MATHEMATICAL STUDY ON POPULATION DYNAMICS WITH ALLEE EFFECT USING FRACTIONAL ORDER DIFFERENTIAL EQUATIONS

Thesis Submitted for the Award of the Degree of

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in
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By

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Declaration of Authorship

I, hereby declare that the presented work in the thesis entitled “Mathematical Study on Population Dynamics with Allee Effect using Fractional Order Differential Equations” in fulfilment of my degree of **Doctor of Philosophy (Ph. D.)** is the outcome of research work carried out by me under the supervision Dr. Preety Kalra, working as Professor, in the Department of Mathematics at School of Chemical Engineering and Physical Sciences of Lovely Professional University, Punjab, India. In keeping with the general practice of reporting scientific observations, due acknowledgments have been made whenever the work described here has been based on the findings of other investigators. This work has not been submitted in part or in full to any other University or Institute for the award of any degree.

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Certificate

This is to certify that the work reported in the Ph. D. thesis entitled “Mathematical Study on Population Dynamics with Allee Effect using Fractional Order Differential Equations” submitted in fulfillment of the requirement for the award of degree of **Doctor of Philosophy (Ph.D.)** in Mathematics, is a research work carried out by Nisha (41900326), is bonafide record of her original work carried out under my supervision and that no part of thesis has been submitted for any other degree, diploma or equivalent course.

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Abstract

This study considers the mathematical as well as numerical analysis of the fractional order single-species and multispecies mathematical models in population dynamics by incorporating the additive Allee effect. This study is done by making use of fractional order derivatives in the Caputo sense. According to the classical law of population dynamics, higher population densities cause competition for resources among species, whereas lower population densities reduce competition and thus promote growth. However, it has been observed that when population density is extremely low, the per capita birth rate in many populations falls. This may result in a zero or negative growth rate. This phenomenon is referred to as Allee's principle, also known as the Allee effect. There are two types of the Allee effect: weak and strong. The Allee effect is said to be strong when the birth rate becomes negative, and weak when the birth rate continues to fall while remaining positive. A population that is experiencing a strong Allee effect requires a minimum population known as the threshold level to survive; the population becomes extinct below this level.

In recent decades, fractional order calculus has garnered significant interest from the scientific community to elucidate real-world phenomena in physics, particularly in the fields of engineering, viscoelasticity, epidemiology, and others. Fractional calculus is an extension of ordinary calculus, where differentiation and integration can be considered of any order, whether it be real or complex. The most important use of fractional calculus is that it provides the solutions dependent on the entire time interval (non-local) considered and the incorporation of memory retention, as each step builds upon the previous one. So, in this study, the single-species and multispecies models already existing in the literature on population dynamics are reconsidered by incorporating the Allee effect and by replacing the ordinary derivatives with Caputo fractional order derivatives. Newly formed models are analyzed mathematically for the uniqueness of the positive and bounded solutions. Local asymptotical stability of all

steady-state solutions is analyzed by considering weak and strong Allee effects. Global stability of all steady-state solutions is analyzed by the Lyapunov direct method. Numerical analysis of the models to validate the findings is done by using Roberto Garrappa created the PECE scheme for fractional differential equations in MATLAB.

The proposed study's findings may help in determining the threshold level of populations suffering from the strong Allee effect, below which the affected species cannot survive. It will also aid in the analysis of the study's parameters, as varying them will allow the Allee effect to be controlled. Because fractional calculus works at the microscopic level and memorizes the entire development process, it assists us in identifying steady states that are unstable in integer-order modeling but can be stabilized using fractional-order modeling. This will assist in developing timely strategies to protect endangered species. Additionally, the obtained results can help to eliminate unwanted species by introducing the Allee effect through artificial strategies. From the above perspective, the mathematical models used in population dynamics, which include single species, multispecies with prey-predator interactions, and competitive interactions, are employed to achieve the following objectives:

1. Study of single species population dynamics with Allee effect using fractional order differential equations.

2. Mathematical modeling and analysis of two interacting species as prey predator population dynamics with Allee effect using fractional order differential equations.

3. Mathematical modeling and analysis of two interacting species as competitive population dynamics with Allee effect using fractional order differential equations.

4. Study and analysis of three interacting species population dynamics including food chain\ food web with Allee effect using fractional order differential equations.

Chapter 1 provides an overview of population ecology and the evolution theory of mathematical models used to study patterns and changes in populations of the same and different species. Following this, there is a general introduction to fractional calculus, its general properties, and a brief review of the development process of fractional calculus. The literature review section discusses notable research on population

dynamics using various types of fractional derivatives. After conducting an extensive review of the literature, the research gap is identified, which leads to the study's objectives. Various mathematical definitions, theorems, and lemmas used in the study are discussed. Finally, a summary of the research work done is provided.
Chapter 2 analyzes a single-species logistic growth model where the species is exposed to the Allee effect. Mathematical analysis is done by replacing the ordinary derivatives with Caputo's definition of the fractional derivative. The Allee effect is studied additively. The model is evaluated for the uniqueness of positive and bounded solutions. The preconditions for local asymptotic stability and global stability of all steady-state solutions are determined. The threshold level of populations experiencing the strong Allee effect is calculated mathematically. All mathematical findings are validated through numerical simulations in MATLAB.

Chapter 3 examines the fractional order Lotka-Volterra model in which the prey population is growing logistically and exposed to the Allee effect, which is studied additively by making use of the Caputo fractional order derivative. All basic theorems of population dynamics, like the existence of unique, positive, and bounded solutions, are discussed. The effect of prey capture rate by the predator has been studied numerically. Hopf bifurcation is studied at the coexistence point when populations are subjected to weak and strong Allee effects, with fractional order and capture rate of prey by predator as bifurcation parameters. This study shows that fractional order models offer wider stable regions by identifying equilibrium points that are unstable in ordinary integer order models, but are stable in fractionally ordered models.

Chapter 4 examines the Lotka-Volterra competition model with the logistic growth of both populations and the additive Allee effect in one of the species, using both fractional and integer order modeling. The fractional derivative is taken in the Caputo sense. This study discovers that the incorporation of the Allee effect into the classical Lotka-Volterra competition model can result in multistable steady states, depending on the potential of the Allee effect experienced by the population in different competition scenarios. According to the classical Lotka-Volterra competition model, competitive species can coexist only if intraspecific competition is stronger than interspecific competition. However, the inculcation of the Allee effect into the fractionally ordered

model allows the coexistence of the species even in the case of stronger interspecific competition. Moreover, fractional derivatives enrich the population dynamics by stabilizing the model at different coexistence points with different orders of the fractional derivatives. Different intervals for Allee's constant are discovered in various competition scenarios to perform the stability analysis of the obtained equilibrium points. Furthermore, it has been observed that with the same value of Allee’s constant, different solutions exhibit stability depending on the value of the fractional order, beginning with the same populations. Additionally, the values of Allee’s constant are identified to determine the severity of the Allee effect, that is, when the Allee effect will influence weakly and strongly in the presence of competition. The model is evaluated for the uniqueness of positive and bounded solutions. The theoretical results obtained are also verified numerically.

Chapter 5 delves into Caputo’s definition-based fractional order food chain model of three species that includes a prey, and two predators (intermediate predator, and top predator). In the proposed model, prey grows logistically while experiencing the Allee effect. Holing type II functional responses are used to represent interactions between prey and intermediate predators, as well as between intermediate predators and top predators. Numerical simulations of the proposed model are performed in traditional integer order models along with fractional order models, with and without exposing the prey population to the Allee effect. It is found that the exposure of prey to the weak Allee effect can play a significant (positive) role in the coexistence of all species under the restricted conditions of the parameters. It is observed through an example that, when prey are not exposed to the Allee effect, all populations in the proposed model show chaotic behavior in the integer order model, but exposure of the prey population to a weak Allee effect allows the stable coexistence of all populations in fractional as well as in integer-order models. It is also observed that the populations that fail to coexist in the integer model can coexist in the fractional order model even when prey are subjected to a strong Allee effect. Theorems like possessing unique, positive, and uniformly bounded solutions are derived. The global and local asymptotic stability requirements for each equilibrium are derived.

Chapter 6 highlights the key findings of the research in the form of a conclusion and outlines the future scope of the research.

The bibliography in the final section provides support for the issues examined in the thesis.

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Chapter 1

General Introduction

1.1 Introduction

Ecology is an exploration of living organisms and how they interact among themselves and with their surroundings, such as climate, soil, and availability of water. This field of study involves examining life processes, adaptations, biodiversity, and the distribution and abundance of organisms. An ecosystem can be defined as a geographic area where biotic and abiotic factors work together to produce life. Biotic factors are living parts such as plants, animals, and other living organisms, whereas abiotic factors are non-living things such as the climate, temperature, humidity, water, sunlight, soil, etc. Population ecology is a specialized branch of ecology that specifically studies the patterns and changes in populations of different species, as well as how these populations interact with their surrounding environment. This field is the scientific examination of variables like rates of births and deaths, immigration, and emigration, and how these variables impact population size, growth, and lifespan patterns. Population ecology has three characteristics.

1. Population size: It represents the total number of individuals. A larger population size depicts greater genetic variations, long-term survival, and overuse of resources, resulting in a population crash.
2. Population density: It represents the total number of individuals per unit area. Lower density lowers the competition for survival, and high density leads to greater competition for survival.
3. Population dispersion: It represents the nature of interaction within species and how and why species are dispersed [1].

A population is defined as an assemblage of individuals from the same species who can coexist and interbreed in an ecosystem. A community is an assembly of populations of different species coexisting and interacting with each other. Interactions can be within the same species as well as among different species. Interactions within the same species are known as intraspecific interactions, and interactions among different species of a community are known as interspecific interactions. The intraspecific interactions are competitive and density-dependent. In densely populated ecosystems, the organisms of the same species compete for resources, whereas in sparsely populated ecosystems, organisms compete for mates to reproduce. These interactions are essential in determining the composition and organization of ecological communities, and they have a significant impact on determining the dynamics and structuring of these communities. The Interspecific interactions are mainly of three types: competition, predation, and symbiosis. In interspecific competition, two or more species compete over the same resources. In predation, one species, known as the predator, kills another species known as prey to get food, and in symbiosis, two species help each other to live together [2]. These biological interactions many times lead to the exploitation of resources, which can be further harmful to the living organisms in an ecosystem, even sometimes resulting in the extinction of some species, which can destroy the food chains existing in nature. Human activities are also responsible for the damage to natural ecosystems, so to save the species from extinction, it becomes very important to study the biological interactions within species.

Mathematical biology is the study of biological interactions using mathematics. Mathematical biology is concerned with modeling, studying, analyzing, and interpreting biological phenomena such as evolution, coexistence, and interaction with different species [3]. To study biological interactions, mathematical modeling is an effective tool, as it saves time, energy, and resources. Mathematical models are designed accurately to represent biological interactions. The modeling process requires carefully designing the variables, identifying the governing laws, and making some assumptions to make the model tractable [4]. Work on population dynamics using mathematical modeling started long ago. In the year 1202, Leonardo of Pisa wrote a book on arithmetic in which he used a mathematical model for growing the rabbit population [2]. After that, many mathematical models were suggested to understand population dynamics. The very first useful work was done by Malthus in 1798 by developing an exponential growth model for a single species [5]. This model was practically impossible as it considers unlimited resources to survive. In the year 1838, Verhulst modified the exponential growth by introducing the logistic model in which initially population grows exponentially, but as saturation begins, the growth slows down, and at the level of maturity, growth stops [2]. Later, an improved model known as the Hutchison model was suggested, which considers the fact that the birth rate cannot act instantaneously, and there is a possibility of a delay in reaching maturity. This model is also known as the delay model [6]. All these models were used to describe the evolutionary progression of a single species. Vito Volterra, an Italian mathematician, in the year 1926, proposed a two-species model describing a prey-predator community, and at the same time, Lotka, an American ecologist and mathematician, proposed the same model, which is known as the Lotka-Volterra model [2,5]. In this model, prey grows at a constant rate in the absence of predators, but decreases linearly as a function of the density of the predators in their presence. The Lotka–Volterra model's structure was oversimplified because it makes numerous assumptions that are not entirely true. The assumptions include an uninterrupted pattern of prey species' food availability, the absence of adaptation of all interacting species, and the initial relationship between the size of every species and its rate of growth, which leads to exponential expansion if the species remains unaffected by others. To make this model more realistic, various types of functional responses that explain mathematically how predators grow on prey were suggested. A few of them are known as Holling type I, Holling type II, Holling type III, Holling type IV, Beddington-DeAngelis-type, Ratio-dependent type, Hassel-Verley-type, etc [7]. Apart from the prey-predator model, there are many other types of Lotka-Volterra models, such as the competition and cooperation model. In the year 1926, McKendrich proposed the first continuous age-dependent model (classical model of linear age-dependent population model) known as the Sharpe-Lotka-McKendrich model. In the year 1974, Gurtin and McCamy first introduced the non-linear age-structured population model [8].

Until now, there has been the development of various types of models with minor or major changes in the pre-existing models, but the commonality in all the models was the use of traditional, integer-order calculus. These traditional evolutionary theories frequently presume individuals have no memories of previous events or experiences. However, it is found that many species demonstrate the behavior, indicating that they preserve the memory of the past. Memory can affect an individual's fitness, reproductive success, and ability to adjust to changing situations [9]. It is observed that the fractional derivatives, the derivatives of arbitrary orders, real or complex, unlike traditional derivatives, are non-local, consider entire data of the past when evaluated, and hence solutions provided are dependent on the past; hence, fractional calculus helps in retaining memory. Therefore, modeling the natural phenomena using fractional calculus exhibits long-term memory [10]. The birth of fractional calculus is assumed to have been more than 300 years ago. In 1695, the famous mathematician Leibnitz wrote a letter to L'Hôpital in which he raised the question, “Can the meaning of derivatives of integer order be generalized to the derivative with non-integer order?” L’Hôpital replied, “What if the order will be ½?” Then Leibnitz replied, “It will lead to a paradox, from which one day the useful consequences will be drawn.” So, fractional order calculus is not much older than conventional calculus. In 1730, Euler, in 1772 J.L. Lagrange, in 1812 Laplace, in 1819 Lacrox, and in 1822 J.L.J. Fourier made a major contribution to the development of fractional calculus. In 1832, J. Liouville made a major contribution to fractional calculus, and he applied his definition to the problem. In 1867, A.K. Grunwald, and in 1892, G.F. B Riemann contributed to the fractional calculus. From 1868 to 1872, A.V. Letnikov wrote many papers on fractional calculus [11]. From 1900 to 1970, H.H. Hardy, S. Samko, H. Weyl, M. Riesz, S. Blair, and from 1970 to date, J. Spanier, K.B. Oldham, B. Ross, K. Nishimoto, O. Marichev, A. Kilbas, H.M. Srivastava, R. Bagley, K.S. Miller, M. Caputo, I. Podlubny, and many others made noteworthy contributions in the development of fractional calculus [12].

Fractional calculus is inextricably linked to the memory systems seen in numerous real-world systems, and fractional derivatives are defined globally rather than locally [13,14]. The scientific community in recent years has changed its focus to describing real-world problems using fractional calculus since traditional derivatives cannot capture the memory effect, but fractional differential operators do. Furthermore, various published studies have proved that fractional-order systems provide better insight into real-world phenomena as compared to ordinary integer-order systems [15-17]. Hereditary properties due to memory preservation, and flexibility in degrees of freedom, fractional calculus have piqued the interest of the scientific community, particularly in different fields of physics, viscoelasticity, engineering, signal processing, etc. [18-24]. In the field of epidemiology, fractional order models are widely used nowadays, as results obtained using fractional order models fit better with the real world [25-32].

Recent fractional derivatives include Riemann-Liouville, Caputo, Weyl, Jumarie, Hadamard, Davidson, Essex, Riesz, Erdelyi-Kober, and Coimbra. Furthermore, each definition has advantages and disadvantages when used in modeling real-life problems. One of the disadvantages of the Riemann-Liouville fractional derivative is that it does not assign a value of zero to a constant. Furthermore, if a function results in a constant value at the origin, then its fractional derivative will have a singularity at the origin, for instance, the exponential and Mittag-Leffler functions. The Riemann-Liouville fractional derivative has limited applications due to its disadvantages. Caputo's definition of a fractional derivative requires a higher level of regularity for differentiability. For a function to have a fractional derivative in the Caputo sense, it should be differentiable. A function without a first-order derivative should have a Riemann-Liouville fractional derivative of an order less than one. If a function is not continuous at the origin, the calculation of the Jumarie fractional derivative is not possible. The Weyl fractional derivative has a significant disadvantage: the integral used to define it is improper, requiring additional restrictions on the function. Furthermore, the Weyl derivative of a constant has not been defined [33,34]. To address these limitations, Caputo and Fabrizio (CF) and Atangana and Baleanu (AB) proposed improved definitions of fractional-order operators that are dependent upon the exponential kernel and generalized Mittag-Leffler function, respectively. However, it is revealed that out of these, Caputo's fractional derivative is preferred for studying the practical problems of the real world as it allows the inclusion of traditional boundary conditions while formulating the problem mathematically, assigns value zero to the derivative of a constant, and is very simple to apply. Researchers have shown that while studying co-infection models through different fractional derivatives, Caputo derivatives give a better fit with real-world data [35-36].

1.2 Review of Literature

It is observed that history plays a crucial role in the evolution process of biological systems [9], and fractional derivatives have the property of retaining memory of the past, which is very helpful in understanding the complexity of the dynamical systems [10,13]. This property of fractional calculus motivated ecologists and mathematicians to analyze the mathematical models in population dynamics using fractional derivatives.

Early work using fractional order derivatives in the research area of population dynamics was the study of the Lotka-Volterra predator-prey model by Ahmed et al. in 2007. The author analyzed the fractional-order Lotka-Volterra model and the rabies model. The author compared the results obtained with the ordinary integer derivative model and found that for a particularly selected value of one parameter, one of the steady state solutions (equilibrium point) was the center of oscillation in the case of the integer order model, but when that solution was considered using fractional order model it was asymptotically stable and the results were verified by numerical simulations too [37]. El Sayed et al. published a paper analyzing the fractional-order logistic equation by taking different values of the fractional order. In the paper author established the stability conditions of all steady state solutions [38]. Das et al. studied the fractional order Lotka-Volterra model numerically using the Homotopy perturbation method. The author's findings include that populations in lower fractional order derivative models stabilize faster than in the higher fractional order derivative models [39]. Another theory in the same direction was on the same model by Adriana et al., however, by using the Caputo fractional derivatives. The author compared the results with the classical integer order model and found that in fractional-order models, solutions move more slowly than those in the integer order models. It was concluded that, in the fractional order models, populations take more time to reach the steady state solutions in comparison to those in the integer order model [40].

One of the important factors in ecology is how populations interact with each other. These interactions can be represented in the form of a mathematical function. A function that describes how many prey are killed, consumed, and converted to predators (in terms of reproduction) by predators in a unit of time for the given densities of the populations is known as a functional response. In the ecological literature, many researchers have suggested different types of functional responses representing predator-prey interactions. Some of the important functional responses are Holling type I, Holling type II, Holling type III, Holling type IV, Beddington-DeAngelis-type, Ratio-dependent type, Hassel-Verley-type, etc [7]. Classical Lotka-Volterra models are unrealistic as they do not consider the competition for resources due to overcrowding. The introduction of functional responses in the prey-predator models made the models more reliable for the future prediction of the interactive species. To study such interactions, George et al. published a paper on prey-predator interactions in which they used the Caputo fractional order derivative and functional response, Holling type II. The author established the prerequisites for the uniqueness, boundedness, and stability of the solutions [41]. Die et al. studied a stage-structured prey-predator, fractional-order model with functional response of Holling type III using modified Riemann-Liouville fractional derivatives. By constructing a suitable Lyapunov function author established the existence, uniqueness, and asymptotic stability of the positive equilibrium points [42]. Several studies proved that the interactions between predators and prey may depend upon the prey-predator ratio in the ecosystem, too. Such functional responses are known as ratio-dependent functional responses. Suryanto et al. studied a fractional-order predator-prey system incorporating linear harvesting of both populations and ratio-dependent functional response by using Caputo's definition of fractional-order derivatives. Apart from establishing the conditions for stability of all steady state solutions, Hopf bifurcation is also explored by taking the order of the fractional derivative and the prey population's harvesting rate as parameters [43].

With the growing population of mankind, the exploitation of the ecosystem has increased. To fulfill the needs of human beings, harvesting of natural resources like fisheries and wild animals is done on a large scale. While harvesting a population or a resource, it is important to consider both human needs and the sustainability of the ecosystem. Javidi and Nyamoradi published their study on the prey-predator system using Caputo's definition of fractional-order derivative, incorporating the functional response of Holling type II and harvesting of predators. The authors investigated the stability conditions of all steady state solutions (equilibrium points) and observed that in the fractional-order model, all populations reach their equilibrium points with time. Through various numerical simulations, the author proved that the system has stable cycle trajectories. It was shown using fractional derivatives that the optimal rate of harvesting can fulfill the food requirements of the populations, as well as ecosystems can be saved from exploitation [44]. Mandal et al. studied the fractional-order system of prey-predator incorporating harvesting of both populations and considering the interaction between them of Holling type II. The author studied the uniqueness, boundedness, and non-negativity of the solutions of the commensurate system as well as the incommensurate system. Stability conditions of solutions for both systems were established. Prerequisites for the system to show a Hopf bifurcation were also derived. Numerical simulations proved that the intrinsic growth rate of the prey population, biomass conversion rate, half-saturation constant, and harvesting of both populations can play an important role in controlling the populations [45]. Another study in this direction was published by Yavuz and Sene. In this paper, the author analyzed the fractional predator-prey system with the harvesting of the predator population by taking the Caputo fractional derivative. This model considered the harvesting rate proportional to the predator density until it reaches a threshold value, and after that harvesting rate assumes a constant value. Numerical simulations have shown that the harvesting rate plays a crucial role in determining the stability of the populations [46].

It has been recognized that the interaction between predators and prey cannot be instantaneous. Contact of the predator with prey should be retarded due to a gestation period or due to some other factors; to make prey-predator systems more realistic, a time delay is incorporated. Song et al. published a paper on a fractional-order delayed predator-prey system with harvesting of both populations taking delay as a parameter, using Caputo fractional-order derivatives. The results obtained were compared with the results of the non-delayed model, and it was found that fractional-order delayed models are more realistic, and the order of fractional derivatives affects the stability as well as the stability switching time. The theory established proved that by controlling the harvesting rates and using fractional order systems, populations moving towards extinction can be stabilized [47]. Rajivganthi and Rihan studied a predator-prey system with time delay and Holling type IV interaction using Caputo fractional order derivative. The author justified the delay by dividing the predators into two groups: juvenile (immature) and adult (mature) predators. The obtained results were compared with the non-delayed ordinary integer order model, and noted that fractional-order models provide more stable regions. Also, the solutions that were unstable in the model of integer order became stable when the model was considered by taking a fractional order. The theory established that the memory-retaining property of the fractional order derivative can help in preventing the oscillatory behaviors of the populations [48].

The herd behavior or social behavior of the prey population plays a significant role in their interactions with predators. In ecosystems, various prey populations are observed to be moving in groups to protect the offspring and the oldest ones from predation. Herd shape depicts the defense strategy adopted by prey against predators. This behavior ensures the increase of birth rate, smooth growth of the youngest prey, and long life for the oldest and weakest population. While moving in groups, the strongest prey remains at the outer bound of the pack, and others remain at the center of the pack. In this situation, predators can interact with some of the prey, but not with all. Such interactions cannot be explained with the help of Holling types I, II, III, or IV. Arjaldi et al. investigated such types of interactions and introduced some new functional responses representing the interactions between different populations moving in search of food using herd strategies [49]. Ghanbari and Djilali studied such type of interaction using Caputo fractional-order derivatives-based models. In this paper, the author considered one prey population showing herd behavior and two predators. One of the predators consumes only the prey population, while the other can consume prey as well as the predator. Through numerical simulations, the author proved that lesser values of the order of fractional derivatives help in the early stabilization of the populations, and the type of interaction plays a significant role in the coexistence of both populations. The author concluded that increasing the birth rate of prey populations by providing better living conditions and using suitable strategies for hunting can help the survival of all species [50]. Another study in this sequence was by Bingnan Tang. In his paper, the author studied the dynamics of the fractional-order predator-prey model incorporating a group defense mechanism and Michaelis-Menten type (Holling type II) harvesting. This study was mainly focused on finding the critical value of the order of the fractional derivative below which the system can be stabilized, and after crossing that, the system shows a Hopf bifurcation [51]. Alidousti and Ghanfari studied the dynamic behavior of a one-prey and two-predator system with group defense mechanisms adopted by populations considering the interactions of the Monod-Haldane kind using Caputo's fractional-order derivative-based model. On comparing the classical integer order model results with the fractional order model results it was noticed that the prey population shows chaotic character for the lesser value of carrying capacity in the classical integer order model and in the fractional-order system value of carrying capacity obtained for showing chaotic behavior was relatively higher, hence it was concluded that fractional-order models provide more stable regions. Similar kinds of results were obtained if the intrinsic growth of prey and the half-saturation constant of the middle predator were taken as parameters. Therefore, it can be concluded that memory helps in achieving better stability conditions [52].

The refuge is an interesting concept in population dynamics. It is the defensive strategy adopted by prey to protect themselves from predators. It has been observed that prey populations hide themselves in an area that is inaccessible to predators. Studies have shown that the existence of refuge helps the populations in coexistence. Li et al. published their study on the prey-predator model considering constant prey refuge along with feedback control employing Caputo's definition of fractional-order derivatives. On comparing the obtained results with the results of classical integer-order modeling, it was found that the fractional-order models provide larger stability regions than the classical model. Higher-order models converge to equilibrium points more speedly than lower-order models. Therefore, lowering the order of the fractional models helps to control the populations at an early stage. The study proved that the prey refuge mechanism and feedback control can play a positive role in the co-existence of populations [53]. Another study in this direction was published by Das and Samanta. The authors studied Caputo's fractional-order derivative definition-based food chain model consisting of one prey, one middle predator, and one super predator. Holling type I interactions between prey and middle predator, as well as between middle predator and super predator, along with the fear effect of predation felt by the prey population, were incorporated. The author investigated the non-delayed as well as the delayed model. In the non-delayed model with higher orders of fractional derivative (close to 1) and higher prey refuge leads to the extinction of super predators, and on increasing the fear factor, both populations, prey and super predators, move towards extinction, and no Hopf bifurcation was observed. However, in the delayed model, the authors found a critical value of the delay at which stability switches to Hopf bifurcation. It is observed that the critical value of time delay depends upon the order of fractional derivatives. Lowering the order of fractional derivatives gives a higher value of critical delays and hence provides more stability regions [54]. Barman et al. published their research on fractional-order prey-predator interactions considering fear induced by predators on prey population and prey refuge using Caputo fractional derivatives and Holling II type functional response. Apart from establishing the existence, positivity, boundedness, and stability, it was found that fractional-order models provide wider stable regions than integer-order models. Hopf bifurcation was studied around the coexistence equilibrium point by taking the fear factor and prey refuge as parameters. It was found that when fear factor and prey refuge are taken at lower levels, the system moves towards stability by decreasing the order of the fractional derivative continuously, so it can be judged that memory plays an important role in the coexistence of all populations [55].

To fulfill the needs of the rising population, the exploitation of natural resources has increased. The construction of factories, highways, development of tourism destroyed the natural habitats of wildlife. To survive, the wildlife has dispersed into different patches, so to study the population dynamics more realistically, the dispersion of populations is also considered by many researchers. Gao and Zhao published a study using the Caputo fractional derivative-based model of a single species, taking the dispersal of the population into account. The authors studied existence, positivity, and boundedness and laid conditions for the stability of all solutions [56]. Li et al. published another paper analyzing the fractional-order single species system incorporating diffusion, using Caputo's fractional order derivative. The author considered the dispersal of species in several patches. The study was done to see the impact of fractional derivatives on the population dynamics of a single species and observed that higher-order derivative models converge to equilibrium points with more speed as compared to the lower fractional-order model, so it was concluded that the order of fractional derivatives can help in controlling the population and dispersal of populations helps in coexistence [57]. Xie et al. analyzed the prey-predator system considering prey refuge and dispersal of the population using Caputo's fractional order derivatives' definition. By taking populations in two patches, the author proved that the fractional models provide stable solutions. Prey refuge in one patch helps the population in other patches from extinction, and diffusion rates also help in the survival of populations in different patches [58].

In ecosystems, some species can survive independently and benefit each other by living together. Such interactions are known as facultative mutualism. An example of such interaction can be observed between herbivorous crabs and coralline algae. Supajaidee and Moonchai studied facultative mutualism interactions in the fractional-order two-species model, with harvesting taking Caputo's definition of the fractional derivative. The author mainly studied the conditions required for coexistence points to exhibit globally stable behavior [59]. Amirian et al. published their study considering a fractional-order model using Caputo fractional order derivative, consisting of two predators behaving in a mutualistic way (Dolphins and yellowfin tuna) and one prey (lanternfish) with the harvesting of all three populations. Numerical simulations of the model proved that the lower values of the order of the fractional derivative help in early stabilization of populations, and the fractional order helps to control the fluctuations in populations. Using the fractional-order model author established that economic harvesting helps in the coexistence of populations [60]. Tang proposed and studied the fractional-order competitor- competitor-mutualistic Lotka–Volterra system, taking the same feedback time delays in all species using Caputo derivative. The author established the conditions for existence, uniqueness, Hopf bifurcation, and local stability of non-negative solutions and proved that time delay, along with fractional order, has a significant effect in controlling the stability [61].

Populations not only have to protect themselves from predation, but they must also struggle to get food when there are limited resources or overcrowding of populations. So, interspecific competition plays a crucial role in studying population dynamics. Wang et al. published a paper using a fractional-order delayed model consisting of one prey and two predators with interspecific competition using a fractional-order derivative. The author proved that the combined effects of gestation delay and fractional order give stable solutions. The author calculated the critical value of delay below which the system always remains stable and above which Hopf bifurcation appears. The study proved that a fractional-order derivative helps in finding the conditions for the co-existence of populations within competition too [62].

Eco-epidemiological models are very helpful in studying the interactions between prey and predator populations in which one or both populations suffer from infectious diseases. The main objective of studying such models is to control the spread of infection among populations. Many researchers studied such interactions using fractional order models where the fractional derivatives were taken in the Caputo sense. Moustafa et al. studied the eco-epidemiological system in which the prey population was exposed to an infectious disease, incorporating the attack rate of the predator and half-saturation constant as parameters. The authors successfully calculated the value of the order of the fractional derivative. Below that order, the disease-free equilibria were stable, and crossing that order system showed a Hopf bifurcation around that equilibrium point [63]. Lemnaour et al. studied the prey-predator interactions with disease in predators along with competition and toxicity using Caputo's definition-based fractional order derivative model. The prey population was considered in two different areas: a reserve area and a harvest area. Results obtained were compared with the model of integer order, and it was found that fractional-order derivatives help in stability by damping the oscillations at lower values of the fractional derivatives [64]. Rihan and Rajivganthi studied the fractional-order delayed model of prey-predator in which predators were exposed to an infectious disease along with Holling type III interaction between both populations. It was observed that if no delay was incorporated, interior equilibrium is always unstable, but in the delayed model, there was always a possibility of finding a critical value of delay, below which all populations in the model can be stabilized, and for different orders of fractional derivatives, different critical delays can be possible [65]. Perumal et al. considered a prey-predator system with infection in both populations using fractional order derivatives and compared the obtained results with those obtained by employing the classical integer model and found that the fractional derivatives of order less than one can minimize the risk of extinction of the species and help in damping of oscillations of the populations in relatively less time. It was spotted that in the fractional-order model, the disease-free equilibrium point was globally asymptotically stable [66]. Ghanbari and Djilali considered a fractional-order prey-predator system in which the prey population was behaving socially and predators were suffering from an infectious disease. In the fractional-order model, it was observed that an endemic equilibrium is possible with a suitable order of fractional derivative, and the value of the fractional-order can be found, which can help in the eradication of the disease. Moreover, it was concluded that herd shape cannot help in the elimination of disease, but it can help in reducing the infection level [67]. Djilali and Ghanbari published another study to see the effect of non-fatal infectious disease developed in prey on the population dynamics of prey-predator, with the help of Caputo fractional order derivative. The prey population was considered in two compartments: susceptible and infected. Predators cooperated while hunting, and infection was supposed to be non-vertical. The authors observed that the disease-free equilibrium point is unstable in the case of the ordinary integer order model, but stable in the case of the fractional-order model, and proved that all species can coexist without extinction in the fractional-order model. It was also noticed that lower values of fractional derivatives helped in damping the oscillations of the population [68].

It has been recognized that individuals of many species can benefit from the presence of conspecifics [69]. This is simply contrary to the classical law of population dynamics, which states that a higher population density causes competition for resources, whereas a lower population density reduces competition and thus promotes growth. However, it has been observed that when population density is critically low, per capita growth rates in many populations decline. This may lead to a zero-growth rate or even a negative growth rate. The primary reason behind this may be the possibility of not finding a mate for reproduction when the population is quite low. This results in low reproductivity, which can lead to the extinction of the population. It is well observed that for a population to grow, a minimum density of that population is required, which is known as the critical density population or threshold level, below which that population cannot exist [70]. An American biologist, Eugene Odum, named this phenomenon Allee's principle in honor of its founder, Warder Clyde Allee, which is now acknowledged as the Allee effect [71]. The Allee effect can be caused not only due to disability to find mates at low population density but also due to social dysfunctioning due to low population density, inbreeding depression, swamping of enemies, allelic diversity, and food exploitation [72]. The Allee effect can affect population growth strongly or weakly depending upon its intensity [73]. The Allee effect is said to be strong when the birth rate becomes negative, and weak when the birth rate continues to decrease but remains positive. However, when a population is exposed to a strong Allee effect, it will require a minimum population known as the threshold level to survive; below this level, the population will become extinct [74-76]. Several studies have examined the Allee effect in various population systems and concluded that it can significantly affect population dynamics [77,78]. Moreover, the introduction of the Allee effect can change the system’s dynamic behavior by stabilizing or destabilizing it, or by switching the time to reach the equilibrium point [79]. Various published studies have addressed the Allee effect using integer order modeling. For instance in studying the stability of the prey predator interactions [80], dynamics of an age structured population with harvesting [81], prey predator model with Hawk and Dove tactics [82], predator prey system involving distributed delays [83], dynamics of a predator prey system with diffusion [84], dynamics of a predator prey system involving ratio dependent interactions [85], dynamics of a predator prey system when prey subject to disease [86], in studying prey predator model in which predators were experiencing component Allee effect for reproduction [87], the complex dynamics of a prey predator system with diffusion [88], predator prey population dynamics involving intraspecific competition [89], deterministic and stochastic predator prey models with herd behavior [90], considering Holling type II interactions in predator prey model with disease in prey incorporating prey refuge a [91], in studying the dynamical analysis of model consisting of two prey and one predator [92], in studying the intraspecific competition among predators and their influence on a Gause type predation model [93], predator prey model with Holling type I functional response [94], in comparative analysis of predator prey model when prey experienced fear effect of predation using Holling type I and II interactions [95], while studying prey predator model incorporating fear of predation among prey along with employing the strategy of prey refuge [96], to simplify the prey predator model [97], stability analysis along with Hopf bifurcation analysis with time delays in a three species system [98], the stability analysis and bifurcation analysis with fear factor in a predator-prey system [99].

While studying population dynamics using mathematical modeling, the Allee effect can be studied in two ways, additively and multiplicatively. Bazykin [100] introduced the multiplicative Allee effect for a single species logistic growth model as,

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right)(x - m), \tag{1.1}$$

where x is the population at any time $t. r, K, m$ are all positive constants defined as, r population's per capita intrinsic growth rate, K is the maximum population that the environment can support, and m is Allee's threshold. The growth rate of the population will remain positive if $0 < m < x < K$ and will be negative outside this interval. Another single-species logistic growth model was introduced by Brian Dennis [101], in which he introduced the additive Allee effect. The author considered the growth equation as,

$$\frac{dx}{dt} = \left[r\left(1 - \frac{x}{K}\right) - \frac{\alpha a}{x + a}\right]x. \tag{1.2}$$

Here, x is the population at any time $t. r, K, m$ are all positive constants defined as, r population's per capita intrinsic growth rate, K is the maximum population that the environment can support, and a is the population density at which fitness is half of its maximum value. Here, fitness is considered in terms of reproduction efficiency. The factor $a \backslash x + a$ measures the relative fitness of the population due to the Allee effect. Therefore, a greater value of a will reduce the fitness of the population due to the Allee effect. The term α, which is the constant of proportionality, denotes the severity of the Allee effect. By taking $\alpha a = m$, equation (1.2) can be written as

$$\frac{dx}{dt} = \left[r\left(1 - \frac{x}{K}\right) - \frac{m}{x + a}\right] x. \tag{1.3}$$

Here m and a depict the degree of the Allee effect. According to this model, if $0 < m < ar$, then the population is exposed to a weak Allee effect, and if $m > ar$, then the population is exposed strong Allee effect. Syed Abbas et al. [102] published a paper analyzing the fractional-order modified logistic equation of a single species incorporating the multiplicative Allee effect using Riemann-Liouville fractional derivatives and integrals. The author found that the population can grow only if the initial population lies between the Allee effect threshold parameter and the maximum population that can be supported by the environment. An initial population below the threshold parameter leads to extinction. It was observed that decreasing the order of fractional derivatives leads to an increase in the time taken to reach the steady state by the population. Using AB fractional derivatives, Nisar et al. [103] studied the fractional order food chain system with Holling Type II interactions among populations by exposing the prey population to the additive Allee effect. Given an extensive review of the literature, it is observed that the study of population dynamics incorporating the additive Allee effect is still unexplored using fractional order derivatives.

1.3 Objectives of the proposed work

Based on the literature review done above, and the found research gap, it is observed that there is a lot of scope to study the population dynamics incorporating the Allee effect using fractional order differential equations. The same can be achieved using the following objectives:

1. Study of single species population dynamics with Allee effect using fractional order differential equations.

2. Mathematical modeling and analysis of two interacting species as prey predator population dynamics with Allee effect using fractional order differential equations.

3. Mathematical modeling and analysis of two interacting species as competitive population dynamics with Allee effect using fractional order differential equations.

4. Study and analysis of three interacting species population dynamics including food chain\ food web with Allee effect using fractional order differential equations.

1.4 Mathematical Preliminaries

Definition 1.4.1 Autonomous and non-autonomous systems [104]

An autonomous system of a fractional order differential equation is defined as

$$D^{\alpha}x(t) = f(x), \qquad x(0) = x_0.$$

Here function does not depend upon the independent variable, whereas in non-autonomous systems

$$D^{\alpha}x(t) = f(t, x), \qquad x(0) = x_0.$$

The function depends upon the independent variable. Where $\alpha \in (0,1)$ is the fractional order.

Definition 1.4.2 Equilibrium point [104]

A constant x_* is said to be the equilibrium point of the fractional order differential equation $D^{\alpha}x(t) = f(t, x)$, iff $f(t, x_*) = 0$. Where $\alpha \in (0,1)$ is the fractional order.

Definition 1.4.3 Stability of solution [104]

(i) The solution x_* of a fractional order system $D^{\alpha}x(t) = f(t, x)$, is considered to be stable if for any $t_0 \in \mathbb{R}$ and any $\epsilon > 0$, we can find $\delta = \delta(t_0, \epsilon) > 0$, so that whenever we take $\|x(t_0) - x_*(t_0)\| < \delta$ it will imply $\|x(t) - x_*(t)\| < \epsilon$ for all $t > t_0$.

(ii) The solution x_* of a fractional order system $D^{\alpha}x(t) = f(t, x)$, is considered asymptotically stable if the solution is stable, and for any $t_0 \in \mathbb{R}$ we can find $\delta = \delta(t_0) > 0$, so that whenever we take $\|x(t_0) - x_*(t_0)\| < \delta$ it will imply $\operatorname*{Lim}_{t \longrightarrow \infty} \|x(t) - x_*(t)\| = 0$.

(iii) The solution x_* of a fractional order system $D^{\alpha}x(t) = f(t, x)$, is considered uniformly (globally) stable if the solution is stable and if for any $t_0 \in \mathbb{R}$ and any $\epsilon > 0$, we can find $\delta = \delta(\epsilon) > 0$, so that whenever we take $\|x(t_0) - x_*(t_0)\| < \delta$ it will imply $\|x(t) - x_*(t)\| < \epsilon$ for all $t > t_0$.

(iv) A solution is said to be unstable if it is not stable.

Definition 1.4.4 Caputo fractional derivative [104]

If $\alpha > 0$. Caputo's fractional order derivative of a function $f(x)$ is expressed as

$$D^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} f^n(t)\, dt, \tag{1.4}$$

where $n - 1 < \alpha < n.$ If, $0 < \alpha < 1$,

$$D^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-t)^{-\alpha} f'(t)\, dt. \tag{1.5}$$

Definition 1.4.5 Mittag-Leffler function [104]

The Mittag-Leffler function of a one-parameter family is defined as,

$$E_{\alpha}(z) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(i\alpha + 1)},$$

where $\alpha > 0$, and z is a complex number. Mittag-Leffler function of the two-parameter family is defined as,

$$E_{\alpha,\beta}(z) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(i\alpha + \beta)},$$

where $\alpha > 0, \beta > 0$, and z is a complex number.

Lemma 1.4.6 Lipschitz condition [54]

Consider the fractional order differential equations $D^{\alpha}x(t) = f(t, x)$, where $\alpha \in (0,1]$, with initial condition $x(t = t_0) = x_0 > 0$. Here $f : [t_0, \infty) \times \mathrm{D} \rightarrow \mathbb{R}$ is a function. Then $f(x, t)$ is said to satisfy the Lipschitz condition w. r. t. variable x in $[t_0, \infty) \times \mathrm{D}$, if there exists some real constant $L > 0$, such that $\left\| f\left(t, X(t)\right) - f\left(t, Y(t)\right) \right\| \leq L \| X(t) - Y(t) \|$,

where L is independent of t, X, and Y, and $\mathrm{D} = \{ x \in \mathbb{R} : |x| \leq M \}$ and M is a positive finite real constant.

Lemma 1.4.7 Solution of fractional order initial value problem [105]

Consider the fractional order differential equation

$D^{\alpha}x(t) = \ f(t, x), x(t = t_0) = x_0,$

where $f : [t_0, T] \times \mathrm{D} \ \rightarrow \mathbb{R}$, is a continuous function. Then every solution of the given initial value problem also satisfies

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s))\, ds, \tag{1.6}$$

and vice versa, where $t \in [t_0, T]$, $T < \infty$ and $\mathrm{D} = \{ x \in \mathbb{R} : |x| \leq M \}$ and M is a positive finite real constant.

Theorem 1.4.8 Existence and uniqueness theorem [102]

Consider the fractional order differential equation

$D^{\alpha}x(t) = \ f(t, x), x(t = t_0) = x_0.$

Define $\|x\| = \sup_{\mathrm{t}} |e^{-St}\, x|$, $S > 0$. $t \in [t_0, T]$, $T < \infty$. Clearly $\|x\| = \sup_{\mathrm{t}} |x|$. If $f(t, x)$ satisfies the Lipschitz condition and $\mathcal{F}(t)$ be the solution of the given system. If the solution $\mathcal{F}(t)$ satisfies the inequality $\|\mathcal{F}(x) - \mathcal{F}(y)\| \leq \frac{L}{s^{\alpha}} \|x - y\|$ with $\frac{L}{s^{\alpha}} < 1$, then the given fractional order differential equation has a unique solution.

Lemma 1.4.9 [54] Let us assume that $\alpha \in (0,1]$ and consider that the function $f(t)$ and $D^{\alpha}f(t) \in \mathrm{C}\,[\mathrm{a}, \mathrm{b}]$ for all $\mathrm{t} \in [\mathrm{a}, \mathrm{b}]$, where $\mathrm{C}\,[\mathrm{a}, \mathrm{b}]$ is the class of continuous functions on $[\mathrm{a}, \mathrm{b}]$ and D^{α} stands for the fractional order Caputo derivative. The function $f(t)$ is said to be non-decreasing on $[\mathrm{a}, \mathrm{b}]$ if $D^{\alpha}f(t) \geq 0$, and the function $f(t)$ is said to be non-increasing on $[\mathrm{a}, \mathrm{b}]$ and if $D^{\alpha}f(t) \leq 0$.

Theorem 1.4.10 [54] Let $\alpha > 0$, $n - 1 < \alpha < n$, where $n \in \mathbb{N}$. Assume $U(t)$ is n times continuously differentiable function and $D^{\alpha}U(t)$ is piecewise continuous on $[t_0, \infty)$, then

$$\mathcal{L}\{D^{\alpha}U(t)\} = s^{\alpha}\, \mathcal{F}(s) - \sum_{j=0}^{n-1} s^{\alpha-j-1} U^{j}(t_0), \tag{1.7}$$

where $\mathcal{F}(s) = \mathcal{L}\{U(t)\}$ is the Laplace transform of U(t), D^{α} is the Caputo fractional order derivative.

Theorem 1.4.11 [54] Let C be the complex plane. For each $m > 0,\ p > 0, k \in C^{n\times n}$. The Laplace transform of $t^{p-1}\, E_{m,p}(\, k\, t^{m}\,)$ is defined as,

$$\mathcal{L}\{\, t^{p-1}\, E_{m,p}(\, k\, t^{m}\,)\} = \frac{s^{m-p}}{s^{m}\ - k}, \tag{1.8}$$

$\Re\,(s) > ||k||^{1\backslash m},$ where $\Re\,(s)$ is the real part of the complex number s, and $E_{m,p}$ is the Mittag–Leffler function defined as $E_{m,p}(z) = \sum_{n=0}^{\infty}\frac{z^{n}}{\Gamma(\mathrm{mn+p})}$, and $\boldsymbol{\Gamma}$ is the Gamma function.

Theorem 1.4.12 Global Stability Theorem (Lyapunov Direct Method) [54]

Let $D^{\alpha}\, x(t) = f(t,x),$ where $x(t_0) > 0$ be the non-autonomous system and $x_* \in \Lambda \subset \mathbb{R}^{n}$ be the equilibrium point of the system. If $F\ (\,t,x)$: $[\,0,\infty\,) \times\ \Lambda \to \mathbb{R}$ represents a continuous and differentiable function satisfying the conditions

$D^{\alpha}f(t,x)\ \leq -\,\Omega_1(x),$

along with

$\Omega_2(x) \leq\ f(t,x)\ \leq\ \Omega_3(x),$

for $\alpha \in (0,1)$ and for all $x(t) \in\ \Lambda$, then the equilibrium point x_* becomes globally asymptotically stable, where Ω_i , $(\ i = 1,2,3\ )$ are positive definite continuous functions on Λ.

Lemma 1.4.13 [54] If $x(t)$: $\mathbb{R} \longrightarrow \mathbb{R}^{+}$ be a continuous and differentiable function. Then any time for $t > t_0$, $x_* \in\ \mathbb{R}$,

$$D^{\alpha}\left(\, x(t) - x_* - x_* \ln\frac{x(t)}{x_*}\right) \leq \left(\ \frac{x - x_*}{x}\right) D^{\alpha}x(t). \tag{1.9}$$

Lemma 1.4.14 [104] If J(X) denotes the Jacobian matrix of a fractional order system at the equilibrium point X and λ_i $(i = 1,2,3 \ldots n)$ be the respective eigenvalues of the

Jacobian matrix J(X). Then the equilibrium point X is locally asymptotically stable if $|\arg\lambda_i| > \frac{\alpha\pi}{2}$ for all $\lambda_i\ (i = 1,2,3 \dots n)$.

1.4.15 Routh–Hurwitz conditions [106]

If $P(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n$, is the characteristic polynomial of the Jacobian matrix with all coefficients real. Then the Hurwitz matrix of order n can be written as

$$H_n = \begin{bmatrix} a_1 & 1 & 0 & 0 & 0 & 0 & 0 \dots . & 0 \\ a_3 & a_2 & a_1 & 1 & 0 & 0 & 0 \dots . & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 & 0 \dots . & 0 \\ . & . & . & . & . & . & \dots . & . \\ . & . & . & . & . & . & \dots . & . \\ . & . & . & . & . & . & \dots . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \dots . & a_n \end{bmatrix} \tag{1.10}$$

.

Let us define $D_1 = |a_1|,\ D_2 = \begin{vmatrix} a_1 & 1 \\ a_3 & a_2 \end{vmatrix}, \dots, D_n = \det(H_n)$

For $\alpha = 1$, the Routh-Hurwitz conditions for a polynomial equation to satisfy

$|\arg\lambda| > \alpha\frac{\pi}{2}$ are $D_1 > 0, D_2 > 0, \dots D_n > 0$.

For fractional order $\alpha \in [0,1)$ Routh-Hurwitz conditions for a polynomial equation to satisfy $|\arg\lambda| > \alpha\frac{\pi}{2}$ are given below.

(i) When n = 1, the condition is $a_1 > 0$.

