.3942415	0.5923709	0.3975857

We can conclude from the tables that the preferences of various combinations are based on different parameters. Below, we present the most preferred approach for various parameters, which preferred maximizing the level of satisfaction and difference between satisfaction and dissatisfaction levels but minimizing the level of dissatisfaction:

- Satisfaction level (0.7162414): Exponential v/s Exponential
- Dissatisfaction level (0.09692229): Exponential v/s Sigmoidal
- Difference between satisfaction and dissatisfaction level (0.5923709): Exponential v/s Sigmoidal.

4.7 Conventional Dual Hesitant Fuzzy Approach with Linear Association Functions for MOLPP

The distance function will be the same as defined in eq (4.14) of this chapter. Here, the number of membership or non-membership functions for an objective function will not be single due to the presence of hesitant properties, according to which the membership and non-membership values at a point will be set. We provide a set of membership and non-membership functions below for your reference:

$$\acute{m}^{e}(Ð_i(x)) = \begin{cases} 0 & if\ Ð_i(x) \geq \bar{Ð}_i \\ \omega_e\left(\frac{\left(\bar{Ð}_i - Ð_i(x)\right)}{D}\right) & if\ 0 < Ð_i(x) < \bar{Ð}_i \\ 1 & if\ Ð_i(x) \leq 0 \end{cases} \quad (4.79)$$

$$\acute{n}^{e}(Ð_i(x)) = \begin{cases} 1 & if\ Ð_i(x) \geq \bar{Ð}_i{}' \\ \omega_e\left(1 - \frac{\left(\bar{Ð}_i{}' - Ð_i(x)\right)}{\bar{Ð}_i{}'}\right) & if\ 0 < Ð_i(x) < \bar{Ð}_i \\ 0 & if\ Ð_i(x) \leq 0 \end{cases} \quad (4.80)$$

Here $0 \leq ω_e \leq 1$, which is provided by the e^{th} expert, are a total of n in numbers. Similar to the intuitionistic approach, we define two parameters for each expert, denoted by γ^e and δ^e, and provide their mathematical framework as follows:

$$\gamma^e \leq ω_e \left(\frac{\left(\bar{Đ}_i - Đ_i(x) \right)}{\bar{Đ}_i} \right) \tag{4.81}$$

$$\delta^e \geq ω_e \left(1 - \frac{\left(\bar{Đ}_i{}' - Đ_i(x) \right)}{\bar{Đ}_i{}'} \right) \tag{4.82}$$

The issue has been simplified to a single goal LPP, as demonstrated in:

$$\left. \begin{array}{c} \max \left(\left(\frac{\gamma^1 + \gamma^2 + .. + \gamma^n}{n} \right) - \left(\frac{\delta^1 + \delta^2 + .. + \delta^n}{n} \right) \right) \\ \text{sub. to: } \gamma^e \leq ω_e \left(\frac{\left(\bar{Đ}_i - Đ_i(x) \right)}{\bar{Đ}_i} \right) \\ \delta^e \geq ω_e \left(1 - \frac{\left(\bar{Đ}_i{}' - Đ_i(x) \right)}{\bar{Đ}_i{}'} \right) \\ c_k^j x_k \leq d^j \\ x_k \geq 0 \\ 0 \leq \gamma^p + \delta^p \leq 1 \\ \gamma^p \geq \delta^p \end{array} \right\} \tag{4.83}$$

4.8 Conventional Dual Hesitant Fuzzy Approach with Non-linear Association Functions for MOLPP

Due to the development of dual hesitant sets with the aid of intuitionistic sets. As they involve the properties of intuitionistic sets, we provide a set of membership and non-membership functions that are preferred based on the maximum number of parameters from the study of the intuitionistic approach, which can be seen from Figure 4.17. Functions are defined below:

$$ḿ^e(Đ_i(x)) = \begin{cases} 0 & if\ Đ_i(x) \geq \bar{Đ}_i \\ ω_e 1.2 \left[1 - exp\left(-ln\left(\frac{1}{6}\right) \left(\frac{|\bar{Đ}_i - Đ_i(x)|}{(\bar{Đ}_i)} \right) \right) \right] & if\ 0 < Đ_i(x) < \bar{Đ}_i \\ 1 & if\ Đ_i(x) \leq 0 \end{cases} \tag{4.84}$$

$$
ń^{e}(Ð_i(x)) = \begin{cases} 1 & if\ Ð_i(x) \geq \bar{Ð}_i{}' \\ ω_e\left(\dfrac{1}{1+(0.001001001e^{13.813\left(\frac{Ð_i(x)}{(\bar{Ð}'_i)}\right)}}\right) & if\ 0 < Ð_i(x) < \bar{Ð}_i{}' \\ 1 & if\ Ð_i(x) \leq 0 \end{cases} \quad (4.85)
$$

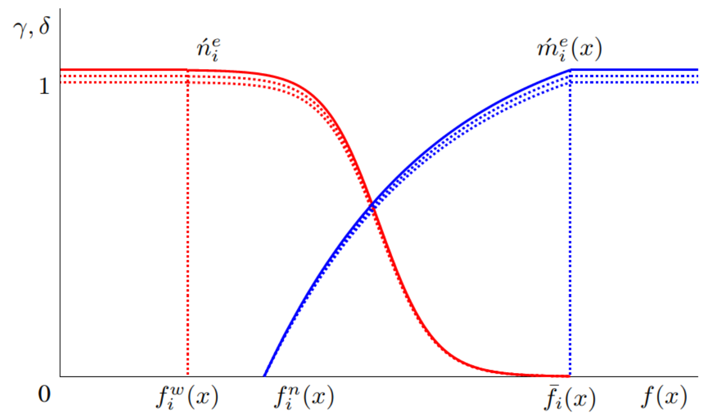


Figure 4.17: Membership and non-membership functions' graphical representation

These association functions assume additional parameters as discussed in the previous section. The ultimate single-objective programming challenge will be:

$$
\left.\begin{array}{c}
max\ \left(\dfrac{\gamma^1+\gamma^2+..+\gamma^n}{n}\right)-\left(\dfrac{\delta^1+\delta^2+..+\delta^n}{n}\right) \\
sub.to: \gamma^e \leq ω_e 1.2\left[1-exp\left(-ln\left(\dfrac{1}{6}\right)\left(\dfrac{|\bar{Ð}_i - Ð_i(x)|}{(\bar{Ð}_i)}\right)\right)\right] \\
\delta^e \geq ω_e\left(\dfrac{1}{1+(0.001001001e^{13.813\left(\frac{Ð_i(x)}{(\bar{Ð}'_i)}\right)}}\right) \\
c_k^j x_k \leq\ d^j \\
x_k \geq 0 \\
0 \leq \gamma^p + \delta^p \leq 1 \\
\gamma^p \geq \delta^p
\end{array}\right\} \quad (4.86)
$$

4.9 Conclusion

Finally, a strong and promising way to manage traffic signs is to use fuzzy inference rules to figure out how long a green light lasts, as well as the Analytic Hierarchy Process (AHP) to figure out the choice of criteria. The AHP-provided hierarchical decision process allows for an open and methodical evaluation of criteria, laying the groundwork for future adaptation using fuzzy logic. To optimize traffic flow and blockages in metropolitan areas, the fuzzy inference system improves the adaptability and understanding of traffic signal control by handling imprecise input and adjusting to fluctuating traffic patterns.

In the present chapter, we looked at the environmental benefits of using inventory and production management technology. We present a multi-objective, single-item inventory model that at the same time maximizes profit with total back-ordered quantity, optimizes costs associated with inventory holding, accounts for different emission costs, and takes steps to mitigate any environmental damage that may result from managing inventory production. To enhance the results' realism, we formalize the suggested model using fuzzy goal programming and the distance function. A numerical study then quantitatively demonstrates the results, and LINGO 18.0×64 optimization software solves the model. We can find an efficient and optimal trade-off between its financial advantages (profit) and its contribution to the environment by minimizing the costs of environmental pollution, electricity usage throughout manufacturing, and emissions resulting from the transportation of final products.

The primary objective of our study was to explore the potential applications of linear, hyperbolic, parabolic, exponential, and sigmoidal functions in capturing vagueness and ambiguity due to multiplicity of objectives in decision-making. Our research has helped us understand the rationale behind selecting specific membership and non-membership functions in specific scenarios. In cases where uncertainty follows a smooth and consistent pattern, linear functions are a suitable fit due to their simplicity and interpretability. For a global approximation of the situation for non-linear association functions, hyperbolic, parabolic, exponential, and sigmoidal natures are taken into consideration. This is because they can show maximum areas or sudden changes in uncertainty degrees. The comparative study, with the help of a financial case study, concludes that the membership function's hyperbolic nature provides the best result.

In this chapter, we solve the transportation planning problem using an intuitionistic fuzzy programming method. We use a hypothetical situation to highlight the innovative nature of the intuitionistic method. In the results section, we observe that intuitionistic fuzzy programming produces results that are closer to ideal ones, and it also leads to an increase in satisfaction levels. The integration of fuzzy programming into the transportation problem's solution enhances its realism. This chapter goes into great detail about how important it is to choose the right membership and non-membership functions for intuitionistic fuzzy sets (IFS) based on different parameters in a wide range of real-life situations that give the best mix of exponential and sigmoidal nature for membership and non-membership functions, respectively, on the basis of the maximum number of parameters.

There is a new way to solve multi-objective linear optimization problems in this study. It uses a set to find the value of the association function at a certain point. This is called a dual-hesitant fuzzy optimization method. The inclusion of non-linear behavior in the best combination of these functions makes the study more robust.

4.9.1 Major findings

- The study provides the fundamental information to handle linguistic variables for further research, and it also provides information about how to define and handle membership functions.
- The min-max fuzzy model offers a productive and encouraging optimal solution incorporating comprehensive success values that satisfy decision-makers with multiple objectives and adapt to changing variables.
- A comprehensive comparative analysis of all combinations of linear and non-linear association functions across various fuzzy extensions revealed the most effective combination, enabling the selection of the best-suited nature of association functions for improved solution quality in multi-objective optimization problems.
- Improved capacity to manage complicated multi-objective decision-making issues was shown by extended fuzzy models (intuitionistic/ dual hesitant) with modified association functions.

Chapter 5

Enhanced Approaches with Normalized Distance Function Including Various Membership Functions and Fuzzy Numbers

As established in the previous chapter, the extension of fuzzy sets and the non-linear behavior of association functions play a crucial role in enhancing the modeling capability of fuzzy multi-objective optimization problems. In real-world applications, large variations in the scale of objective function values often create imbalance and hinder accurate comparison across objectives. To address this, the normalized distance function is introduced in this chapter as a robust tool to standardize these values, ensuring fair and consistent evaluation. We begin by incorporating this function within the simple fuzzy approach, which provides a fundamental framework to validate its effectiveness in a controlled setting. Once its benefits are established, we progressively apply it to more advanced extensions of fuzzy sets—such as intuitionistic and dual hesitant fuzzy models—to explore its performance in more complex decision-making environments. Similarly non-linearity is also introduced in all these approaches. This stepwise integration as shown in Figure 5.1 not only strengthens the theoretical foundation but also highlights the adaptability and robustness of the normalized distance function across varying levels of uncertainty.

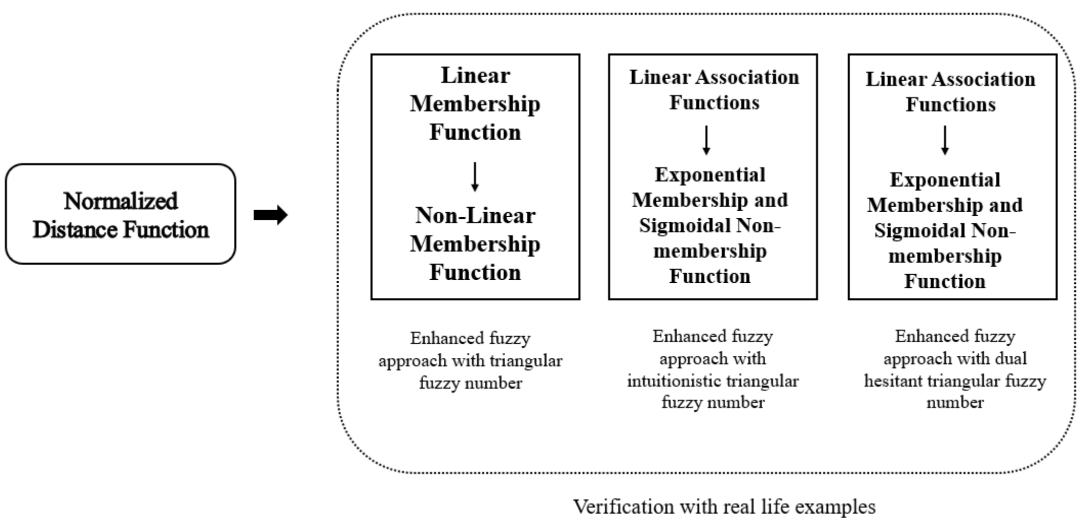


Figure 5.1: Conceptual flow and organization of the chapter

5.1 Introduction

Several studies exist that provide applications of Zimmerman's fuzzy approach in different real-life sectors that we discussed in the previous chapter. But in most cases, the range of functions varies widely, so existing methods don't treat all functions equally. This chapter introduces a fuzzy approach that uses the normalized distance parameter for an association function. This approach produces superior outcomes in multi-objective linear programming problems in terms of the satisfaction level value and the total normalized distance between the resultant values and their ideal values. A major step forward in solving the complicated decision-making difficulties encountered in real life has been using non-linear participation in fuzzy approaches for multiple goal optimization techniques. The non-linear nature of the links between the different objectives means that traditional linear membership functions may not be able to show how choice variables and goals are interdependent in many real-world situations [103].

Fuzzy and intuitionistic fuzzy optimization methods are enhanced by non-membership measures and the non-linear behaviour of assignment functions, respectively, and can better depict complicated decision spaces with non-linear interactions [62]. In real-world queries regarding optimization, inconsistencies and ambiguities are inherent. Non-linear measures allow for more flexible and nuanced membership responsibilities, which better capture these issues. This improvement results in better and more reliable decision-making, allowing users to investigate and take advantage of non-linear trade-offs and correlations among various goals [65]. The present study enhances the intuitionistic approach with the help of the normalized distance function. A new approach is then developed using the optimal combination of non-linear association functions identified in the previous chapter.

As discussed in the previous chapter, the dual hesitant approach involves the properties of intuitionistic as well as hesitant fuzzy sets and improves the situations associated with the intuitionistic fuzzy approach. So, here we have developed a dual hesitant fuzzy approach with the help of an improved intuitionistic fuzzy approach. The linear and non-linear behaviour are discussed separately for this approach.

5.2 Enhanced Fuzzy Multi-Objective Optimization Approach with Triangular Fuzzy Number

To enhance the fuzzy approach, we have used the normalized distance function [156]. For the normalized fuzzy approach, the normalized distance function for the objective $f_{\mathrm{i}}(x)$ is defined as follows:

$$Đ_i(x) = \frac{\left|\overline{f}_{\mathrm{i}} - f_i(x)\right|}{\sqrt{\sum_{k=1}^{r} {p_k^i}^2}} \tag{5.1}$$

Here, the coefficients linked to the k^{th} decision variables and the i^{th} objective function, as described by equation 1.35, are denoted by p_k^i. If we are given some constraints, we may determine the maximum value of a single objective using the notation $\overline{f}_i$ = positive ideal value of $f_i(x)$. The formula of the distance function for the largest possible $f_i(x)$ is:

$$\overline{f}_i - f_i(x) = 0 \tag{5.2}$$

The function that characterizes the separation of any two hyperplanes can be expressed as follows:

$$Max\ Đ_i(x) = \frac{\left|\overline{f}_i - f_i^n\right|}{\sqrt{\sum_{k=1}^{r} {p_k^i}^2}},\ \forall\ i = 1,2, \ldots l \tag{5.3}$$

Here f_{i}^n is the nadir point of the n^{th} function. Let $\overline{Đ}_i = \{\{max\ Đ_i(x)\}; \forall\ i = 1,2, \ldots l\}$ and for a common limiting value of membership functions, we consider $D = sup\ \{(max\ Đ_i(x)); \forall\ i = 1,2, \ldots l\}$. The next step is to figure out the best way to set up a distance relationship function, which can have different behaviour.

5.2.1 Approach with linear membership function

Here, the membership function is used as a linear function, which is the simplest form. Membership's value increases constantly as the distance between its functional value and its ideal value decreases, as defined below:

$$\acute{m}(Đ_i(x)) = \begin{cases} 0 & if\ Đ_i(x) \geq D \\ \dfrac{(D - Đ_i(x))}{D} & if\ 0' < Đ_i(x) < D \\ 1 & if\ Đ_i(x) \leq 0 \end{cases} \tag{5.4}$$

A new type of characteristic represents the connection between our participation functions. In the direction of the intersection of all functions, the minimum operator is defined as follows:

$$\gamma \leq \frac{(D - Đ_i(x))}{D} \tag{5.5}$$

This statement reduces the situation to a single goal.

$$\left.\begin{array}{c} \text{Max}\ \gamma \\ \text{Subjected to:}\ -f_{\text{i}}(x) + D\left(\sqrt{\sum_{k=1}^{r} {p_k^i}^2}\right)\gamma \leq D\left(\sqrt{\sum_{k=1}^{r} {p_k^i}^2}\right) - \bar{f}_{\text{i}} \\ c_k^j x_k \leq d^{\text{j}} \\ x_k \geq 0 \end{array}\right\} \tag{5.6}$$

These can be solved using the standard optimization tools and techniques.

5.2.1.1 Numerical Experiment:

Here a hypothetical example of MOLPP is considered:

$$\left.\begin{array}{c} max\ f_1 = 2.5x_1 + 4x_2 \\ max\ f_2 = 7x_1 + 2.75x_2 \\ sub\ to\ 2x_1 + 3x_2 \leq 3 \\ 2x_1 + 1x_2 \leq 7 \\ 5x_1 + 4x_2 \leq 7 \\ 3x_1 + 4x_2 \leq 9 \\ 7x_1 + 6x_2 \leq 10 \\ 4x_1 + 6x_2 \leq 11 \\ x_{1,}x_2 \geq 0 \end{array}\right\} \tag{5.7}$$

Let us take both objective functions individually and find their maximal values with the graphical method under the given constraint:

At point (0, 1): $\bar{f}_1 = 4$

At point (1.4, 0): $\overline{f}_2 = 9.8$

5.2.1.2 Solution

Now, distance functions are provided as follows in accordance with the model:

$$Đ_1(x) = \left|\frac{\overline{f}_1 - f_1(x)}{\sqrt{\sum_{k=1}^{r} {p_k^i}^2}}\right| = \frac{2.5x_1 + 4x_2 - 4}{4.72} \tag{5.8}$$

$$Đ_2(x) = \left|\frac{\overline{f}_2 - f_2(x)}{\sqrt{\sum_{k=1}^{r} {p_k^i}^2}}\right| = \frac{7x_1 + 2.75x_2 - 9.8}{7.52} \tag{5.9}$$

The following are the maximum separations for two goal functions:

$$\text{Max } Đ_1(x) = \frac{\overline{f}_1 - f_1^n}{\sqrt{\sum_{k=1}^{r} {p_k^i}^2}} = \frac{0.5}{4.72} \tag{5.10}$$

$$\text{Max } Đ_2(x) = \frac{7.05}{7.52} \tag{5.11}$$

As: $\text{Max } Đ_2(x) > \text{Max } Đ_1(x)$

$$=> \text{D} = \frac{7.05}{7.52} \tag{5.12}$$

Now, membership functions will be:

$$ḿ(Đ_1(x)) = \begin{cases} 0 & \text{if } Đ_1(x) \geq \frac{7.05}{7.52} \\ \dfrac{\left(\frac{7.05}{7.52} - \frac{2.5x_1 + 4x_2 - 4}{4.72}\right)}{\frac{7.05}{7.52}} & \text{if } 0' < Đ_1(x) < \frac{7.05}{7.52} \\ 1 & \text{if } Đ_1(x) \leq 0 \end{cases} \tag{5.13}$$

$$ḿ(Đ_2(x)) = \begin{cases} 0 & \text{if } Đ_2(x) \geq \frac{7.05}{7.52} \\ \dfrac{\left(\frac{7.05}{7.52} - \frac{7x_1 + 2.75x_2 - 9.8}{7.52}\right)}{\frac{7.05}{7.52}} & \text{if } 0' < Đ_2(x) < \frac{7.05}{7.52} \\ 1 & \text{if } Đ_2(x) \leq 0 \end{cases} \tag{5.14}$$

So, the new single-objective formed LPP is given as:

$$
\left.\begin{array}{c}
\text{Max } \gamma \\
\text{Subjected to } 4.42\gamma \leq 2.5x_1 + 4x_2 + 0.42 \\
7.52\gamma \leq 7x_1 + 2.75x_2 - 2.28 \\
2x_1 + 3x_2 \leq 3 \\
2x_1 + 1x_2 \leq 7 \\
5x_1 + 4x_2 \leq 7 \\
3x_1 + 4x_2 \leq 9 \\
7x_1 + 6x_2 \leq 10 \\
4x_1 + 6x_2 \leq 11 \\
x_{1,} x_2 \geq 0
\end{array}\right\} \quad (5.15)
$$

Since it defines membership degree, the range of γ will be the same as that of membership degree, which is between 0 and 1, since it is the minimum of all membership functions associated with each objective function. By using the simplex method, the value of γ =0.9484 and the point of maxima is (1.2912, 0.136). At this point, the values of the objective functions are f_1=3.772 and f_2= 9.5824.

