FIXED POINTS AND STABILITY OF THE FUNCTIONAL EQUATIONS IN ABSTRACT SPACES

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By

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DECLARATION

I, hereby declared that the presented work in the thesis entitled **"Fixed Points and Stability of the Functional Equations in Abstract Spaces"** in fulfilment of degree of **Doctor of Philosophy (Ph.D.)** is outcome of research work carried out by me under the supervision of **Dr. Deepak Kumar**, working as **Professor and Assistant Dean**, in the **Department of Mathematics** in Lovely Professional University, Punjab, India. In keeping with general practice of reporting scientific observations, due acknowledgements have been made whenever work described here has been based on findings of other investigator. This work has not been submitted in part or full to any other University or Institute for the award of any degree.

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<u>CERTIFICATE</u>

This is to certify that the work reported in the Ph.D. thesis entitled **"Fixed Points and Stability of the Functional Equations in Abstract Spaces"** submitted in fulfillment of the requirement for the award of degree of **Doctor of Philosophy (Ph.D.)** in the **Department of Mathematics**, is a research work carried out by **Kapil, 12105734**, is bonafide record of his original work carried out under my supervision and that no part of thesis has been submitted for any other degree, diploma or equivalent course.

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Abstract

Fixed point theory has long been an important area of research in mathematical analysis, with applications ranging from mathematical modelling to graph theory, optimization, approximation theory, and multidisciplinary disciplines such as economics and physics. The constructive proofs of fixed point theorems and iterative approaches for determining the fixed points of self mappings form the foundation of the theory. Furthermore, the concept of common fixed points is especially important in the study of problems involving pairs of self mappings, broadening the theory's applicability to increasingly complex contexts. The stability of a functional equation is concerned with the existence of an exact solution that is close to the functional equation's approximate solution. The study of stability has numerous applications in dynamical systems and control theory, where the sensitivity of solutions to small changes in initial conditions is of critical importance.

Despite significant progress in fixed point theory, many problems in metric spaces and its generalized structures remain unexplored. The existing theory do not always ensure the existence of a fixed point or the stability of solutions to functional equations. Exploring the relation between fixed points and the stability of functional equations is an interesting area of study offering substantial potential for future advancements in both theoretical and applied mathematics.

The main objectives of this research are to investigate the stability of generalized functional equations in abstract spaces, to check the existence and uniqueness of fixed points of mappings using various contraction principles, and to examine the existence and uniqueness of common fixed points of self mappings in abstract spaces. The research further aims to introduce a generalized metric space and analyze the existence and uniqueness of fixed points for different contraction mappings within this framework. As application, we claim the existence of solution to Fredholm integral equation, initial value problem and operator equation.

In the first chapter, we begin with a brief introduction to the research work along with some notations and definitions that are used throughout the thesis. The chapterwise summary of all the subsequent chapters is also given at the end.

In the second chapter, we establish the results on the stability of quadratic and quartic type functional equations. Some illustrations are presented to demonstrate the significance of the assumption made in the proved results. Also, the stability of generalized quartic function equation using a fixed point approach and a conventional approach in n-$\mathcal{BS}$ and non- Archimedean n-$\mathcal{BS}$ is discussed.

In the third chapter, we introduce the concept of C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-MS, which is a generalization of both C^*_{AV}-$\mathfrak{m}$-MS and $\mathcal{R}$-MS. The first section presents the definition of C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-MS, along with its intrinsic properties and several illustrative examples. The second section focuses on the existence and uniqueness of fixed points

within C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-MS, using the concept of C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-contraction mappings. The results established in this chapter extend and generalize several well-known fixed point theorems found in the literature. Also, as an application, the existence and uniqueness of the solution to the operator equation is presented.

In the fourth chapter, we introduce the concept of a generalized distance function, referred to as the multiplicative $\mathfrak{m}$-metric. The first section presents the basic definition and intrinsic properties of the multiplicative $\mathfrak{m}$-MS, along with illustrative examples. The second section discusses various fixed point results for self mappings within the framework of multiplicative $\mathfrak{m}$-MS, using different contractions. Illustrative examples are provided to discuss the existence of fixed points for discontinuous self mappings. In the third section, common fixed point results for a pair of self mappings are explored using generalized contraction conditions. An illustrative example, involving discontinuous self mappings, is discussed, along with numerical iterations to approximate the common fixed point, supported by graphical representations. The fourth section presents fixed point and common fixed point results using a three-point analogue of contraction mappings. Additionally, as applications, the existence and uniqueness of solutions to the initial value problem and a system of integral equations are discussed.

In the fifth chapter, we present several common fixed point results for self mappings in PMS using the (ϕ, ψ)-Wardowski type contraction. Furthermore, some fixed point results are proven using generalized cyclic contractions, followed by illustrative examples. As an application, the existence of a fractal set for the Hutchinson-Barnsley operator is established using the fixed point results proved in the chapter. Finally, some iterations for generating fractal sets are presented, along with the resulting fractals.

In the sixth chapter, we discuss some common fixed point results for self mapping in b-MS using relation theoretic and α-admissible generalized contractions. As applications of the proved results, the existence of solution to a class of non-linear functional integral equation and an operator equation are established.

In the last chapter, we introduce several fixed point results within the framework of $\mathfrak{m}$-MS using contraction mappings. The continuity conditions of self mappings are not essential in the results proved, unlike those in existing literature. The chapter discusses examples where well-known contractions in metric spaces do not guarantee the existence of a fixed point, but their generalizations within $\mathfrak{m}$-MS yield the desired outcome. These examples are validated through graphical visualizations of contraction mappings, which help in understanding their behavior and highlight the distinctions between metric spaces and $\mathfrak{m}$-MS. The main sections present fixed point and common fixed point results using various contractions. Finally, some numerical iterations for approximating the common fixed point are provided, accompanied by graphs that visually demonstrate the results.

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This work is dedicated to my family, whose unwavering love and support have been the foundation of my success. To my grandparents, whose wisdom and belief in me have inspired me throughout this journey. To my parents, for their endless encouragement and sacrifices, which have shaped me into who I am today. To my siblings, for their constant support and for always being there when I needed a boost. To my wife, for the care, love, and understanding that have been a continual source of strength. I am truly grateful to each of you for your presence and love.

Thank you all for being a part of this achievement.

Kapil

Contents

List of Abbreviations

$\mathbb{N}$	Set of natural numbers
$\mathbb{R}$	Set of real numbers
$\mathbb{Z}$	Set of integers
$\mathbb{N}_0$	Set of non negative integers
$\exists$	there exists
VS	Vector space
$\mathcal{NS}$	Normed space
$\mathcal{BS}$	Banach space
MS	Metric space
C^*_{AV}	C^*-algebra valued
P.M.S.	Partial metric space
$\forall$	for all
C_{seq}	Cauchy sequence
FE	functional equation
w.r.t.	with respect to
c.t.b.	called to be
s.t.	such that

List of Figures

Chapter 1

General Introduction

1.1 Introduction

Fixed point theory is a fundamental branch of mathematical analysis that has numerous applications in mathematical modeling, graph theory, optimization, approximation theory, and interdisciplinary areas like economics and physics. It comprises algebraic, topological and geometrical aspects of mathematical analysis. The theory deals with the existence of atleast one point that remains invariant under the given transformation. Consider a self mapping Γ defined on a non-empty set Ω. A point $\varrho \in \Omega$ is called a fixed point of the self mapping Γ if it satisfies the condition $\Gamma(\varrho) = \varrho$. The existence of a fixed point relies not only on the behaviour of mapping but also on the algebraic and topological properties of the domain. A mapping may or may not possess a fixed point within a specified domain. Moreover, if it has a fixed point, it may not be unique. For example, consider the mappings $\Gamma_1, \Gamma_2, \Gamma_3 : \mathbb{R} \to \mathbb{R}$ defined as

(i) $\Gamma_1(\varrho) = \varrho + 3$;

(ii) $\Gamma_2(\varrho) = \frac{\varrho}{5}$;

(iii) $\Gamma_3(\varrho) = \varrho^2$.

We observe that Γ_1 has no fixed point, Γ_2 has a unique fixed point $\varrho = 0$ and Γ_3 has multiple fixed points $\varrho = 0$ and $\varrho = 1$.
Graphically, the fixed points are the point of intersection of the graphs $y = \Gamma(\varrho)$ and $y = \varrho$ (see Figure 1.1).
Fixed point theory deals with the development of novel approaches for proving

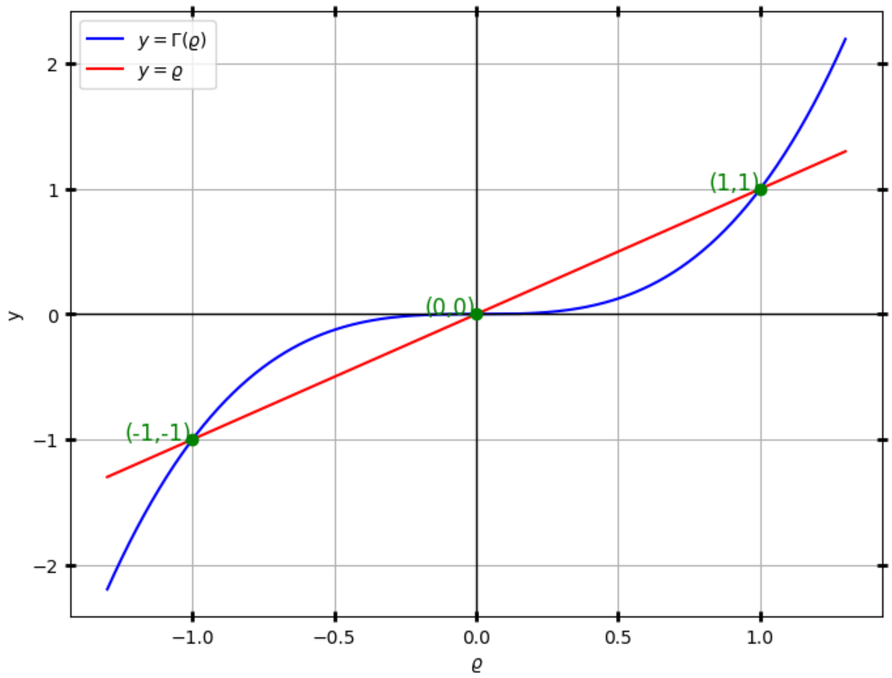

Figure 1.1: Graphical representation of the fixed point

the existence of fixed points. The origins of fixed point theory can be traced to the method of successive approximation Liouville (1837); Picard (1890), initially employed to establish the existence of solutions to differential equations. Picard (1890) introduced the iterative scheme ($\varrho_{n+1} = \Gamma\varrho_n$, where Γ is a self mapping defined on a non-empty set Ω and $\varrho_0 \in \Omega$ is the initial point of the scheme) to approximate the fixed point of the mapping. Brouwer (1912) established the fixed point result "Every continuous self-map of a closed unit ball centered at the origin in $\mathbb{R}^n$ has a fixed point." The result is considered as a key contribution to the theory of fixed points.

Banach (1922) introduced the contraction principle known as the Banach Contraction Principle that has become a cornerstone of nonlinear analysis and has found numerous applications in ensuring the existence of solutions to differential equations, integral equations, optimization problems, etc.

Theorem 1.1.1. *Banach (1922) Consider a complete MS* (Ω, d)*. Then, a mapping* $\Gamma : \Omega \to \Omega$ *has a unique fixed point if* $\exists k \in [0, 1)$ *s.t.*

$$d(\Gamma\varrho, \Gamma\vartheta) \leq kd(\varrho, \vartheta)\ \forall \varrho, \vartheta \in \Omega.$$

The key limitation of the principle was the condition of continuity of the mapping. It restricts the applicability of contraction in certain scenarios where mappings may be discontinuous or defined piecewise. To address this limitation, Kannan (1968) introduced the generalized contraction condition that ensures the existence of a fixed point even for discontinuous mappings.

Theorem 1.1.2. *Kannan (1968) Consider a complete MS* (Ω, d). *Then, a mapping* $\Gamma : \Omega \to \Omega$ *has a unique fixed point if* $\exists k \in [0, 1/2)$ *s.t.*

$$d(\Gamma\varrho, \Gamma\vartheta) \leq k\,(d(\varrho, \Gamma\varrho) + d(\vartheta, \Gamma\vartheta))\ \forall \varrho, \vartheta \in \Omega.$$

Subsequently, many researchers have independently generalized this contraction in their own ways (for references, see Edelstein (1962); Rakotch (1962); Ćirić (1974); Reich (1971); Sehgal (1971); Bianchini (1971); Chatterjea (1972); Zamfirescu (1972); Wardowski (2012); Wardowski and Dung (2014); Imdad et al. (2018); Pasupathi et al. (2020); Chanda et al. (2021); Nazir et al. (2021) and many more.)

Frechet (1906) gave the framework of metric space to explore topology using distance notion. In a MS, the distance function is well defined and satisfies key properties such as non-negativity, symmetry, and the triangle inequality. However, in some scenarios, these characteristics may be relaxed, prompting the establishment of a more generalized notion. Czerwik (1993) introduced the notion of b-metric space(b-MS) as an generalization of MS. Matthews (1994), introduced the notion of partial metric space (PMS), where the self-distance of a point may be non-zero, offering a new oversight in studying fixed points within various functional spaces. Asadi et al. (2014) further extended the concept of partial metric spaces by introducing $\mathfrak{m}$-metric space ($\mathfrak{m}$-MS), offering an even more versatile framework for the fixed point theory. Some other generalization of metric space can be seen in Wilson (1931); Karapınar et al. (2013); Ma et al. (2014); Shukla (2014); Gupta and Gautam (2015); Alsamir et al. (2019); Asim et al. (2019); Jleli and Samet (2018); Chandok et al. (2019); Khalehoghli et al. (2020); M. Joshi et al. (2021); Khalil et al. (2021); Malhotra et al. (2022).

Grossman and Katz (1972) contributed significantly to non-Newtonian calculus, building on Robinson (1966) foundational development of non-standard analysis. Their work introduced a comprehensive framework based on ultrapowers and hyperreals, providing a rigorous structure for non-Newtonian calculus that aligns with conventional mathematics. Stanley (1999) made significant contributions to the field of "multiplicative calculus", also known as the "geometric calculus". Bashirov et al. (2008) introduced the notion of a distance function in multiplicative calculus, using multiplicative absolute values, and laid the foundation of a

multiplicative MS as an alternative to the MS.
Ozavsar and Cevikel (2012) proved some fixed point results using Banach-type, Kannan-type, and Chhaterjea-type contractions in multiplicative MS. Subsequently, many researchers have investigated the fixed points of various nonlinear contractive mappings in multiplicative MS.
Functional equations(FEs) are equations with functions as unknown variables instead of conventional variables. The general FE can be represented as

$$g(\mathcal{F}_1, \mathcal{F}_2, \dots) = 0,$$

where $\mathcal{F}_i$ are functions of finite variables.
Some illustrations of the FE along with their solutions are given below.

(i) Cauchy- FE, $\mathcal{F}(\varrho.\vartheta) = \mathcal{F}(\varrho) + \mathcal{F}(\vartheta)$, satisfied by $\mathcal{F}(\varrho) = \log(\varrho)$.

(ii) $\mathcal{F}(\varrho + T) = \mathcal{F}(\varrho)$, satisfied by periodic function $\mathcal{F}$ with period T.

(iii) Jensen linear FE, $2\mathcal{F}(\frac{\varrho+\vartheta}{2}) = \mathcal{F}(\varrho) + \mathcal{F}(\vartheta)$, satisfied by $\mathcal{F}(\varrho) = \varrho$.

(iv) Quadratic FE, $\mathcal{F}(\varrho+\vartheta)+\mathcal{F}(\varrho-\vartheta) = 2\mathcal{F}(\varrho)+2\mathcal{F}(\vartheta)$, satisfied by $\mathcal{F}(\varrho) = \varrho^2$.

The concept of stability of FE was posed by Ulam in 1940, in his talk at the University of Wisconsin. The open problem was posed as

> *"Suppose $\mathcal{F}(\varrho)$ satisfies the linear equation approximately. Does there exist a linear function that approximate $\mathcal{F}(\varrho)$?"*

More precisely the problem is stated as follows
Ulam's Problem: Let G_1 be a group with binary operation $*_1$ and G_2 be a metric group with metric d and binary operation $*_2$. Does for given $\epsilon > 0$, $\exists \delta > 0$ s.t. if for $\mathcal{F} : G_1 \to G_2$

$$d\left(\mathcal{F}(\varrho *_1 \vartheta), \mathcal{F}(\varrho) *_2 \mathcal{F}(\vartheta)\right) \leq \delta,$$

then $\exists$ a homomorphism $g : G_1 \to G_2$ s.t.

$$d\left(\mathcal{F}(\varrho) *_2 g(\varrho)\right) \leq \epsilon \; \forall \varrho \in G_1.$$

Hyers (1941) provided the first solution to the stability problem of Ulam for $\delta-$linear transformation on Banach spaces. He proved that if $\mathcal{F}(\varrho)$ is a δ-linear function on $\mathcal{BS}$, then $\exists$ a unique linear map $l(\varrho)$ that approximates $\mathcal{F}(\varrho)$, i.e.,

$\|\digamma(\varrho) - l(\varrho)\| \leq \delta$. This solution was stated as Ulam-Hyers stability of Cauchy FE. Since then many researchers proved the stability results for higher order functional equations as well as generalized the Ulam-Hyers stability result for various functional equations.
Consider Cauchy FE

$$\digamma(\varrho + \vartheta) = \digamma(\varrho) + \digamma(\vartheta).$$

Later, Aoki (1950) proved the same result with unbounded Cauchy difference. T. M. Rassias (1978), proved the stability result for the Cauchy FE and gave an affirmative solution to Ulam's problem. He stated the result as

Theorem 1.1.3. *T. M. Rassias (1978) Let E and E' be $\mathcal{BS}$s. Let $\digamma : E \to E'$ be a transformation on E s.t. $\digamma(t\varrho)$ is continuous in t for fixed ϱ. If $\exists k \geq 0$ and $0 < p < 1$ s.t. $\digamma$ satisfies the following condition*

$$\frac{\|\digamma(\varrho + \vartheta) - \digamma(\varrho) - \digamma(\vartheta)\|}{\|\varrho\|^p + \|\vartheta\|^p} \leq k \ \forall \varrho, \vartheta \in E.$$

Then, $\exists$ a linear function ϕ s.t.

$$\frac{\|\digamma(\varrho) - \phi(\varrho)\|}{\|\varrho\|^p} \leq \frac{2k}{2 - 2^p}.$$

Gajda (1991) proved the result of T. M. Rassias (1978) for $p > 1$. He also proved that the stability result is not valid for $p = 1$. Gavruta (1994) introduced a generalized Ulam-Hyers-Rassias stability result for an approximately additive mapping by relaxing the domain of mapping to the Abelian group. He stated the result as