(ii) When n = 2, the conditions are either Routh–Hurwitz conditions, or $a_1 < 0, 4a_2 > (a_1^2)$, $\left|\sqrt{\tan^{-1}(4a_2 - a_1^2)}\ /a_1\right| > \alpha\frac{\pi}{2}$.

(iii) When n = 3, first, we define the discriminant of $P(\lambda)$ is defined as

$$D(p) = 18a_1a_2a_3 + (a_1a_2)^2 - 4a_3(a_1)^3 - 4(a_2)^3 - 27(a_3)^2. \tag{1.11}$$

- If $D(p) > 0$, then $a_1 > 0, a_3 > 0, a_1a_2 > a_3$.
- If $D(p) < 0, a_1 \geq 0, a_2 \geq 0, a_3 > 0, \alpha < {}^2/_3$.
- If $D(p) < 0, a_1 > 0, a_2 > 0, a_1a_2 = a_3$ for all $\alpha \in [0,1)$.

1.4.16 Predictor-Corrector method to solve the Caputo fractional order differential equations [107]

For numerical investigations, we have used the Adam-Bashforth-Moulton predictor-corrector technique, and Roberto Garrappa developed the PECE scheme for fractional differential equations in MATLAB [108]. This is a very effective method to solve fractional order differential equations, and this applies to both linear and non-linear fractional order differential equations. This technique can be extended to solve the fractional differential equation with more than one differential operator. This method has an error bound $O(h^q)$, where

$$q = min = \{1+\alpha, 2\} = \left\{\begin{matrix} 2, & if\ \alpha \geq 1, \\ (1+\alpha), & if\ \alpha < 1. \end{matrix}\right\}. \tag{1.12}$$

The stability analysis has been done in [109]. Here we are giving the formula for the same. Consider an initial value fractional order differential equation

$$D^{\alpha}x(t) = f\big(t, x(t)\big),$$

with the initial condition $x(0) = x_0 > 0$, and $t \in (0, T]$. Consider the set of points $\{x_i, t_i\}$,

where $x_i(t_i) = x_i,\ t_i = ih,\ i = 1,2,3 \ldots..\ N\ (\,integer),\ T = Nh.$

The general formula for the Predictor-Corrector method is

$$x_{n+1} = \sum_{j=0}^{\lceil\alpha\rceil-1} \frac{t_{n+1}^{j}}{j!} x_0^{j} + \frac{h^{\alpha}}{\Gamma(\alpha+1)} \sum_{k=0}^{n} \gamma_{k,n+1}\, f(t_k, x_k) + \frac{h^{\alpha}}{\Gamma(\alpha+2)} \gamma_{n+1,n+1} f(t_{n+1}, x_{n+1}^{P}). \tag{1.13}$$

Where,

$$\gamma_{k,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha}, & if\ k = 0, \\ (n-k+2)^{\alpha+1} + (n-k)^{\alpha+1} - 2(n-k+1)^{\alpha+1}, & if\ 1 \leq k \leq n, \\ 1, & if\ k = n+1, \end{cases} \tag{1.14}$$

$$x_{n+1}^{P} = \sum_{j=0}^{\lceil\alpha\rceil-1} \frac{t_{n+1}^{j}}{j!} x_0^{j} + \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} \delta_{k,n+1}\, f(t_k, x_k), \tag{1.15}$$

$$\delta_{k,n+1} = \frac{h^{\alpha}}{\alpha}\{(n+1-k)^{\alpha} - (n-k)^{\alpha}\}. \tag{1.16}$$

1.5 Main Terms Used in the Thesis

- **Population Dynamics:** Population dynamics reveals critical insights into the changing patterns of living populations, their growth, and the environmental influences that affect how they evolve.
- **Single Species:** A species is a collection of organisms that can interbreed or mate to create viable, productive offspring.
- **Prey:** A prey animal is one that another animal hunts, captures, or kills for food.
- **Predator:** A predator is an organism that feeds primarily by killing and consuming other organisms called prey.
- **Food Chain:** A food chain is a cycle by which nutrients and energy in the form of food are transferred from one organism to another.
- **Hopf Bifurcation**: A Hopf bifurcation takes place when altering a parameter switches an equilibrium that was originally a stable spiral to an unstable spiral.
- **Chaotic Behavior:** Chaos is a scenario in which typical differential equation solutions do not converge to a steady or periodic function (of time), but instead continue to display unexpected behavior.

1.6 Summary of the Thesis

The work done in this thesis examines the mathematical and numerical analysis of fractional-order single and multispecies mathematical models in population dynamics, considering the additive Allee effect. The fractional derivative in Caputo's sense is considered for this research work. In this research work, the single-species and multispecies models existing in the literature of population dynamics are reconsidered by incorporating the Allee effect and replacing ordinary derivatives with Caputo fractional order derivatives. Newly formed models are mathematically analyzed to determine the uniqueness of positive and bounded solutions. Asymptotic stability analysis of all steady-state solutions is done. Prerequisites for global stability of all equilibrium points are established using the direct Lyapunov method. To validate the findings, numerical analysis of the models is done by using Roberto Garrappa's PECE scheme for fractional differential equations in MATLAB.

Key findings of the research include how exposure of the populations to the Allee effect affects their dynamics. The study's findings may help determine the threshold level of populations suffering from the Allee effect, below which the affected species cannot survive. It can additionally help in the study of the different parameters used in the study and how changing them will allow the Allee effect to be controlled. Because fractional calculus operates at the microscopic level and remembers the entire development process, it helps us identify steady states that are unstable in integer-order modeling but can be stabilized using fractional-order modeling. This will help us in developing timely strategies to protect endangered species. Furthermore, we can use the findings to eliminate unwanted species by introducing the Allee effect using an artificial strategy.

The first chapter provides an overview of population ecology and the theory of mathematical models for studying patterns and changes in populations of the same and different species. Following that, a general introduction to fractional calculus and a summary of the development process of fractional calculus are given. The literature review section examines notable studies on population dynamics that employ various types of fractional derivatives. After a thorough review of the literature, the research gap is identified, resulting in the study's objectives. Several mathematical definitions, theorems, and lemmas used in the research are discussed. Finally, a summary of the research results is provided.

Chapter 2 examines the mathematically and numerically, a single-species model with logistic population growth that incorporates the additive Allee effect, considering Caputo's fractional derivative. The model is examined for the uniqueness of positive and bounded solutions. The preconditions for the local asymptotic and global stability of all steady-state solutions are established. The threshold level of populations with a strong Allee effect is calculated below which a suffering population will be unable to survive. Different values of the Allee's constant are found, determining the severity of the Allee effect. All mathematically established results are validated through MATLAB by running numerical simulations.

Chapter 3 investigates the Lotka-Volterra prey-predator system incorporating logistic growth to the prey population and exposing it to the additive Allee effect using Caputo's fractional order derivative. Apart from all basic population dynamics theorems, and stability analysis of all steady state solutions, the Hopf bifurcation is also studied at the coexistence point using fractional order and capture rate of prey by predator as bifurcation parameters. The impact of the prey's capture rate by the predator is investigated numerically. It is found that the Allee effect can be controlled through optimal harvesting. This study identifies equilibrium points that are stable in the fractional order model but remain unstable in the traditional integer order model, implying that fractional order models provide wider stability regions.

Chapter 4 investigates the Lotka-Volterra competition system incorporating an additive Allee effect in one of the species, through both fractional and integer order modeling. For the study, Caputo's definition-based fractional derivative is used. Different intervals for Allee's constant are discovered in various competition scenarios to investigate the stable coexistence of competing species. Furthermore, Allee's constants are calculated to determine the severity of the Allee effect, when it is weak, and when it is strong in the presence of competition. Moreover, in some cases, it is found that with the same value of Allee's constant and the same initial conditions, different solutions exhibit stability depending upon the value of the fractional order. In other words, it is possible to find the value of fractional order below which one solution is stable, and after crossing that other solution shows stability, but initial conditions remain the same. Using lower fractional order models, it is discovered that both species can coexist despite stronger interspecific competition. The uniqueness of positive and bounded solutions is investigated. The theoretical results obtained have also been numerically validated.

Chapter 5 examines a Caputo fractional order three-species food chain comprising a prey, an intermediate predator, and a top predator. The prey population grows logistically while being exposed to the additive Allee effect. Holing type II functional responses are incorporated to depict interactions between prey, intermediate predators, and between intermediate predators and top predators. Uniqueness of positive and boundedness of solutions is established. All equilibrium points are examined for their local and global stability. Numerical analysis is done with and without the Allee effect in the integer order model as well as in the fractional order model. It is found that when the prey are not exposed to the Allee effect, all populations show chaotic behavior in the integer order model, whereas in the fractional models, populations can coexist stably. The introduction of the weak Allee effect in the prey population allows the stable coexistence of all populations in integer as well as fractional order models. However, when the prey population is subjected to the strong Allee effect, populations can coexist stably in the fractional model, but in the integer order model, all populations die out.

Chapter 6 highlights the key findings of the research in the form of a conclusion and outlines the future scope of the research.

Chapter 2

Dynamical Study of a Single Species through the Fractional Order Logistic Equation with Additive Allee Effect

2.1 Introduction

Many researchers revealed that the single-species modeling approach can help in ecosystem modeling, to provide practical ecosystem-based management in fisheries, pest management, epidemic avoidance, and management, optimal biological resource planning, cellular growth regulation, and many more [110]. There are many case studies in wildlife management where the single-species conservation approach helped to save biodiversity [111-113]. In population dynamics, there are three popular single-species models: the exponential growth model, the logistic growth model, and the delay model. Many researchers have used quantitative analysis to obtain some advantageous characteristics of these models, and their findings help us to anticipate and control the actual production [114,115]. Various studies on the single-species logistic growth model for optimal harvesting [116,117], the extinction and permanence of single species by considering the logistic growth of the population in a polluted environment [118], the dynamical behavior of stochastic single species with Allee effect [119], Hopf bifurcation with delay [120], the stochastic growth of single species with limited resources [121] are witnessed in the literature.

During the past few years, the scientific community has shifted its focus to explaining real-world problems using fractional calculus. Fractional calculus is inextricably linked with the memory systems seen in numerous real-world systems [13,14]. As discussed in Chapter 1, various published studies have proved that fractional order models provide a more appropriate explanation to real-world phenomena than those of ordinary integer-order systems. Sayed et al. established the prerequisites for the uniqueness of the solution and laid conditions for the stability of the steady state solutions of a fractional order single species logistic equation [38]. The authors applied Caputo's definition of fractional derivatives. Gao and Zhao used Caputo's definition-based fractional derivative to investigate the single-species model with species dispersion in n patches as a coupled system on a network [56]. Li et al. used Caputo fractional differential equations to study the single-species model incorporating population dispersion in several patches [57]. Abbas et al. used Riemann-Liouville's definition-based fractional order differential equations to analyze the logistic equation incorporating the multiplicative Allee effect [102]. In this chapter, we are studying the fractional order single species logistic equation with the additive Allee effect by using Caputo's definition of the fractional derivative. The analysis of the model is done by exposing the population to the strong Allee effect as well as the weak Allee effect, and to find a threshold level for a population suffering from a strong Allee effect, mathematically and numerically.

2.2 Mathematical Model

The logistical growth equation of a single species when it is exposed to the additive Allee effect can be written as eq. (1.3). Replacing the ordinary derivative with the Caputo fractional order derivative, the model equation (1.3) can be represented as

$$D^{\alpha}x(t) = \left[r\left(1 - \frac{x}{K}\right) - \frac{m}{x + a}\right]x, \tag{2.1}$$

subject to the starting condition $x(t = t_0) = x_0 > 0$. Here $x, r, K, a,$ and m are all positive, depicting the number of individuals in the population, the species' intrinsic growth rate, the maximum population of the species that can be supported by the environment (carrying capacity), the population density of the species having fitness half of its extreme value, and the Allee's constant respectively. D^{α} denotes Caputo fractional order derivative, and $\alpha \in (0,1]$ represents the fractional order.

2.3 Primary Evaluation of the Model

In this section, an analysis of the proposed model will be conducted to establish the properties of its solutions, including existence, uniqueness, positivity, and uniform boundedness.

2.3.1 Existence of Unique Solution

Theorem 2.1: Consider the interval I = $[\ t_0,\ T\ ]$, $T < \infty$ and the region $D = \{\, x \in \mathbb{R} : |x| \leq M \,\}$, here M is a positive constant. Let C(I) denote a class of real-valued functions defined on I that have continuous first-order derivatives on C(I). Then the initial value problem $D^{\alpha}x(t) = f(t\,,x)$, where $f(t\,,x) = x\left[r\left(1-\frac{x}{K}\right)-\frac{m}{x+a}\right]$, with initial condition $x(\,t = t_0) = x_0$ and $f : I \times D \rightarrow \mathbb{R}$; $\alpha \in (0,1]$ satisfies Lipschitz condition w.r.t the second variable x and has a unique solution.

Proof. Consider a mapping $G : D \rightarrow \mathbb{R}$, where $G(x) = x\left[r\left(1-\frac{x}{K}\right)-\frac{m}{x+a}\right]$.

Define $\|x\| = \sup_{t} |\, x e^{-St}|$, $S > 0$, $t \in [\,t_0,\ T\,]$, $T < \infty$.

Clearly $\|x\| = \sup_{t} |x|$. Let $x\,,\ y \in D$. Consider

$$\|G(x) - G(y)\| = |G(x) - G(y)|, \tag{2.2}$$

$$= \left| r\,x - \frac{r\,x^2}{K} - \frac{mx}{x+a} - ry + \frac{ry^2}{K} + \frac{my}{y+a} \right|,$$

$$= \left| r\,(x-y) - \frac{r}{K}\,(\,x^2 - y^2) - m\left(\frac{x}{x+a} - \frac{y}{y+a}\right) \right|,$$

$$\leq |r\,(x-y)| + \left|\frac{r}{K}\,(\,x^2 - y^2)\right| + \left| m\left(\frac{x}{x+a} - \frac{y}{y+a}\right) \right|,$$

$$= r\,|(x-y)| + \frac{r}{K}\,|\,(x-y)|\,|(x+y)| + m\left| \frac{x\,(\,y+a) - y\,(\,x+a\,)}{(\,x+a\,)(y+a)} \right|,$$

$$\leq r\,|(x-y)| + 2\,\frac{Mr}{K}\,|(x-y)| + \;ma|(\,x-y)|,$$

$$= |(x-y)|\left(r+\;2\,\frac{Mr}{K}+ma\right) = \;L\,|(x-y)|. \tag{2.3}$$

Where,

$$L = r+\;2\,\frac{Mr}{K}+ma.$$

Therefore, G satisfies the Lipschitz condition.

Let H (x) be the solution of the system (2.1). By Eq. (1.6),

$$\mathrm{H}\,(x) = x-\;x_0 = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,G\big(x(s)\big)\,ds. \tag{2.4}$$

Consider

$$\mathrm{H}(x)-\;\mathrm{H}(y) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\;\;\{G\big(x(s)\big)-G\big(y(s)\big)\}\,ds. \tag{2.5}$$

$$|\mathrm{H}(x)\;-\;\mathrm{H}(y)\,| = \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\;\;\{G\big(x(s)\big)-G\big(y(s)\big)\}\,ds\right|,$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,\big|G\big(x(s)\big)-G\big(y(s)\big)\big|\,ds. \tag{2.6}$$

Now consider

$$e^{-pt}\,\big(\mathrm{H}(x)-\;\mathrm{H}(y)\big) \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\;\{G\big(x(s)\big)-G\big(y(s)\big)\}\,e^{-pt}ds,$$

$$= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\;\;e^{-p(t-s)}\{G\big(X(s)\big)-G\big(Y(s)\big)\}\,e^{-ps}\,ds.$$

$$\|\mathrm{H}(X)-\;\mathrm{H}(Y)\| \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\;\;e^{-p(t-s)}\big\|G\big(X(s)\big)-G\big(Y(s)\big)\big\|\,ds,$$

By using Eq. (2.3)

$$\|\mathrm{H}(X)-\mathrm{H}(Y)\| \leq \frac{\left(r+2\frac{Mr}{K}+ma\right)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ e^{-p(t-s)}\|x-y\|\,ds,$$

$$=\frac{\left(r+2\frac{Mr}{K}+ma\right)}{\Gamma(\alpha)}\|x-y\|\int_0^t (t-s)^{\alpha-1}\ e^{-p(t-s)}\,ds,$$

$$\leq \frac{\left(r+2\frac{Mr}{K}+ma\right)}{\Gamma(\alpha)}\|x-y\|\frac{\Gamma(\alpha)}{p^{\alpha}}=\frac{\left(r+2\frac{Mr}{K}+ma\right)}{p^{\alpha}}\|x-y\|.$$

Choosing p sufficiently large so that, $\frac{\left(r+2\frac{Mr}{K}+ma\right)}{p^{\alpha}}<1$, then by Theorem (1.4.8), the model given by (2.1) has a unique solution.

2.3.2 Positivity of the Solutions

Theorem 2.3 Solutions to the given system (2.1) starting in $\mathbb{R}^{+}$are non-negative, where $\mathbb{R}^{+}$ is the set of positive ordered reals including zero.

Proof. Let $x(t_0)=x_0\in\mathbb{R}^{+}$ be the starting solution of the given fractional order system (2.1). Let $t>t_0$ and we are to show that $x(t)\geq 0$ for all $t\geq t_0$. Let us suppose that it does not hold. It means that there are some t_1 such that $t_1>t_0$ but

$$\left\{\begin{array}{c} x(t)>0 \quad \text{when } t_0\leq t<t_1, \\ x(t_1)=0, \\ x(t)<0 \quad \text{when } t_1\leq t<t^{*}. \end{array}\right\}. \tag{2.7}$$

Where $t^{*}$is sufficiently close to t_1. Now $x(t_1)=0$ gives $D^{\alpha}x(t_1)=0$.

Case 1. If $D^{\alpha}x(t)\geq 0$, for all $t\in(t_1,t^{*}]$.

Now, from Eq. (2.1), we can observe that

$$D^{\alpha}x(t)>r\,x(t). \tag{2.8}$$

Applying the Laplace transform and using Eq. (1.7), we have

$$s^{\alpha}X(s)-s^{\alpha-1}x(t_0)\geq r\,X(s),$$

where, $X(s)=\mathcal{L}\{x(t)\}$. This further gives,

$(s^{\alpha} - r)\,X(s) \geq s^{\alpha-1}\,x(t_0),$

$$X(s) \geq s^{\alpha-1}\frac{x(t_0)}{(s^{\alpha} - r)}\,. \qquad (2.9)$$

By applying the inverse Laplace transform to (2.9) and using Eq. (1.8), we have,

$$x(t) \geq \mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{(s^{\alpha} - r)}x(t_0)\right\} = x(t_0)E_{\alpha,1}\{r\,(\mathrm{t} - t_0)^{\alpha}\}. \qquad (2.10)$$

Therefore, we have,

$$x(t) \geq x(t_0)E_{\alpha,1}\{r\,(\mathrm{t} - t_0)^{\alpha}\} \geq 0. \qquad (2.11)$$

Therefore, we have $x(t) \geq 0$. It contradicts the assumption $x(t) < 0$ for all $t \in (t_1, t^*]$.

Case 2. If $D^{\alpha}x(t) < 0$ for all $t \in (t_1, t^*]$.

This implies $x(t)$ is a non-increasing function for all $t \in (t_1, t^*]$.

Consider

$$D^{\alpha}x(t) = \left[r\left(1 - \frac{x}{k}\right) - \frac{m}{x+a}\right]x = \left(r - \frac{m}{x+a}\right)x - \frac{x^2 r}{K}.$$

From this, we can have

$$x\left[r\left(1 - \frac{x}{K}\right) - \frac{m}{x+a}\right] > x\,\rho. \qquad (2.12)$$

Here ρ is the minimum value of $r\left(1 - \frac{x}{K}\right) - \frac{m}{x+a}$, $\forall\, t \in (t_1, t^*]$.

This implies that for $t_1 < t < t^*$, $D^{\alpha}x \geq \rho\,x$.

By taking the Laplace transform and using Eq. (1.7), we have

$s^{\alpha}\,X(s) - s^{\alpha-1}\,x(t_0) \geq \rho\,X(s),$

where, $X(s) = \mathcal{L}\{x(t)\}$. This further gives,

$(s^{\alpha} - \rho)\,X(s) \geq s^{\alpha-1}\,x(t_0),$

$$X(s) \geq s^{\alpha-1}\frac{x(t_0)}{(s^{\alpha}-\rho\,)}, \tag{2.13}$$

where, $X(s) = \mathcal{L}\{x(t)\}$.

By taking the inverse Laplace transform and using Eq. (1.8), we have,

$$x(t) \geq x(t_0)E_{\alpha,1}\{\,\rho\,(\mathrm{t}-t_0)^{\alpha}\} \geq 0. \tag{2.14}$$

Therefore, again we have $x(t) \geq 0$, for all $t_1 < t < t^*$. This again contradicts the assumption $x(t) < 0$ for all $t \in (\,t_1, t^*]$. Hence, all solutions begin in $\mathbb{R}^+$ are non-negative.

2.3.3 Uniform Boundedness of the Solutions

Theorem 2.4 All non–negative solutions of the system (2.1) are uniformly bounded.

Proof. Consider $D^{\alpha}F(t) + \frac{F(t)}{m}$,

where, $F(t) = x(t)$.

$$D^{\alpha}F(t) + \frac{F(t)}{m} = rx - \frac{rx^2}{K} - \frac{mx}{x+a} + \frac{x}{m}, \tag{2.15}$$

$$\leq x\left(r+\frac{1}{m}\right) - \frac{r}{K}x^2 = -\frac{r}{K}x^2 + \left(r+\frac{1}{m}\right)x,$$

$$= -\frac{r}{K}\left(x^2 - \frac{K}{r}\left(r+\frac{1}{m}\right)x\right) = -\frac{r}{K}\left[x^2 - \left(K+\frac{K}{rm}\right)x\right],$$

$$= -\frac{r}{K}\left[\left(x-\frac{L}{2}\right)^2 - \frac{L^2}{4}\right] = \frac{rL^2}{4K} - \frac{r}{K}\left(x-\frac{L}{2}\right)^2,$$

$$\leq \frac{rL^2}{4K} = R.$$

Where, $L = \left(K + \frac{K}{rm}\right)$. Therefore, we have

$$D^{\alpha}F(t) + \frac{F(t)}{m} \leq R. \tag{2.16}$$

By taking the Laplace transform and using Eq. (1.7), we have

$$s^{\alpha}\,G(s) - s^{\alpha-1}\,F(t_0) + \frac{1}{m}\,G(s) \;\leq\; \frac{R}{s},$$

where $G(s) = \mathcal{L}\{F(t)\}$,

$$\left(s^{\alpha} + \frac{1}{m} \right) G(s) \leq \frac{R}{s} + s^{\alpha-1} F(t_0).$$

This implies

$$G(s) \leq \frac{R}{s\left(s^{\alpha} + \frac{1}{m} \right)} + \frac{s^{\alpha-1} F(t_0)}{\left(s^{\alpha} + \frac{1}{m} \right)}$$

$$= \frac{s^{\alpha-1} F(t_0)}{\left(s^{\alpha} + \frac{1}{m} \right)} + \frac{s^{\alpha-(\alpha+1)}}{\left(s^{\alpha} + \frac{1}{m} \right)} R. \tag{2.17}$$

By taking the inverse Laplace transform and using Eq. (1.8) we have,

$$F(t) \leq F(t_0)\, \mathcal{L}^{-1}\left\{ \frac{s^{\alpha-1}}{\left(s^{\alpha} + \frac{1}{m} \right)} \right\} + R\, \mathcal{L}^{-1}\left\{ \frac{s^{\alpha-(\alpha+1)}}{\left(s^{\alpha} + \frac{1}{m} \right)} \right\},$$

$$= F(t_0)\, E_{\alpha,1}\left\{ -\frac{(t-t_0)^{\alpha}}{m} \right\} + R\, t^{\alpha}\, E_{\alpha,\alpha+1}\left\{ -\frac{(t-t_0)^{\alpha}}{m} \right\}. \tag{2.18}$$

Now using the recurrence formula for Mittag-Leffler's function,

$$E_{\alpha,\beta}\{z\} = z\, E_{\alpha,\alpha+\beta}\{z\} + \frac{1}{\Gamma(\beta)}, \tag{2.19}$$

we have

$$E_{\alpha,1}\left\{ -\frac{(t-t_0)^{\alpha}}{m} \right\} = -\frac{(t-t_0)^{\alpha}}{m} E_{\alpha,\alpha+1}\left\{ -\frac{(t-t_0)^{\alpha}}{m} \right\} + \frac{1}{\Gamma(1)}. \tag{2.20}$$

Which implies

$$(t-t_0)^{\alpha} E_{\alpha,\alpha+1}\left\{ -\frac{(t-t_0)^{\alpha}}{m} \right\} = -m\left[E_{\alpha,1}\left\{ -\frac{(t-t_0)^{\alpha}}{m} \right\} - 1 \right]. \tag{2.21}$$

Therefore, Eq. (2.18) implies,

$$F(t) \leq F(t_0)E_{\alpha,1}\left\{-\frac{(t-\ t_0)^{\alpha}}{m}\right\} - Rm\left[\ E_{\alpha,1}\left\{-\frac{(t-\ t_0)^{\alpha}}{m}\right\} - 1\right],$$

$$= (\ F(t_0) - Rm\ )E_{\alpha,1}\left\{-\frac{(t-\ t_0)^{\alpha}}{m}\right\} + Rm. \qquad (2.22)$$

Now as t⟶ ∞ , $E_{\alpha,1}\left\{-\frac{(t-\ t_0)^{\alpha}}{m}\right\} \longrightarrow 0$.

Therefore, we see that,

$$F(t) \leq\ Rm = \frac{rL^2m}{4K}\ . \qquad (2.23)$$

Therefore, all non–negative solutions of the given system (2.1) lie in the region

$$\left\{\bar{x}\ \in\ \mathbb{R}^+ : \bar{x}\ \leq \frac{rL^2m}{4K} + v,\ v > 0\ \right\}.$$

2.3.4 Equilibrium Points and Conditions for the Existence

First, we define some mathematical representations for Allee’s constant that will be used for the whole analysis.

$$\begin{cases} m_w = ar, \\ m_{cr} = \frac{Kr}{4}\left(1 + \ \frac{a}{K}\right)^2. \end{cases} \qquad (2.24)$$

The steady-state solutions (equilibria) of system (2.1) are given by

1. $x_e = 0$, and this point exists without any condition.
2. $x_s = \frac{K}{2}\left[\left(1 - \frac{a}{K}\right) + \sqrt{\left(1 + \frac{a}{K}\right)^2 - \frac{4m}{Kr}}\right]$, provided $m\ < m_{cr}$ (See Eq. (2.24)).

3. $x_t = \frac{K}{2}\left[\left(1 - \frac{a}{K}\right) - \sqrt{\left(1 + \frac{a}{K}\right)^2 - \frac{4m}{Kr}}\right]$, provided $m_w < m < m_{cr}$ (See Eq. (2.24)).

Remark: Proofs of the existence conditions are given along with the stability proofs in Theorems 2.5, 2.6, and 2.7.

2.3.5 Examination of the Equilibrium Points for Local Asymptotic Stability

Firstly, we will explain the criteria to be used to inspect the local stability of all equilibria.

Let $x = x_*$ be the equilibrium point of the fractional order system $D^\alpha x(t) = f(x(t))$.

Then $f(x_*) = 0$.

To evaluate the asymptotical stability, let us perturb the equilibrium point by adding $\epsilon(t)$.

Let $x(t) = x_* + \epsilon$, then we have

$$D^\alpha x(t) = D^\alpha (x_* + \epsilon) = f(x_* + \epsilon). \qquad (2.25)$$

Therefore,

$$D^\alpha (x_* + \epsilon) = f(x_* + \epsilon). \qquad (2.26)$$

By using Taylor's formula,

$$f(x_* + \epsilon) \approx f(x_*) + f'(x_*)\,\epsilon + \cdots \qquad (2.27)$$

As $f(x_*) = 0$. Therefore, we have

$$f(x_* + \epsilon) \approx f'(x_*)\epsilon.$$

Also,

$$D^\alpha(x_* + \epsilon) = D^\alpha(x_*) + D^\alpha(\epsilon) = D^\alpha(\epsilon). \qquad (2.28)$$

Therefore,

$$D^\alpha(\epsilon) = f'(x_*)\,\epsilon. \qquad (2.29)$$

The solution of the above equation is given by,

$$\varepsilon(t) = \epsilon(t_0)\, E_{\alpha,1}(f'(x_*)t^{\alpha}). \tag{2.30}$$

Therefore, from Eqs. (2.25) to (2.29), we can have

$$D^{\alpha}x(t) = D^{\alpha}(\epsilon(t)\,) = \; f'(x_*)\epsilon(t_0)\, E_{\alpha,1}(f'(x_*)t^{\alpha}). \tag{2.31}$$

If $\epsilon(t)$ increases or $D^{\alpha}(\epsilon(t)\,) > 0$, then the equilibrium point becomes unstable, and if $\epsilon(t)$) decreases or $D^{\alpha}(\epsilon(t)\,) < 0$, the equilibrium point becomes stable. Therefore, stability or instability depends upon the sign of $f'(x_*)$. Therefore, the equilibrium point x_* will be stable or unstable according to the condition $f'(x_*) < 0$, or $f'(x_*) > 0$ [38].

Theorem 2.5 If $m > m_{cr}$ (See Eq. (2.24)), then the system (2.1) has only one asymptotically stable (locally) equilibrium point x_e.

Proof. To find equilibria of the fractional order system (2.1), we set to zero eq. (2.1).

$$\left[r\left(1-\frac{x}{k}\right)-\frac{m}{x+a}\right] x = 0.$$

From above, either $x = 0$ or $r\left(1-\frac{x}{k}\right)-\frac{m}{x+a} = 0.$

This, on further simplification, gives

$$\frac{1}{K}\, x^2 - \left(1-\frac{a}{K}\right) x + \frac{m}{r} - a = 0. \tag{2.32}$$

This is a quadratic equation, and its discriminant is given by

$$\mathbb{D} = \left(1-\frac{a}{K}\right)^2 - \frac{4}{K}\left(\frac{m}{r}-a\right) = \left(1+\frac{a}{K}\right)^2 - \frac{4m}{Kr}. \tag{2.33}$$

To have only one solution $x_e = 0$, the condition to be satisfied is given by $\left(1+\frac{a}{K}\right)^2 - \frac{4m}{Kr} < 0.$

This further gives

$$m > \frac{Kr}{4}\left(1+\frac{a}{K}\right)^2 = m_{cr} \quad \text{(See Eq. (2.24))}.$$

For stability analysis, consider the given system (2.1) as

$$D^{\alpha}x = f(x),$$

where

$$f(x) = \left[r\left(1 - \frac{x}{k}\right) - \frac{m}{x + a}\right] x.$$

Its differential coefficient is given by

$$f'(x) = r - \frac{2rx}{K} - \frac{ma}{(x + a)^2}. \tag{2.34}$$

Now $f'(x_e) = r - \frac{m}{a} < 0$ gives $m > ar = m_w$. Hence, combining the existence and stability conditions, we conclude that if $m > \max\{m_{cr}, m_w\} = m_{cr}$ (See Eq. (2.24)), the only equilibrium points of (2.1) are x_e, which is asymptotically stable.

Theorem 2.6 If $0 < m < m_w$ (See Eq. (2.24)), in other words, when a species is exposed to the weak Allee effect, then system (2.1) has two equilibrium points x_e, and x_s. Where x_e is always unstable and x_s is asymptotically stable.

Proof. To have non-zero solutions along with $x_e = 0$, the condition to be satisfied by the system (2.1) can be obtained from Eq. (2.32) as,

$$\left(1 + \frac{a}{K}\right)^2 - \frac{4m}{Kr} > 0.$$

This on solving gives,

$$m < \frac{Kr}{4}\left(1 + \frac{a}{K}\right)^2 = m_{cr} \text{ (See Eq. (2.24))}$$

On solving Eq. (2.32), we have

$$x_s = \frac{K}{2}\left[\left(1 - \frac{a}{K}\right) + \sqrt{\left(1 + \frac{a}{K}\right)^2 - \frac{4m}{Kr}}\right], \tag{2.35}$$

$$x_t = \frac{K}{2}\left[\left(1 - \frac{a}{K}\right) - \sqrt{\left(1 + \frac{a}{K}\right)^2 - \frac{4m}{Kr}}\right]. \tag{2.36}$$

As $x_s > 0$, and for $x_t > 0$, the condition to be satisfied is given by

$$\left(1-\frac{a}{K}\right) > \sqrt{\left(1+\frac{a}{K}\right)^2-\frac{4m}{Kr}}.$$

This, on further simplification, gives $m > m_w$. Therefore, $m < m_w$ implies $x_t < 0$. As we are interested only in non-negative solutions, system (2.1) has two feasible equilibrium points x_e and x_s when $m < m_w$. Now for the stability of the point x_s, from Theorem 2.5, the differential coefficient of $f(x)$ is calculated as,

$$f'(x) = r - \frac{2rx}{K} - \frac{ma}{(x+a)^2}. \tag{2.37}$$

Substituting for x_s in the above equation, we can have

$$f'(x_s) = r - \frac{2rx_s}{K} - \frac{ma}{(x_s+a)^2}. \tag{2.38}$$

Now

$$x_s + a = \frac{K}{2}\left[\left(1+\frac{a}{K}\right) + \sqrt{\left(1+\frac{a}{K}\right)^2-\frac{4m}{Kr}}\right], \tag{2.39}$$

$$(x_s+a)^2 = \frac{K^2}{4}\left[2\left(1+\frac{a}{K}\right)^2 - \frac{4m}{Kr} + 2\left(1+\frac{a}{K}\right)\sqrt{\left(1+\frac{a}{K}\right)^2-\frac{4m}{Kr}}\right]. \tag{2.40}$$

$$r\left(1-\frac{2x_s}{K}\right) = r\left[\frac{a}{K} - \sqrt{\left(1+\frac{a}{K}\right)^2-\frac{4m}{Kr}}\right]. \tag{2.41}$$

On multiplying (2.40) and (2.41), we have,

$$r\left(1-\frac{2x_s}{K}\right)(x_s+a)^2 = \frac{rK^2}{2}\left[\frac{a}{K} - \sqrt{\left(1+\frac{a}{K}\right)^2-\frac{4m}{Kr}}\right] \times$$

$$\left[\left(1+\frac{a}{K}\right)^2-\frac{2m}{Kr}+\left(1+\frac{a}{K}\right)\sqrt{\left(1+\frac{a}{K}\right)^2-\frac{4m}{Kr}}\right],$$

$$=\frac{rK^2}{2}\left[\frac{a}{K}-A\right]\left[\left(1+\frac{a}{K}\right)^2-\frac{2m}{Kr}+\left(1+\frac{a}{K}\right)\mathrm{A}\right].$$

Where,

$$\mathrm{A}=\sqrt{\left(1+\frac{a}{K}\right)^2-\frac{4m}{Kr}}.$$

Therefore,

$$r\left(1-\frac{2x_s}{K}\right)(x_s+a)^2$$

$$=\frac{rK^2}{2}\left[\left(\frac{2m}{Kr}-1-\frac{a}{K}\right)A+\frac{2m}{Kr}\left(2+\frac{a}{K}\right)-\left(1+\frac{a}{K}\right)^2\right]. \qquad (2.42)$$

Further considering

$$r\left(1-\frac{2x_s}{K}\right)(x_s+a)^2-am$$

$$=\frac{rK^2}{2}\left[\left(\frac{2m}{Kr}-1-\frac{a}{K}\right)A+\frac{4m}{Kr}-\left(1+\frac{a}{K}\right)^2\right]. \qquad (2.43)$$

Now asymptotical stability of x_s, requires $f'(x_s) < 0$. Applying this condition, we can have

$$\left(\frac{2m}{Kr}-1-\frac{a}{K}\right)A+\frac{4m}{Kr}-\left(1+\frac{a}{K}\right)^2<0. \qquad (2.44)$$

This simplification gives

$$\left(1+\frac{a}{K}\right)^2-\frac{4m}{Kr}>\left(\frac{2m}{Kr}-1-\frac{a}{K}\right)A.$$

This further implies

$$A^2>\left(\frac{2m}{Kr}-1-\frac{a}{K}\right)A.$$

Which further gives,

$$A > \left(\frac{2m}{Kr} - 1 - \frac{a}{K}\right).$$

Squaring both sides and substituting the value of A, it is observed that

$$\left(1 + \frac{a}{K}\right)^2 - \frac{4m}{Kr} > \frac{4m^2}{K^2r^2} + \left(1 + \frac{a}{K}\right)^2 - \frac{4m}{Kr}\left(1 + \frac{a}{K}\right).$$

On solving, we get,

$$m(m - ar) < 0. \tag{2.45}$$

This further gives, $0 < m < ar = m_w$. From the stability analysis of x_e, we found that $f'(x_e) < 0$ gives $m > m_w$, which proves that x_e is unstable, and $f'(x_s) < 0$ gives $0 < m < m_w$. Hence stability of x_s is established when $0 < m < m_w$.

Theorem 2.7 If $m_w < m < m_{cr}$ (See Eq, (2.24)), there will be three equilibrium points of the system (2.1) x_e, x_s, x_t. Out of which x_e is stable, and x_s, x_t both are unstable.

Proof. Following the last theorem 2.6 if $m < m_{cr}$, system (2.1) has three steady-state solutions x_e , x_s, x_t. From Theorem 2.5,

$$f(x) = \left[r\left(1 - \frac{x}{k}\right) - \frac{m}{x + a}\right] x.$$

$$f'(x) = r - \frac{2rx}{K} - \frac{ma}{(x + a)^2}.$$

$f'(0) < 0$ gives $m > m_w$ (See Theorem 2.5).

$f'(x_s) < 0$ gives $0 < m < m_w$ (See Theorem 2.6).

$$f'(x_t) == r - \frac{2rx_t}{K} - \frac{ma}{(x_t + a)^2}. \tag{2.46}$$

Consider

$$x_t + a = \frac{K}{2}\left[\left(1 + \frac{a}{K}\right) - \sqrt{\left(1 + \frac{a}{K}\right)^2 - \frac{4m}{Kr}}\right]. \tag{2.47}$$

$$(x_t+a)^2=\frac{K^2}{4}\left[2\left(1+\frac{a}{K}\right)^2-\frac{4m}{Kr}-2\left(1+\frac{a}{K}\right)\sqrt{\left(1+\frac{a}{K}\right)^2-\frac{4m}{Kr}}\right]. \quad (2.48)$$

Again consider

$$r\left(1-\frac{2x_t}{K}\right)=r\left[\frac{a}{K}+\sqrt{\left(1+\frac{a}{K}\right)^2-\frac{4m}{Kr}}\right]. \quad (2.49)$$

On multiplying (2.47) and (2.48), we have

$$r\left(1-\frac{2x_t}{K}\right)(x_t+a)^2=\frac{rK^2}{2}\left[\frac{a}{K}+\sqrt{\left(1+\frac{a}{K}\right)^2-\frac{4m}{Kr}}\right]\times$$

$$\left[\left(1+\frac{a}{K}\right)^2-\frac{2m}{Kr}-\left(1+\frac{a}{K}\right)\sqrt{\left(1+\frac{a}{K}\right)^2-\frac{4m}{Kr}}\right].$$

$$=\frac{rK^2}{2}\left[\frac{a}{K}+A\right]\left[\left(1+\frac{a}{K}\right)^2-\frac{2m}{Kr}-\left(1+\frac{a}{K}\right)\mathrm{A}\right].$$

Here, $\mathrm{A}=\sqrt{\left(1+\frac{a}{K}\right)^2-\frac{4m}{Kr}}$.

Therefore,

$$r\left(1-\frac{2x_t}{K}\right)(x_t+a)^2$$

$$=\frac{rK^2}{2}\left[\left(\frac{2m}{Kr}-1-\frac{a}{K}\right)A+\frac{2m}{Kr}\left(2+\frac{a}{K}\right)-\left(1+\frac{a}{K}\right)^2\right]. \quad (2.50)$$

$$r\left(1-\frac{2x_t}{K}\right)(x_t+a)^2-am=\frac{rK^2}{2}\left[\left(\frac{2m}{Kr}-1-\frac{a}{K}\right)A+\frac{4m}{Kr}-\left(1+\frac{a}{K}\right)^2\right],$$

$$=\frac{rK^2}{2}\left[\left(1+\frac{a}{K}-\frac{2m}{Kr}\right)A+\frac{4m}{Kr}-\left(1+\frac{a}{K}\right)^2\right]. \quad (2.51)$$

Now asymptotical stability of x_t, requires $f'(x_t)<0$. From this condition, we can have

$$\left(1+\frac{a}{K}-\frac{2m}{Kr}\right)A+\frac{4m}{Kr}-\left(1+\frac{a}{K}\right)^2<0.$$

This implies

$$\left(1+\frac{a}{K}\right)^2-\frac{4m}{Kr}>\left(1+\frac{a}{K}-\frac{2m}{Kr}\right)A.$$

This implies

$$A^2>\left(1+\frac{a}{K}-\frac{2m}{Kr}\right)A.$$

This further gives,

$$A>\left(1+\frac{a}{K}-\frac{2m}{Kr}\right).$$

Squaring both sides

$$\left(1+\frac{a}{K}\right)^2-\frac{4m}{Kr}>\frac{4m^2}{K^2r^2}+\left(1+\frac{a}{K}\right)^2-\frac{4m}{Kr}\left(1+\frac{a}{K}\right).$$

On solving, we get,

$$m(m-ar)<0.$$

This further gives, $0<m<ar=m_w$. This is contradictory to the existence condition of the point. Hence $x_e=0$ is stable, but x_s and x_t are unstable.

2.3.6 Examination of the Equilibrium Points for Global Stability

Theorem 2.8 If $r-\frac{m}{M+a}+\frac{rx_*}{K}<0$ and $\frac{mx_*}{a}-rx_*<0$, where $|x|\leq M$, then equilibria x_s is globally stable where M is a positive finite real constant.

Proof. Consider a function

$$V(x)=\left(x-x_*-x_*\ln\frac{x}{x_*}\right), \tag{2.52}$$

where $x(t)=f(x)$, and $x_*=x_s$.

Again consider,

$$D^{\alpha} V(x) = D^{\alpha}\left(x - x_* - x_* \ln\frac{x}{x_*}\right). \tag{2.53}$$

Using Eq. (1.9), we have,

$$D^{\alpha} V(x) \le \left(\frac{x - x_*}{x}\right) D^{\alpha}x, \tag{2.54}$$

substituting the value of $D^{\alpha}x$, we can see

$$D^{\alpha} V(x) \le rx - \frac{mx}{x+a} - rx_* + \frac{rxx_*}{K} + \frac{mx_*}{x+a},$$

$$D^{\alpha} V(x) \le x\left[r - \frac{m}{M+a} + \frac{rx_*}{K}\right] + \frac{mx_*}{a} - rx_* \le 0,$$

if $r - \frac{m}{M+a} + \frac{rx_*}{K} < 0$ and $\frac{mx_*}{a} - rx_* < 0$, then V becomes a positive definite function (Lyapunov function) and hence the point $x_* = x_s$ is globally stable.

2.4 Numerical Analysis

Table 2.1: Table for Values of Parameters Used for Numerical Simulations

Parameters	**Value in Case 1**	**Value in Case 2**	**Value in Case 3**	**Value in Case 4**	**Source**
K	10	10	10	10	[102]
r	0.5	0.5	0.5	0.5	[102]
a	2	2	2	2	Assumed
m	0	0.8	1.5	1.9	Assumed

From these values of Allee's constants are obtained as,

$m_w = 1$, $m_{cr} = 1.8$ (See Eq. (2.24))

Case1.When Allee's constant $m = 0$

Firstly, we consider the population without the Allee effect. By taking Allee's constant $m = 0$, system (2.1) results in two equilibrium points as $x = 0, and\ K = 10$. Figure 2.1 is plotted by taking the initial population of 0.01 in various fractional order models and traditional models. In all models, populations are observed to be moving towards the equilibrium point $x = 10$.

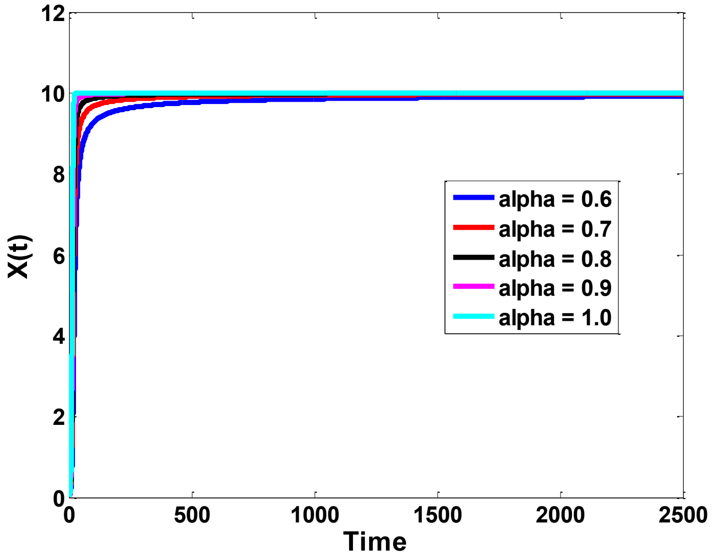


Figure 2.1 Showing local asymptotic stability of equilibrium point $x_s = 10$ in different models when population is not experiencing the Allee effect.

Case 2. When Allee's constant $0 < m < m_w$ (See Eq. (2.24))

By taking Allee's constant $m = 0.8$ we obtain the value of the equilibrium point x_s= 8.47. By Theorem 2.6, the stability criteria of the point x_s is also satisfied. Figure 2.2 is plotted by taking an initial population of 1.5 and different fractional order models, along with the traditional model of order 1, and populations are observed to be moving towards the equilibrium point x_s. Figure 2.3 is plotted in the fractional model of order 0.9, with different initial populations, and the equilibrium point x_s is observed to be globally stable in the fractional order model. So, it is proven numerically that when the population is exposed to a weak Allee effect, no minimum population will be required for its survival.

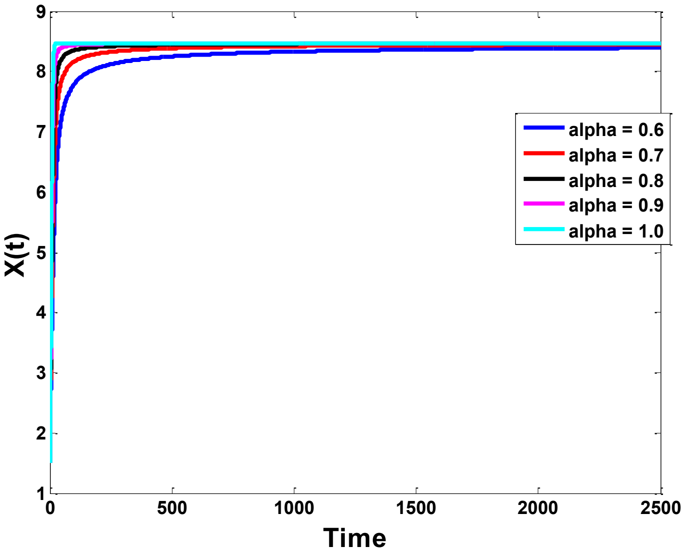


Figure 2.2 Showing local asymptotic stability of the equilibrium point $x_s = 8.47$ in different models when populations are subjected to weak Allee effect $(0 < m < m_w)$

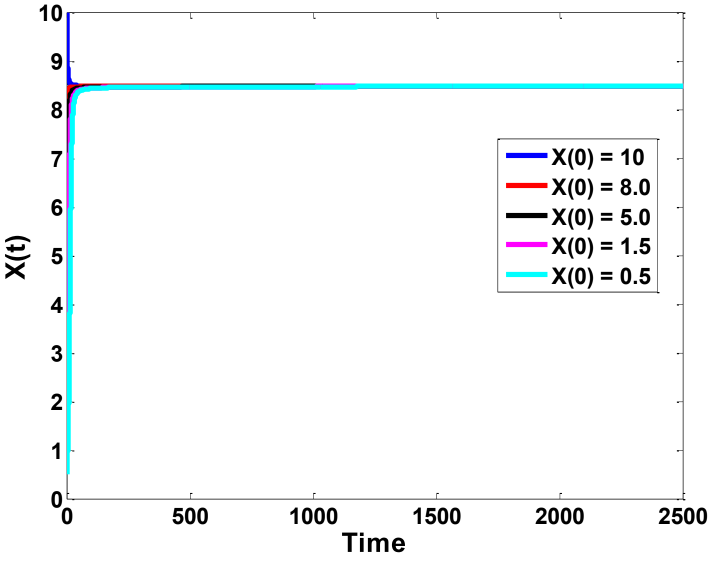


Figure 2.3 Showing global stability of equilibrium point $x_s = 8.47$ in fractional model of order 0.9 when populations are subjected to weak Allee effect ($0 < m < m_w$)

Case 3. When Allee's constant $m_w < m < m_{cr}$ (See Eq. (2.24))

By taking Allee's constant $m = 1.5$ we obtain the value of the equilibrium point x_s= 6.45, and $x_t = 1.55$. Figure 2.4 is plotted by taking an initial population of $1.5 < x_t$, and Figure 2.5 is plotted by taking an initial population size of $2 > x_t$ respectively in various fractional models along with the traditional model of order 1. It is observed that in Figure 2.4, populations are moving towards the extinction point x_e . From Figure 2.5, it is observed that in each model, populations are increasing and converging to a point $x_s = 6.45$. Thus, it is found numerically that when the population is exposed to a strong Allee effect, whenever the initial population is more than the value of x_t, the population will converge to x_s and whenever the initial population is less than x_t, the population will become extinct.