5.2.1.3 Results and Comparative analysis

By using Zimmerman's method, the value of γ =0.9082 and the point of maxima is (1.21835, 0.2271). At this point, the values of the objective functions are f_1=3.9542 and f_2= 9.152776.

Table 5.1: Values of different parameters for the Zimmerman's and proposed approach

Parameter	**Ideal value for 1st function**	**Ideal value for 2nd function**	**Zimmerman's approach**	**Proposed approach**
γ	---	---	0.9082	0.9484
Optimal point	(0,1)	(1.4, 0)	(1.21835, 0.2271)	(1.2912, 0.136)
f_1	4	3.5	3.9542	3.772
f_2	2.75	9.8	9.152776	9.5824
Total functional value	6.75	13.3	13.106976	13.3544
Deviation	---	---	0.09577041	0.07724125

As shown in Table 5.1, the proposed approach decreases the total deviation from ideal values while increasing both the satisfaction level and the total functional value that we desire.

5.2.2 Approach with non-linear membership function

The membership function after taking the non-linear nature, which is described in Chapter 4, and taking D in place of $\overline{Đ_i}(x)$, is defined below [157]:

a) Hyperbolic Membership Function:

$$ḿ(Đ_i(x)) = \begin{cases} 1 & if\ Đ_i(x) \leq 0 \\ (\frac{1}{2} tanh\left(\left(-Đ_i(x) + \frac{D}{2}\right)\delta\right) + \frac{1}{2}) & if 0 < Đ_i(x) < D \\ 0 & if\ Đ_i(x) \geq D \end{cases} \quad (5.16)$$

Here, $\delta = |\frac{6}{\bar{f}_i(x)+f_i^n(x)}|$.

b) *Parabolic membership function:*

$$ḿ(Đ_i(x)) = \begin{cases} 1 & if Đ_i(x) \leq 0 \\ \frac{(D - Đ_i(x))^2}{(D)^2} & if\ 0 < Đ_i(x) < D \\ 0 & if\ Đ_i(x) \geq D \end{cases} \quad (5.17)$$

c) *Exponential membership function:*

$$ḿ(Đ_i(x)) = \begin{cases} 1 & if Đ_i(x) \leq 0 \\ \eta\left[1 - exp\left\{-\rho \frac{|D - Đ_i(x)|}{(D)}\right\}\right] & if\ 0 < Đ_i(x) < D \\ 0 & if\ Đ_i(x) \geq D \end{cases} \quad (5.18)$$

d) *Sigmoidal membership function**:***

$$ḿ(Đ_i(x)) = \begin{cases} 1 & if Đ_i(x) \leq 0 \\ 1 - \left(\frac{1}{1 + Be^{\alpha\left(\frac{D - Đ_i(x)}{(D)}\right)}}\right) & if\ 0 < Đ_i(x) < D \\ 0 & if\ Đ_i(x) \geq D \end{cases} \quad (5.19)$$

The parametric values are used the same as described in the previous chapter. After taking these membership functions, the approach will go in the same sense as described in the above section described by eq (5.5), (5.6).

5.3 Enhanced Fuzzy Multi-Objective Optimization Approach with Intuitionistic Triangular Fuzzy Number

Here, apart from the conventional approach, the distance function is defined with normalization.

$$Đ_i(x) = \frac{|\overline{f}_i - f_i(x)|}{\sqrt{\sum_{k=1}^{r} {p_k^i}^2}} \tag{5.20}$$

Both association functions, listed below, will have a different maximum value for the distance operator:

a) *For membership function:*

$$Max\ Đ_i(x) = \frac{|\overline{f}_i - f_i^n|}{\sqrt{\sum_{k=1}^{r} {p_k^i}^2}},\ \ \forall\, i = 1,2,\dots l \tag{5.21}$$

b) *For non-membership function:*

$$Max\ Đ'_i(x) = \frac{|\overline{f}_i - f_i^w|}{\sqrt{\sum_{k=1}^{r} {p_k^i}^2}},\ \ \forall\, i = 1,2,\dots l \tag{5.22}$$

Where f_i^n and f_i^w are the nadir and worst values of functions whose difference defines the hesitancy level. Let $D = sup\{(Max\ Đ_i(x));\ \forall\, i = 1,2,\dots l, D' = sup\{(Max\ Đ'_i(x));\ \forall\, i = 1,2,\dots l\}$. The next step is to define membership and non-membership functions with linear and non-linear behaviour based on the best results from the previous section.

5.3.1 Approach with linear association function

We treat both membership and non-membership functions here as linear functions that vary continuously with the normalized distance.

$$ḿ(Đ_i(x)) = \begin{cases} 0 & if\ Đ_i(x) \geq D \\ \dfrac{(D - Đ_i(x))}{D} & if\ 0 < Đ_i(x) < D \\ 1 & if\ Đ_i(x) \leq 0 \end{cases} \tag{5.23}$$

$$\acute{n}(Đ_i(x)) = \begin{cases} 1 & if\ Đ_i(x) \geq D' \\ 1 - \dfrac{(D' - Đ_i(x))}{D'} & if\ 0 < Đ_i(x) < D' \\ 0 & if\ Đ_i(x) \leq 0 \end{cases} \tag{5.24}$$

New characteristics represent the relationship between participation and non-participation functions. We aim to enhance the value of membership and diminish the value of non-membership functions. So, we have determined two distinct parameters for each function.

$$\gamma \leq \frac{(D - Đ_i(x))}{D} \tag{5.25}$$

$$\delta \geq 1 - \frac{(D' - Đ_i(x))}{D'} \tag{5.26}$$

This statement reduces the situation to a single goal.

$$\left.\begin{array}{c} Max\ \gamma - \delta \\ Subjected\ to: \gamma \leq \dfrac{(D - Đ_i(x))}{D} \\ \delta \geq 1 - \dfrac{(D' - Đ_i(x))}{D'} \\ c_k^j x_k \leq d^j \\ x_k \geq 0 \\ 0 \leq \gamma, \delta \leq 1 \\ \gamma + \delta \leq 1 \\ 0 \leq \gamma - \delta \leq 1 \end{array}\right\} \tag{5.27}$$

This statement reduces the situation to a single goal.

5.3.2 Approach with non-linear association functions

We look at both membership and non-membership functions as non-linear functions based on the earlier chapter's comparison and the normalized distance.

$$\acute{m}(Đ_i(x)) = \begin{cases} 0 & if\ Đ_i(x) \geq D \\ 1.2\left[1 - exp\left(-ln\left(\dfrac{1}{6}\right)\left(\dfrac{\left(D - Đ_i(x)\right)}{D}\right)\right)\right] & if\ 0 < Đ_i(x) < D \\ 1 & if\ Đ_i(x) \leq 0 \end{cases} \tag{5.28}$$

$$
\acute{n}(\text{Đ}_i(x)) = \begin{cases} 1 & if\ \text{Đ}_i(x) \geq D' \\ \left(\dfrac{1}{1 + \left(0.001001001 e^{13.813\left(\frac{D' - \text{Đ}_i(x)}{D'}\right)} \right)} \right) & if\ 0 < \text{Đ}_i(x) < D' \\ 1 & if\ \text{Đ}_i(x) \leq 0 \end{cases} \qquad (5.29)
$$

Figure 5.2 illustrates the geometric behaviour of these association functions in detail. The method follows the same steps as the previous approach using linear functions, resulting in a single objective LPP:

$$
\left.\begin{array}{c} Max\ \gamma - \delta \\ Subjected\ to{:}\ \gamma \leq 1.2\left[1 - exp\left(-ln\left(\frac{1}{6}\right)\left(\frac{(D - \text{Đ}_i(x))}{D}\right)\right)\right] \\ \delta \geq \left(\dfrac{1}{1 + \left(0.001001001 e^{13.813\left(\frac{D' - \text{Đ}_i(x)}{D'}\right)} \right)} \right) \\ c_k^j x_k \leq d^j \\ x_k \geq 0 \\ 0 \leq \gamma, \delta \leq 1 \\ \gamma + \delta \leq 1 \\ 0 \leq \gamma - \delta \leq 1 \end{array}\right\} \qquad (5.30)
$$

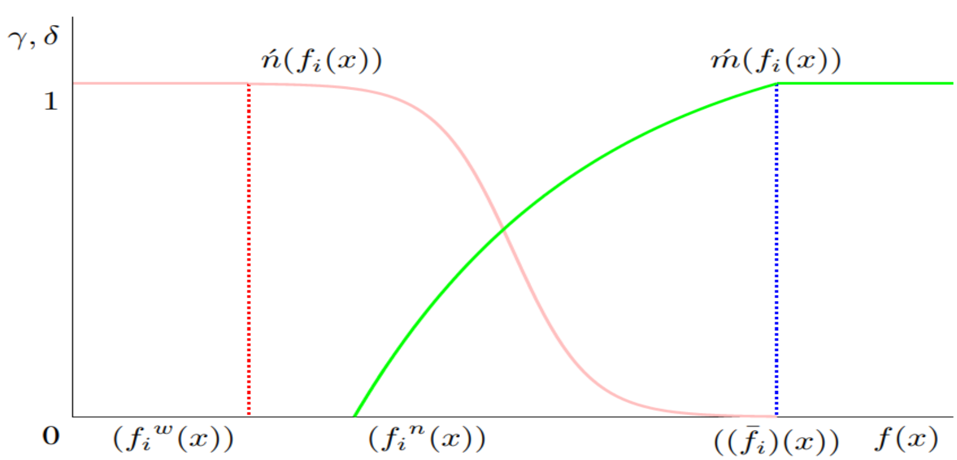


Figure 5.2: Graphical representation of exponential membership and sigmoidal non-membership function

5.3.2.1 Numerical Experiment:

Here a hypothetical problem of MOLPP is considered, given below:

$$\begin{cases} Max\ f_1 = 2x_1 - 3x_2 \\ Max\ f_2 = x_1 + 5x_2 \\ \text{s.t.}\ x_1 - 3x_2 \leq 15 \\ 2x_1 + x_2 \leq 25 \\ 1x_1 + 5x_2 \leq 20 \\ 2x_1 + 3x_2 \leq 22 \\ x_1, x_2 \geq 0 \end{cases} \tag{5.31}$$

To solve both functions separately using LINGO 18.0 × 64:

Max $f_1 = \overline{f_1}$=22; at the optimal point (11,0)

Max $f_2 = \overline{f_2}$=20; at the optimal point (7.14,2.57)

Nadir point of $f_1 = {f_1}^n$ = 6.57; at the point (7.14,2.57).

Nadir point of $f_2 = {f_2}^n$=11; at the point (11,0).

Minimum point of $f_1 = {f_1}^w$= -12

Minimum point of $f_2 = {f_2}^w$= 0

5.3.2.2 Solution

The expected first- and second-function levels increased simultaneously. The first function's nadir and ideal values are 6.57 and 22, respectively, which form the basis for a participation function with an exponential character. The ideal value remains constant, and the non-membership measure exhibits a sigmoidal pattern. However, intuitionistic characteristics lead to an even worse minimum value of -12 when considering the supremum normalized distance. Membership is defined here as a value that ranges from zero at the bottom to the number one at the top; non-membership is the opposite. In the second objective, 20 is the best, -28.08 is the worst, and -1.823 is the nadir number, as shown in Figure 5.3. The goal of this strategy is to widen the difference between the two functions' satisfaction and dissatisfaction ratings. We have now reduced the problem to a single-objective LPP:

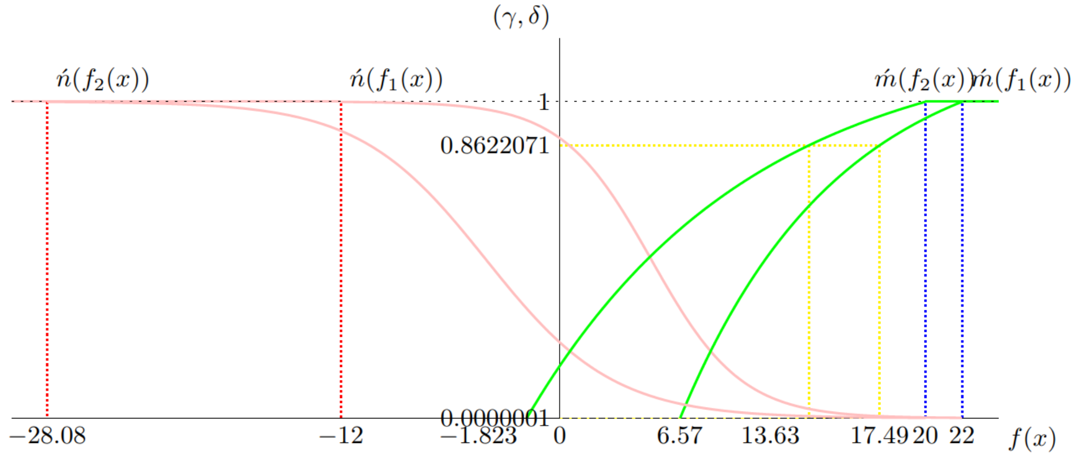


Figure 5.3: Results with assignment function according to non-linear and normalized intuitionistic fuzzy approach

$$\left.\begin{array}{c}
\text{Max}\ \gamma-\delta \\
\text{s.t.}\ \gamma \le 1.2\left[1-exp\left\{-1.79\dfrac{\dfrac{15.43}{\sqrt{13}}-\left(\dfrac{22-(2x_1-3x_2)}{\sqrt{13}}\right)}{\dfrac{15.43}{\sqrt{13}}}\right\}\right] \\
\gamma \le 1.2\left[1-exp\left\{-1.79\dfrac{\dfrac{15.43}{\sqrt{13}}-\left(\dfrac{20-(x_1+5x_2)}{\sqrt{26}}\right)}{\dfrac{15.43}{\sqrt{13}}}\right\}\right. \\
\delta \ge \dfrac{1}{1+0.001001001\exp\left(13.813\times\dfrac{\dfrac{34}{\sqrt{13}}-\left(\dfrac{22-(2x_1-3x_2)}{\sqrt{13}}\right)}{\dfrac{34}{\sqrt{13}}}\right)} \\
\delta \ge \dfrac{1}{1+0.001001001\exp\left(13.813\times\dfrac{\dfrac{34}{\sqrt{13}}-\left(\dfrac{20-(x_1+5x_2)}{\sqrt{26}}\right)}{\dfrac{34}{\sqrt{13}}}\right)} \\
x_1-3x_2 \le 15 \\
2x_1+x_2 \le 25 \\
1x_1+5x_2 \le 20 \\
2x_1+3x_2 \le 22 \\
x_1, x_2 \ge 0 \\
0 \le \gamma+\delta \le 1 \\
\gamma \ge \delta
\end{array}\right\} \quad (5.32)$$

The range of γ and δ, which are the lowest and maximum of all membership and non-membership functions associated with each objective function, respectively, will be from 0 to 1, because these functions essentially determine the degrees of membership and non-membership.

5.3.2.3 Results and Comparative Analysis

LINGO 18.0×64 is used to obtain the optimal point and the level of satisfaction and dissatisfaction based on the problem findings of the numerical trial. We determine the values of multiple variables based on this optimum point, listed in Table 5.2.

Table 5.2: Various parameter's values after optimization with different techniques for an illustrative example

Parameter	**IFA (Linear, Non-normalized)**	**IFA (Linear, Normalized)**	**IFA (Non-linear, Normalized)**
γ	0.5000231	0.7419355	0.8622071
δ	0.2269013	0.1827957	0.0000001
$\gamma - \delta$	0.2731219	0.5591398	0.862207
Optimal point	(9.071, 1.286)	(9.873,0.751)	(9.873,0.751)
f_1	14.284	17.493	17.493
f_2	15.501	13.63	13.63
Total functional value	29.785	31.123	31.123
Deviation	3.02236	2.4993	2.4993

Table 5.2 displays the intended outcome of our study, which shows a 13.95% increase in satisfaction levels compared to linear techniques. 35.2% and 68.3% increment in the gap between satisfaction and dissatisfaction levels compared to the current linearly normalized and non-normalized techniques, respectively. Improving overall functional value and decreasing normalized operational value disparity from ideal locations are our primary goals. The Table 5.2 shows that the total functional values from using linear and non-linear normalized methods were 4.3% higher than those from the non-normalized findings. The normalized distance decreased by 17.31% compared to the non-normalized method.

5.4 Enhanced Fuzzy Multi-Objective Optimization Approach with Dual Hesitant Triangular Fuzzy Number

The normalized distance function will be the same as defined in earlier chapters. Here, the number of membership or non-membership functions for an objective function is not single due to the presence of hesitant properties, according to which the membership and non-membership values at a point form a set. We provide a set of membership and non-membership functions below for your reference. The analysis shows that normalized techniques are better to calculate functional and deviational values. Non-linearity with normalization enhances the satisfaction level with decreased dissatisfaction value.

5.4.1 Approach with linear association functions

The membership functions are linear in accordance with the normalized distance function. The membership function declines with an increase in the distance function, while the non-membership function rises for any expert, as illustrated below:

$$ḿ^e(Đ_i(x)) = \begin{cases} 0 & if\ Đ_i(x) \geq D \\ ω_e\left(\frac{(D - Đ_i(x))}{D}\right) & if\ 0 < Đ_i(x) < D \\ 1 & if\ Đ_i(x) \leq 0 \end{cases} \quad (5.33)$$

$$ń^e(Đ_i(x)) = \begin{cases} 1 & if\ Đ_i(x) \geq D' \\ ω_e\left(1 - \frac{(D' - Đ_i(x))}{D'}\right) & if\ 0 < Đ_i(x) < D' \\ 0 & if\ Đ_i(x) \leq 0 \end{cases} \quad (5.34)$$

Here $0 \leq ω_e \leq 1$, which is provided by the e^{th} expert, which are total n in numbers. Like the intuitionistic approach, we define two parameters for each expert, denoted by γ^e and δ^e, and provide their mathematical framework as follows:

$$\gamma^e \leq ω_e\left(\frac{(D - Đ_i(x))}{D}\right) \quad (5.35)$$

$$\delta^e \geq ω_e\left(1 - \frac{(D' - Đ_i(x))}{D'}\right) \quad (5.36)$$

The issue has been simplified to a single goal LPP, as demonstrated in:

$$\left.\begin{array}{c} \max \left(\frac{\gamma^1+\gamma^2+..+\gamma^n}{n}\right)-\left(\frac{\delta^1+\delta^2+..+\delta^n}{n}\right) \\ \text{sub. to: } \gamma^e \le ա_e\left(\frac{\left(D-Đ_i(x)\right)}{D}\right) \\ \delta^e \ge ա_e\left(1-\frac{\left(D'-Đ_i(x)\right)}{D'}\right) \\ c_k^j x_k \le d^j \\ x_k \ge 0 \\ 0 \le \gamma^p+\delta^p \le 1 \\ \gamma^p \ge \delta^p \end{array}\right\} \tag{5.37}$$

5.4.2 Approach with non-linear association function

Due to the involvement of intuitionistic fuzzy sets into dual hesitant fuzzy sets, the study provides a set of membership and non-membership functions with the same non-linear behaviour, best concluded in Chapter 4 from a comparison study of different association functions. Functions are defined below:

$$ḿ^e(Đ_i(x)) = \begin{cases} 0 & if\ Đ_i(x) \ge D \\ ա_e 1.2\left[1-exp\left(-ln\left(\frac{1}{6}\right)\left(\frac{\left(D-Đ_i(x)\right)}{D}\right)\right)\right] & if\ 0 < Đ_i(x) < D \\ 1 & if\ Đ_i(x) \le 0 \end{cases} \tag{5.38}$$

$$ń^e(Đ_i(x)) = \begin{cases} 1 & if\ Đ_i(x) \ge D' \\ ա_e\left(\frac{1}{1+(0.001001001e^{13.813\left(\frac{D'-\left(Đ_i(x)\right)}{D'}\right)}}\right) & if\ 0 < Đ_i(x) < D' \\ 1 & if\ Đ_i(x) \le 0 \end{cases} \tag{5.39}$$

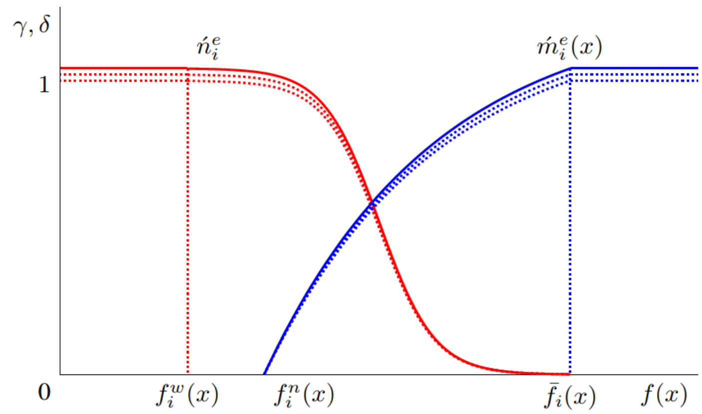


Figure 5.4: Membership and non-membership functions' graphical representation

These association functions incorporate additional parameters known as satisfaction and dissatisfaction levels, which can be seen geometrically from Figure 5.4. For a particular expert, satisfaction is the minimum of all membership degrees, and dissatisfaction is the maximum of all non-membership degrees. Now the aim is to maximize the difference between average satisfaction and dissatisfaction levels. This will be the ultimate single-objective programming challenge:

$$\left.\begin{array}{c} max\left(\frac{\gamma^1+\gamma^2+..+\gamma^n}{n}\right)-\left(\frac{\delta^1+\delta^2+..+\delta^n}{n}\right) \\ sub.to: \gamma^e \leq \omega_e 1.2\left[1-exp\left(-ln\left(\frac{1}{6}\right)\left(\frac{\left(D-Đ_i(x)\right)}{D}\right)\right)\right] \\ \delta^e \geq \omega_e\left(\frac{1}{1+(0.001001001e^{13.813\left(\frac{D'-\left(Đ_i(x)\right)}{D'}\right)}}\right) \\ c_k^j x_k \leq d^j \\ x_k \geq 0 \\ 0 \leq \gamma^p+\delta^p \leq 1 \\ \gamma^p \geq \delta^p \end{array}\right\} \quad (5.40)$$

The problem can be solved by conventional simplex method.