Theorem 1.1.4. *Gavruta (1994) Let $(G, +)$ be an Abelian group and Ω be a $\mathcal{BS}$. Let $\phi : G \times G \to [0, \infty)$ be a mapping on $G \times G$ s.t.*

$$\overline{\phi}(\varrho, \vartheta) = \sum_{k=0}^{\infty} 2^{-k} \phi(2^k \varrho, 2^k \vartheta) < \infty \quad \forall \ \varrho, \vartheta \in G.$$

If $\digamma : G \to \Omega$ be a mapping s.t.

$$\|\digamma(\varrho + \vartheta) - \digamma(\varrho) - \digamma(\vartheta)\| \leq \phi(\varrho, \vartheta) \quad \forall \ \varrho, \vartheta \in G.$$

Then, $\exists$ a unique linear mapping $T : G \to \Omega$ s.t.

$$\|\digamma(\varrho) - T(\varrho)\| \leq \frac{1}{2}\overline{\phi}(\varrho, \vartheta) \quad \forall \ \varrho \in G.$$

Another generalization of Hyers' findings was given by J. M. Rassias and Kim (2009). He stated the results as

Theorem 1.1.5. *J. M. Rassias and Kim (2009) Let* Ω *be a* $\mathcal{NS}$ *and* Y *be a* $\mathcal{BS}$*. Let* $c \geq 0$ *and* $p, q \in \mathbb{R}$ *s.t.* $p + q \in [0, 1)$*. Consider a mapping* $\digamma : \Omega \to Y$ *that satisfies the following*

$$\|\digamma(\varrho + \vartheta) - \digamma(\varrho) - \digamma(\vartheta)\| \leq c\|\varrho\|^p \|\vartheta\|^q \; \forall \varrho, \vartheta \in \Omega \sim \{0\}.$$

Then, $\exists$ *a linear mapping* $T : \Omega \to Y$ *s.t.*

$$\|\digamma(\varrho) - T(\varrho)\| \leq \frac{c\|\varrho\|^{p+q}}{2 - 2^{p+q}}.$$

Brzdek (2014) extended the result of J. M. Rassias and Kim (2009) by proving the above stability result for $p + q < 0$, using fixed point approach in extended MS and called it as Hyper stability of Cauchy FE. He stated the result as

Theorem 1.1.6. *Brzdek (2014) Let* Ω *be a* $\mathcal{NS}$ *and* Y *be a* $\mathcal{BS}$*. Let* $c \geq 0$ *and* p, q *be real number s.t.* $p + q < 0$*. Consider a mapping* $\digamma : \Omega \to Y$ *that satisfies the following*

$$\|\digamma(\varrho + \vartheta) - \digamma(\varrho) - \digamma(\vartheta)\| \leq c\|\varrho\|^p \|\vartheta\|^q \; \forall \varrho, \vartheta \in \Omega \sim \{0\}.$$

Then, $\digamma$ *is a linear mapping. i.e.,*

$$\digamma(\varrho + \vartheta) = \digamma(\varrho) + \digamma(\vartheta) \; \forall \varrho, \vartheta \in \Omega.$$

Moghimi and Najati (2022) proved the above result using a new approach and also, proved some results on hyper stability and super stability result for the Cauchy FE and Jensen FE on the restricted and unrestricted domains.
Numerous results related to fixed point theory and the stability of functional equations can be found in various books and monographs, see (Hutchinson (1981); Istratescu (1981); Dugundji and Granas (1982); M. C. Joshi and Bose (1985); Zeidler (1986); Geobel and Kirk (1990); Murphy (1990); Rudin (1991); Rosen (1991); Davidson (1996); Jungck and Rhoades (1998); Agarwal et al. (2001); Kirk and Khamsi (2001); William and Brailey (2001); Agarwal et al. (2009); Chandok (2015)), Oltra and Valero (2004); Valero (2005); Romaguera (2009a); Berenguer et al. (2009); Altun et al. (2010); Altun and Sadarangani (2011); Abbas et al. (2012); Aydi et al. (2012); Minirani and Mathew (2014); Sintunavarat (2016); Zada and Sarwar (2019); Petruşel and Petruşel (2019); Altun et al. (2021); Choudhury and Chakraborty (2022). In the subsequent section, we will outline the definitions and results that are utilized in the later chapters, followed by a chapterwise summary in the last section.

1.2 Notations and Definitions

Throughout the thesis, the symbols $\mathbb{R}, \mathbb{Z}, \mathbb{N}$ have their usual meaning, Ω, Y, X are non empty sets, $\mathcal{R}$ is a binary relation, $\mathbb{B}$ is a C^*-algebra with zero element $\theta_{\mathbb{B}}$ and identity element $I_{\mathbb{B}}$

Definition 1.2.1. *Frechet (1906) A mapping $d : \Omega \times \Omega \to [0, \infty)$ is termed as a **metric** if it satisfies :*

(i) $d(\varrho, \vartheta) \geq 0$ *and* $d(\varrho, \vartheta) = 0 \Leftrightarrow \varrho = \vartheta$;

(ii) $d(\varrho, \vartheta) = d(\vartheta, \varrho)$;

(iii) $d(\varrho, \vartheta) \leq d(\varrho, \zeta) + d(\zeta, \vartheta)$,

$\forall \varrho, \vartheta, \zeta \in \Omega$. *Moreover,* (Ω, d) *is termed as a **Metric Space (MS)**.*

Definition 1.2.2. *Rudin (1991) Let X be a VS over the field F. A mapping $\|.\| : X \to [0, \infty)$ is termed as **norm** if it satisfies :*

(i) $\|\varrho\| \geq 0$ *and* $\|\varrho\| = 0 \Leftrightarrow \varrho = 0$;

(ii) $\|\alpha\varrho\| = |\alpha|\|\varrho\|$;

(iii) $\|\varrho + \vartheta\| \leq \|\varrho\| + \|\vartheta\|$,

$\forall\ \varrho, \vartheta \in X$ *and* $\alpha \in F$. *Moreover,* $(X, \|.\|)$ *is termed as a **Normed Space** ($\mathcal{NS}$).*

Definition 1.2.3. *Matthews (1994) A mapping $\wp : \Omega \times \Omega \to [0, \infty)$ is termed as a **partial metric** if it satisfies :*

(i) $\wp(\varrho, \vartheta) = \wp(\vartheta, \vartheta) = \wp(\varrho, \varrho) \Leftrightarrow \varrho = \vartheta$;

(ii) $\wp(\varrho, \varrho) \leq \wp(\varrho, \vartheta)$;

(iii) $\wp(\varrho, \vartheta) = \wp(\vartheta, \varrho)$;

(iv) $\wp(\varrho, \vartheta) \leq \wp(\varrho, \zeta) + \wp(\zeta, \vartheta) - \wp(\zeta, \zeta)$,

$\forall \varrho, \vartheta, \zeta \in \Omega$. *Moreover,* $(\Omega, \wp)$ *is termed as a **Partial Metric Space (PMS)**.*

Definition 1.2.4. *Matthews (1994); Romaguera (2009b) Let* $(\Omega, \wp)$ *be a PMS and* $\{\varrho_\hbar\}$ *be a sequence in* Ω. *Then,*

(i) *(a)* $\{\varrho_\hbar\}$ *is* ***convergent*** *if* $\exists \varrho \in \Omega$ *s.t.* $\wp(\varrho_\hbar, \varrho) - \wp(\varrho, \varrho) \to 0$ *as* $\hbar \to \infty$;

(b) $\{\varrho_\hbar\}$ *is* ***Cauchy*** *if* $\lim_{\hbar,m\to\infty} \wp(\varrho_\hbar, \varrho_m)$ *exists finitely;*

(c) $\{\varrho_\hbar\}$ *is* **0*-Cauchy*** *if* $\lim_{\hbar,m\to\infty} \wp(\varrho_\hbar, \varrho_m) = 0$;

(ii) $(\Omega, \wp)$ *is* ***complete*** *if every partial* C_{seq} $\{\varrho_\hbar\}$ *is convergent in* Ω;

(iii) $(\Omega, \wp)$ *is* **0*-complete*** *if for every* 0-C_{seq} $\{\varrho_\hbar\}$, $\exists \varrho \in \Omega$ *s.t.* $\lim_{\hbar,m\to\infty} \wp(\varrho_\hbar, \varrho) = \wp(\varrho, \varrho) = 0$.

Definition 1.2.5. *Asadi et al. (2014) A mapping* $\varpi : \Omega \times \Omega \to [0, \infty)$ *is termed as* $\mathfrak{m}$*-metric if it satisfies:*

(i) $\varpi(\varrho, \vartheta) = \varpi(\vartheta, \vartheta) = \varpi(\varrho, \varrho) \Leftrightarrow \varrho = \vartheta$;

(ii) $\varpi_{\varrho\vartheta} \leq \varpi(\varrho, \vartheta)$;

(iii) $\varpi(\varrho, \vartheta) = \varpi(\vartheta, \varrho)$;

(iv) $\varpi(\varrho, \vartheta) - \varpi_{\varrho\vartheta} \leq \varpi(\varrho, \zeta) - \varpi_{\varrho\zeta} + \varpi(\zeta, \vartheta) - \varpi_{\zeta\vartheta}$,

where $\varpi_{\varrho\vartheta} = \min\{\varpi(\varrho, \varrho), \varpi(\vartheta, \vartheta)\}$ *and* $M_{\varrho\vartheta} = \max\{\varpi(\varrho, \varrho), \varpi(\vartheta, \vartheta)\}$ $\forall \varrho, \vartheta, \zeta \in \Omega$. *Moreover,* (Ω, ϖ) *is termed as* $\mathfrak{m}$***-Metric Space***$(\mathfrak{m} - MS)$.

Example 1.2.6. *Asadi et al. (2014) Let* $\Omega = \mathbb{R}_+ \cup \{0\}$ *and* $\varpi(\varrho, \vartheta) = \dfrac{\varrho + \vartheta}{2}$. *Then,* (Ω, ϖ) *is an* $\mathfrak{m}$*-MS.*

Definition 1.2.7. *Asadi et al. (2014) A sequence* $\{\varrho_\hbar\} \in (\Omega, \varpi)$ *is c.t.b.*

(i) ***convergent*** *if* $\exists \varrho \in \Omega$ *s.t.* $\varpi(\varrho_\hbar, \varrho) - \varpi_{\varrho_\hbar\varrho} \to 0$ *as* $\hbar \to \infty$;

(ii) $\mathfrak{m}$-***Cauchy*** *if* $\lim_{\hbar,\ell\to\infty} \varpi(\varrho_\hbar, \varrho_\ell) - \varpi_{\varrho_\hbar\varrho_\ell}$ *and* $\lim_{\hbar,\ell\to\infty} M_{\varrho_\hbar\varrho_\ell} - \varpi_{\varrho_\hbar\varrho_\ell}$ *exist finitely.*

Moreover, if every $\mathfrak{m}$-C_{seq} $\{\varrho_\hbar\}$ *is convergent in* Ω, *i.e.,* $\exists \varrho \in \Omega$ *s.t.*

$$\varpi(\varrho_\hbar, \varrho) - \varpi_{\varrho_\hbar\varrho} \to 0 \quad \text{and} \quad M_{\varrho_\hbar\varrho} - \varpi_{\varrho_\hbar\varrho} \to 0 \text{ as } \hbar \to \infty.$$

Then, (Ω, ϖ) *is a complete* $\mathfrak{m}$*-MS.*

Lemma 1.2.8. *Asadi et al. (2014) Let $\{\varrho_\hbar\}$ and $\{\vartheta_\hbar\}$ be two sequences in (Ω, ϖ) s.t. $\varrho_\hbar \to \varrho$ and $\vartheta_\hbar \to \vartheta$. Then,*

$$\lim_{\hbar\to\infty} \varpi(\varrho_\hbar, \vartheta_\hbar) - \varpi_{\varrho_\hbar\vartheta_\hbar} = \varpi(\varrho, \vartheta) - \varpi_{\varrho\vartheta}.$$

Lemma 1.2.9. *Asadi et al. (2014) Let (Ω, ϖ) be a $\mathfrak{m}$-MS. Then, $\forall\ \varrho, \vartheta, \zeta \in \Omega$, we have*

(i) $0 \leq \left| M_{\varrho,\vartheta} - \varpi_{\varrho,\vartheta} \right| = \left| \varpi(\varrho, \varrho) - \varpi(\vartheta, \vartheta) \right|$;

(ii) $M_{\varrho,\vartheta} + \varpi_{\varrho,\vartheta} = \varpi(\varrho, \varrho) + \varpi(\vartheta, \vartheta)$;

(iii) $M_{\varrho,\vartheta} - \varpi_{\varrho,\vartheta} \leq M_{\varrho,\zeta} - \varpi_{\varrho,\zeta} + M_{\zeta,\vartheta} - \varpi_{\zeta,\vartheta}$

Lemma 1.2.10. *Asadi et al. (2014) Let (Ω, ϖ) be a $\mathfrak{m}$-MS and $\{\varrho_\hbar\}$ be a sequence in Ω s.t.*

$$\varpi(\varrho_{\hbar+1}, \varrho_\hbar) \leq r.\ \varpi(\varrho_\hbar, \varrho_{\hbar-1})\ \forall \hbar \in \mathbb{N},\ \textit{where}\ r \in [0, 1). \tag{1.2.1}$$

Then,

(i) $\lim_{\hbar\to\infty} \varpi(\varrho_{\hbar+1}, \varrho_\hbar) = 0$;

(ii) $\lim_{\hbar\to\infty} \varpi(\varrho_\hbar, \varrho_\hbar) = 0$;

(iii) $\lim_{\hbar,m\to\infty} \varpi_{\varrho_\hbar\varrho_m} = 0$;

(iv) $\{\varrho_\hbar\}$ *is a* $\mathfrak{m}$-C_{seq}.

Definition 1.2.11. *Khalehoghli et al. (2020) Let $\mathcal{R}$ be a binary relation on Ω and (Ω, d) be a MS. Then, the triplet $(\Omega, d, \mathcal{R})$ is termed as a **$\mathcal{R}$-Metric Space** ($\mathcal{R}$-MS).*

Definition 1.2.12. *Khalehoghli et al. (2020) Let $\{\varrho_\hbar\}$ is a sequence in $(\Omega, d, \mathcal{R})$. Then, $\{\varrho_\hbar\}$ is c.t.b. an $\mathcal{R}$-sequence if $(\varrho_\hbar, \varrho_{\hbar+k}) \in \mathcal{R}$, $\forall \hbar, k \in \mathbb{N}$.*

Definition 1.2.13. *Khalehoghli et al. (2020) Let $(\Omega, d, \mathcal{R})$ be an $\mathcal{R}$-MS and $\{\varrho_\hbar\}$ be an $\mathcal{R}$-sequence in Ω. Then, $\{\varrho_\hbar\}$ is c.t.b.*

(i) ***$\mathcal{R}$-convergent*** *to ϱ, if $\forall \epsilon > 0$, $\exists K \in \mathbb{N}$ s.t.*

$$d(\varrho_\hbar, \varrho) \leq \epsilon\ \ \forall \hbar \geq K;$$

(ii) $\mathcal{R}$***-Cauchy****, if* $\forall \epsilon > 0$, $\exists K \in \mathbb{N}$ *s.t.*

$$d(\varrho_{\hbar}, \varrho_m) \leq \epsilon \quad \forall \hbar, m \geq K.$$

Moreover, the triplet $(\Omega, d, \mathcal{R})$ *is c.t.b.* $\mathcal{R}$***-complete****, if every* $\mathcal{R}$*-*C_{seq} *in* $(\Omega, d, \mathcal{R})$ *is convergent to some* $\varrho \in \Omega$*.*

Definition 1.2.14. *Khalehoghli et al. (2020) Let* $(\Omega, d, \mathcal{R})$ *be an* $\mathcal{R}$*-MS, and* Γ *be a self mapping on* Ω*. Then,* Γ *is c.t.b.* $\mathcal{R}$*-continuous at* $\varrho \in \Omega$ *if for every* $\mathcal{R}$*-sequence* $\{\varrho_n\}$ *that converges to* ϱ*, the sequence* $\{\Gamma\varrho_n\}$ *converges to* $\Gamma\varrho$*. Additionally,* Γ *is considered as* $\mathcal{R}$*-continuous on* Ω *if it is* $\mathcal{R}$*-continuous at every point* $\varrho \in \Omega$*.*

Definition 1.2.15. *Alsamir et al. (2019) Let* $\mathbb{B}$ *be a unital* C^* *algebra with unit* $I_{\mathbb{B}}$ *and zero element* $\theta_{\mathbb{B}}$*. A mapping* $\varpi : \Omega \times \Omega \to \mathbb{B}_+$ *is termed as* C^****-Algebra Valued*** $\mathfrak{m}$***-metric****, if it satisfies :*

(i) $\varpi(\varrho, \vartheta) = \varpi(\varrho, \varrho) = \varpi(\vartheta, \vartheta) \Leftrightarrow \varrho = \vartheta$*;*

(ii) $\varpi(\varrho, \varrho)$ *and* $\varpi(\vartheta, \vartheta)$ *are comparable;*

(iii) $\varpi_{\varrho\vartheta} \preceq \varpi(\varrho, \vartheta)$*, ;*

(iv) $\varpi(\varrho, \vartheta) = \varpi(\vartheta, \varrho)$*;*

(v) $\varpi(\varrho, \vartheta) - \varpi_{\varrho,\vartheta} \preceq \varpi(\varrho, \zeta) - \varpi_{\varrho\zeta} + \varpi(\zeta, \vartheta) - \varpi_{\zeta\vartheta}$*,*

$\forall \varrho, \vartheta, \zeta \in \Omega$*, where* $\varpi_{\varrho\vartheta} = \min\Big\{\varpi(\varrho, \varrho), \varpi(\vartheta, \vartheta)\Big\}$*. Moreover,* $(\Omega, \mathbb{B}, \mathfrak{m})$ *is termed as* C^****-Algebra Valued*** $\mathfrak{m}$***-Metric Space*** *(*C^**-AV-*$\mathfrak{m}$*-MS). The other topological aspects of the space can be seen in Alsamir et al. (2019).*

Remark 1.2.16. *Alsamir et al. (2019) Let* $(\Omega, \mathbb{B}, \varpi)$ *be a* C^**-AV-*$\mathfrak{m}$*-MS. Then,* $\forall \varrho, \vartheta, z \in \Omega$*,*

(i) $\theta \preceq M_{\varrho\vartheta} + \varpi_{\varrho\vartheta} = \varpi(\varrho, \varrho) + \varpi(\vartheta, \vartheta)$*;*

(ii) $\theta \preceq M_{\varrho\vartheta} - \varpi_{\varrho\vartheta} = \Big(\varpi(\varrho, \varrho) - \varpi(\vartheta, \vartheta)\Big) \vee \Big(\varpi(\vartheta, \vartheta) - \varpi(\varrho, \varrho)\Big)$*;*

(iii) $M_{\varrho\vartheta} - \varpi_{\varrho\vartheta} \preceq M_{\varrho\zeta} - \varpi_{\varrho\zeta} + M_{\zeta\vartheta} - \varpi_{\zeta\vartheta}$*,*

where $\varpi_{\varrho\vartheta} = \min\Big\{\varpi(\varrho, \varrho), \varpi(\vartheta, \vartheta)\Big\}$ *and* $M_{\varrho\vartheta} = \max\Big\{\varpi(\varrho, \varrho), \varpi(\vartheta, \vartheta)\Big\}$*.*

Definition 1.2.17. *Czerwik (1993) A mapping* $d_b : \Omega \times \Omega \to [0, \infty)$ *is termed as a **b-metric**, if for some* $s \geq 1$*, it satisfies:*

(i) $d_b(\varrho, \vartheta) = 0 \Leftrightarrow \varrho = \vartheta$*;*

(ii) $d_b(\varrho, \vartheta) = d_b(\vartheta, \varrho)$*;*

(iii) $d_b(\varrho, \vartheta) \leq s(d_b(\varrho, \zeta) + d_b(\zeta, \vartheta))$,

$\forall \varrho, \vartheta, \zeta \in \Omega$*. Moreover,* (Ω, d_b) *is termed as **b-Metric Space**(b-MS).*

Example 1.2.18. $(\mathbb{R}, d_b)$ *is a b-MS with* $s = 2$ *and* $d_b(\varrho, \vartheta) = |\varrho - \vartheta|^2$.