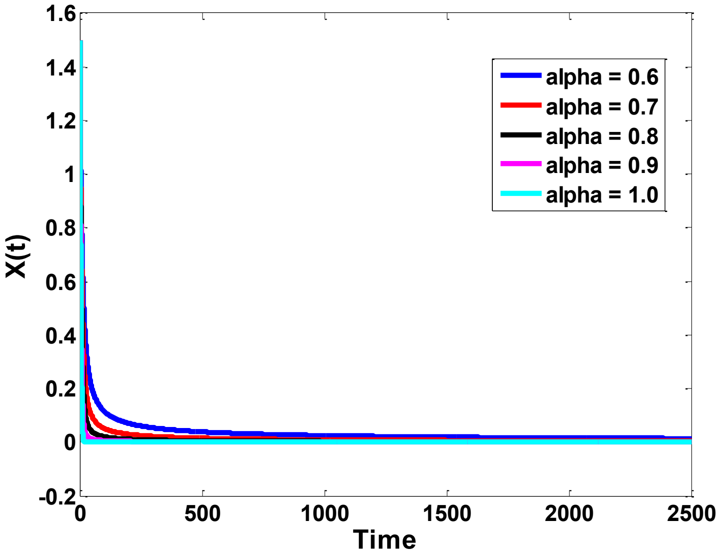


Figure 2.4 Showing the bevior of population when initial population is less than $x_t = 1.55$ in the case when population is subjected to strong Allee effect $(m_w < m < m_{cr})$

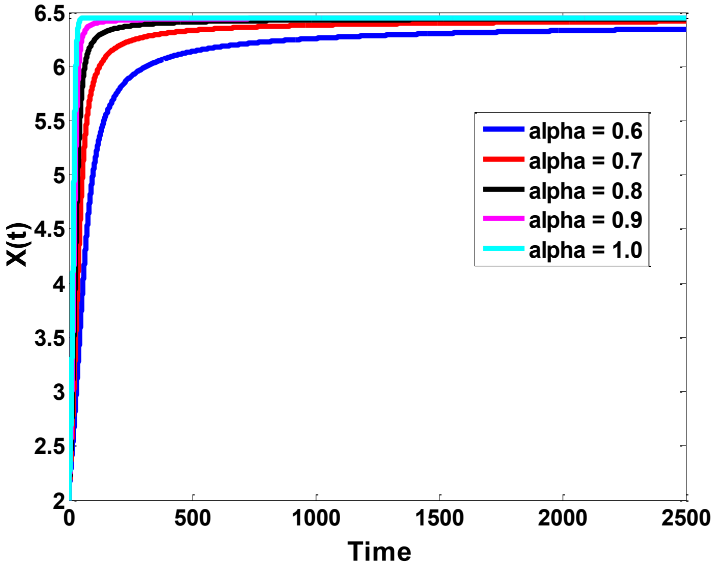


Figure 2.5 Showing the stability of point $x_s = 6.45$ when initial population is more than $x_t = 1.55$ and population is subjected to strong Allee effect $(m_w < m < m_{cr})$

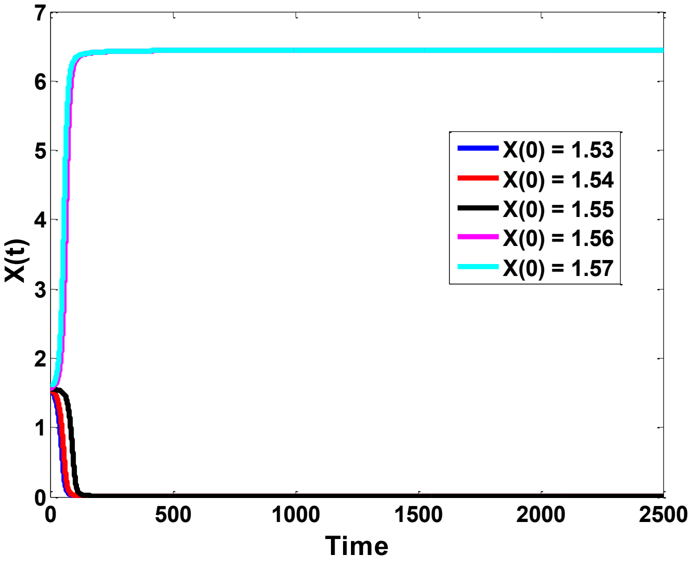


Figure 2.6 Showing x_t is the threshold level for the populations suffering from strong Allee effect

Figure 2.6 is plotted by taking the fractional order as 0.9 and with different initial populations, and it is observed that when the initial population is more than 1.55, the populations move toward $x_s = 6.45$, and when the initial population is below 1.55, populations move toward extinction. Therefore, through numerical examples, it is shown that the populations exposed to the strong Allee effect require a threshold level to grow.

Case 4. When Allee's constant $m > m_{cr}$ (See Eq. (2.24))

For the next simulation, we have considered $m = 1.9 > m_{cr}$. Figure 2.7 is plotted by taking different fractional order models along with an integer order model and an initial population of 10, which is equal to the carrying capacity. From the figure, it is observed that the populations are moving towards extinction in each model, both in fractional as well as in integer order models. Therefore, the results obtained in Theorem 2.5 are verified. It has been proved mathematically as well as numerically that when the population is exposed to such a high degree of Allee effect, even the strong enough populations can move towards extinction.

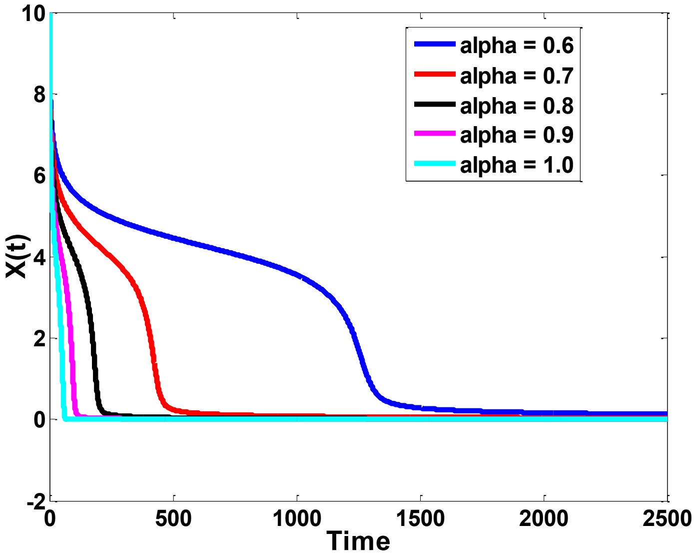


Figure 2.7 Showing the stability of the extinction point when populations are subjected to strong Allee effect $(m > m_{cr})$

2.5 Conclusion

In this chapter, while investigating the fractional order single species logistic equation with additive Allee effect, it is found that if there is no Allee effect or populations are experiencing a weak Allee effect, i.e., when $0 \leq m < m_w$ (See Eq. (2.24)), the population will not die, whatever may be the initial population, and their existence will remain globally stable (Figs. 2.1,2.2,2.3). However, when the population is under the strong Allee effect or when $m_w < m < m_{cr}$ (See Eq. (2.24)), the threshold level of the population is given by x_t (Figs. 2.4,2.5,2.6). But if the level of the Allee effect exceeds $m > m_{cr}$ (See Eq. (2.24)), all populations will die irrespective of the initial population (Fig. 2.7). Therefore, this research established the result that the populations exposed to strong Allee effect require a minimum population for their survival, as already suggested by many researchers mathematically as well as theoretically [69-78,100-

102]. Abbas et al. [102] also worked on a fractional order logistic equation incorporating the multiplicative Allee effect by using Riemann-Liouville's definition-based fractional order differential equations. In his research, the authors found a threshold population numerically that depends only upon the carrying capacity of the environment for that population. However, the proposed research has found a mathematical expression for Allee's constant ($m_{cr)}$, which is the upper bound of the Allee effect. Whenever the Allee effect is less than this value, then there will always be a threshold population that will be required for the survival of the species, and crossing that level of the Allee's constant survival of that population is not possible even if the initial population is equal to the carrying capacity. Moreover, it is also found that by incorporating the additive Allee effect, the threshold value of the population depends on all parameters used for the study. Once knowing the values of all parameters for a specific spatial distribution of a vulnerable species, a threshold level beyond which survival of that species is not possible can be calculated using the expression for x_t(See Eq. (2.36)). Once knowing the threshold level of the population prone to the strong Allee effect, timely strategies can help in saving that population from extinction. The strong Allee effect notion is frequently used in pest control [77,78]. To eliminate the unwanted pest population, a strong Allee effect can be introduced in the pest population by reducing its number below the threshold level (independent of the method used), which can be computed. Therefore, it is observed that exposure to the Allee effect of the population has introduced several equilibrium points depending upon the level of the Allee effect, and hence, we can have rich population dynamics in terms of several stable equilibrium points. The behavior of the fractional order derivatives is observed while studying the population dynamics. It is observed in the fractional order model that as the order of fractional derivatives is reduced, the time required for populations to reach the equilibrium point increases; hence, we found wider stability regions. Therefore, it is discovered that replacing the ordinary derivative with a fractional derivative does not affect the value of the equilibrium points and their stability; rather, it provides greater flexibility in the stability region of the equilibrium points. Therefore, fractional derivatives are relatively better as compared with integer order derivatives. According to Du et al. [13], fractional derivatives have learning and forgetting stages. Populations take less time to learn the behavior in lower fractional order models and more time to learn the behavior in the higher order fractional derivative model. As a result, lower-order fractional derivatives preserve higher levels of memory, whereas integer-order models do not exhibit such behaviors. Thus, the use of fractional order models contributes to warning us of impending extinction at an early stage. Therefore, fractional order models provide opportunities to prevent population extinction by providing flexible stability regions.

Chapter 3

Analyzing the Influence of Additive Allee Effect in Prey Population in Lotka-Volterra Predator-Prey System through Fractional Order Derivative

3.1 Introduction

All populations in an ecosystem have interspecific as well as intraspecific relationships known as prey-predatory, competitive, mutualism, and parasitism [122]. The prey-predator relationship in the natural world is the most common and has thus been studied extensively by ecologists [123]. The famous Lotka-Volterra model was developed for the first time by biophysicist A.J. Lotka [124] and mathematician V. Volterra [125]. The conventional Lotka-Volterra model structure is oversimplified because it makes numerous assumptions that are not precisely true. This includes an uninterrupted pattern of prey species' food availability, the absence of adaptation of all interacting species, and the initial relationship between the size of every species and its rate of growth, leading to exponential expansion if the species remains unaffected by others. Regardless of these assumptions, the Lotka-Volterra model has successfully explained the dynamics of a wide variety of biological interaction systems, including interactions between numerous species of fish, bacterial lake communities, and freshwater ciliates. In the past few years, Lotka-Volterra systems have proven to be an excellent analytical tool and, indeed, the standard for quantifying interactions within massive microbial communities. Lotka-Volterra models have also been presented in non-ecological contexts, such as to address pollution issues. Although Lotka-Volterra models are frequently regarded as archaic and insufficient for much more modern systems biology techniques, it has been discovered that the Lotka-Volterra model structure is remarkably rich and capable of capturing any differentiable nonlinearities if the required number of supplementary variables is introduced into the model. Despite its simplicity and intuitive nature, the Lotka-Volterra model has proven to be an excellent baseline model for comparison with modern complex models. Lotka-Volterra models have been used in nearly 500 PubMed-listed studies in the last decade alone [126].

For the first time using the Riemann Liouville fractional derivative, Ahmed et.al. studied the Lotka-Volterra prey-predator model and rabies model [37]. Since then, using fractional order differential equations, various studies have been reported on the modeling of the Lotka-Volterra system. Using Holling type III interactions, Xie et al. studied a fractional-order Lotka-Volterra system by incorporating logistic growth to the prey population and discontinuous predator harvesting [127]. Kumar et al. studied the Lotka-Volterra system with two different numerical schemes: the Adams-Bashforth-Moulton scheme and the Haar wavelet scheme by considering the model with Caputo's fractional order derivative [128]. Das et al. applied the Homotopy Perturbation method to solve the fractional order derivatives based Lotka-Volterra system [39]. Elsonbaty and Elsadany analyzed a discrete, fractionally ordered Lotka-Volterra model considering the logistic growth of the prey population [129]. Mesady et al. investigated how immigration affected an extended Lotka-Volterra model using Caputo's fractional order derivative [130]. Panigoro et al. examined Allee effects multiplicatively with Michaelis–Menten harvesting in a prey-predator model using fractional order differential equations [131]. Xie and Zhang investigated a fractional order prey-predator system by exposing prey to the Allee effect and fear effect of predation, incorporating prey refuge [132]. So far, we have not witnessed any study on the Lotka-Volterra model incorporating the additive Allee effect. However, Kalra and Malhotra studied a logistic equation incorporating the additive Allee effect to the growth equation of the species by employing Caputo's definition of fractional derivative [133]. Nisar et al. studied an AB fractional derivative-based food chain model by incorporating the additive Allee effect on prey populations and Holling Type II interaction among species [103]. Therefore, in this chapter, we are going to analyze the Lotka-Volterra model in which prey are growing logistically and are exposed to an additive Allee effect using Caputo's definition based fractional order derivative. Model behavior will be observed by exposing the prey to weak Allee effect as well as the strong Allee effect, along with changing the predation rate of prey in fractionally ordered as well as in integer order-based models.

3.2 Mathematical model

The Lotka-Volterra model with logistic growth and additive Allee effect in the prey population by taking a fractional order derivative can be written as,

$$\begin{cases} D^{\alpha}\, x_1 = x_1 \left[r \left(1 - \frac{x_1}{K} \right) - \frac{m}{x_1 + a} \right] - f\, x_1 x_2, \\ \qquad D^{\alpha}\, x_2 = f\, c\, x_1 x_2 - e\, x_2, \end{cases} \tag{3.1}$$

with initial conditions $x_1(0) \geq 0, x_2(0) \geq 0$. Here $x_1 > 0, x_2 > 0$ respectively represents the prey's population and predator's population at time t. All parameters r, K, m, a, f, c, e are all positive depicting prey's intrinsic growth rate, maximum prey population that can supported by the environment (carrying capacity), Allee's constant, prey population having fitness half of its maximum value, the prey's capture rate by predator, the prey's conversion rate to predators, and predators' intrinsic death rate respectively. D^{α} is the time-fractional Caputo derivative, where $\alpha \in (0,1]$.

These equations are based on a few assumptions:

1. In the absence of predators and the Allee effect, the prey grows exponentially when its population is far below the carrying capacity, as it reaches near the carrying capacity, growth slows down and becomes stable at the carrying capacity.
2. Predators die of starvation in the absence of the prey population,
3. Predators can consume any number of prey.
4. Environmental conditions are homogeneous for both populations.

3.3 Primary Evaluation of the Model

In this section, an analysis of the proposed model will be conducted to establish the properties of its solutions, including existence, uniqueness, positivity, and uniform boundedness.

3.3.1 Existence of Unique Solution

Theorem 3.1 Let us consider the interval $I = [\, t_0,\, T\,],\ T < \infty$. Suppose that C(I) represents the class containing all real-valued functions having continuous first-order derivatives on I. Consider the region $D = \ (x_1, x_2) \in \mathbb{R}^2 : |x_1|, |x_2| \leq M\,\}$. Here M is a positive constant. Then the problem $D^{\alpha} x_1(t) = h(\, x_1\, x_2, t)$, $D^{\alpha} x_2(t) = g(\, x_1\, x_2, t)$, with initial condition $x_1(\, t = t_0) = x_{1_0}$, $x_2(\, t = t_0) = x_{2_0}$ and $f : I \times D \rightarrow \mathbb{R}^2$; $\alpha \in (0,1]$ have a unique solution $X(x_1(t), x_2(t)) \in D$, with initial condition $X(x_1(t_0\,), x_2(t_0)) = X_{t_0}$, where,

$$h(\, x_1\, x_2, t) = x_1 \left[r\left(1 - \tfrac{x_1}{k}\right) - \tfrac{m}{x_1 + a} \right] - f\, x_1 x_2\ ,\ g(\, x_1\, x_2, t) = f\, c\, x_1 x_2 - e\, x_2.$$

Proof. Consider a mapping $F : D \rightarrow \mathbb{R}^2$ by $F(X) = \big(\, F_1(X), F_2(X)\big)$,

where

$$F_1(X) = x_1[\left[r\left(1 - \frac{x_1}{k}\right) - \frac{m}{x_1 + a} \right] - f\, x_1 x_2, \qquad F_2(X) = f\, c\, x_1 x_2 - e\, x_2$$

Define $\|F(x)\| = \sup\limits_{t} |e^{-St}\, F(x)|,\ S > 0.\ t \in [\, t_0,\, T\,],\ T < \infty.$

Clearly $\|F(x)\| = \sup\limits_{t} |F(x)|$.

For any $X, Y \in$ D; Let $X = (x_1, x_2\,)$ and $Y = (y_1, y_2\,)$.

Consider

$$\|F(X) - F(Y)\| = |F_1(X) - F_1(Y)| + |F_2(X) - F_2(Y)|, \tag{3.2}$$

$$= \left| x_1 \left[r\left(1 - \frac{x_1}{k}\right) - \frac{m}{x_1 + a} \right] - f\, x_1 x_2 - y_1 \left[r\left(1 - \frac{y_1}{k}\right) - \frac{m}{y_1 + a} \right] + f\, y_1 y_2 \right|$$
$$+\ |f\, c\, x_1 x_2 - e\, x_2 - f\, c\, y_1 y_2 + e\, y_2|,$$

$$= \left|rx_1 - \frac{r}{K}\, x_1^2 - \frac{mx_1}{x_1 + a} - f\, x_1 x_2 - ry_1 + \frac{r}{K} y_1^2 + \frac{my_1}{y_1 + a} + fy_1 y_2\right|$$

$$+ \ |fcx_1x_2 - ex_2 - fcy_1y_2 + ey_2|, \tag{3.3}$$

$$\leq \ |r(x_1 - y_1)| + \left|\frac{r}{K}(x_1^2 - y_1^2)\right| + \left|m \left(\frac{x_1}{x_1 + a} - \frac{y_1}{y_1 + a}\right)\right| + \ |f\ (x_1x_2 - y_1y_2)|$$

$$+ |fc\ (x_1x_2 - y_1y_2)| + \ |e\ (x_2 - y_2)|$$

,

$$\leq \ |(x_1 - y_1| \left|r + \frac{r}{K}(x_1 + y_1) + \frac{ma}{(x_1 + a) + (y_1 + a)}\right|$$

$$+ \ f\ |x_1(x_2 - y_2) + y_2(x_1 - y_1)|$$

$$+ fc|x_1(x_2 - y_2) + y_2(x_1 - y_1)| + e|x_2 - y_2|,$$

$$\leq |(x_1 - y_1| \left(r + \frac{2rM}{K} + \ ma \right) + \ f\ M|(x_2 - y_2)| + \ fM\ |(x_1 - y_1|$$

$$+ \ fcM|(x_2 - y_2)| + \ fcM\ |(x_1 - y_1| + e\ |(x_2 - y_2)|,$$

$$= |(x_1 - y_1| \left(r + \frac{2rM}{K} + \ ma + fM + \ fcM\right) + \ |x_2 - y_2|(fM + fcM + e),$$

$$= K_1|x_1 - y_1| + K_2|x_2 - y_2| \ \leq \ \mathcal{K}\ \|x - y\|. \tag{3.4}$$

Where,

$$K_1 = \ r + \frac{2rM}{K} + \ ma + fM + \ fcM, \ \ K_2 = \ (fM + fcM + e),$$

and $\mathcal{K} \ = \ \max\{K_1, K_2\}$.

Therefore, F satisfies the Lipschitz condition.

Let $G(X)$ be the solution of the given system. Then by Eq. (.16),

$$G(X) = X - \ X(0) = \frac{1}{\Gamma(\alpha)} \int_0^t F\big(X(s)\big)\,(t - s)^{\alpha - 1}\ ds. \tag{3.5}$$

Consider

$$G(X) - G(Y) = \frac{1}{\Gamma(\alpha)} \int_0^t \{F(X(s)) - F(Y(s))\} (t-s)^{\alpha-1} \, ds. \tag{3.6}$$

$$|G(X) - G(Y)| = \left| \frac{1}{\Gamma(\alpha)} \int_0^t \{F(X(s)) - F(Y(s))\} (t-s)^{\alpha-1} \, ds \right|, \tag{3.7}$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t |F(X(s)) - F(Y(s))| (t-s)^{\alpha-1} \, ds.$$

Now

$$e^{-pt} \left(G(X) - G(Y)\right) = \frac{1}{\Gamma(\alpha)} \int_0^t e^{-pt} \{F(X(s)) - F(Y(s))\} (t-s)^{\alpha-1} \, ds,$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^t e^{-p(t-s)} e^{-ps} \{F(X(s)) - F(Y(s))\} (t-s)^{\alpha-1} \, ds. \tag{3.8}$$

$$\|G(X) - G(Y)\| \leq \frac{1}{\Gamma(\alpha)} \int_0^t e^{-p(t-s)} \|F(X(s)) - F(Y(s))\| (t-s)^{\alpha-1} \, ds,$$

$$\leq \frac{\mathcal{K}}{\Gamma(\alpha)} \int_0^t e^{-p(t-s)} \|x - y\| \, (t-s)^{\alpha-1} \, ds, \qquad \text{(By Eq. (3.4))}$$

$$= \frac{\mathcal{K}}{\Gamma(\alpha)} \|x - y\| \int_0^t e^{-p(t-s)} \, (t-s)^{\alpha-1} \, ds,$$

$$\leq \frac{\mathcal{K}}{\Gamma(\alpha)} \|x - y\| \frac{\Gamma(\alpha)}{p^\alpha} = \frac{\mathcal{K}}{p^\alpha} \|x - y\|. \tag{3.9}$$

Choosing p sufficiently large so that $\frac{\mathcal{K}}{p^\alpha} < 1$, then by Theorem (1.4.8), the model given by (3.1) has a unique solution.

3.3.2 Positivity of the Solutions

Theorem 3.2 Solutions to the given system (3.1) that start in $\mathbb{R}_+^2$ are non-negative, where $\mathbb{R}_+^2$ represents the set containing positive ordered reals including zero.

Proof. Let us consider that $X(\, t_0) = (x_1(t_0), x_2(t_0)) \in \mathbb{R}_+^2$ is the initial solution of the system (3.1). Let $t > t_0$ and we claim that $X(t) \; \geq 0$ for all $t \geq t_0$. To establish the claim, we suppose that there exists some solution $X(t')$ which do not lie in $\mathbb{R}_+^2$ where $t' \; > t_0$. This implies that either $X(t)$ crosses $x_1$axis or x_2 axis.

If the solution crosses x_2 axis, then we can find some t_* so that $t_* \geq \; t_0$ and $x_1(t_*) = 0$, and some $t' > t_*$ (t' is sufficiently close to t_*) but $x_1(t) < 0$ for all $t_* < t \leq t'$. Now there are two possibilities.

Case1. If $D^\alpha \, x_1(t) < 0 \;\; \forall \, t_* < t \leq t'$.

This implies

$$x_1 \left[r \left(1 - \frac{x_1}{k} \right) - \frac{m}{x_1 + a} \right] - f \, x_1 x_2 < 0. \tag{3.10}$$

But

$$x_1 \left[r \left(1 - \frac{x_1}{k} \right) - \frac{m}{x_1 + a} \right] - f \, x_1 x_2 > x_1 \, \rho \qquad \forall \, t \; \in (t_*, t']. \tag{3.11}$$

Here ρ is the minimum value of

$$r \left(1 - \frac{x_1}{K} \right) - \frac{m}{x_1 + a} - f \, x_2 \;\; \forall \;\; t_* < t \leq t'.$$

Hence, we have

$$D^\alpha \, x_1(t) \; > \; \rho \, x_1. \tag{3.12}$$

By taking the Laplace transform and using Eq. (1.7), we have

$$s^\alpha \, X_1(s) - s^{\alpha-1} \, x_1(t_0) \geq \; \rho \, X_1(s),$$

where $X_1(s) = \; \mathcal{L} \, \{x_1(t)\}$.

$$(s^\alpha \; - \; \rho \,) X_1(s) \; \geq s^{\alpha-1} \, x_1(t_0).$$

$$X_1(s) > \frac{s^{\alpha-1} \, x_1(t_0)}{(s^\alpha \; - \; \rho \,)}. \tag{3.13}$$

On applying the inverse Laplace transform and using Eq. (1.8) we have,

$$x_1(t) \geq \mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{(s^{\alpha} - \rho)} x_1(t_0)\right\}$$
$$= x_1(t_0)E_{\alpha,1}\{\rho\,(\mathrm{t} - t_0)^{\alpha}\}. \tag{3.14}$$

Therefore, we have $x_1(t) \geq 0$ for all $t \in (t_*, t']$.

This contradicts the assumption that $x_1(t) < 0,\ \forall\, t \in (t_*, t']$.

Case 2. If $D^{\alpha}\, x_1(t) > 0\ \forall\, t\ \in (t_*, t']$.

This implies

$$x_1\left[r\left(1 - \frac{x_1}{k}\right) - \frac{m}{x_1 + a}\right] - f\,x_1x_2 > 0. \tag{3.15}$$

But

$$x_1\left[r\left(1 - \frac{x_1}{k}\right) - \frac{m}{x_1 + a}\right] - f\,x_1x_2 > rx_1. \tag{3.16}$$

This gives

$$D^{\alpha}\, x_1(t) > rx_1.$$

By applying the Laplace transform,

$$s^{\alpha}\, X_1(s) - s^{\alpha-1}\, x_1(t_0) \geq r\, X_1(s),$$

where $X_1(s) = \mathcal{L}\{x_1(t)\}$.

$$(s^{\alpha} - r)X_1(s) \geq s^{\alpha-1}\, x_1(t_0).$$

$$X_1(s) > \frac{s^{\alpha-1}\, x_1(t_0)}{(s^{\alpha} - r)}. \tag{3.17}$$

By applying the inverse Laplace transform we have,

$$x_1(t) \geq \mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{(s^{\alpha} - \mathrm{r})} x_1(t_0)\right\} = x_1(t_0)E_{\alpha,1}\{r\,(\mathrm{t} - t_0)^{\alpha}\} \geq 0. \tag{3.18}$$

This is a contradiction to the supposition that $x_1(t) < 0$ for all $t_* < t \leq t'$. Similarly, if we take $x_2(t) < 0$ for some $\mathrm{t} > t_0$, we arrive at a contradiction. Hence all solutions start in $\mathbb{R}^2_+$ lies in $\mathbb{R}^2_+$.

3.3.3 Uniform Boundedness of the Solutions

Theorem 3.3 All non–negative solutions to the system (3.1) are uniformly bounded.

Proof. Consider a function

$$F(t) = x_1(t) + \frac{1}{c}\, x_2(t). \tag{3.19}$$

Now

$$D^\alpha F(t) = D^\alpha \left[x_1(t) + \frac{1}{c}\, x_2(t)\right], \tag{3.20}$$

$$= D^\alpha x_1(t) + \frac{1}{c} D^\alpha x_2(t),$$

$$= x_1 \left[r\left(1 - \frac{x_1}{k}\right) - \frac{m}{x_1 + a}\right] - f\, x_1 x_2 + f\, x_1 x_2 - \frac{e}{c} x_2,$$

$$= r x_1 - \frac{r}{K}\, x_1^2 - \frac{m}{x_1 + a} x_1 - \frac{e}{c} x_2, \tag{3.21}$$

$$< r x_1 - \frac{r}{K}\, x_1^2 - \frac{m}{K + a} x_1 - \frac{e}{c} x_2,$$

$$= \left(r - \frac{m}{K + a}\right) x_1 - \frac{r}{K}\, x_1^2 - \frac{e}{c} x_2,$$

$$= \left(r - \frac{m}{K + a}\right) x_1 - \frac{r}{K}\, x_1^2 - e\, F(t) + e x_1. \text{ (By Eq. (3.19))}$$

Therefore, we found that

$$D^\alpha F(t) + e\, F(t) < \left(r + e - \frac{m}{K+a}\right) x_1 - \frac{r}{K}\, x_1^2, \tag{3.22}$$

$$= M_1 x_1 - M_2 x_1^2,$$

where, $M_1 = r + e - \frac{m}{K+a}$ and $M_2 = \frac{r}{K}$.

Therefore,

$$D^\alpha F(t) + e\, F(t) < -M_2 \left(x_1^2 - \frac{M_1}{M_2}\, x_1 + \frac{M_1^2}{4M_2^2}\right) + \frac{M_1^2}{4M_2},$$

$$= \frac{M_1^2}{4M_2} - M_2 \left(x_1 - \frac{M_1}{2M_2}\right)^2 \le \frac{M_1^2}{4M_2} = M_3.$$

Where $M_3 = \frac{M_1^2}{4M_2}$.

Therefore, we have

$$D^\alpha F(t) + e\,F(t) < M_3. \tag{3.23}$$

By taking the Laplace transform and using Eq. (1.7), we have

$$s^\alpha\, G(s) - s^{\alpha-1}\, F(t_0) + e\, G(s) \le \frac{M_3}{s}. \tag{3.24}$$

Where $G(s) = \mathcal{L}\{F(t)\}$, this gives

$$(\, s^\alpha + e\,)\, G(s) \le \frac{M_3}{s} + s^{\alpha-1}\, F(t_0),$$

which gives,

$$G(s) \le \frac{M_3}{s\,(\,s^\alpha + e\,)} + \frac{s^{\alpha-1}\, F(t_0)}{(\,s^\alpha + e)},$$

$$= \frac{s^{\alpha-1}\, F(t_0)}{(\,s^\alpha + e)} + \frac{s^{\alpha-(\,\alpha+1\,)}}{(\,s^\alpha + e)}\, M_3. \tag{3.25}$$

On applying the inverse Laplace transform and using Eq. (1.8) we have,

$$F(t) \le F(t_0)\, \mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{(\,s^\alpha + e\,)}\right\} + M_3\, \mathcal{L}^{-1}\left\{\frac{s^{\alpha-(\alpha+1)}}{(\,s^\alpha + e\,)}\right\}$$

$$= F(t_0)\, E_{\alpha,1}\{-e(\mathrm{t}-t_0)^\alpha\} + M_3\, (\mathrm{t}-t_0)^\alpha\, E_{\alpha,\alpha+1}\{-e(\mathrm{t}-t_0)^\alpha\}. \tag{3.26}$$

Using

$$E_{\alpha,\beta}\{z\} = z\, E_{\alpha,\alpha+\beta}\{z\} + \frac{1}{\Gamma_\beta}.$$

We have

$$F(t) \le F(t_0) E_{\alpha,1}\{-e(\mathrm{t}-t_0)^\alpha\} - \frac{M_3}{e}\left[\, E_{\alpha,1}\{-e(\mathrm{t}-t_0)^\alpha\} - 1\,\right],$$

$$= \left(\, F(t_0) - \frac{M_3}{e}\,\right) E_{\alpha,1}\{-e(\mathrm{t}-t_0)^\alpha\} + \frac{M_3}{e}. \tag{3.27}$$

Now as t⟶ ∞ , $E_{\alpha,1}\{-e(\mathrm{t}-t_0)^\alpha\} \longrightarrow 0$. Therefore, all solutions to system (3.1) starting in $\mathbb{R}_+^2$ lies in region

$$\left\{X \in \mathbb{R}_+^2 : X \leq \frac{M_3}{e} + v,\ v > 0\right\}.$$

3.3.4 Equilibrium Points and Conditions for the Existence

First, we define some mathematical representations for Allee's constant that are going to be used for the whole analysis.

$$\begin{cases} m_w = ar \\ m_{co} = \dfrac{r}{fc}\left(1 - \dfrac{e}{fcK}\right)(\,e + afc) \\ m_{cr} = \dfrac{Kr}{4}\left(1 + \dfrac{a}{K}\right)^2 \end{cases} \tag{3.28}$$

The evaluation of equilibrium points of the fractional order system (3.1) is done by setting equal to zero both equations of the model (3.1).

$$x_1\left[r\left(1 - \frac{x_1}{K}\right) - \frac{m}{x_1 + a}\right] - f\,x_1x_2 = 0. \tag{3.29}$$

$$f\,c\,x_1x_2 - e\,x_2 = 0. \tag{3.30}$$

Now Eq. (3.29) gives either $x_1 = 0$ or $f\,x_2 = \left[r\left(1 - \frac{x_1}{K}\right) - \frac{m}{x_1 + a}\right]$.

Eq. (3.30) gives either $x_2 = 0$ or $x_1 = \frac{e}{cf}$.

Solving $x_2 = 0$ and the expression $f\,x_2 = \left[r\left(1 - \frac{x_1}{K}\right) - \frac{m}{x_1 + a}\right]$,

we can have

$$r\left(1 - \frac{x_1}{K}\right) - \frac{m}{x_1 + a} = 0.$$

Which can be represented as

$$\frac{1}{K}\,x_1^2 - \left(1 - \frac{a}{K}\right)x_1 + \frac{m}{r} - a = 0. \tag{3.31}$$

Which is the same as Eq. (2.32) in Chapter 2. As solved this equation in Chapter 2, its positive solutions are given by,

$x_1' = \frac{K}{2}\left[\left(1 - \frac{a}{K}\right) + \sqrt{\left(1 + \frac{a}{K}\right)^2 - \frac{4m}{Kr}}\right]$, provided $m \leq m_{cr}$ (See Eq. (3.28))

$x_1'' = \frac{K}{2}\left[\left(1 - \frac{a}{K}\right) - \sqrt{\left(1 + \frac{a}{K}\right)^2 - \frac{4m}{Kr}}\right]$, provided $m_w \leq m \leq m_{cr}$ (See Eq. (3.28))

On substituting $x_1 = \frac{e}{cf}$ in $fx_2 = \left[r\left(1 - \frac{x_1}{K}\right) - \frac{m}{x_1 + a}\right]$, we obtain

$x_2 = \frac{r}{f} - \frac{re}{cf^2K} - \frac{mc}{e + afc}$.

For x_2 to be nonnegative, $\frac{r}{f} - \frac{re}{cf^2K} - \frac{mc}{e + afc} \geq 0$.

This on simplification gives $m < m_{co}$ (See Eq. (3.28)), provided $\frac{e}{fcK} < 1$.

Therefore, the given fractional order system (3.1) has,

1. $E_0\,(0,0)$ as the equilibrium point without any condition,
2. $E_1\,(x_1', 0)$ as the equilibrium point if $m \leq m_{cr}$ (See Eq. (3.28))
3. $E_2(x_1'', 0)$ as equilibrium points, if $m_w \leq m \leq m_{cr}$ (See Eq. (3.28))
4. $E_3\,(x_{1s}, x_{2s})$ as equilibrium point if $m < m_{co}$ and $\frac{e}{fcK} < 1$ (See Eq. (3.28))

Where,

$$x_1' = \frac{K}{2}\left[\left(1 - \frac{a}{K}\right) + \sqrt{\left(1 + \frac{a}{K}\right)^2 - \frac{4m}{Kr}}\right],$$

$$x_1'' = \frac{K}{2}\left[\left(1 - \frac{a}{K}\right) - \sqrt{\left(1 + \frac{a}{K}\right)^2 - \frac{4m}{Kr}}\right],\ x_{1s} = \frac{e}{cf},\ x_{2s} = \frac{r}{f} - \frac{re}{cf^2K} - \frac{mc}{e + afc}.$$

3.3.5 Examination of the Equilibrium Points for Local Asymptotical Stability

To analyze the local asymptotic stability of equilibrium points, we will first evaluate the Jacobian matrix at each point and then find the eigenvalues of the Jacobian matrix.

The given system of equations can be expressed as,

$$D^{\alpha}\, x_1 = x_1 \left[r\left(1-\frac{x_1}{k}\right) - \frac{m}{x_1+a}\right] - f\, x_1x_2 = h(\, x_1, x_2).$$

$$D^{\alpha}\, x_2 = f\; c\, x_1x_2 - e\, x_2 = g(\, x_1, x_2\,).$$

$$\frac{\partial h}{\partial x_1} = r - \frac{2rx_1}{K} - \frac{ma}{(x_1+a)^2} - fx_2, \quad \frac{\partial h}{\partial x_2} = -fx_1,$$
$$\frac{\partial g}{\partial x_1} = fcx_2, \qquad \frac{\partial g}{\partial x_2} = fcx_1 - e.$$

Jacobian matrix of the model (3.1) is calculated as below,

$$\mathbb{J}\,(\,h,g) = \begin{bmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \end{bmatrix} = \begin{bmatrix} r - \frac{2rx_1}{K} - \frac{ma}{(x_1+a)^2} - fx_2 & -fx_1 \\ fcx_2 & fcx_1 - e \end{bmatrix}. \qquad (3.32)$$

Theorem 3.4 The condition $m > m_{cr}$ (see Eq. (3.28)), guarantees the existence as well as the local asymptotic stability of the extinction point E_0 (0,0).

Proof. Substituting the point E_0(0,0) in $\mathbb{J}\,(\,h,g)$, the value of the Jacobian matrix is found as,

$$\mathbb{J}\,(\,0{,}0) = \begin{bmatrix} r - \frac{m}{a} & 0 \\ 0 & -e \end{bmatrix} \text{ (see Eq. (3.32))}\,.$$

Being a diagonal matrix, its eigenvalues are given by $\lambda_1 = r - \frac{m}{a}$, $\lambda_2 = -e < 0$. Now point E_0 (0,0) will be asymptotically stable if $|\arg \lambda_{1,2}| > \frac{\alpha\pi}{2}$. As eigenvalues are real, therefore $|\arg \lambda_{1,2}| > \frac{\alpha\pi}{2}$ will be satisfied iff both eigenvalues are negative. This will be possible only if $r - \frac{m}{a} < 0$. This gives $r < \frac{m}{a}$ or $m > m_w$(see Eq. (3.28)). However, if $m > m_{cr}$ the eq. (3.31) will not have any real solution. Therefore, on taking $m >$ max $\{m_w, m_{cr}\} = m_{cr}$, point E_0 (0,0) will be the only equilibrium point of the model (3.1), and it will be locally asymptotically stable.

Theorem 3.5 Whenever $m_{co} < m < m_w$, (See Eq. (3.28)) where $m > 0$, the equilibrium point, $E_1(x_1', 0)$ will be asymptotically stable.

Proof. Substituting the point E₁$(x_1', 0)$ in $\mathbb{J}\ (h, g)$, the value of the Jacobian matrix is given as,

$$\mathbb{J}\ (x_1', 0) = \begin{bmatrix} r - \dfrac{2rx_1'}{K} - \dfrac{ma}{(x_1' + a)^2} & -fx_1' \\ 0 & fcx_1' - e \end{bmatrix} \text{(see Eq. (3.32))} .$$

Eigenvalues of the matrix are given by $\lambda_1 = r - \frac{2rx_1'}{K} - \frac{ma}{\left(x_1'+a\right)^2}$, $\lambda_2 = fcx_1' - e$. The point E₁$(x_1', 0)$ will be stable if $\lambda_1 < 0$ and $\lambda_2 < 0$, as eigenvalues are real. This further gives $r - \frac{2rx_1'}{K} - \frac{ma}{\left(x_1'+a\right)^2} < 0$ and $fcx_1' - e < 0$. On solving the inequation $r - \frac{2rx_1'}{K} - \frac{ma}{\left(x_1'+a\right)^2} < 0$, we have $0 < m < m_w$, (see Theorem 2.6). Now the inequality $fcx_1' - e < 0$, gives, $\left(1 - \frac{a}{K}\right) + \sqrt{\left(1 + \frac{a}{K}\right)^2 - \frac{4m}{Kr}} < \frac{2e}{Kfc}$. This on further solving gives, $\left(1 + \frac{a}{K}\right)^2 - \frac{4m}{Kr} < \left(\frac{2e}{Kfc} + \frac{a}{K} - 1\right)^2$. On simplification, we can see

$$m > \frac{r}{fc}\left(1 - \frac{e}{fcK}\right)(\ e + afc) = m_{co}. \tag{3.33}$$

Hence predator-free equilibrium point E₁$(\ x_1', 0)$ will be asymptotically locally stable if $0 < m < m_w$ and $m > m_{co}$(see Eq. (3.28)). Combining the both obtained results, it is concluded that point E₁$(x_1', 0)$ will exhibit asymptotically locally stable behavior whenever $m_{co} < m < m_w$.

Theorem 3.6 The predator free equilibria $\mathrm{E}_2(x_1'', 0)$ is always unstable.

Proof. Substituting the point $\mathrm{E}_2(x_1'', 0)$ in $\mathbb{J}\ (h, g)$, the value of the Jacobian matrix is given as,

$$\mathbb{J}\ (x_1'', 0) = \begin{bmatrix} r - \dfrac{2rx_1''}{K} - \dfrac{ma}{(x_1'' + a)^2} & -fx_1'' \\ 0 & fcx_1'' - e \end{bmatrix} \left(\text{see Eq. (3.32)}\right).$$

Eigenvalues of the matrix are given by $\lambda_1 = r - \frac{2rx_1''}{K} - \frac{ma}{\left(x_1''+a\right)^2}$, $\lambda_2 = fcx_1'' - e$. The point $\mathrm{E}_2(x_1'', 0)$ will be stable if $\lambda_1 < 0$ and $\lambda_2 < 0$. This gives $r - \frac{2rx_1''}{K} - \frac{ma}{\left(x_1'+a\right)^2} < 0$ and $fcx_1'' - e < 0$. On solving the inequality $r - \frac{2rx_1''}{K} - \frac{ma}{\left(x_1''+a\right)^2} < 0$, it is found that $0 < m < m_w$ (see Theorem 2.7). This is contrary to the existence condition of $(x_1'', 0)$ which is given by $m \geq m_w$. Therefore, the point $E_2(x_1'', 0)$ is always unstable.

Theorem 3.7 If any of the following conditions are satisfied, the coexistence point, E_3 (x_{1s}, x_{2s}) will be locally asymptotically stable.

i. $T^2 - 4\Delta \geq 0$, and $T < 0$.
ii. $T^2 - 4\Delta < 0$, and $T < 0$.
iii. $T^2 - 4\Delta < 0, T > 0$, and $0 < \alpha < \frac{2}{\pi}\tan^{-1}\left|\frac{\sqrt{4\,\Delta - T^2}}{T}\right|$.
iv. $T = 0$, and $\Delta > 0$.

Where, $T = \frac{mecf}{(e+afc)^2} - \frac{re}{Kcf}$ and $\Delta = e\left[r - \frac{re}{cfK} - \frac{mcf}{e+afc}\right]$.

Proof. The value of the Jacobian matrix at E_3 (x_{1s}, x_{2s}) is as given below.

$$\mathbb{J}\,(x_{1s}, x_{2s}) = \begin{bmatrix} r - \dfrac{2rx_{1s}}{K} - \dfrac{ma}{(x_{1s}+a)^2} - fx_{2s} & -fx_{1s} \\ fcx_{2s} & fcx_{1s} - e \end{bmatrix} \text{(see Eq. (3.32))}$$

.

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

$$a_{11} = r - \frac{2rx_{1s}}{K} - \frac{ma}{(x_{1s}+a)^2} - fx_{2s}, \qquad a_{12} = -fx_{1s},$$
$$a_{21} = fcx_{2s}, \qquad a_{22} = fcx_{1s} - e = 0.$$

Now characteristic equation of $\mathbb{J}\,(x_{1s}, x_{2s})$ can be written as,

$\lambda^2 - T\lambda + \Delta = 0.$

Here

$$T = a_{11} + a_{22} = a_{11} = r - \frac{2rx_{1s}}{K} - \frac{ma}{(x_{1s}+a)^2} - fx_{2s} = \frac{mecf}{(e+afc)^2} - \frac{re}{Kcf}$$

$$\Delta = a_{11}a_{22} - a_{12}a_{21} = -\,a_{12}a_{21} = -\,f^2c\,x_{1s}\,x_{2s} = e\left[r - \frac{re}{cfK} - \frac{mcf}{e+afc}\right].$$

Eigenvalues of the Jacobian matrix are given by

$$\lambda_i = \frac{T \pm \sqrt{T^2 - 4\,\Delta}}{2}\ , i = 1, 2.$$

Case 1. If $T^2 - 4\Delta \geq 0$

i. If $T > 0$. Here both eigenvalues will be positive reals and hence $|\arg(\lambda_1)| = |\arg(\lambda_2)| = 0 < \alpha\,\frac{\pi}{2}$. Therefore E3 (x_{1s}, x_{2s}) will be unstable.

ii. If $T < 0$, then both eigenvalues will be negative reals, and hence $|\arg(\lambda_1)| = |\arg(\lambda_1)| = \pi > \alpha\,\frac{\pi}{2}$. Therefore E3 (x_{1s}, x_{2s}) will be locally stable.

Case 2. If $T^2 - 4\Delta < 0$

i. If $T > 0$. Then $\lambda_1 = \frac{T + i\sqrt{4\,\Delta - T^2}}{2}$, $\lambda_2 = \frac{T - i\sqrt{4\,\Delta - T^2}}{2}$ will be complex conjugates of one another, with their real parts positive. Therefore, $\left|\arg(\lambda_{1,2})\right| = \left|\tan^{-1}(\frac{\sqrt{4\,\Delta - T^2}}{T})\right| = \tan^{-1}\left|\frac{\sqrt{4\,\Delta - T^2}}{T}\right|$. If it is possible to find the value of fractional order α, so that $\tan^{-1}\left|\frac{\sqrt{4\,\Delta - \gamma^2}}{T}\right| > \alpha\frac{\pi}{2}$, only then E3 (x_{1s}, x_{2s}) will be stable. In other words, if it is possible to find $0 < \alpha < \frac{2}{\pi}\tan^{-1}\left|\frac{\sqrt{4\,\Delta - T^2}}{T}\right|$, the equilibrium point will be stable where $\alpha \in (0,1]$. .

ii. If $T < 0$. Here again, both eigenvalues will be complex conjugates of each other with negative real parts. Here,

$$\left|\arg(\lambda_{1,2})\right| = \left|-\pi + \tan^{-1}(\frac{\sqrt{4\,\Delta - T^2}}{T})\right| = \left|\pi - \tan^{-1}(\frac{\sqrt{4\,\Delta - T^2}}{T})\right| > \alpha\,\frac{\pi}{2}.$$

Hence, equilibrium point E3 (x_{1s}, x_{2s}) will be stable.

Case 3. If $T = 0$, then $\left(\lambda_{1,2}\right) = \frac{\pm\sqrt{-4\Delta}}{2} = \pm\sqrt{-\Delta}$.

i. Now if $\Delta > 0$. Then λ_1 and λ_2 will be the complex conjugate of each other and $\left|\arg(\lambda_{1,2})\right| = \frac{\pi}{2} > \alpha\,\frac{\pi}{2}$, the equilibrium point will be stable.

ii. If $\Delta < 0$, then λ_1 and λ_2 will be real numbers with one of the eigenvalues positive, say $\lambda_1 > 0$ and $|\arg(\lambda_1)| = 0 < \alpha\,\frac{\pi}{2}$. So, the equilibrium point E3 (x_{1s}, x_{2s}) will be unstable.

3.3.6 Bifurcation analysis

Theorem 3.8 If $T^2 - 4\Delta < 0$, $T > 0$, and when the fractional order α passes its critical value, which is given by $\alpha_0 = \frac{2}{\pi}\tan^{-1}\left|\frac{\sqrt{4\,\Delta - T^2}}{T}\right|$, the given fractional order system shows Hopf bifurcation around coexistence equilibria E$_3$ (x_{1s}, x_{2s}).

Proof. Following case 2 of the Theorem 3.7, the eigenvalues of the system (3) at E$_3$ (x_{1s}, x_{2s}) are given by,

$$\lambda_i = \frac{T \pm i\sqrt{4\,\Delta - T^2}}{2} = \frac{T}{2} \pm i\,\frac{\sqrt{4\,\Delta - T^2}}{2} = u \pm i\,v.$$

Here $u = \frac{T}{2} > 0,\ \ v = \frac{\sqrt{4\,\Delta - T^2}}{2}$.

Therefore, we have two eigenvalues that are complex conjugates of each other with positive real parts.

Now $\min\left|\arg(\lambda_{1,2})\right| = \tan^{-1}\left(\frac{v}{u}\right)$.