5.5 Conclusion

When analysing all the characteristics with a normalized separation value, the proposed fuzzy method becomes much more robust and dependable. Such an approach is critical when making decisions in real life, where scale parameter variations play a significant role. However, the data we looked at shows that non-linearity doesn't change the normalized methods used to rank objective function results and normalized distance parameters. However, in this scenario, the level of satisfaction is higher.

The study's results highlight the importance of using modern computer techniques to tackle MOLPP problems, such as standard separation tasks and non-linearity in an intuitionistic fuzzy framework. The proposed approach effectively incorporates the inherent inconsistency and unpredictability of everyday decision-making scenarios, providing decision-makers across numerous domains with a practical and efficient solution. Since it can generate ideas that are close to the intended outcomes while managing several competing goals, this technique is appropriate and pertinent to challenging optimization situations. Analysing the most preferred non-linear association function from the above comparative study with a normalized separation value significantly enhances the robustness and reliability of the proposed method. However, looking at the data shows that non-linearity does not change the normalized methods used to rank the results of the objective function and the normalized distance parameters. By this approach, both levels of satisfaction and the difference between satisfaction and dissatisfaction are higher. The computational technique outperforms traditional methods and gives a comprehensive solution to optimization problems with many objectives.

When faced with the complexity and unpredictability of industrial processes, the suggested dual-hesitant fuzzy optimization method, which incorporates both membership and non-membership functions performed better, as discussed in the previous chapter. Using a normalized distance operation, which places all objective functions on the same scale, increases the approach's trustworthiness and inclusion of hesitant sets with the intuitionistic approach, making them more realistic with several experts' opinions.

5.5.1 Major Findings

- The strategy ensures consistent and comparable findings by eliminating scale dependability, allowing decision-makers to make intelligent decisions regardless of the size of the factors.

- Results confirm the distinctiveness and efficiency of the proposed methodologies when compared to existing methodologies.
- The data analysis shows that the placement of results acquired using normalized methods applied to the distance parameters is unaffected by the inclusion of non-linearity.

Chapter 6

Enhanced Technologies in Various Real-Life Circumstances

Continuing from the last chapter, which highlighted the importance of normalized distance functions and fuzzy set extensions for improving multi-objective optimization, this chapter shows how these tools can be used in various real-world contexts. Starting with biomimetic systems and smart city transportation, we applied the improved fuzzy method using triangular fuzzy numbers, gradually moving from linear to non-linear membership functions to see how they affect the system. In addition, the method is expanded to include titanium alloy composition-related material science applications. After that, the chapter delves into how to apply dual hesitant fuzzy approaches to complicated production challenges and then how to apply enhanced triangular intuitionistic fuzzy optimization to the manufacturing sector. This methodical procedure as shown in Figure 6.1 verifies the efficacy of each approach in practical settings and permits comparative analysis among fuzzy extensions.

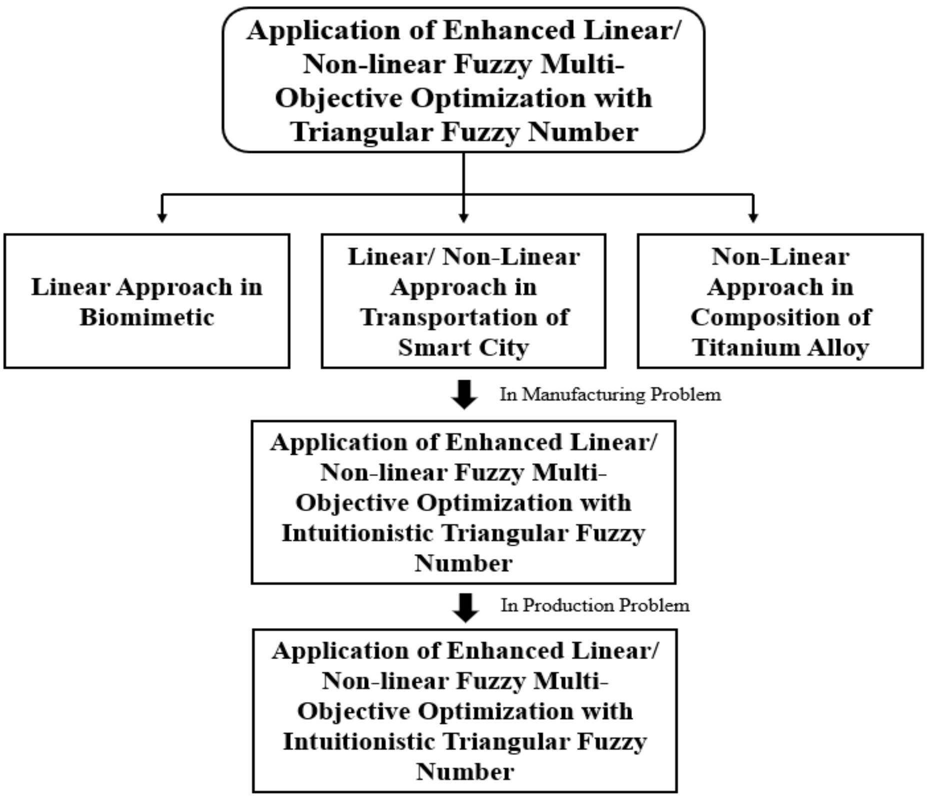


Figure 6.1: Conceptual flow and organisation of the chapter

6.1 Introduction

The focus of this study into the possible uses of applied systems and MOLPP in the field of tissue engineering is on wound healing and the problems that come with having serious skin injuries. The use of scaffold-based 3D bioprinting, an additive manufacturing technique well-known for producing cell-laden structures, is revolutionizing traditional tissue engineering. A multifaceted approach is required to tackle the intricacies of living tissues and the mechanical characteristics they possess. We can use several different bioprinting methods to repair damaged tissues [152]. Their results are based on a variety of criteria. The aim is to determine the bioprinting technology that is superior based on a variety of criteria. For the most appropriate choice of method, we need to optimize these factors. Therefore, we employ multi-objective optimization to address this problem. According to section 4.1, fuzzy theory generates fuzzy numbers to deal with the parameter's linguistic outcomes. The enhanced fuzzy approach with linear quality is then applied to this problem.

We conduct a comparative analysis to pinpoint the optimal non-linear membership function, considering the transportation issues in smart cities. We apply the best approach to a real-life case study to determine the optimal composition of titanium alloy. Identifying the significant relationships between mechanical properties and chemical component proportions can enhance our understanding of the material's behaviour. We can achieve these objectives by applying linear regression analysis. Since fuzzy-based optimization is adapted to deal with imprecise and ambiguous data, it can help with a complex decision-making process that aims to find a middle ground between competing goals. By optimizing component proportions, this study provides material engineers with a workable framework for improving mechanical performance in a variety of contexts.

Our research utilizes an advanced intuitionistic fuzzy technique with a normalized distance function, utilizing both linear and non-linear association functions to simultaneously optimize multiple objectives in a real-life manufacturing management case study. This approach leverages the most efficient (exponential v/s sigmoidal) behaviour of associated functions. The intuitionistic fuzzy approach is a flexible and useful way to solve multi-purpose optimization problems in manufacturing planning and administration. It does this by using a normalized distance function, membership and non-membership operations with exponential and sigmoidal (non-linear) behaviour and combining them in a way that makes sense.

An enhanced dual-hesitant fuzzy approach analyses the manufacturing problem based on a real-life case study. We assign both linear and non-linear association functions to find the optimal compromise solution. A comparative study has been done to find the effects of improved approaches in relation to conventional approaches.

6.2 Application of Enhanced Linear/Non-linear Fuzzy Multi-Objective Optimization with Triangular Fuzzy Number

As in Chapter 5, we enhance the fuzzy approach with a triangular fuzzy approach utilizing a normalized distance function. We will discuss three different case studies: one focusing on the linear behaviour of the membership function, another on the comparative analysis of various non-linear functions, and the final one focusing on the resulting non-linear membership function.

6.2.1 Linear approach in biomimetic

We are using an improved fuzzy method with triangular fuzzy numbers for a biomimetic application, including the straight-line shape of the membership function from section 5.2.1 of Chapter 5.

Inkjet, laser, extrusion, stereolithography, and microfluidic bioprinting are the most common forms of contemporary skin 3D bioprinting technologies. Inkjet bioprinting procedures expel liquid drops containing biomaterials and cells from the nozzle tip [158]. By shining high-energy pulses of laser light on a thin surface covered with laser-absorbing substances, as in laser bioprinting [159], the bioink particles force the biomaterial and cells to separate from the backing plate and place on the surface of the platform. In extrusion bioprinting, either air pressure or a machinery-driven nozzle deposits bioink on a platform to form a two-dimensional structure. As the nozzle or generating platform moves along the z-axis, bioink builds in layers to produce a three-dimensional structure [160]. In stereolithography bioprinting [161], a UV lamp or laser casts light onto a polymer solution, polymerizing it into the desired shape. Collecting potatoes from mashed potatoes, diced potatoes, filar potatoes, and sheet potatoes, respectively, is analogous to established 3D bioprinting techniques such as laser, inkjet, extrusion, and stereolithography bioprinting. In contrast to conventional bioprinters, the microprinting apparatus used in micro bioprinting may produce artificial skin in a shorter amount of time [162]. Table 6.1 provides data through a secondary source [163] about the technologies.

Table 6.1: Parametric responses for various techniques of bioprinting

Parameter	Inkjet (I)	Laser (L)	Extrusion (E)	Stereolithography (S)	Microfluidic (M)
Cost (C)	Low	High	Medium	Low	Low
Cell viability(V)	>85%	>95%	40–80%	>85%	>80%
Print speed(S)	Fast	Medium	Slow	Fast	Fast
Resolution(R)	High	High	Medium	High	High
Cell density(D)	Low	Medium	High	Medium	High

a) Fuzzification of linguistic variables:

We define fuzzy numbers for linguistic variables for cost, resolution, cell density, print speed, and cell viability, as described in Table 6.2, with the help of triangular fuzzy numbers:

Table 6.2**:** Fuzzy numbers corresponding to linguistic variables

Criteria	Fuzzy number	Linguistic variable
	(0,1,2)	Low(L)
Cost (C), Resolution (R), Cell density (D)	(2,3,4)	Medium(M)
	(4,5,6)	High(H)
	(0,1,2)	Slow(S)
Print speed (S)	(2,3,4)	Medium(M)
	(4,5,6)	Fast(F)
	(0,1,2)	40-80%
Cell viability(V)	(2,3,4)	>80%
	(4,5,6)	>85%
	(6,7,8)	>95%

Now we have to minimize cost, maximize resolution, cell density, print speed, and cell viability. We now present the updated Table 6.3, obtained by fuzzifying the data presented in Table 6.1:

Table 6.3: New table with numerical responses of various techniques

Parameters	I	L	E	S	M
C	(0,1,2)	(4,5,6)	(2,3,4)	(0,1,2)	(0,1,2)
S	(4,5,6)	(2,3,4)	(0,1,2)	(4,5,6)	(4,5,6)
V	(4,5,6)	(6,7,8)	(0,1,2)	(4,5,6)	(2,3,4)
R	(4,5,6)	(4,5,6)	(2,3,4)	(4,5,6)	(4,5,6)
D	(0,1,2)	(2,3,4)	(4,5,6)	(2,3,4)	(4,5,6)

b) Defuzzification of fuzzy numbers:

After applying α- cut technique of defuzzification, the resulting values of Table 6.3 are given as in Table 6.4:

Table 6.4: Values of parameters after defuzzification

Parameters	**I**	**L**	**E**	**S**	**M**
C	1	5	3	1	1
S	5	3	1	5	5
V	5	7	1	5	3
R	5	5	3	5	5
D	1	3	5	3	5

6.2.1.1 Problem formulation

We treat the methods as choice variables that can take on just two values: 1 if we use the method and 0 if we don't. Here is a description of how to formulate and resolve the resulting multi-objective optimization problem:

a) Decision variables:

Here, we need to decide which bioprinting technique to employ. Therefore, we will use techniques (I, L, E, S, M) as decision variables to determine their adoption or rejection. We will take the value of the decision variable as 1 for the presence of the technique and 0 for its absence.

b) Objective Functions:

This section presents five equations that outline the objectives of the model under optimization. The initial goal is to maximize the total resolution parameter. The second goal is to maximize cell density; the third goal is to maximize cell viability; the fourth goal is to optimize printing speed; and the final is to minimize the technique's cost. Here are the corresponding mathematical equations for each goal:

Resolution $$Max\ R = 5I + 5L + 3E + 5S + 5M \quad (6.1)$$

Cell density $$Max\ D = I + 3L + 5E + 3S + 5M \quad (6.2)$$

Cell viability $$Max\ V = 5I + 7L + 1E + 5S + 5M \quad (6.3)$$

Speed $$Max\ S = 5I + 3L + 1E + 5S + 5M \quad (6.4)$$

Cost $$Min\ C = 1I + 5L + 3E + 1S + 1M \quad (6.5)$$

c) Constraints associated:

This section outlines the numerous constraints or limitations required to accomplish the multi-objective optimization discussed in the previous section. There are a total of two equations. The first constraint specifies that we can only use one technique at a time, so the sum of all decision variables should be 1. The second constraint states that the value of variables should be between 0 and 1. The last one shows that the values for decision variables can be 0 or 1. The limitations are shown in the following mathematical form:

$$\left.\begin{array}{c} I+L+E+S+M=1 \\ 0 \leq I,L,E,S,M \leq 1 \\ I,L,E,S,M \in \{0,1\} \end{array}\right\} \tag{6.6}$$

6.2.1.2 Solution

LINGO 18.0 × 64 software calculates the best possible objective function values on the basis of the constraints provided above.

Max $R = \bar{R} = 5$ at multiple points, nadir point of $R = R^n = 3$ at point (0,0,1,0,0)

Max $D = \bar{D} = 5$ at multiple points, nadir point of $D = D^n = 1$ at point (1,0,0,0,0)

Max $V = \bar{V} = 7$ at point (0,1,0,0,0), nadir point of $V = V^n = 1$ at point (0,0,1,0,0)

Max $S = \bar{S} = 5$ at multiple points, nadir point of $S = S^n = 1$ at point (0,0,1,0,0)

Min $C = \bar{C} = 1$ at multiple points, nadir point of $C = C^n = 5$ at point (0,1,0,0,0).

Now, the model's predicted distance functions are as follows:

$$Đ_R(x) = \left|\frac{\bar{R} - R(x)}{(109)^{1/2}}\right| = \frac{5 - (5I + 5L + 3E + 5S + 5M)}{\sqrt{109}} \tag{6.7}$$

$$Đ_D(x) = \left|\frac{\bar{D} - D(x)}{(69)^{1/2}}\right| = \frac{5 - (I + 3L + 5E + 3S + 5M)}{\sqrt{67}} \tag{6.8}$$

$$Đ_V(x) = \left|\frac{\bar{V} - V(x)}{(125)^{1/2}}\right| = \frac{7 - (5I + 7L + 1E + 5S + 5M)}{\sqrt{125}} \tag{6.9}$$

$$Đ_S(x) = \left|\frac{\bar{S} - S(x)}{(85)^{1/2}}\right| = \frac{5 - (5I + 3L + 1E + 5S + 5M)}{\sqrt{85}} \tag{6.10}$$

$$Đ_C(x) = \left|\frac{\bar{C} - C(x)}{(37)^{1/2}}\right| = \frac{-1 + (1I + 5L + 3E + 1S + 1M)}{\sqrt{37}} \tag{6.11}$$

Maximum separations for two objectives can be expressed as:

$$\overline{Đ_R}(x) = \frac{2}{\sqrt{109}}, \overline{Đ_D}(x) = \frac{4}{\sqrt{67}}, \overline{Đ_V}(x) = \frac{6}{\sqrt{125}}, \tag{6.12}$$

$$\overline{Đ_S}(x) = \frac{4}{\sqrt{85}}, \overline{Đ_C}(x) = \frac{4}{\sqrt{37}}$$

$$=> Max\ \{\overline{Đ_R}(x), \overline{Đ_D}(x), \overline{Đ_V}(x), \overline{Đ_S}(x), \overline{Đ_C}(x)\} = \overline{Đ_C}(x) \tag{6.13}$$

$$=> \ D = \frac{4}{\sqrt{37}} = 0.657 \tag{6.14}$$

After applying the enhanced fuzzy approach with a triangular fuzzy number, the final single-objective problem is given by:

$$\begin{cases} Max\ \gamma \\ \text{Sub. to. } 2\gamma \le 5I + 5L + 3E + 5S + 5M - 3 \\ 4\gamma \le I + 3L + 5E + 3S + 5M - 1 \\ 6\gamma \le 5I + 7L + 1E + 5S + 5M - 1 \\ 4\gamma \le 5I + 3L + 1E + 5S + 5M - 1 \\ 4\gamma \le -(1I + 5L + 3E + 1S + 1M) + 5 \\ I + L + E + S + M = 1 \\ 0 \le I, L, E, S, M \le 1 \\ I, L, E, S, M \ \in \{0,1\} \end{cases} \tag{6.15}$$

6.2.1.3 Results

After solving eq (6.7) with LINGO 18.0 × 64, the value of γ = 0.7277059 at point (0,0,0,0,1). The values of goals at this point are R = 5, D = 5, V = 5, S = 5, and C = 1. Table 6.5 provides the parametric values corresponding to this point. Based on these findings, it's clear that the microfluidic technique would come out on top in a comparison of bioprinting methods if only the five criteria are considered.

Table 6.5: Parametric values according to proposed and conventional approaches

Parameter	Proposed Approach	Conventional Approach
γ	0.7277059	0.667
δ	0.272294	0.333
γ-δ	0.455412	0.334
Optimal Point	(0,0,0,0,1)	(0,0,0,0,1)
Total Functional Value	21	21

Total Normalized Distance	0.179	0.179

6.2.1.4 Comparative Analysis

Figure 6.2 shows that our proposed methodology provides more satisfaction and less dissatisfaction than the existing one. Other parameters will remain the same as the conventional approach, but improvement in satisfaction level make it more useful to apply in such types of studies.

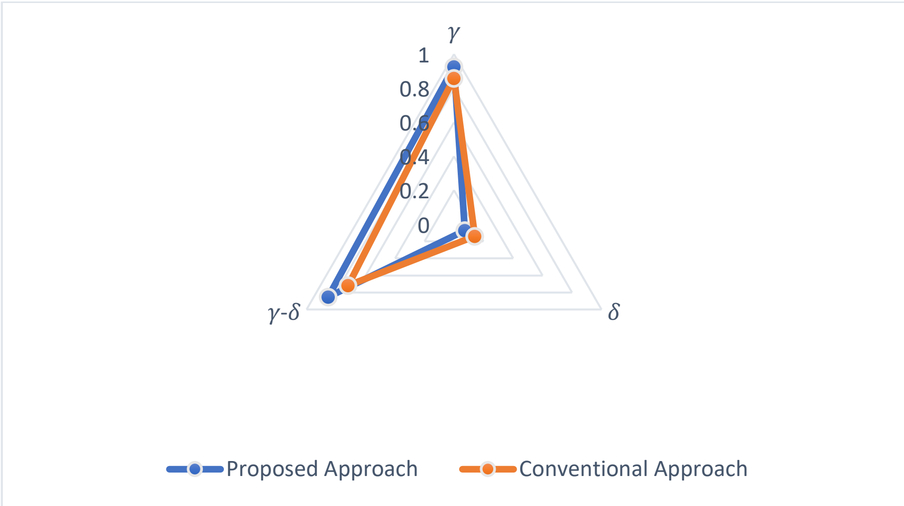


Figure 6.2: Comparative graph of parametric values with enhanced and conventional approaches

6.2.2 Linear/Non-linear approach in transportation of smart city

The enhanced fuzzy approach with triangular fuzzy numbers including the linear/non-linear nature of the membership function [157] provided in section 5.2.2 of chapter 5 is used here for the transportation application explained below:

Let vehicle types 1 and 2 be present at a location that corresponds to their capacities and availability. The numerical comfort level of car no. 1 is 2 units, while that of car no. 2 is 3. After deducting all the ride's expenses, vehicle 1 can earn \$3.5 per rider, whereas car 2 can lose \$4. Then, after imposing some constraints provided in eq (6.18) on the vehicles available, we need to determine how many will make the final roster for the journey.