Definition 1.2.19. *Czerwik (1993) Let* (Ω, d_b) *be a a b-MS and* $\{\varrho_\hbar\}$ *be a sequence in* Ω*. Then,* $\{\varrho_\hbar\}$ *is c.t.b.*

*(i) **convergent** if* $\exists \varrho \in \Omega$ *s.t.* $d_b(\varrho_\hbar, \varrho) \to 0$ *as* $\hbar \to \infty$;

(ii) a $\boldsymbol{C_{seq}}$ *in* Ω *if* $\lim_{\hbar, m \to \infty} d_b(\varrho_\hbar, \varrho_m) = 0$.

Moreover, If every C_{seq} $\{\varrho_\hbar\}$ *is convergent in* Ω*. Then,* (Ω, d_b) *is complete.*

Definition 1.2.20. *Alam and Imdad (2015, 2017) Let* $\mathcal{R}$ *be a binary relation on MS* (Ω, d)*,* $\{\varrho_\hbar\}$ *be a sequence in* Ω *and* $\Gamma : \Omega \to \Omega$ *be a mapping. Then,*

(i) $\{\varrho_\hbar\}$ *is* $\boldsymbol{\mathcal{R}}$***-preserving** if* $(\varrho_\hbar, \varrho_{\hbar+1}) \in \mathcal{R}$ $\forall \hbar \in \mathbb{N}$*;*

(ii) (Ω, d) *is* $\boldsymbol{\mathcal{R}}$***-complete** if every* $\mathcal{R}$*-preserving* C_{seq} *in* Ω *is convergent in* Ω*;*

(iii) $\mathcal{R}$ *is* $\boldsymbol{\Gamma}$***-closed** if for* $(\varrho, \vartheta) \in \mathcal{R}$*, we have* $(\Gamma\varrho, \Gamma\vartheta) \in \mathcal{R}$*;*

(iv) $\mathcal{R}$ *is **d-self-closed** if for every* $\mathcal{R}$*-preserving sequence* $\{\varrho_\hbar\} \to \varrho$*,* $\exists$ *a subsequence* $\varrho_{\hbar_k}$ *s.t.* $[\varrho_{\hbar_k}, \varrho] \in \mathcal{R}$ *i.e.,* $\Big((\varrho_{\hbar_k}, \varrho) \in \mathcal{R}$ *or* $(\varrho, \varrho_{\hbar_k}) \in \mathcal{R}\Big)$*;*

(v) Γ *is* $\boldsymbol{\mathcal{R}}$***-continuous** at* $\varrho \in \Omega$ *if for every* $\mathcal{R}$*-preserving sequence* $\{\varrho_\hbar\} \to \varrho$*, we have* $\{\Gamma\varrho_\hbar\} \to \Gamma\varrho$.

Definition 1.2.21. *Zada and Sarwar (2019) Let* $\mathcal{R}$ *be a binary relation on set* Ω *and* $S, \Gamma : \Omega \to \Omega$ *be self mapings. Then,* $\mathcal{R}$ *is* $(\mathbf{S}, \boldsymbol{\Gamma})$***-regular closed** if*

$$(S\varrho, \Gamma\vartheta) \in \mathcal{R} \text{ and } (\Gamma\vartheta, S\varrho) \in \mathcal{R}, \text{ whenever } (\varrho, \vartheta) \in \mathcal{R}.$$

Definition 1.2.22. *Samet et al. (2012) Let* $\alpha : \Omega \times \Omega \to [0, \infty)$ *and* $\Gamma : \Omega \to \Omega$ *be mappings. Then,* Γ *is* ***α-admissible*** *if*

$$\alpha(\varrho, \vartheta) \geq 1 \Rightarrow \alpha(\Gamma\varrho, \Gamma\vartheta) \geq 1 \; \forall \varrho, \vartheta \in \Omega.$$

Definition 1.2.23. *Sintunavarat (2016) Let* $\alpha : \Omega \times \Omega \to [0, \infty)$ *and* $\Gamma : \Omega \to \Omega$ *be mappings. Then,* Γ *is c.t.b.* ***α-admissible of type s*** *if*

$$\alpha(\varrho, \vartheta) \geq s \Rightarrow \alpha(\Gamma\varrho, \Gamma\vartheta) \geq s \; \forall \varrho, \vartheta \in \Omega.$$

Definition 1.2.24. *Let* $\alpha : \Omega \times \Omega \to [0, \infty)$ *and* $S, \Gamma : \Omega \to \Omega$ *be mappings. Then,* (S, Γ) *is c.t.b.* ***generalized α-admissible of type*** s *if*

$$\alpha(\varrho, \vartheta) \geq s \Rightarrow \alpha(S\varrho, \Gamma\vartheta) \geq s \text{ and } \alpha(\Gamma\vartheta, S\varrho) \geq s \; \forall \varrho, \vartheta \in \Omega.$$

Definition 1.2.25. *Samet et al. (2012) Let* Ψ *be the collection of all functions* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfying:*

(i) ψ *is monotonically increasing i.e.,* $\psi(z_1) \leq \psi(z_2) \Leftrightarrow z_1 \leq z_2$*;*

(ii) $\lim_{\hbar \to \infty} \psi^{\hbar}(z) = 0 \; \forall z > 0$*, where* $\psi^{\hbar}$ *is the* $\hbar^{th}$ *iteration, i.e.,* $\psi(z) < z \;\; \forall z > 0$.

Definition 1.2.26. *Liu et al. (2016) Let* Φ *be the collection of all continuous functions* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfying:*

(i) ϕ *is monotonically increasing function i.e.,* $\phi(t_1) \leq \phi(t_2) \Leftrightarrow t_1 \leq t_2$*;*

(ii) $\lim_{\hbar \to \infty} \phi(t_\hbar) = 0 \Leftrightarrow \lim_{\hbar \to \infty} t_\hbar = 0$*, for any sequence* $t_\hbar \in (0, \infty)$.

Lemma 1.2.27. *Miculescu and Mihail (2017) Let* (Ω, d_b) *be a b-MS and* $\{\varrho_\hbar\}$ *be a sequence in* Ω*. If* $\exists \lambda \in [0, 1)$ *s.t.*

$$d_b(\varrho_{\hbar+1}, \varrho_\hbar) \leq \lambda d_b(\varrho_\hbar, \varrho_{\hbar-1}) \; \forall \hbar \in \mathbb{N}.$$

Then, $\{\varrho_\hbar\}$ *is a* C_{seq}.

Definition 1.2.28. *Stanley (1999) Let* $g : \mathbb{R} \to \mathbb{R}^+$ *be a positive function. Then, the multiplicative derivative and integral of* g *are defined as*

$$\frac{d^* \, g(\varrho)}{d\varrho} = g^*(\varrho) = \lim_{h \to 0} \left(\frac{g(\varrho + h)}{g(\varrho)} \right)^{\frac{1}{h}}.$$

$$\int_a^b g(\varrho)^{d\varrho} = e^{\int_a^b ln(g(\varrho)) \, d\varrho}.$$

Theorem 1.2.29. *Grossman and Katz (1972) Let f and g are two multiplicative integral function on $[a, b]$. Then, we have*

(i) $\int_a^b (f(\varrho).g(\varrho))^{d\varrho} = \int_a^b f(\varrho)^{d\varrho}. \int_a^b g(\varrho)^{d\varrho}$;

(ii) $\int_a^b \frac{f(\varrho)}{g(\varrho)}^{d\varrho} = \frac{\int_a^b f(\varrho)^{d\varrho}}{\int_a^b g(\varrho)^{d\varrho}}$;

(iii) $\int_a^b ((g(\varrho))^{\hbar})^{d\varrho} = (\int_a^b g(\varrho)^{d\varrho})^{\hbar}$;

(iv) $\left|\int_a^b g(\varrho)^{d\varrho}\right| \leq \int_a^b |g(\varrho)|^{d\varrho}$.

Definition 1.2.30. *Ozavsar and Cevikel (2012) A mapping $u : \Omega \times \Omega \to [1, \infty)$ is termed as **Multiplicative metric**, if it satisfies :*

(i) $u(\varrho, \vartheta) \geq 1$;

(ii) $u(\varrho, \vartheta) = 1 \Leftrightarrow \varrho = \vartheta$;

(iii) $u(\varrho, \vartheta) = u(\vartheta, \varrho)$;

(iv) $u(\varrho, \vartheta) \leq u(\varrho, \zeta).u(\zeta, \vartheta)$,

$\forall \varrho, \vartheta, \zeta \in \Omega$. *Moreover, (Ω, u) is termed as a **multiplicative metric space**(multiplicative MS).*

Example 1.2.31. *Ozavsar and Cevikel (2012)* $u(\varrho, \vartheta) = \left|\frac{\varrho_1}{\vartheta_1}\right|_* . \left|\frac{\varrho_2}{\vartheta_2}\right|_*$, *where* $\varrho = (\varrho_1, \varrho_2)$, $\vartheta = (\vartheta_1, \vartheta_2) \in \mathbb{R}^2_+$ *and* $|a|_* = \begin{cases} a, a \geq 1; \\ \frac{1}{a}, a < 1. \end{cases}$ *is a multiplicative metric on* $\Omega = \mathbb{R}^2_+$.

Definition 1.2.32. *Ozavsar and Cevikel (2012) Let (Ω, u) be a multiplicative MS and $\{\varrho_\hbar\}$ be a sequence in Ω. Then, $\{\varrho_\hbar\}$ is c.t.b.*

(i) ***convergent*** *if* $u(\varrho_\hbar, \varrho) \to 1$ *as* $\hbar \to \infty$;

(ii) ***Cauchy*** *if* $u(\varrho_\hbar, \varrho_m) \to 1$ *as* $\hbar, m \to \infty$.

Moreover, if every C_{seq} $\{\varrho_\hbar\} \in \Omega$ converges to some $\varrho \in \Omega$, then (Ω, u) is multiplicative complete.

Definition 1.2.33. *Ozavsar and Cevikel (2012) A self mapping Γ on (Ω, u) is c.t.b. multiplicative continuous at $\varrho \in \Omega$ if for every multiplicative convergent sequence $\{\varrho_\hbar\} \to \varrho$ implies $\{\Gamma\varrho_\hbar\} \to \Gamma\varrho$, i.e.,*

$$u(\varrho_\hbar, \varrho) \to 1 \textit{ implies } u(\Gamma\varrho_\hbar, \Gamma\varrho) \to 1.$$

Γ is c.t.b. multiplicative continuous on Ω, if Γ is multiplicative continuous at every $\varrho \in \Omega$.

Lemma 1.2.34. *Ma et al. (2014) Let $\mathbb{B}$ be a unital C^*-algebra with unit $I_{\mathbb{B}}$, zero element $\theta_{\mathbb{B}}$. Then, we have*

(i) if $\alpha \in \mathbb{B}_+$ with $\|\alpha\| < \frac{1}{2}$, then $I_{\mathbb{B}} - \alpha$ is invertible and $\|\alpha(I_{\mathbb{B}} - \alpha)^{-1}\| < 1$;

(ii) $\forall \alpha, \beta \in \mathbb{B}_+$ with $\alpha, \beta \succeq \theta_{\mathbb{B}}$ and $\alpha\beta = \beta\alpha$, then $\alpha\beta \succeq \theta_{\mathbb{B}}$;

(iii) if $\alpha \in \mathcal{B}'$ and $\beta, \gamma \in \mathbb{B}$ where $\beta \succeq \gamma \succeq \theta_{\mathbb{B}}$ and $I_{\mathbb{B}} - \alpha \in \mathbb{B}'_+$ is invertible operator then

$$(I_{\mathbb{B}} - \alpha)^{-1}\beta \succeq (I_{\mathbb{B}} - \alpha)^{-1}\gamma,$$

where $\mathcal{B}_+ = \{\alpha \in \mathbb{B} : \alpha \succeq \theta_{\mathbb{B}}\}$ and $\mathcal{B}' = \{\alpha \in \mathbb{B} : \alpha\beta = \beta\alpha \ \ \forall \beta \in \mathbb{B}\}$.

Definition 1.2.35. *Ma et al. (2014) Let $\mathbb{B}$ be an unital C^*-algebra and A mapping $d : \Omega \times \Omega \to \mathbb{B}$ is c.t.b. a C^*-algebra valued-metric, if it satisfies*

(i) $d(\varrho, \vartheta) \preceq \theta_{\mathbb{B}}$, and $d(\varrho, \vartheta) = \theta_{\mathbb{B}} \Leftrightarrow \varrho = \vartheta$;

(ii) $d(\varrho, \vartheta) = d(\vartheta, \varrho)$

(iii) $d(\varrho, \vartheta) = d(\varrho, \zeta) + d(\zeta, \vartheta)$,

$\forall \vartheta, \varrho, \zeta \in \Omega$. Then, $(\Omega, d, \mathbb{B})$ is c.t.b a C^*-algebra valued-metric space (C^*_{AV}-MS).

Definition 1.2.36. *Malhotra et al. (2022)* $(\Omega, d, \mathbb{B}, \mathbb{R})$ *is c.t.b. a C^*-algebra valued $\mathcal{R}$-metric space (C^*_{AV}-$\mathcal{R}$-MS) if it satisfies:*

(i) $(\Omega, d, \mathbb{B})$ is a C^*_{AV}-MS;

(ii) $\mathcal{R}$ is a reflexive binary relation on Ω.

Definition 1.2.37. *Misiak (1989) Let Ω be a $\mathcal{VS}$ with dimension atleast n, for some $n \in \mathbb{N}$. A mapping $\|.,\ldots,.\| : \Omega^n \to [0, \infty)$ is termed as an **n-norm**, if it satisfies :*

(i) $\|\varrho_1, \varrho_2, \ldots, \varrho_n\| = 0 \Leftrightarrow \varrho_i$ *and* ϱ_j *are linearly independent for* $1 \leq i \neq j \leq n$*;*

(ii) $\|\varrho_1, \varrho_2, \ldots, \varrho_n\|$ *is invariant under the permutations of* $\varrho_1, \varrho_2, \ldots, \varrho_n$*;*

(iii) $\|\alpha\varrho_1, \varrho_2, \ldots, \varrho_n\| = |\alpha|\|\varrho_1, \varrho_2, \ldots, \varrho_n\|$*;*

(iv) $\|\varrho_1 + \vartheta_1, \varrho_2, \ldots, \varrho_n\| \leq \|\varrho_1, \varrho_2, \ldots, \varrho_n\| + \|\vartheta_1, \varrho_2, \ldots, \varrho_n\|$,

$\forall \varrho_i \in \Omega$. *Moreover,* $(\Omega, \|.,\ldots,.\|)$ *is termed as n-* $\mathcal{NS}$.

Example 1.2.38. *Misiak (1989) Consider* $\Omega = \mathbb{R}^n$ *with usual inner product. Then,* $\|.,\ldots,.\| : \Omega^n \to [0,\infty)$ *defined as* $\|\varrho_1, \varrho_2, \ldots, \varrho_n\| = \left|\det(\varrho_{ij})\right|$, *where* $\det(\varrho_{ij}) = \begin{vmatrix} \varrho_{11} & \varrho_{12} & \cdot & \varrho_{1n} \\ \varrho_{21} & \varrho_{22} & \cdot & \varrho_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \varrho_{n1} & \varrho_{n2} & \cdot & \varrho_{nn} \end{vmatrix}$ *is an* $n-norm$.

Definition 1.2.39. *Misiak (1989) Let* $(\Omega, \|.,\ldots,.\|)$ *be an n-*$\mathcal{NS}$ *and* $\{\varrho_n\}$ *be a sequence in* Ω. *Then,* $\{\varrho_n\}$ *is c.t.b. a* C_{seq} *if*

$$\lim_{k,l\to\infty} \|\varrho_k - \varrho_l, z_2, \ldots, z_n\| = 0 \ \forall z_2, z_3, \ldots, z_n \in \Omega.$$

Definition 1.2.40. *Misiak (1989) Let* $(\Omega, \|.,\ldots,.\|)$ *be an n-*$\mathcal{NS}$ *and* $\{\varrho_n\}$ *be a sequence in* Ω. *Then,* $\{\varrho_n\}$ *is c.t.b. convergent if*

$$\lim_{k\to\infty} \|\varrho_k - \varrho, z_2, \ldots, z_n\| = 0 \ \forall z_2, z_3, \ldots, z_n \in \Omega.$$

Definition 1.2.41. *Misiak (1989)* $(\Omega, \|.,\ldots,.\|)$ *is c.t.b. an n-* $\mathcal{BS}$ *if every* C_{seq} *in* Ω *is convergent in* Ω.

Lemma 1.2.42. *Xu and Rassias (2012) Let* $(\Omega, \|.,\ldots,.\|)$ *be an n-*$\mathcal{NS}$ *and* $\{\varrho_k\}$ *be a convergent sequence in* Ω. *Then,*

$$\lim_{k\to\infty} \|\varrho_k, z_2, z_3, \ldots, z_n\| = \left\|\lim_{k\to\infty} \varrho_k, z_2, z_3, \ldots, z_n\right\|, \ \forall z_i \in \Omega, \ \textit{where } 1 < i \leq n.$$

Lemma 1.2.43. *Xu and Rassias (2012) Let* $(\Omega, \|.,\ldots,.\|)$ *be an n-*$\mathcal{NS}$ *and*

$$\|\varrho, \vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}\| = 0 \ \forall \vartheta_i \in \Omega.$$

Then, $\varrho = 0$.

Theorem 1.2.44. *Diaz and Margolis (1968) Consider a generalized complete MS* (Ω, d) *and a self mapping* $\Gamma : \Omega \to \Omega$ *satisfying:*

$$d(\Gamma\varrho, \Gamma\vartheta) \leq kd(\varrho, \vartheta).$$

for $k \in [0, 1)$, *whenever* $d(\varrho, \vartheta) < \infty$. *Then,*
either $d(\Gamma^n(\varrho), \Gamma^{n+1}(\varrho)) = \infty$ *or the following assertions holds:*

(i) $\lim_{n\to\infty} \Gamma^n\varrho = \varrho^*$, *where* ϱ^* *is the fixed point of* Γ*;*

(ii) $d(\varrho, \varrho^*) \leq \dfrac{1}{1-k} d(\vartheta, \Gamma\vartheta)$.