Now consider a function

$$\mathrm{F}(\alpha) = \frac{\pi\alpha}{2} - \min\left|\arg(\lambda_{1,2})\right| = \frac{\pi\alpha}{2} - \tan^{-1}\left(\frac{v}{u}\right).$$

Now $\mathrm{F}(\alpha) = 0$ gives $\alpha = \frac{2}{\pi}\tan^{-1}\left(\frac{v}{u}\right)$, and $\mathrm{F}'\left(\frac{2}{\pi}\tan^{-1}\left(\frac{v}{u}\right)\right) = \frac{\pi}{2} \neq 0$.

Therefore, when α crosses the critical value $\frac{2}{\pi}\tan^{-1}\left(\frac{v}{u}\right)$, the given fractional-order system will show a Hopf bifurcation near the coexistence point.

3.3.7 Examination of the Equilibrium Points for Global Stability

Theorem 3.9 If $r - \frac{m}{\mathbb{M}+a} + \frac{r\,x_{1s}}{K} - f x_{2s} < 0$, and $\frac{e}{c}x_{2s} + \frac{m x_{1s}}{a} - r\,x_{1s} < 0$, the equilibrium point E$_3$ (x_{1s}, x_{2s}) is globally asymptotically stable, where $\max\{|x_1|, |x_2|\} \leq \mathbb{M}$.

Proof. Consider the positive definite function

$$V(x_1,x_2) = \left(x_1 - x_{1s} - x_{1s}\ln\frac{x_1}{x_{1s}} \right) + \frac{1}{c}\left(x_2 - x_{2s} - x_{2s}\ln\frac{x_2}{x_{2s}} \right). \qquad (3.34)$$

$$D^\alpha V(x_1,x_2) = D^\alpha \left[\left(x_1 - x_{1s} - x_{1s}\ln\frac{x_1}{x_{1s}} \right) + \frac{1}{c}\left(x_2 - x_{2s} - x_{2s}\ln\frac{x_2}{x_{2s}} \right)\right], \qquad (3.35)$$

$$= D^\alpha\left(x_1 - x_{1s} - x_{1s}\ln\frac{x_1}{x_{1s}} \right) + \frac{1}{c}\, D^\alpha\left(x_2 - x_{2s} - x_{2s}\ln\frac{x_2}{x_{2s}} \right),$$

Using Eq. (1.9)

$$D^\alpha V(x_1,x_2) \le \left(\frac{x_1 - x_{1s}}{x_1} \right) D^\alpha x_1 + \frac{1}{c}\left(\frac{x_2 - x_{2s}}{x_2} \right) D^\alpha x_2, \qquad (3.36)$$

$$= (x_1 - x_{1s})\left[r\left(1 - \frac{x_1}{k}\right) - \frac{m}{x_1 + a} - f\,x_2 \right] + \frac{1}{c}\,(x_2 - x_{2s})\,(fcx_1 - e),$$

$$= rx_1 - \frac{rx_1^2}{K} - \frac{mx_1}{x_1 + a} - fx_1x_2 - r\,x_{1s} + \frac{r\,x_1x_{1s}}{K} + \frac{mx_{1s}}{x_1 + a} + f\,x_{1s}x_2$$
$$+ \frac{1}{c}\,(fcx_1x_2 - ex_2 - fcx_1x_{2s} + e\,x_{2s}),$$

$$\le rx_1 - \frac{mx_1}{x_1 + a} - r\,x_{1s} + \frac{r\,x_1x_{1s}}{K} + \frac{mx_{1s}}{x_1 + a} + f\,x_{1s}x_2$$
$$+ \frac{1}{c}\,(-ex_2 - fcx_1x_{2s} + e\,x_{2s}),$$

$$\le rx_1 - \frac{mx_1}{\mathbb{M} + a} - r\,x_{1s} + \frac{r\,x_1x_{1s}}{K} + \frac{mx_{1s}}{a} + f\,x_{1s}x_2 - \frac{e}{c}x_2 - fx_1x_{2s} + \frac{e}{c}x_{2s},$$

$$= x_1\left(r - \frac{m}{\mathbb{M} + a} + \frac{r\,x_{1s}}{K} - fx_{2s} \right) + x_2\left(fx_{1s} - \frac{e}{c} \right) + \frac{e}{c}x_{2s} + \frac{mx_{1s}}{a} - r\,x_{1s}.$$

Using $x_{1s} = \frac{e}{cf}$, we have

$$D^\alpha V(x_1,x_2) \le x_1\left(r - \frac{m}{\mathbb{M}+a} + \frac{r\,x_{1s}}{K} - fx_{2s} \right) + \frac{e}{c}x_{2s} + \frac{mx_{1s}}{a} - r\,x_{1s} \le 0$$

if $r - \frac{m}{\mathbb{M}+a} + \frac{r\,x_{1s}}{K} - fx_{2s} < 0$, $\frac{e}{c}x_{2s} + \frac{mx_{1s}}{a} - r\,x_{1s} < 0$, then V becomes a positive definite function (Lyapunov function), and therefore, the point (x_{1s}, x_{2s}) will be globally stable.

Theorem 3.10 If $r - \frac{m}{\mathbb{M}+a} + \frac{r\,x_1'}{K} < 0$, $f\,x_1' - \frac{e}{c} < 0$ and $\frac{mx_1'}{a} - rx_1' < 0$, then equilibrium point $(x_1', 0)$ will be globally asymptotically stable where max $\{|x_1|\} \leq \mathbb{M}$.

Proof: Consider the positive definite function

$$V'(x_1, x_2) = \left(x_1 - x_1' - x_1' \ln \frac{x_1}{x_1'} \right) + \frac{1}{c}\, x_2. \tag{3.37}$$

$$D^\alpha V'(x_1, x_2) = D^\alpha \left[\left(x_1 - x_1' - x_1' \ln \frac{x_1}{x_1'} \right) + \frac{1}{c}\, x_2 \right], \tag{3.38}$$

$$= D^\alpha \left(x_1 - x_1' - x_1' \ln \frac{x_1}{x_1'} \right) + \frac{1}{c}\, D^\alpha\, x_2, \tag{3.39}$$

$$\leq \left(\frac{x_1 - x_1'}{x_1} \right) D^\alpha x_1 + \frac{1}{c}\, D^\alpha\, x_2, \text{(using Eq. (1.9))}$$

$$= (x_1 - x_1') \left[r\left(1 - \frac{x_1}{k}\right) - \frac{m}{x_1 + a} - f\, x_2 \right] + \frac{1}{c}\, (f\, c\, x_1 x_2 - e\, x_2), \tag{3.40}$$

$$= rx_1 - \frac{rx_1^2}{K} - \frac{mx_1}{x_1 + a} - fx_1x_2 - r\,x_1' + \frac{r\,x_1x_1'}{K} + \frac{mx_1'}{x_1 + a} + f\,x_1'x_2 + fx_1x_2 - \frac{e}{c}\, x_2,$$

$$\leq rx_1 - \frac{mx_1}{x_1 + a} - rx_1' + \frac{r\,x_1x_1'}{K} + \frac{mx_1'}{x_1 + a} + f\,x_1'x_2 - \frac{e}{c}\, x_2,$$

$$\leq rx_1 - \frac{mx_1}{\mathbb{M} + a} - rx_1' + \frac{r\,x_1x_1'}{K} + \frac{mx_1'}{a} + f\,x_1'x_2 - \frac{e}{c}\, x_2,$$

$$= x_1 \left(r - \frac{m}{\mathbb{M} + a} + \frac{r\,x_1'}{K} \right) + x_2 \left(f\,x_1' - \frac{e}{c} \right) + \frac{mx_1'}{a} - rx_1' \leq 0,$$

if $r - \frac{m}{\mathbb{M}+a} + \frac{r\,x_1'}{K} < 0$, $f\,x_1' - \frac{e}{c} < 0$, $\frac{mx_1'}{a} - rx_1' < 0$, then V' becomes a positive definite function (Lyapunov function) and hence the point $(x_1', 0)$ is globally stable.

3.4 Numerical Analysis

Table 3.1: Table for Values of Parameters Used for Numerical Simulations

Parameters	**Value in Case 1**	**Value in Case 2**	**Value in Case 3**	**Value in Case 4**	**Source**
K	1	1	1	1	[51]
r	1	1	1	1	[51]
a	0.2	0.2	0.2	0.2	Assumed
m	0	0.1	0.25	0.37	Assumed
f	2,1, 0.5,0.1	2, 1, 0.1	2, 1, 0.5	2,1, 0.5,0.1	Assumed
c	0.5	0.5	0.5	0.5	[131]
e	0.1	0.1	0.1	0.1	[132]

The values obtained for Allee's constant (see Eq. (3.28))

$$m_w = 0.2, \qquad m_{cr} = 0.36$$

Case 1. When prey are not exposed to the Allee effect

By selecting $m = 0$, the prey population will not be experiencing the Allee effect. By selecting $f = 2$, the coexistence point obtained is E_3 (0.1,0.45). Numerical simulations are run by taking the initial population (0.2,0.5) in various fractional order models and integer order models of order 1, and Figure 3.1 is plotted, and it is found that populations are moving towards and stabilizing at E_3(0.1,0.45).

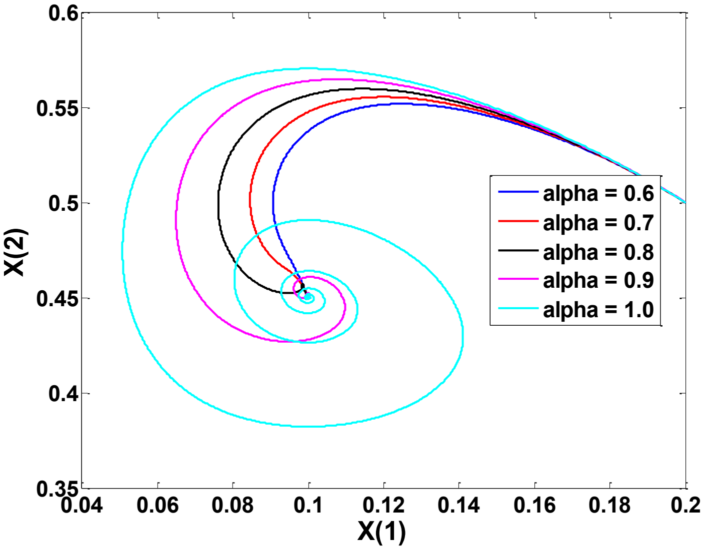


Figure 3.1 Stability analysis of the coexistence point (0.1,0.45) in various models without Allee effect

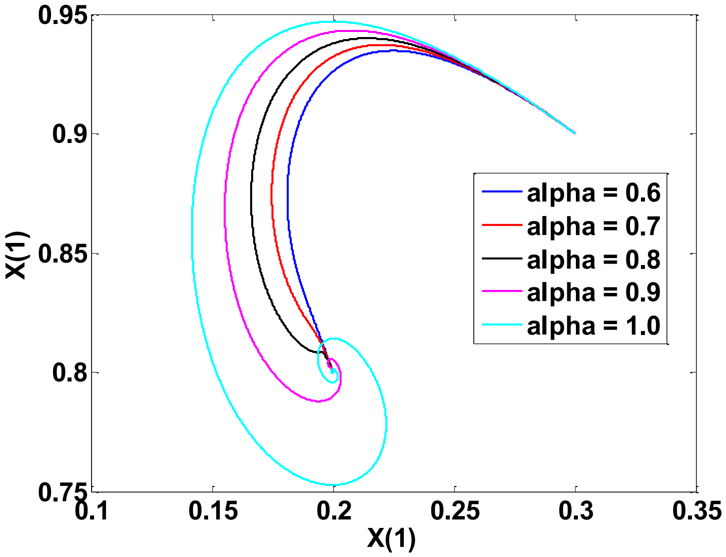


Figure 3.2 Stability analysis of the coexistence point (0.2,0.8) in various models without Allee effect

On taking $f = 1$, and keeping all other parameters the same as in the previous simulation, the value of the coexistence point obtained is E_3 (0.2,0.8). It is observed that the densities of the prey population as well as the predator population both rise. Figure 3.2 is plotted in fractional as well as in integer order model of order 1 by taking the initial population (0.3, 0.9) to show that populations are stabilizing at the coexistence point E_3 (0.2,0.8). On further decreasing the $f = 0.5$, the value of the coexistence point was found to be E_3 (0.4, 1.2). Figure 3.3 shows that populations are stable around the coexistence point in all models, integer order as well as in fractional order. On further decreasing the $f = 0.1$, it is found that predators fail to exist, as the equilibrium point obtained is E_1 (1,0). Figure 3.4 shows the stability of the predator-free point in various fractional order models and integer order models.

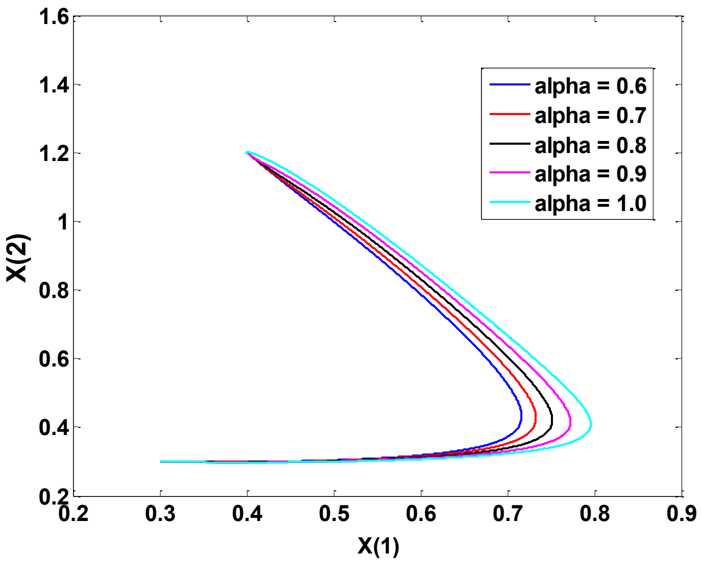


Figure 3.3 Stability analysis of the coexistence point (0.4, 1.2) in various models without Allee effect

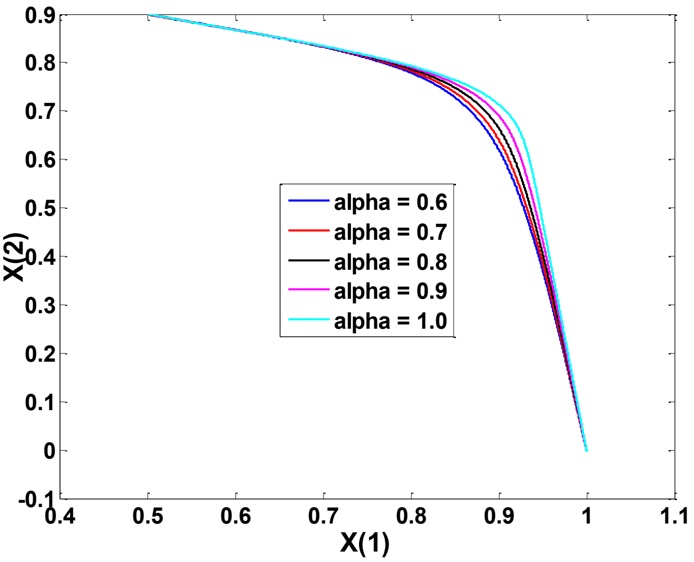


Figure 3.4 Stability analysis of the axial point (1, 0) in various models without Allee effect

Case 2. When Allee's constant $m < m_w$ (see Eq. (3.28))

By selecting $m = 0.1$, prey are exposed to the weak Allee effect. By selecting $f = 2$, the coexistence point obtained is $E_3(0.1,0.3)$. Various values to check stability are given below.

$$T = \frac{1}{90} > 0, \Delta = \frac{17}{300}, T^2 - 4\,\Delta = -\frac{1835}{8100} < 0.$$

By Theorem 3.7 (iii), the critical value of fractional order was found to be $\alpha = \frac{2}{\pi}\tan^{-1}\left|\frac{\sqrt{4\,\Delta - T^2}}{T}\right| = 0.985$. Figures 3.5, 3.6, 3.7, and 3.8 are plotted by taking alpha = 0.97, 0.98, 0.99, and 1respectively, with initial populations (0.2, 0.5).

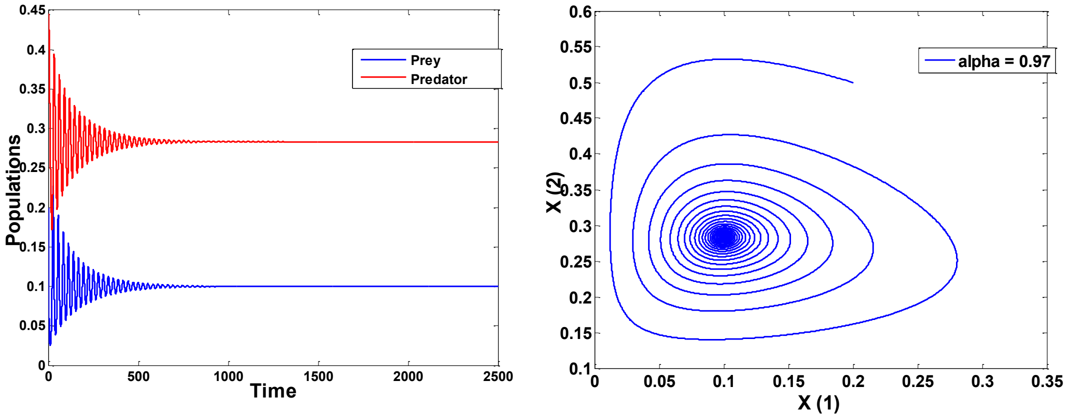


Figure 3.5 Stability analysis of the coexistence point (0.1,0.3) in fractional model with fractional order alpha = 0.97

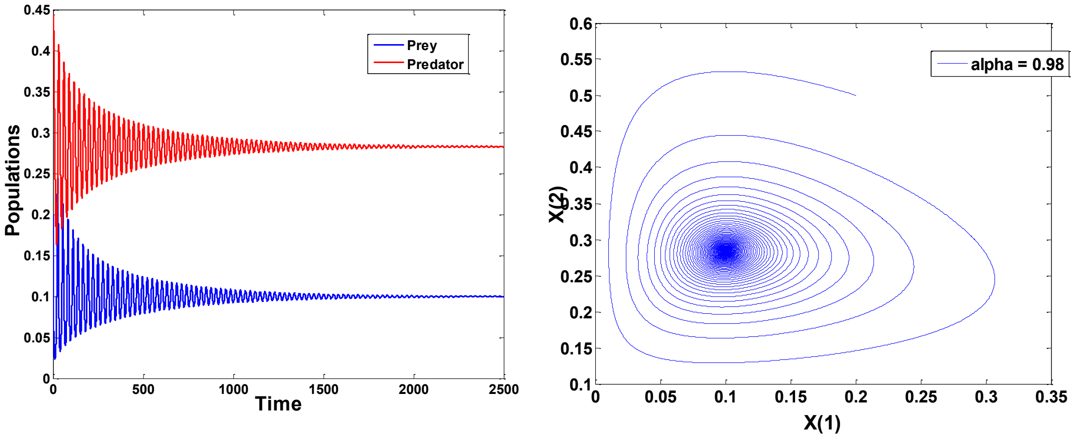


Figure 3.6 Stability analysis of the coexistence point (0.1,0.3) in fractional model with fractional order alpha = 0.98

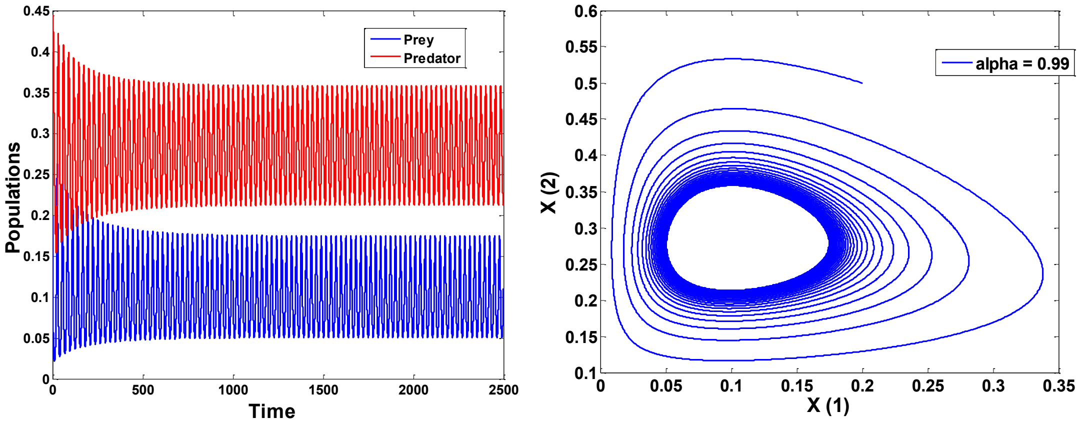


Figure 3.7 Stability analysis of the coexistence point (0.1,0.3) in fractional model with fractional order alpha = 0.98

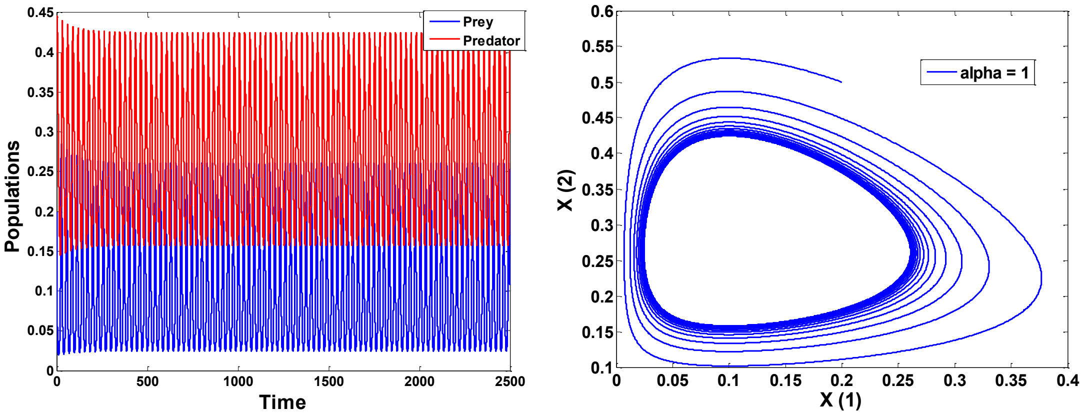


Figure 3.8 Stability analysis of the coexistence point (0.1,0.3) in integer model

It is observed that when alpha is less than its critical value of 0.985, the populations stabilize (Figs. 3.5,3.6) after fluctuating. As alpha crosses 0.985, populations start fluctuating and Hopf bifurcation appears (Fig. 3.7). In the integer order model size of fluctuations increases (Fig. 3.8).

On decreasing the level of $f = 1$, and keeping all other parameters the same the value of the coexistence point was found to be E3 (0.2,0.55). It is observed that prey density as well as predator density increase.

$T = -\frac{3}{40} > 0, \Delta = 0.055, T^2 - 4\,\Delta = -0.214375 < 0.$

Therefore, by Theorem 3.7, conditions for the stability of the point E3 (0.2,0.55) are satisfied. Figure 3.9 is plotted to show the stability of obtained equilibrium point E3 (0.2,0.55) in fractional models and integer model.

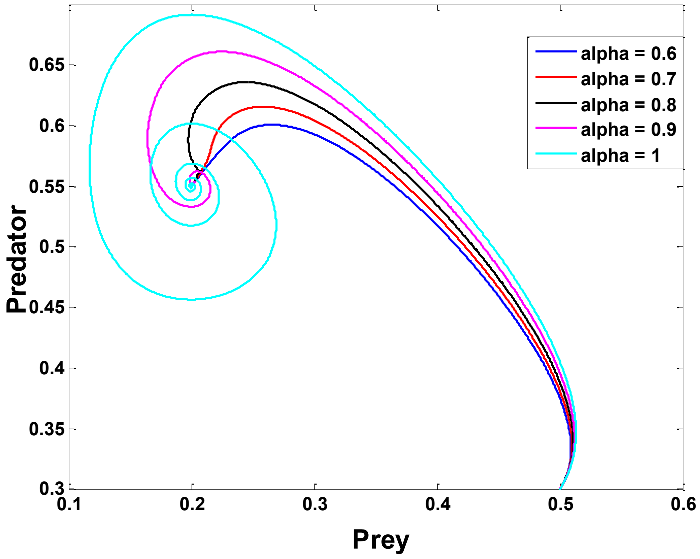


Figure 3.9 Stability analysis of the coexistence point (0.2,0.55) in different models

On further decreasing the $f = 0.1$, it is found that predators fail to exist, as existence conditions of the axial equilibrium point E$_1$ are satisfied. The equilibrium point obtained is E$_1$ (0.91,0). In this case value of $m_{co} = -2.2$ and $m = 0.1$ is considered, and therefore, by Theorem 3.5, stability conditions of equilibria E$_1$ are satisfied. Figure 3.10 shows the stable behavior of the predator-free point in various fractional-order and integer-order models.

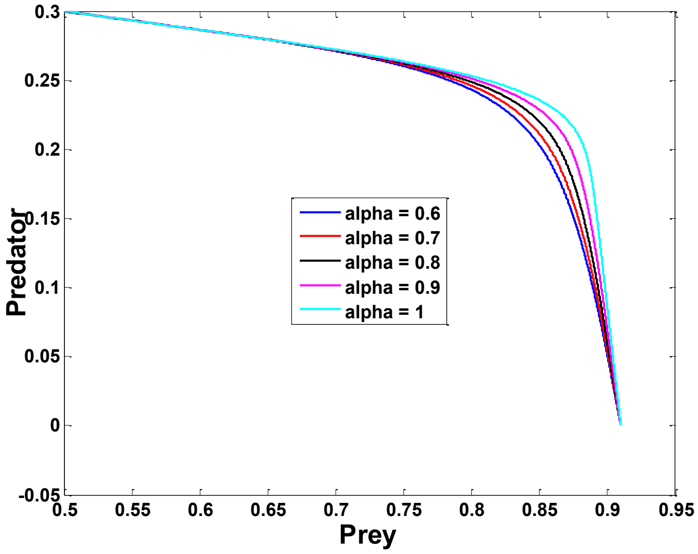


Figure 3.10 Stability analysis of the predator free point (0.91,0) in different models

Case 3. When Allee's constant $m_w < m < m_{cr}$ (See Eq. (3.28))

By selecting $m = 0.25$ the prey population is exposed to a strong Allee effect. By selecting $f = 2$, the coexistence point obtained is E$_3$(0.1,0.03). Mathematically, there is no change observed in the prey population as compared to the situation when it was experiencing a weak Allee effect; however, the predator population density significantly decreases. We obtain

$$T = -\frac{13}{180} < 0, \qquad \Delta = \frac{1}{150}, \quad T^2 - 4\,\Delta = -0.0214 < 0.$$

By Theorem 3.7, the point E3(0.1,0.03) satisfies all conditions of local stability. Figure 3.11 is plotted by taking different fractional orders and it is observed that in lower fractional order models' population stabilizes at different coexistence points. As fractional order takes the value of 0.6, and higher both populations fail to exist. However, if the attack rate is decreased to $f = 1$. The value of the coexistence point was found to be E3(0.2,0.175). We obtain

$$T = 0.1125 > 0, \Delta = 0.0175, T^2 - 4\,\Delta = -0.05734375 < 0\,,$$

$$\frac{2}{\pi}\tan^{-1}\left|\frac{\sqrt{4\,\Delta - T^2}}{T}\right| = 0.7204.$$

By Theorem 3.7 (iii), the critical value of fractional order was found to be $\alpha = 0.7204$. Figure 3.12 is drawn to show the behavior of populations at point E3 when the initial population (0.5,0.2). It is found that the populations stabilize at the coexistence point in the fractional model of order 0.72. As fractional order takes a value of 0.73, a Hopf bifurcation (Theorem 3.8) appears around the coexistence point, and populations fail to exist when the fractional order takes a value of 0.74.

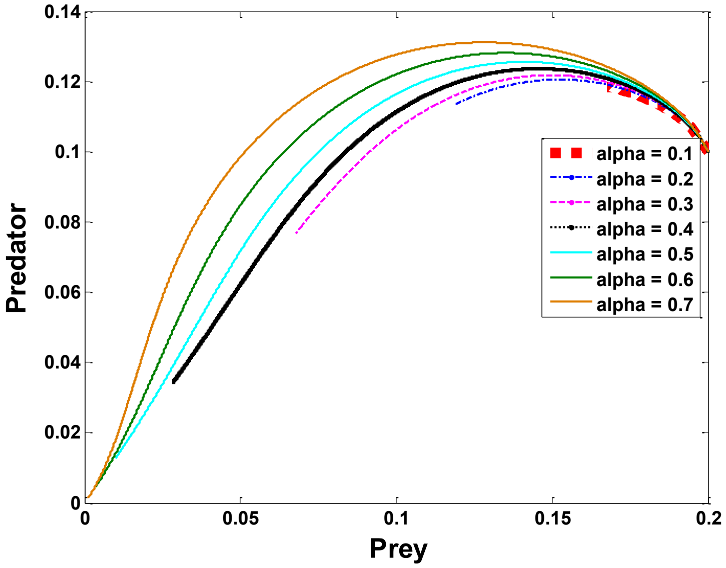


Figure 3.11 Stability analysis of the coexistence point (0.1,0.03) in different fractional models

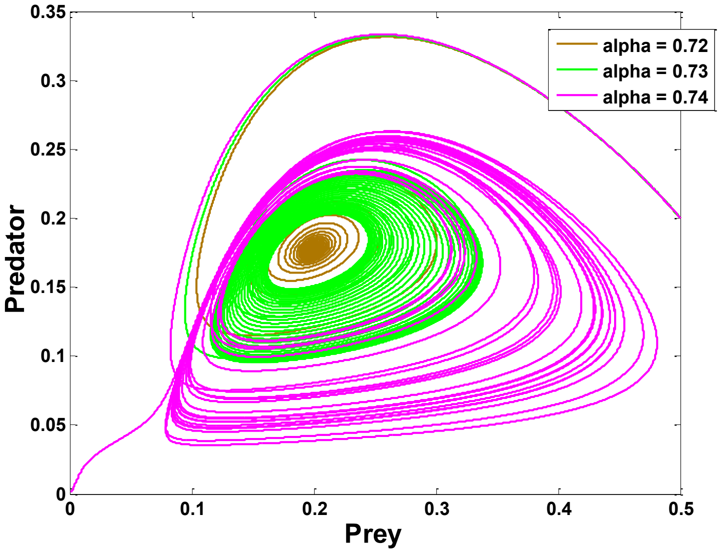


Figure 3.12 Stability analysis of the coexistence point (0.2,0.175) in different fractional model

To check the threshold level of the prey population simulation is run by taking the fractional model of order 0.73 with different initial populations, and it is found that for a given predator population of 0.2, the minimum prey population must be 0.5. Figure 3.13 is drawn to show the threshold level desired by the prey population.

On further decreasing the capture rate to $f = 0.5$, the value of the coexistence point was found to be E3 (0.4, 0.37). We obtain

$T = -\frac{1}{15} < 0, \Delta = \frac{11}{600}, T^2 - 4\,\Delta = -61\backslash 900 < 0.$

Therefore, the stability conditions of E3 are satisfied. Figure 3.14 shows that populations are stable at coexistence points in all models' integer order as well as fractional orders.

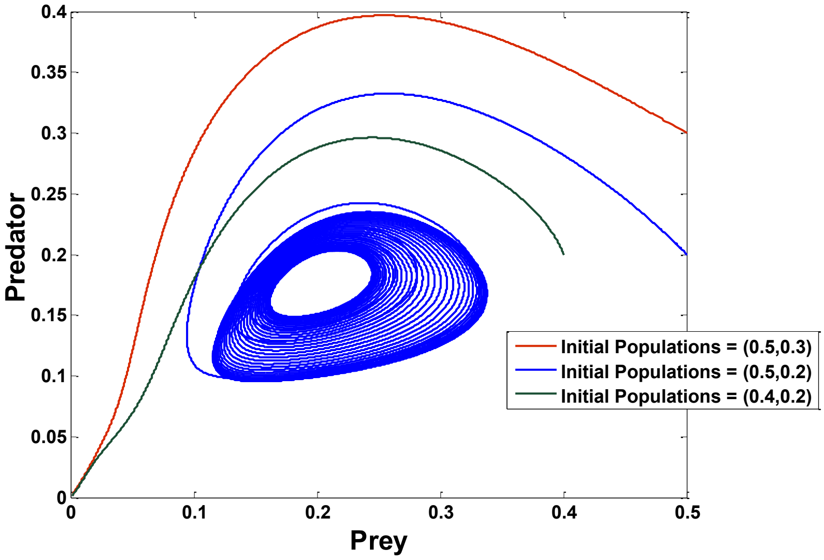


Figure 3.13 Stability analysis of the coexistence point (0.2,0.175) in fractional model of order 0.73 with different initial populations

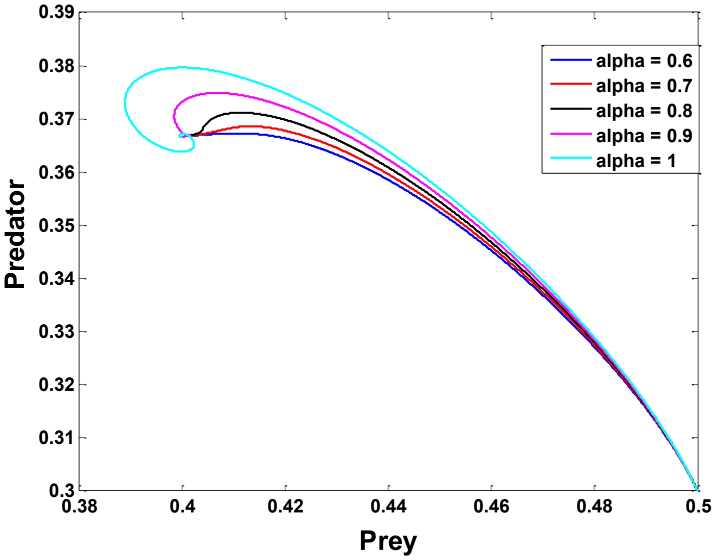


Figure 3.14 Stability analysis of the coexistence point (0.4,0.37) in different models

Case 4. When Allee's constant $m > m_{cr}$ (See Eq. (3.28))

By selecting $m = 0.37$ the prey population is exposed to the strong Allee effect of a very high degree. Stability conditions of $E_0(0,0)$ are satisfied by Theorem 3.4 Simulations are run by taking attack rates of predators $f = 2, 1, 0.5, 0.1$ as shown in Figures 3.15, 3.16, 3.17, and 3.18 respectively with an initial population (1,0.5), and it is found that even prey population is at carrying capacity initially, populations are unable to survive whatever rate of predation may be.

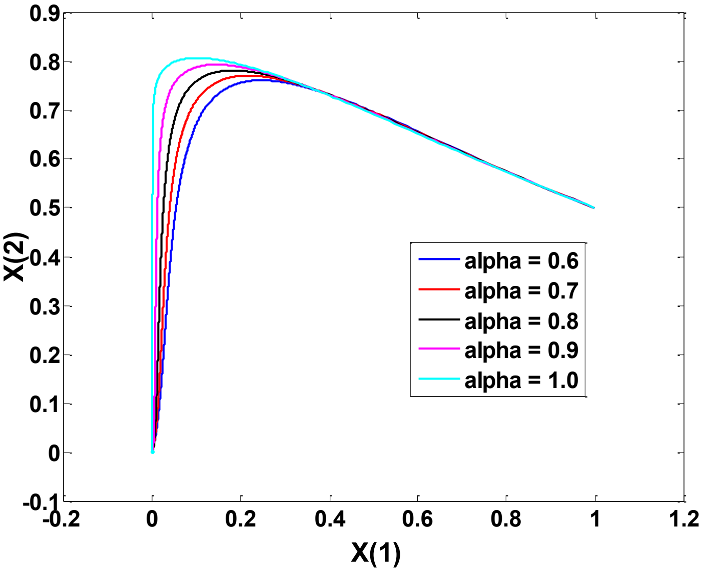


Figure 3.15 Behavior of the populations when prey population is experiencing Allee effect $m > m_{cr}$ and $f = 2$

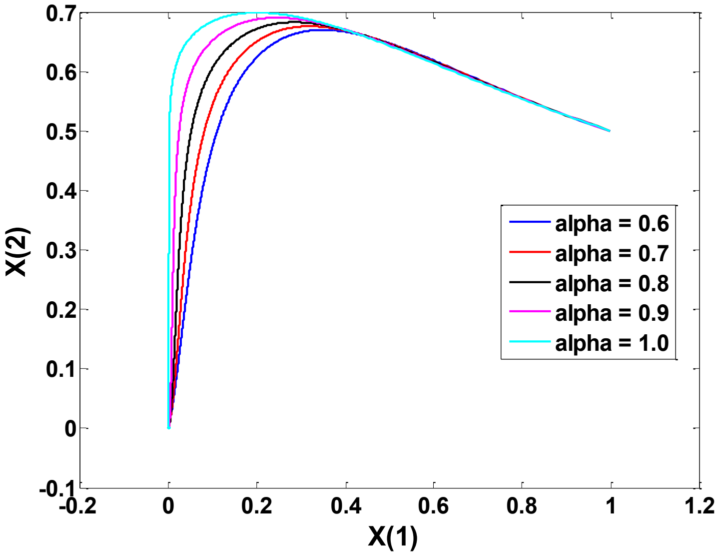


Figure 3.16 Behavior of the populations when prey population is experiencing Allee effect $m > m_{cr}$ and $f = 1$

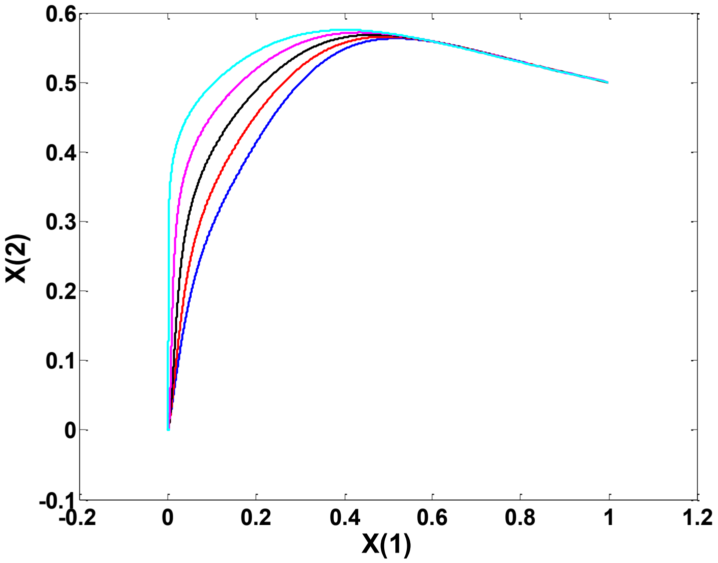


Figure 3.17 Behavior of the populations when prey population is experiencing Allee effect $m > m_{cr}$ and $f = 0.5$

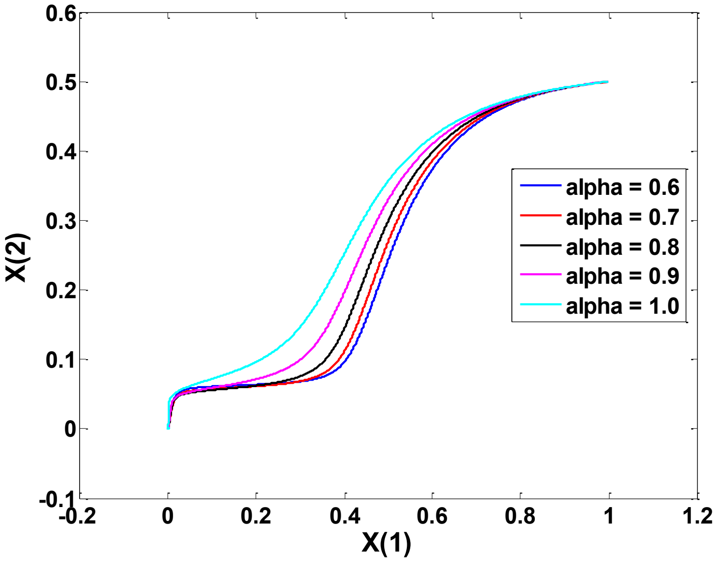


Figure 3.18 Behavior of the populations when prey population is experiencing Allee effect $m > m_{cr}$ and $f = 0.1$

3.5 Conclusion

This chapter analyses Caputo's definition-based fractional order Lotka-Volterra prey-predator model in which the prey population is growing logistically and is exposed to the additive Allee effect. All basic theorems, like the existence of unique, positive, and bounded solutions, are derived. Conditions required for the existence and local asymptotic stability of all steady state solutions are derived. Global stability of the predator-free and coexistence points is also discussed. It is observed that when the prey population is not experiencing the Allee effect, both populations can coexist stably; however, their densities depend upon the rate of predation (Figs. 3.1,3.2,3.3). However, when the rate of predation falls to a considerably low value, then predators fail to exist (Fig. 3.4) and the prey reaches its maximum population (carrying capacity). In case the prey population is exposed to a weak Allee effect along with a higher attack rate of predators, the populations can oscillate through Hopf bifurcation (Figs. 3.5,3.6,3.7,3.8). However, the optimal rate of predation helps in the coexistence of both populations stably (Fig. 3.9). A very low rate of predation can make predators extinct but the prey population never reaches to the carrying capacity (Fig. 3.10). It is also observed that when prey population is exposed to strong Allee effect, the suitably chosen attack rate of predators can help in stable coexistence of both populations (Fig. 3.14). However, when prey populations are subjected to a high rate of predation along with the strong Allee effect, both populations can coexist stably in fractional order models whenever the fractional order is less than the critical value (Fig., 3.12). It is also observed that in this situation requirement of the minimum population is felt by the prey to survive (Fig. 3.13). As Fig. 3.11 shows, lower fractional order models are converging to different coexistence points with the same values of parameters. Therefore, fractional order models provide early insight into the trends that populations will observe over time. Timely strategies or artificial strategies can help to save the population. Also, it is found that when prey populations are subjected to the weak Allee effect or strong Allee effect, or there is no Allee effect, the attack rate of predators plays a very important role in preserving both populations, but the Allee effect crosses the critical value m_{cr}, (see Eq. (3.28) extinction of populations cannot be prevented even through a low attack rate of predators (Figs. 3.15,3.16,3.17,3.18).

The Allee effect can make prey populations vulnerable to extinction, as their ability to recover from low densities is impaired. This can also make predator populations vulnerable to extinction as their food source becomes scarce. Therefore, the Allee effect can lead to unstable prey-predator systems, potentially causing oscillations or even the complete collapse of the system. Understanding the Allee effect can inform conservation and management strategies. By recognizing the potential for population collapse due to low densities, managers can implement interventions to support prey populations and mitigate the risk of extinction for both populations.

Chapter 4

Impact of Additive Allee Effect on Fractional Order Lotka-Volterra Competition Model

4.1 Introduction

A competition is an interaction between two or more members of the same species or between two or more species where one species adversely affects the other's access to a limited resource. Competition is classified into two types: intraspecific competition and interspecific competition. Intraspecific competition is among individuals from the same species. Competition among the individuals of the same species has negative as well as positive impacts. The negative impact is observed when a species in which all individuals consume the same amount of a limited resource till that resource is depleted, and all members of that population may die of starvation. And impact is positive if an individual competes and wins over a resource, and then it survives by exploiting that resource. Interspecific competition refers to the competition between individuals from different species. Competition between different populations affects both populations' fitness. Competition reduces access to resources and harms the population's reproductive success. When one population dominates over the availability of a limited resource entirely or partially, the availability of that resource for its competitor is reduced. This affects the persistence of the other species [134]. The mechanism to study how populations evolve through both space and time when subjected to such interactions is known as population dynamics. As discussed in Chapter 1, mathematical models are frequently used by ecologists to provide insight into population dynamics. Mathematical models are often presented in the form of differential equations, which describe how populations evolve through space and time or different phases of development. Lotka [124] and Volterra [125] in the year 1925 proposed for the first time a mathematical model depicting the relationship between two species competing for a limited resource, which is well known as the famous Lotka-Volterra competition model. Its model equations are given as,

$$\begin{cases} \dfrac{dx_1}{dt} = x_1\left(r_1\left(1 - \dfrac{(x_1 + a_{12}x_2)}{K_1}\right)\right) \\ \dfrac{dx_2}{dt} = x_2\left(r_2\left(1 - \dfrac{(x_2 + a_{21}x_1)}{K_2}\right)\right) \end{cases} \tag{4.1}$$

Here $x_1 > 0$ and $x_2 > 0$ represents two competing species with intrinsic growth rates r_1 and r_2 respectively. K_1 and K_2 are their maximum populations respectively which environment can support in the absence of competition. a_{12} represents the inhibiting impact that species x_2 has on the population growth of x_1, and a_{21} represents the inhibiting impact that species x_1 has on the population growth of x_2.

These equations are based on a few assumptions:

1. All competition constants, intrinsic growth rates, and carrying capacities are assumed to be constant.
2. Population diversification is not permitted.
3. Every individual of each category is identical.
4. The habitat is uniform.

These equations are solved under four different scenarios:

Scenario I: $\frac{K_2}{a_{21}} < K_1, \frac{K_1}{a_{12}} > K_2$

Scenario II: $\frac{K_2}{a_{21}} > K_1, \frac{K_1}{a_{12}} < K_2$

Scenario III: $\frac{K_2}{a_{21}} < K_1, \frac{K_1}{a_{12}} < K_2$

Scenario IV: $\frac{K_2}{a_{21}} > K_1, \frac{K_1}{a_{12}} > K_2$

In scenario I, the species (x_1) competitively exclude the second species (x_2). Scenario II is exactly opposite to the previous one, where the species (x_2) will competitively exclude the first species (x_1). In scenario III, the coexistence of both species is possible when the growth rate of both species becomes zero. However, this coexistence is unstable, as any environmental changes may trigger the extinction of one species. More explicitly, if the species x_1 reaches its carrying capacity K_1 and $K_1 > K_2/a_{21}$, the species x_2 will move towards extinction, and if x_2 touch its carrying capacity K_2 and $K_2 > K_1/a_{12}$, the species $x_1$will move towards extinction. In the last scenario IV, both species reach a coexistence point when the population growth rate becomes zero, and both populations coexist stably regardless of the initial population size of both populations. In this circumstance, the intraspecific competition feels more potent than the interspecific competition [134].

Despite being simple, the Lotka-Volterra competition models stand out as a crucial tool for studying the coexistence of interacting species. There are various published studies on the population dynamics where the Lotka Volterra competition model turned out quite helpful, for example, in clarifying the dynamics and assessing coexistence in traditional two-species systems [135,136], population dynamics of river ecology [137], global dynamics with asymmetric dispersal of the two-species patch model [138], coexistence of multiple species with crowding effects [139], to predict the yields in multispecies [140], global competition among diverse taxa [141], in studying plant interspecific interactions to predict optimal combinations [142], in investigating the coexistence of diversified natural groups with size structure [143]. In the literature survey of fractional modeling in population dynamics, Zibaei and Namjoo investigated a discrete fractional order Lotka-Volterra competition model using a non-standard finite difference (NSFD) scheme [144], and Hassani et al. explored a Caputo fractional derivative-based breast cancer competition model [145]. After conducting a thorough literature review, we observed that the Lotka-Volterra competition model is not explored much using fractional derivatives. Also, there are published studies on the Allee effect using fractional order derivatives [102,103,131,132,133] in population dynamics, but the Lotka-Volterra competition model incorporating the additive Allee effect is still unexplored. Motivated by the studies on population dynamics using fractional order derivatives and the importance of the traditional Lotka Volterra competition model in various fields of population ecology and the Allee effect, we are studying in this chapter the classical Lotka Volterra competition model when one of the competing species is exposed to additive Allee effect by using fractional order derivative in Caputo sense.

4.2 Mathematical model

Lotka-Volterra's competition model, by taking fractional order derivatives, can be written as,

$$\begin{cases} D^{\alpha}x_1 = x_1\left[r_1\left(1 - \dfrac{(x_1 + a_{12}x_2)}{K_1}\right)\right], \\ D^{\alpha}x_2 = x_2\left[r_2\left(1 - \dfrac{(x_2 + a_{21}x_1)}{K_2}\right)\right]. \end{cases} \tag{4.2}$$

Studying the Allee effect in population dynamics by introducing a negative factor $m/(x+a)$ in one of the species' growth equations is defined as the additive Allee effect. When one of the species in the model (4.2) say x_1 is subjected to the Allee effect additively, then the model equations can be written as,

$$\begin{cases} D^{\alpha}x_1 = x_1\left[r_1\left(1 - \dfrac{(x_1 + a_{12}x_2)}{K_1}\right) - \dfrac{m}{x_1 + a}\right], \\ D^{\alpha}x_2 = x_2\left[r_2\left(1 - \dfrac{(x_2 + a_{21}x_1)}{K_2}\right)\right], \end{cases} \tag{4.3}$$

with initial conditions $x_1(0) \geq 0, x_2(0) \geq 0$. Here $x_1 > 0$ and $x_2 > 0$ represents two competing species with intrinsic growth rates r_1 and r_2 respectively. K_1 and K_2 are their maximum populations respectively which environment can support in the absence of competition. a_{12} represents the inhibiting impact that species x_2 has on the population growth of x_1, and a_{21} represents the inhibiting impact that species x_1 has on the population growth of x_2. D^{α} is the Caputo fractional derivative and $\alpha \in (0,1]$.