6.2.2.1 Problem Formulation

The problem encompasses two objectives for optimization under linear constraints, resulting in MOLPP. We define the problem's components as follows:

a) Decision variables:

We need to determine the number of vehicles for each type. So, the decision variables will be the number of these vehicles denoted by x_1 and x_2 for types 1 and 2 respectively.

b) Objective functions:

There are two set objectives: We should pursue two primary goals simultaneously: (a) enhancing the comfort level as much as possible and (b) maximizing profits.

$$Max\ f_1 = 2x_1 + 3x_2 \tag{6.16}$$

$$Max\ f_2 = 3.5x_1 - 4x_2 \tag{6.17}$$

c) Constraints:

We define the constraints based on their budget and space capacity, as shown below:

$$\left.\begin{array}{c} sub.to.: -x_1 + 2x_2 \leq 5 \\ 2x_1 + 5x_2 \leq 10 \\ 5x_1 - 4x_2 \leq 4 \\ 3x_1 + 4x_2 \leq 6 \\ x_{1,}x_2 \geq 0 \end{array}\right\} \tag{6.18}$$

6.2.2.2 Solution

The optimal value of the objective functions subject to the specified restrictions are determined graphically.

Max $f_1 = \overline{f}_1 = 4.5$ at point (0,1.5) and nadir point of $f_1 = f_1^n = 1.6$ at point (0.8,0).

Max $f_2 = \overline{f}_2 = 2.8$ at point (0.8,0) and nadir point of $f_2 = f_2^n = -6$ at point (0,1.5).

Now, the model-predicted distance functions are as follows:

$$Đ_1(x) = \left|\frac{\overline{f}_1 - f_1(x)}{\sqrt{\sum_{k=1}^{2} {p_k^1}^2}}\right| = \frac{4.5 - (2x_1 + 3x_2)}{\sqrt{13}} \tag{6.19}$$

$$\text{Đ}_2(x) = \left| \frac{\overline{f}_2 - f_2(x)}{\sqrt{\sum_{k=1}^{2} p_k^{2\,2}}} \right| = \frac{2.8 - (3.5x_1 - 4x_2)}{\sqrt{28.25}} \tag{6.20}$$

Maximum distance functions for two objectives can be expressed as:

$$Max\ \text{Đ}_1(x) = \frac{2.9}{\sqrt{13}}, Max\ \text{Đ}_2(x) = \frac{8.8}{\sqrt{28.25}} \tag{6.21}$$

As

$$Max\ \text{Đ}_1(\text{ɣ}) < Max\ \text{Đ}_2(\text{ɣ}) \tag{6.22}$$

$$=> \ \text{D} = \frac{8.8}{\sqrt{28.25}} = 1.66 \tag{6.23}$$

The curve of satisfaction level increases with the increment of the first functional values, and it increases from 0 to 1 for values -1.47 to 4.5. For second function, it increases from -6 to 2.8.

Eq (6.24)-(6.28) provide the final problem with a single objective function that includes all membership functions, and Figures 6.3–6.7 provide a geometrical representation of them:

a) *With linear membership function:*

$$\left.\begin{array}{c} Max\ \gamma \\ Sub.to.:\ 31.73\gamma \le 10.62x_1 + 15.95x_2 + 7.81 \\ 8.8\gamma \le 6 + 3.5x_1 - 4x_2 \\ -x_1 + 2x_2 \le 5 \\ 2x_1 + 5x_2 \le 10 \\ 5x_1 - 4x_2 \le 4 \\ 3x_1 + 4x_2 \le 6 \\ x_1, x_2 \ge 0 \end{array}\right\} \tag{6.24}$$

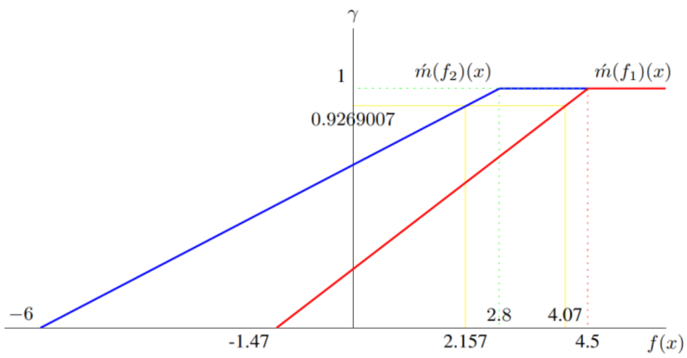


Figure 6.3: Linear membership (satisfaction) function corresponding to both objective functions

b) *With Parabolic membership function:*

$$\left.\begin{array}{c} Max\ \gamma \\ Sub.to.:\ 1006.72\gamma \le (10.63x_1 + 15.95x_2 + 7.811)^2 \\ 2187.68\gamma \le (31.89 + 18.60x_1 - 21.26x_2)^2 \\ -x_1 + 2x_2 \le 5 \\ 2x_1 + 5x_2 \le 10 \\ 5x_1 - 4x_2 \le 4 \\ 3x_1 + 4x_2 \le 6 \\ x_1, x_2 \ge 0 \end{array}\right\} \quad (6.25)$$

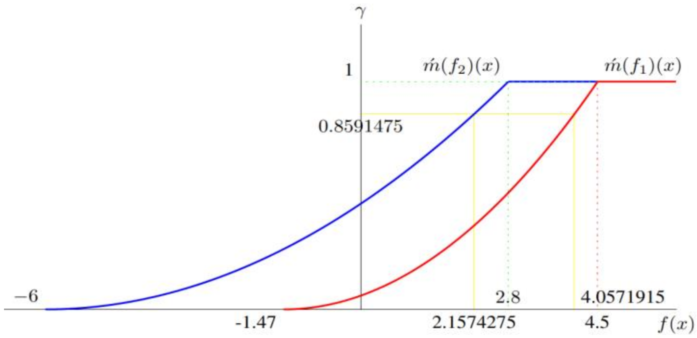


Figure 6.4: Parabolic membership (satisfaction) function corresponding to both objective functions

c) With Hyperbolic membership Function:

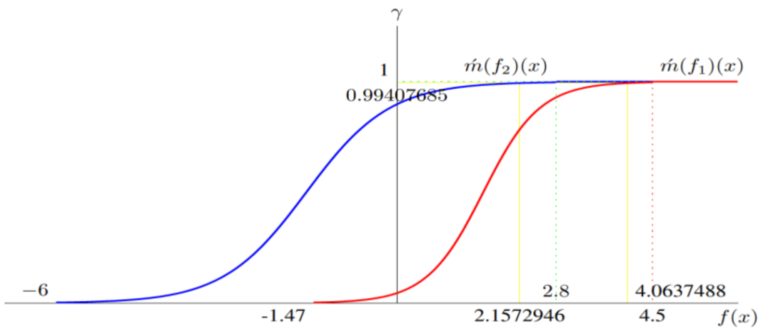


Figure 6.5: Hyperbolic membership (satisfaction) function corresponding to both objective functions

$$\left.\begin{array}{c} Max\ \gamma \\ Sub.to.:\ \gamma \le \frac{1}{2}\tanh\left((3.624)\left(\frac{4.4}{\sqrt{28.25}} - \frac{4.5 - 2x_1 - 3x_2}{\sqrt{13}}\right)\right) + \frac{1}{2} \\ \gamma \le \frac{1}{2}\tanh\left((3.624)\left(\frac{4.4}{\sqrt{28.25}} - \frac{2.8 - 3.5x_1 + 4x_2}{\sqrt{28.25}}\right)\right) + \frac{1}{2} \\ -x_1 + 2x_2 \le 5 \\ 2x_1 + 5x_2 \le 10 \\ 5x_1 - 4x_2 \le 4 \\ 3x_1 + 4x_2 \le 6 \\ x_1, x_2 \ge 0 \end{array}\right\} \quad (6.26)$$

d) With Exponential Membership Function:

$$\left.\begin{array}{c} Max\ \gamma \\ Sub.to.:\ \gamma \leq (1.2)(1 - \exp((-0.0564)(10.63x_1 + 15.95x_2 + 7.811)) \\ \gamma \leq (1.2)(1 - \exp((-0.2034)(31.89 + 18.60x_1 - 21.26x_2) \\ -x_1 + 2x_2 \leq 5 \\ 2x_1 + 5x_2 \leq 10 \\ 5x_1 - 4x_2 \leq 4 \\ 3x_1 + 4x_2 \leq 6 \\ x_{1,}x_2 \geq 0 \end{array}\right\} \quad (6.27)$$

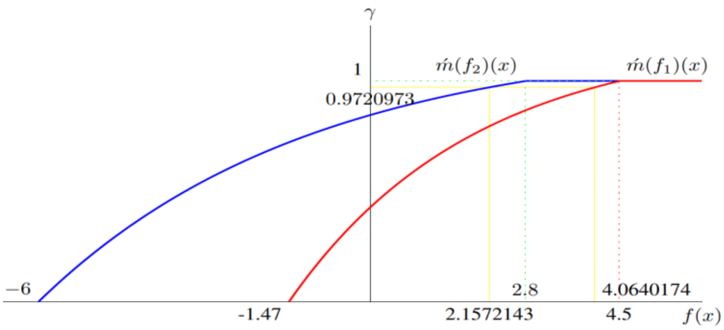


Figure 6.6: Exponential membership (satisfaction) function corresponding to both objective functions

e) With Sigmoidal Membership Function:

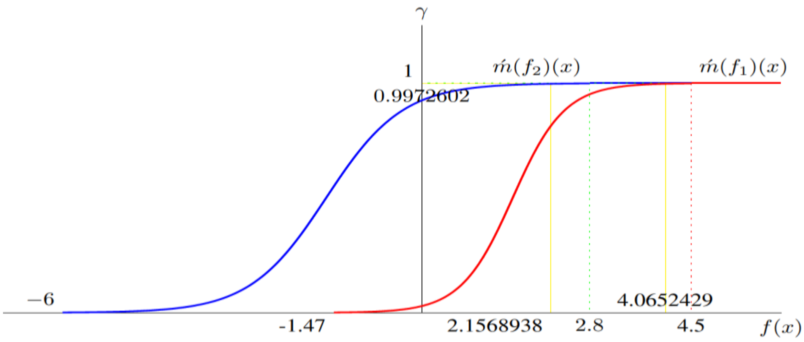


Figure 6.7: Sigmoidal membership (satisfaction) function corresponding to both objective functions

$$\left.\begin{array}{c} Max\ \gamma \\ Sub.to.:\ \gamma \le 1 - \left(\dfrac{1}{1+0.001001e^{8.343\left(\frac{8.8}{\sqrt{28.25}}-\frac{4.5-2x_1-3x_2}{\sqrt{13}}\right)}}\right) \\ \gamma \le 1 - \left(\dfrac{1}{1+0.001001e^{8.343\left(\frac{8.8}{\sqrt{28.25}}-\frac{\mathbf{2.8-3.5}x_1+4x_2}{\sqrt{\mathbf{28.25}}}\right)}}\right) \\ -x_1+2x_2 \le 5 \\ 2x_1+5x_2 \le 10 \\ 5x_1-4x_2 \le 4 \\ 3x_1+4x_2 \le 6 \\ x_{1,}x_2 \ge 0 \end{array}\right\} \quad (6.28)$$

6.2.2.3 Results

Once we input all these problems into the LINGO 18.0 × 64 software, we obtain the optimal points, which serve as the basis for the values of the various parameters, as shown in Table 6.6.

Table 6.6: Values of different parameters for various membership functions of enhanced fuzzy approach

Parameter	Linear	Parabolic	Hyperbolic	Exponential	Sigmoidal
γ	0.9269007	0.8591475	0.99407685	0.9720973	0.9972602
δ	0.0730993	0.1408525	0.00592315	0.0279027	0.0027398
γ-δ	0.8538014	0.718295	0.9881537	0.9441946	0.9945204
Optimal point	(1.228849, 0.5360615)	(1.228381,0.5 354765)	(1.228478, 0.5355976)	(1.228525,0.5 356558)	(1.228738, 0.5359223)
f_1	4.0658825	4.0571915	4.0637488	4.0640174	4.0652429
f_2	2.1567255	2.1574275	2.1572946	2.1572143	2.1568938
Total functional value	6.222068	6.214619	6.2210434	6.2212317	6.2221367
Deviation	0.24163088046	0.2437092	0.2419156	0.2418562	0.2415804

6.2.2.4 Comparative Analysis

Here, we examine various membership functions by comparing them with various parameters. Some of these parameters are functional values, satisfaction and dissatisfaction levels, and value of normalized distance. These are properly explained below:

a) Values of goal functions:

In consideration of the goal functions, we received three parameters, the values of which we compared with the comparative graph in Figure 6.7.

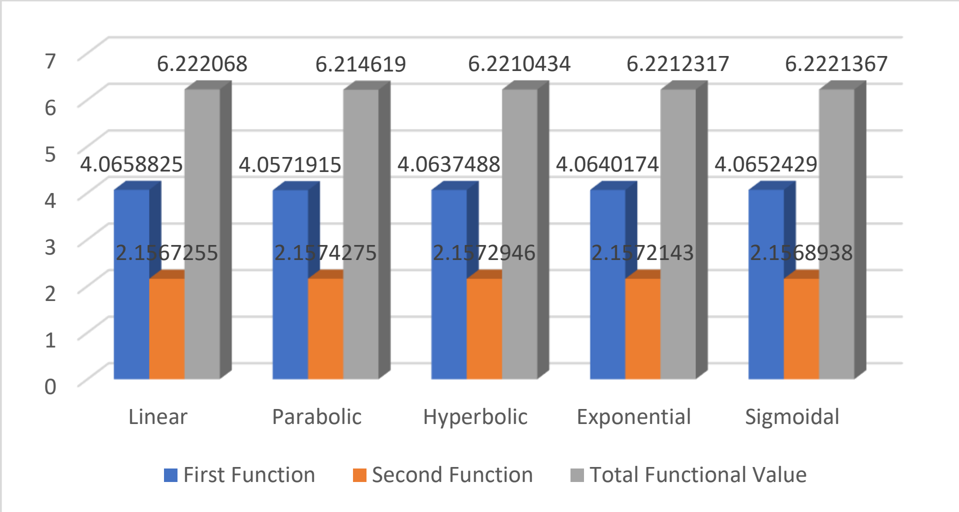


Figure 6.8: Comparative graph of functional values with fuzzy approach having various membership functions

The graph in Figure 6.8 demonstrates that the sigmoidal membership function increases the total value of all goals.

*b) **δ** and **γ** values:*

The satisfaction and dissatisfaction levels are important parameters for the comparison of fuzzy approaches, which can be calculated as:

$$\gamma = \min_{i \in \{1,2\}} \acute{m}_i(x) \tag{6.29}$$

$$\delta = 1 - \gamma \tag{6.30}$$

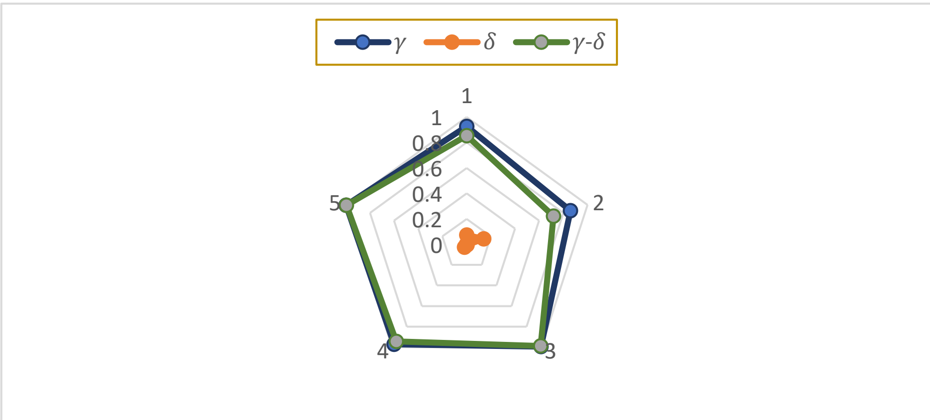


Figure 6.9: Comparative graph of satisfaction values with fuzzy approach having various membership functions

This comparative analysis by Figure 6.9 shows that the technique with the sigmoidal membership function improves outcomes because we need to increase the value of the degree of fulfilment and the difference while decreasing the value of the discontent level.

c) Value of normalized distance from ideal points:

We can calculate the normalized distance from ideal points of functions using the following formula:

$$Đ(x) = \sum_{i=1}^{2} \left(\frac{\overline{f}_i - f_i(x)}{\sqrt{\sum_{k=1}^{2} {p_k^i}^2}} \right) \tag{6.31}$$

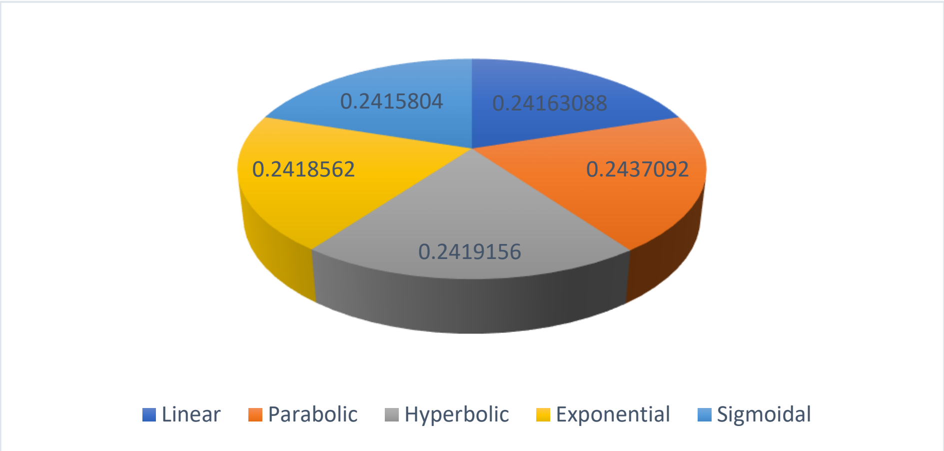


Figure 6.10: Comparative graph of normalized distance values with fuzzy approach having various membership functions

Figure 6.10 demonstrates how the normalized distance from ideal points of goal functions and the normalization procedure successfully reduces it.

The analysis reveals that the sigmoidal membership function provides the lowest deviation value and the highest satisfaction (total functional value). Therefore, we will prefer the sigmoidal non-linear function for further calculation.

6.2.3 Non-linear approach in composition of titanium alloy

In the last section, we compared different methods using a better fuzzy approach with a triangular fuzzy number, looking at both linear and non-linear membership functions. The results indicated that the sigmoidal function yielded the best results. Therefore, we have to apply this approach to the material science application [164].

Material science and engineering are based on mechanical properties [165]. Understanding how materials respond to mechanical forces is paramount to designing and developing structures and components that can withstand a wide range of loads, from the weight of an aircraft to the forces applied to a medical implant. These properties encompass a multitude of characteristics, such as strength, modulus of elasticity, and elongation, each of which is pivotal in determining how a material will perform under various conditions. In common usage, the word "modulus" denotes the stiffness measurement known as Young's modulus or modulus of elasticity. The stress-strain ratio is a mechanical characteristic that characterizes a material's behaviour during deformation [166]. Tensile strength is the ultimate stress that a material can withstand before giving way under controlled stretching or pulling. A material begins to deform plastically when it reaches its yield point, a characteristic known as yield strength or yield stress. A mechanical component's maximum allowed load is often determined by calculating its yield strength, which is the highest force that can be applied without permanently deforming the component. A material's elongation is its measurably measured propensity to lengthen under stress [167]. One material that has captured the attention of engineers and researchers alike for its exceptional mechanical properties is titanium, both in its pure form and as a central component in a myriad of titanium alloys. Chemical composition, temperature, and pressure all have an impact on these characteristics. This investigation, however, centres on the concentration of chemical components. Research [168], shows that one mechanical property of unalloyed titanium improves as the concentration of a specific component declines at different chemical concentrations. Because it is too expensive and takes too long to do the experiments needed to look at these mechanical properties at different chemical concentrations, we need to come up with a way to use computers to find the relationship between these properties and the best condition for all the mechanical qualities to be at their best at the same time. Our research provides a solution to this constraint. Through an in-depth examination of the mechanical properties, their significance, and their role in chemical component selection and design, we aim to provide a comprehensive understanding of the pivotal role that these properties play in materials science and engineering.

a) Data Collection:

We have two data sets: one shows the metal materials' percentages, and the other shows their mechanical properties. Table 6.7 describes the total information.