Definition 1.2.45. *Yang et al. (2015) Let* Ω *be a* $\mathcal{VS}$ *over a scalar field* K *with a non-Archimedean nontrivial valuation* $|\cdot|$ *and* $dim\Omega \geq n$. *Then, a mapping* $\|., \ldots, .\| : \Omega^n \to [0, \infty)$ *is termed as a* ***non-Archimedean n-norm****, if it satisfies :*

(i) $\|\varrho_1, \varrho_2, \ldots, \varrho_n\| = 0 \Leftrightarrow \varrho_i$ *and* ϱ_j *are linearly independent* $\forall\ 1 \leq i \neq j \leq n$*;*

(ii) $\|\varrho_1, \varrho_2, \ldots, \varrho_n\|$ *is invariant under the permutations of* $\varrho_1, \varrho_2, \ldots, \varrho_n$*;*

(iii) $\|\alpha\varrho_1, \varrho_2, \ldots, \varrho_n\| = |\alpha|\|\varrho_1, \varrho_2, \ldots, \varrho_n\|$*;*

(iv) $\|\varrho_1 + \vartheta_1, \varrho_2, \ldots, \varrho_n\| \leq \max\{\|\varrho_1, \varrho_2, \ldots, \varrho_n\|, \|\vartheta_1, \varrho_2, \ldots, \varrho_n\|\}$,

$\forall \varrho_i \in \Omega$. *Moreover,* $(\Omega, \|., \ldots, .\|)$ *is termed as non-Archimedean n-* $\mathcal{NS}$.

Definition 1.2.46. *Yang et al. (2015) Let* Ω *be a non-Archimedean n-* $\mathcal{NS}$ *and* $\{\varrho_\hbar\}$ *be a sequence in* Ω. *Then,* $\{\varrho_\hbar\}$ *is c.t.b. Cauchy* $\Leftrightarrow \{\varrho_{\hbar+1} - \varrho_\hbar\} \to 0$, *as* $\hbar \to \infty$.

Definition 1.2.47. *J. M. Rassias and Kim (2009) Consider a function* $\Phi : \Omega \to Y$, *where* Ω *and* $(Y, \leq)$ *are closed under addition. Then,* Φ *is c.t.b.*

(i) sub additive if $\Phi(\varrho + \vartheta) \leq \Phi(\varrho) + \Phi(\vartheta)$*;*

(ii) contractively sub additive if $\exists\lambda \in [0, 1)$ *s.t.* $\Phi(\varrho + \vartheta) \leq \lambda (\Phi(\varrho) + \Phi(\vartheta))$*;*

(iii) expansively super additive if $\exists\lambda \in [0, 1)$ *s.t.* $\Phi(\varrho + \vartheta) \geq \frac{1}{\lambda} (\Phi(\varrho) + \Phi(\vartheta))$,

$\forall \varrho, \vartheta \in \Omega$.

1.3 Chapterwise Summary

In this section, we provide a brief summary of the results proved in the later chapters of the thesis.
Chapter 2, deals with the results on the stability of quadratic and quartic type functional equations. Some illustrations are presented to demonstrate the significance of the assumption made in proved results. Also, the stability of a generalized quartic function equation using fixed point approach and conventional approach in n-$\mathcal{BS}$ and non- Archimedean n-$\mathcal{BS}$ is discussed.

Chapter 3, introduces the concept of C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-MS, which is a generalization of both C^*_{AV}-$\mathfrak{m}$-MS and $\mathcal{R}$-MS. The first section presents the definition of C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-MS, along with its intrinsic properties and several illustrative examples. The second section focuses on the existence and uniqueness of fixed points within C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-MS, using the concept of C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-contraction mappings. The results established in this chapter extend and generalize several well-known fixed point theorems found in the literature. Also, as an application the existence and uniqueness of the solution to the operator equation is presented.

Chapter 4, introduces the concept of a generalized distance function, referred to as the multiplicative $\mathfrak{m}$-metric. The first section presents the basic definition and intrinsic properties of the multiplicative $\mathfrak{m}$-MS, along with illustrative examples. The second section discusses various fixed point results for self mappings within the framework of multiplicative $\mathfrak{m}$-MS, using different contractions. Illustrative examples are provided to discuss the existence of fixed points for discontinuous self mappings. In the third section, common fixed point results for a pair of self mappings are explored using generalized contraction conditions. An illustrative example, involving discontinuous self mappings, is discussed, along with numerical iterations to approximate the common fixed point, supported by graphical representations. The fourth section presents fixed point and common fixed point results using a three-point analogue of contraction mappings. Additionally, as applications, the existence and uniqueness of solutions to the initial value problem and a system of integral equations are discussed.

Chapter 5, presents several common fixed point results for self mappings in PMS using the (ϕ, ψ)-Wardowski type contraction. Furthermore, some fixed point results are proven using generalized cyclic contractions, followed by illustrative examples. As an application, the existence of a fractal set for the Hutchinson-Barnsley operator is established using the fixed point results proved in the chapter. Finally, some iterations for generating fractal sets are presented, along with the resulting fractals.

Chapter 6, discusses some common fixed point results for self mapping in b-MS using relation theoretic and α-admissible generalized contractions. As applications of the proved results the existence of solution to a class of non-linear functional integral equation and an operator equation are established.

Chapter 7 introduces several fixed point results within the framework of $\mathfrak{m}$-MS using contraction mappings. The continuity conditions of self mappings are not essential in the results proved, unlike those in existing literature. The chapter discusses examples where well-known contractions in metric spaces do not guarantee the existence of a fixed point, but their generalizations within $\mathfrak{m}$-MS yield the desired outcome. These examples are validated through graphical visualizations of contraction mappings, which help in understanding their behavior and highlight the distinctions between metric spaces and $\mathfrak{m}$-MS. The main sections present fixed point and common fixed point results using various contractions. Finally, some numerical iterations for approximating the common fixed point are provided, accompanied by graphs that visually demonstrate the results.

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Chapter 2

Some Stability and Hyperstability Results for Functional Equations

2.1 Introduction

The concept of stability of functional equations(FEs) began with Ulam's problem, raised at the University of Wisconsin, which sought to determine when approximate solutions to FEs are close to exact solutions. Hyers provided the first significant response to this problem, known as Ulam-Hyers stability, which offers a framework for understanding the conditions under which approximate solutions to FEs are close to the exact solution.

Following Hyers' work, numerous generalizations and extensions of this theory have been developed, expanding its applicability to various functional equations and contexts, for reference see (Cholewa (1984); Brzdek and Pietrzyk (2008); Bahyrycz et al. (2013); Brzdek and Cieplinski (2013); Bahyrycz and Piszczek (2014); Brzdek (2014); Bahyrycz and Olko (2016); Brzdek et al. (2016); Alessa et al. (2021); Ciepliński (2021); Bahyrycz and Sikorska (2022); Jeyaraman et al. (2022); Aderyani et al. (2023); Benzarouala, Brzdek, et al. (2023); Benzarouala, Brzdęk, and Oubbi (2023); Park and Senasukh (2023); Jin and Lee (2024) and the referenced cited therein).

The present chapter of the thesis explores some stability results for FEs mainly quadratic type and quartic type FEs. In the first section, we establish some stability results for classical quadratic FE $\digamma(\varrho+\vartheta)+\digamma(\varrho-\vartheta) = 2\digamma(\varrho)+2\digamma(\vartheta)$ and Jensen type quadratic FE $2\digamma\left(\frac{\varrho+\vartheta}{2}\right)+2\digamma\left(\frac{\varrho-\vartheta}{2}\right)=\digamma(\varrho)+\digamma(\vartheta)$ using a well established approach given by Moghimi and Najati (2022). Additionally, some hyperstability results for quadratic type FEs along with some instances that illustrate the necessity of the assumptions made to establish the stability results on quadratic type FEs.

In the second section, inspired by the stability results in literature, we present the stability result for the generalized quartic FE of the form

$$\begin{aligned}\digamma(a\varrho+b\vartheta)+\digamma(a\varrho-b\vartheta) &= 2a^2(a^2-b^2)\digamma(\varrho)+2b^2(b^2-a^2)\digamma(\vartheta)\\ &\quad + a^2b^2\left(\digamma(\varrho+\vartheta)+\digamma(\varrho-\vartheta)\right), \quad (2.1.1)\end{aligned}$$

where a, b $(a \neq b)$ are positive integers. Also, we explore the Ulam-Hyers-Rassias stability results in n-Banach space using a fixed point technique. Eventually, some stability results are established using the conventional approach with contractively and expansively sub additive control functions. At last, the stability results of the quartic FE in non-Archimedean n-Banach space are established. The results of this chapter are presented in [1] [2].

2.2 Some Results on Stability of Quadratic type Functional Equations in Banach Space

In this section, we explore some hyperstability results for quadratic type FEs in the context of $\mathcal{BS}$s.

Theorem 2.2.1. *Let V and W be two $\mathcal{NS}$s and $U \subset V - \{0\}$ be a non-empty subset. Choose $\epsilon \geq 0$ and real numbers p_1, p_2 and p_3 satisfying $p_1+p_2+p_3<0$. Consider that $\forall\ \varrho \in U$, $\exists\ \hbar_\varrho \in \mathbb{N}$ s.t. $\hbar\varrho \in U\ \forall\ \hbar \geq \hbar_\varrho$. If a function $\digamma : V \to W$*

[1] Yadav, K., & Kumar, D. (2024). Some hyperstability results for quadratic type functional equations. *Applied Mathematics E-Notes*, *24*, 212–227.

[2] Yadav, K., & Kumar, D. Stability analysis of a generalized quartic functional equation in n-Banach space, (Communicated).

defined on V satisfies

$$\|\digamma(\varrho+\vartheta+\zeta)+\digamma(\varrho)+\digamma(\vartheta)+\digamma(\zeta)-\digamma(\varrho+\vartheta)-\digamma(\vartheta+\zeta)-\digamma(\zeta+\varrho)\| \leq \epsilon\|\varrho\|^{p_1}\|\vartheta\|^{p_2}\|\zeta\|^{p_3}, \quad (2.2.1)$$

$\forall \varrho, \vartheta, \zeta \in U, \varrho+\vartheta, \vartheta+\zeta, \varrho+\zeta, \varrho+\vartheta+\zeta \in U$.

Then, $\digamma$ is a quadratic type FE satisfying

$$\digamma(\varrho+\vartheta+\zeta)+\digamma(\varrho)+\digamma(\vartheta)+\digamma(\zeta)=\digamma(\varrho+\vartheta)+\digamma(\vartheta+\zeta)+\digamma(\zeta+\varrho)$$

$\forall \varrho, \vartheta, \varrho+\vartheta, \vartheta+\zeta, \varrho+\zeta, \varrho+\vartheta+\zeta \in U$.

Proof. It is given that $p_1+p_2+p_3<0$. Therefore, without loss of generality, let $p_2+p_3<0$. Let $\varrho, \vartheta \in U$ with $\varrho+\vartheta,\ \vartheta+\zeta,\ \zeta+\varrho \in U$. Hence, by the given hypothesis $\exists$ a natural number k s.t. $\hbar\varrho, \hbar\vartheta, \hbar\zeta, \hbar(\varrho+\vartheta), \hbar(\vartheta+\zeta), \hbar(\varrho+\zeta)$ and $\hbar(\varrho+\vartheta+\zeta) \in U\ \forall\ \hbar \geq \mathsf{k}$.

Substituting $\varrho=\varrho$, $\vartheta=\hbar\varrho$ and $\zeta=\hbar\varrho$ in (2.2.1), we have

$$\left\|\digamma\Big((1+2\hbar)\varrho\Big)+\digamma(\varrho)+2\digamma(\hbar\varrho)-2\digamma\Big((1+\hbar)\varrho\Big)-\digamma(2\hbar\varrho)\right\| \leq \epsilon\hbar^{p_2+p_3}\|\varrho\|^{p_1+p_2+p_3}.$$

Similarly, we have

$$\left\|\digamma\Big((1+2\hbar)\zeta\Big)+\digamma(\zeta)+2\digamma(\hbar\zeta)-2\digamma\Big((1+\hbar)\zeta\Big)-\digamma(2\hbar\zeta)\right\| \leq \epsilon\hbar^{p_2+p_3}\|\zeta\|^{p_1+p_2+p_3},$$

$$\left\|\digamma\Big((1+2\hbar)\vartheta\Big)+\digamma(\vartheta)+2\digamma(\hbar\vartheta)-2\digamma\Big((1+\hbar)\vartheta\Big)-\digamma(2\hbar\vartheta)\right\| \leq \epsilon\hbar^{p_2+p_3}\|\vartheta\|^{p_1+p_2+p_3},$$

$$\Big\|\digamma\Big((1+2\hbar)(\varrho+\vartheta)\Big)+\digamma(\varrho+\vartheta)+2\digamma\Big(\hbar(\varrho+\vartheta)\Big)-2\digamma\Big((1+\hbar)(\varrho+\vartheta)\Big) - \digamma\Big(2\hbar(\varrho+\vartheta)\Big)\Big\| \leq \epsilon\hbar^{p_2+p_3}\|\varrho+\vartheta\|^{p_1+p_2+p_3},$$

$$\Big\|\digamma\Big((1+2\hbar)(\vartheta+\zeta)\Big)+\digamma(\vartheta+\zeta)+2\digamma\Big(\hbar(\vartheta+\zeta)\Big)-2\digamma\Big((1+\hbar)(\vartheta+\zeta)\Big) - \digamma\Big(2\hbar(\vartheta+\zeta)\Big)\Big\| \leq \epsilon\hbar^{p_2+p_3}\|\vartheta+\zeta\|^{p_1+p_2+p_3},$$

$$\left\| F\Big((1+2\hbar)(\varrho+\zeta)\Big) + F(\varrho+\zeta) + 2F\Big(\hbar(\varrho+\zeta)\Big) - 2F\Big((1+\hbar)(\varrho+\zeta)\Big) - F\Big(2\hbar(\varrho+\zeta)\Big)\right\| \leq \epsilon \hbar^{p_2+p_3} \|\varrho+\zeta\|^{p_1+p_2+p_3}$$

and

$$\left\| F\Big((1+2\hbar)(\varrho+\vartheta+\zeta)\Big) + F(\varrho+\vartheta+\zeta) + 2F\Big(\hbar(\varrho+\vartheta+\zeta)\Big) - 2F\Big((1+\hbar)(\varrho+\vartheta+\zeta)\Big) - F\Big(2\hbar(\varrho+\vartheta+\zeta)\Big)\right\| \leq \epsilon \hbar^{p_2+p_3} \|\varrho+\vartheta+\zeta\|^{p_1+p_2+p_3}.$$

Since $p_2+p_3<0$, taking limit as $\hbar \to \infty$ in above inequalities, we have

$$F(\varrho) = \lim_{\hbar\to\infty}\left(2F\Big((1+\hbar)(\varrho)\Big) + F(2\hbar\varrho) - F\Big((1+2\hbar)\varrho\Big) - 2F(\hbar\varrho)\right),$$

$$F(\vartheta) = \lim_{\hbar\to\infty}\left(2F\Big((1+\hbar)(\vartheta)\Big) + F(2\hbar\vartheta) - F\Big((1+2\hbar)\vartheta\Big) - 2F(\hbar\vartheta)\right),$$

$$F(\zeta) = \lim_{\hbar\to\infty}\left(2F\Big((1+\hbar)(\zeta)\Big) + F(2\hbar\zeta) - F\Big((1+2\hbar)\zeta\Big) - 2F(\hbar\zeta)\right),$$

$$F(\varrho+\vartheta) = \lim_{\hbar\to\infty}\left(2F\Big((1+\hbar)(\varrho+\vartheta)\Big) + F\Big(2\hbar(\varrho+\vartheta)\Big) - F\Big((1+2\hbar)(\varrho+\vartheta)\Big) - 2F\Big(\hbar(\varrho+\vartheta)\Big)\right),$$

$$F(\vartheta+\zeta) = \lim_{\hbar\to\infty}\left(2F\Big((1+\hbar)(\vartheta+\zeta)\Big) + F\Big(2\hbar(\vartheta+\zeta)\Big) - F\Big((1+2\hbar)(\vartheta+\zeta)\Big) - 2F\Big(\hbar(\vartheta+\zeta)\Big)\right),$$

$$F(\varrho+\zeta) = \lim_{\hbar\to\infty}\left(2F\Big((1+\hbar)(\varrho+\zeta)\Big) + F\Big(2\hbar(\varrho+\zeta)\Big) - F\Big((1+2\hbar)(\varrho+\zeta)\Big) - 2F\Big(\hbar(\varrho+\zeta)\Big)\right)$$

and

$$F(\varrho+\vartheta+\zeta) = \lim_{\hbar\to\infty}\left(2F\Big((1+\hbar)(\varrho+\vartheta+\zeta)\Big) + F\Big(2\hbar(\varrho+\vartheta+\zeta)\Big) - F\Big((1+2\hbar)(\varrho+\vartheta+\zeta)\Big) - 2F\Big(\hbar(\varrho+\vartheta+\zeta)\Big)\right).$$