4.3 Primary Evaluation of the Model

In this section, an analysis of the proposed model will be conducted to establish the properties of its solutions, including existence, uniqueness, positivity, and uniform boundedness.

4.3.1 Existence of Unique Solution

Theorem 4.1 The initial value problem $D^{\alpha}x_1(t) = h(x_1\, x_2, t)$, $D^{\alpha}x_2(t) = g(x_1\, x_2, t)$, where $h(x_1\, x_2, t) = x_1\left[r_1\left(1 - \frac{(x_1 + a_{12}x_2)}{K_1}\right) - \frac{m}{(x_1 + a)}\right]$, and $g(x_1\, x_2, t) = x_2\left[r_2\left(1 - \frac{(x_2 + a_{21}x_1)}{K_2}\right)\right]$, with initial condition $x_1(t = t_0) = x_{1_0} > 0$, $x_2(t = t_0) = x_{2_0} > 0$ and $f : I \times D \to \mathbb{R}^2$; $\alpha \in (0,1]$ have a unique solution $X(x_1(t), x_2(t)) \in D$, with initial condition $X(x_1(t_0), x_2(t_0)) = X_{t_0}$. Here $f \in C\,(I)$, where $C\,(I)$ is a class of continuously differentiable functions and $I = [\,t_0,\, T\,]$, $T < \infty$ and $D = (x_1, x_2) \in \mathbb{R}^2 : |x_1|, |x_2| \le M\,\}$. M is a positive constant.

Proof. Consider a mapping $F : D \to \mathbb{R}^2$ by $F(X) = \left(F_1(X), F_2(X)\right)$, where $X = (x_1, x_2)$, $Y = (y_1, y_2)$, $F_1(X) = h(x_1\, x_2, t)$, and $F_2(X) = g(x_1\, x_2, t)$.

Define $\|F(x)\| = \sup_{t} |e^{-St}\, F(x)|$, $S > 0$. $t \in [\,t_0,\, T\,]$, $T < \infty$.

Clearly $\|F(x)\| = \sup_{t} |F(x)|$.

Consider

$$\|F(X) - F(Y)\| = |F_1(X) - F_1(Y)| + |F_2(X) - F_2(Y)|, \qquad (4.4)$$

$$= \left|\left[r_1\left(1 - \frac{(x_1 + a_{12}x_2)}{K_1}\right) - \frac{m}{(x_1 + a)}\right]x_1 - \left[r_1\left(1 - \frac{(y_1 + a_{12}y_2)}{K_1}\right) - \frac{m}{(y_1 + a)}\right]y_1\right| + \left|\left[r_2\left(1 - \frac{(x_2 + a_{21}x_1)}{K_2}\right)\right]x_2 - \left[r_2\left(1 - \frac{(y_2 + a_{21}y_1)}{K_2}\right)\right]y_2\right|,$$

$$
\begin{aligned}
&= \left|(x_1 - y_1)r_1 - \left(\frac{x_1}{K_1}\,(x_1 + a_{12}x_2) - \frac{y_1}{K_1}\,(y_1 + a_{12}y_2)\right)r_1\right. \\
&\qquad\qquad \left. - m\left(\frac{x_1}{x_1 + a} - \frac{y_1}{y_1 + a}\right)\right| \\
&\qquad\qquad + \left|(x_2 - y_2)r_2 - \left(\frac{x_2}{K_2}\,(x_2 + a_{21}x_1) - \frac{y_2}{K_2}\,(y_2 + a_{21}y_1)\right)r_2\right|,
\end{aligned}
$$

$$
\begin{aligned}
&= \left|(x_1 - y_1)r_1 - \frac{r_1}{K_1}(\,x_1^2 - y_1^2\,) - \frac{a_{12}r_1}{K_1}\,(x_1x_2 - y_1y_2) - m\frac{a(x_1 - y_1)}{(x_1 + a\,)(y_1 + a\,)}\right| \\
&\qquad\qquad + \left|(x_2 - y_2)r_2 - \frac{r_2}{K_2}(\,x_2^2 - y_2^2\,) - \frac{a_{21}r_2}{K_2}\,(x_1x_2 - y_1y_2)\right|,
\end{aligned}
$$

$$
\begin{aligned}
&= \left|(x_1 - y_1)r_1 - \frac{r_1}{K_1}(\,x_1^2 - y_1^2\,) - \frac{a_{12}r_1}{K_1}\,(x_1x_2 - x_1y_2 + x_1y_2 - y_1y_2)\right. \\
&\qquad\qquad \left. - m\frac{a(x_1 - y_1)}{(x_1 + a\,)(y_1 + a\,)}\right| \\
&\qquad\qquad + \left|(x_2 - y_2)r_2 - \frac{r_2}{K_2}(\,x_2^2 - y_2^2\,)\right. \\
&\qquad\qquad \left. - \frac{a_{21}r_2}{K_2}\,(x_1x_2 - x_1y_2 + x_1y_2 - y_1y_2)\right|,
\end{aligned}
$$

$$
\begin{aligned}
&\leq |(x_1 - y_1)r_1| + \left|\frac{r_1}{K_1}(\,x_1^2 - y_1^2\,)\right| + \left|\frac{a_{12}r_1}{K_1}\,(x_1x_2 - x_1y_2 + x_1y_2 - y_1y_2)\right| \\
&\qquad\qquad + \left|m\frac{a(x_1 - y_1)}{(x_1 + a\,)(y_1 + a\,)}\right| + |(x_2 - y_2)r_2\,| + \left|\frac{r_2}{K_2}(\,x_2^2 - y_2^2\,)\right| \\
&\qquad\qquad + \left|\frac{a_{21}r_2}{K_2}\,(x_1x_2 - x_1y_2 + x_1y_2 - y_1y_2)\right|,
\end{aligned}
$$

$$
\begin{aligned}
&= |(x_1 - y_1)r_1| + \frac{r_1}{K_1}\,|(x_1 + y_1)||(x_1 - y_1)| + \left|\frac{a_{12}r_1}{K_1}\right|\,|x_1(x_2 - y_2) + y_2(\,x_1 - y_1)| \\
&\qquad\qquad + ma\,\left|\frac{(x_1 - y_1)}{(x_1 + a\,)(y_1 + a\,)}\right| + |(x_2 - y_2)r_2\,| \\
&\qquad\qquad + \frac{r_2}{K_2}|(x_2 + y_2)||(x_2 - y_2)| + \left|\frac{a_{21}r_2}{K_2}\right|\,|x_1(x_2 - y_2) + y_2(\,x_1 - y_1)|,
\end{aligned}
$$

$$\leq r_1|(x_1-y_1)| + \frac{2Mr_1}{K_1}\,|(x_1-y_1)| + \frac{a_{12}r_1M}{K_1}\,(|(x_2-y_2)| + |(\,x_1-y_1)|)$$

$$+\, ma|(x_1-y_1)| + r_2|(x_2-y_2)| + \frac{2Mr_2}{K_2}\,|(x_2-y_2)|$$

$$+\, \frac{a_{21}r_2\,M}{K_2}\,(|(x_2-y_2)| + |(\,x_1-y_1)|),$$

$$= \ |(x_1-y_1)|\left[r_1 + ma\ +\ \frac{2Mr_1}{K_1} + \frac{a_{12}r_1M}{K_1} + \frac{a_{21}r_2\,M}{K_2}\right]$$

$$+\,|(x_2-y_2)|\left[r_2 + ma + \frac{2Mr_2}{K_2} + \frac{a_{21}r_2\,M}{K_2} + \frac{a_{12}r_1M}{K_1}\right],$$

$$= M_1|(x_1-y_1)| + M_2|(x_2-y_2)| \leq \mathbb{M}\|(x-y)\|. \tag{4.5}$$

Where,

$$M_1 = \ r_1 + ma\ +\ \frac{2Mr_1}{K_1} + \frac{a_{12}r_1M}{K_1} + \frac{a_{21}r_2\,M}{K_2}, \tag{4.6}$$

$$M_2 = \ \ r_2 + ma + \frac{2Mr_2}{K_2} + \frac{a_{21}r_2\,M}{K_2} + \frac{a_{12}r_1M}{K_1}, \tag{4.7}$$

and $\mathbb{M} = \max\{M_1, M_2\}$. Therefore, F satisfies the Lipschitz condition.

Let $G(X)$ be the solution of the given system. Then,

By Eq. (1.6)

$$G(X) = X -\ X(t_0) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,F\big(X(s)\big)\,ds. \tag{4.8}$$

Consider

$$G(X) -\ G(Y) = \frac{1}{\Gamma(\alpha)}\int_0^t \big\{F\big(X(s)\big) - F\big(Y(s)\big)\big\}\ \ (t-s)^{\alpha-1}\,ds.$$

$$|G(X) -\ G(Y)\,| = \left|\frac{1}{\Gamma(\alpha)}\int_0^t \big\{F\big(X(s)\big) - F\big(Y(s)\big)\big\}\ \ (t-s)^{\alpha-1}ds\right|,$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_0^t \big|F\big(X(s)\big) - F\big(Y(s)\big)\big|\,(t-s)^{\alpha-1}\,ds. \tag{4.9}$$

Now

$$e^{-pt}\left(G(X)-\ G(Y)\right)=\frac{1}{\Gamma(\alpha)}\int_0^t\{F\left(X(s)\right)-F\left(Y(s)\right)\}\ \ (t-s)^{\alpha-1}\,e^{-pt}ds,$$

$$=\frac{1}{\Gamma(\alpha)}\int_0^t\{F\left(X(s)\right)-F\left(Y(s)\right)\}\ \ (t-s)^{\alpha-1}e^{-p(t-s)}\,e^{-ps}\,ds. \tag{4.10}$$

$$\|G(X)-\ G(Y)\|\le\frac{1}{\Gamma(\alpha)}\int_0^t\|F\left(X(s)\right)-F\left(Y(s)\right)\|\ \ (t-s)^{\alpha-1}e^{-p(t-s)}ds,$$

$$\le\frac{\mathbb{M}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\ \ e^{-p(t-s)}\|x-y\|\,e^{-p(t-s)}\,ds,\ \ (\text{ By Eq. } (4.4))$$

$$=\frac{\mathbb{M}}{\Gamma(\alpha)}\|x-y\|\int_0^t e^{-p(t-s)}\ \ (t-s)^{\alpha-1}ds,$$

$$\le\frac{\mathbb{M}}{\Gamma(\alpha)}\|x-y\|\frac{\Gamma(\alpha)}{p^{\alpha}}=\frac{\mathbb{M}}{p^{\alpha}}\ \|x-y\|. \tag{4.11}$$

Choosing p sufficiently large so that $\frac{\mathbb{M}}{p^{\alpha}}<1$, then by Theorem (1.4.8), there exists a unique solution to the model given by (4.3).

4.3.2 Positivity of the Solutions

Theorem 4.2 Solutions to the given system (4.3) starting in $\mathbb{R}_+^2$ are non-negative, where $\mathbb{R}_+^2$ represents the set containing positive ordered reals, including zero.

Proof. Let $X(\,t_0)=(x_1(t_0),x_2(t_0))\in\mathbb{R}_+^2$ be the initial solution of system (4.3). Let $t>t_0$ and we claim that $X(t)\ \geq 0$ for all $t\geq t_0$. To establish the claim, we suppose that there exists some solution $X(t')$ which do not lie in $\mathbb{R}_+^2$ where $t'\ >t_0$. This implies that either $X(t)$ crosses $x_1$axis or x_2 axis.

Case 1. If the solution crosses x_2 axis, then we can find some t_* satisfying the conditions, $t_*\geq\ t_0$ and $x_1(t_*)=0$, and there is some t' is sufficiently close to t_* and $t'>t_*$, but $x_1(t)$ is less than zero for all $t_*<t\leq t'$.

Now there are two possibilities;

Subcase (i) If $D^{\alpha}\,x_1(t)<0\ \forall\ t\ \in(t_*,t']$.

This implies

$$x_1\left[r_1\left(1-\frac{(x_1+a_{12}x_2)}{K_1}\right)-\frac{m}{(x_1+a\,)}\right]\le 0. \tag{4.12}$$

But

$$\left[r_1\left(1-\frac{(x_1+a_{12}x_2)}{K_1}\right)-\frac{m}{(x_1+a)}\right]x_1>\ \rho x_1 \quad \forall\, t\ \in (t_*,t']. \tag{4.13}$$

Here ρ is the minimum value of

$$r_1\left(1-\frac{(x_1+a_{12}x_2)}{K_1}\right)-\frac{m}{(x_1+a)}\ \ \forall t\in(t_*,t'].$$

By applying the Laplace transform and using Eq. (1.7), we have

$$s^{\alpha}\,X_1(s)-s^{\alpha-1}\,x_1(t_0)\ge \rho\,X_1(s), \tag{4.14}$$

where, $X_1(s)=\ \mathcal{L}\{x_1(t)\}$. This gives

$$(s^{\alpha}\ -\ \rho\,)\,X_1(s)\ge s^{\alpha-1}\,x_1(t_0).$$

This implies

$$X_1(s)>\ \frac{s^{\alpha-1}\,x_1(t_0)}{(s^{\alpha}\ -\rho\,)}. \tag{4.15}$$

By taking the inverse Laplace transform and using Eq. (1.8), we have,

$$x_1(t)\ge\ \mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{(\,s^{\alpha}\ -\rho)}x_1(t_0)\right\}=x_1(t_0)\,E_{\alpha,1}\,\{\rho\,(\mathrm{t}-t_0)^{\alpha}\}. \tag{4.16}$$

From this, we have

$$x_1(t)\ge x_1(t_0)\,E_{\alpha,1}\,\{\rho\,(\mathrm{t}-t_0)^{\alpha}\}\ge 0. \tag{4.17}$$

Therefore, again, we have $x_1(t)\ \ge\ 0$, for all $t_*\ <\ t\ <\ t'$. This contradicts the assumption that $x_1(t)<0\ \forall\ t\ \in(t_*,t')$.

Subcase (ii) If $D^{\alpha}\,x_1(t)\ge 0\ \forall t\in(t_*,t']$.

Now

$$D^{\alpha}\,x_1(t)=\left[r_1\left(1-\frac{(x_1+a_{12}x_2)}{K_1}\right)-\frac{m}{(x_1+a\,)}\right]x_1\ge r_1x_1. \tag{4.18}$$

Therefore, we have

$$D^{\alpha}\, x_1(t) \ \geq x_1 r_1. \tag{4.19}$$

By taking the Laplace transform on both sides and proceeding as in subcase (i), we have

$$x_1(t) \geq x_1(t_0) E_{\alpha,1}\{r_1(\mathrm{t}-t_0)^{\alpha}\} \geq 0. \tag{4.20}$$

This contradicts our assumption that $x_1(t) < 0,$ for all $t \in (t_*, t']$.

Case 2. If the solution crosses x_1 axis, then proceeding as in case 1, we will arrive at a contradiction. Hence, all solutions start in $\mathbb{R}_+^2$ lies in $\mathbb{R}_+^2$.

4.3.3. Uniform Boundedness of Solutions

Theorem 4.3 All non–negative solutions to the system (4.3) are uniformly bounded.

Proof. By considering a function

$$F(t) = \frac{K_1}{r_1} x_1(t) + \frac{K_2}{r_2}\, x_2(t). \tag{4.21}$$

Taking Caputo's fractional derivative on both sides,

$$D^{\alpha} F(t) = \ D^{\alpha}\left[\frac{K_1}{r_1} x_1(t) + \frac{K_2}{r_2}\, x_2(t)\right] = \frac{K_1}{r_1}\ D^{\alpha} x_1(t) + \ \frac{K_2}{r_2}\, D^{\alpha} x_2(t), \tag{4.22}$$

$$= \ \frac{K_1}{r_1} x_1 \left[r_1 \left(1 - \frac{(x_1 + a_{12} x_2)}{K_1} \right) - \frac{m}{(x_1 + a\,)} \right] + \frac{K_2}{r_2}\, x_2 \left[r_2 \left(1 - \frac{(x_2 + a_{21} x_1)}{K_2} \right) \right],$$

$$= K_1 x_1 - x_1^2 - a_{12} x_1 x_2 - \frac{m K_1 x_1}{r_1 (x_1 + a)} + K_2 x_2 - x_2^2 - a_{21} x_1 x_2,$$

$$< K_1^2 + K_2^2 - \frac{m K_1 x_1}{r_1 (K_1 + a)} - x_2^2 = K_1^2 + K_2^2 - x_2^2 - \frac{m K_1}{r_1 (K_1 + a)} \left[\frac{r_1}{K_1} F - \frac{r_1 K_2}{r_2 K_1} x_2 \right].$$

Therefore,

$$D^{\alpha}F(t) + \frac{m}{(K_1 + a)}F \; < \; K_1^2 + K_2^2 - x_2^2 - \frac{mK_2}{r_2(K_1 + a)}x_2,$$

$$= \; K_1^2 + K_2^2 - \left(x_2^2 - \frac{mK_2}{r_2(K_1 + a)}x_2 + \frac{m^2K_2^2}{4r_2^2(K_2 + a)^2}\right) + \; \frac{m^2K_2^2}{4r_2^2(K_2 + a)^2},$$

$$= K_1^2 + K_2^2 \; + \; \frac{m^2K_2^2}{4r_2^2(K_2 + a)^2} - \; \left(x_2 - \frac{mK_2}{2r_2(K_1 + a)}\right)^2,$$

$$= \mathrm{A} - \; \left(x_2 - \frac{mK_2}{2r_2(K_1 + a)}\right)^2 \; < \; \mathrm{A}.$$

Where,

$$\mathrm{A} = \; K_1^2 + K_2^2 \; + \; \frac{m^2K_2^2}{4r_2^2(K_2 + a)^2} \; .$$

Therefore,

$$D^{\alpha}F(t) + \frac{m}{(K_1 + a)}F \; < \; \mathrm{A}.$$

$$D^{\alpha}F(t) + fF < \; \mathrm{A}\,. \tag{4.23}$$

Where,

$$f = \frac{m}{(K_1 + a)}.$$

By taking the Laplace transform on both sides of (4.22) and using Eq. (1.7), we have

$$s^{\alpha}\, G(s) - s^{\alpha-1}\, F(t_0) + \; fG(s) \; \leq \; \frac{\mathrm{A}}{s}.$$

Where, $G(s) = \; \mathcal{L}\{F(t)\}$. This gives

$$(\, s^{\alpha} + \, f\,)G(s) \; \leq \; \frac{\mathrm{A}}{s} + \; s^{\alpha-1}\, F(t_0).$$

This implies

$$G(s) \; \leq \; \frac{A}{s\,(\, s^{\alpha} + \, f\,)} + \; \frac{s^{\alpha-1}\, F(t_0)}{(\, s^{\alpha} + \, f)} \; = \; \frac{s^{\alpha-1}\, F(t_0)}{(\, s^{\alpha} + \, f)} + \; \frac{s^{\alpha-(\,\alpha+1\,)}}{(\, s^{\alpha} + \, f)}\mathrm{A}. \tag{4.24}$$

By taking the inverse Laplace transform and using Eq. (1.8) we have,

$$F(t) \leq F(t_0)\, \mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{(s^{\alpha}+f)}\right\} + \mathrm{A}\mathcal{L}^{-1}\left\{\frac{s^{\alpha-(\alpha+1)}}{(s^{\alpha}+f)}\right\},$$

$$= F(t_0)\, E_{\alpha,1}\{-f(t-t_0)^{\alpha}\} + \mathrm{A}(t-t_0)^{\alpha}\, E_{\alpha,\alpha+1}\{-f(t-t_0)^{\alpha}\}. \quad (4.25)$$

Using $E_{\alpha,\beta}\{z\} = z\, E_{\alpha,\alpha+\beta}\{z\} + \frac{1}{\Gamma_{\mathrm{A}}}$,

we can have

$$E_{\alpha,1}\{-f(t-t_0)^{\alpha}\} = -f(t-t_0)^{\alpha} E_{\alpha,\alpha+1}\{-f(t-t_0)^{\alpha}\} + \frac{1}{\Gamma_1}.$$

This implies

$$(t-t_0)^{\alpha} E_{\alpha,\alpha+1}\{-f(t-t_0)^{\alpha}\} = -\frac{1}{f}\left[E_{\alpha,1}\{-f(t-t_0)^{\alpha}\} - 1\right]. \quad (4.26)$$

Therefore,

$$F(t) \leq F(t_0) E_{\alpha,1}\{-f(t-t_0)^{\alpha}\} - \frac{\mathrm{A}}{f}\left[\, E_{\alpha,1}\{-f(t-t_0)^{\alpha}\} - 1\,\right],$$

$$= \left(F(t_0) - \frac{\mathcal{B}}{f} \right) E_{\alpha,1}\{-f(t-t_0)^{\alpha}\} + \frac{A}{f}\,. \quad (4.27)$$

Now as t⟶ ∞ , $E_{\alpha,1}\{-f(t-t_0)^{\alpha}\} \longrightarrow 0$. Therefore, all solutions that start in $\mathbb{R}_+^2$ of the proposed system (4.3) lies in the region

$$\left\{X \in \mathbb{R}_+^2 : X \leq \frac{A}{f} + v,\ v > 0\right\}.$$

4.3.4 Equilibrium Points and Conditions for the Existence

First, we define some mathematical expressions for Allee's constant that will be used for the whole analysis.

$$\begin{cases} m_w = a r_1 \\ m_t = a r_1 \left(1 - \frac{a_{12}K_2}{K_1}\right) \\ m_e = r_1\left(a + \frac{K_2}{a_{21}}\right)\left(1 - \frac{K_2}{a_{21}K_1}\right) \end{cases}$$

$$\begin{cases} m_{co} = \frac{r_1\psi^2}{4K_1(1-a_{12}a_{21})} \\ m_{cr} = \frac{K_1 r_1}{4}\left(1+\frac{a}{K_1}\right)^2 \end{cases} \tag{4.28}$$

To evaluate equilibrium points of the proposed system (4.3), we will equate to zero both equations given by (4.3).

$$x_1\left[r_1\left(1-\frac{(x_1+a_{12}x_2)}{K_1}\right)-\frac{m}{(x_1+a\,)}\right]=0. \tag{4.29}$$

$$x_2\left[r_2\left(1-\frac{(x_2+a_{21}x_1)}{K_2}\right)\right]=0. \tag{4.30}$$

Solving (4.28), either $x_1 = 0$ or $r_1\left(1-\frac{(x_1+a_{12}x_2)}{K_1}\right)-\frac{m}{(x_1+a\,)}=0.$

Solving (4.29), either $x_2 = 0$ or $\left(1-\frac{(x_2+a_{21}x_1)}{K_2}\right)=0.$

On substituting $x_1 = 0$ in $1-\frac{(x_2+a_{21}x_1)}{K_2}=0$, we found that $x_2 = K_2$.

On substituting $x_2 = 0$ in $r_1\left(1-\frac{(x_1+a_{12}x_2)}{K_1}\right)-\frac{m}{(x_1+a\,)}=0$, we can have

$r\left(1-\frac{x_1}{K_1}\right)-\frac{m}{x_1+a}=0.$

Which can be represented as

$$\frac{1}{K_1}x_1^2-\left(1-\frac{a}{K_1}\right)x_1+\frac{m}{r}-a=0. \tag{4.31}$$

This is the same as Eq. (2.32) in Chapter 2. As we solved this equation in Chapter 2, its positive solutions are given by,

$x_e = \frac{K_1}{2}\left[\left(1-\frac{a}{K_1}\right)+\sqrt{\left(1+\frac{a}{K_1}\right)^2-\frac{4m}{K_1r_1}}\right]$, provided $m \le m_{cr}$ (See Eq. (4.28)).

$x_e' = \frac{K_1}{2}\left[\left(1-\frac{a}{K_1}\right)-\sqrt{\left(1+\frac{a}{K_1}\right)^2-\frac{4m}{K_1r_1}}\right]$, provided $m_w \leq m \leq m_{cr}$ (See Eq. (4.28)).

On solving $1 - \frac{(x_2+a_{21}x_1)}{K_2} = 0$, we have $x_2 = K_2 - a_{21}x_1$. Put this value in

$r_1\left(1 - \frac{(x_1+a_{12}x_2)}{K_1}\right) - \frac{m}{(x_1+a)} = 0.$

On solving this, we have,

1) $x_1^* = mK_1 \frac{\varphi + \sqrt{\psi^2-4\phi}}{2\phi r_1}$, provided $\psi^2 - 4\phi \geq 0$; $\sqrt{\psi^2 - 4\phi} > \varphi$, when $\varphi < 0$; $(1 - a_{12}a_{21}) > 0$.

2) $x_1^{**} = mK_1 \frac{\varphi - \sqrt{\psi^2-4\phi}}{2\phi r_1}$, provided $\psi^2 - 4\phi \geq 0$; $\sqrt{\psi^2 - 4\phi} < \varphi$, when $\varphi > 0$; $(1 - a_{12}a_{21}) > 0$.

Where,

$$\begin{cases} \varphi = a(a_{12}a_{21} - 1) + (K_1 - a_{12}K_2), \\ \psi = (a + K_1) - a_{12}(aa_{21} + K_2), \\ \phi = \dfrac{mK_1(1 - a_{12}a_{21})}{r_1}. \end{cases} \tag{4.32}$$

The equilibrium points along with conditions of existence of the fractional order system (4.3) are given as,

1. $E_0(0,0), E_1(0,K_2)$, without any conditions.
2. $E_2(x_e, 0)$, where $x_e = \frac{K_1}{2}\left[\left(1 - \frac{a}{K_1}\right) + \sqrt{\left(1 + \frac{a}{K_1}\right)^2 - \frac{4m}{K_1 r_1}}\right]$,

   provided $m < m_{cr}$. (See Eq. (4.28))

3. $E_3(x_e', 0)$, where $x_e' = \frac{K_1}{2}\left[\left(1 - \frac{a}{K_1}\right) - \sqrt{\left(1 + \frac{a}{K_1}\right)^2 - \frac{4m}{K_1 r_1}}\right]$,

   provided $m_w < m < m_{cr}$. (See Eq. (4.28))

4. $E_4(x_1^*, x_2^*)$, where $x_1^* = mK_1 \frac{\varphi + \sqrt{\psi^2-4\phi}}{2\phi r_1}$, $x_2^* = K_2 - a_{21}x_1^*$,

   provided $\psi^2 - 4\phi \geq 0$; $\sqrt{\psi^2 - 4\phi} > \varphi$, when $\varphi < 0$; $(1 - a_{12}a_{21}) > 0$;

$K_2 - a_{21}x_1^* > 0.$

5. $E_5(x_1^{**}, x_2^{**})$, where $x_1^{**} = mK_1 \frac{\varphi - \sqrt{\psi^2 - 4\phi}}{2\phi r_1}$, $x_2^{**} = K_2 - a_{21}x_1^{**}$,

provided $\psi^2 - 4\phi \geq 0$; $\sqrt{\psi^2 - 4\phi} < \varphi$, when $\varphi > 0$; $(1 - a_{12}a_{21}) > 0$;

$K_2 - a_{21}x_1^{**} > 0.$

For φ, ψ, ϕ see Eq. (4.31), and $\psi^2 - 4\phi \geq 0$ implies $m < m_{co}$.

4.3.5 Examination of the Equilibrium Points for Local Asymptotic Stability

To analyze the local asymptotic stability of all equilibrium points, first, we will calculate the eigenvalues of the Jacobian matrix at these points. Representing the given system of equations as follows,

$$D^\alpha x_1(t) = \left[r_1 \left(1 - \frac{(x_1 + a_{12}x_2)}{K_1} \right) - \frac{m}{(x_1 + a)} \right] x_1 = h(x_1, x_2),$$

$$D^\alpha x_2(t) = \left[r_2 \left(1 - \frac{(x_2 + a_{21}x_1)}{K_2} \right) \right] x_2 = g(x_1, x_2),$$

$$\frac{\partial h}{\partial x_1} = r_1 - \frac{r_1}{K_1}(2x_1 + a_{12}x_2) - \frac{ma}{(x_1 + a)^2}, \quad \frac{\partial h}{\partial x_2} = -\frac{r_1}{K_1} a_{12}x_1,$$

$$\frac{\partial g}{\partial x_1} = -\frac{r_2}{K_2} a_{21}x_2, \quad \frac{\partial g}{\partial x_2} = r_2 - \frac{r_2}{K_2}(2x_2 + a_{21}x_1).$$

Now Jacobian matrix of the fractional order system (4.3) can be calculated as,

$$\mathbb{J}(h, g) = \begin{bmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \end{bmatrix},$$

$$= \begin{bmatrix} r_1 - \frac{r_1}{K_1}(2x_1 + a_{12}x_2) - \frac{ma}{(x_1 + a)^2} & -\frac{r_1}{K_1} a_{12}x_1 \\ -\frac{r_2}{K_2} a_{21}x_2 & r_2 - \frac{r_2}{K_2}(2x_2 + a_{21}x_1) \end{bmatrix}. \tag{4.33}$$

Theorem 4.4 The extinction point E_0, is always unstable**.**

Proof. Substituting the point E₀(0,0) in $\mathbb{J}\,(h, g)$, the value of the Jacobian matrix is found as,

$$\mathbb{J}\,(0{,}0) = \begin{bmatrix} r_1 - \dfrac{m}{a} & 0 \\ 0 & r_2 \end{bmatrix} \text{ (See Eq. (4.33))} .$$

Being a diagonal matrix, its eigenvalues are given by $\lambda_1 = (r_1 - \frac{m}{a})$ and $\lambda_2 = r_2$. As $r_2 > 0$. Therefore | arg (λ_2)| = 0 < απ /2. So, the extinction of both species is unstable.

Theorem 4.5 The equilibrium point $\mathrm{E}_2(x_e, 0)$ is stable if $m < \ min\,\{m_w, m_e, m_{cr}\}$, (see Eq. (4.28)) and $K_1 \ > K_2/a_{21}$.

Proof. Substituting the point $\mathrm{E}_2(x_e, 0)$ in $\mathbb{J}\,(h, g)$, the value of the Jacobian matrix is given as,

$$\mathbb{J}(x_e, 0) = \begin{bmatrix} r_1 - \dfrac{r_1}{K_1}(2x_e) - \dfrac{ma}{(x_e + a)^2} & -\dfrac{r_1}{K_1}\,a_{12}x_e \\ 0 & r_2 - \dfrac{r_2}{K_2}(a_{21}x_e) \end{bmatrix} \text{ (See Eq. (4.33))} .$$

Its eigenvalues are given by $\lambda_1 = r_1 - \frac{r_1}{K_1}(2x_e) - \frac{ma}{(x_e+a)^2}$ and $\lambda_2 = r_2 - \frac{r_2}{K_2}(a_{21}x_e)$.

Now the condition, |arg $(\lambda_{1,2})$| > απ /2 will be fulfilled if $r_1 - \frac{r_1}{K_1}(2x_e) - \frac{ma}{(x_e+a)^2} < 0$, and $r_2 - \frac{r_2}{K_2}(a_{21}x_e) < 0$. Solving $r_1 - \frac{r_1}{K_1}(2x_e) - \frac{ma}{(x_e+a)^2} < 0$, as in Theorem 2.6, we have $0 < m \ < m_w$. And on solving $r_2 - \frac{r_2}{K_2}(a_{21}x_e) < 0$, we have $\frac{K_2}{a_{21}} < \frac{K_1}{2}\left[\left(1 - \frac{a}{K_1}\right) + \sqrt{\left(1 + \frac{a}{K_1}\right)^2 - \frac{4m}{K_1 r_1}}\right]$. This further gives $\frac{2K_2}{a_{21}K_1} - \left(1 - \frac{a}{K_1}\right) < \sqrt{\left(1 + \frac{a}{K_1}\right)^2 - \frac{4m}{K_1 r_1}}$. Squaring both sides and solving, we have $m < r_1\left(a + \frac{K_2}{a_{21}}\right)\left(1 - \frac{K_2}{a_{21}K_1}\right) = m_e$, provided $1 - \frac{K_2}{a_{21}K_1} > 0$, i.e. $K_1 > K_2/a_{21}$. But the existence of the equilibrium point requires that $m < m_{cr}$. Hence the point $(x_e, 0)$ will be stable if $m < \{m_w, m_e, m_{cr}\}$ (See Eq. (4.28)) and $K_1 > K_2/a_{21}$.

Theorem 4.6 The axial equilibrium point $E_3(x'_e, 0)$ is inevitably unstable.

Proof. Substituting the point $E_3(x_e, 0)$ in $\mathbb{J}\,(h, g)$, the value of the Jacobian matrix is given as,

$$\mathbb{J}(x'_e, 0) = \begin{bmatrix} r_1 - \frac{r_1}{K_1}(2x'_e) - \frac{ma}{(x'_e+a)^2} & -\frac{r_1}{K_1}\, a_{12}x'_e \\ 0 & r_2 - \frac{r_2}{K_2}(a_{21}x'_e) \end{bmatrix} \text{ (See Eq. (4.33)).}$$

Its eigenvalues are given by $\lambda_1 = r_1 - \frac{r_1}{K_1}(2x'_e) - \frac{ma}{(x'_e+a)^2}$ and $\lambda_2 = r_2 - \frac{r_2}{K_2}(a_{21}x'_e)$.

Now $|\arg(\lambda_{1,2})| > \alpha\pi/2$ will be satisfied if, $r_1 - \frac{r_1}{K_1}(2x'_e) - \frac{ma}{(x'_e+a)^2} < 0$ and $r_2 - \frac{r_2}{K_2}(a_{21}x'_e) < 0$. Now solving $r_1 - \frac{r_1}{K_1}(2x'_e) - \frac{ma}{(x'_e+a)^2} < 0$ as in Theorem 2.7, it is found that $0 < m < m_w$ (See Eq. (4.28)) . This is contrary to the existence condition of the point $(x'_e, 0)$. Therefore $(x'_e, 0)$ is inevitably unstable.

Theorem 4.7 The equilibrium point $E_4(x_1^*, x_2^*)$ will be locally asymptotically stable whenever any of the conditions given below are satisfied:

i. $\gamma^2 - 4\Delta \geq 0$ and $\gamma < 0$.
ii. $\gamma^2 - 4\Delta < 0$ and $\gamma < 0$.
iii. $\gamma^2 - 4\Delta < 0, \gamma > 0$, and $0 < \alpha < \frac{2}{\pi}\tan^{-1}\left|\frac{\sqrt{4\,\Delta-\gamma^2}}{\gamma}\right|$.
iv. $\gamma = 0$, and $\Delta > 0$.

Here γ and Δ respectively are the trace and determinant of the Jacobian matrix at the equilibrium point (x_1^*, x_2^*).

Proof. Substituting the point $E_4(x_1^*, x_2^*)$ in $\mathbb{J}\,(h, g)$, the value of the Jacobian matrix is given as,

$$\mathbb{J}(x_1^*, x_2^*)$$

$$= \begin{bmatrix} \frac{r_1}{K_1}(\,K_1 - 2x_1^* - a_{12}x_2^*) - \frac{ma}{(x_1^*+a)^2} & -\frac{r_1}{K_1}\,a_{12}x_1^* \\ -\frac{r_2}{K_2}\,a_{21}x_2^* & \frac{r_2}{K_2}(K_2 - 2x_2^* - a_{21}x_1^*) \end{bmatrix} \text{(See Eq. (4.33))},$$

$$= \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}.$$

Where,

$$c_{11} = \frac{r_1}{K_1}(\,K_1 - 2x_1^* - a_{12}x_2^*) - \frac{ma}{(x_1^*+a)^2}, \qquad c_{12} = -\frac{r_1}{K_1}\,a_{12}x_1^*,$$

$$c_{21} = -\frac{r_2}{K_2}\,a_{21}x_2^*, \qquad c_{22} = \frac{r_2}{K_2}(K_2 - 2x_2^* - a_{21}x_1^*),$$

$$x_1^* = m\,\frac{\varphi \; + \; \sqrt{\psi^2 - 4\phi}}{4\phi r_1}, \quad x_2^* = \; K_2 - a_{21}x_1^*,$$

$$\begin{cases} \varphi = a(a_{12}a_{21} - 1) + (K_1 - a_{12}K_2\,) \\ \psi = (a + K_1) - a_{12}(aa_{21} + K_2) \\ \phi = \dfrac{mK_1(1 - a_{12}a_{21})}{r_1}. \end{cases} \tag{4.34}$$

Provided, $\psi^2 - 4\phi \geq 0,\ (1 - a_{12}a_{21}) > 0,\ K_2 - a_{21}x_1^* > 0$.

Now writing the characteristic equation of $\mathbb{J}(x_1^*, x_2^*)$ as, $\lambda^2 - \gamma\lambda + \Delta = 0$.

Here $\gamma = c_{11} + c_{22}$, $\delta = c_{11}c_{22} - c_{12}c_{21}$. Eigenvalues of $\mathbb{J}(x_1^*, x_2^*)$ are given by $\lambda_i = \frac{\gamma \pm \sqrt{\gamma^2 - 4\,\Delta}}{2}, i = 1, 2.$

Case 1. If $\gamma^2 - 4\Delta \geq 0$

i. If $\gamma > 0$. Here, both eigenvalues will be positive reals and hence $|\arg(\lambda_1)| = |\arg(\lambda_2)| = 0 < \alpha\,\frac{\pi}{2}$. Therefore (x_1^*, x_2^*) will be unstable.

ii. If $\gamma < 0$, then both eigenvalues will be negative reals, and hence $|\arg(\lambda_1)| = |\arg(\lambda_2)| = \pi > \alpha\,\frac{\pi}{2}$. Therefore (x_1^*, x_2^*) will be locally stable.

Case 2. If $\gamma^2 - 4\Delta < 0$

i. If $\gamma > 0$. Then $\lambda_1 = \frac{\gamma + i\sqrt{4\,\Delta - \gamma^2}}{2}$, $\lambda_2 = \frac{\gamma - i\sqrt{4\,\Delta - \gamma^2}}{2}$. Therefore, both eigenvalues are complex conjugates with their real parts positive. Therefore, $|\arg(\lambda_{1,2})| = \left|\tan^{-1}\left(\frac{\sqrt{4\,\Delta-\gamma^2}}{\gamma}\right)\right| = \tan^{-1}\left|\frac{\sqrt{4\,\Delta-\gamma^2}}{\gamma}\right|$. If it is possible to find the value of fractional order α, so that $\tan^{-1}\left|\frac{\sqrt{4\,\Delta-\gamma^2}}{\gamma}\right| > \alpha\frac{\pi}{2}$, only then (x_1^*, x_2^*) will be stable. In other words, if it is possible to find $0 < \alpha < \frac{2}{\pi}\tan^{-1}\left|\frac{\sqrt{4\,\Delta-\gamma^2}}{\gamma}\right|$, the equilibrium point will be stable.

ii. If $\gamma < 0$. Here again, both eigenvalues will be complex conjugates of each other with negative real parts. Here,

$$|\arg(\lambda_{1,2})| = \left|-\pi + \tan^{-1}\left(\frac{\sqrt{4\,\Delta-\gamma^2}}{\gamma}\right)\right| = \left|\pi - \tan^{-1}\left(\frac{\sqrt{4\,\Delta-\gamma^2}}{\gamma}\right)\right| > \alpha\,\frac{\pi}{2}.$$

Hence equilibrium point will be stable.

Case 3. If $\gamma = 0$, then $(\lambda_{1,2}) = \frac{\pm\sqrt{-4\Delta}}{2} = \pm\sqrt{-\Delta}$.

i. Now if $\Delta > 0$. Then λ_1 and λ_2 will be the complex conjugate of each other and $|\arg(\lambda_{1,2})| = \frac{\pi}{2} > \alpha\,\frac{\pi}{2}$, the equilibrium point will be stable.

ii. If $\Delta < 0$, then λ_1 and λ_2 will be real numbers with one of the eigenvalues positive, say $\lambda_1 > 0$ and $|\arg(\lambda_1)| = 0 < \alpha\,\frac{\pi}{2}$. So, the equilibrium point will be unstable.

A similar analysis can be done to establish the local asymptotic stability of the other coexistence equilibrium point $E_5(x_1^{**}, x_2^{**})$.

Theorem 4.8 The axial equilibria E_1 (0, K_2) will be stable whenever $m > m_t$ (See Eq. (4.28))

Proof. Substituting the point E_1 (0, K_2) in $\mathbb{J}\,(h, g)$, the value of the Jacobian matrix is given as,

$$\mathbb{J}\,(0, K_2) = \begin{bmatrix} r_1 - \frac{r_1}{K_1}(a_{12}K_2) - \frac{m}{a} & 0 \\ -r_2\,a_{21} & -r_2 \end{bmatrix} \text{ (See Eq. (4.33))} .$$

Its eigenvalues are given by $\lambda_1 = r_1 - \frac{r_1}{K_1}(a_{12}K_2) - \frac{m}{a}$ and $\lambda_2 = -r_2$.

As $|\arg(\lambda_2)| = \pi > \alpha\pi/2$. Therefore, the point $(0, K_2)$ will be stable if and only if $r_1 - \frac{r_1}{K_1}(a_{12}K_2) - \frac{m}{a} < 0$. Upon simplification, it turns out that $m > ar_1\left(1 - \frac{a_{12}K_2}{K_1}\right) = m_t$.

4.3.6. Examination of the Equilibrium Points for Global Stability

Theorem 4.9 The equilibrium point $E_4(x_1^*, x_2^*)$ will be globally stable if, $r_1 + \frac{x_1^* r_1}{K_1} - \frac{m}{(M+a)} + a_{21}\frac{r_1 x_2^*}{K_1} < 0$, $\frac{2r_1K_2}{K_1} - \frac{a_{21}x_1^*}{K_1} + \frac{a_{12}x_1^* r_1}{K_1} < 0$, and $\frac{mx_1^*}{a} - \frac{r_1K_2}{K_1}x_2^* - r_1x_1^* < 0$, where $\max\{|x_1|, |x_2|\} \leq M$.

Proof: Consider the positive definite function

$$V(x_1, x_2) = \left(x_1 - x_1^* - x_1^* \ln\frac{x_1}{x_1^*}\right) + \frac{r_1K_2}{r_2K_1}\left(x_2 - x_2^* - x_2^* \ln\frac{x_2}{x_2^*}\right). \tag{4.35}$$

$$D^\alpha V(x_1, x_2) = D^\alpha\left[\left(x_1 - x_1^* - x_1^* \ln\frac{x_1}{x_1^*}\right) + \frac{r_1K_2}{r_2K_1}\left(x_2 - x_2^* - x_2^* \ln\frac{x_2}{x_2^*}\right)\right],$$

$$= D^\alpha\left(x_1 - x_1^* - x_1^* \ln\frac{x_1}{x_1^*}\right) + \frac{r_1K_2}{r_2K_1} D^\alpha\left(x_2 - x_2^* - x_2^* \ln\frac{x_2}{x_2^*}\right), \tag{4.36}$$

By using Eq. (1.9)

$$\leq \left(\frac{x_1 - x_1^*}{x_1}\right) D^\alpha x_1 + \frac{r_1K_2}{r_2K_1}\left(\frac{x_2 - x_2^*}{x_2}\right) D^\alpha x_2, \tag{4.37}$$

$$= (x_1 - x_1^*)\left[\left(1 - \frac{(x_1 + a_{12}x_2)}{K_1}\right) r_1 - \frac{m}{(x_1 + a)}\right] + \frac{r_1K_2}{r_2K_1}(x_2 - x_2^*)\left[\left(1 - \frac{(x_2 + a_{21}x_1)}{K_2}\right) r_2\right],$$

$$\leq (x_1 - x_1^*) r_1 + \frac{(x_1 + a_{12}x_2)}{K_1} x_1^* r_1 - \frac{mx_1}{(M + a)} + \frac{mx_1^*}{a} + \frac{r_1K_2}{K_1}(x_2 - x_2^*) + \frac{r_1x_2^*}{K_1}(x_2 + a_{21}x_1) - \frac{r_1(a_{12} + a_{21})}{K_1} x_1x_2,$$

$$\leq x_1\left(r_1+\frac{x_1^*\,r_1}{K_1}-\frac{m}{(M+a\,)}+a_{21}\frac{r_1x_2^*}{K_1}\right)$$

$$+x_2\left(\frac{r_1K_2}{K_1}+\frac{r_1x_2^*}{K_1}+a_{12}\frac{x_1^*\,r_1}{K_1}-\frac{r_1(a_{12}+a_{21})}{K_1}x_1\right)$$

$$+\frac{mx_1^*}{a}-\frac{r_1K_2}{K_1}x_2^*-r_1x_1^*,$$

$$\leq x_1\left(r_1+\frac{x_1^*\,r_1}{K_1}-\frac{m}{(M+a\,)}+a_{21}\frac{r_1x_2^*}{K_1}\right)+x_2\left(\frac{2r_1K_2}{K_1}-\frac{a_{21}x_1^*}{K_1}+\frac{a_{12}x_1^*\,r_1}{K_1}\right)+\frac{mx_1^*}{a}$$

$$-\frac{r_1K_2}{K_1}x_2^*-r_1x_1^*\leq 0,$$

if $r_1+\frac{x_1^*\,r_1}{K_1}-\frac{m}{(M+a\,)}+a_{21}\frac{r_1x_2^*}{K_1}<0, \frac{2r_1K_2}{K_1}-\frac{a_{21}x_1^*}{K_1}+\frac{a_{12}x_1^*\,r_1}{K_1}<0, \frac{mx_1^*}{a}-\frac{r_1K_2}{K_1}x_2^*-r_1x_1^*\leq 0$, then V becomes a positive definite function (Lyapunov function) and hence the point $(\,x_1^*,x_2^*)$ becomes globally stable.

Theorem 4.10 The equilibrium point $E_2(x_e,0)$ will be globally stable if, $r_1+\frac{x_er_1}{K_1}-\frac{m}{(M+a\,)}<0$, and $\frac{Mr_1}{K_1}(K_2+a_{12}x_e)-x_er_1+\frac{mx_e}{a}<0$, where max $\{|x_1|,|x_2|\}\leq M$

Proof. Consider the positive definite function

$$V'(\,x_1,x_2)=\left(\,x_1-x_e-x_e\ln\frac{x_1}{x_e}\right)+\frac{r_1K_2}{r_2K_1}x_2. \tag{4.38}$$

$$D^\alpha V'(\,x_1,x_2)=D^\alpha\left[\left(\,x_1-x_e-x_e\ln\frac{x_1}{x_e}\right)+\frac{r_1K_2}{r_2K_1}x_2\right], \tag{4.39}$$

$$=\;D^\alpha\left(\,x_1-x_e-x_e\ln\frac{x_1}{x_e}\right)+\frac{r_1K_2}{r_2K_1}D^\alpha x_2, \tag{4.40}$$

By using Eq. (1.9)

$$\leq\left(\frac{x_1-x_e}{x_1}\right)D^\alpha\,x_1+\frac{r_1K_2}{r_2K_1}D^\alpha x_2, \tag{4.41}$$

$$= (x_1 - x_e)\left[r_1\left(1 - \frac{(x_1 + a_{12}x_2)}{K_1}\right) - \frac{m}{(x_1 + a\ )}\right]$$

$$+\frac{r_1K_2}{r_2K_1}x_2\left[r_2\left(1 - \frac{(x_2 + a_{21}x_1)}{K_2}\right)\right],$$

$$\leq (x_1 - x_e)\,r_1 + x_e r_1\frac{(x_1 + a_{12}x_2)}{K_1} - \frac{mx_1}{(x_1 + a\ )} + \frac{mx_e}{(x_1 + a\ )} + \frac{r_1K_2}{K_1}x_2,$$

$$\leq x_1\left(r_1 + \frac{x_e r_1}{K_1} - \frac{m}{(M + a\ )}\right) + x_2\left(\frac{r_1K_2}{K_1} + \frac{x_e r_1 a_{12}}{K_1}\right) - x_e r_1 + \frac{mx_e}{a},$$

$$\leq x_1\left(r_1 + \frac{x_e r_1}{K_1} - \frac{m}{(M + a\ )}\right) + \frac{Mr_1}{K_1}(K_2 + a_{12}x_e) - x_e r_1 + \frac{mx_e}{a} \leq 0, \tag{4.42}$$

if $r_1 + \frac{x_e r_1}{K_1} - \frac{m}{(M+a\ )} < 0, \frac{Mr_1}{K_1}(K_2 + a_{12}x_e) - x_e r_1 + \frac{mx_e}{a} < 0$, then V' becomes a positive definite function (Lyapunov function) and hence the point $(\ x_e, 0)$ becomes globally stable.

Theorem 4.11 The equilibrium point $E_1(0, K_2)$ will be globally stable if, $r_1 - \frac{m}{(M+a\ )} + \frac{a_{21}r_1}{K_1} < 0, M(K_2 + 1) - K_2^2 < 0$, where max $\{|x_1|, |x_2|\} \leq M$.