Table 6.7: Data of unalloyed titanium

Grade	Tensile strength(T)	Yield strength(Y)	Elongation (E)	Modulus (M)	O_2	N	Fe	C	H	Ti
grade1	240	170	24	102	0.18	0.03	0.2	0.08	0.015	99.495
grade2	345	275	20	102	0.25	0.03	0.3	0.08	0.015	99.325
grade3	450	380	18	102	0.35	0.05	0.3	0.08	0.015	99.205
grade4	550	483	15	104	0.4	0.05	0.5	0.08	0.015	98.955

b) Data Pre-processing:

The multiple linear regression process eliminates the component proportions of carbon and hydrogen, which are the same in all four grades. Thus, the resulting data will be as shown in Table 6.8:

Table 6.8: Processed data of unalloyed titanium

Grade	T	Y	E	M	O_2	N	Fe	Ti
grade1	240	170	24	102	0.18	0.03	0.2	99.495
grade2	345	275	20	102	0.25	0.03	0.3	99.325
grade3	450	380	18	102	0.35	0.05	0.3	99.205
grade4	550	483	15	104	0.4	0.05	0.5	98.955

To process the data in this study, we have used R Studio. Our model, known as multiple linear regression, assumes that the dependent variable linearly depends on the compositions of chemical components. We assume the intercept is zero since the mechanical property is zero without metal or chemicals. We adopt the following mathematical form:

$$f = \beta 1 * O_2 + \beta 2 * N + \beta 3 * Fe + \beta 4 * Ti \qquad (6.32)$$

c) Multiple Linear Regression Analysis:

According to the data provided in Table 6.8, all mechanical properties do not follow the same pattern with similar chemical components. As we can see, when nitrogen is fixed and the other two chemical components proportions increase, then the modulus remains the same, tensile and yield strength increase, but elongation decreases. When ferrous remains the same proportion but the other two chemical components increase, then the modulus remains the same, tensile and yield strength increase, but elongation decreases, but not in the same proportion as in the earlier case. In the last case, when all three components increase, then modulus, yield strength, and tensile strength increase, but elongation decreases. Considering all this analysis, we can conclude that the coefficients in linear regression do not exhibit a positive trend.

d) Interpretation of Regression Results:

$$\left.\begin{aligned} T &= 1222.29173\, O_2 - 861.04160\, N + 194.51395\, Fe + 0.06951\, Ti \\ Y &= 1188.2210\, O_2 - 695.1123\, N + 217.1099\, Fe - 0.6679\, Ti \\ E &= -55.2664\, O_2 + 178.0669\, N - 0.8219\, Fe + 0.2892\, Ti \\ M &= -21.2257\, O_2 + 112.1076\, N + 16.5521\, Fe + 0.9965\, Ti \end{aligned}\right\} \quad (6.33)$$

6.2.3.1 Problem formulation

The problem provides 4 goals and 5 limitations with a linear nature, which means it involves the application of MOLPP, whose components are given below:

a) Decision Variables:

Oxygen, nitrogen, ferrous, and titanium are the four variables whose values determine the values of goal functions, which means they will act as decision variables.

b) Objective function formation:

This section's four equations outline the model's optimization goals. First and foremost, we want to maximize the tensile strength of the metal. Keeping the yield strength to a minimum is the second objective; maximizing the elongation factor is the third; and maximizing the modulus is the fourth. We evaluate these objectives.

c) Constraints associated:

The maximum amount of the phase components should not exceed the maximum amount of that component in the given data set, and the total proportion of all the components should be equal to100%:

$$\left.\begin{aligned} &\text{Oxygen proportion: } 0.18 <= O_2 <= 0.4 \\ &\text{Nitrogen proportion: } 0.03 <= N <= 0.05 \\ &\text{Ferrous proportion: } 0.2 <= Fe <= 0.5 \\ &\text{Titanium proportion: } 98.955 \leq Ti \leq 99.495 \\ &\text{Total: } O_2 + N + Fe + Ti = 99.915 \end{aligned}\right\} \quad (6.34)$$

The resulting MOLPP by using eq (6.22) and (6.23) as objectives and constraints, respectively, are given below:

$$\left.\begin{array}{c}
\text{Max T} = 1222.29173\ O_2\ - 861.04160\ N\ + 194.51395\ Fe\ + 0.06951\ Ti \\
\text{Max Y} = 1188.2210\ O_2 - 695.1123\ N\ + 217.1099\ Fe - 0.6679\ Ti \\
\text{Max E} = -\ 55.2664\ O_2\ + 178.0669\ N - 0.8219\ Fe + 0.2892\ Ti \\
\text{Max M} = -\ 21.2257\ O_2\ + 112.1076\ N\ + 16.5521\ Fe\ + 0.9965\ Ti \\
\text{Sub. to. } 0.18 <= O_2 <= 0.4 \\
0.03 <= N <= 0.05 \\
0.2 <= Fe <= 0.5 \\
98.955 \le Ti \le 99.495 \\
O_2 + N + Fe + Ti = 99.915
\end{array}\right\} \quad (6.35)$$

6.2.3.1 Solution

LINGO 18.0 × 64 finds the optimal values for the objective functions based on the constraints we specify.

Max $M=\overline{M}$=108.8987 at point (0.18,0.05,0.5,99.185) and nadir point M^n =100.99 at point (0.4,0.03,0.5,98.985).

Max T= $\overline{T}$= 567.2229 at point (0.4,0.03,0.5,98.985) and nadir point T^n =222.76 at point (0.18,0.05,0.2,99.485).

Max Y=$\overline{Y}$= 496.8779 at point (0.4,0.03,0.5,98.985) and nadir point Y^n=156.1 at point (0.18,0.05,0.2,99.485).

Max E=$\overline{E}$= 27.56208 at point (0.18,0.05,0.2,99.485) and nadir point E^n =11.45 at point (0.4,0.03,0.5,98.985).

Here are the distance measures that the model has predicted:

$$Đ_M(x) = \left|\frac{\overline{M} - M}{\sqrt{\sum_{k=1}^{4} p_k^{M^2}}}\right| = \frac{108.8987 - (-21.2257\ O_2 + 112.1076\ N + 16.5521\ Fe + 0.9965\ Ti)}{115.3} \quad (6.36)$$

$$Đ_T(x) = \left|\frac{\overline{T} - T}{\sqrt{\sum_{k=1}^{4} p_k^{T^2}}}\right| = \frac{567.2229 - (1222.29173\ O_2\ - 861.0416\ N + 194.51395\ Fe + 0.06951\ Ti)}{1507.72} \quad (6.37)$$

$$Đ_Y(x) = \left|\frac{\overline{Y} - Y}{\sqrt{\sum_{k=1}^{4} p_k^{Y^2}}}\right| = \frac{496.8779 - (1188.221\ O_2 - 695.1123\ N + 217.1099\ Fe - 0.6679\ Ti)}{1393.62} \quad (6.38)$$

$$Đ_E(x) = \left|\frac{\overline{E} - E}{\sqrt{\sum_{k=1}^{4} p_k^{E^2}}}\right| = \frac{27.56208 - (-55.2664\ O_2 + 178.0669\ N - 0.8219\ Fe + 0.2892\ Ti)}{186.45} \quad (6.39)$$

We can express the maximum differences for each goal as:

$$\bar{Đ}_M(x) = 0.069, \bar{Đ}_T(x) = 0.228, \bar{Đ}_Y(x) = 0.245, \bar{Đ}_E(x) = 0.0864 \quad (6.40)$$

$$=> Max\,\{\bar{Đ}_E(x), \bar{Đ}_M(x), \bar{Đ}_T(x), \bar{Đ}_Y(x)\} = \bar{Đ}_Y(x) \quad (6.41)$$

$$=> D = 0.245 \quad (6.42)$$

After applying sigmoidal membership function new LPP issue with a single objective:

$$\left.\begin{array}{c} max\,\gamma \\ sub.to.\ \gamma \le 1 - \left(\dfrac{1}{1 + 0.001001e^{56.38\left(\frac{-80.65+(-21.2257\ O_2+112.1076\ N+16.5521\ Fe+0.9965\ Ti)}{115.3}\right)}}\right) \\ \gamma \le 1 - \left(\dfrac{1}{1 + 0.001001e^{56.38\left(\frac{-197.83+(1222.29173\ O_2\ -861.0416\ N+194.51395\ Fe+0.06951\ Ti)}{1507.72}\right)}}\right) \\ \gamma \le 1 - \left(\dfrac{1}{1 + 0.001001e^{56.38\left(\frac{-155.441+(1188.221\ O_2-695.1123\ N+217.1099\ Fe-\ 0.6679\ Ti)}{1393.62}\right)}}\right) \\ \gamma \le 1 - \left(\dfrac{1}{1 + 0.001001e^{56.38\left(\frac{18.12+(-55.2664\ O_2+178.0669\ N-0.8219\ Fe+0.2892\ Ti)}{186.45}\right)}}\right) \\ 0.18 <= O_2 <= 0.4 \\ 0.03 <= N <= 0.05 \\ 0.2 <= Fe <= 0.5 \\ 98.955 \le Ti \le 99.495 \\ O_2 + N + Fe + Ti = 99.915 \\ 0 \le \gamma \le 1 \end{array}\right\} \quad (6.43)$$

6.2.3.2 Results

We create a sigmoidal participation function (where 0 is the lowest level and 1 is the highest) to find the best values for modulus (80.65, 108.8987), tensile strength (-197.832, 567.2229), yield strength (156.1, 496.88), and elasticity (-18.12, 27.56). The response to the previously described LPP, with LINGO 18.0 × 64 assistance, is as follows:

The value of $\gamma = 0.9810621$ at point (0.3501639, 0.05, 0.5, 99.01484). At this point, values of M=105.12, T=489.09, Y=423.74, and E=17.78.

Table 6.9: Normalized distance parameter values for various optimal points

Function	**Ideal values**	**At point (0.18,0.05,0.2,99.485)**	**At point (0.4,0.03,0.5,98.985)**	**At point (0.18,0.05,0.5,99.185)**	**Proposed Approach**
Modulus	108.8987	0.0405	0.068	0	0.033
Tensile strength	567.2229	0.23	0	0.19	0.052
Yield strength	496.8779	0.24	0	0.197	0.052
Elongation	27.56208	0	0.086	0.0018	0.052
Total		0.5105	0.1544	0.3888	0.137

According to Table 6.9, the total normalized distance of the objective functions from their ideal points is the minimum for our proposed approach in comparison to single-objective results.

The study analyses the optimal values of chemical components in Table 6.10.

Table 6.10: The optimal proportions of chemical components

Chemical Component	Optimal Proportion
Nitrogen	0.05
Oxygen	0.3501639
Ferrous	0.5
Carbon	0.8
Hydrogen	0.015

6.3 Application of Linear/Non-Linear Enhanced Fuzzy Multi-Objective Optimization with Triangular Intuitionistic Fuzzy Number in Manufacturing

As in Chapter 5, we enhance the fuzzy approach with an intuitionistic triangular fuzzy approach utilizing a normalized distance function. Two different case studies will be discussed here: one for the linear behaviour of the membership function and the second one for the resulting non-linear membership function from the case study of Section 5.3.2. Below is a real-life case study for the manufacturing problem, which will serve as the basis for a comparative analysis of various techniques.

6.3.1 Manufacturing problem

The data provided below is derived from a secondary source [169] with certain parameters and is sourced from the Chocoman firm in the United States. This company uses a variety of raw materials and procedures to manufacture chocolate bars, candies, and wafers. The Chocoman company's manufacturing process produced eight distinct chocolate items, requiring the blending of eight different raw materials in varying proportions and the use of nine different procedures. The primary concern in the initial instance is resolving an optimization problem with multiple goals. By the time the procedure for solving the problem reaches the optimal set of variables, the issue's multi-objective functions will have 8 parameters that need optimization and 29 restrictions that must be satisfied.

6.3.1.1 Problem Formation

Here is the complete mathematical description of the problem previously discussed as a multi-objective linear programming problem:

a) Decision variable:

We must identify the following products, known as decision variables, to optimize the values of objective functions:

Number of units	Product (chocolate type)
x_1	Milk (250g)
x_2	Milk (100g)
x_3	Crunchy (250g)
x_4	Crunchy(100g)
x_5	With nuts (250 g)
x_6	With nuts (100g)
x_7	Candy
x_8	Wafer

b) Objectives Formulation:

The functions that we have to optimize with the help of products are called objectives. Below, we describe five functions in total for optimization:

Revenue
$$Max\ f_1 = 375x_1 + 150x_2 + 400x_3 + 160x_4 + 420x_5 + 175x_6 + 400x_7 + 150x_8 \quad (6.44)$$

Profit
$$Max\ f_2 = 180x_1 + 83x_2 + 153x_3 + 72x_4 + 130x_5 + 70x_6 + 208x_7 + 83x_8 \quad (6.45)$$

Market share
$$Max\ f_3 = 0.25x_1 + 0.1x_2 + 0.25x_3 + 0.1x_4 + 0.25x_5 + 0.1x_6 + 0x_7 + 0x_8 \quad (6.46)$$

Production
$$Max\ f_4 = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 \quad (6.47)$$

Plant Utilization
$$Max\ f_5 = 1.65x_1 + 0.9x_2 + 1.975x_3 + 1.03x_4 + 1.75x_5 + 0.94x_6 + 4.2x_7 + 1.006x_8 \quad (6.48)$$

c) Constraints associated:

The conditions that are applied to products on the basis of different criteria are taken as restrictions, which are defined below:

$$x_1 - 0.6x_2 \leq 0$$
$$x_3 - 0.6x_4 \leq 0 \qquad (6.49)$$
$$x_5 - 0.6x_6 \leq 0$$
$$-56.25x_1 - 22.5x_2 - 60x_3 - 24x_4 - 63x_5 - 26.25x_6 + 400x_7 \cdot$$

Eatable items usage

$$87.5x_1 + 35x_2 + 75x_3 + 30x_4 + 50x_5 + 20x_6 + 70x_7 + 12x_8 :$$
$$62.5x_1 + 25x_2 + 50x_3 + 20x_4 + 50x_5 + 20x_6 + 30x_7 + 12x_8 : \qquad (6.50)$$
$$0x_1 + 0x_2 + 37.5x_3 + 15x_4 + 75x_5 + 30x_6 + 0x_7 + 0x_8 \leq$$
$$100x_1 + 40x_2 + 87.5x_3 + 35x_4 + 75x_5 + 30x_6 + 210x_7 + 24x_8$$
$$72x_8 \leq 200000$$

Packing items usage

$$500x_1 + 500x_3 + 250x_8 \leq 500000$$
$$450x_1 + 450x_3 + 450x_5 \leq 500000 \qquad (6.51)$$
$$60x_1 + 120x_2 + 60x_3 + 120x_4 + 60x_5 + 120x_6 + 1600x_7 + 25$$

Facility usage

$$0.5x_1 + 0.2x_2 + 0.425x_3 + 0.17x_4 + 0.35x_5 + 0.14x_6 + 0.6x_7 +$$
$$0.15x_3 + 0.06x_4 + 0.25x_5 + 0.1x_6 \leq 200$$
$$0.75x_1 + 0.3x_2 + 0.75x_3 + 0.3x_4 + 0.75x_5 + 0.3x_6 + 0.9x_7 +$$
$$0.25x_3 \leq 200 \qquad (6.52)$$
$$0.3x_8 \leq 100$$
$$0.5x_1 + 0.1x_2 + 0.1x_3 + 0.1x_4 + 0.1x_5 + 0.1x_6 + 0.2x_7 +$$
$$0.25x_1 + 0.25x_3 + 0.25x_5 + 0.1x_8 \leq 400$$
$$0.05x_1 + 0.3x_2 + 0.05x_3 + 0.3x_4 + 0.05x_5 + 0.3x_6 + 2.5x_7 +$$

Labour

$$0.3x_1 + 0.3x_2 + 0.05x_3 + 0.3x_4 + 0.3x_5 + 0.3x_6 + 2.5x_7 + 0.25x_8 \leq 1000 \qquad (6.53)$$

Demand

$$\left.\begin{aligned} x_1 &\leq 500 \\ x_2 &\leq 800 \\ x_3 &\leq 400 \\ x_4 &\leq 600 \\ x_5 &\leq 300 \\ x_6 &\leq 500 \\ x_7 &\leq 200 \\ x_8 &\leq 400 \end{aligned}\right\} \qquad (6.54)$$

Non-negativity

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \geq 0 \ \& \ \in Z \qquad (6.55)$$

By solving all the functions separately by LINGO 18.0 × 64:

Max $f_1=\overline{f_1}$=614613.2; optimal point= (2.52,800,260,600,300,500,75.42,333.33)

Max $f_2=\overline{f_2}$=267187.7; optimal point = (2.52, 800,260,600,300,500,75.42,333.33)

Max $f_3=\overline{f_3}$=357.1429; optimal point= (257.14,428.57,300,500,300,500,0,0)

Max $f_4=\overline{f_4}$= 2871.274; optimal point = (2.52, 800,260,600,300,500,75.42,333.33)

Max $f_5=\overline{f_5}$=3519.757; optimal point= (0,769.29,260,600,300,500,111.27,232.18)

Nadir point of $f_1 = f_1{}^n$ =574160.5; optimal point = (257.14,428.57,300,500,300,500,0,0).

Nadir point of $f_2 = f_2{}^n$=237756.51; optimal point= (257.14,428.57,300,500,300,500,0,0).

Nadir point of $f_3 = f_3{}^n$ =326.929; optimal point = (2.52, 800,260,600,300,500,75.42,333.33).

Nadir point of $f_4 = f_4{}^n$=2285.71; optimal point= (257.14,428.57,300,500,300,500,0,0).

Nadir point of $f_5 = f_5{}^n$ =2912.494; optimal point = (257.14,428.57,300,500,300,500,0,0).

Minimum point of $f_1 = f_1{}^w$=0

Minimum point of $f_2 = f_2{}^w$=0

Minimum point of $f_3 = f_3{}^w$=0

Minimum point of $f_4 = f_4{}^w$=0

Minimum point of $f_5 = f_5{}^w$=0

$$Đ_1(d) = \frac{614613.2 - (375x_1 + 150x_2 + 400x_3 + 160x_4 + 420x_5 + 175x_6 + 400x_7 + 150x_8)}{859.215} \quad (6.56)$$

$$Đ_2(d) = \frac{267187.7 - (180x_1 + 83x_2 + 153x_3 + 72x_4 + 130x_5 + 70x_6 + 208x_7 + 83x_8)}{373.945} \quad (6.57)$$

$$Đ_3(d) = \frac{357.1429 - (0.25x_1 + 0.1x_2 + 0.25x_3 + 0.1x_4 + 0.25x_5 + 0.1d_6 + 0x_7 + 0x_8)}{0.2175} \quad (6.58)$$

$$Đ_4(d) = \frac{2871.274 - (x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8)}{2.828} \quad (6.59)$$

$$Đ_5(d) = \frac{3519.757 - (1.65x_1 + 0.9x_2 + 1.975x_3 + 1.03x_4 + 1.75x_5 + 0.94x_6 + 4.2x_7 + 1.006x_8)}{31.092} \quad (6.60)$$

When the normalised distance parameter is set to its maximum value, the results are:

$$\bar{Đ}_1(x) = 47.081, \bar{Đ}_2(x) = 78.705, \bar{Đ}_3(x) = 138.914, \bar{Đ}_4(x) = 207.059, \bar{Đ}_5(x) = 19.55 \quad (6.61)$$

$$=> D = \frac{585.564}{2.828} \quad (6.62)$$

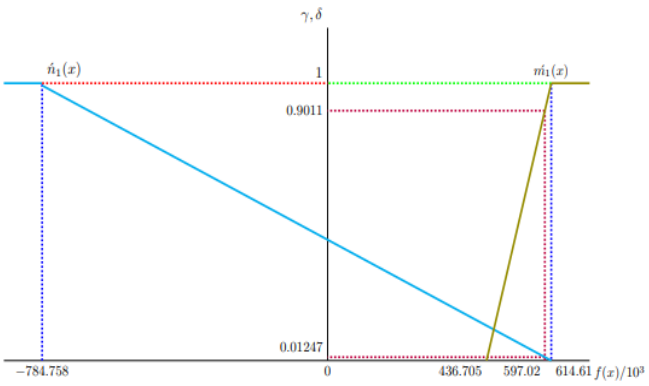
$$\bar{Đ}'_1(x) = 715.32, \bar{Đ}'_2(x) = 714.51, \bar{Đ}'_3(x) = 1642.04, \bar{Đ}'_4(x) = 1015.3, \bar{Đ}'_5(x) = 113.21 \quad (6.63)$$

$$=> D' = \frac{357.1429}{0.2175} \quad (6.64)$$

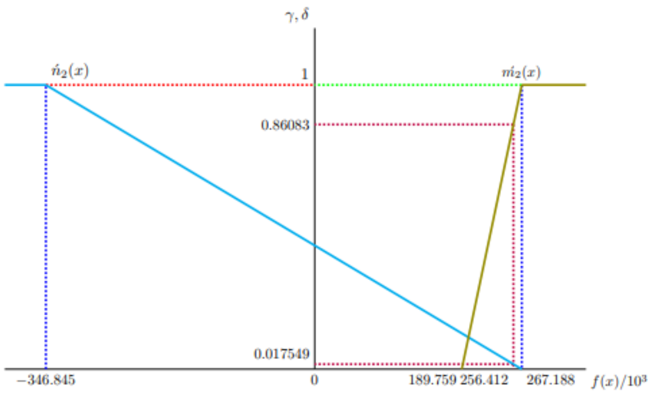
6.3.1.2 Solution by proposed linear approach

Here, the expected all-function levels increase simultaneously, as provided in Figure 6.11. The first function's nadir and ideal values, 436705.001 and 574160.5, respectively, form the basis for a participation function with a linear character. Similarly, nadir points for second, third, fourth, and fifth functions are 237756.51, 326.929, 2285.71, and 2912.494, respectively, and ideal values are 267187.7, 357.1429, and 3519.757, respectively. The ideal values stay the same, and the non-membership measure is also linear, but due to intuitionistic characteristics, the minimum value gets even worse: -796217.83, -346843.45, 0, -1772.404, and -47534.43, respectively, when the supremum normalized distance is taken into consideration. Membership is defined as a value from zero to one; lack of membership is the opposite. The goal of this strategy is to widen the difference between the two functions' resulting satisfaction and dissatisfaction ratings. The best solution was found with the following parameters resulting from LINGO 18.0×64 (76.85,800,260,600,300,500,0,231.31) and a difference level of 0.92405.