Now,

$$
\begin{aligned}
& \Big\|F(\varrho+\vartheta+\zeta)+F(\varrho)+F(\vartheta)+F(\zeta)-F(\varrho+\vartheta)-F(\vartheta+\zeta)-F(\varrho+\zeta)\Big\| \\
= \; & \lim_{\hbar\to\infty}\bigg\|\bigg(2F\Big((1+\hbar)(\varrho+\vartheta+\zeta)\Big)+F\Big(2\hbar(\varrho+\vartheta+\zeta)\Big)-F\Big((1+2\hbar)(\varrho+\vartheta+\zeta)\Big) \\
& -2F\Big(\hbar(\varrho+\vartheta+\zeta)\Big)\bigg)+\bigg(2F\Big((1+\hbar)(\varrho)\Big)+F(2\hbar\varrho)-F\Big((1+2\hbar)\varrho\Big)-2F(\hbar\varrho)\bigg) \\
& +\bigg(2F\Big((1+\hbar)(\vartheta)\Big)+F(2\hbar\vartheta)-F\Big((1+2\hbar)\vartheta\Big)-2F(\hbar\vartheta)\bigg) \\
& +\bigg(2F\Big((1+\hbar)(\zeta)\Big)+F(2\hbar\zeta)-F\Big((1+2\hbar)\zeta\Big)-2F(\hbar\zeta)\bigg) \\
& -\bigg(2F\Big((1+\hbar)(\varrho+\vartheta)\Big)+F\Big(2\hbar(\varrho+\vartheta)\Big)-F\Big((21+\hbar)(\varrho+\vartheta)\Big)-2F\Big(\hbar(\varrho+\vartheta)\Big)\bigg) \\
& -\bigg(2F\Big((1+\hbar)(\vartheta+\zeta)\Big)+F\Big(2\hbar(\vartheta+\zeta)\Big)-F\Big((1+2\hbar)(\vartheta+\zeta)\Big)-2F\Big(\hbar(\vartheta+\zeta)\Big)\bigg) \\
& -\bigg(2F\Big((1+\hbar)(\varrho+\zeta)\Big)+F\Big(2\hbar(\varrho+\zeta)\Big)-F\Big((1+2\hbar)(\varrho+\zeta)\Big)-2F\Big(\hbar(\varrho+\zeta)\Big)\bigg)\bigg\| \\
\leq \; & \lim_{\hbar\to\infty} 2\bigg\|F\Big((1+\hbar)(\varrho+\vartheta+\zeta)\Big)+F\Big((1+\hbar)\varrho\Big)+F\Big((1+\hbar)\vartheta\Big)+F\Big((1+\hbar)\zeta\Big) \\
& -F\Big((1+\hbar)(\varrho+\vartheta)\Big)-F\Big((1+\hbar)(\vartheta+\zeta)\Big)-F\Big((1+\hbar)(\varrho+\zeta)\Big)\bigg\| \\
& +\bigg\|F\Big(2\hbar(\varrho+\vartheta+\zeta)\Big)+F(2\hbar\varrho)+F(\hbar\vartheta)+F(2\hbar\zeta)-F\Big(2\hbar(\varrho+\vartheta)\Big)-F(2\hbar(\vartheta+\zeta)) \\
& -F\Big(2\hbar(\varrho+\zeta)\Big)\bigg\| \\
& +\bigg\|F\Big((1+2\hbar)(\varrho+\vartheta+\zeta)\Big)+F\Big((1+2\hbar)\varrho\Big)+F\Big((1+2\hbar)\vartheta\Big)+F\Big((21+\hbar)\zeta\Big) \\
& -F\Big((1+2\hbar)(\varrho+\vartheta)\Big)-F\Big((1+2\hbar)(\vartheta+\zeta)\Big) \\
& -F\Big((1+2\hbar)(\varrho+\zeta)\Big)\bigg\|+2\bigg\|F\Big(\hbar(\varrho+\vartheta+\zeta)\Big)+F(\hbar\varrho)+F(\hbar\vartheta)+F(\hbar\zeta) \\
& -F\Big(\hbar(\varrho+\vartheta)\Big)-F\Big(\hbar(\vartheta+\zeta)\Big)-F\Big(\hbar(\varrho+\zeta)\Big)\bigg\| \\
\leq \; & \lim_{\hbar\to\infty}\epsilon\bigg(2(1+\hbar)^{p_1+p_2+p_3}+(2\hbar)^{p_1+p_2+p_3}+(1+2\hbar)^{p_1+p_2+p_3}+2\hbar^{p_1+p_2+p_3}\bigg) \\
& .\|\varrho\|^{p_1}\|\vartheta\|^{p_2}\|\zeta\|^{p_3} \\
= \; & 0.
\end{aligned}
$$

Hence,

$$\digamma(\varrho+\vartheta+\zeta)+\digamma(\varrho)+\digamma(\vartheta)+\digamma(\zeta)=\digamma(\varrho+\vartheta)+\digamma(\vartheta+\zeta)\digamma(\varrho+\zeta),$$

$\forall \varrho, \vartheta, \varrho+\vartheta, \vartheta+\zeta, \varrho+\zeta, \varrho+\vartheta+\zeta \in U$. □

Remark 2.2.2. *The condition $p_1+p_2+p_3<0$ specified in Theorem 2.2.1 is crucial for the hyperstability result. The example 2.2.3 illustrates that if the criterion $p_1+p_2+p_3<0$ is not met, then the function may not be of quadratic type.*

Example 2.2.3. *Let $\digamma:\mathbb{R}\to\mathbb{R}$ be defined as $\digamma(\varrho)=\varrho^3$. Then we have*

$$\begin{aligned}\Big\|\digamma(\varrho+\vartheta+\zeta) &+ \digamma(\varrho)+\digamma(\vartheta)+\digamma(\zeta)-\digamma(\varrho+\vartheta)-\digamma(\vartheta+\zeta)-\digamma(\zeta+\varrho)\Big\| \\ &= \Big|(\varrho+\vartheta+\zeta)^3+\varrho^3+\vartheta^3+\zeta^3-(\varrho+\vartheta)^3-(\vartheta+\zeta)^3-(\zeta+\varrho)^3\Big| \\ &= |6\varrho\vartheta\zeta| \le 6|\varrho||\vartheta||\zeta|.\end{aligned}$$

The hypothesis (2.2.1) holds, for $p_1=1,\ p_2=1,\ p_3=1,\ \epsilon=6$, but $\digamma$ is not a quadratic function.

Theorem 2.2.4. *Let V and W be two $\mathcal{NS}$s and $U\subset V-\{0\}$ be a non-empty subset. Choose $\epsilon\ge 0$ and real numbers p_1 and p_2 satisfying $p_1+p_2<0$. Consider that $\forall\ \varrho\in U,\ \exists\ \hbar_\varrho\in\mathbb{N}$ s.t. $\hbar\varrho\in U\ \forall\ \hbar\ge\hbar_\varrho$. If a function $\digamma:V\to W$ defined on V satisfies*

$$\|\digamma(\varrho+\vartheta)+\digamma(\varrho-\vartheta)-2\digamma(\varrho)-2\digamma(\vartheta)\|\le\epsilon\|\varrho\|^{p_1}\|\vartheta\|^{p_2}, \tag{2.2.2}$$

$\forall \varrho, \vartheta, \varrho+\vartheta, \varrho-\vartheta \in U$. Then, $\digamma$ is quadratic on U, i.e.,

$$\digamma(\varrho+\vartheta)+\digamma(\varrho-\vartheta)=2\digamma(\varrho)+2\digamma(\vartheta),$$

$\forall \varrho, \vartheta, \varrho+\vartheta, \varrho-\vartheta \in U$.

Proof. The result is analogous to Theorem 2.2.1. □

Remark 2.2.5. *The condition $p_1+p_2<0$ specified in Theorem 2.2.4 is crucial for the hyperstability result. The example 2.2.6, illustrates that if the criterion $p_1+p_2<0$ is not met, then the function may not be quadratic.*

Example 2.2.6. *Let $\digamma : \mathbb{R} \to \mathbb{R}$ be defined as $\digamma(\varrho) = \varrho^4$. Then, we have*

$$\begin{aligned} \left\| \digamma(\varrho+\vartheta) + \digamma(\varrho-\vartheta) - 2\digamma(\varrho) - 2\digamma(\vartheta) \right\| &= \left| (\varrho+\vartheta)^4 + (\varrho-\vartheta)^4 - 2\varrho^4 - 2\vartheta^4 \right| \\ &= \left| 12\varrho^2\vartheta^2 \right| \leq 12|\varrho|^2|\vartheta|^2. \end{aligned}$$

The hypothesis (2.2.2) holds for $\epsilon = 12$ and $p_1 = p_2 = 2$, but $\digamma$ is not quadratic.

Theorem 2.2.7. *Let V and W be two $\mathcal{NS}s$ and $U \subset V - \{0\}$ be a non-empty subset. Choose $\epsilon \geq 0$ and real numbers p_1 and p_2 satisfying $p_1 + p_2 < 0$. Consider that $\forall\ \varrho \in U$, $\exists\ \hbar_\varrho \in \mathbb{N}$ s.t. $\frac{\hbar\varrho}{2} \in U\ \forall\ \hbar \geq \hbar_\varrho$. If a function $\digamma : V \to W$ defined on V satisfies*

$$\left\| 2\digamma\left(\frac{\varrho+\vartheta}{2}\right) + 2\digamma\left(\frac{\varrho-\vartheta}{2}\right) - \digamma(\varrho) - \digamma(\vartheta) \right\| \leq \epsilon \|\varrho\|^{p_1} \|\vartheta\|^{p_2}, \tag{2.2.3}$$

$\forall \varrho, \vartheta, \frac{\varrho+\vartheta}{2}, \frac{\varrho-\vartheta}{2} \in U$. Then, $\digamma$ is quadratic of Jensen type on U, i.e.,

$$2\digamma\left(\frac{\varrho+\vartheta}{2}\right) + 2\digamma\left(\frac{\varrho-\vartheta}{2}\right) = \digamma(\varrho) + \digamma(\vartheta),$$

$\forall \varrho, \vartheta, \frac{\varrho+\vartheta}{2}, \frac{\varrho-\vartheta}{2} \in U$.

Proof. The result is analogous to Theorem 2.2.1. □

Remark 2.2.8. *The condition $p_1+p_2 < 0$ specified in Theorem 2.2.7 is crucial for the hyperstability result. In Example 2.2.9, we have illustrated that if the criterion $p_1 + p_2 < 0$ is not met, then the function may not be quadratic of Jensen type.*

Example 2.2.9. *Let $\digamma : \mathbb{R} \to \mathbb{R}$ be defined as $\digamma(\varrho) = \varrho^4$ and let $U = [1, \infty)$. Then, we have*

$$\begin{aligned} \left\| 2\digamma\left(\frac{\varrho+\vartheta}{2}\right) + 2\digamma\left(\frac{\varrho-\vartheta}{2}\right) - \digamma(\varrho) - \digamma(\vartheta) \right\| &= \left| 2\left(\frac{\varrho+\vartheta}{2}\right)^4 + 2\left(\frac{\varrho-\vartheta}{2}\right)^4 - \varrho^4 - \vartheta^4 \right| \\ &= \frac{3}{4}\left|\varrho^2 - \vartheta^2\right|^2 \leq \frac{3}{2}\left|\varrho^2\right|^2\left|\vartheta^2\right|^2 \\ &= \frac{3}{2}|\varrho|^4|\vartheta|^4. \end{aligned}$$

The hypothesis (2.2.3) holds for $\epsilon = \frac{3}{2}$, and $p_1 = p_2 = 4$, but $\digamma$ is not quadratic of Jensen type.

Theorem 2.2.10. *Let V and W be two $\mathcal{NS}$s and $U \subset V - \{0\}$ be a non-empty subset. Choose $\epsilon \geq 0$ and real numbers p_1, p_2, p_3 and p_4 satisfying $p_1+p_2+p_3+p_4 < 0$. Consider that $\forall \varrho \in U$, $\exists\, \hbar_\varrho \in \mathbb{N}$ s.t. $\hbar\varrho \in U\ \forall \hbar \geq \hbar_\varrho$. If a function $F : V \to W$ defined on V satisfies*

$$\|F(\varrho+\vartheta+\zeta)+F(\varrho)+F(\vartheta)+F(\zeta)-F(\varrho+\vartheta)-F(\vartheta+\zeta)-F(\zeta+\varrho)\| \\ \leq \|\varrho\|^{p_1}\|\vartheta\|^{p_2}\|\zeta\|^{p_3}\epsilon\|\varrho+\vartheta+\zeta\|^{p_4}, \quad (2.2.4)$$

$\forall \varrho, \vartheta, \zeta, \varrho+\vartheta, \vartheta+\zeta, \varrho+\zeta, \varrho+\vartheta+\zeta \in U$. Then, F is a quadratic type FE satisfying

$$F(\varrho+\vartheta+\zeta)+F(\varrho)+F(\vartheta)+F(\zeta) = F(\varrho+\vartheta)+F(\vartheta+\zeta)+F(\zeta+\varrho),$$

$\forall\ \varrho,\ \vartheta,\ \zeta,\ \varrho+\vartheta,\ \vartheta+\zeta,\ \varrho+\zeta,\ \varrho+\vartheta+\zeta \in U$.

Proof. The result is analogous to Theorem 2.2.1. □

Remark 2.2.11. *The condition $p_1+p_2+p_3+p_4 < 0$ specified in Theorem 2.2.10 is crucial for the hyperstability result. The example 2.2.12 illustrates that if the criterion $p_1+p_2+p_3+p_4 < 0$ is not met, then the function may not be of quadratic type.*

Example 2.2.12. *Let $F : \mathbb{R} \to \mathbb{R}$ defined as $F(\varrho) = \varrho^4$. Then, we have*

$$\begin{aligned}\Big\|F(\varrho+\vartheta+\zeta) &+ F(\varrho)+F(\vartheta)+F(\zeta)-F(\varrho+\vartheta)-F(\vartheta+\zeta)-F(\zeta+\varrho)\Big\| \\ &= \left|(\varrho+\vartheta+\zeta)^4+\varrho^4+\vartheta^4+\zeta^4-(\varrho+\vartheta)^4-(\vartheta+\zeta)^4-(\varrho+\zeta)^4\right| \\ &= \left|8\left(\varrho^2yz+xy^2\zeta+\varrho\vartheta\zeta^2\right)\right| \\ &\leq 8|\varrho\vartheta\zeta||\varrho+\vartheta+\zeta| \\ &\leq 8|\varrho||\vartheta||\zeta||\varrho+\vartheta+\zeta|\,.\end{aligned}$$

The hypothesis (2.2.4) holds for $p_1 = p_2 = p_3 = p_4 = 1\&\epsilon = 8$, but F is not a quadratic function.

Remark 2.2.13. *The results proved in Theorem 2.2.1 and Theorem 2.2.10 is associated to various hyperstability results concerning linear and quadratic FEs* *in the existing literature. For instance, see Proposition 1.10 of Zhang (2015), Corollary 3.7, Corollary 3.10, Corollary 3.12 of Bahyrycz and Olko (2016).*

Theorem 2.2.14. *Let V and W be two $\mathcal{NS}$s and $U \subset V - \{0\}$ be a non-empty subset. Choose $\epsilon, \theta \geq 0$ and real numbers p_1, p_2 and p_3 satisfying $p_1 + p_2 + p_3 < 0$ and $p_1 + p_2 + 2p_3 < 0$. Consider that $\forall \varrho \in U$, $\exists$ an $\hbar_\varrho \in \mathbb{N}$ s.t. $\hbar\varrho \in U \ \forall\ \hbar \geq \hbar_\varrho$. If a function $\digamma : V \to W$ defined on V satisfies*

$$\|\digamma(\varrho+\vartheta)+\digamma(\varrho-\vartheta)-2\digamma(\varrho)-2\digamma(\vartheta)\| \leq \|\varrho\|^{p_1}\|\vartheta\|^{p_2}(\epsilon\|\varrho+\vartheta\|^{p_3}+\theta\|\varrho-\vartheta\|^{p_3}), \quad (2.2.5)$$

$\forall \varrho, \vartheta, \varrho+\vartheta, \varrho-\vartheta \in U$, then $\digamma$ is quadratic on U, i.e.,

$$\digamma(\varrho+\vartheta)+\digamma(\varrho-\vartheta) = 2\digamma(\varrho)+2\digamma(\vartheta)\ \forall \varrho, \vartheta, \varrho+\vartheta, \varrho-\vartheta \in U.$$

Proof. The result is analogous to Theorem 2.2.1. □

Remark 2.2.15. *The conditions $p_1 + p_2 + p_3 < 0$ and $p_1 + p_2 + 2p_3 < 0$ specified in Theorem 2.2.14 are crucial for the hyperstability result. In Example 2.2.16, we have illustrated that if the criteria ($p_1 + p_2 + p_3 < 0$ and $p_1 + p_2 + 2p_3 < 0$) is not met, then the function may not be quadratic.*

Example 2.2.16. *Let $\digamma : \mathbb{R} \to \mathbb{R}$ be defined as $\digamma(\varrho) = \varrho^3$ and let $U = [1, \infty)$. Then, we have*

$$\begin{aligned}
\Big\|\digamma(\varrho+\vartheta)+\digamma(\varrho-\vartheta)-2\digamma(\varrho)-2\digamma(\vartheta)\Big\| &= \Big|(\varrho+\vartheta)^3+(\varrho-\vartheta)^3-2\varrho^3-2\vartheta^3\Big| \\
&= \Big|6\varrho\vartheta^2-2\vartheta^3\Big| \\
&= 2|\vartheta|^2|3\varrho-\vartheta| \\
&= 2|\vartheta|^2|3\varrho-\varrho+\varrho-\vartheta| \\
&\leq 2|\vartheta|^2\Big(|2\varrho|+|\varrho-\vartheta|\Big) \\
&\leq |\varrho|^1|\vartheta|^2\Big(4|\varrho+\vartheta|+2|\varrho-\vartheta|\Big).
\end{aligned}$$

The hypothesis (2.2.5) holds for $p_1 = 1, p_2 = 2, p_3 = 1, \epsilon = 4\&\theta = 2$, but $\digamma$ is not a quadratic function.