Proof. Consider the positive definite function

$$V''(\ x_1, x_2) = x_1 + \frac{r_1K_2}{r_2K_1}\left(\ x_2 - K_2 - K_2\ln\frac{x_2}{K_2}\right). \tag{4.43}$$

$$D^\alpha V''(\ x_1, x_2) = D^\alpha\left[x_1 + \frac{r_1K_2}{r_2K_1}\left(\ x_2 - K_2 - K_2\ln\frac{x_2}{K_2}\right)\right],$$

$$= D^\alpha x_1 + \frac{r_1K_2}{r_2K_1}D^\alpha\left(\ x_2 - K_2 - K_2\ln\frac{x_2}{K_2}\right), \tag{4.44}$$

By using Eq. (1.9)

$$\leq D^\alpha x_1 + \frac{r_1K_2}{r_2K_1}\left(\frac{x_2 - K_2}{x_2}\right)D^\alpha x_2, \tag{4.45}$$

$$= x_1\left[r_1\left(1-\frac{(x_1+a_{12}x_2)}{K_1}\right)-\frac{m}{(x_1+a\,)}\right]$$

$$+\frac{r_1K_2}{r_2K_1}(x_2-K_2)\left[r_2\left(1-\frac{(x_2+a_{21}x_1)}{K_2}\right)\right],$$

$$\le r_1x_1-\frac{mx_1}{(M+a\,)}+\frac{r_1K_2}{K_1}x_2-\frac{r_1K_2^2}{K_1}+\frac{r_1}{K_1}(x_2+a_{21}x_1),$$

$$\le x_1\left(r_1-\frac{m}{(M+a\,)}+\frac{a_{21}r_1}{K_1}\right)+\frac{r_1}{K_1}(M(K_2+1)-K_2^2)\le 0, \qquad (4.46)$$

if $r_1-\frac{m}{(M+a\,)}+\frac{a_{21}r_1}{K_1}<0,\ M(K_2+1)-K_2^2<0$, then V'' becomes a positive definite function (Lyapunov function), and therefore the point $(0,K_2)$ becomes globally stable.

4.4 Numerical Analysis

Scenario I: $\frac{K_2}{a_{21}} < K_1,\ \frac{K_1}{a_{12}} > K_2$

Table 4.1: Table for Values of Parameters Used for Numerical Simulations

Parameters	Value in Case 1	Value in Case 2	Value in Case 3	Value in Case 4	Source
K_1	49	49	49	49	[146]
K_2	37.5	37.5	37.5	37.5	[146]
r_1	0.074	0.074	0.074	0.074	[146]
r_2	0.075	0.075	0.075	0.075	[146]
a_{12}	0.6	0.6	0.6	0.6	[146]
a_{21}	0.95	0.95	0.95	0.95	[146]
a	19.5	19.5	19.5	19.5	Assumed
m	0.7	0.83	1	1.8	Assumed

The values obtained for Allee’s constant

$m_t = 0.78, \ \ m_e = 0.85, \ m_{co} = 1.07, \ m_w = 1.443, \ m_{cr} = 1.77.$ (See Eq. (4.28))

Case 1. When Allee's constant $m < m_t$ (See Eq. (4.28))

By taking Allee's constant $m = 0.7 < m_t$, we obtain $E_2(41.3,0)$ as an equilibrium point. By Theorem 4.5, stability criteria are also satisfied. Figure 4.1 is plotted to show the local stability of the point $E_2(41.3,0)$ in integer order and fractional order models, and Figure 4.2 shows the global stability of the point $E_2(41.3,0)$ in the fractional model of order 0.9.

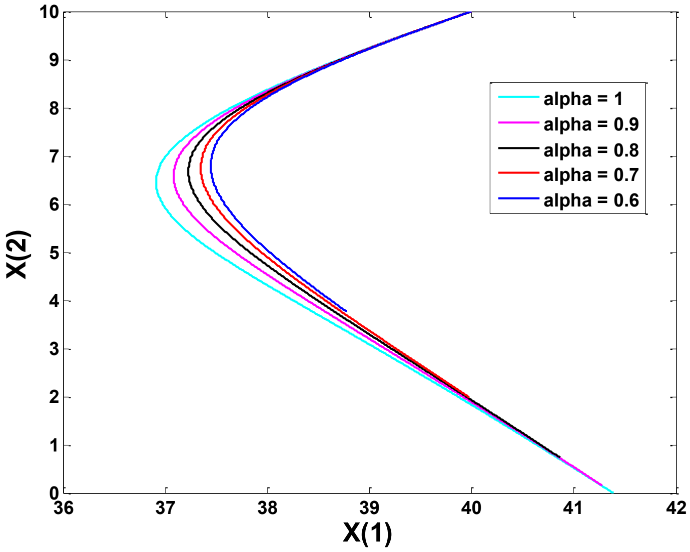


Figure 4.1 Behavior of populations in different models at the equilibrium point (41.3,0) when $m = 0.7 < (m_t)$, with initial populations (40,10)

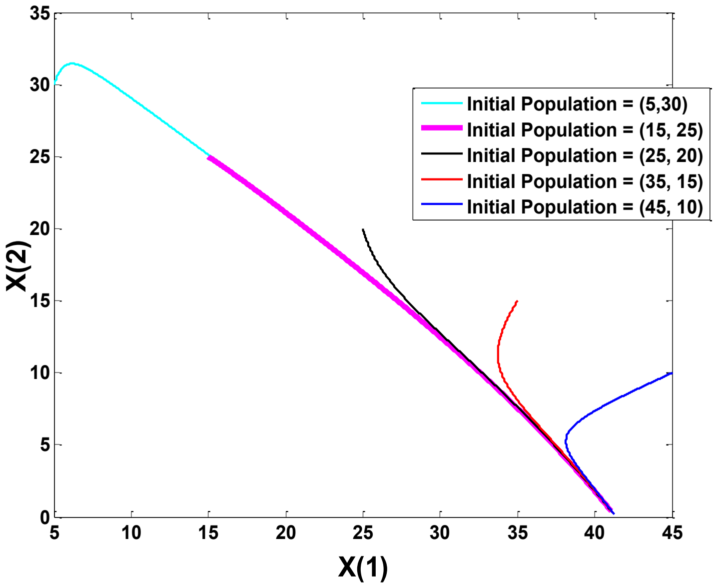


Figure 4.2 Behavior of populations at equilibrium point (41.3,0) in fractional model of order 0.9 when $m = 0.7$ ($< m_t$), with different initial populations

Case 2. When Allee's constant $m \in (m_t, m_e)$ (See Eq. (4.28))

By taking $m = 0.83$, the equilibrium point obtained is $E_2(39.6,0)$. By Theorem 4.5, the stability criteria are satisfied. But it is found numerically that the stability of this solution is subject to the initial population. Figures 4.3 and 4.4 are plotted by taking the integer order model and the fractional model of order 0.9 to show that if the initial population of the second species is 37, then the minimum population required for the first species to reach the equilibrium point is 2.

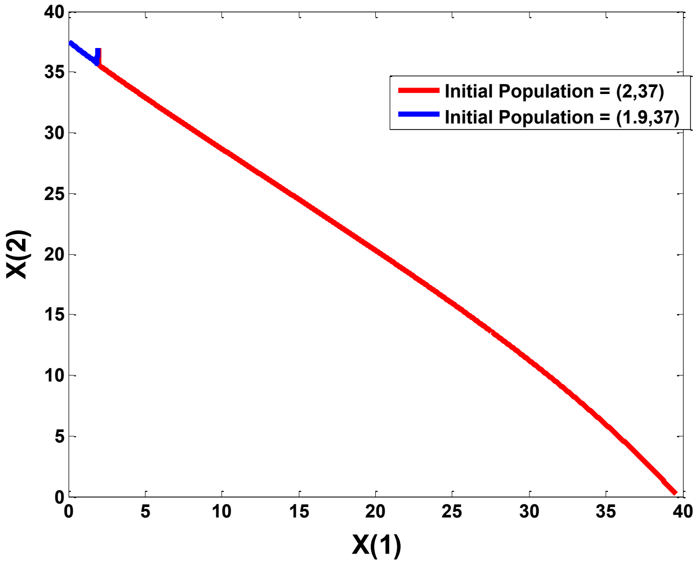


Figure 4.3 Behavior of populations at the equilibrium point (39.6,0) in integer order model when $m = 0.83$ $(m_t < m < m_e)$

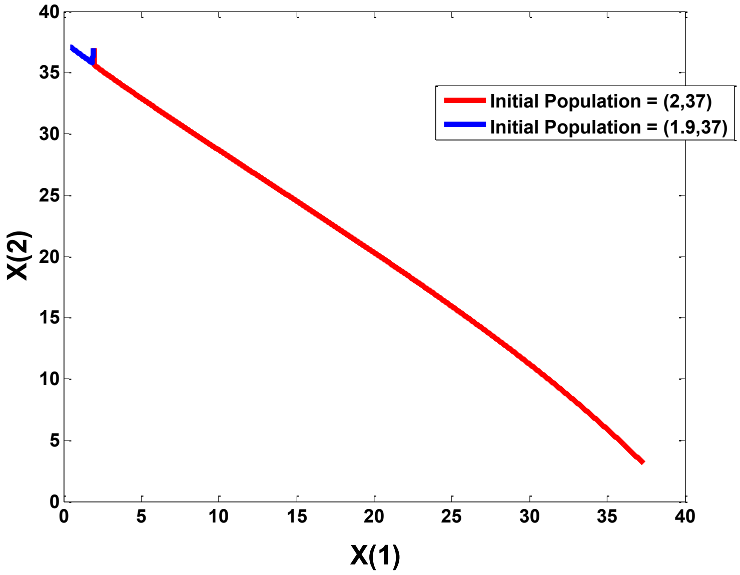


Figure 4.4 Behavior of populations at the equilibrium point (39.6,0) in fractional model of order 0.9 when $m = 0.83$ $(m_t < m < m_e)$

Case 3. When Allee's constant $m \in (m_e, m_{co})$ (See Eq. (4.28))

By taking Allee's constant $m = 1$, the value of the equilibrium point is found to be (31.36,7.7). At this point, we have $\gamma^2 - 4\Delta = 0.0021 > 0$, $\gamma = -0.051 < 0$. Therefore, the conditions of the local asymptotic stability (Theorem 4.6) are satisfied. Figure 4.5 is plotted by taking the initial population (25,10) and different fractional orders and integer order 1, to show that populations are converging to (31.36,7.7). Furthermore, the requirement of the minimum population for the survival of the first species for a given population of the second species is observed numerically. Figures 4.6 (a) and 4.6 (b) are plotted by taking the initial population (12,30) and (11,30), respectively. It is shown that populations starting from (12,30) are converging to (31.36,7.7) and those starting from (11,30) are converging to (0,37.5) in the fractional order and integer order model as well.

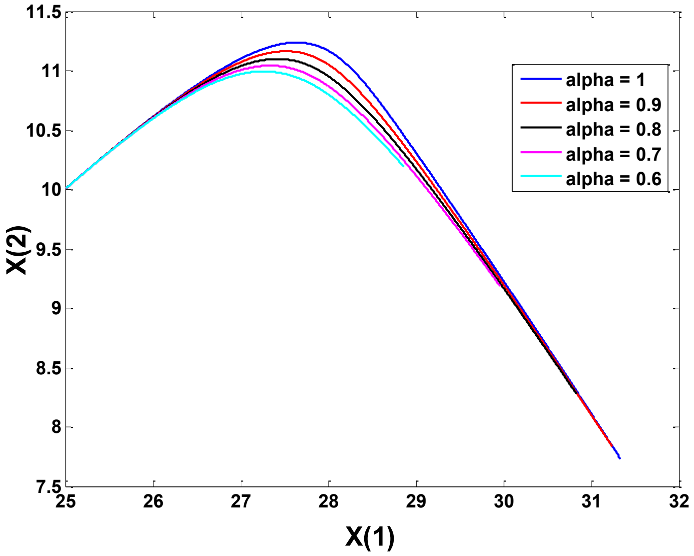


Figure 4.5 Behavior of populations in different models at the equilibrium point (31.36,7.7) by taking $m = 1$ ($m_e < m < m_{co}$) and the initial population (25,10)

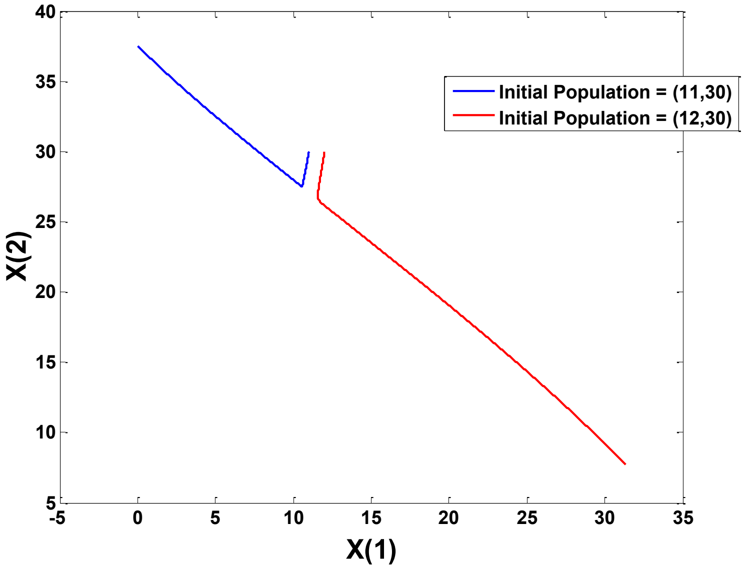


Figure 4.6(a) Behavior of populations in integer model at the equilibrium point (31.36,7.7) by taking $m = 1\ \ (m_e < m < \ m_{co})$

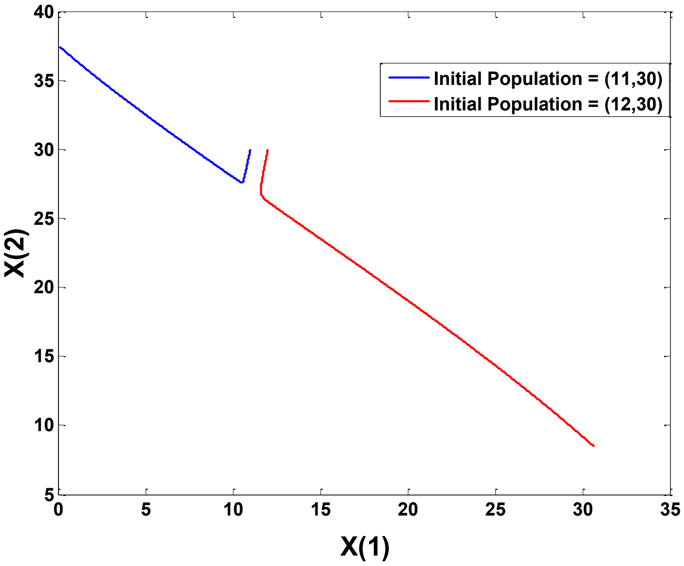


Figure 4.6(b) Behavior of populations in fractional model of the order 0.9 at the equilibrium point (31.36,7.7) by taking $m = 1\ \ (\ m_e < m < \ m_{co})$

Case 4. When Allee’s constant $m > m_{cr}$

By taking Allee’s constant m =1.8, and the initial population (49,5), simulations are run in fractional as well as integer-order models. Figure 4.7 shows that when Allee’s constant $m > m_{cr}$, the extinction of the first species cannot be prevented.

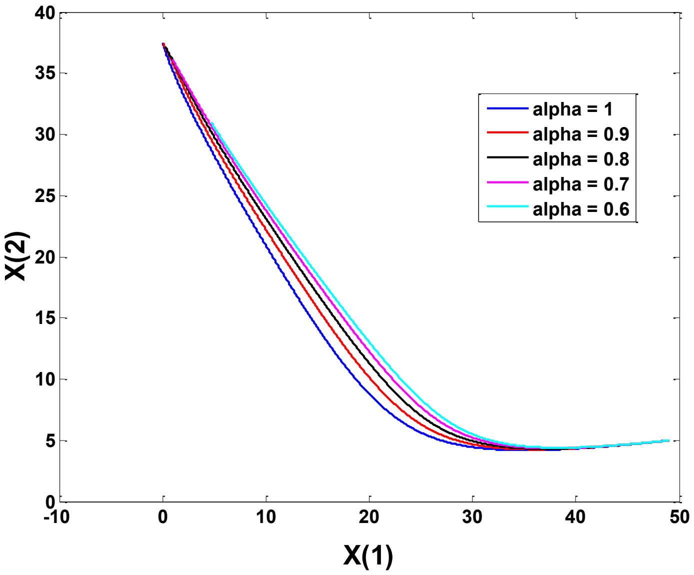


Figure 4.7 Behavior of populations in different models when $m = 1.8$ ($m > m_{cr}$) and the initial population (49, 5)

Scenario II: $\frac{K_2}{a_{21}} > K_1, \frac{K_1}{a_{12}} < K_2$

Table 4.2: Table for Values of Parameters Used for Numerical Simulations

Parameters	**Value**	**Source**
K_1	37.5	[146]
K_2	49	[146]
r_1	0.075	[146]
r_2	0.074	[146]
a_{12}	0.95	[146]
a_{21}	0.6	[146]
a	15	Assumed
m	0.01	Assumed

The values obtained for Allee's constant are as,

$m_w = 1.125$, $m_t = -0.2715$, $m_{cr} = 1.38$, $m_{co} = 0.08$, $m_e = -8.56$. (See Eq. (4.28))

In this case, weaker species are exposed to the Allee effect. The only possible solution, which is locally as well as globally stable, should be the equilibrium point $E_1(0, 49)$ in traditional modeling. However, when Figure 4.8 is plotted by taking m = 0.01 and the initial population (20,15) with different values of fractional orders, it is observed that lower fractional order models move slowly, stabilize faster, and converge to different coexistence points. As we increase the value of the fractional order from 0.2 to 0.8, the chances of coexistence start decreasing. When alpha takes a value of 0.9, the populations stabilize at $E_1(0,49)$. Therefore, lower fractional-order models help in the coexistence of both species even when the first species is subjected to the Allee effect.

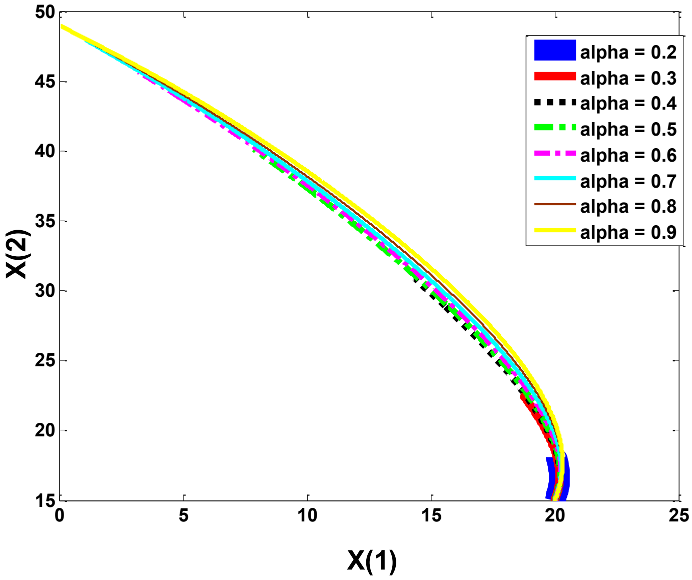


Figure 4.8 Dynamic analysis of the populations in different fractional models by taking $m = 0.01$, and initial population (20,15)

Scenario III: $\frac{K_2}{a_{21}} < K_1, \ \frac{K_1}{a_{12}} < K_2$

Table 4.3: Table for Values of Parameters Used for Numerical Simulation

Parameters	**Value in 1st Example**	**Source**	**Value in 2nd Example**	**Source**
K_1	37	[146]	300	Assumed
K_2	42	[146]	310	Assumed
r_1	0.062	[146]	0.6	[59]
r_2	0.069	[146]	0.4	[59]
a_{12}	1.1	[146]	1.2	Assumed
a_{21}	1.15	[146]	1.1	[146]
a	14.8	Assumed	25	Assumed
m	0.003	Assumed	2	Assumed

The values obtained for Allee's constant in the 1st example are as,

$$m_{co} = -21.25, \qquad m_t = -0.28, \qquad m_e = 0.041, m_w = 0.92, \qquad m_{cr} = 1.17$$

(See Eq. (4.28)).

By taking $m = 0.003$, the equilibrium point was found $E_2(36.9,0)$. In this scenario, the stability conditions of both equilibrium points $E_2(36.9,0)$ (see Theorem 4.5) and $E_1(0, K_2)$ (see Theorem 4.8) are satisfied. Figure 4.9 (a) is plotted by taking the initial population (36,1) in the integer model as well as in the fractional model, and it is observed that populations are moving towards $E_2(36.9,0)$. Taking (36,2) as the initial population, the coexistence of populations is observed in Figure 4.9 (b) but with a very low density of the second species, and when the initial population is considered as (36,3) then populations in all models start moving towards the point (0,49) as shown in Figure 4.9 (c).

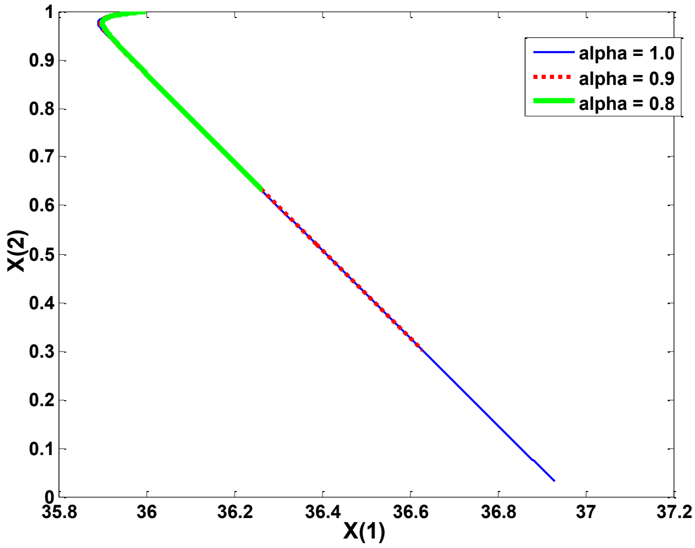


Figure 4.9(a) Dynamic behavior of populations when $m = 0.003$ with initial population (36,1) in different models

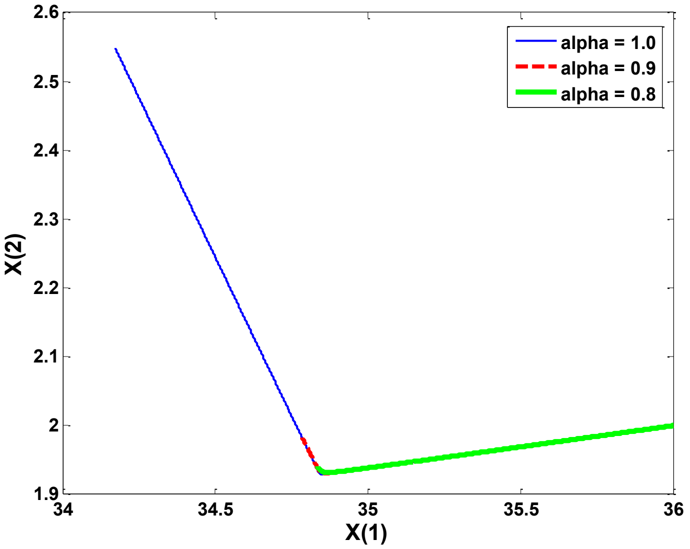


Figure 4.9(b) Dynamic behavior of populations when $m = 0.003$ with initial population (36,2) in different models

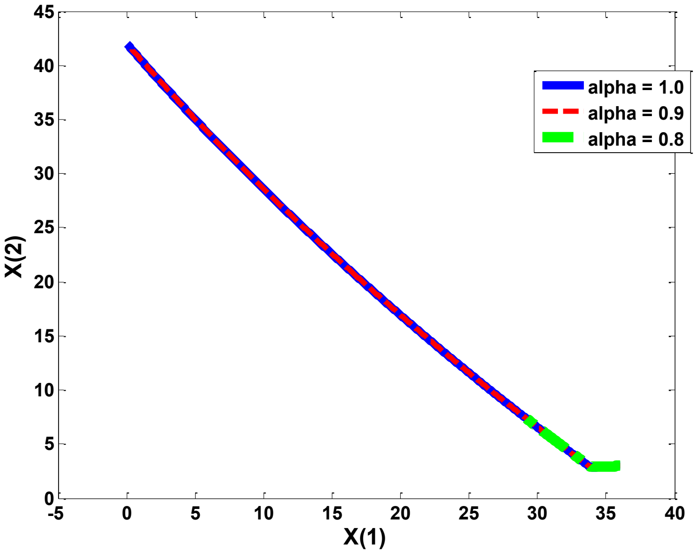


Figure 4.9(c) Dynamic behavior of populations when $m = 0.003$ with initial population (36,3) in different models.

The values obtained for Allee's constant in the 2nd example are as,

$m_{co} = -16.67, \qquad m_t = -3.6, \qquad m_e = 11.16, \qquad m_w = 15, \qquad m_{cr} = 52.8$

(See Eq. (4.28)).

Here existence conditions of the coexistence point are not satisfied. By taking $m = 2$, the equilibrium point was found E_2(296.7,0). In this scenario, the stability conditions of both equilibrium points E_2(296.7,0) (see Theorem 4.5) and $E_1(0, K_2)$ (see Theorem 4.8) are satisfied. By taking values of fractional order as 0.2,0.3,0.4,0,0.5,0.6,0.7,0.8,0.9, and integral value 1, Figure 4.10 is plotted by taking the initial population as (23,10). It is seen that lower fractional order models (α = 0.2,0.3,0,4,0.5) help in the coexistence of both species, as the order of the fractional derivative increases, populations start converging to the axial point (296.7,0), but in the ordinary integer order model populations stabilize at (0,310). Figure 4.11 is drawn to find the threshold value of the fractional order that can support the stability of two equilibria E_2(296.7,0) and $E_1(0, K_2)$. By taking the initial population as (22,10). Numerical simulations are run by taking different values of fractional order between 0.8 and 0.9, and it is found that populations are moving towards (296.7,0) initially, as alpha takes the value 0.8972, populations start moving towards (0,310). Therefore, the threshold memory (alpha) is 0.8972, which can support the affected species (from the Allee effect), after crossing that affected species will be unable to survive.

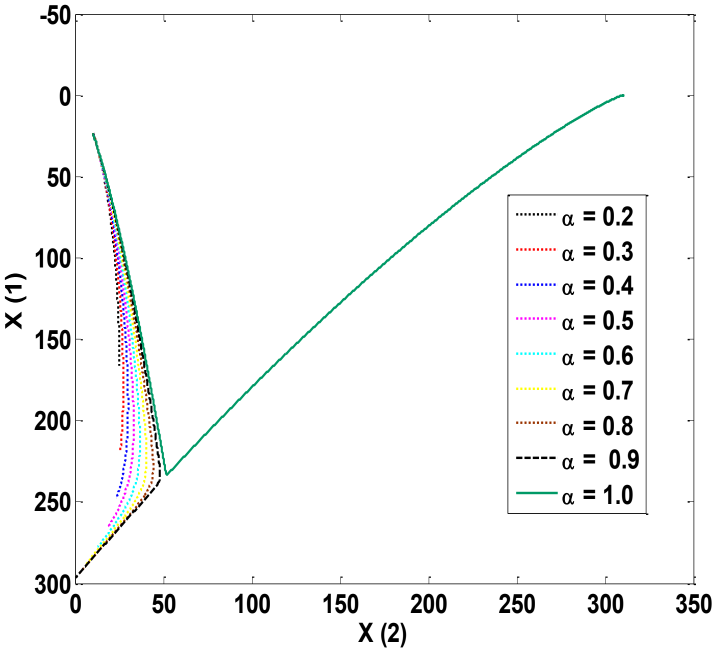


Figure 4.10 Dynamic behavior of populations in various models when first species is exposed to Allee effect in scenario $\frac{K_2}{a_{21}} > K_1, \ \frac{K_1}{a_{12}} < K_2$

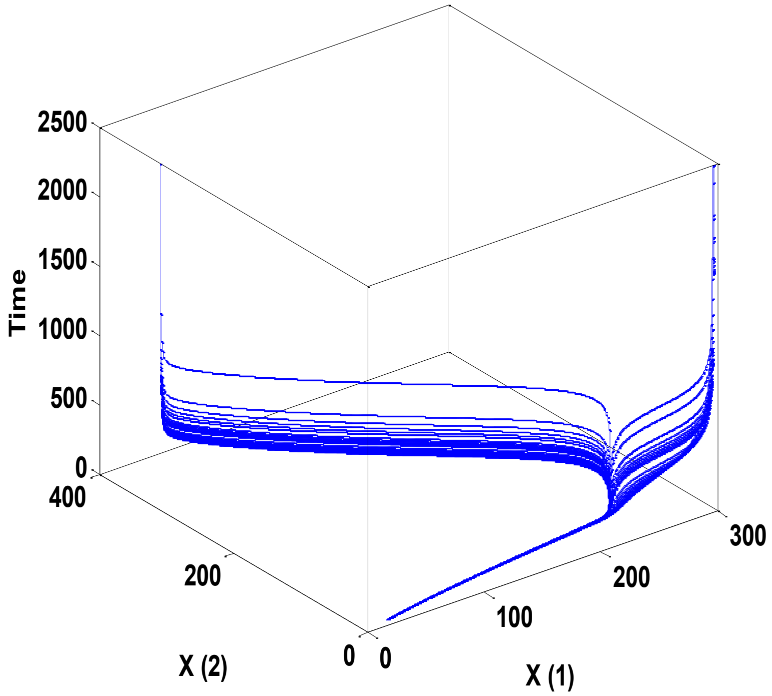


Figure 4.11. Showing the critical value of fractional order (α = 0.8972) at which one equilibrium is destabilized, and stability switches to other equilibria when the first species is exposed to the Allee effect in the scenario $\frac{K_2}{a_{21}} > K_1, \frac{K_1}{a_{12}} < K_2$,where α ∈ [0.8,0.9]

Scenario IV: $\frac{K_2}{a_{21}} > K_1, \frac{K_1}{a_{12}} > K_2$

Table 4.4: Table for Values of Parameters Used for Numerical Simulations

Parameters	Value in Case 1	Value in Case 2	Value in Case 3	Source
K_1	49	49	49	[146]
K_2	37.5	37.5	37.5	[146]
r_1	0.074	0.074	0.074	[146]
r_2	0.075	0.075	0.075	[146]
a_{12}	0.95	0.95	0.95	[146]
a_{21}	0.6	0.6	0.6	[146]
a	19.5	19.5	19.5	Assumed
m	0.3	0.4	0.5	Assumed

The values obtained for Allee's constant are as follows, $m_e = -1.67$, $m_t = 0.394$, $m_{co} = 0.416, m_w = 1.443$. (See Eq. (4.28))

Case 1. When Allee's constant $m < m_t$ (See Eq. (4.28))

By taking Allee's constant $m = 0.3$, we have a coexistence point $E_4(19.1,26.04)$. The values of $\gamma^2 - 4\Delta = 0.0045 > 0, \gamma = -0.007 < 0$ are obtained. Therefore, the stability conditions of the coexistence point are satisfied (see Theorem 4.7). Figure 4.12 is plotted by taking the initial population (20,25) to show the stability of the coexistence point in fractional as well as integer-order models.

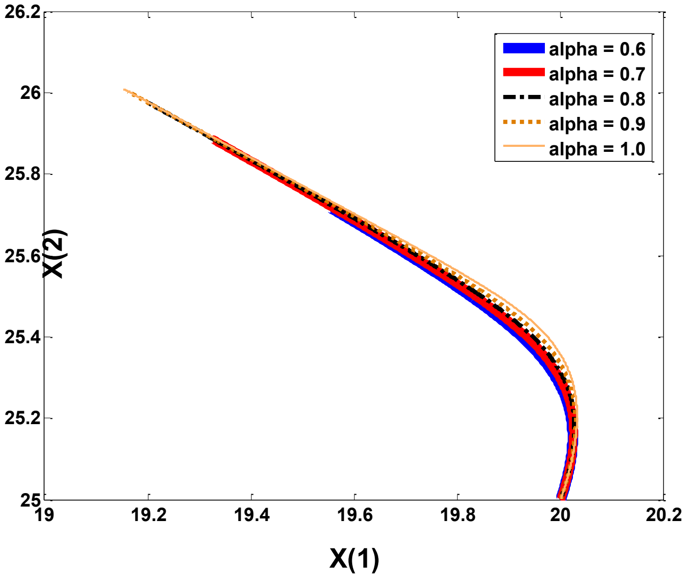


Figure 4.12 Behavior of populations at the equilibrium point (19.1,26.04) when $\mathrm{m} = 0.3$ ($\mathrm{m} < \mathrm{m_t}$) with initial population (20,25)

Case 2. When Allee's constant $m \in (m_t, m_{co})$ (See Eq. (4.28))

By taking Allee's constant $m = 0.4$, we have two coexistence points $E_4(10.73,31.06)$ and $E_5(0.87,37)$. At (10.73,31.06), the value of $\gamma = -0.078 < 0$, and $\gamma^2 - 4\Delta = 0.003 > 0$. At $E_5(0.87,37)$, the value of $\gamma = -0.054 < 0$, and $\gamma^2 - 4\Delta = 0.009 > 0$. Therefore stability conditions of the points $E_4(10.73,31.06)$ and E_5 (0.87,37) are satisfied (see Theorem 4.7). Figure 4.13 is plotted by taking the initial population (15,35) to show the stability of the coexistence point in fractional as well as integer-order models.

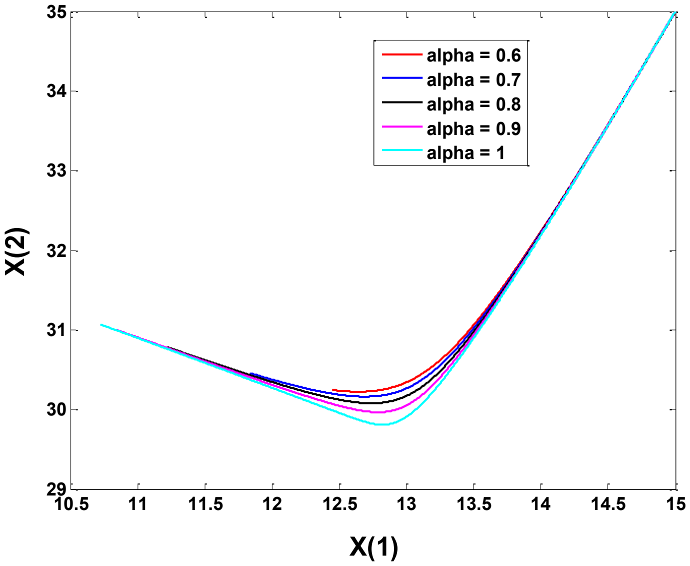


Figure 4.13 Behavior of the populations at the equilibrium point (10.73,31.06) taking $m = 0.4$ $(m_t < m < m_{co})$ with the initial population (15,25) in various models

Figure. 4.14 is plotted to show that the equilibrium point $E_5(0.87,37)$ is numerically unstable in integer order models as well as in fractional order models. Figure 4.14 (a) is plotted by taking the initial population (0.84,35) and (0.85,35). It is found that in both cases, when the initial population is (0.84,35), populations move towards $E_5(0.87,37)$. However, when the initial population is considered (0.85,35), the populations move away from $E_5(0.87,37)$ and similar behavior is observed in fractional models of orders 0.9,0.8, and 0.7 too, as shown in Figures 4.14 (b), 4.14 (c), and 4.14 (d). This shows that point $E_5(0.87,37)$ is stable but not asymptotically.

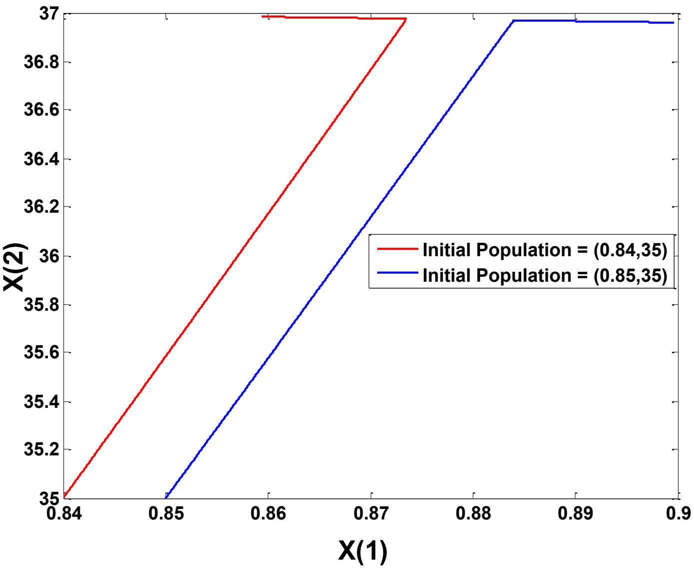


Figure 4.14(a) Stability analysis of the equilibrium point (0.87,37) in integer model $m = 0.4, when\ (m_t < m < m_{co})$

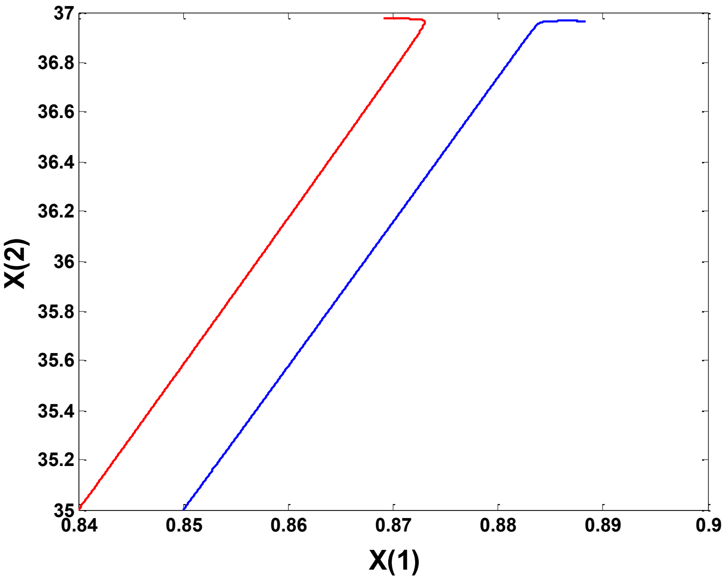


Figure 4.14(b) Stability analysis of the equilibrium point (0.87,37) in fractional model of order 0.9, when $m = 0.4\ (m_t < m < m_{co})$

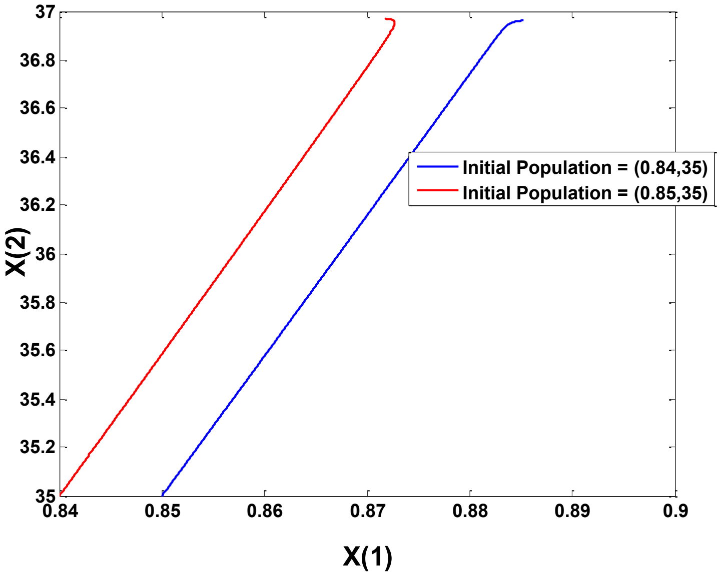


Figure 4.14 (c) Stability analysis of the equilibrium point (0.87,37) in fractional model of order 0.8, when $m = 0.4$ $(m_t < m < m_{co})$

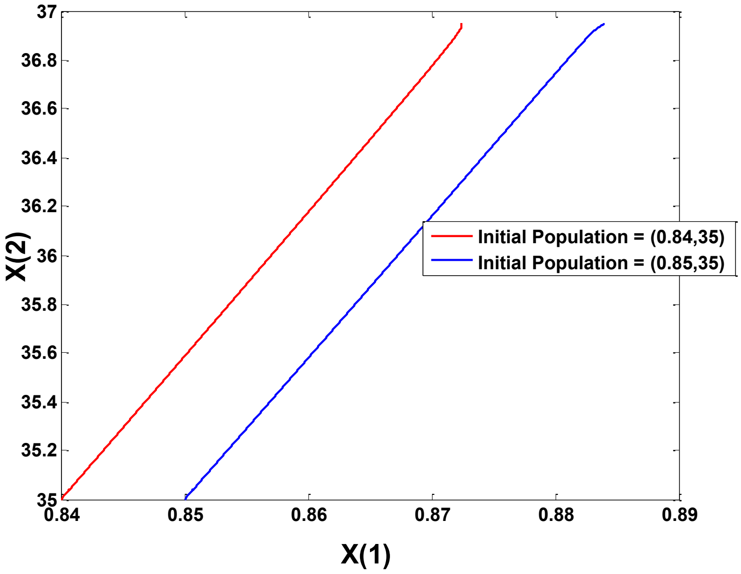


Figure 4.14 (d) Stability analysis of the equilibrium point (0.87,37) in fractional model of order 0.7, when $m = 0.4, (m_t < m < m_{co})$

Case 3. When Allee's constant $m \in (m_{co}, m_w)$ (See Eq. (4.28))

By taking Allee's constant $m = 0.5$, stability conditions of the point $(0, K_2)$ are satisfied (see Theorem 4.8). It is observed that in the integer order model, populations are moving toward the equilibrium point $(0, K_2)$, whereas in the fractional models of different orders, populations are stabilizing at different coexistence points, as shown in Figure 4.15.

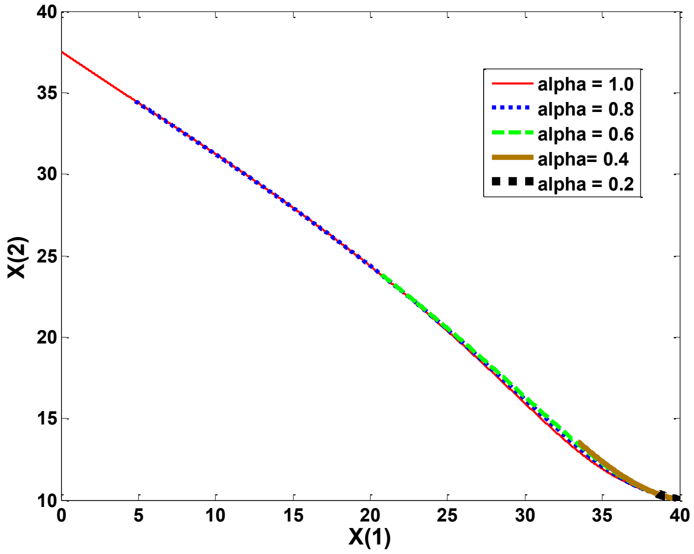


Figure 4.15 Behavior of the populations by taking $m = 0.5$ $(m_{co} < m < m_w)$ with the initial population (40, 10) in various models

4.5 Conclusion

This study examines a fractional order Lotka-Volterra competition model exposed to an additive Allee effect to one of the species, where the fractional derivative in the Caputo sense is used. The mathematical analysis of the model is done to confirm the uniqueness, positivity, and uniform boundedness of solutions. All necessary preconditions required for the existence and stability of the different equilibrium points are determined. The numerical analysis of the model is performed in each competitive scenario.

In scenario I, there is competitive exclusion of the second species by the first species in the classical Lotka-Volterra competition model. However, the proposed model shows multistable behavior depending upon the different values taken by Allee's constant. For the same values of parameters, as the value of Allee's constant increases, the proposed model stabilizes at different equilibrium points. Initially, by taking Allee's constant $m(< m_t)$ (See Eq. (4.28)), the proposed model behaves similarly to the traditional Lotka-Volterra model. The solution obtained as an equilibrium point E_2 , which is found to be locally stable in integer as well as in fractional models (Figs. 4.1, 4.2). This shows that the first species (x_1) can competitively exclude the second species (x_2) regardless of the initial population. On increasing the value of Allee's constant $m,$ so that $m_t < m < m_e,$ (See Eq. (4.28)), the equilibrium point obtained again E_2. However, it is found that on increasing the value of Allee's constant, the population density of the first species (x_1) decreases, but it can still exclude the second species. Through numerical examples as shown in Figs. 4.3, 4.4, it is found that in this situation, the minimum initial population density of the first species (x_1) is required to exclude the second species (x_2), otherwise the second species (x_2) excludes the first by stabilizing at the equilibrium point $E_1(0, K_2)$. On further increasing the value of Allee's constant $m,$ so that $m_e < m < m_{co}$ (See Eq. (4.28)), the equilibrium point obtained E_2. Through numerical simulation, it is observed that both populations stabilize at E_2 in fractional and integer order models (Fig. 4.5). However, at high densities of the second species (x_2), the first species (x_1) requires a minimum population to survive (Figs. 4.6(a), 4.6(b)). On further increasing the value of Allee's constant m, so that $m > m_{co}$ (See Eq. (4.28)), the coexistence of the species destabilizes, and the second species (x_2) excludes the first species (x_1) competitively (Fig.4.7).

In competitive scenario II, the first species (x_1) is a weaker one, and the value of Allee's constant m is found to be greater than m_t (See Eq. (4.28)) always. In the integer order LVCM with an additive Allee effect, the first species fails to exist. However, when simulations are run in the fractional order model, it is noticed that populations can coexist (Fig. (4.8). Also, it is found that with the increase in the fractional order from 0.2 to 0.8, the first species (x_1) keep on decreasing, whereas the second species (x_2) keep on increasing. When the fractional order takes a value of 0.9, the populations stabilize at $(0, K_2)$. Therefore, on changing the order in the fractional model, both populations stabilize at different coexistence points, starting from the same initial population. Therefore, on increasing the value of fractional order from 0.2 to 0.9, the memory of the system decreases, leading to a decrease in the chances of coexistence due to the Allee effect and competition imposed by the second species on the first species (x_1). Therefore, lower fractional-order models help in the coexistence of both species even when interspecific competition is stronger and the first species (x_1) is exposed to the Allee effect.

In scenario III, according to the classical LVCM, either of the populations can survive stably depending on the initial populations. If the first species (x_1) is near its carrying capacity, it will exclude the second species (x_2) competitively and vice versa. However, it is found by taking a numerical example that exposure to the Allee effect of the first species (x_1) can exclude the second species, even if the density of the second species is near to carrying capacity. Figs. 4.9 (a), 4.9 (b), and 4.9 (c) show that when the first species (x_1) is exposed to the Allee effect, it can survive by eliminating the second species (x_2) only if the initial population density of the second species (x_2) is below 3 in the integer order model, even if the initial density of the first species is very near its carrying capacity. As the initial population of the second species rises to 3, the first species (x_1) move toward extinction. In other words, a very small population density of the second species (x_2) can competitively exclude the first species (x_1) when the first species is exposed to the Allee effect. However, in fractional order models, both species can coexist, but with a very low density of the second species. The role of the memory (fractional order) is also observed in preserving the populations for a given initial population in this scenario. It is possible to find the value of fractional order (threshold) at which stability of the system switches, or in other words, whenever the fractional order is below that value system stabilizes at one equilibrium point, and after crossing that system stabilizes at other equilibria, as shown in Figs. 4.10, 4.11.

In scenario IV, in classical LVCM, both species can coexist stably. However, in the proposed model, multistability is observed depending upon the different values taken by Allee's constant. When $m < m_t$ (See Eq. (4.28)), the value of the equilibrium point obtained as E_4. The population density of the first species (x_1) is lower than that of the second species (x_2) because of the Allee effect and stronger intraspecific competition. However, this coexistence is stable in integer order and fractional order models (Fig. 4.10). On increasing the value of Allee's constant $m > m_t$ (See Eq. (4.28)), we find two equilibrium points E_4 and E_5. In both equilibrium points, the population of the second species (x_2) is quite high as compared to the first species (x_1). It is found numerically that both populations are stabilizing at the first equilibrium point E_4 in both types of models, integer order as well as in fractional order models (Fig. 4.13). However, both species always deviate from the second coexistence equilibrium point, as shown in Figs. 4.14 (a), 4.14 (b), 4.14 (c), and 4.14 (d). On increasing the value of Allee's constant $m > m_{co}$ (See Eq. (4.28)), first species (x_1) in the integer order model moves towards extinction. However, in fractional models with different orders, populations stabilize at different coexistence points as shown in Fig. 4.15. Hence, in this scenario, the proposed model has multistable behavior.