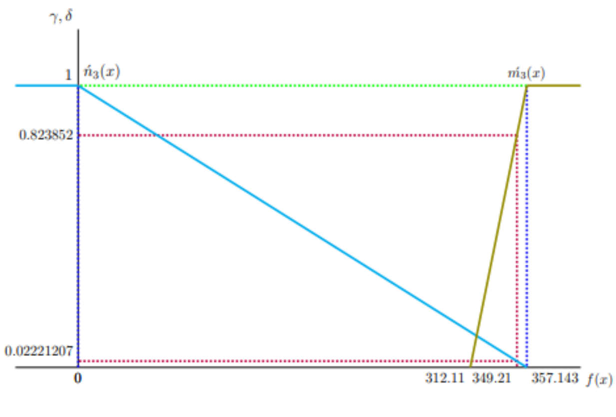
The problem has now been reduced to a single objective LPP with constraints defined in eq (6.49)-(6.55):



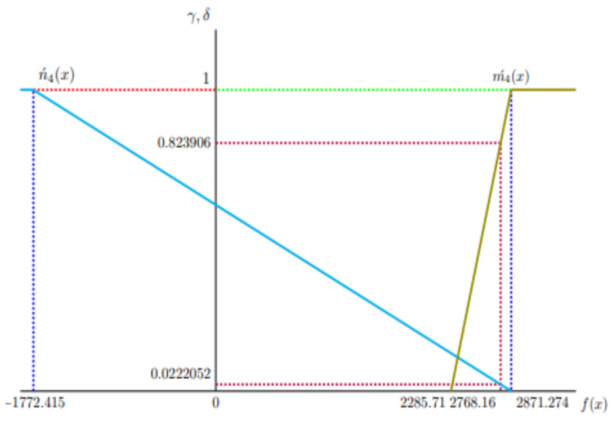
(a)



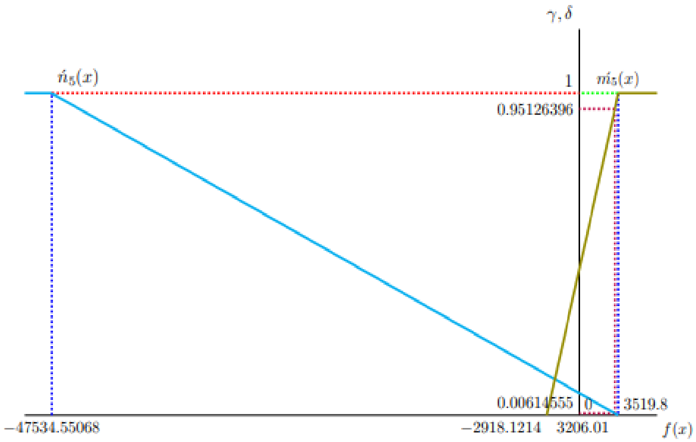
(b)



(c)



(d)



(e)

Figure 6.11: Different satisfaction and dissatisfaction functions with linear association functions: (a) Revenue function (b) Profit function (c) Market share function (d) Production function (e) Plant utilization function

$$\left.\begin{array}{c}
max\ \gamma - \delta \\
sub.to{:}\ \gamma \le \dfrac{\dfrac{585.564}{2.828} - \left(\dfrac{614613.2 - (375x_1 + 150x_2 + 400x_3 + 160x_4 + 420x_5 + 175x_6 + 400x_7 + 150x_8)}{859.215}\right)}{\dfrac{585.564}{2.828}} \\
\gamma \le \dfrac{\dfrac{585.564}{2.828} - \left(\dfrac{267187.7 - (180x_1 + 83x_2 + 153x_3 + 72x_4 + 130x_5 + 70x_6 + 208x_7 + 83x_8)}{373.945}\right)}{\dfrac{585.564}{2.828}} \\
\gamma \le \dfrac{\dfrac{585.564}{2.828} - \left(\dfrac{357.1429 - (0.25x_1 + 0.1x_2 + 0.25x_3 + 0.1x_4 + 0.25x_5 + 0.1d_6 + 0x_7 + 0x_8)}{0.2175}\right)}{\dfrac{585.564}{2.828}} \\
\gamma \le \dfrac{\dfrac{585.564}{2.828} - \left(\dfrac{2871.274 - (x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8)}{2.828}\right)}{\dfrac{585.564}{2.828}} \\
\gamma \le \dfrac{\dfrac{585.564}{2.828} - \left(\dfrac{3519.757 - (1.65x_1 + 0.9x_2 + 1.975x_3 + 1.03x_4 + 1.75x_5 + 0.94x_6 + 4.2x_7 + 1.006x_8)}{31.092}\right)}{\dfrac{585.564}{2.828}} \\
\delta \ge \dfrac{\left(\dfrac{614613.2 - (375x_1 + 150x_2 + 400x_3 + 160x_4 + 420x_5 + 175x_6 + 400x_7 + 150x_8)}{859.215}\right)}{\dfrac{357.1429}{0.2175}} \\
\delta \ge \dfrac{\left(\dfrac{267187.7 - (180x_1 + 83x_2 + 153x_3 + 72x_4 + 130x_5 + 70x_6 + 208x_7 + 83x_8)}{373.945}\right)}{\dfrac{357.1429}{0.2175}} \\
\delta \ge \dfrac{\left(\dfrac{357.1429 - (0.25x_1 + 0.1x_2 + 0.25x_3 + 0.1x_4 + 0.25x_5 + 0.1d_6 + 0x_7 + 0x_8)}{0.2175}\right)}{\dfrac{357.1429}{0.2175}} \\
\delta \ge \dfrac{\left(\dfrac{2871.274 - (x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8)}{2.828}\right)}{\dfrac{357.1429}{0.2175}} \\
\delta \ge \dfrac{\left(\dfrac{3519.757 - (1.65x_1 + 0.9x_2 + 1.975x_3 + 1.03x_4 + 1.75x_5 + 0.94x_6 + 4.2x_7 + 1.006x_8)}{31.092}\right)}{\dfrac{357.1429}{0.2175}} \\
\gamma \ge \delta, 0 \le \gamma + \delta \le 1
\end{array}\right\} \quad (6.65)$$

6.3.1.3 Solution by proposed non-linear approach

The limiting points for the objective value remain the same in both linear and non-linear cases. Here, the expected membership functions level increase exponentially, and non-membership functions decrease simultaneously as provided in Figure 6.12. The problem has now been reduced to a single objective LPP with constraints defined in eq (6.49)-(6.55):

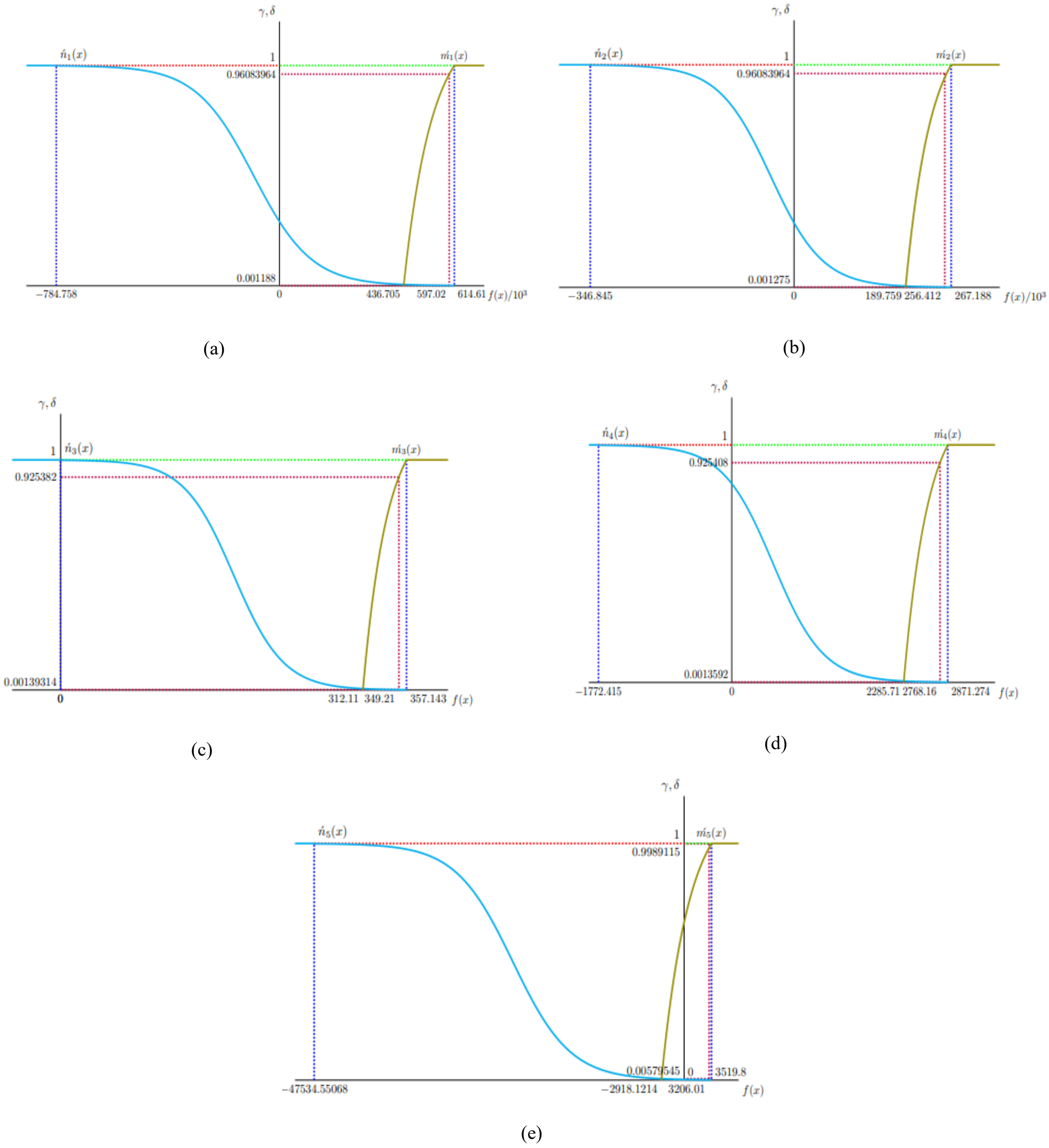


Figure 6.12: Different satisfaction and dissatisfaction functions with non-linear association functions: (a) Revenue function (b) Profit function (c) Market share function (d) Production function (e) Plant utilization function

$$
\left.
\begin{array}{l}
\max \gamma - \delta \\
\text{sub. to: } \gamma \le 1.2[1 - exp\left\{-1.79\dfrac{\dfrac{585.564}{2.828} - (\dfrac{614613.2 - (375x_1 + 150x_2 + 400x_3 + 160x_4 + 420x_5 + 175x_6 + 400x_7 + 150x_8)}{859.215})}{\dfrac{585.564}{2.828}}\right\} \\
\gamma \le 1.2[1 - exp\left\{-1.79\dfrac{\dfrac{585.564}{2.828} - (\dfrac{267187.7 - (180x_1 + 83x_2 + 153x_3 + 72x_4 + 130x_5 + 70x_6 + 208x_7 + 83x_8)}{373.945})}{\dfrac{585.564}{2.828}}\right\} \\
\gamma \le 1.2[1 - exp\left\{-1.79\dfrac{\dfrac{585.564}{2.828} - (\dfrac{357.1429 - (0.25x_1 + 0.1x_2 + 0.25x_3 + 0.1x_4 + 0.25x_5 + 0.1d_6 + 0x_7 + 0x_8)}{0.2175})}{\dfrac{585.564}{2.828}}\right\} \\
\gamma \le 1.2[1 - exp\left\{-1.79\dfrac{\dfrac{585.564}{2.828} - (\dfrac{2871.274 - (x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8)}{2.828})}{\dfrac{585.564}{2.828}}\right\} \\
\gamma \le 1.2[1 - exp\left\{-1.79\dfrac{\dfrac{585.564}{2.828} - (\dfrac{3519.757 - (1.65x_1 + 0.9x_2 + 1.975x_3 + 1.03x_4 + 1.75x_5 + 0.94x_6 + 4.2x_7 + 1.006x_8)}{31.092})}{\dfrac{585.564}{2.828}}\right\} \\
\delta \ge \dfrac{1}{1 + Bexp\,(13.813 \times \dfrac{\dfrac{357.1429}{0.2175} - (\dfrac{614613.2 - (375x_1 + 150x_2 + 400x_3 + 160x_4 + 420x_5 + 175x_6 + 400x_7 + 150x_8)}{859.215})}{\dfrac{357.1429}{0.2175}})} \\
\delta \ge \dfrac{1}{1 + Bexp\,(13.813 \times \dfrac{\dfrac{357.1429}{0.2175} - (\dfrac{267187.7 - (180x_1 + 83x_2 + 153x_3 + 72x_4 + 130x_5 + 70x_6 + 208x_7 + 83x_8)}{373.945})}{\dfrac{357.1429}{0.2175}})} \\
\delta \ge \dfrac{1}{1 + Bexp\,(13.813 \times \dfrac{\dfrac{357.1429}{0.2175} - (\dfrac{357.1429 - (0.25x_1 + 0.1x_2 + 0.25x_3 + 0.1x_4 + 0.25x_5 + 0.1d_6 + 0x_7 + 0x_8)}{0.2175})}{\dfrac{357.1429}{0.2175}})} \\
\delta \ge \dfrac{1}{1 + Bexp\,(13.813 \times \dfrac{\dfrac{357.1429}{0.2175} - (\dfrac{2871.274 - (x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8)}{2.828})}{\dfrac{357.1429}{0.2175}})} \\
\delta \ge \dfrac{1}{1 + Bexp\,(13.813 \times \dfrac{\dfrac{357.1429}{0.2175} - (\dfrac{3519.757 - (1.65x_1 + 0.9x_2 + 1.975x_3 + 1.03x_4 + 1.75x_5 + 0.94x_6 + 4.2x_7 + 1.006x_8)}{31.092})}{\dfrac{357.1429}{0.2175}})} \\
\gamma \ge \delta, 0 \le \gamma + \delta \le 1
\end{array}
\right\} \quad (6.66)
$$

6.3.1.4 Results

For the optimal point and level of fulfilment or discontent, LINGO 18.0×64 is used to obtain conclusion from case study regarding manufacturing strategy. Table 6.11 presents the values of various variables derived from this improved point. The outcomes of the case study of the manufacturing problem are identical in the case of total functional value and deviation but better for the non-linear nature in the case of satisfaction and discontent levels.

Table 6.11: Various parameters' values after optimization with different techniques for case study

Parameter	IFA (Linear, Normalized)	IFA (Non-linear, Normalized)
γ	0.8239022	0.9254065
δ	0.02220574	0.001359175
$\gamma - \delta$	0.8016965	0.92405
Optimal point	(76.85,800,260,600,300,500,0,231.31)	(76.85,800,260,600,300,500,0,231.31)

f_1	597015.25	597015.25
f_2	256411.73	256411.73
f_3	349.21	349.21
f_4	2768.16	2768.16
f_5	3206.0003	3206.0003
Total functional value	859750.3503	859750.3503
Deviation	132.324	132.324

The proposed approach successfully resolves the manufacturing problem. Table 6.11 displays our study's intended outcome, which is a 10.9% increase in satisfaction levels compared to linear techniques and a 13.24% increase in the gap between satisfaction and dissatisfaction levels compared to the current linear normalized technique.

6.3.1.5 Comparative Analysis

Here, we examine various intuitionistic approaches by comparing them with the following parameters:

a) Value of goal functions:

The values of different objective functions and total values of them are taken from Table 6.11.

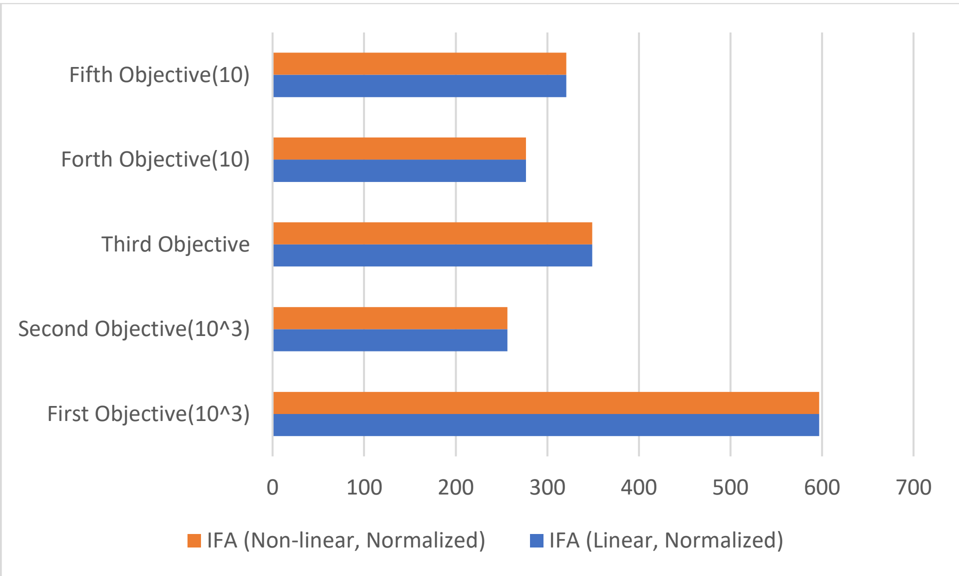


Figure 6.13: Comparison of various normalized techniques by using functional values via case-study of manufacturing

When comparing various normalized methods, the graph in Figure 6.13 shows that the values remain the same for both normalized techniques.

b) ***δ*** *and* ***γ*** *values*

Which can be calculated as:

$$\gamma = \min_{i=1 to 5} \acute{m}_i \tag{6.67}$$

$$\delta = \max_{i=1 \; to \; 5} \acute{n}_i \tag{6.68}$$

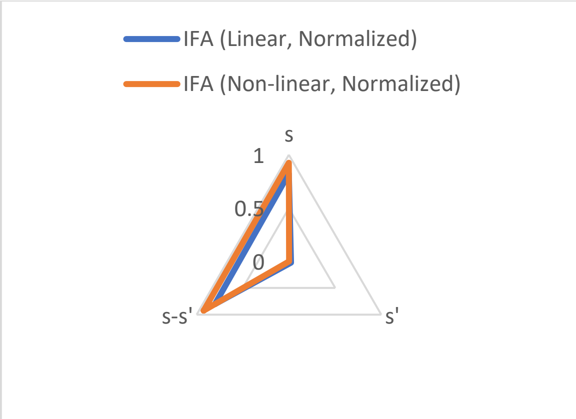


Figure 6.14: Satisfaction, dissatisfaction and their separation corresponding to various techniques for case-study

In Figure 6.14, we can see the results of comparing the level of fulfilment and displeasure levels, as well as the variances between them, for the two approaches we used in the research we conducted for the numerical issue and the real-world investigation, respectively. Since we need to raise the value of the degree of fulfilment and the difference while lowering the value of the discontent level, the non-linear methodology improves the outcomes, as displayed in these comparative graphs.

c) *Value of normalized distance from ideal points:*

Normalized distance from ideal points of goal functions can be calculated by the following formula:

$$Đ(x) = \sum_{i=1}^{2} \left(\frac{\overline{f}_i - f_i(x)}{\sqrt{\sum_{k=1}^{2} {p_k^i}^2}} \right) \tag{6.69}$$

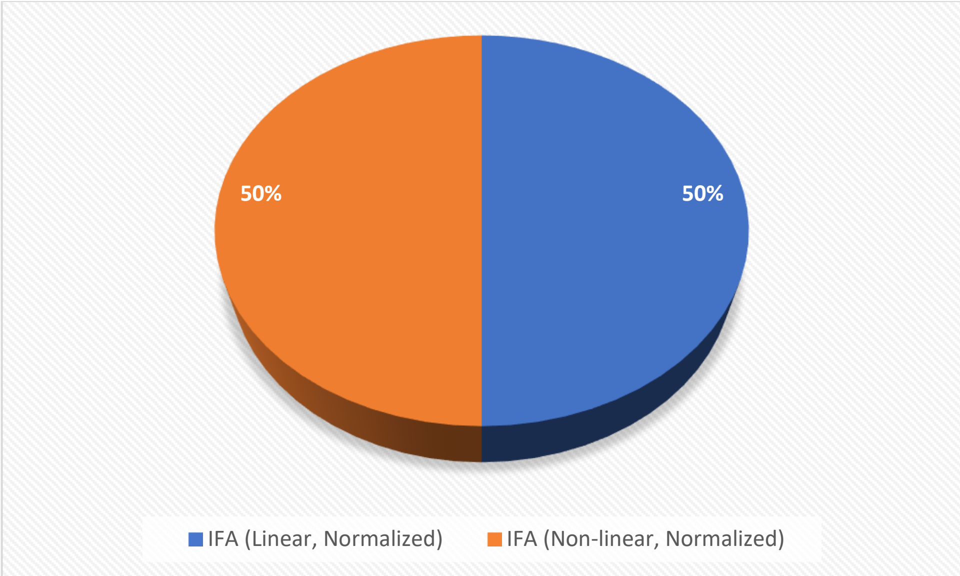


Figure 6.15: Total normalized deviation for various normalized approaches

Figure 6.15 demonstrates how the normalized distance from ideal points of goal functions for normalized procedures will be the same for every technique.