Remark 2.2.17. *The result proved in Theorem 2.2.4 and Theorem 2.2.14 is associated to various hyperstability results concerning linear and quadratic FEs in the existing literature. For instance, see Theorem 2 of Brzdek et al. (2016).*

Theorem 2.2.18. *Let V and W be two $\mathcal{NS}$s and $U \subset V - \{0\}$ be a non-empty subset. Choose $\epsilon, \theta \geq 0$ and real numbers p_1, p_2 and p_3 satisfying $p_1 + p_2 + p_3 < 0$ and $p_1 + p_2 + 2p_3 < 0$. Consider that $\forall \varrho \in U$, $\exists$ an $\hbar_\varrho \in \mathbb{N}$ s.t. $\hbar\varrho \in U$ $\forall \hbar \geq \hbar_\varrho$. If a function $F : V \to W$ defined on V satisfies*

$$\left\| 2F\left(\frac{\varrho+\vartheta}{2}\right) + 2F\left(\frac{\varrho-\vartheta}{2}\right) - F(\varrho) - F(\vartheta) \right\| \leq \|\varrho\|^{p_1}\|\vartheta\|^{p_2}(\epsilon\|\varrho+\vartheta\|^{p_3} + \theta\|\varrho-\vartheta\|^{p_3}), \tag{2.2.6}$$

$\forall \varrho, \vartheta, \frac{\varrho+\vartheta}{2}, \frac{\varrho-\vartheta}{2} \in U$, then F is quadratic of Jensen type on U, i.e.,

$$2F\left(\frac{\varrho+\vartheta}{2}\right) + 2F\left(\frac{\varrho-\vartheta}{2}\right) = F(\varrho) + F(\vartheta) \ \forall \varrho, \vartheta, \frac{\varrho+\vartheta}{2}, \frac{\varrho-\vartheta}{2} \in U.$$

Proof. The result is analogues to Theorem 2.2.1. □

Remark 2.2.19. *The conditions $p_1 + p_2 + p_3 < 0$ and $p_1 + p_2 + 2p_3 < 0$ specified in Theorem 2.2.18 are crucial for the hyperstability result. In Example 2.2.20, we have illustrated that if the criteria $p_1 + p_2 + p_3 < 0$ and $p_1 + p_2 + 2p_3 < 0$ is not met, then the function may not be quadratic of Jensen type.*

Example 2.2.20. *Let $F : \mathbb{R} \to \mathbb{R}$ be defined as $F(\varrho) = \varrho^3$, and let $U = [1, \infty)$. Then, we have*

$$\begin{aligned}
\left\| 2F\left(\frac{\varrho+\vartheta}{2}\right) + 2F\left(\frac{\varrho-\vartheta}{3}\right) - F(\varrho) - F(\vartheta) \right\| &= \left| 2\left(\frac{\varrho+\vartheta}{2}\right)^3 + 2\left(\frac{\varrho-\vartheta}{2}\right)^3 - \varrho^3 - \vartheta^3 \right| \\
&= \left| \frac{1}{4}\left(-2\varrho^3 + 6\varrho\vartheta^2 - 4\vartheta^3\right) \right| \\
&= \frac{1}{2}\left|\varrho^3 - 3\varrho\vartheta^2 + 2\vartheta^3\right| \\
&= \frac{1}{2}\left|\varrho^3 - \varrho\vartheta^2 + 2\vartheta^3 - 2\varrho\vartheta^2\right| \\
&= \frac{1}{2}\left|\varrho(\varrho^2 - \vartheta^2) + 2\vartheta(\varrho^2 - \vartheta^2)\right| \\
&\leq \frac{1}{2}\left|\varrho^2 - \vartheta^2\right|\left|\varrho + 2\vartheta\right| \\
&\leq |\varrho|^2|\vartheta|^2\left(2|\varrho+\vartheta| + |\varrho-\vartheta|\right).
\end{aligned}$$

The hypothesis (2.2.6) holds for $p_1 = 2, p_2 = 2, p_3 = 1, \epsilon = 2, \&\theta = 1,$ *but* $\digamma$ *is not a quadratic type Jensen function.*

Remark 2.2.21. *The result proved in Theorem 2.2.7 and Theorem 2.2.18 is associated to various hyperstability results concerning linear and quadratic FEs in the existing literature. For instance, see Corollary 1, Corollary 2 and Corollary 3 of Brzdek et al. (2016), Theorem 2 and Theorem 5 of Bahyrycz and Piszczek (2014).*

Theorem 2.2.22. *Let* V *be a* $\mathcal{NS}$ *over field* $\mathbb{F}$ *and* W *be a* $\mathcal{NS}$ *over field* $\mathbb{K}$*. Let* $p, q \in \mathbb{F} - \{0\}$ *and* $\phi : X \times X \to [0, \infty)$ *be a mapping s.t.*

$$\lim_{\hbar\to\infty} \phi(p^{-1}(1+\hbar)\varrho, -q^{-1}\hbar\varrho) = 0, \ \lim_{\hbar\to\infty} \phi(\hbar\varrho, \hbar\vartheta) = 0 \ \forall \varrho, \vartheta \in V - \{0\}.$$

Let $P, Q \in \mathbb{K}$ *and* $R \in W$*. If* $\digamma : V \to W$ *satisfies*

$$\|\digamma(p\varrho + q\vartheta) + \digamma(p\varrho - q\vartheta) - P\digamma(\varrho) - Q\digamma(\vartheta) - R\| \leq \phi(\varrho, \vartheta), \tag{2.2.7}$$

$\forall\ \varrho, \vartheta \in \{\zeta \in V : \|\zeta\| \geq d\}$ *for some* $d > 0$*, then*

$$\digamma(p\varrho + q\vartheta) + \digamma(p\varrho - q\vartheta) = P\digamma(\varrho) + Q\digamma(\vartheta) + R \ \forall \varrho, \vartheta \in V.$$

Proof. By substituting $\varrho = \frac{(1+\hbar)\varrho}{p}$ and $\vartheta = \frac{\hbar\varrho}{q}$ in (2.2.7), we have

$$\begin{aligned}\Big\|\digamma(\varrho) + \digamma\left((1+2\hbar)\varrho\right) - P\digamma\left(p^{-1}(1+\hbar)\varrho\right) - Q\digamma\left(-q^{-1}\hbar\varrho\right) - R\Big\| \\ \leq \phi\left(p^{-1}(1+\hbar)\varrho, -q^{-1}\hbar\varrho\right),\end{aligned}$$

$\forall\ \varrho \in V - \{0\}$ and $\hbar \in \mathbb{N}$, where $p^{-1}(1+\hbar)\varrho, q^{-1}\hbar\varrho \in \{\zeta \in V : \|\zeta\| \geq d\}$.
Taking limit as $\hbar \to \infty$ on both sides, we have

$$\begin{aligned}\lim_{\hbar\to\infty}\Big\|\digamma(\varrho) + \digamma\left((1+2\hbar)\varrho\right) - P\digamma\left(p^{-1}(1+\hbar)\varrho\right) - Q\digamma\left(-q^{-1}\hbar\varrho\right) - R\Big\| \\ \leq \lim_{\hbar\to\infty} \phi\left(p^{-1}(1+\hbar)\varrho, -q^{-1}\hbar\varrho\right) = 0.\end{aligned} \tag{2.2.8}$$

or

$$\digamma(\varrho) = \lim_{\hbar\to\infty}\Big(P\digamma\left(p^{-1}(1+\hbar)\varrho\right) + Q\digamma\left(-q^{-1}\hbar\varrho\right) + R - \digamma\left((1+2\hbar)\varrho\right)\Big).$$

Similarly, we have

$$\digamma(\vartheta)=\lim_{\hbar\to\infty}\bigg(P\digamma\left(p^{-1}(1+\hbar)\vartheta\right)+Q\digamma\left(-q^{-1}\hbar\vartheta\right)+R-\digamma\left((1+2\hbar)\vartheta\right)\bigg),$$

$$\begin{aligned}\digamma(p\varrho+q\vartheta)=\lim_{\hbar\to\infty}\bigg(&P\digamma\left(p^{-1}(1+\hbar)(p\varrho+q\vartheta)\right)\\&+Q\digamma\left(-q^{-1}\hbar(p\varrho+q\vartheta)\right)+R-\digamma\left((1+2\hbar)(p\varrho+q\vartheta)\right)\bigg),\end{aligned}$$

and

$$\begin{aligned}\digamma(p\varrho-q\vartheta)=\lim_{\hbar\to\infty}\bigg(&P\digamma\left(p^{-1}(1+\hbar)(p\varrho-q\vartheta)\right)\\&+Q\digamma\left(-q^{-1}\hbar(p\varrho-q\vartheta)\right)+R-\digamma\left((1+2\hbar)(p\varrho-q\vartheta)\right)\bigg).\end{aligned}$$

Now,

$$\begin{aligned}&\|\digamma(p\varrho+q\vartheta)+\digamma(p\varrho-q\vartheta)-P\digamma(\varrho)-Q\digamma(\vartheta)-R\|\\&=\lim_{\hbar\to\infty}\bigg\|\bigg(P\digamma\left(p^{-1}(1+\hbar)(p\varrho+q\vartheta)\right)+Q\digamma\left(-q^{-1}\hbar(p\varrho+q\vartheta)\right)+R\\&-\digamma\left((1+2\hbar)(p\varrho+q\vartheta)\right)\bigg)+\bigg(P\digamma\left(p^{-1}(1+\hbar)(p\varrho-q\vartheta)\right)+Q\digamma\left(-q^{-1}\hbar(p\varrho-q\vartheta)\right)\\&+R-\digamma\left((1+2\hbar)(p\varrho-q\vartheta)\right)\bigg)\\&-P\bigg(P\digamma\left(p^{-1}(1+\hbar)\varrho\right)+Q\digamma\left(-q^{-1}\hbar\varrho\right)+R-\digamma\left((21+\hbar)\varrho\right)\bigg)\\&-Q\bigg(P\digamma\left(p^{-1}(1+\hbar)\vartheta\right)+Q\digamma\left(-q^{-1}\hbar\vartheta\right)+R-\digamma\left((21+\hbar)\vartheta\right)\bigg)-R\bigg\|\\ \leq\ &\lim_{\mathsf{k}\to\infty}P\bigg\|\digamma\left(p^{-1}(1+\hbar)(p\varrho+q\vartheta)\right)+\digamma\left(p^{-1}(1+\hbar)(p\varrho-q\vartheta)\right)-P\digamma(p^{-1}(1+\hbar)\varrho)\\&-Q\digamma(p^{-1}(1+\hbar)\vartheta)-R\bigg\|+Q\bigg\|\digamma\left(-q^{-1}\hbar(p\varrho+q\vartheta)\right)+\digamma\left(-q^{-1}\hbar(p\varrho-q\vartheta)\right)\\&-P\digamma\left(-q^{-1}\hbar\varrho\right)-Q\digamma\left(-q^{-1}\hbar\vartheta\right)-R\bigg\|+\bigg\|\digamma\left((1+2\hbar)(p\varrho+q\vartheta)\right)\\&+\digamma\left((1+2\hbar)(p\varrho-q\vartheta)\right)-P\digamma\left((1+2\hbar)\varrho\right)-Q\digamma\left((1+2\hbar)\vartheta\right)-R\bigg\|\\ \leq\ &\lim_{\mathsf{k}\to\infty}\bigg(P\phi\left(p^{-1}(1+\hbar)\varrho,p^{-1}(1+\hbar)\vartheta\right)+Q\phi\left(-q^{-1}\hbar\varrho,-q^{-1}\hbar\vartheta\right)+\\&\phi\left((1+2\hbar)\varrho,(1+2\hbar)\vartheta\right)\bigg)=0.\end{aligned}$$

Therefore,

$$\digamma(p\varrho + q\vartheta) + \digamma(p\varrho - q\vartheta) = P\digamma(\varrho) + Q\digamma(\vartheta) + R\ \forall \varrho, \vartheta \in V.$$

□

Remark 2.2.23. *The result proved in Theorem 2.2.22 is associated to various hyperstability results concerning linear and quadratic FEs in the existing literature. For instance, see Theorem 2, Corollary 3 of Piszczek (2014), Theorem 2.1, Theorem 2.3 of Piszczek (2015), Theorem 2.1 of Phochai and Saejung (2019), Theorem 2 of Brzdek et al. (2016).*

2.3 Stability Analysis of a Generalized Quartic Functional Equation

In this section, we explore hyperstability results for quartic FE in n-$\mathcal{NS}$ using both conventional as well as fixed point approach.

Remark 2.3.1. *Consider two $\mathcal{VS}$s Z and Y over the same field. Assume that the mapping $\digamma : Z \to Y$ satisfies (2.1.1). Then, $\forall\ \varrho, \vartheta \in Z$, we have*

(i) On substituting $\varrho = \vartheta = 0$ in (2.1.1), we have $\digamma(0) = 0$;

(ii) On substituting $\vartheta = 0$ in (2.1.1), we have $\digamma(a\varrho) = a^4\digamma(\varrho)$;

(iii) On substituting $\varrho = 0$ in (2.1.1), we have $\digamma(-\vartheta) = \digamma(\vartheta)$.

2.3.1 Ulam-Hyers-Rassias Stability in n-Banach Space using Fixed Point Approach

In this section, we use the fixed point result of Diaz and Margolis (1968) on generalized MS to examine the stability of the quartic FE (2.1.1) on an n-$\mathcal{NS}$. Throughout the section, $(Z, \|.,.\|_Z)$ represents an n-$\mathcal{NS}$ and $(Y, \|.,.\|_Y)$ represents an n-$\mathcal{BS}$. Also, let

$$G_{a,b}F(\varrho,\vartheta) = F(a\varrho+b\vartheta)+F(a\varrho-b\vartheta)-2a^2(a^2-b^2)F(\varrho)-2b^2(b^2-a^2)F(\vartheta) - a^2b^2(F(\varrho+\vartheta)+F(\varrho-\vartheta)),$$

$\forall\ \varrho,\vartheta \in Z$ and positive integers a and b with $a \neq b$ represent the quartic FE and $\|\vartheta, z\|_Y = \|\vartheta, z_2, z_3, ..., z_n\|$, where $\vartheta \in Y$, $z \in Y^{n-1}$ and Y^n denotes $Y \times Y \times ... \times Y$, n times.

Theorem 2.3.2. *Let Z be an n- $\mathcal{NS}$ and Y be an n-$\mathcal{BS}$ over the same field. Let $F : Z \to Y$ be a mapping s.t. $F(0) = 0$ and $\Xi : Z \times Z \to [0,\infty)$ satisfying:*

$$\Xi(a\varrho, a\vartheta) \leq \lambda a^4 \Xi(\varrho,\vartheta), \tag{2.3.1}$$

$$\|G_{a,b}F(\varrho,\vartheta), z\|_Y \leq \Xi(\varrho,\vartheta), \tag{2.3.2}$$

$\forall\ \varrho,\vartheta \in Z$ and $z \in Y^{n-1}$ and $\lambda \in [0,1)$. Then, $\exists$ a unique quartic map $\Theta : Z \to Y$ satisfying (2.1.1) and

$$\|F(\varrho) - \Theta(\varrho), z\|_Y \leq \frac{1}{2a^4(1-\lambda)}\Xi(\varrho, 0),\ \forall \varrho \in Z.$$

Proof. Consider,

$$\chi = \{f : Z \to Y | f(0) = 0\}.$$

Let $d : \chi \times \chi \to [0,\infty]$ be a mapping defined as

$$d(f,g) = \inf\{\mu \geq 0 : \|f(\varrho)-g(\varrho), z\|_Y \leq \mu\Xi(\varrho,0)\ \forall\varrho \in Z \text{ and } z \in Y^{n-1}\}. \tag{2.3.3}$$

Then, we have

(i) $d(f,g) = 0$

$$\begin{aligned}
&\Leftrightarrow\ \inf\left\{\mu \geq 0 | \|f(\varrho)-g(\varrho), z\|_Y \leq \mu\Xi(\varrho,0),\ \forall\varrho \in Z \text{ and } z \in Y^{n-1}\right\} = 0\\
&\Leftrightarrow\ \|f(\varrho)-g(\varrho), z\|_Y = 0,\ \forall\varrho \in Z \text{ and } z \in Y^{n-1}\\
&\Leftrightarrow\ f(\varrho)-g(\varrho) = 0,\ \forall\varrho \in Z\\
&\Leftrightarrow\ f = g.
\end{aligned}$$

(ii) Symmetric $d(f,g) = d(g,f)$.

(iii) For triangle inequality

$$\begin{aligned}\|f(\varrho) - g(\varrho), z\|_Y &\leq \|f(\varrho) - h(\varrho), z\|_Y + \|h(\varrho) - g(\varrho), z\|_Y \\ &\leq d(f,h)\phi(\varrho,0) + d(h,g)\phi(\varrho,0) \\ &= (d(f,h) + d(h,g))\,\phi(\varrho,0).\end{aligned}$$

Thus,

$$d(f,g) \leq d(f,h) + d(h,g).$$

Hence, (χ, d) is a generalized MS.

Now, consider, a C_{seq} $\{f_\hbar\} \subseteq \chi$ i.e. $\lim\limits_{\hbar,l\to\infty} d(f_\hbar, f_l) = 0$. Then, from (2.3.3), we have

$$\lim_{\hbar,l\to\infty} \|f_\hbar(\varrho) - f_l(\varrho), z\|_Y \leq \lim_{\hbar,l\to\infty} d(f_\hbar, f_l)\phi(\varrho,0) = 0 \;\forall z \in Y^{n-1}.$$

This implies $\{f_\hbar(\varrho)\}$ is a C_{seq} in $(Y, \|.,.\|_Y)$.

Since, $(Y, \|.,.\|_Y)$ is an n-$\mathcal{BS}$, therefore $\exists$ some $\vartheta \in Y$ s.t. $\lim_{\hbar\to\infty} f_\hbar(\varrho) = \vartheta$. We define a function $f : Z \to Y$ s.t. $f(\varrho) = \vartheta = \lim\limits_{\hbar\to\infty} f_\hbar(\varrho)$.

As $\{f_\hbar\}$ is a C_{seq} in (χ, d), therefore for each $\epsilon > 0$ $\exists$ $\hbar_0$ s.t. $d(f_\hbar, f_l) \leq \epsilon$, for every $\hbar, l \geq \hbar_0$ i.e.,

$$\|f_\hbar(\varrho) - f_l(\varrho), z\|_Y \leq d(f_\hbar, f_l)\; \Xi(\varrho, 0) \;\forall \varrho \in Z. \tag{2.3.4}$$

Taking limit as $l \to \infty$ in (2.3.4), we have

$$\|f_\hbar(\varrho) - f(\varrho), z\|_Y \leq \epsilon \Xi(\varrho,0) \;\forall \hbar \geq \hbar_0,$$

or

$$d(f_\hbar, f) \leq \epsilon \;\forall \hbar \geq \hbar_0.$$

Therefore, $\{f_\hbar\} \to f$. Also, $f(0) = 0$ as $f_\hbar(0) = 0$, $\forall\ \hbar \in \mathbb{N}$. Hence, (χ, d) is complete.

Define $F : \chi \to \chi$ as

$$F(\digamma)(\varrho) = \frac{\digamma(a\varrho)}{a^4}.$$

Consider,

$$\begin{aligned}
\|Tq(\varrho) - Tg(\varrho), z\|_Y &= \left\| \frac{F(a\varrho)}{a^4} - \frac{g(a\varrho)}{a^4}, z \right\|_Y \\
&\leq \frac{1}{a^4} \|F(a\varrho) - g(a\varrho), z\|_Y \\
&\leq \frac{1}{a^4} d(F, g)\Xi(a\varrho, 0) \\
&\leq \frac{d(F, g)}{a^4} \lambda a^4 \Xi(\varrho, 0) \\
&\leq \lambda d(F, g)\Xi(\varrho, 0),
\end{aligned}$$

or

$$d(Tq, Tg) \leq \lambda d(F, g) \ \forall F, g \in \chi.$$

Since $\lambda \in [0, 1)$, by Theorem 2.3.3, we have

(i) either $d(F^{\hbar}F, F^{1+\hbar}F) = \infty$;

(ii) or

  (i) $\lim_{\hbar \to \infty} F^{\hbar}(F) = \Theta$, where Θ is the fixed point of mapping F.

  (ii) $d(F, \Theta) \leq \frac{1}{1-\lambda} d(F, Tq)$.

On substituting $\vartheta = 0$ in (2.3.2), we have

$$\left\| \frac{F(a\varrho)}{a^4} - F(\varrho), z \right\|_Y \leq \frac{1}{2a^4} \Xi(\varrho, 0). \tag{2.3.5}$$

On substituting $\varrho = a^{\hbar}\varrho$ in (2.3.5), we have

$$\left\| \frac{F(a^{1+\hbar}\varrho)}{a^4} - F(a^{\hbar}\varrho), z \right\|_Y \leq \frac{1}{2a^4} \Xi(a^{\hbar}\varrho, 0).$$

To show $d(F^{\hbar}F, F^{1+\hbar}F) < \infty$.