It is observed that in the case of competition, whenever Allee's constant $m < m_t$ (See Eq. (4.28)), equilibrium points obtained in each scenario remain stable without any requirement for the minimum population of the first species (x_1). However, when Allee's constant $m > m_t$, a minimum population of the first species (x_1) is required for its survival. Therefore, the Allee effect can be classified as weak if $m < m_t$ and strong if $m > m_t$ in the presence of competition. Moreover, it can be concluded that when one of the species in the classical Lotka-Volterra competition model is exposed to the Allee effect, the coexistence is promoted even in the case of stronger interspecific competition when the system is reconsidered by making use of Caputo's fractional order differential equations.

Chapter 5

Dynamical Study of Fractional Order One Prey Two Predator Food Chain Model with Allee Effect on Prey

5.1 Introduction

A wide range of species have become extinct, with others alarmingly close to extinction, due to a variety of factors, including but not limited to overexploitation, environmental pollution, mismanagement of natural resources, and excessive predation. The coexistence of all species within ecological systems is of the utmost importance for resource management and long-term survival forecasting for each species [147]. In studying ecology, mathematical models can be used to describe a typical natural phenomenon: the interaction between different species. Food chain analysis is a fundamental aspect of ecology, and the application of mathematical models to food chains has significantly contributed to advancements in contemporary ecological research [148]. The initial food chain mathematical model was formulated by US physical chemist Alfred James Lotka [124] and Italian mathematician Vito Volterra [125]. The Lotka–Volterra model has been modified many times. To develop more accurate and believable mathematical models, it is crucial to carefully analyze factors like competition among species, prey or predator harvesting, and especially predator functional responses, which represent mathematically the average consumption rate of prey by predators. Holling classified functional responses into three categories based on their effects on the rate of prey death: Holling type I, Holling type II, and Holling type III. Type I is the situation where the quantity of prey consumed varies directly with the available concentration of prey population, and type II is the situation where the consumption rate of each consumer falls as the prey density rises, eventually leveling off at a constant value. Type III behavior, characterized by a sigmoidal pattern linked to learning behavior in the predator population, occurs when the slope of the curve rises and then falls with increasing prey density [149].

Most mathematical models often consist of differential and integral equations with integer order. It is observed that fractional calculus is an effective instrument for describing systems or processes that have memory and hereditary properties. Most biological mathematical models have long-term memory, so fractional differential equations are more appropriate to describe the actual dynamic processes accurately and reliably [150,151]. Das and Samanta used Caputo's fractional derivative to explore the stability of populations in a delayed as well as non-delayed three-species food web model incorporating a fear effect and refuge in the prey population [54]. Nisar et al. studied a delayed food chain model incorporating the Allee effect in the prey population, using a fractional derivative in the Atangana-Baleanu-Caputo sense. The authors used the fixed-point theorem to investigate the existence and stability of the system solution and used the Adams-Bashforth-Moulton method for numerical analysis [103]. Matouk et al. studied the behavior of the fractional order Hastings-Powell food chain system with a new discretization technique [152]. A fractional order food chain system comprising small herbivores, medium omnivores, and large carnivores (Hastings-Powell) is examined by Liu et.al. by using Caputo's fractional derivative [153]. Using ordinary differential equations, Manna et al. investigated a food chain model comprising a prey, two predators, an intermediate, and a top predator [154]. In this study, the prey population is exposed to the strong Allee effect while growing logistically, where the Allee effect is considered a multiplicative type. Interactions between prey and intermediate predator, as well as between Intermediate predator and top predator, were considered Holling type II. The model equations are given as

$$\begin{cases} \dfrac{dx}{dt} = x\left(r\left(1-\dfrac{x}{K}\right)\left(\dfrac{x}{m}-1\right)-\dfrac{by}{x+p}\right), x(0)>0, \\ \dfrac{dy}{dt} = y\left(-c+\dfrac{bdx}{p+x}-\dfrac{ez}{q+y}\right), y(0)>0 \\ \dfrac{dz}{dt} = z\left(-f+\dfrac{gey}{q+y}\right), z(0)>0. \end{cases} \tag{5.1}$$

Here $x, y, and\ z$ are respectively prey density, intermediate predator density, and top predator density. r represents the prey's intrinsic growth rate, K is the maximum prey population supported by the environment (carrying capacity). $b, c, d,$ respectively, are the prey's rate of predation by the intermediate predator, intermediate predator's natural decay rate, and intermediate predator's conversion rate from prey to predator. e, f, g respectively are the intermediate predator's predation rate by top predator, top predator's natural decay rate, and top predator's conversion rate from intermediate predator to top predator. p and $q,$ respectively, are half-saturation constants for the intermediate and top predator. m is the minimum population of prey that is required for the survival of that population.

In this chapter, we are studying Caputo's definition-based fractional order three-species food chain model that includes a prey, an intermediate predator, and a top predator. The interactions among species are of Holing type II, and prey are exposed to an additive Allee effect.

5.2 Mathematical Model Description

Three-species food chain model in which prey are exposed to an additive Allee effect, and considering Holling type II interactions between the intermediate predator and prey population, as well as between the top predator and intermediate predator, using Caputo's fractional derivative can be represented as,

$$\begin{cases} D^{\alpha}x = \left(\left(1-\dfrac{x}{K}\right)r - \dfrac{m}{x+a} - \dfrac{by}{x+p}\right)x, \\ D^{\alpha}y = \left(-c + \dfrac{bdx}{p+x} - \dfrac{ez}{q+y}\right)y, \\ D^{\alpha}z = \left(-f + \dfrac{gey}{q+y}\right)z. \end{cases} \tag{5.2}$$

Here $x, y, and\ z$ are respectively prey density, intermediate density, and top predator density. r is the prey's intrinsic birth rate, K is the maximum prey population that the environment can support (carrying capacity). $b, c, d,$ respectively, are the prey's rate of predation by the intermediate predator, intermediate predator's natural decay rate, and intermediate predator's conversion rate. e, f, g respectively are the intermediate predator's predation rate by the top predator, the top predator's natural decay rate, and the top predator's conversion rate. p and q are the half-saturation constants for the intermediate and top predators, respectively. a is the population density of prey at which mating is reduced to half of the maximum. $m > 0$ is Allee's constant. If $m < ar$, then the Allee effect is considered to be weak, and if $m > ar$, then the Allee effect is considered to be strong. D^{α} is the Caputo fractional derivative, and $\alpha \in (0, 1]$.

5.3 Primary Evaluation of the Model

In this section, an analysis of the proposed model will be conducted to establish the properties of its solutions, including existence, uniqueness, positivity, and uniform boundedness.

5.3.1. Existence of Unique Solution

Theorem 5.1 The fractional order system given by (5.2) possesses a unique solution.

Proof: Consider a mapping $F : H \rightarrow \mathbb{R}^3$ by defining $F(X) = (F_1(X), F_2(X), F_3(X))$, where

$$\begin{cases} F_1(X) = \left[\left(1 - \dfrac{x}{K}\right)r - \dfrac{m}{x+a} - \dfrac{by}{x+p}\right]x, \\ F_2(X) = \left[-c + \dfrac{bdx}{p+x} - \dfrac{ez}{q+y}\right]y, \\ F_3(X) = \left[-f + \dfrac{gey}{q+y}\right]z. \end{cases} \tag{5.3}$$

Let $X, Y \in H$ be any two arbitrary elements. Where $X = (x_1, y_1, z_1)$ and $Y = (x_2, y_2, z_2)$.

Define $\|X\| = \sup\limits_{t}|e^{-St}\, X|$, $S > 0$. $t \in [\, t_0, T\,]$, $T < \infty$. Clearly $\|X\| = \sup\limits_{t}|X|$.

Consider

$$\|F(X) - F(Y)\| = |F_1(X) - F_1(Y)| + |F_2(X) - F_2(Y)| + |F_3(X) - F_3(Y)|,$$

$$\begin{aligned}
= \; & \left|\left[\left(1 - \frac{x_1}{K}\right) r - \frac{m}{x_1 + a} - \frac{by_1}{x_1 + p}\right] x_1 - \left[\left(1 - \frac{x_2}{K}\right) r - \frac{m}{x_2 + a} - \frac{by_2}{x_2 + p}\right] x_2\right| \\
& + \left|\left[-c + \frac{bdx_1}{p + x_1} - \frac{ez_1}{q + y_1}\right] y_1 - \left[-c + \frac{bdx_2}{p + x_2} - \frac{ez_2}{q + y_2}\right] y_2\right| \\
& + \left|\left[-f + \frac{gey_1}{q + y_1}\right] z_1 - \left[-f + \frac{gey_2}{q + y_2}\right] z_2\right|,
\end{aligned}$$

$$\begin{aligned}
\leq \; & |r(x_1 - x_2)| + \left|\frac{r}{K}(x_1^2 - x_2^2)\right| + \left|\left(\frac{x_1}{x_1 + a} - \frac{x_2}{x_2 + a}\right) m\right| \\
& + \left|\left(\frac{x_1 y_1}{x_1 + p} - \frac{x_2 y_2}{x_2 + p}\right) b\right| + |\,(y_1 - y_2)c| + \left|bd\left(\frac{x_1 y_1}{x_1 + p} - \frac{x_2 y_2}{x_2 + p}\right)\right| \\
& + \left|e\left(\frac{y_1 z_1}{q + y_1} - \frac{y_2 z_2}{q + y_2}\right)\right| + f|z_1 - z_2| + \left|ge\left(\frac{y_1 z_1}{q + y_1} - \frac{y_2 z_2}{q + y_2}\right)\right|,
\end{aligned}$$

$$\begin{aligned}
\leq \; & |x_1 - x_2|\left|r + \frac{r}{K}(\, x_1 + x_2) + \frac{ma}{c(x_1 + a)(x_2 + a)} + \frac{y_2 bp(1 + d)}{(x_1 + p)(x_2 + p)}\right| \\
& + |y_1 - y_2|\left|c + \frac{x_1 x_2 b(1 + d)}{(x_1 + p)(x_2 + p)} + \frac{x_1 bp(1 + d)}{(x_1 + p)(x_2 + p)} + \frac{z_2 eq(1 + g)}{(y_1 + q)(y_2 + q)}\right| \\
& + |z_1 - z_2|\left|f + \frac{y_1 y_2 e(1 + q)}{(y_1 + q)(y_2 + q)} + \frac{y_1 ge(1 + q)}{(y_1 + q)(y_2 + q)}\right|,
\end{aligned}$$

$$\begin{aligned}
\leq \; & |x_1 - x_2|\left(r + \frac{2rM}{K} + ma + Mbp(1 + d)\right) \\
& + |y_1 - y_2|\big(c + b(1 + d) + Mbp(1 + d) + Meq(1 + g)\big) \\
& + |z_1 - z_2|\big(f + e(1 + q) + Mge(1 + q)\big), \\
= \; & |x_1 - x_2| L_1 + |y_1 - y_2| L_2 + |z_1 - z_2| L_3 \leq L\, \|x - y\|.
\end{aligned} \tag{5.4}$$

Where,

$$\begin{cases} L_1 = \left(r + \dfrac{2rM}{K} + \ ma \ + Mbp(1+d) \right), \\ L_2 = \ \left(c + b(1+d) + Mbp(1+d) + Meq(1+g) \right)' \\ L_3 = \ \left(f + e(1+q) + Mge(1+q) \right), \end{cases} \tag{5.5}$$

and $L = max\{L_1, L_2, L_3\}$. Therefore, F satisfies the Lipschitz condition.

Let the solution of the system (5.2) be $G(X)$.

Therefore,

$$G(X) = X - \ X(0) = \frac{1}{\Gamma(\alpha)} \int_0^t F\big(X(s)\big) \ (t-s)^{\alpha-1} ds. \tag{5.6}$$

Consider

$$G(X) - \ G(Y) = \frac{1}{\Gamma(\alpha)} \int_0^t \big\{F\big(X(s)\big) - F\big(Y(s)\big)\big\} \ (t-s)^{\alpha-1} ds. \tag{5.7}$$

$$|G(X) - \ G(Y)\,| = \left| \tfrac{1}{\Gamma(\alpha)} \textstyle\int_0^t \big\{F\big(X(s)\big) - F\big(Y(s)\big)\big\} \ \ (t-s)^{\alpha-1} ds \right|,$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t \big|F\big(X(s)\big) - F\big(Y(s)\big)\big| \ (t-s)^{\alpha-1} ds. \tag{5.8}$$

Now

$$e^{-pt} \left(G(X) - \ G(Y) \right) = \frac{1}{\Gamma(\alpha)} \int_0^t e^{-pt} \big\{F\big(X(s)\big) - F\big(Y(s)\big)\big\} (t-s)^{\alpha-1} \, ds. \tag{5.9}$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^t e^{-ps} \ e^{-p(t-s)} \big\{F\big(X(s)\big) - F\big(Y(s)\big)\big\} \, (t-s)^{\alpha-1} ds. \tag{5.10}$$

$$\|G(X) - \ G(Y)\| \leq \frac{1}{\Gamma(\alpha)} \int_0^t e^{-p(t-s)} \ \big\|F\big(X(s)\big) - F\big(Y(s)\big)\big\| \, (t-s)^{\alpha-1} ds,$$

(By Eq. (5.4))

$$\|G(X) - G(Y)\| \leq \frac{L}{\Gamma(\alpha)} \int_0^t e^{-p(t-s)} \ \|x - y\|(t - s)^{\alpha-1} \, ds,$$

$$= \frac{L}{\Gamma(\alpha)} \|x - y\| \int_0^t e^{-p(t-s)} \ (t - s)^{\alpha-1} ds, \tag{5.11}$$

$$\leq \frac{L}{\Gamma(\alpha)} \|x - y\| \frac{\Gamma(\alpha)}{p^\alpha} = \frac{L}{p^\alpha} \|x - y\|. \tag{5.12}$$

Choosing p sufficiently large so that $\frac{L}{p^\alpha} < 1$, then by Theorem (1.4.8), the model given by (5.2) has a unique solution.

5.3.2. Positivity of the Solutions

Theorem 5.2 Solutions to the given system (5.2) start in $\mathbb{R}_+^3$ are nonnegative, where $\mathbb{R}_+^3$ is set of positive triplets including zero.

Proof. Let $X(t_0) = \big(x(t_0), y(t_0), z(t_0)\big) \in \mathbb{R}_+^3$ be the initial solution of the given system, let $t > t_0$ and we are to show that $X(t) \geq 0$ for all $t \geq t_0$. Let us suppose that there exists some solution $X(t_1)$ which do not lie in $\mathbb{R}_+^3$ where $t_1 > t_0$. Let us suppose that, there exists some t^* very near to t_1 s.t.

$$\begin{cases} x(t) > 0, for\ t_0 < t < t_1, \\ \qquad x(t_1) = 0, \\ x(t) < 0, for\ t_1 < t \leq t^*. \end{cases} \tag{5.13}$$

There are two possibilities:

Case 1. If $D^\alpha x(t) \geq 0 \ \forall \ t \in [t_1, t^*]$. (5.14)

Now

$$D^\alpha x(t) = \left[\left(1 - \frac{x}{K}\right) r - \frac{m}{x + a} - \frac{by}{x + p}\right] x \geq rx.$$

We have

$$D^{\alpha}\, x(t) \; \geq rx. \tag{5.15}$$

By applying the Laplace transform,

$$s^{\alpha}\, \mathrm{X}(s) - s^{\alpha-1}\, x(t_0) \geq \; \mathrm{X}(s)\, r, \tag{5.16}$$

where $\mathrm{X}(s) = \; \mathcal{L}\{x(t)\}$

$$\mathrm{X}(s)\; (s^{\alpha} - r) \; \geq s^{\alpha-1}\, x(t_0). \tag{5.17}$$

We can have from this,

$$\mathrm{X}(s) \geq \; \frac{s^{\alpha-1}}{(s^{\alpha} - r)}\; x(t_0). \tag{5.18}$$

By applying the inverse Laplace transform,

$$x(t) \geq \; \mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{(\, s^{\alpha} \; - r)} x(t_0)\right\} = x(t_0)\; E_{\alpha,1}\, \{r(t - t_0)^{\alpha}\, \} \geq 0. \tag{5.19}$$

This is a contradiction to the supposition that $x(t) < 0$ for all $t \in (t_1, t^*]$.

Case 2. If $\quad D^{\alpha}\, x(t) < 0 \; \forall \; t \; \in (t_1, t^*]. \qquad (5.20)$

This further implies,

$$\left[\left(1 - \frac{x}{K}\right) r - \frac{m}{x + a} - \frac{by}{x + p}\right] x < 0. \tag{5.21}$$

But

$$\left[\left(1 - \frac{x}{K}\right) r - \frac{m}{x + a} - \frac{by}{x + p}\right] x > x\, \rho. \tag{5.22}$$

Here ρ is the minimum value of $\left(\, 1 - \frac{x}{K}\right) r - \frac{m}{x+a} - \frac{by}{x+p} \;\forall t \in [t_1, t^*]$.

Taking the Laplace transform on both sides and proceeding as above,

$$x(t) \geq \; \mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{(\, s^{\alpha} \; - r)} x(t_0)\right\} = x(t_0)\; E_{\alpha,1}\, \{\rho\; (t - t_0)^{\alpha}\, \} \geq 0. \tag{5.23}$$

This again contradicts the assumption that $x(t) < 0$ for all $t \in (t_1, t^*]$.

Therefore, for $t \; > t_0\, , x(t) > 0$. Similarly, we can establish the positivity of $y(t)$, and $z(t)$.

5.3.3. Uniform Boundedness of the Solutions

Theorem 5.3 All non–negative solutions to the system (5.2) are uniformly bounded.

Proof: Consider a function

$$F(t) = x + \frac{y}{d} + \frac{z}{dg}. \tag{5.24}$$

$$D^{\alpha}F(t) = \left[\left(1 - \frac{x}{K}\right)r - \frac{m}{x+a} - \frac{by}{x+p}\right]x + \frac{y}{d}\left[-c + \frac{bdx}{p+x} - \frac{ez}{q+y}\right]$$

$$+\frac{z}{dg}\left[-f + \frac{gey}{q+y}\right], \tag{5.25}$$

$$= rx - \frac{rx^2}{K} - \frac{mx}{(x+a)} - \frac{yc}{d} - \frac{zf}{dg}. \tag{5.26}$$

$$D^{\alpha}F(t) + \frac{yc}{d} + \frac{zf}{dg} = rx - \frac{rx^2}{K} - \frac{mx}{(x+a)} \leq rx - \frac{rx^2}{K}. \tag{5.27}$$

Further solving, we get

$$D^{\alpha}F(t) + F(t) \leq x\,(1+r) - \frac{rx^2}{K}, \tag{5.28}$$

$$= \frac{K(1+r)^2}{4r} - \left\{x^2 - \frac{K(1+r)}{r}x + \frac{K^2(1+r)^2}{4r^2}\right\}\frac{r}{K}. \tag{5.29}$$

Therefore, (5.26) implies

$$D^{\alpha}F(t) + F(t) \leq \frac{(1+r)^2}{4r}K = Q. \tag{5.30}$$

By taking the Laplace transform and using Eq. (1.7), we have

$$s^{\alpha}\,G(s) - s^{\alpha-1}\,F(t_0) + G(s) \leq \frac{Q}{s}, \tag{5.31}$$

where $G(s) = \mathcal{L}\{F(t)\}$.

This implies,

$$G(s)\,(s^{\alpha}+1) - F(t_0)\,s^{\alpha-1} \leq \frac{Q}{s},$$

$$G(s)(s^{\alpha}+1) < \frac{Q}{s} + s^{\alpha-1}\,F(t_0). \tag{5.32}$$

This implies,

$$G(s) < \frac{s^{\alpha-1}\,F(t_0)}{(s^{\alpha}+1)} + \frac{Q}{s(s^{\alpha}+1)},$$

$$G(s) < \frac{s^{\alpha-1}\, F(t_0)}{(\, s^{\alpha} +\, 1)} + \frac{s^{\alpha-(\,\alpha+1\,)}}{(\, s^{\alpha} +\, 1)}\, Q. \tag{5.33}$$

By taking the inverse Laplace transform and using Eq. (1.8), we have,

$$F(t) \leq\ F(t_0)\, \mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{(\, s^{\alpha}+\, 1)}\right\} + Q\, \mathcal{L}^{-1}\left\{\frac{s^{\alpha-(\alpha+1)}}{(\, s^{\alpha}+\, 1)}\right\},$$

$$= F(t_0)\, E_{\alpha,1}\{-(t-t_0)^{\alpha}\} + Q\,(t-t_0)^{\alpha}\, E_{\alpha,\alpha+1}\{-(t-t_0)^{\alpha}\}. \tag{5.34}$$

Using $E_{\alpha,\beta}\{z\} = z\, E_{\alpha,\alpha+\beta}\{z\} + \frac{1}{\Gamma_{\beta}}$, we have

$$F(t) \leq F(t_0) E_{\alpha,1}\{-(t-t_0)^{\alpha}\} - Q\big[\, E_{\alpha,1}\{-(t-t_0)^{\alpha}\} - 1\,\big], \tag{5.35}$$

$$=\ (\, F(t_0) - Q\,) E_{\alpha,1}\{-(t-t_0)^{\alpha\alpha}\} + Q,$$

$$= (\, F(t_0) - Q\,) E_{\alpha,1}\{-(t-t_0)^{\alpha}\} + Q. \tag{5.36}$$

Now as t⟶ ∞ , $E_{\alpha,1}\{-(t-t_0)^{\alpha}\}\ \longrightarrow 0$.

Therefore, all solutions to the given system start in $\mathbb{R}^3_+$ lies in the region $\{X\ \in\ \mathbb{R}^3_+ : X\ \leq Q + v,\ v > 0\,\}$.

5.3.4. Equilibrium Points and Conditions for the Existence

First, we define some mathematical representations for Allee’s constant that are going to be used for the whole analysis.

$$\begin{cases} m_w = ar, \\ m_{cr} = \dfrac{Kr}{4}\left(1 +\ \dfrac{a}{K}\right)^2 . \end{cases} \tag{5.37}$$

To find equilibrium points of the system (5.2), equate to zero all equations of system (5.2),

$$D^{\alpha}x = \left[\left(\,1 - \frac{x}{K}\right) r - \frac{m}{x+a} - \frac{by}{x+p}\right] x = 0, \tag{5.38}$$

$$D^{\alpha}y = \left[-c + \frac{bdx}{p+x} - \frac{ez}{q+y}\right] y = 0, \tag{5.39}$$

$$D^{\alpha} z = \left[-f + \frac{gey}{q+y}\right] z = 0. \tag{5.40}$$

From Eq. (5.38), either $x = 0$ or $\left(1 - \frac{x}{K}\right) r - \frac{m}{x+a} = \frac{by}{x+p}$.

From Eq. (5.39), either $y = 0$ or $\frac{bdx}{p+x} - \frac{ez}{q+y} = c$.

From Eq. (5.40), either $z = 0$ or $y = \frac{fq}{(ge-f)}$.

i) One of the obtained solutions is (0,0,0).

ii) For the next solution, we put $z = 0$, in $\frac{bdx}{p+x} - \frac{ez}{q+y} = c$, we have $x = \frac{pc}{bd-c}$, and $y = \frac{rpd(bd-(p+1)c)}{(bd-c)^2} - \frac{mpd}{pc+a(bd-c)}$ is obtained by substituting the value of x in the expression, $\left(1 - \frac{x}{K}\right) r - \frac{m}{x+a} = \frac{by}{x+p}$.

iii) For another solution, put $y = 0$, in the equation $\left(1 - \frac{x}{K}\right) r - \frac{m}{x+a} - \frac{by}{x+p} = 0$. From this, we find $\left(1 - \frac{x}{K}\right) r - \frac{m}{x+a} = 0$, which can be represented as

$$\frac{x^2}{K} - x\left(1 - \frac{a}{K}\right) + \frac{m}{r} - a = 0. \tag{5.41}$$

This equation is the same as Eq. (2.32) in Chapter 2. As we solved this equation in Chapter 2, its positive solutions are given by,

$x_1' = \frac{K}{2}\left[\left(1 - \frac{a}{K}\right) + \sqrt{\left(1 + \frac{a}{K}\right)^2 - \frac{4m}{Kr}}\right]$, provided $m \leq m_{cr}$ (See Eq. (5.37)).

$x_1'' = \frac{K}{2}\left[\left(1 - \frac{a}{K}\right) - \sqrt{\left(1 + \frac{a}{K}\right)^2 - \frac{4m}{Kr}}\right]$, provided $m_w \leq m \leq m_{cr}$(See Eq. (5.37)).

iv) For the next solution, substitute $y = \frac{fq}{(ge-f)}$, in the expression $\left(1 - \frac{x}{K}\right)r - \frac{m}{x+a} = \frac{by}{x+p}$, to find the value of x. Then, to evaluate z, substitute the obtained nonnegative value of x and y in the expression $\frac{bdx}{p+x} - \frac{ez}{q+y} = c$.

Therefore, the equilibrium points along with the prerequisites for their existence are as follows:

1. E$_0$(0,0,0), without any condition.
2. E$_1$$(x_1', 0{,}0)$, $x_1' = \frac{K}{2}\left[\left(1 - \frac{a}{K}\right) + \sqrt{\left(1 + \frac{a}{K}\right)^2 - \frac{4m}{Kr}}\right]$, provided $m \le m_{cr}$ (See Eq. (5.37)).
3. $\mathrm{E}_1'(x_1'', 0{,}0)$, $x_1'' = \frac{K}{2}\left[\left(1 - \frac{a}{K}\right) - \sqrt{\left(1 + \frac{a}{K}\right)^2 - \frac{4m}{Kr}}\right]$, provided $m_w \le m \le m_{cr}$ (See Eq. (5.37)).
4. E$_2$$(x_2, y_2. 0)$, where $x_2 = \frac{pc}{bd-c}$, $y_2 = \frac{(x_2+b)}{b}\left[r\left(1 - \frac{x_2}{K}\right) - \frac{m}{x_2+a}\right]$, provided $bd - c > 0, m < r\left(a + \frac{pc}{bd-c}\right)\left(1 - \frac{pc}{K(bd-c)}\right), bdK > (K+p)c$.
5. E$_3$$(x_3, y_3, z_3)$, where $y_3 = \frac{fq}{(ge-f)}$, x_3 is the solution of the equation $\frac{fq}{(ge-f)} = \left[r\left(1 - \frac{x_3}{K}\right) - \frac{m}{x_3+a}\right]\left(1 + \frac{x_3}{b}\right)$, $z_3 = \left(\frac{q+y_3}{e}\right)\left[\frac{bdx_3}{(p+x_3)} - c\right]$,

   provided $m < a\left(r - \frac{fq}{ge-f}\right)$, $(bd-c)\, x_3 > pc, ge - f > 0$.

5.3.5. Examination of the Equilibrium Points for Local Asymptotic Stability

Representing the given system of equations as follows,

$$D^{\alpha}x = x\left[r\left(1 - \frac{x}{K}\right) - \frac{m}{x+a} - \frac{by}{x+p}\right] = f_1(x,y,z).$$

$$D^{\alpha}y = y\left[-c + \frac{bdx}{p+x} - \frac{ez}{q+y}\right] = f_2(x,y,z).$$

$$D^{\alpha}z = z\left[-f + \frac{gey}{q+y}\right] = f_3(x,y,z).$$

$$\frac{\partial f_1}{\partial x} = r - \frac{2rx}{K} - \frac{ma}{(x+a)^2} - \frac{bpy}{(p+x)^2};\ \frac{\partial f_1}{\partial y} = -\frac{bx}{(p+x)};\ \frac{\partial f_1}{\partial z} = 0;$$

$$\frac{\partial f_2}{\partial x} = \frac{bdpy}{(p+x)^2};\ \frac{\partial f_2}{\partial y} = -c + \frac{bdx}{(p+x)} - \frac{eqz}{(q+y)^2};\ \frac{\partial f_2}{\partial z} = -\frac{ey}{(q+y)};$$

$$\frac{\partial f_3}{\partial x} = 0;\ \frac{\partial f_3}{\partial y} = \frac{geqz}{(q+y)^2};\ \frac{\partial f_3}{\partial z} = -f + \frac{gey}{(q+y)}.$$

$$\mathbb{J}\,(f_1, f_2, f_3) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{bmatrix},$$

$$= \begin{bmatrix} r - \frac{2rx}{K} - \frac{ma}{(x+a)^2} - \frac{bpy}{(p+x)^2} & -\frac{bx}{(p+x)} & 0 \\ \frac{bdpy}{(p+x)^2} & -c + \frac{bdx}{(p+x)} - \frac{eqz}{(q+y)^2} & -\frac{ey}{(q+y)} \\ 0 & \frac{geqz}{(q+y)^2} & -f + \frac{gey}{(q+y)} \end{bmatrix}. \tag{5.42}$$

Theorem 5.4 The condition $m > m_w$ (See Eq. (5.37)), ensures the local asymptotic stability of the extinction point $E_0(0,0,0)$.

Proof: Substituting the point $E_0(0,0,0)$ in $\mathbb{J}\,(f_1, f_2, f_3)$, the value of the Jacobian matrix is given as,

$$\mathbb{J}\,(0,0,0) = \begin{bmatrix} r - \frac{m}{a} & 0 & 0 \\ 0 & -c & 0 \\ 0 & 0 & -f \end{bmatrix} \text{(See Eq. (5.42))}.$$

.

Its eigenvalues are given by $\lambda_1 = r - \frac{m}{a}, \lambda_2 = -c,\ \lambda_3 = -f$. The $|\arg(\lambda_i)| > \alpha\pi/2$ requires that all eigenvalues should be negative. Therefore, $r - \frac{m}{a} < 0$. This gives $m > ar = m_w$. Hence the proof.

Theorem 5.5 The equilibrium point $\mathrm{E}_1(x_1', 0,0)$ will be asymptotically stable, provided $0 < m < m_w$ (See Eq. (5.42)), and $x_1' < \frac{pc}{bd-c}$.

Proof: Substituting the point $\mathrm{E}_1(x_1', 0,0)$ in $\mathbb{J}\,(f_1, f_2, f_3)$, the value of the Jacobian matrix obtained as,

$$\mathbb{J}\,(x_1', 0,0) = \begin{bmatrix} r - \dfrac{2rx_1'}{K} - \dfrac{ma}{(x_1'+a)^2} & -\dfrac{bx_1'}{(p+x_1')} & 0 \\ 0 & -c + \dfrac{bdx_1'}{(p+x_1')} & 0 \\ 0 & 0 & -f \end{bmatrix} \text{(See Eq. (5.42))}.$$

Its eigenvalues are given by, $\lambda_1 = r - \frac{2rx_1'}{K} - \frac{ma}{(x_1'+a)^2}$, $\lambda_2 = -c + \frac{bdx_1'}{(p+x_1')}$, $\lambda_3 = -f < 0$. $|\arg(\lambda_i)| > \alpha\pi/2$ requires $\lambda_i < 0, i = 1,2,3$. Now $\lambda_1 < 0$, gives $r - \frac{2rx_1'}{K} - \frac{ma}{(x_1'+a)^2} < 0$. Solving this inequality as in Theorem 2.6, we found that $0 < m < m_w$ (See Eq. (5.37)). By taking $\lambda_2 < 0$, we have $\frac{bdx_1'}{(p+x_1')} < c$. This implies that $x_1' < \frac{pc}{bd-c}$. Hence the proof.

Theorem 5.6 The axial equilibria $\mathrm{E}_1'(x_1'', 0,0)$ is always unstable.

Proof. Substituting the point $\mathrm{E}_1'(x_1'', 0,0)$ in $\mathbb{J}\,(f_1, f_2, f_3)$, the value of the Jacobian matrix is given as,

$$\mathbb{J}\,(x_1'', 0,0) = \begin{bmatrix} r - \dfrac{2rx_1''}{K} - \dfrac{ma}{(x_1''+a)^2} & -\dfrac{bx_1''}{(p+x_1'')} & 0 \\ 0 & -c + \dfrac{bdx_1''}{(p+x_1'')} & 0 \\ 0 & 0 & -f \end{bmatrix} \text{(See Eq. (5.42))}.$$

Its eigenvalues are given by, $\lambda_1 = r - \frac{2rx_1''}{K} - \frac{ma}{(x_1''+a)^2}$, $\lambda_2 = -c + \frac{bdx_1''}{(p+x_1'')}$, $\lambda_3 = -f < 0$. $|\arg(\lambda_i)| > \alpha\pi/2$ requires $\lambda_i < 0, i = 1,2,3$. Now $\lambda_1 < 0$, gives $r - \frac{2rx_1''}{K} - \frac{ma}{(x_1''+a)^2} < 0$. Solving this inequality as in Theorem 2.7, in Chapter 2, we found that

$0 < m < m_w$ (See Eq. (5.37)). This is contrary to the existence condition of the point $\mathrm{E}_1'(x_1'', 0,0)$. Therefore, the proof is accomplished.

Theorem 5.7 The planar equilibria E2$(x_2, y_2, 0)$ will be locally asymptotically stable if $f > \frac{gey_2}{(q+y_2)}$, and any of the conditions given below are satisfied:

i. $\varphi_1^2 - 4\varphi_2 \geq 0, \varphi_1 < 0$.

ii. $\varphi_1^2 - 4\varphi_2 < 0, \varphi_1 < 0$.

iii. $\varphi_1^2 - 4\varphi_2 < 0, \varphi_1 > 0, 0 < \alpha < \frac{2}{\pi}\tan^{-1}\left|\frac{\sqrt{4\,\varphi_2 - \varphi_1^2}}{\varphi_1}\right|$.

Where φ_1 and φ_2 are described in the proof.

Proof. Substituting the point E2$(x_2, y_2, 0)$ in $\mathbb{J}\,(f_1, f_2, f_3)$, the value of the Jacobian matrix is given as,

$$\mathbb{J}\,(x_2, y_2, 0) = \begin{bmatrix} r - \dfrac{2rx_2}{K} - \dfrac{ma}{(x_2+a)^2} - \dfrac{bpy_2}{(p+x_2)^2} & -\dfrac{bx_2}{(p+x_2)} & 0 \\ \dfrac{bdpy_2}{(p+x_2)^2} & -c + \dfrac{bdx_2}{(p+x_2)} & -\dfrac{ey_2}{(q+y_2)} \\ 0 & 0 & -f + \dfrac{gey_2}{(q+y_2)} \end{bmatrix}.$$

The characteristic equation is given as,

$$(\lambda^2 - \varphi_1\lambda + \varphi_2)\left(\lambda + f - \frac{gey_2}{(s+y_2)}\right) = 0.$$

Where,

$$\varphi_1 = r - \frac{2rx_2}{K} - \frac{ma}{(x_2+a)^2} - \frac{bpy_2}{(p+x_2)^2} - c + \frac{bdx_2}{(p+x_2)},$$

$$\varphi_2 = \left(r - \frac{2rx_2}{K} - \frac{ma}{(x_2+a)^2} - \frac{bpy_2}{(p+x_2)^2}\right)\left(-c + \frac{bdx_2}{(p+x_2)}\right) + \left(\frac{bx_2}{(p+x_2)}\right)\left(\frac{bdpy_2}{(p+x_2)^2}\right)$$

Its eigenvalues are given by,

$$\lambda_1 = \frac{gey_2}{(q+y_2)} - f, \qquad \lambda_{2,3} = \frac{\varphi_1 \pm \sqrt{\varphi_1^2 - 4\varphi_2}}{2}.$$

One of the conditions to be satisfied by the equilibrium point $(x_2, y_2, 0)$ to be asymptotically stable is $\lambda_1 < 0$, which implies $f > \frac{gey_2}{(q+y_2)}$.

Case 1. If $\varphi_1^2 - 4\varphi_2 \geq 0, \varphi_1 < 0$

Then, both eigenvalues given by $\lambda_{2,3}$ are negative but real, which gives $|\arg(\lambda_{2,3})| = \pi > \alpha\frac{\pi}{2}$. Therefore, the equilibrium point will be stable.

Case 2. If $\varphi_1^2 - 4\varphi_2 < 0, \varphi_1 < 0$

Here, both eigenvalues will be complex conjugates of each other with negative real parts. Therefore,

$$|\arg(\lambda_{2,3})| = \left|-\pi + \tan^{-1}\left(\frac{\sqrt{4\,\varphi_2 - \varphi_1^2}}{\varphi_1}\right)\right|$$

$$= \left|\pi - \tan^{-1}\left(\frac{\sqrt{4\,\varphi_2 - \varphi_1^2}}{\varphi_1}\right)\right| > \alpha\,\frac{\pi}{2}.$$

Hence, stability of the equilibrium point is established.

Case 3. If $\varphi_1^2 - 4\varphi_2 < 0, \varphi_1 > 0, 0 < \alpha < \frac{2}{\pi}\tan^{-1}\left|\frac{\sqrt{4\,\varphi_2 - \varphi_1^2}}{\varphi_1}\right|$.

Here stability criterion is automatically satisfied.

Theorem 5.8 The equilibrium point E3(x_3, y_3, z_3) will be stable iff one of the conditions given below is satisfied:

1. $\Theta(P) > 0, \beta_1 > 0, \beta_3 > 0, \ \beta_1\beta_2 > \ \beta_3$.
2. $\Theta(P) < 0, \beta_1 \geq 0, \beta_2 \geq 0, \beta_3 > 0, \alpha < \frac{2}{3}$.
3. $\Theta(P) < 0, \beta_1 > 0, \beta_2 > 0, \beta_1\beta_2 > \ \beta_3, \alpha \in (0,1)$.

Where, $\beta_1, \beta_2, \beta_3$, and $\Theta(P)$ are defined below.

Proof: The Jacobian matrix at point E3(x_3, y_3, z_3) is given as,

$$\mathbb{J}\,(x_3, y_3, z_3)$$
$$= \begin{bmatrix} r - \dfrac{2rx_3}{K} - \dfrac{ma}{(x_3+a)^2} - \dfrac{bpy_3}{(p+x_3)^2} & -\dfrac{bx_3}{(p+x_3)} & 0 \\ \dfrac{bdpy_3}{(p+x_3)^2} & -c + \dfrac{bdx_3}{(p+x_3)} - \dfrac{esz_3}{(q+y)^2} & -\dfrac{ey_3}{(q+y_3)} \\ 0 & \dfrac{gesz_3}{(q+y_3)^2} & -f + \dfrac{gey_3}{(q+y_3)} \end{bmatrix},$$
$$= \begin{bmatrix} \alpha_{11} & \alpha_{12} & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ 0 & \alpha_{32} & \alpha_{33} \end{bmatrix} \text{(See Eq. (5.42))}\,.$$

$$\alpha_{11} = r - \frac{2rx_3}{K} - \frac{ma}{(x_3+a)^2} - \frac{bpy_3}{(p+x_3)^2}, \qquad \alpha_{12} = -\frac{bx_3}{(p+x_3)},$$
$$\alpha_{21} = \frac{bdpy_3}{(p+x_3)^2}, \qquad \alpha_{22} = -c + \frac{bdx_3}{(p+x_3)} - \frac{eqz_3}{(q+y_3)^2},$$
$$\alpha_{23} = -\frac{ey_3}{(q+y_3)}, \alpha_{33} = -f + \frac{gey_3}{(q+y_3)}, \alpha_{32} = \frac{geqz_3}{(q+y_3)^2}\,.$$

Therefore, writing the characteristic equation of $\mathbb{J}\,(x_3, y_3, z_3)$ as,

$$\lambda^3 + \beta_1\lambda^2 + \beta_2\lambda + \beta_3 = 0. \tag{5.43}$$

$$\beta_1 = -(\alpha_{11} + \alpha_{22} + \alpha_{33}), \tag{5.44}$$

$$\beta_2 = (\alpha_{11}\alpha_{22} + \alpha_{22}\alpha_{33} + \alpha_{33}\alpha_{11} - \alpha_{12}\alpha_{21} - \alpha_{23}\alpha_{32}), \tag{5.45}$$

$$\beta_3 = (\alpha_{11}\alpha_{23}\alpha_{32} + \alpha_{12}\alpha_{21}\alpha_{33} - \alpha_{11}\alpha_{22}\alpha_{33}), \tag{5.46}$$

$$\Theta(P) = 18\,\beta_1\beta_2\beta_3 + (\beta_1\beta_2)^2 - 4\beta_1^2\beta_3 - 4\beta_2^2 - 27\beta_3^2. \tag{5.47}$$

Now, by using the Routh–Hurwitz conditions, the equilibrium point (x_3, y_3, z_3) will be stable iff one of the conditions given below is satisfied;

1. $\Theta(P) > 0, \beta_1 > 0, \beta_3 > 0, \ \beta_1\beta_2 > \beta_3$.
2. $\Theta(P) < 0, \beta_1 \geq 0, \beta_2 \geq 0, \beta_3 > 0, \alpha < \frac{2}{3}$.

3. $\Theta(P) < 0, \beta_1 > 0, \beta_2 > 0, \beta_1\beta_2 = \beta_3, \alpha \in (0,1).$ (5.48)

5.3.6. Examination of the Equilibrium Points for Global Stability

Theorem 5.9 If $r - \frac{m}{\mathbb{M}+a} + \frac{rx_3}{K} - \frac{by_3}{\mathbb{M}+p} < 0, \frac{bx_3}{p} - \frac{c}{d} - \frac{ez_3}{d(q+\mathbb{M})} < 0,$ $\frac{ey_3}{dq} - \frac{f}{dg} < 0,$ and $\frac{mx_3}{a} - rx_3 + \frac{cy_3}{d} + \frac{fz_3}{gd} < 0,$ the equilibrium point E₃(x_3, y_3, z_3) is globally stable, where the max $\{|x|, |y|, |z|\} \leq \mathbb{M}$.

Proof: Consider the positive definite function

$$V(x,y,z) = \left(x - x_3 - x_3 \ln\frac{x}{x_3}\right) + \frac{1}{d}\left(y - y_3 - y_3 \ln\frac{y}{y_3}\right) + \frac{1}{gd}\left(z - z_3 - z_3 \ln\frac{z}{z_3}\right). \tag{5.49}$$

$$D^{\alpha}V(x,y,z) = D^{\alpha}\left[\left(x - x_3 - x_3 \ln\frac{x}{x_3}\right) + \frac{1}{d}\left(y - y_3 - y_3 \ln\frac{y}{y_3}\right) + \frac{1}{gd}\left(z - z_3 - z_3 \ln\frac{z}{z_3}\right)\right],$$

$$= D^{\alpha}\left(x - x_3 - x_3 \ln\frac{x}{x_3}\right) + \frac{1}{d}D^{\alpha}\left(y - y_3 - y_3 \ln\frac{y}{y_3}\right) + \frac{1}{gd}D^{\alpha}\left(z - z_3 - z_3 \ln\frac{z}{z_3}\right)$$

,

$$\leq \left(\frac{x - x_3}{x}\right)D^{\alpha}x + \frac{1}{d}\left(\frac{y - y_3}{y}\right)D^{\alpha}y + \frac{1}{gd}\left(\frac{z - z_3}{z}\right)D^{\alpha}z, \tag{5.50}$$

$$= (x - x_3)\left[\left(1 - \frac{x}{K}\right)r - \frac{m}{x+a} - \frac{by}{x+p}\right] + \frac{(y - y_3)}{d}\left(-c + \frac{bdx}{p+x} - \frac{ez}{q+y}\right) + \frac{(z - z_3)}{gd}\left[-f + \frac{gey}{q+y}\right],$$

$$= xr - \frac{rx^2}{K} - \frac{mx}{x+a} - rx_3 + \frac{rxx_3}{K} + \frac{mx_3}{x+a} + \frac{bx_3y}{x+p} - \frac{cy}{d} + \frac{cy_3}{d} - \frac{bxy_3}{p+x} + \frac{ey_3z}{d(q+y)} - \frac{fz}{dg} + \frac{fz_3}{gd} - \frac{eyz_3}{d(q+y)},$$

$$\leq x\left(r-\frac{m}{\mathbb{M}+a}+\frac{rx_3}{K}-\frac{by_3}{\mathbb{M}+p}\right)+y\left(\frac{bx_3}{p}-\frac{c}{d}-\frac{ez_3}{d(q+\mathbb{M})}\right)+z\left(\frac{ey_3}{dq}-\frac{f}{dg}\right)$$
$$+\frac{mx_3}{a}-rx_3+\frac{cy_3}{d}+\frac{fz_3}{gd}\leq 0, \tag{5.51}$$

if $r-\frac{m}{\mathbb{M}+a}+\frac{rx_3}{K}-\frac{by_3}{\mathbb{M}+p}<0$, $\frac{bx_3}{p}-\frac{c}{d}-\frac{ez_3}{d(q+\mathbb{M})}<0$, $\frac{ey_3}{dq}-\frac{f}{dg}<0$, $\frac{mx_3}{a}-rx_3+\frac{cy_3}{d}+\frac{fz_3}{gd}<0$, then V becomes the Lyapunov function, the equilibrium point E3(x_3, y_3, z_3) become globally stable.

Theorem 5.10 If $r-\frac{m}{\mathbb{M}+a}+\frac{rx_2}{K}-\frac{by_2}{\mathbb{M}+p}<0, \frac{bx_2}{p}-\frac{c}{d}<0, \frac{ey_2}{dq}-\frac{f}{dg}<0$, and $\frac{cy_2}{d}-rx_2+\frac{mx_2}{a}<0$, the equilibrium point E2$(x_2, y_2. 0)$ is globally stable, where the max $\{|x|,|y|,|z|\}\leq\mathbb{M}$.

Proof. Consider a positive definite function

$$V'^{(x,y,z)}=\left(x-x_2-x_2\ln\frac{x}{x_2}\right)+\frac{1}{d}\left(y-y_2-y_2\ln\frac{y}{y_2}\right)+\frac{z}{gd}. \tag{5.52}$$

$$D^{\alpha}V'(x,y,z)=D^{\alpha}\left[\left(x-x_2-x_2\ln\frac{x}{x_2}\right)+\frac{1}{d}\left(y-y_2-y_2\ln\frac{y}{y_2}\right)+\frac{z}{gd}\right],$$

$$=D^{\alpha}\left(x-x_2-x_2\ln\frac{x}{x_2}\right)+\frac{1}{d}D^{\alpha}\left(y-y_2-y_2\ln\frac{y}{y_2}\right)+\frac{1}{gd}D^{\alpha}z,$$

$$\leq\left(\frac{x-x_2}{x}\right)D^{\alpha}x+\frac{1}{d}\left(\frac{y-y_2}{y}\right)D^{\alpha}y+\frac{1}{gd}D^{\alpha}z, \tag{5.53}$$

$$=(x-x_2)\left[\left(1-\frac{x}{K}\right)r-\frac{m}{x+a}-\frac{by}{x+p}\right]+\frac{(y-y_2)}{d}\left(-c+\frac{bdx}{p+x}-\frac{ez}{q+y}\right)$$
$$+\frac{z}{gd}\left[-f+\frac{gey}{q+y}\right],$$

$$=xr-\frac{rx^2}{K}-\frac{mx}{x+a}-rx_2+\frac{rxx_2}{K}+\frac{mx_2}{x+a}+\frac{bx_2y}{x+p}-\frac{cy}{d}+\frac{cy_2}{d}-\frac{bxy_2}{p+x}$$
$$+\frac{ey_2z}{d(q+y)}-\frac{fz}{dg},$$

$$\leq x\left(r-\frac{m}{\mathbb{M}+a}+\frac{rx_2}{K}-\frac{by_2}{\mathbb{M}+p}\right)+y\left(\frac{bx_2}{p}-\frac{c}{d}\right)+z\left(\frac{ey_2}{dq}-\frac{f}{dg}\right)+\ \frac{cy_2}{d}-rx_2$$
$$+\frac{mx_2}{a}\leq 0, \tag{5.54}$$

if $r-\frac{m}{\mathbb{M}+a}+\frac{rx_2}{K}-\frac{by_2}{\mathbb{M}+p}\leq 0$, $\frac{bx_2}{p}-\frac{c}{d}\leq 0$, $\frac{ey_2}{dq}-\frac{f}{dg}\leq 0$, $\frac{cy_2}{d}-rx_2+\frac{mx_2}{a}\leq 0$, then V' becomes the Lyapunov function, the equilibrium point E2$(x_2, y_2.0)$ become globally stable.

Theorem 5.11 If $r-\frac{m}{\mathbb{M}+a}+\frac{rx_1}{K}<0, \frac{bx_1}{p}-\frac{c}{d}<0$, and $m<ar$, the equilibrium point E1$(x_1,0,0)$ will be globally stable, where the max $\{|x|,|y|,|z|\}\leq\mathbb{M}$.