6.4 Application of Linear/Non-linear Dual Hesitant Fuzzy Multi-Objective Optimization in Production

The resulting problem from Section 7 of Chapter 3 is used here as a multi-objective optimization problem, which can be formulated in standard form as:

6.4.1 Production Problem

Here is the complete mathematical description of the problem previously discussed as a multi-objective linear modelling problem:

a) Decision variable:

The products that we have to find out are decision variables, which are given below:

Number of units	Product
x_1	Product 1
x_2	Product 2
x_3	Product 3

b) Objectives:

The functions that we have to optimize with the help of products are called objectives. Here, total five functions are described for optimization, which are given below:

Profit	$Max\ f_1 = 52x_1 + 104x_2 + 18x_3$	(6.70)
Quality	$Max\ f_2 = 95x_1 + 77.5x_2 + 52x_3$	(6.71)
Worker Satisfaction	$Max\ f_3 = 26x_1 + 104x_2 + 77.5x_3$	(6.72)

c) Constraints associated:

The conditions that are applied to products on the basis of different criteria are taken as restrictions, which are defined below:

$$\left.\begin{array}{c} 12.5x_1 + 17.5x_2 + 0x_3 \leq 1410 \\ 3.1x_1 + 9.255x_2 + 8.25x_3 \leq 1007.5 \\ 10.4x_1 + 13.5x_2 + 15.5x_3 \leq 1762.5 \\ 6.2x_1 + 0x_2 + 16.5x_3 \leq 1335 \\ 0x_1 + 12.5x_2 + 7.25x_3 \leq 907.5 \\ 9.75x_1 + 9.75x_2 + 4.1x_3 \leq 1082.5 \\ x_1, x_2, x_3 \geq 0 \end{array}\right\} \quad (6.73)$$

To solve all the functions separately by LINGO 18.0 × 64, the ideal points of all functions are calculated, and with the help of all these points, the functional values at these points are found out to find the nadir points of functions, which are given in Table 6.12:

$\overline{f_1}$= 8130.720; at point (11.16,72.6,0)

$\overline{f_2}$= 11108.36; at point (91.45,0,46.54)

$\overline{f_3}$=9423.908; at point (44.74,48.62,41.35)

Table 6.12: Calculation of ideal, nadir and minimal points of various functions

Function	**(11.16,72.6,0)**	**(91.45,0,46.54)**	**(44.74,48.62,41.35)**	**Nadir values**	**Worst values**
f_1	8130.720	5593.12	8127.26	5593.12	0
f_2	6686.7	11108.36	10168.55	6686.7	0
f_3	7840.56	5984.55	9423.908	5984.55	0

By using the formula for normalized distance function:

$$Đ_1(x) = \frac{8130.720 - (52x_1 + 104x_2 + 18x_3)}{117.66} \quad (6.74)$$

$$Đ_2(x) = \frac{11108.36 - (95x_1 + 77.5x_2 + 52x_3)}{133.17} \quad (6.75)$$

$$Đ_3(x) = \frac{9423.908 - (26x_1 + 104x_2 + 77.5x_3)}{132.28} \quad (6.76)$$

At its highest value, the normalised distance parameter produces:

$$max\, Đ_1(x) = 21.57, max\, Đ_2(x) = 33.2, max\, Đ_3(x) = 26 \quad (6.77)$$

$$max\, Đ'_1(x) = 69.1, max\, Đ'_2(x) = 83.41, max\, Đ'_3(x) = 71.24 \quad (6.78)$$

The values of the greatest distance variables for these two positions are:

$$D = \frac{4421.66}{133.17}, D' = \frac{11108.36}{133.17} \quad (6.79)$$

With $ɯ_1, ɯ_2,$ and $ɯ_3$ as 1,0.98, and 0.96, respectively, the membership and non-membership functions are given as linear and non-linear types in the following subsections:

6.4.1.1 Solution by proposed linear approach

The problem has now been reduced to a single objective LPP with constraints defined in eq (6.59) with the following objective shown in eq (6.80) and additional constraints generated from the membership function by eq (6.81), due to non-membership functions by eq (6.82):

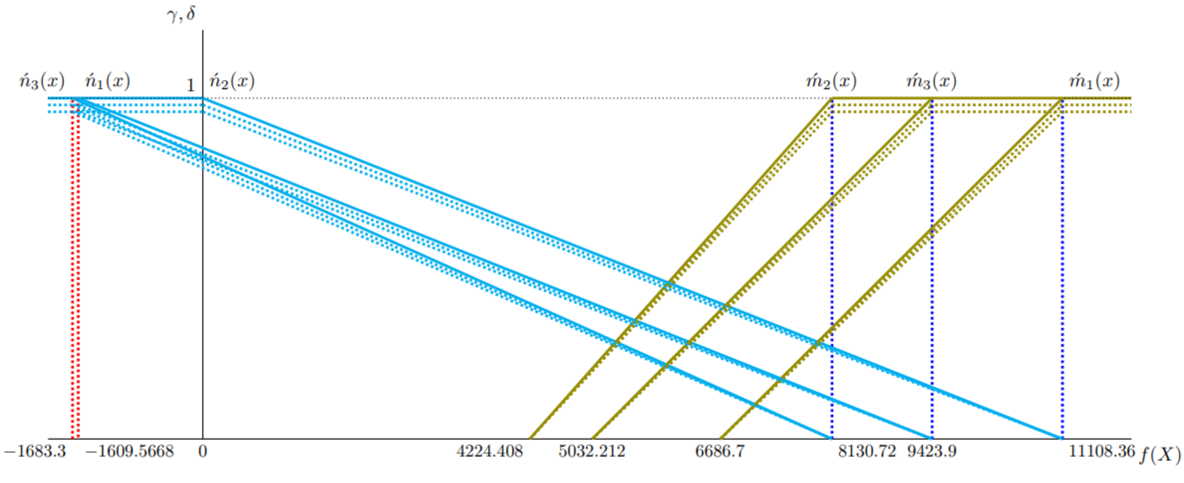


Figure 6.16: Geometrical representation of various linear association functions according to various decision makers

$$max \left(\frac{\gamma^1 + \gamma^2 + \gamma^3}{3}\right) - \left(\frac{\delta^1 + \delta^2 + \delta^3}{3}\right) \quad (6.80)$$

$$
\left.
\begin{aligned}
sub.to: \gamma^1 &\le \frac{\frac{4421.66}{133.17} - \left(\frac{8130.720 - (52x_1 + 104x_2 + 18x_3)}{117.66}\right)}{\frac{4421.66}{133.17}} \\
\gamma^2 &\le 0.98\left(\frac{\frac{4421.66}{133.17} - \left(\frac{8130.720 - (52x_1 + 104x_2 + 18x_3)}{117.66}\right)}{\frac{4421.66}{133.17}}\right) \\
\gamma^3 &\le 0.96\left(\frac{\frac{4421.66}{133.17} - \left(\frac{8130.720 - (52x_1 + 104x_2 + 18x_3)}{117.66}\right)}{\frac{4421.66}{133.17}}\right) \\
\gamma^1 &\le \frac{\frac{4421.66}{133.17} - \left(\frac{11108.36 - (95x_1 + 77.5x_2 + 52x_3)}{133.17}\right)}{\frac{4421.66}{133.17}} \\
\gamma^2 &\le 0.98\left(\frac{\frac{4421.66}{133.17} - \left(\frac{11108.36 - (95x_1 + 77.5x_2 + 52x_3)}{133.17}\right)}{\frac{4421.66}{133.17}}\right) \\
\gamma^3 &\le 0.96\left(\frac{\frac{4421.66}{133.17} - \left(\frac{11108.36 - (95x_1 + 77.5x_2 + 52x_3)}{133.17}\right)}{\frac{4421.66}{133.17}}\right) \\
\gamma^1 &\le \frac{\frac{4421.66}{133.17} - \left(\frac{9423.908 - (26x_1 + 104x_2 + 77.5x_3)}{132.28}\right)}{\frac{4421.66}{133.17}} \\
\gamma^2 &\le 0.98\left(\frac{\frac{4421.66}{133.17} - \left(\frac{9423.908 - (26x_1 + 104x_2 + 77.5x_3)}{132.28}\right)}{\frac{4421.66}{133.17}}\right) \\
\gamma^3 &\le 0.96\left(\frac{\frac{4421.66}{133.17} - \left(\frac{9423.908 - (26x_1 + 104x_2 + 77.5x_3)}{132.28}\right)}{\frac{4421.66}{133.17}}\right)
\end{aligned}
\right\} \tag{6.81}
$$

In this case, we see the anticipated simultaneous rise in all function levels, as shown in Figure 6.16. A linear participation function is based on the first function's minimum and ideal values, which are 4224.8 and 8130.720, respectively. Similarly, the second function has a minimum point of 6686.7, while the third function has 5032.212 and an ideal value of 11108.36 and 9423.908, respectively. For the linear non-membership measure, the ideal values stay the same. However, when the supremum normalized distance is taken into account, the lowest value for all functions becomes zero. This is because of intuitionistic features. However, the minimal values decrease more for the maximum distance chosen in the case of the worst points, as shown in Figure 6.16. We can describe a membership status here as a number between zero and one, where zero represents the most extreme form of membership and one represents the least extreme form. The plan's objective is to make the gap between the two functions' ultimate ratings of happiness and discontentment wider.

$$\left.\begin{array}{l}
\delta^1 \geq \dfrac{\left(\dfrac{8130.720-(52x_1+104x_2+18x_3)}{117.66}\right)}{\dfrac{11108.36}{133.17}} \\
\delta^2 \geq 0.98\left(\dfrac{\left(\dfrac{8130.720-(52x_1+104x_2+18x_3)}{117.66}\right)}{\dfrac{11108.36}{133.17}}\right) \\
\delta^3 \geq 0.96\left(\dfrac{\left(\dfrac{8130.720-(52x_1+104x_2+18x_3)}{117.66}\right)}{\dfrac{11108.36}{133.17}}\right) \\
\delta^1 \geq \dfrac{\left(\dfrac{11108.36-(95x_1+77.5x_2+52x_3)}{133.17}\right)}{\dfrac{11108.36}{133.17}} \\
\delta^2 \geq 0.98\left(\dfrac{\left(\dfrac{11108.36-(95x_1+77.5x_2+52x_3)}{133.17}\right)}{\dfrac{11108.36}{133.17}}\right) \\
\delta^3 \geq 0.96\left(\dfrac{\left(\dfrac{11108.36-(95x_1+77.5x_2+52x_3)}{133.17}\right)}{\dfrac{11108.36}{133.17}}\right) \\
\delta^1 \geq \dfrac{\left(\dfrac{9423.908-(26x_1+104x_2+77.5x_3)}{132.28}\right)}{\dfrac{11108.36}{133.17}} \\
\delta^2 \geq 0.98\left(\dfrac{\left(\dfrac{9423.908-(26x_1+104x_2+77.5x_3)}{132.28}\right)}{\dfrac{11108.36}{133.17}}\right) \\
\delta^3 \geq 0.96\left(\dfrac{\left(\dfrac{9423.908-(26x_1+104x_2+77.5x_3)}{132.28}\right)}{\dfrac{11108.36}{133.17}}\right)
\end{array}\right\} \quad (6.82)$$

\

$$\left.\begin{array}{c}
0 \leq \gamma^p + \delta^p \leq 1 \ \forall\ p = 1,2,3 \\
\gamma^p \geq \delta^p \ \forall p = 1,2,3
\end{array}\right\} \quad (6.83)$$

6.4.1.2 Solution by proposed non-linear approach:

The limiting points for the objective value remain the same in both linear and non-linear cases. Here, the expected membership function levels increase exponentially, and non-membership functions decrease simultaneously as provided in Figure 6.17.

The problem has now been reduced to a single objective LPP with constraints defined in eq (6.59):

$$max\left(\frac{\gamma^1+\gamma^2+\gamma^3}{3}\right)-\left(\frac{\delta^1+\delta^2+\delta^3}{3}\right) \quad (6.84)$$

$$\left.\begin{array}{l}
sub.to{:}\,\gamma^1 \le 1.2\left[1-exp\left\{-1.79\dfrac{\frac{4421.66}{133.17}-\left(\frac{8130.720-(52x_1+104x_2+18x_3)}{117.66}\right)}{\frac{4421.66}{133.17}}\right\}\right] \\
\gamma^2 \le 0.98*1.2\left[1-exp\left\{-1.79\dfrac{\frac{4421.66}{133.17}-\left(\frac{8130.720-(52x_1+104x_2+18x_3)}{117.66}\right)}{\frac{4421.66}{133.17}}\right\}\right] \\
\gamma^3 \le 0.96*1.2\left[1-exp\left\{-1.79\dfrac{\frac{4421.66}{133.17}-\left(\frac{8130.720-(52x_1+104x_2+18x_3)}{117.66}\right)}{\frac{4421.66}{133.17}}\right\}\right] \\
\gamma^1 \le 1.2\left[1-exp\left\{-1.79\dfrac{\frac{4421.66}{133.17}-\left(\frac{11108.36-(95x_1+77.5x_2+52x_3)}{133.17}\right)}{\frac{4421.66}{133.17}}\right\}\right] \\
\gamma^2 \le 0.98*1.2\left[1-exp\left\{-1.79\dfrac{\frac{4421.66}{133.17}-\left(\frac{11108.36-(95x_1+77.5x_2+52x_3)}{133.17}\right)}{\frac{4421.66}{133.17}}\right\}\right] \\
\gamma^3 \le 0.96*1.2\left[1-exp\left\{-1.79\dfrac{\frac{4421.66}{133.17}-\left(\frac{11108.36-(95x_1+77.5x_2+52x_3)}{133.17}\right)}{\frac{4421.66}{133.17}}\right\}\right] \\
\gamma^1 \le 1.2\left[1-exp\left\{-1.79\dfrac{\frac{4421.66}{133.17}-\left(\frac{9423.908-(26x_1+104x_2+77.5x_3)}{132.28}\right)}{\frac{4421.66}{133.17}}\right\}\right] \\
\gamma^2 \le 0.98*1.2\left[1-exp\left\{-1.79\dfrac{\frac{4421.66}{133.17}-\left(\frac{9423.908-(26x_1+104x_2+77.5x_3)}{132.28}\right)}{\frac{4421.66}{133.17}}\right\}\right] \\
\gamma^3 \le 0.96*1.2\left[1-exp\left\{-1.79\dfrac{\frac{4421.66}{133.17}-\left(\frac{9423.908-(26x_1+104x_2+77.5x_3)}{132.28}\right)}{\frac{4421.66}{133.17}}\right\}\right]
\end{array}\right\} \quad (6.85)$$

$$
\left.\begin{array}{l}
\delta^1 \geq \dfrac{1}{1 + Bexp\left(13.813 \times \dfrac{\frac{11108.36}{133.17} - \left(\frac{8130.720 - (52x_1 + 104x_2 + 18x_3)}{117.66}\right)}{\frac{11108.36}{133.17}}\right)} \\
\delta^2 \geq \dfrac{0.98}{1 + Bexp\left(13.813 \times \dfrac{\frac{11108.36}{133.17} - \left(\frac{8130.720 - (52x_1 + 104x_2 + 18x_3)}{117.66}\right)}{\frac{11108.36}{133.17}}\right)} \\
\delta^3 \geq \dfrac{0.96}{1 + Bexp\left(13.813 \times \dfrac{\frac{11108.36}{133.17} - \left(\frac{8130.720 - (52x_1 + 104x_2 + 18x_3)}{117.66}\right)}{\frac{11108.36}{133.17}}\right)} \\
\delta^1 \geq \dfrac{1}{1 + Bexp\left(13.813 \times \dfrac{\frac{11108.36}{133.17} - \left(\frac{11108.36 - (95x_1 + 77.5x_2 + 52x_3)}{133.17}\right)}{\frac{11108.36}{133.17}}\right)} \\
\delta^2 \geq \dfrac{0.98}{1 + Bexp\left(13.813 \times \dfrac{\frac{11108.36}{133.17} - \left(\frac{11108.36 - (95x_1 + 77.5x_2 + 52x_3)}{133.17}\right)}{\frac{11108.36}{133.17}}\right)} \\
\delta^3 \geq \dfrac{0.96}{1 + Bexp\left(13.813 \times \dfrac{\frac{11108.36}{133.17} - \left(\frac{11108.36 - (95x_1 + 77.5x_2 + 52x_3)}{133.17}\right)}{\frac{11108.36}{133.17}}\right)} \\
\delta^1 \geq \dfrac{1}{1 + Bexp\left(13.813 \times \dfrac{\frac{11108.36}{133.17} - \left(\frac{9423.908 - (26x_1 + 104x_2 + 77.5x_3)}{132.28}\right)}{\frac{11108.36}{133.17}}\right)} \\
\delta^2 \geq \dfrac{0.98}{1 + Bexp\left(13.813 \times \dfrac{\frac{11108.36}{133.17} - \left(\frac{9423.908 - (26x_1 + 104x_2 + 77.5x_3)}{132.28}\right)}{\frac{11108.36}{133.17}}\right)} \\
\delta^3 \geq \dfrac{0.96}{1 + Bexp\left(13.813 \times \dfrac{\frac{11108.36}{133.17} - \left(\frac{9423.908 - (26x_1 + 104x_2 + 77.5x_3)}{132.28}\right)}{\frac{11108.36}{133.17}}\right)}
\end{array}\right\} \quad (6.86)
$$

$$
\left.\begin{array}{c}
0 \leq \gamma^p + \delta^p \leq 1 \ \forall \ p = 1,2,3 \\
\gamma^p \geq \delta^p \ \forall p = 1,2,3
\end{array}\right\} \quad (6.87)
$$

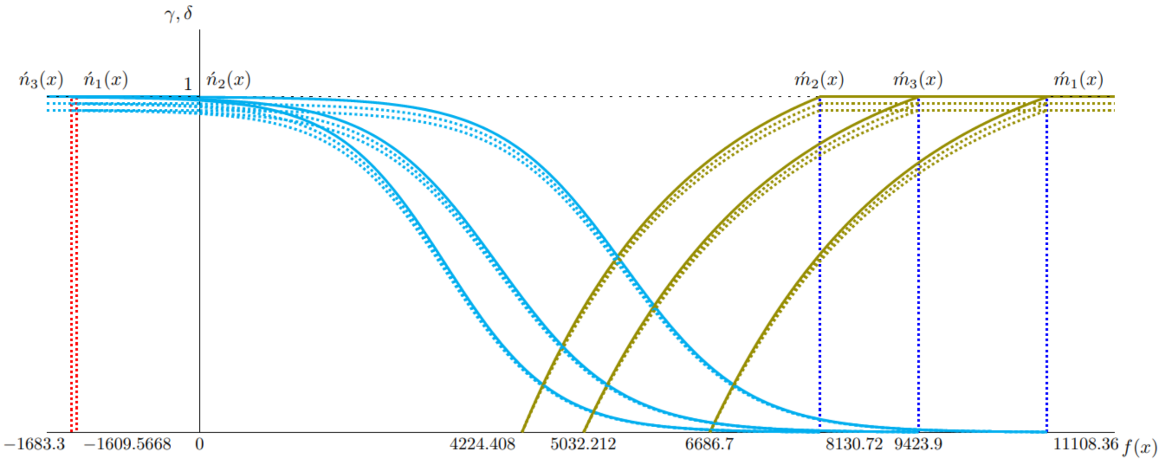


Figure 6.17: Geometrical representation of various non-linear association functions according to various decision makers

6.4.1.3 Results

The optimal solution is defined by the following LINGO 18.0 × 64 parameters: point of solution (44.74, 48.62, 41.35) with a difference level of 0.9913376. The satisfaction levels for the first, second, and third decision makers are 0.9949962, 0.9950963, and 0.9969139, while the dissatisfaction levels are 0.0050038, 0.0049037, and 0.0037861, respectively.