Consider,

$$\begin{aligned}
\|F^{1+\hbar}\digamma(\varrho)-F^{\hbar}\digamma(\varrho),z\|_Y &= \frac{1}{a^{4\hbar}}\left\|\frac{\digamma(a^{1+\hbar}\varrho)}{a^4}-\digamma(a^{\hbar}\varrho),z\right\|_Y\\
&\leq \frac{1}{a^{4\hbar}}\frac{1}{2a^4}\Xi(a^{\hbar}\varrho,0)\\
&\leq \frac{\lambda}{2a^4}\Xi(\varrho,0),
\end{aligned}$$

or

$$d(F^{1+\hbar}\digamma,F^{\hbar}\digamma)<\frac{\lambda}{2a^4}<\infty,\ \text{where } F^{\hbar}\digamma(\varrho)=\frac{\digamma(a^{\hbar}\varrho)}{a^{4\hbar}}.$$

Also,

$$\|F(\digamma(\varrho))-\digamma(\varrho),z\|_Y=\left\|\frac{\digamma(a\varrho)}{a^4}-\digamma(\varrho),z\right\|_Y\leq\frac{1}{2a^4}\Xi(\varrho,0),$$

or

$$d(Tq,\digamma)\leq\frac{1}{2a^4}.$$

Therefore, F has a fixed point Θ. Also,

$$d(\digamma,\Theta)\leq\frac{1}{1-\lambda}d(\digamma,Tq),$$

implies

$$\|\digamma(\varrho)-\Theta(\varrho),z\|_Y\leq\frac{1}{1-\lambda}d(\digamma,Tq)\Xi(\varrho,0)\leq\frac{1}{2a^4(1-\lambda)}\Xi(\varrho,0).$$

Now, by replacing ϱ with $a^{\hbar}\varrho$, ϑ with $a^{\hbar}\vartheta$ in (2.3.2) and dividing by $a^{4\hbar}$, we have

$$\begin{aligned}
&\left\|\frac{\digamma(a^{\hbar}(a\varrho+b\vartheta))}{a^{4\hbar}}+\frac{\digamma(a^{\hbar}(a\varrho-b\vartheta))}{a^{4\hbar}}-2a^2(a^2-b^2)\frac{\digamma(a^{\hbar}\varrho)}{a^{4\hbar}}-2b^2(b^2-a^2)\frac{\digamma(a^{\hbar}\vartheta)}{a^{4\hbar}}\right.\\
&\left.-a^2b^2\left(\frac{\digamma(a^{\hbar}(\varrho+\vartheta))}{a^{4\hbar}}+\frac{\digamma(a^{\hbar}(\varrho-\vartheta))}{a^{4\hbar}}\right),z\right\|_Y\leq\frac{\Xi(a^{\hbar}\varrho,a^{\hbar}\vartheta)}{a^{4\hbar}}. \qquad (2.3.6)
\end{aligned}$$

Taking limit as $\hbar\to\infty$ in (2.3.6), we have

$$\begin{aligned}
\Big\|\Theta(a\varrho+b\vartheta) &+ \Theta(a\varrho-b\vartheta)-2a^2(a^2-b^2)\Theta(\varrho)-2b^2(b^2-a^2)\Theta(\vartheta)\\
&-a^2b^2(\Theta(\varrho+\vartheta)+\Theta(\varrho-\vartheta)),z\Big\|_Y\\
&\leq \lim_{\hbar\to\infty}\frac{\Xi(a^{\hbar}\varrho,a^{\hbar}\vartheta)}{a^{4\hbar}}\\
&\leq \lim_{\hbar\to\infty}\lambda^{\hbar}\Xi(\varrho,\vartheta)=0.
\end{aligned}$$

Therefore, Θ is a quartic mapping.

Uniqueness: If possible, let Θ' be another quartic mapping. Then, from Remark (2.3.1), we have

$$\Theta(a^{\hbar}\varrho) = a^{4\hbar}\Theta(\varrho) \text{ and } \Theta'(a^{\hbar}\varrho) = a^{4\hbar}\Theta'(\varrho),$$

or

$$\Theta(\varrho) = \frac{\Theta(a^{\hbar}\varrho)}{a^{4\hbar}} \text{ and } \Theta'(\varrho) = \frac{\Theta'(a^{\hbar}\varrho)}{a^{4\hbar}} \text{ for } \hbar \in \mathbb{N}.$$

Consider,

$$\begin{aligned}
\left\|\Theta(\varrho) - \Theta'(\varrho), z\right\|_Y &= \left\|\frac{\Theta(a^{\hbar}\varrho)}{a^{4\hbar}} - \frac{\Theta'(a^{4\hbar}\varrho)}{a^{4\hbar}}, z\right\|_Y \\
&= \left\|\frac{\Theta(a^{\hbar}\varrho)}{a^{4\hbar}} - \frac{F(a^{\hbar}\varrho)}{a^{4\hbar}} + \frac{F(a^{\hbar}\varrho)}{a^{4\hbar}} - \frac{\Theta'(a^{\hbar}\varrho)}{a^{4\hbar}}, z\right\|_Y \\
&\leq \frac{1}{a^{4\hbar}}\left(\left\|\Theta(a^{\hbar}\varrho) - F(a^{\hbar}\varrho), z\right\|_Y + \left\|\Theta'(a^{\hbar})\varrho - F(a^{\hbar}\varrho), z\right\|_Y\right) \\
&\leq \frac{1}{a^{4\hbar}}\frac{1}{(1-\lambda)a^4}\Xi(a^{\hbar}\varrho, 0) \\
&\leq \frac{\lambda^{\hbar}}{a^4(1-\lambda)}\Xi(\varrho, 0).
\end{aligned}$$

Taking limit as $\hbar \to \infty$, we have

$$\|\Theta(\varrho) - \Theta'(\varrho), z\|_Y = 0, \ \forall \varrho \in Z \text{ and } z \in Y^{n-1}.$$

Therefore, $\Theta = \Theta'$. $\square$

2.3.2 Ulam-Hyers-Rassias Stability in n-Banach Space using Conventional Approach

In this section, to examine the stability of the quartic FE (2.1.1) on n-$\mathcal{NS}$ the conventional approach is implemented.

Theorem 2.3.3. *Let Z be an n-$\mathcal{NS}$ and Y be an n-$\mathcal{BS}$ over the same field. Let $F : Z \to Y$ be a mapping s.t. $F(0) = 0$. If for a mapping $\Xi : Z \times Z \to [0, \infty)$, we have*

$$\bar{\Xi}(\varrho, \vartheta) = \sum_{\hbar=0}^{\infty} \frac{1}{a^{4\hbar}}\Xi(a^{\hbar}\varrho, a^{\hbar}\vartheta) < \infty, \tag{2.3.7}$$

$$\|G_{a,b}F(\varrho,\vartheta), z\|_Y \leq \Xi(\varrho,\vartheta), \tag{2.3.8}$$

$\forall\ \varrho, \vartheta \in Z$ and $z \in Y^{n-1}$. Then, $\exists$ a unique quartic mapping $\Theta : Z \to Y$ s.t.

$$\|F(\varrho) - \Theta(\varrho), z\|_Y \leq \frac{1}{2a^4}\bar{\Xi}(\varrho, 0),\ \forall \varrho \in Z.$$

Proof. On substituting $\vartheta = 0$ in (2.3.8), we have

$$\|2F(a\varrho) - 2a^4F(\varrho), z\|_Y \leq \Xi(\varrho, 0),$$

or

$$\left\|\frac{F(a\varrho)}{a^4} - F(\varrho), z\right\|_Y \leq \frac{1}{2a^4}\Xi(\varrho, 0). \tag{2.3.9}$$

On substituting $\varrho = a\varrho$ in (2.3.9), we have

$$\left\|\frac{F(a^2\varrho)}{a^4} - F(a\varrho), z\right\|_Y \leq \frac{1}{2a^4}\Xi(a\varrho, 0). \tag{2.3.10}$$

Using (2.3.9) and (2.3.10), we have

$$\left\|\frac{F(a^2\varrho)}{(a^4)^2} - F(\varrho), z\right\|_Y \leq \frac{1}{2a^4}\left(\Xi(\varrho, 0) + \frac{1}{a^4}\Xi(a\varrho, 0)\right).$$

On generalizing, we have

$$\left\|\frac{F(a^\hbar\varrho)}{a^{4\hbar}} - F(\varrho), z\right\|_Y \leq \frac{1}{2a^4}\sum_{i=0}^{\hbar-1}\frac{\Xi(a^i\varrho, 0)}{a^{4i}}. \tag{2.3.11}$$

Now, replacing $\varrho = a^\hbar\varrho$ in (2.3.9) and dividing by $a^{4\hbar}$, we have

$$\left\|\frac{F(a^{1+\hbar}\varrho)}{a^{4(\hbar+1)}} - \frac{F(a^\hbar\varrho)}{a^{4\hbar}}, z\right\|_Y \leq \frac{1}{2a^4}\frac{\Xi(a^\hbar\varrho, 0)}{a^{4\hbar}}. \tag{2.3.12}$$

On taking limit as $\hbar \to \infty$ in (2.3.12) and using (2.3.7), we have

$$\lim_{\hbar\to\infty}\left\|\frac{F(a^{1+\hbar}\varrho)}{a^{4(1+\hbar)}} - \frac{F(a^\hbar\varrho)}{a^{4\hbar}}, z\right\|_Y = 0, \tag{2.3.13}$$

i.e., $\left\{\frac{F(a^\hbar\varrho)}{a^{4\hbar}}\right\}$ is a C_{seq} in $(Y, \|.,.\|_Y)$. Also, as $(Y, \|.,.\|_Y)$ is a n-$\mathcal{BS}$, therefore we can define a mapping $\Theta : Z \to Y$ s.t. $\Theta(\varrho) = \lim_{\hbar\to\infty}\frac{F(a^\hbar\varrho)}{a^{4\hbar}}$.
On taking limit as $\hbar \to \infty$ in (2.3.11), we have

$$\|F(\varrho) - \Theta(\varrho), z\|_Y \leq \lim_{\hbar\to\infty}\frac{1}{2a^4}\sum_{i=0}^{\hbar-1}\frac{\Xi(a^i\varrho)}{a^{4i}} = \frac{1}{2a^4}\bar{\Xi}(\varrho, 0).$$

On similar line of Theorem 2.3.2, it can be shown that Θ is a quartic mapping.

Uniqueness: If possible, let Θ' be an another quartic map.

Then,

$$\begin{aligned}
\left\|\Theta(\varrho) - \Theta'(\varrho), z\right\|_Y &= \left\|\frac{\Theta(a^\hbar\varrho)}{a^{4\hbar}} - \frac{\Theta'(a^\hbar\varrho)}{a^{4\hbar}}, z\right\|_Y \\
&= \left\|\frac{\Theta(a^\hbar\varrho)}{a^{4\hbar}} - \frac{F(a^\hbar\varrho)}{a^{4\hbar}} + \frac{F(a^\hbar\varrho)}{a^{4\hbar}} - \frac{\Theta'(a^\hbar\varrho)}{a^{4\hbar}}, z\right\|_Y \\
&\leq \frac{1}{a^{4\hbar}}\left(\left\|\Theta(a^\hbar\varrho) - F(a^\hbar\varrho), z\right\|_Y + \left\|\Theta'(a^\hbar)\varrho - F(a^\hbar\varrho), z\right\|_Y\right) \\
&\leq \frac{1}{a^{4(1+\hbar)}}\bar{\Xi}(a^\hbar\varrho, 0) \\
&= \frac{1}{a^{4(1+\hbar)}}\sum_{i=0}^{\infty}\frac{\Xi(a^{(i+\hbar)}\varrho)}{a^{4i}} \\
&\leq \frac{1}{a^4}\sum_{i=0}^{\infty}\frac{\Xi(a^{(i+\hbar)}\varrho)}{a^{4(i+\hbar)}},
\end{aligned}$$

or

$$\left\|\Theta(\varrho) - \Theta'(\varrho), z\right\|_Y \leq \frac{1}{a^4}\sum_{i=0}^{\infty}\frac{\Xi(a^{(i+\hbar)}\varrho)}{a^{4(i+\hbar)}}. \tag{2.3.14}$$

Taking limit as $\hbar \to \infty$ in (2.3.14), we have $\Theta = \Theta'$. □

Theorem 2.3.4. *Let Z be an n-$\mathcal{NS}$ and Y be an n-$\mathcal{BS}$ over the same field. Let $F : Z \to Y$ be a mapping with $F(0) = 0$. If $\exists$ a mapping $\Xi : Z \times Z \to [0, \infty)$ s.t.*

$$\bar{\Xi}(\varrho, \vartheta) = \sum_{\hbar=0}^{\infty} a^{4\hbar}\Xi\left(\frac{\varrho}{a^\hbar}, \frac{\vartheta}{a^\hbar}\right) < \infty, \tag{2.3.15}$$

$$\|G_{a,b}F(\varrho, \vartheta), z\|_Y \leq \Xi(\varrho, \vartheta)\ \forall \varrho, \vartheta \in Z \text{ and } z \in Y^{n-1}. \tag{2.3.16}$$

Then, $\exists$ a unique quartic map $\Theta : Z \to Y$ s.t.

$$\|F(\varrho) - \Theta(\varrho), z\|_Y \leq \frac{1}{2a^4}\bar{\Xi}(\varrho, 0).$$

Proof. On substituting $\vartheta = 0$ in (2.3.16), we have

$$\|F(a\varrho) - a^4F(\varrho), z\|_Y \leq \frac{1}{2}\Xi(\varrho, 0). \tag{2.3.17}$$

On substituting $\varrho = \dfrac{\varrho}{a}$ in (2.3.17), we have

$$\left\|F(\varrho) - a^4F\left(\frac{\varrho}{a}\right), z\right\|_Y \leq \frac{1}{2}\Xi\left(\frac{\varrho}{a}, 0\right). \tag{2.3.18}$$

Again on substituting $\varrho = \frac{\varrho}{a}$ again in (2.3.18), we have

$$\left\| F\left(\frac{\varrho}{a}\right) - a^4 F\left(\frac{\varrho}{a^2}\right), z \right\|_Y \leq \frac{1}{2}\Xi\left(\frac{\varrho}{a^2}, 0\right). \tag{2.3.19}$$

From (2.3.18) and (2.3.19), we have

$$\left\| F(\varrho) - (a^4)^2 F\left(\frac{\varrho}{a^2}\right), z \right\|_Y \leq \frac{1}{2}\left(\Xi\left(\frac{\varrho}{a}, 0\right) + a^4\Xi\left(\frac{\varrho}{a^2}, 0\right)\right). \tag{2.3.20}$$

On generalizing, we have

$$\left\| F(\varrho) - a^{4\hbar} F\left(\frac{\varrho}{a^\hbar}\right), z \right\|_Y \leq \frac{1}{2a^4}\sum_{i=1}^{\hbar} a^{4i}\Xi\left(\frac{\varrho}{a^i}, 0\right). \tag{2.3.21}$$

Now, on substituting $\varrho = \dfrac{\varrho}{a^\hbar}$ in (2.3.18) and multiplying it by $a^{4\hbar}$, we have

$$\left\| a^{4\hbar} F\left(\frac{\varrho}{a^\hbar}\right) - a^{4(1+\hbar)} F\left(\frac{\varrho}{a^{1+\hbar}}\right), z \right\|_Y \leq \frac{1}{2a^4} a^{4(1+\hbar)}\Xi\left(\frac{\varrho}{a^{1+\hbar}}, 0\right). \tag{2.3.22}$$

On taking limit as $\hbar \to \infty$ in (2.3.22) and using (2.3.15), we have

$$\lim_{\hbar\to\infty} \left\| a^{4\hbar} F\left(\frac{\varrho}{a^\hbar}\right) - a^{4(1+\hbar)} F\left(\frac{\varrho}{a^{1+\hbar}}\right), z \right\|_Y = 0, \tag{2.3.23}$$

i.e., $\left\{a^{4\hbar} F\left(\frac{\varrho}{a^\hbar}\right)\right\}$ is a C_{seq} in $(Y, \|., .\|_Y)$. Also, as $(Y, \|., .\|_Y)$ is an n-$\mathcal{BS}$, therefore we can define a mapping $\Theta : Z \to Y$ as $\Theta(\varrho) = \lim_{\hbar\to\infty} a^{4\hbar} F(\dfrac{\varrho}{a^\hbar})$. The remaining part of the proof is analogues to Theorem 2.3.3. □

Theorem 2.3.5. *Let Z be an n-$\mathcal{NS}$ and Y be an n-$\mathcal{BS}$ over the same field. Let $F : Z \to Y$ be a mapping with $F(0) = 0$. If for a contractively sub additive mapping $\Xi : Z \times Z \to [0, \infty)$ with contractive constant λ having $a^{-3}\lambda < 1$, we have*

$$\|G_{a,b}F(\varrho, \vartheta), z\|_Y \leq \Xi(\varrho, \vartheta)\ \forall \varrho, \vartheta \in Z \text{ and } z \in Y^{n-1}. \tag{2.3.24}$$

Then, $\exists$ a unique quartic map $\Theta : Z \to Y$ s.t.

$$\|F(\varrho) - \Theta(\varrho), z\|_Y \leq \frac{1}{2}\frac{\Xi(\varrho, 0)}{a^4 - a\lambda}.$$

Proof. On substituting $\vartheta = 0$ in (2.3.24), we have

$$\|2F(a\varrho) - 2a^4 F(\varrho), z\|_Y \leq \Xi(\varrho, 0),$$

or

$$\left\| \frac{F(a\varrho)}{a^4} - F(\varrho), z \right\|_Y \leq \frac{1}{2a^4}\Xi(\varrho, 0). \tag{2.3.25}$$

Now, on substituting $\varrho = a^\hbar \varrho$ in (2.3.25) and dividing it by $a^{4\hbar}$, we have

$$\left\| \frac{F(a^{1+\hbar}\varrho)}{a^{4(1+\hbar)}} - \frac{F(a^\hbar \varrho)}{a^{4\hbar}}, z \right\|_Y \leq \frac{1}{2a^4}\frac{\Xi(a^\hbar \varrho, 0)}{a^{4\hbar}}.$$

As Ξ is contractively sub additive function i.e., $\Xi(a^i\varrho, 0) \leq (a\lambda)^i \Xi(\varrho, 0)$.

Therefore,

$$\left\| \frac{F(a^{1+\hbar}\varrho)}{a^{4(1+\hbar)}} - \frac{F(a^\hbar \varrho)}{a^{4\hbar}}, z \right\|_Y \leq \frac{1}{2a^4}\frac{(a\lambda)^\hbar}{a^{4\hbar}}\Xi(\varrho, 0) = \frac{1}{2a^4}(a^{-3}\lambda)^\hbar \Xi(\varrho, 0). \tag{2.3.26}$$

On taking limit as $\hbar \to \infty$ in (2.3.26), we have

$$\lim_{\hbar\to\infty} \left\| \frac{F(a^{1+\hbar}\varrho)}{a^{4(1+\hbar)}} - \frac{F(a^\hbar \varrho)}{a^{4\hbar}}, z \right\|_Y = 0, \tag{2.3.27}$$

i.e., $\left\{ \frac{F(a^\hbar \varrho)}{a^{4\hbar}} \right\}$ is a C_{seq} in $(Y, \|., .\|_Y)$. Also, as $(Y, \|., .\|_Y)$ is an n-$\mathcal{BS}$. Therefore, $\Theta : Z \to Y$ can be defined as $\Theta(\varrho) = \lim_{\hbar\to\infty} \frac{F(a^\hbar \varrho)}{a^{4\hbar}}$.