Proof. Consider a positive definite function

$$V"(x,y,z)=\left(x-x_1-x_1\ln\frac{x}{x_1}\right)+\frac{y}{d}+\frac{z}{gd}. \tag{5.55}$$

$$D^\alpha V"(x,y,z)=D^\alpha\left[\left(x-x_1-x_1\ln\frac{x}{x_1}\right)+\frac{y}{d}+\frac{z}{gd}\right],$$

$$\leq\left(\frac{x-x_1}{x}\right)D^\alpha x+\ \frac{1}{d}\ D^\alpha y+\frac{1}{gd}D^\alpha z, \tag{5.56}$$

$$=(x-x_1)\left[r\left(1-\frac{x}{K}\right)-\frac{m}{x+a}-\frac{by}{x+p}\right]+\frac{y}{d}\left[-c+\frac{bdx}{p+x}-\frac{ez}{q+y}\right]$$
$$+\frac{z}{gd}\left[-f+\frac{gey}{q+y}\right],$$

$$=xr-\frac{rx^2}{K}-\ \frac{mx}{x+a}-rx_1+\ \frac{rxx_1}{K}+\frac{mx_1}{x+a}+\frac{bx_1y}{x+p}-\frac{yc}{d}-\frac{zf}{gd},$$

$$\leq x\left(r-\frac{m}{\mathbb{M}+a}+\frac{rx_1}{K}\right)+y\left(\frac{bx_1}{p}-\frac{c}{d}\right)-\frac{zf}{gd}+\frac{mx_1}{a}-rx_1\leq 0, \tag{5.57}$$

if $r-\frac{m}{\mathbb{M}+a}+\frac{rx_1}{K}<0$, $\frac{bx_1}{p}-\frac{c}{d}<0$, $m<ar$, then $V"$ becomes the Lyapunov function and equilibrium point E1$(x_1,0{,}0)$ become globally stable.

5.4 Numerical Analysis

Table 5.1: Table for Values of Parameters Used for Numerical Simulations

Parameters	**Values in Case 1**	**Values in Case 2**	**Values in Case 3**	**Sources**
K	1	1	1	[154]
r	0.5	0.5	0.5	[154]
a	0.3	0.3	0.3	Assumed
m	0	0.1	0.16	Assumed
b	0.5	0.5	0.5	[154]
c	0.01	0.01	0.01	[50]
p	0.5	0.5	0.5	[58]
d	0.9	0.9	0.9	Assumed
g	0.8	0.8	0.8	Assumed
e	0.16	0.16	0.16	Assumed
f	0.025	0.025	0.025	Assumed
q	0.9	0.9	0.9	Assumed

The values obtained for Allee's constant are as follows,

$$m_w = 0.15,\ \ m_{cr} = 0.21 \text{ (see Eq. 5.37)}$$

Case 1. When Allee's constant $m = 0$

To analyze the system without the Allee effect, we select $m = 0$. Solving food chain system the value of the coexistence point obtained is E_3 (0.84,0.22,1.9). To check the stability of the equilibria E_3, the various values are as follows (see Theorem 5.8).

$$\beta_1 = 0.263 > 0, \qquad \beta_2 = 0.034 > 0, \qquad \beta_3 = 0.001 > 0,$$
$$\Theta(P) = -0.005 < 0, \qquad \beta_1\beta_2 > \beta_3.$$

Therefore, by Theorem 5.8, the stability conditions of the coexistence point are satisfied for $\alpha \in (0,1)$. Figures 5.1 and 5.2 are drawn by taking the fractional order from 0.8 and

0.85; it is observed that populations stabilize after small oscillations. However, for Figures 5.3 and 5.4, on increasing the fractional order to 0.9 and 0.95, respectively, the size of the oscillations increases. Figure 5.5 is plotted by taking the integer order model, and populations show the chaotic behavior in the traditional model.

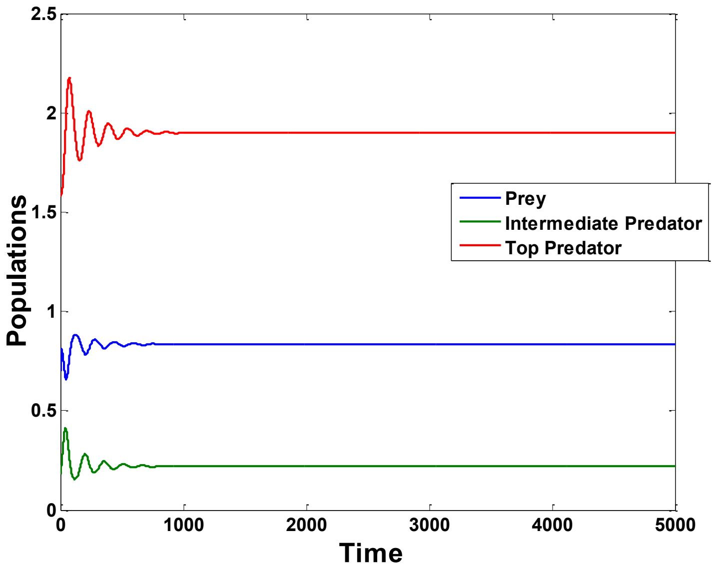


Figure 5.1 Stability analysis of the coexistence point (0.84,0.22,1.9) in fractional model of order 0.8 without Allee effect

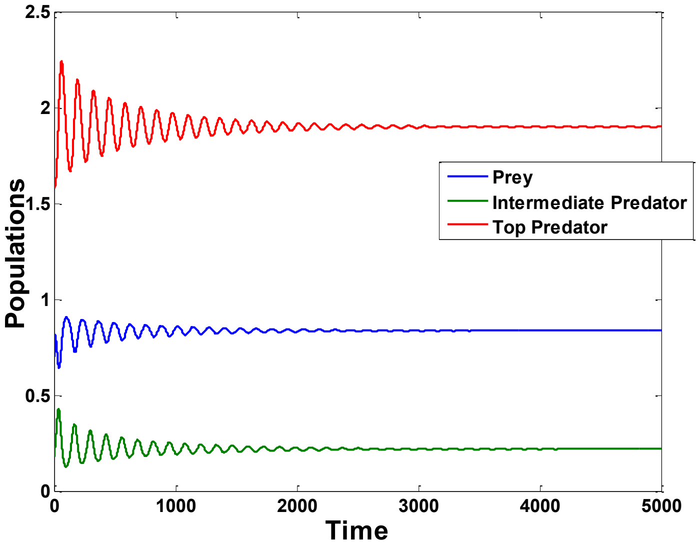


Figure 5.2 Stability analysis of the coexistence point (0.84,0.22,1.9) in fractional model of order 0.85 without Allee effect

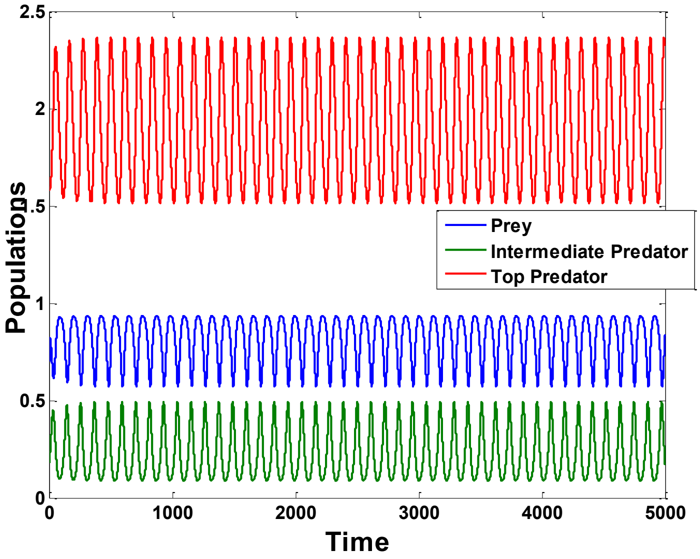


Figure 5.3 Stability analysis of the coexistence point (0.84,0.22,1.9) in fractional model of order 0.9 without Allee effect

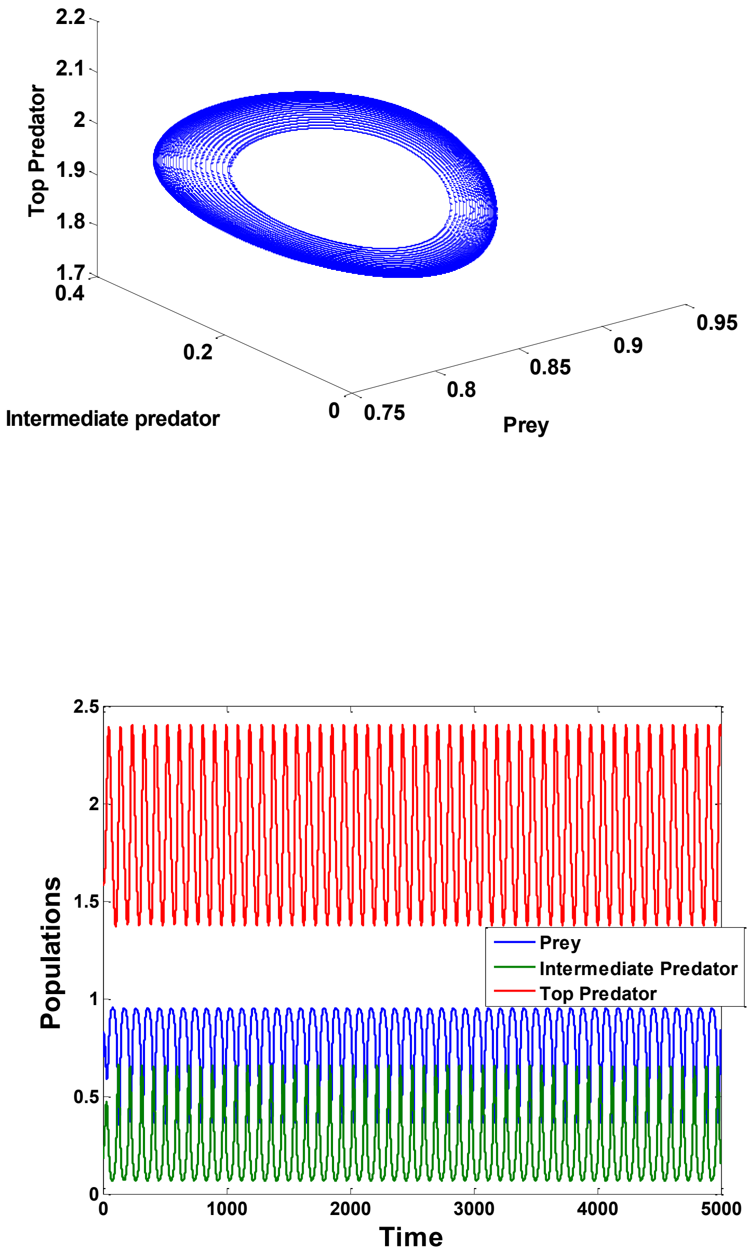
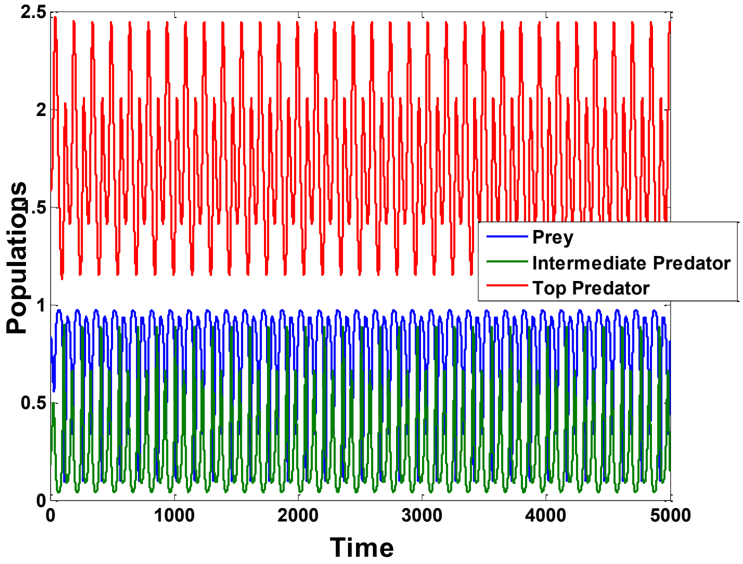


Figure 5.4 Stability analysis of the coexistence point (0.84,0.22,1.9) in fractional model of order 0.95 without Allee effect



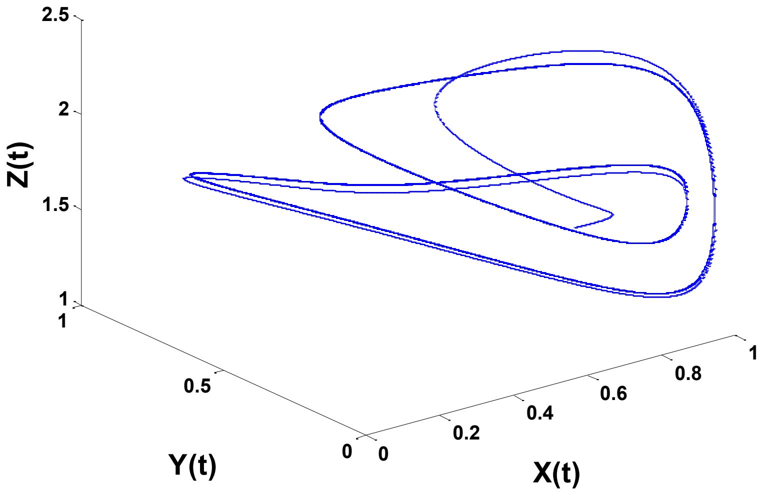


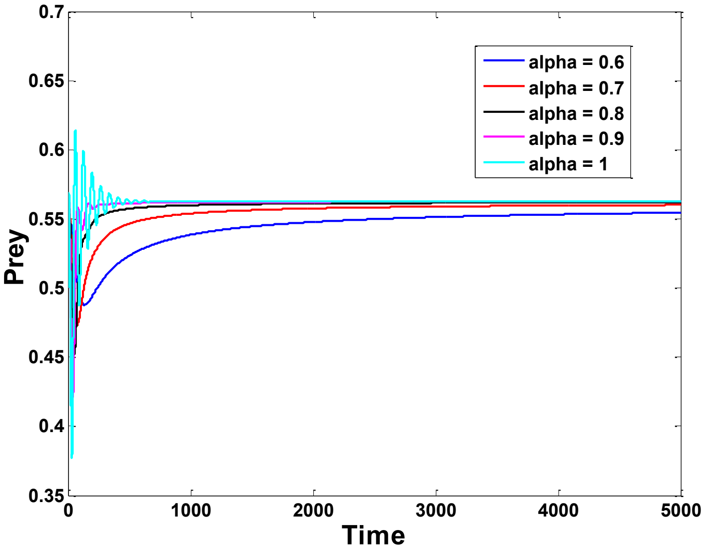
Figure 5.5 Stability analysis of the coexistence point (0.84,0.22,1.9) in integer model without Allee effect

Case 2. When Allee’s constant $0 < m < m_w$

By selecting Allee’s constant $m = 0.1$, the prey are exposed to a weak Allee effect, and solving the food chain system, the equilibrium point obtained is (0.58,0.22,1.68). It is observed that when prey are exposed to a weak Allee effect, the population density of the prey population as well as that of the top predator decreases, whereas the population density of the intermediate predator increases. The various values according to Theorem 5.8, to check the stability of the equilibria, are as follows.

$$\beta_1 = 0.1246 > 0, \qquad \beta_2 = 0.0088 > 0, \qquad \beta_3 = 0.0008 > 0,$$
$$\Theta(P) = -0.0008 < 0, \qquad \beta_1\beta_2 > \beta_3.$$

Therefore, stability conditions of the coexistence point are satisfied in $\alpha \in (0,1)$ (see Theorem 5.8). To check the stability of populations in a fractional order model and integer order models, simulations are run by taking different values of fractional orders and integer orders equal to one. Figure 5.6 is plotted, and it is observed that populations are stabilizing at the (0.56,0.22,1.68) in all models. Therefore, it is found that when the prey population was not experiencing the Allee effect, populations failed to coexist whenever fractional order takes values greater than 0.85, but exposure to weak Allee effect of prey population is helpful for all populations to coexist in each model.



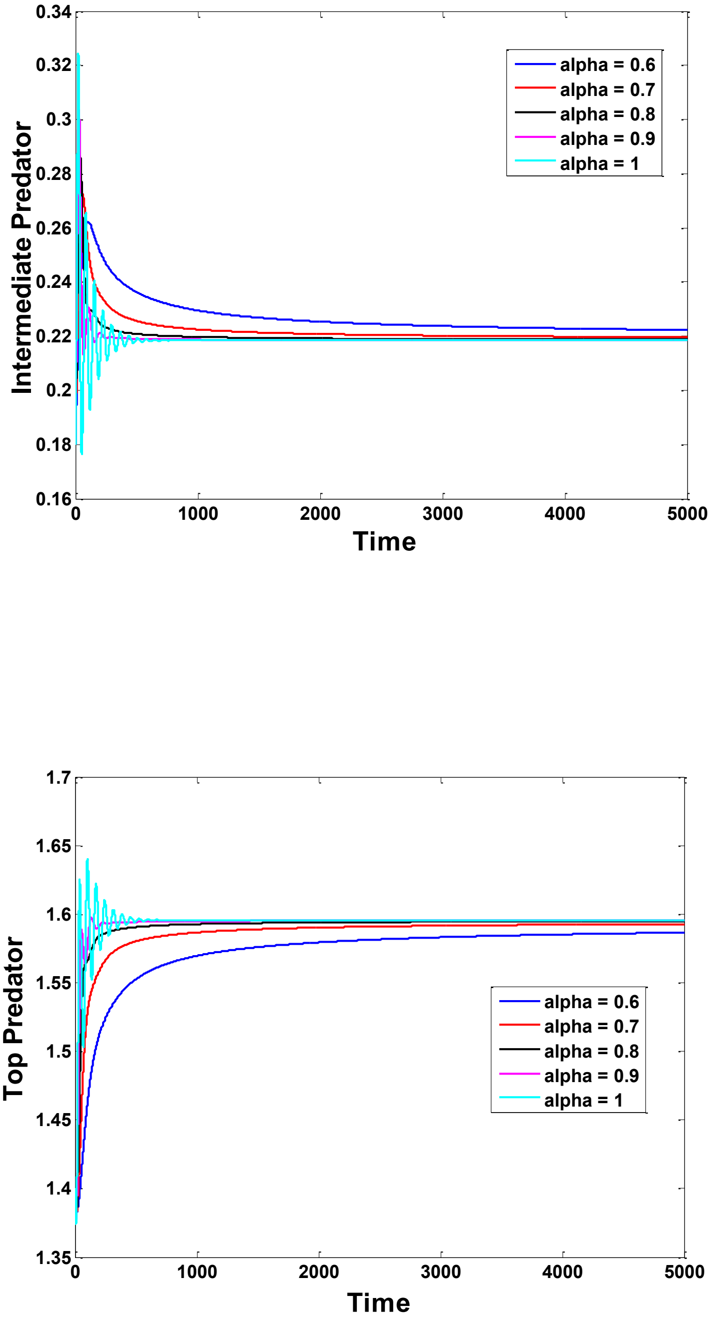


Figure 5.6 Stability analysis of the populations in different models when prey is subjected to the weak Allee effect

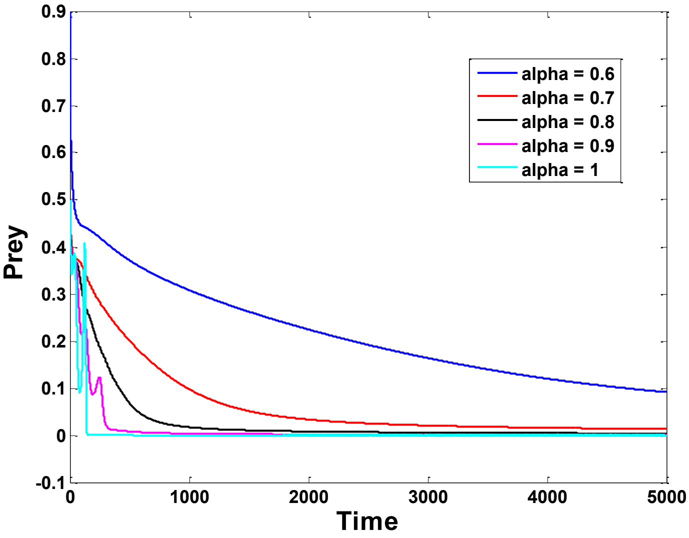
Case 3. When Allee's constant $m_w < m < m_{cr}$

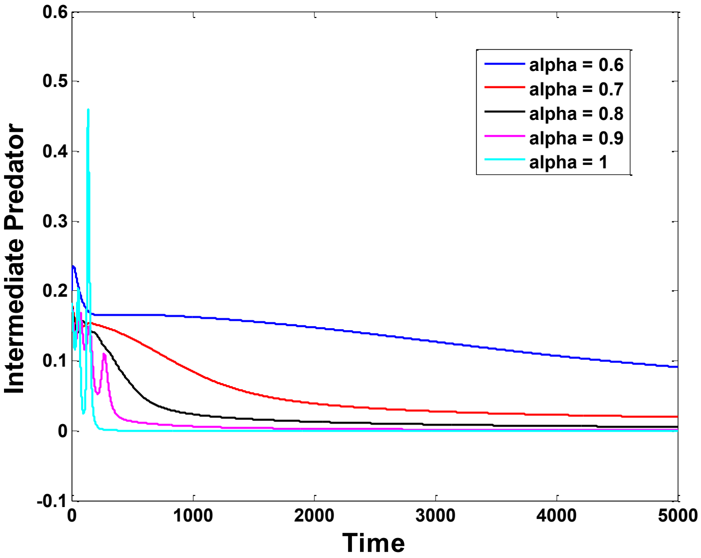
By changing $m = 0.1$ to 0.16, the prey are exposed to a strong Allee effect. Here

$$m = 0.16 > a\left(r - \frac{fq}{ge - f}\right) = 0.0845,$$

$$m = 0.16 > r\left(a + \frac{pc}{bd - c}\right)\left(1 - \frac{pc}{K(bd - c)}\right) = 0.154.$$

Therefore, from section 5.3.4, the only possible solution is E1$(x_1', 0,0) = (0.67,0,0)$. But by Theorem 5.5, the stability conditions of this solution are not satisfied. Moreover, by Theorem 5.4, the stability conditions of the extinction point are satisfied. By taking the initial population as (0.9,0.2,1.5), simulations are run in different fractional order models and integer order models. Figure 5.7 is plotted, and it is found that all populations are moving toward extinction. However, the lower fractional order models are slow, showing that coexistence is possible even if the prey population is experiencing critical depensation (Strong Allee effect).





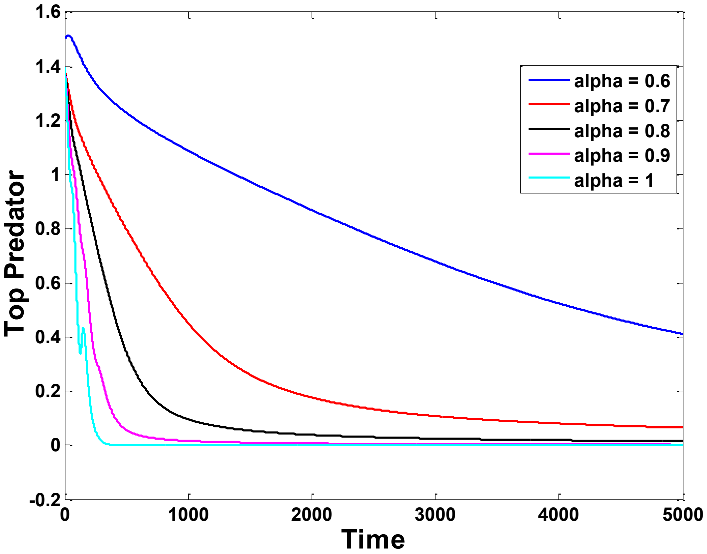


Figure 5.7 Stability analysis of the populations in different models when prey are exposed to strong Allee effect

Table 5.2: Table for Values of Parameters Used for Numerical Simulations

Parameters	**Values**	**Sources**
K	1	[154]
r	0.5	[154]
a	0.3	Assumed
m	0.1	Assumed
b	0.5	[154]
c	0.05	Assumed
p	0.5	[58]
d	0.9	Assumed
g	0.5	Assumed
e	0.2	Assumed
f	0.03	Assumed
q	0.9	Assumed

The values obtained for Allee's constant are as follows,

$$m_w = 0.15, \quad m_{cr} = 0.21 \text{ (see Eq. 5.37)}$$

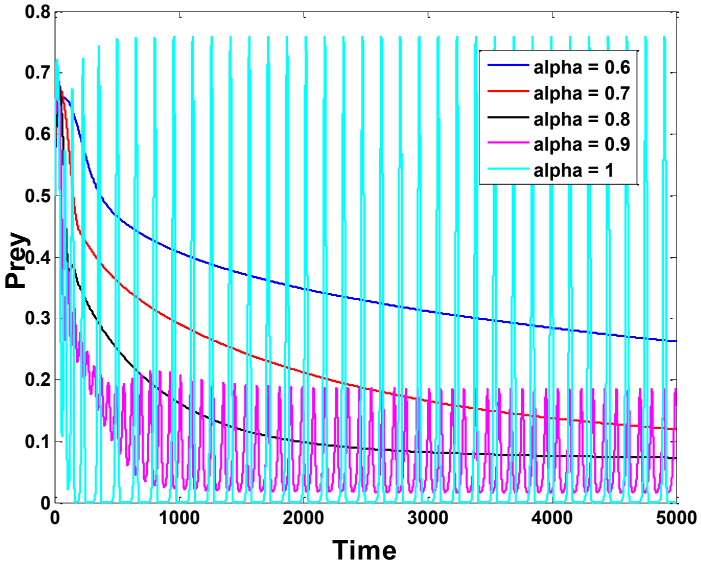
In this example, the prey population is under a weak Allee effect as $m < m_w$ and natural decay rate of intermediate predator is increased from 0.01 to 0.05 and of the top predator from 0.025 to 0.03, the rate of predation of the top predator is also increased from 0.16 to 0.2 and the conversion efficiency of the top predator is decreased from 0.8 to 0.5. It is observed that top predators move towards extinction, as the solution of the model comes out to be top predator-free as equilibrium point (0.0625,0.217,0). The various values to check the stability of the equilibria according to Theorem 5.7 are as follows.

$f = 0.03, \qquad \frac{gey_2}{(q+y_2)} = 0.016,\ \varphi_1 = 0.0377 > 0, \qquad \varphi_2 = 0.0086.$

Clearly

$$f > \frac{gey_2}{(q+y_2)}, \varphi_1^2 - 4\varphi_2 = -0.033 < 0, \alpha = \frac{2}{\pi}\tan^{-1}\left|\frac{\sqrt{4\,\varphi_2 - \varphi_1^2}}{\varphi_1}\right| = 0.87.$$

Therefore, by Theorem 5.7 stability conditions of the obtained equilibrium point are satisfied whenever $\alpha < 0.87$. By taking the initial population as (0.4,0.4,1.2), simulations are run in different models. Figure 5.8 is plotted in various models and it is found that prey and intermediate predator populations show stability around equilibrium point whenever we take fractional order less than 0.87, as fractional order crosses the value 0.87 prey population and the intermediate predator start oscillating around the equilibrium point and shows Hopf bifurcation and in integer order model size of the oscillations increases. However, the top predator always moves towards extinction in all models.



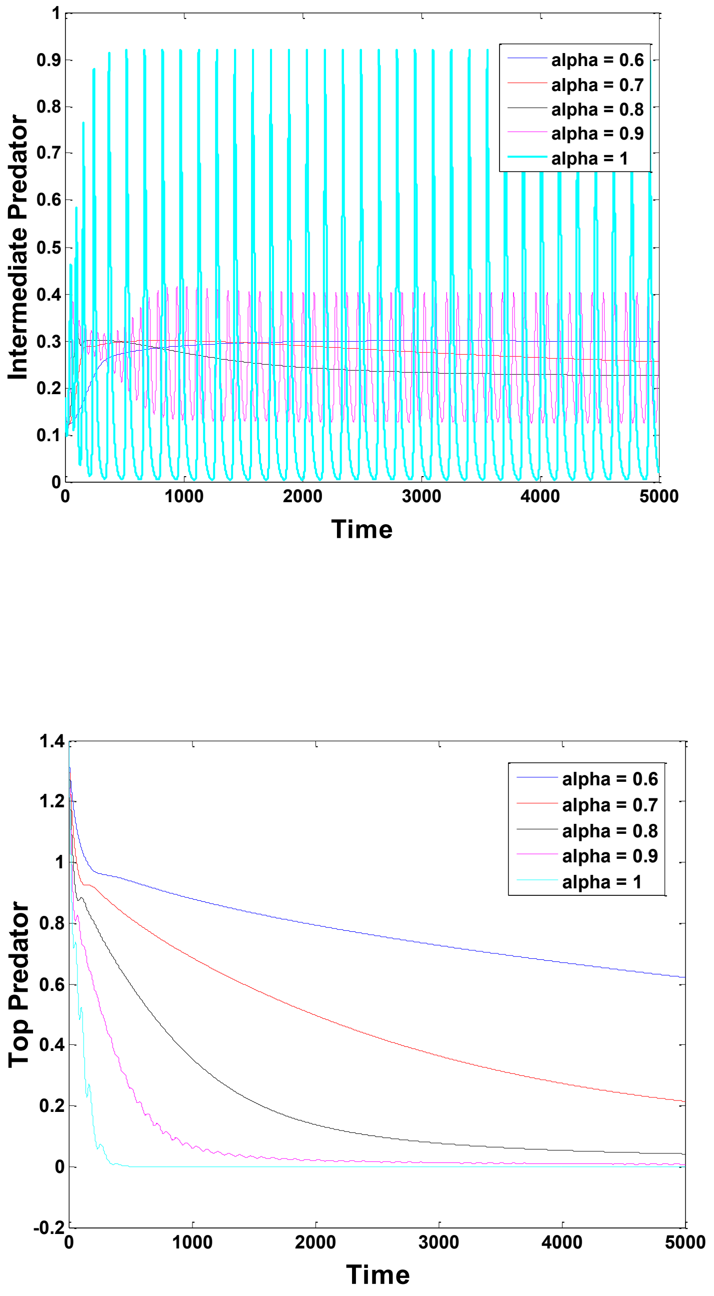


Figure 5.8 Stability analysis of the populations in different models at the equilibrium point (0.0625,0.217,0) when prey are exposed to weak Allee effect

Table 5.3: Table for Values of Parameters Used for Numerical Simulations

Parameters	**Values**	**Sources**
K	1	[154]
r	0.5	[154]
a	0.3	Assumed
m	0.1	Assumed
b	0.2	Assumed
c	0.05	Assumed
p	0.5	[58]
d	0.3	Assumed
g	0.5	Assumed
e	0.2	Assumed
f	0.03	Assumed
q	0.9	Assumed

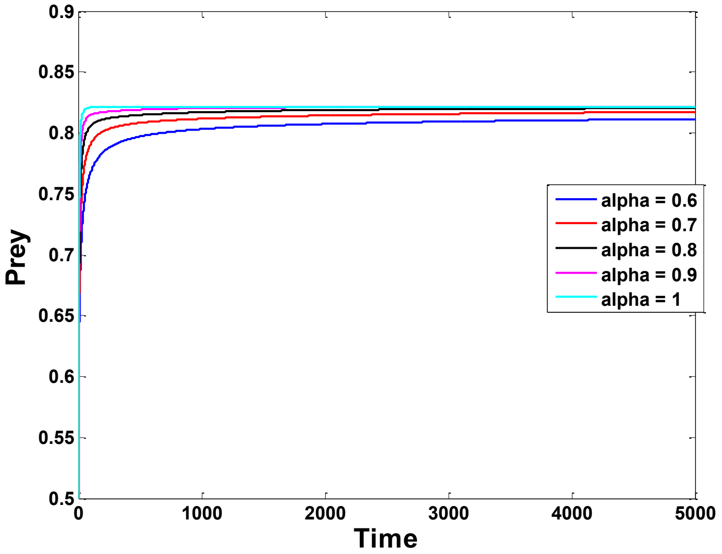
The values obtained for Allee's constant are as follows,

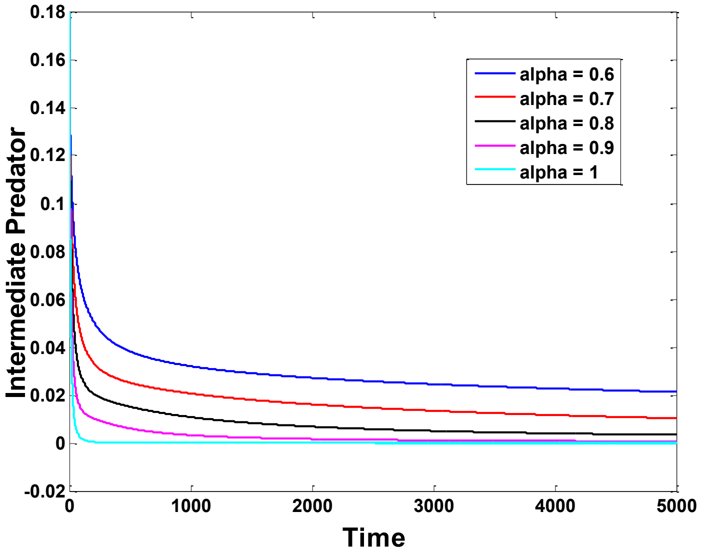
$$m_w = 0.15, \quad m_{cr} = 0.21 \ \text{(see Eq. 5.37)}.$$

In this example, prey are exposed to a weak Allee effect as $m < m_w$ and prey's rate of predation by intermediate predator has been decreased from 0.5 to 0.2 and the conversion efficiency of the intermediate predator is decreased from 0.9 to 0.3. Solving the above equations we have an equilibrium point (0.82,0,0) which shows that the intermediate as well as the top predator fails to exist. The various values to check the stability of the equilibria are as follows.

$$x_1' = 0.82, \qquad \frac{pc}{bd - c} = 2.5, \qquad m = 0.1.$$

Clearly $x_1' < \frac{pc}{bd-c}$, and $m < m_w$. Therefore, by Theorem 5.5 stability conditions of the obtained equilibria are satisfied. To do the stability analysis numerically of the obtained solution, simulations are run in fractional as well as integer models by taking the initial population as (0.7,0.4,0.2). In all models, populations are converging toward the equilibrium point as shown in Figure 5.9. Being the slow movement of the populations in fractional order derivative models, the populations of intermediate and top predators can be protected by using the fractional order modeling, though the population density of both predators will be quite low.





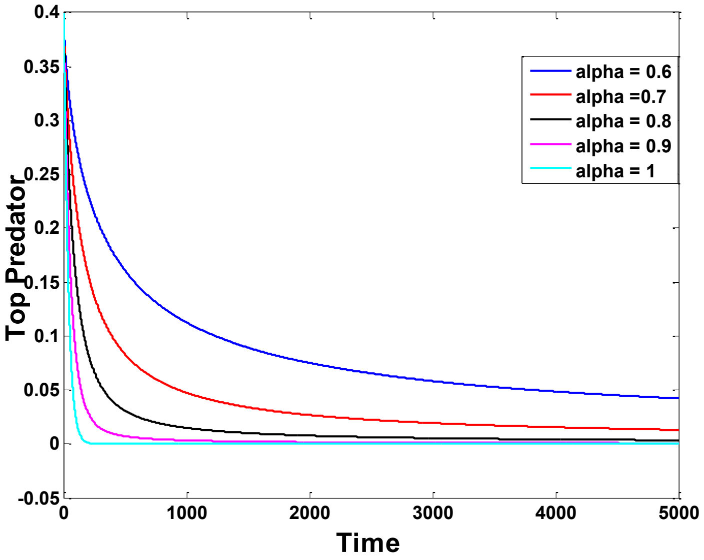


Figure 5.9 Stability analysis of the populations in different models at the equilibrium point (0.82,0,0) when prey is subjected to the weak Allee effect

5.5 Conclusion

This study examines a food chain model comprising a prey exposed to the Allee effect and growing logistically, an intermediate predator, and a top predator, with the help of Caputo fractional order differential equations. The Holling type-II interactions are considered to characterize the relationship between prey and intermediate predator, as well as between top and intermediate predators. The solutions of the model are investigated concerning their existence, uniqueness, nonnegativity, and boundedness. Discussion is given on the local stability and global stability of every steady state point. It is noticed that exposing the prey population to the Allee effect can impact the survival of all populations of the considered system negatively as well as positively. As it is seen that when prey are not exposed to Allee effect, all populations in the food chain model (Table 5.1) coexist stably in lower fractional models (Figs.5.1,5.2). However, population behavior is observed to be oscillatory and chaotic in the case of higher fractional orders and integer models, respectively (Figs. 5.3,5.4, 5.5). When the prey are exposed to the weak Allee effect, all populations coexist stably in all models; fractional order and integer order models (Fig.5.6). This shows that the exposure of the prey population to weak Allee effect can help in stable coexistence of the desired populations. When prey population is exposed to strong Allee effect, then all populations of model move towards extinction in all models fractional as well as in integer order models (Fig.5.7). However, it is also noticed that, when prey are exposed to weak Allee effect, the existence of top predator is quite sensitive to natural decay rate of both predators, attack rate and conversion efficiency of top predator. On increasing attack rate of top predator and natural death rate of both predators and decreasing the conversion efficiency of top predators we found that top predators are getting extinct in integer model and higher fractional order models. However, the prey and intermediate predators can coexist stably in lower fractional order models, oscillate in higher fractional order models, and turns chaotic in integer order models (Fig.5.8). On further decreasing the attack rate and conversion efficiency of the intermediate predator, it is found that intermediate predator along with top predator also moves towards the extinction in all models (Fig. 5.9). However, being slow movement of the populations in fractional order derivative models, the intermediate and the top predators' population can be protected by using the fractional order modeling, though the population density of the both predators will remain quite low. Hence it can be judged that the fractional order models are better for studying population dynamics, as we have seen that via numerical simulations the points that are unstable in the integer order model can be stabilized using fractional order modeling. Lower fractional order models move slowly toward the equilibrium points, which shows that low fractional derivatives preserve the memory of the past. This shows that populations have not adopted the trend fully shown by the traditional modeling, so timely strategies can help in saving the population. As fractional order derivatives do not affect the equilibrium points, therefore assist us by warning at an early stage by demonstrating the trends that populations will exhibit in the future.

The Allee effect is crucial to population ecology and conservation biology because it clarifies how difficult it can be for small populations to endure and expand. The presence of the Allee effect in the prey population can destabilize the stable equilibrium state, and sometimes it can trigger chaotic behavior, where the system exhibits unpredictable and irregular fluctuations. In fact presence of the Allee effect can directly influence the entire food chains existing in nature. With the scarcity of food due to the Allee effect in the prey population, the predators can also move toward extinction. This can destroy the food chains existing in nature.

Chapter 6

Conclusion and Future Scope

In this chapter, we are going to conclude the findings of the research work done and the future scope of the study.

6.1 Conclusion

The work done in the study considers the mathematical and numerical analysis of single-species and multispecies mathematical models in population dynamics, incorporating the additive Allee effect in one of the species by making use of fractional order differential equations, where the fractional derivative is considered in the Caputo sense. Populations that are suffering from the Allee effect are prone to extinction, so findings of the proposed work can help save the populations exposed to the Allee effect. Moreover, the findings of this work can also be exploited to get rid of the unwanted species. The use of fractional derivatives has helped in studying the hereditary properties of populations, which are often overlooked in the traditional models. This is crucial for studying the population dynamics, as past interactions or any environmental changes influence the present behavior.

In Chapter 2, we studied the single-species logistical growth model by incorporating the additive Allee effect. Without the Allee effect, the logistic growth model population stabilizes at its maximum value (carrying capacity) irrespective of the initial population of the species. However, by incorporating the additive Allee effect, populations never reach to the carrying capacity, but in the case of weak Allee effect, depending upon the different values of Allee's constant, the populations can be stabilized at different equilibrium points. These results can help maintain the desired level of the population. When a population is subjected to a strong Allee effect, it requires a minimum population to survive. We have found a mathematical expression for calculating the threshold level of species that are feeling critical depensation (strong Allee effect), which is validated through numerical simulations. Knowing the values of all parameters for a specific spatial distribution of a vulnerable species, a threshold level beyond which the survival of that species is not conceivable can be calculated using the result of calculating the threshold level. Once knowing the threshold level of the population prone to the strong Allee effect, timely strategies can help in saving that population from extinction. The strong Allee effect notion is frequently used in pest control [77,78]. To eliminate the unwanted pest population, a strong Allee effect can be introduced in the pest population by reducing its number below the threshold level (independent of the method used), which can be computed.

In Chapter 3, a prey predator model in which the prey are growing logistically along with suffering from additive Allee effect, is considered. It has been observed that when prey populations are exposed to a weak or strong Allee effect, the prey's predation rate becomes an important factor, as controlling it can help save both populations. In situations when prey are exposed to a weak Allee effect with a higher rate of predation, both populations can oscillate through a Hopf bifurcation. However, the optimal rate of predation helps in the coexistence of both populations stably. A very low rate of predation can make predators extinct. In situations when prey are exposed to a strong Allee effect, a suitably chosen rate of predation can help both populations to coexist stably. This shows that the Allee effect can be controlled with the rate of predation. These findings can help in saving the populations where interactions are prey and predatory in nature. Findings of the study can help in making artificial strategies to save the desired populations.

In Chapter 4, the traditional Lotka Volterra competition model is explored by incorporating the additive Allee effect in one of the species, making use of the Caputo fractional derivative. In this work, very interesting results are found. Exposure to Allee effect of one of the species in the competition model has provided the rich dynamics of populations by establishing the multistability of the populations in various competition scenarios with different levels of the Allee's constant. Moreover, in some cases it is found that with same value of Allee's constant, different solutions exhibit stability depending upon the value of the fractional order. In other words, it is possible to find the value of fractional order below which one solution is stable and after crossing that other solution show stability starting with same conditions. In the traditional model without Allee effect, competing populations can coexist only if intraspecific competition is stronger than the interspecific competition, but in our model, populations can coexist even if interspecific competition is stronger than the intraspecific competition. Furthermore, values of Allee's constant are identified to determine the severity of the Allee effect, that is, when the Allee effect will influence weakly and strongly in the presence of competition.

In Chapter 5, a food chain model comprising a prey, an intermediate predator, and a top predator is explored. In the proposed model the prey population grow logistically and a functional response of Holling type II is observed between intermediate predator and prey as well as between top predator and intermediate predator. It is found that the introduction of the weak Allee effect in prey populations and the usage of fractional derivatives can aid in the stable coexistence of all populations. In one of the examples, when prey population was not exposed to Allee effect, all populations were showing chaotic behavior in integer order models and were oscillating in higher fractional order models, but when prey were subjected to the weak Allee effect, the populations showed stable coexistence in all models. In the situations where prey were influencing the strong Allee effect, populations were coexisting in lower fractional models. However, it is also found that the optimal attack rate, fitness of the predators in terms of conversion efficiency, and death rate can also help in the coexistence of all populations.

In each model, it is found that replacing the ordinary derivative with Caputo fractional derivatives has no effect on the values of equilibrium points, and the points that were stable in ordinary integral models remain stable in the fractional model too. Rather, the usage of fractional modeling provides greater flexibility in the stability region of the equilibrium points. While studying two and three-dimensional models, we found the equilibrium points that were not stable when considered in integer order models but turned out to be stable when considered in fractional order models. This is due to the long-term memory-preserving nature of the Caputo derivative. A lower fractional order model has less memory of the past, therefore stabilizes early, and takes much more time to reach the equilibrium points. However, as the fractional derivative's order rises, the system's strong memory prevents it from forgetting the trend, allowing it to stabilize at the determined equilibrium points in a considerably shorter period. In many cases, populations in different fractional order derivative models can stabilize at different points. Fractional order models can also help in controlling the oscillatory and chaotic behavior of the populations. Therefore, it is always better to study the real-world phenomenon using fractional-order differential equations.

Studying the Allee effect is important because it sheds light on how population size impacts individual fitness and the overall dynamics of the populations, especially at low density. This knowledge is crucial for understanding species' risk of extinction, management of rare and endangered species, and the success of reintroduction programs.

6.2 Future Scope

In this research, we have incorporated the Allee effect (additive) in various fractional-order population models that potentially eradicate the suffering species permanently. This study has the future scope to understand the mechanisms and causes of the Allee effect, such as mating behavior, inbreeding depression, cooperative behavior, and resource limitation. In mathematical modeling, there is scope for developing more accurate and sophisticated models incorporating the Allee effect to understand its nuances in different populations and ecosystems. In biomedical sciences, the study of the Allee effect has the scope to understand and manage the complex biological system, particularly in Cancer research and its treatment [145,155].

This research has been done by making use of the fractional derivative in the Caputo sense, and for numerical analysis, we have used the Adam Bashforth Predictor and Corrector method to solve the models. However, the other fractional derivatives currently in use are Atangana-Baleanu, Caputo-Fabrizio. These fractional derivatives are as suitable and efficient as the Caputo fractional derivative. Various numerical schemes are available to solve fractional-order differential equations. These include but are not limited to Laplace-Adomian decomposition technique (LADM), Homotopy analysis method (HAM), Homotopy perturbation method (HPM), Adomian decomposition method (ADM), Laplace transformation, Variational iteration with Pade approximation, Corrected Fourier series, Natural decomposition method, and Fractional complex transformation, optimal q-HAM, etc. Therefore, the study can be further explored by using the above-mentioned fractional derivatives and the numerical schemes.

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Paper Publications and Presentations

Papers Published and Accepted from the Thesis

1. Kalra, Preety, and Nisha Malhotra. "Modeling and Analysis of Fractional Order Logistic Equation Incorporating Additive Allee Effect." *Contemporary Mathematics* (2024): 380-401. https://doi.org/10.37256/cm.5120243183
2. Kalra, Preety, and Nisha Malhotra. "Dynamical and Numerical Analysis of a Fractional Order Three Species Food Chain Model with Holling type II Functional Response and Allee effect." *Journal of Mathematical Sciences*. (Accepted)

Papers Published in Conference Proceedings from the Thesis

1. Preety Kalra and Nisha Malhotra (2024). "Analysis of modified fractional order Lotka–Volterra model." Advances in Mathematical and Computational Sciences: Proceedings of The ICRTMPCS International Conference 2023: 307-334.Walter de Gruyter GmbH & Co KG.
2. Preety Kalra, and Nisha Malhotra (2025). "Impact of Allee Effect on Prey Predators Populations using Fractional Order Differential Equations." AIP conference proceedings: 3185(1): 020059.

Book Chapters Published from the Thesis

1. Preety Kalra, Nisha, & Sudipa Chauhan (2025). The Fractional Order Lotka-Volterra Competition Model: An Analysis with Additive Allee Effect. In Srivastava, H. M., Arora, G., & Shah, F. (Eds.). *Advances in Computational Methods and Modeling for Science and Engineering (pp 193-216)*. Elsevier. ISBN: 978-0-443-30012-7

Poster Presentations

1. *Modelling on interacting population dynamics using fractional order derivative: A Review,* in the International Conference on "Recent Advances in

Fundamental and Applied Sciences" (RAFAS 2021) held on June 25 – 26, 2021, organized by School of Chemical Engineering and Physical Sciences, Lovely Professional University, Phagwara, Punjab, India.

Paper Presentations

1. *Fractional stability of prey-predator population dynamics with the Allee Effect,* in the 4th National Conference on "Recent Advancement in Physical Sciences" held on December 19-20, 2022, jointly organized by Department of Chemistry, Department of Physics & Department of Mathematics, National Institute of Technology, Uttarakhand.
2. *Lotka Volterra System with Additive Allee Effect in Prey Population,* in the 4th International Conference on "Recent Advances in Fundamental and Applied Sciences" (RAFAS 2023) held on March 24-25, 2023, organized by School of Chemical Engineering and Physical Sciences, Lovely Faculty of Technology and Sciences, Lovely Professional University, Punjab.
3. *Impact of Allee Effect on Prey Predators Populations using Fractional Order Differential Equations,* in 4th International Conference on "Functional Materials, Manufacturing and Performances" (ICFMMP-2023) held on August 25-26, 2023, organized by Department of Research Impact and Outcome, Research and Development Cell, Lovely Professional University, Punjab.
4. *Fractional Order One Prey Two Predator Model with Allee Effect on Prey,* in the 5th International Conference on "Recent Advances in Fundamental and Applied Sciences" (RAFAS-2024) held on April 19 – 20, 2024, organized by School of Chemical Engineering and Physical Sciences, Lovely Faculty of Technology and Sciences, at Lovely Professional University, Punjab.

Workshops, FDPs, Webinars

1. *Workshop on Scientific Writing Using Typesetting Software LaTeX,* held on March 13-14, 2020, organized by Lovely Professional University.

2. *Workshop on Nonlinear Phenomena in Mathematical Biology (WoNPMB-2022)*, held on December 19-23, 2022, organised by ABV- Indian Institute of Information Technology and Management, Gwalior, India.
3. *A Five-Day Online FDP on Mathematical Modeling in Biological Systems,* held on June19-23, 2023, organized by the Division of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Chennai.
4. A Webinar on *Mathematical Modelling perspectives for Prediction off spread and control strategies for COVID-19*, held on June 24, 2020, organized by Jiwaji University, Gwalior, India.
5. A One-day National Webinar on *New Trends in Mathematical Modelling,* held on December 5, 2020, organized by P.G. Department of Mathematics S.S.M. College, Dinanagar.