Table 6.13 shows how we calculate the values of various variables using the average levels of satisfaction, dissatisfaction, and their discrepancies. We also measure all functional values for their total normalized distance from their ideal counterparts.

Table 6.13: Various parameter's values after optimization with different techniques for real-life case study

Parameter	HIFA (Linear, Non-normalized)	DHFA (Linear, Normalized)	DHFA (Non-Linear, Non-Normalized)	DHFA (Non-linear, Normalized)
γ	0.8150149	0.7803881	0.91060373	0.9956688
δ	0.065672	0.1710167	0.0024705	0.004564533
$\gamma - \delta$	0.7493429	0.6093714	0.9081332	0.9913376
point	(52.42, 40.36, 43.38)	(45.81, 47.85, 41.3)	(52.42, 40.36, 43.38)	(44.73, 48.61, 41.34)
f_1	7704.12	8101.92	7704.12	8125.52
f_2	10363.56	10207.925	10363.56	10166.55
f_3	8922.31	9368.21	8922.31	9422.27
Total	26989.99	27678.055	26989.99	27714.34
Deviation	13.01049	7.4273798	13.01049	7.08662685

In comparison to non-linear and non-normalized techniques, our study increases the satisfaction level by 9.34%. Table 6.13 shows that we reached this goal, resulting in a 9.16%

bigger difference between satisfaction and dissatisfaction levels, a 2.68% rise in total functional values, and a 45.53% drop in how far off we are from ideal values. Similarly, compared to the current linear normalized and non-normalized methods, the satisfaction level has risen by 27.59% and 22.17%, while the difference between satisfaction and dissatisfaction levels has increased by 63.68% and 32.29%. The total functional values have increased by 0.13% and 2.684%, while the total deviation from ideal values has decreased by 4.5% and 45.53%, respectively. Our goal is to improve overall functional value and reduce the discrepancy between optimal locations and normalized operational value.

6.4.1.4 Comparative Analysis

Here, we examine various approaches by comparing them with the following parameters:

a) Value of goal functions

The values of objective functions according to the responses corresponding to various approaches are given below:

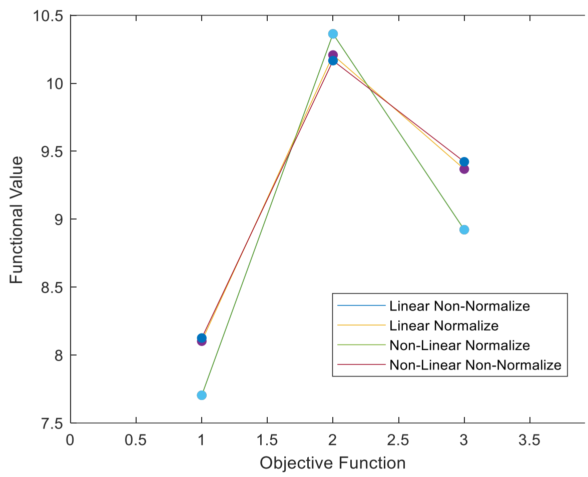


Figure 6.18: Comparison of various techniques by using various functional values

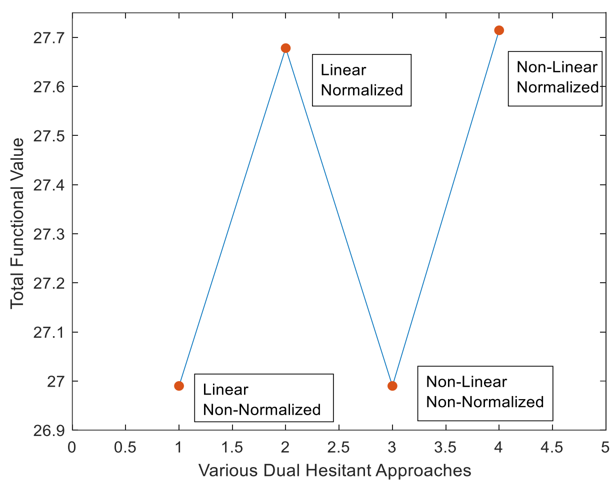


Figure 6.19: Comparison of various techniques by using total functional value

Figures 6.18 and 6.19 show a graph analysing the status existing with normalized approaches; the results show that the second objective function and the total value of all goals are both increased by the normalized procedures. In the instance of non-linear functions, however, the values are greater for both normalized methods.

b) $\boldsymbol{\delta}$ and $\boldsymbol{\gamma}$ values:

Which can be calculated as:

$$\gamma = \frac{\gamma^1 + \gamma^2 + \gamma^3}{3} \tag{6.88}$$

$$\delta = \frac{\delta^1 + \delta^2 + \delta^3}{3} \tag{6.89}$$

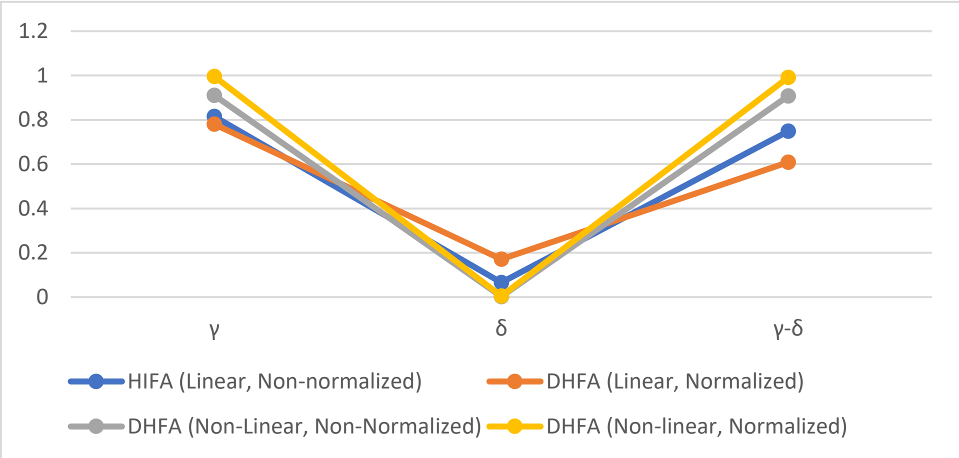


Figure 6.20: Satisfaction, dissatisfaction and their separation corresponding to various techniques for case-study of production sector

Figure 6.20 shows the outcomes of comparing the levels of satisfaction and dissatisfaction, together with the associated variations, for the four methods employed in our research on the real-world investigation. This comparative graph shows that the proposed technique improves the outcomes, which is great because we need to increase the value of the degree of fulfilment and the difference while decreasing the value of the discontent level.

c) Value of normalized distance from ideal points

Our aim is to decrease the normalized distance from ideal points of goal functions, and the normalization procedure decreases it successfully, which can be observed from Table 6.13 and graphically in Figure 6.21.

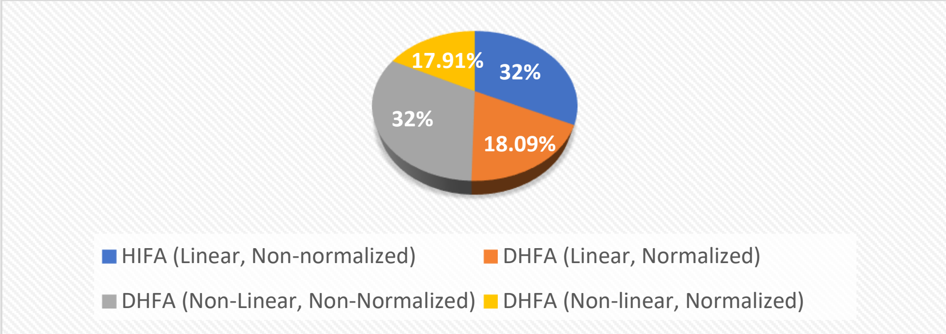


Figure 6.21: Normalized distance for different approaches

Normalized distance from ideal points of goal functions can be calculated by the following formula:

$$Ð(x) = \sum_{i=1}^{2} \left(\frac{\overline{f}_i - f_i(x)}{\sqrt{\sum_{k=1}^{2} p_k^{i\,2}}} \right) \tag{6.90}$$

6.5 Conclusion

In the field of skin tissue engineering, 3D-skin bioprinting using inkjet, laser, extrusion, stereolithography, and microfluidic technologies are crucial processing techniques. However, comparing these methods is challenging as they each optimize unique characteristics. Therefore, this research provides a multi-objective optimization method that can maximize all parameters concurrently and select the most appropriate bioprinting methodology.

The proposed approach effectively incorporates the inherent inconsistency and unpredictability of everyday decision-making scenarios, providing decision-makers across numerous domains with a practical and efficient solution. Since it can generate ideas that are close to the intended outcomes while managing several competing goals, this technique is appropriate and pertinent to challenging optimization situations.

Multiple linear regression analysis provides a quantitative understanding of the chemical characteristics and mechanical properties of unalloyed titanium. We have looked at how nitrogen, oxygen, iron, carbon, and hydrogen levels affect the stretching, stiffness, yield, and strength of titanium. These findings show how important it is to carefully manage the chemical makeup of titanium during production to achieve desired mechanical properties. Furthermore, the regression model provides a framework for predicting mechanical properties based on chemical compositions, which assists in the optimization and design of titanium materials for various engineering applications. We use multi-objective optimization with a fuzzy technique and multiple linear regression to find the best chemical component combinations that improve multiple mechanical qualities at once. We reduce the gap between the best values for mechanical properties by examining different designs, using elongation, modulus, yield, and tensile strength as our targets and chemical components as our choices. Using this approach, we are able to create titanium materials with improved overall performance and get customized solutions that balance conflicting goals. When analysing the most preferred non-linear association function for triangular intuitionistic fuzzy numbers from the comparative study with a normalized separation value, the proposed method becomes much more robust and dependable. The strategy ensures consistent and comparable findings by eliminating scale dependability, allowing decision-makers to make intelligent decisions regardless of the size of the factors. This attribute is crucial when making decisions in real life, where parameter variations in scale play a significant role. Applying the concept to a production problem and seeing positive results further confirms its relevance and feasibility in real-world settings.

The study provides an improved dual-hesitant fuzzy optimization method, which offers a new way to solve multi-objective linear optimization issues in the industrial sector. When faced with the complexity and unpredictability of industrial processes, the suggested dual-hesitant fuzzy optimization method, which incorporates both membership and non-membership functions with multiple experts' opinions, performed better. Using a normalized distance operation, which places all objective functions on the same scale, increases the approach's trustworthiness. The manufacturing sector can greatly profit from applying the dual-hesitant fuzzy technique. These techniques improve both performance and decision-making by helping producers deal with uncertainty more effectively, increase operational efficiency, and produce higher-quality products.

6.5.1 Major Finding

- The improved fuzzy method uses basic and more complex triangular fuzzy numbers to effectively solve multi-objective optimization problems in different real-life fields, such as biomimetics, smart cities, manufacturing, and material science.
- The recommended methods are strong, adaptable, and work well for different industries that have conflicting goals and unclear data, based on the analysis of real-life examples.

Chapter 7

Conclusion

7.1 Findings

The thesis delves deeply into fuzzy multi-objective linear programming problems, illuminating the intricacies and practical uses of fuzzy theory in contexts where decisions are fraught with uncertainty and inaccuracy. Beginning with the basic ideas of fuzzy sets and continuing through the complex methodology of intuitionistic and dual-hesitant fuzzy procedures, the research delves into a wide range of issues before finally applying these theories to practical applications.

- Defuzzification techniques can be fine-tuned by categorizing fuzzy numbers according to the number of components and their degree of symmetry. To address manufacturing issues, three methods were utilized: cantered area, α-cut, and confined area. While the α-cut and centroid of area methods work well with symmetric fuzzy numbers, the bounded area method shows inconsistent results, according to a comparison analysis.
- Using examples from sustainable manufacturing, financial portfolio optimization, and green light control systems, the thesis proves that traditional fuzzy min-max methods work. When dealing with situations with several objectives, these strategies provide balanced trade-offs because binary logic can't handle complex judgments.
- Further investigation on normalized fuzzy multi-objective optimization is underway to resolve the issue of objective functions with different scales. Using distance-based measures to compare results, this is vital. It is demonstrated that non-linear functions provide more modelling flexibility and realism, particularly in areas like as materials science, biomimetic design, and smart transportation, as compared to linear membership functions such as sigmoidal, parabolic, hyperbolic, and exponential.
- The intuitionistic fuzzy method improves decision modelling by combining membership and non-membership functions; it is based on intuitionistic triangular fuzzy numbers. The method's performance in addressing conflicting industrial objectives is greatly improved when supplemented with normalized distance metrics and non-linear association functions.

- An improved dual-hesitant fuzzy method, incorporating aspects of intuitionistic and hesitant fuzzy sets, is also presented in the thesis. Strong trade-off mechanisms for uncertain multi-objective settings are provided by the approach, which incorporates the non-linear functions shown in previous research. The method's ability to achieve sustainable and balanced judgments has been shown by its use in production systems.

A thorough framework for solving fuzzy MOLPPs is presented in the thesis, which begins with a survey of previous methods and the identification of areas where further research is needed. Novel defuzzification approaches are proposed, complicated fuzzy set extensions are explored, and practical normalized fuzzy procedures are developed. These additions not only deepen our theoretical knowledge but also provide useful resources for decision-makers in fields were optimizing many criteria in the face of uncertainty is crucial.

7.2 Areas for further investigation

- Investigate new and improved defuzzification techniques for dealing with complex fuzzy numbers that are not symmetrical, especially in cases where more accuracy and precision are crucial.
- To improve the efficacy and effectiveness of multi-objective optimization models, investigate the combination of the fuzzy approach with other artificial intelligence techniques like neural networks, genetic algorithms, and deep learning.
- Check if hybrid fuzzy models can be used to solve multi-objective optimization problems in various fields. These models can include intuitionistic, dual-hesitant, and other fuzzy set extensions to help people make decisions and deal with uncertainty better.
- To tackle complicated, real-world problems, apply hybrid fuzzy multi-objective optimization methods to new areas, including smart city artificial intelligence, sustainable energy management, and advanced manufacturing processes.

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Appendix A

As α-cut for fuzzy numbers are in the form of intervals for every value of $\alpha \in [0,1]$. So, the interval can be represented in the form of $[L, R]$. Based on these intervals left and right α-cut are defined for the defuzzification approach, mean of α-cut. Here left and right α-cut for every case denoted by L^{-1} and R^{-1} respectively are given below:

Case 1:

$$L^{-1}(\alpha) = t_{k+1} + \left(\frac{\alpha}{2/(n-1)}\right)^{\frac{1}{p}} (t_{k+2} - t_{k+1}) \ \forall \ \alpha \in [\frac{2k}{n-1}, \frac{2(k+1)}{n-1}], k \in \{0,1,..,\frac{n-3}{2}\} \tag{A.1}$$

$$R^{-1}(\alpha) = t_{n-k} - \left(\frac{\alpha}{2/(n-1)}\right)^{\frac{1}{p}} (t_{n-k} - t_{n-(k+1)}) \forall \ \alpha \in [\frac{2k}{n-1}, \frac{2(k+1)}{n-1}], k \in \{0,1,..,\frac{n-3}{2}\} \tag{A.2}$$

Case 2:

$$L^{-1}(\alpha) = t_{k+1} + \left(\frac{\alpha}{2/(n-2)}\right)^{\frac{1}{p}} (t_{k+2} - t_{k+1}) \ \forall \ \alpha \in [\frac{2k}{n-2}, \frac{2(k+1)}{n-2}], k \in \{0,1,..,\frac{n-4}{2}\} \tag{A.3}$$

$$R^{-1}(\alpha) = t_{n-k} - \left(\frac{\alpha}{2/(n-2)}\right)^{\frac{1}{p}} (t_{n-k} - t_{n-(k+1)}) \forall \ \alpha \in [\frac{2k}{n-2}, \frac{2(k+1)}{n-2}], k \in \{0,1,..,\frac{n-4}{2}\} \tag{A.4}$$

Case 3:

$$L^{-1}(\alpha) = t_{2k+1} + \left(\frac{\alpha}{4/n}\right)^{\frac{1}{p}} (t_{2k+2} - t_{2k+1}) \ \forall \ \alpha \in [\frac{4k}{n}, \frac{4(k+1)}{n}], k \in \{0,1,..,\frac{n-4}{4}\} \tag{A.5}$$

$$R^{-1}(\alpha) = t_{n-2k} - \left(\frac{\alpha}{4/n)}\right)^{\frac{1}{p}} (t_{n-2k} - t_{n-(2k+1)}) \forall\, \alpha \in [\frac{4k}{n}, \frac{4(k+1)}{n}], k \in \{0,1,.., \frac{n-4}{4}\} \tag{A.6}$$

List of Publications and Conferences

Journal Articles

1) P. Gulia, R. Kumar, W. Viriyasitavat, A. N. Aledaily, K. Yadav, A. Kaur and G. Dhiman, "A systematic review on Fuzzy-based multi-objective linear programming methodologies: concepts, challenges and applications," *Archives of Computational Methods in Engineering*, vol. 30, no. 8, pp. 4983-5022, Nov. 2022, doi: 10.1007/s11831-023-09966-1. (**I.F.-9.7 / Published**)
2) P. Gulia, R. Kumar, S. Vimal, N. S. Alghamdi, G. Dhiman, S. Pasupathi, A. Sood, W. Viriyasitavat, A. Sapsomboon and A. Kaur, "Artificial intelligence-enabled smart city management using multi-objective optimization strategies," *Expert Systems*, vol. 42, no. 1, p. e13574, Mar. 2024, doi: 10.1111/exsy.13574. (**I.F.-3 / Published**)
3) P. Gulia, R. Kumar, and G. Kaur, "Computational analysis and optimization in chemical components proportions for improved mechanical properties in unalloyed titanium: a fuzzy-based multi-objective approach," *International Journal on Interactive Design and Manufacturing*, vol. 18, no. 6, pp. 4159-4172, Aug. 2024, doi: 10.1007/s12008-024-01912-0. (**I.F.-2.71 / Published**)
4) P. Gulia, R. Kumar, G. Kaur, and S. Suryawanshi, "Numerical modelling using fuzzy multi-objective optimisation for environmental sustainability in green supply chain manufacturing," *International Journal on Interactive Design and Manufacturing*, pp. 1-16, Aug. 2024, doi: 10.1007/s12008-024-02043-2. (**I.F.-2.71 / Published**)
5) P. Gulia, R. Kumar, W. Viriyasitavat, A. Sapsomboon, G. Dhiman, R. Alshahrani, S. Solaiman, R. Choudhary, P. Dey and R. Sivaranjani, " A Symmetric and Comparative Study of Decision Making in Intuitionistic Multi-Objective Optimization Environment: Past, Present and Future." *Archives of Computational Methods in Engineering*, pp. 1-39, Mar. 2025, doi: 10.1007/s11831-025-10243-6. (**I.F.-9.7 / Published**)
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Book Chapters

1) P. Gulia, R. Kumar, A. Kaur, and G. Dhiman, "A Comparative Study of Fuzzy Linear and Multi-Objective Optimization," In *AI-Enabled Multiple-Criteria Decision-Making Approaches for Healthcare Management,* 2022, IGI Global, ch. 7, pp. 117–136. Accessed: Oct. 03. doi: 10.4018/978-1-6684-4405-4.ch007. (**Published**)
2) P. Gulia, R. Kumar, G Kaur, S Singh and M Kapoor. "Application of Fuzzy Multi-Objective Optimization on Existing Technologies and Selection of Optimal Lattice in Artificial Skin Generation", *In Advanced Manufacturing Processes*, 2025, CRC Press, ch. 10, pp. 151-177. (**Published**)
3) P. Gulia, R. Kumar, M. Rakhra, D. Prasharand and S. Jha. "An Enhanced Approach for Soft Computing-Based Manufacturing Optimization Using Non-Linear Intuitionistic Fuzzy Programming Techniques", *In Soft computing in smart manufacturing and materials,* 2025, Elsevier, ch. 2, pp. 11-45. (**Published**)
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Conference Articles

1) P. Gulia, and R. Kumar, "Distance-Based Methods for Multi-Objective Optimization Problems: Fuzzy and Non-Fuzzy Approaches," *in AIP Conference Proceedings*, American Institute of Physics, vol. 2986, no. 1, Feb. 2024. doi: 10.1063/5.0194164. (**Presented, Published**)
2) P. Gulia, R. Kumar, and G. Dhiman, " Various Fuzzy Approaches for Fuzzy Multi-Objective Linear Programming Problems (FMOLPP)," *in International Conference on Communication, Security and Artificial Intelligence (ICCSAI)-2022*, IEEE, Dec. 2022. (**Presented**)
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Workshop Attended

1) International Workshop on Mathematical Computations Using Softwares organised by Department of Mathematics, Akal University, Talwandi Sabo, Bathinda, Punjab (India), (Dated: - April 26-28, 2023). **(Attended)**