On the outlines of Theorem 2.3.3, we have

$$\left\| \frac{F(a^\hbar)}{a^{4\hbar}} - F(\varrho), z \right\|_Y \leq \frac{1}{2a^4} \sum_{i=0}^{-1+\hbar} \frac{\Xi(a^i \varrho, 0)}{a^{4i}}. \tag{2.3.28}$$

As Ξ is a contractively sub additive i.e., $\Xi(a^i\varrho, 0) \leq (a\lambda)^i \Xi(\varrho, 0)$, therefore

$$\begin{aligned} \lim_{\hbar\to\infty} \left\| \frac{F(a^\hbar)}{a^{4\hbar}} - F(\varrho), z \right\|_Y &\leq \lim_{\hbar\to\infty} \frac{1}{2a^4}\Xi(\varrho, 0) \sum_{i=0}^{-1+\hbar} (a^{1-4}\lambda)^i \\ &= \frac{1}{2}\frac{\Xi(\varrho, 0)}{a^4 - a\lambda}, \end{aligned}$$

or

$$\|F(\varrho) - \Theta(\varrho), z\|_Y \leq \frac{1}{2}\frac{\Xi(\varrho, 0)}{a^4 - a\lambda}.$$

The remaining part of the proof is analogues to Theorem 2.3.3.

□

Theorem 2.3.6. *Let Z be an n-$\mathcal{NS}$ and Y be an n-$\mathcal{BS}$ over the same field. Let $F : Z \to Y$ be a mapping with $F(0) = 0$. If for a expansively super additive mapping $\Xi : Z \times Z \to [0, \infty)$ with constant λ having $a^3\lambda < 1$, we have*

$$\|G_{a,b}F(\varrho, \vartheta), z\|_Y \leq \Xi(\varrho, \vartheta)\ \forall \varrho, \vartheta \in Z \text{ and } z \in Y^{n-1}. \tag{2.3.29}$$

Then, $\exists$ a unique quartic mapping $\Theta : Z \to Y$ s.t.

$$\|F(\varrho) - \Theta(\varrho), z\|_Y \leq \frac{1}{2}\frac{\Xi(\varrho, 0)\lambda}{a - a^4\lambda}.$$

Proof. On substituting $\vartheta = 0$ in (2.3.29), we have

$$\|F(a\varrho) - a^4F(\varrho), z\|_Y \leq \frac{1}{2}\Xi(\varrho, 0). \tag{2.3.30}$$

Again on substituting $\varrho = \dfrac{\varrho}{a}$ in (2.3.30), we have

$$\left\|F(\varrho) - a^4F\left(\frac{\varrho}{a}\right), z\right\|_Y \leq \frac{1}{2}\Xi\left(\frac{\varrho}{a}, 0\right). \tag{2.3.31}$$

Now, $\varrho = \dfrac{\varrho}{a^\hbar}$ in (2.3.31) and multiplying by $a^{4\hbar}$, we have

$$\left\|a^{4\hbar}F\left(\frac{\varrho}{a^\hbar}\right) - a^{4(1+\hbar)}F\left(\frac{\varrho}{a^{1+\hbar}}\right), z\right\|_Y \leq \frac{1}{2a^4}a^{4(1+\hbar)}\Xi\left(\frac{\varrho}{a^{1+\hbar}}, 0\right).$$

As Ξ is a expansively super additive, therefore $\Xi\left(\dfrac{\varrho}{a^i}, 0\right) \leq \left(\dfrac{\lambda}{a}\right)^i \Xi(\varrho, 0)$.
Hence,

$$\begin{aligned}
\left\|a^{4\hbar}F\left(\frac{\varrho}{a^\hbar}\right) - a^{4(1+\hbar)}F\left(\frac{\varrho}{a^{1+\hbar}}\right), z\right\|_Y &\leq \frac{1}{2a^4}a^{4(1+\hbar)}\left(\frac{\lambda}{a}\right)^{1+\hbar}\Xi(\varrho, 0) \\
&= \frac{1}{2a^4}(a^3\lambda)^{1+\hbar}\Xi(\varrho, 0).
\end{aligned}$$

On taking limit as $\hbar \to \infty$, we have

$$\lim_{\hbar\to\infty}\left\|a^{4\hbar}F\left(\frac{\varrho}{a^\hbar}\right) - a^{4(1+\hbar)}F\left(\frac{\varrho}{a^{1+\hbar}}\right), z\right\|_Y = 0,$$

i.e., $\left\{a^{4\hbar}F\left(\frac{\varrho}{a^\hbar}\right)\right\}$ is a C_{seq} in $(Y, \|.,.\|_Y)$. Also, as $(Y, \|.,.\|_Y)$ is an n-$\mathcal{BS}$. Therefore, we can define $\Theta : Z \to Y$ as $\Theta(\varrho) = \lim_{\hbar\to\infty} a^{4\hbar}F\left(\dfrac{\varrho}{a^\hbar}\right)$.

On the outlines of Theorem 2.3.4, we have

$$\left\|\digamma(\varrho) - a^{4\hbar}\digamma\left(\frac{\varrho}{a^{\hbar}}\right), z\right\|_Y \leq \frac{1}{2a^4}\sum_{i=1}^{\hbar} a^{4i}\Xi\left(\frac{\varrho}{a^i}, 0\right). \tag{2.3.32}$$

As Ξ is a expansively super additive i.e., $\Xi\left(\frac{\varrho}{a^i}, 0\right) \leq \left(\frac{\lambda}{a}\right)^i \Xi(\varrho, 0)$.
Therefore,

$$\begin{aligned}
\lim_{\hbar\to\infty}\left\|\digamma(\varrho) - a^{4\hbar}\digamma\left(\frac{\varrho}{a^{\hbar}}\right), z\right\|_Y &\leq \frac{1}{2a^4}\Xi(\varrho, 0)\sum_{i=1}^{\infty}(a^3\lambda)^i \\
&= \frac{1}{2a^4}\Xi(\varrho, 0)\frac{a^3\lambda}{1 - a^3\lambda} \\
&= \frac{\lambda}{2(a - a^4\lambda)}\Xi(\varrho, 0),
\end{aligned}$$

or

$$\|\digamma(\varrho) - \Theta(\varrho), z\|_Y \leq \frac{\lambda}{2(a - a^4\lambda)}\Xi(\varrho, 0).$$

The remaining part of the proof is analogues to Theorem 2.3.4. □

2.3.3 Consequences

In this section, we presented the results from the literature that can be deduced from the results proved in previous section.

Corollary 2.3.7. *Let Z be an n-$\mathcal{NS}$ and Y be an n-$\mathcal{BS}$ over the same field. Let $\digamma : Z \to Y$ be a mapping with $\digamma(0) = 0$. If for a real number $p < 4$ and positive real number ϵ, we have*

$$\|G_{a,b}\digamma(\varrho, \vartheta), z\|_Y \leq \epsilon(\|\varrho\|_Z^p + \|\vartheta\|_Z^p)\psi(z)\ \forall \varrho, \vartheta \in Z \text{ and } z \in Y^{n-1}, \tag{2.3.33}$$

where $\psi : Y^{n-1} \to [0, \infty)$. Then, $\exists$ a unique quartic map $\Theta : Z \to Y$ s.t.

$$\|\digamma(\varrho) - \Theta(\varrho), z\|_Y \leq \frac{1}{2a^4(1 - \lambda)}\epsilon\|\varrho\|_Z^p\psi(z).$$

Proof. The result holds as a consequence of Theorem 2.3.2, using $\Xi(\varrho, \vartheta) = 0$ if $\varrho = 0$ or $\vartheta = 0$, and $\Xi(\varrho, \vartheta) = \epsilon(\|\varrho\|_Z^p + \|\vartheta\|_Z^p)\psi(z)$, otherwise.

For $\varrho, \vartheta \in Z$, we have

$$\begin{aligned}\Xi(a\varrho, a\vartheta) &= \epsilon(\|a\varrho\|_Z^p + \|a\vartheta\|_Z^p)\psi(z)\\ &\leq \lambda a^4 \epsilon(\|\varrho\|_Z^p + \|\vartheta\|_Z^p)\psi(z)\\ &= \lambda a^4 \Xi(\varrho, \vartheta),\end{aligned}$$

where $\lambda = a^{p-4} < 1$. □

Corollary 2.3.8. *Let Z be an n-$\mathcal{NS}$ and Y be an n-$\mathcal{BS}$ over the same field. Let $F : Z \to Y$ be a mapping with $F(0) = 0$. If for non negative real numbers ϵ, θ, p and q with p and $q < 4$, we have*

$$\|G_{a,b}F(\varrho, \vartheta), z\|_Y \leq (\epsilon\|\varrho\|_Z^p + \theta\|\vartheta\|_Z^q)\psi(z) \ \forall \varrho, \vartheta \in Z \text{ and } z \in Z^{n-1}, \quad (2.3.34)$$

where $\psi : Y^{n-1} \to [0, \infty)$. Then, $\exists$ a unique quartic map $\Theta : Z \to Y$ s.t.

$$\|F(\varrho) - \Theta(\varrho), z\|_Y \leq \frac{\epsilon\|\varrho\|_Z^p}{2(a^4 - a^p)}\psi(z).$$

Proof. The result holds as a consequence of Theorem 2.3.3, using $\Xi(\varrho, \vartheta) = \left(\epsilon\|\varrho\|_Z^p + \theta\|\vartheta\|_Z^q\right)\psi(z)$. □

Corollary 2.3.9. *Let Z be an n- $\mathcal{NS}$ and Y be an n- $\mathcal{BS}$ over the same field. Let $F : Z \to Y$ be a mapping with $F(0) = 0$. If for non negative real numbers ϵ, θ, p and q with p and $q > 4$, we have*

$$\|G_{a,b}F(\varrho, \vartheta), z\|_Y \leq (\epsilon\|\varrho\|_Z^p + \theta\|\vartheta\|_Z^q)\psi(z) \ \forall \varrho, \vartheta \in Z \text{ and } z \in Y^{n-1}, \quad (2.3.35)$$

where $\psi : Y^{n-1} \to [0, \infty)$. Then, $\exists$ a unique quartic map $\Theta : Z \to Y$ s.t.

$$\|F(\varrho) - \Theta(\varrho), z\|_Y \leq \frac{\epsilon\|\varrho\|_Z^p}{2(a^p - a^4)}\psi(z).$$

Proof. The result holds as a consequence of Theorem 2.3.4, using $\Xi(\varrho, \vartheta) = (\epsilon\|\varrho\|^p + \theta\|\vartheta\|^q)\psi(z)$. □

Corollary 2.3.10. *Let Z be an n-$\mathcal{NS}$ and Y be an n-$\mathcal{BS}$ over the same field. Let $F : Z \to Y$ be a mapping with $F(0) = 0$. If for a positive real number ϵ, we have*

$$\|G_{a,b}F(\varrho, \vartheta), z\|_Y \leq 2\epsilon, \ \forall \varrho, \vartheta \in Z \text{ and } z \in Y^{n-1}. \quad (2.3.36)$$

Then, $\exists$ *a unique quartic map* $\Theta : Z \to Y$ *s.t.*

$$\|F(\varrho) - \Theta(\varrho), z\|_Y \leq \frac{\epsilon}{2(a^4 - 1)}.$$

2.3.4 Ulam-Hyers-Rassias Stability in Non-Archimedean n-Normed Space

In this section, we presents the results on the stability of FE 2.1.1 in Non-Archimedean n- $\mathcal{NS}$.

Theorem 2.3.11. *Let* Z *be a non-Archimedean n-*$\mathcal{NS}$ *and* Y *be a non-Archimedean n-*$\mathcal{BS}$ *over the same field. Let* $F : Z \to Y$ *be a mapping with* $F(0) = 0$*. If for a mapping* $\Xi : Z \times Z \to [0, \infty)$*, we have*

$$\lim_{\hbar \to \infty} \frac{1}{a^{4\hbar}} \Xi(a^\hbar \varrho, a^\hbar \vartheta) = 0, \tag{2.3.37}$$

$$\|G_{a,b}F(\varrho, \vartheta), z\|_Y \leq \Xi(\varrho, \vartheta)\ \forall \varrho, \vartheta \in Z \text{ and } z \in Y^{n-1}. \tag{2.3.38}$$

Then, $\exists$ *a unique quartic map* $\Theta : Z \to Y$ *s.t.*

$$\|F(\varrho) - \Theta(\varrho), z\|_Y \leq \frac{1}{2a^4} \bar{\Xi}(\varrho, 0), \tag{2.3.39}$$

where $\bar{\Xi}(\varrho, 0) = \max\left\{\dfrac{\Xi(a^\hbar \varrho, 0)}{a^{4\hbar}} : i \in \mathbb{N}\right\}$.

Proof. On substituting $\vartheta = 0$ in (2.3.38), we have

$$\|2F(a\varrho) - 2a^4 F(\varrho), z\|_Y \leq \Xi(\varrho, 0),$$

or

$$\left\|\frac{F(a\varrho)}{a^4} - F(\varrho), z\right\|_Y \leq \frac{1}{2a^4} \Xi(\varrho, 0). \tag{2.3.40}$$

Now, on substituting $\varrho = a^\hbar \varrho$ in (2.3.40) and dividing by $a^{4\hbar}$, we have

$$\left\|\frac{F(a^{1+\hbar}\varrho)}{a^{4(1+\hbar)}} - \frac{F(a^\hbar \varrho)}{a^{4\hbar}}, z\right\|_Y \leq \frac{1}{2a^4} \frac{\Xi(a^\hbar \varrho, 0)}{a^{4\hbar}}. \tag{2.3.41}$$

On taking limit as $\hbar \to \infty$ in (2.3.41) and using (2.3.37), we have

$$\lim_{\hbar \to \infty} \left\|\frac{F(a^{1+\hbar}\varrho)}{a^{4(1+\hbar)}} - \frac{F(a^\hbar \varrho)}{a^{4\hbar}}, z\right\|_Y = 0, \tag{2.3.42}$$

i.e., $\left\{\frac{F(a^{\hbar}\varrho)}{a^{4\hbar}}\right\}$ is a C_{seq} in $(Y, \|., .\|_Y)$. Also, as $(Y, \|., .\|_Y)$ is a non-Archimedean n-$\mathcal{BS}$. Therefore, we define $\Theta : Z \to Y$ as $\Theta(\varrho) = \lim_{\hbar\to\infty} \frac{F(a^{\hbar}\varrho)}{a^{4\hbar}}$.

Thus,

$$\begin{aligned}
\lim_{\hbar\to\infty} \left\| F(\varrho) - \frac{F(a^{\hbar}\varrho)}{a^{4\hbar}}, z \right\|_Y &= \lim_{\hbar\to\infty} \left\| \sum_{i=0}^{-1+\hbar} \left(\frac{F(a^{i+1}\varrho)}{a^{4(i+1)}} - \frac{F(a^{i}\varrho)}{a^{4i}} \right), z \right\|_Y \\
&\leq \lim_{\hbar\to\infty} \frac{1}{2a^4} \max \left\{ \frac{\Xi(a^{\hbar}\varrho, 0)}{a^{4\hbar}} : 0 \leq i < \hbar \right\} \\
&= \frac{1}{2a^4} \bar{\Xi}(\varrho, 0),
\end{aligned}$$

where $\bar{\Xi}(\varrho, 0) = \max \left\{ \frac{\Xi(a^{i}\varrho, 0)}{a^{4i}} : i \in \mathbb{N} \right\}$. The remaining part of the proof is analogues to Theorem 2.3.3. $\square$

Theorem 2.3.12. *Let Z be a non-Archimedean n-$\mathcal{NS}$ and Y be a non-Archimedean n-$\mathcal{BS}$ over the same field. Let $F : Z \to Y$ be a mapping with $F(0) = 0$. If for a mapping $\Xi : Z \times Z \to [0, \infty)$, we have*

$$\lim_{\hbar\to\infty} a^{4\hbar} \Xi \left(\frac{\varrho}{a^{\hbar}}, \frac{\vartheta}{a^{\hbar}} \right) = 0, \tag{2.3.43}$$

$$\|G_{a,b}F(\varrho, \vartheta), z\|_Y \leq \Xi(\varrho, \vartheta) \ \forall \varrho, \vartheta \in Z \text{ and } z \in Y^{n-1}. \tag{2.3.44}$$

Then, $\exists$ a unique quartic map $\Theta : Z \to Y$ s.t.

$$\|F(\varrho) - \Theta(\varrho), z\|_Y \leq \frac{1}{2a^4} \bar{\Xi}(\varrho, 0), \tag{2.3.45}$$

where $\bar{\Xi}(\varrho, 0) = \max \left\{ a^{4\hbar} \Xi \left(\frac{\varrho}{a^{\hbar}}, 0 \right) \} : i \in \mathbb{N} \right\}$.

Proof. On substituting $\vartheta = 0$ in (2.3.44), we have

$$\|F(a\varrho) - a^4 F(\varrho), z\|_Y \leq \frac{1}{2} \Xi(\varrho, 0). \tag{2.3.46}$$

On substituting $\varrho = \frac{\varrho}{a}$ in (2.3.46), we have

$$\left\| F(\varrho) - a^4 F\left(\frac{\varrho}{a}\right), z \right\|_Y \leq \frac{1}{2} \Xi \left(\frac{\varrho}{a}, 0 \right). \tag{2.3.47}$$

Again on substituting $\varrho = \dfrac{\varrho}{a^{\hbar}}$ in (2.3.47) and multiplying by $a^{4\hbar}$, we have

$$\left\| a^{4\hbar} F\left(\frac{\varrho}{a^{\hbar}}\right) - a^{4(1+\hbar)} F\left(\frac{\varrho}{a^{1+\hbar}}\right), z \right\|_Y \leq \frac{1}{2a^4} a^{4(1+\hbar)} \Xi\left(\frac{\varrho}{a^{1+\hbar}}, 0\right). \tag{2.3.48}$$

On taking limit as $\hbar \to \infty$ in (2.3.48) and using (2.3.43), we have

$$\lim_{\hbar \to \infty} \left\| a^{4\hbar} F\left(\frac{\varrho}{a^{\hbar}}\right) - a^{4(1+\hbar)} F\left(\frac{\varrho}{a^{1+\hbar}}\right), z \right\|_Y = 0, \tag{2.3.49}$$

i.e., $\left\{ a^{4\hbar} F\left(\dfrac{\varrho}{a^{\hbar}}\right) \right\}$ is a C_{seq} in $(Y, \|., .\|_Y)$. Also, as $(Y, \|., .\|_Y)$ is a non-archimedean n-$\mathcal{BS}$. Therefore, we can define a map $\Theta : Z \to Y$ as $\Theta(\varrho) = \lim\limits_{\hbar \to \infty} a^{4\hbar} F\left(\dfrac{\varrho}{a^{\hbar}}\right)$. The remaining part of the proof is analogues to Theorem 2.3.11. $\square$

2.4 Conclusion

The chapter presents the stability of quadratic and quartic FEs. The analysis includes the stability of classical and Jensen-type quadratic FEs using well-known methods, with a focus on hyperstability and examples that show the importance of the assumptions in the analysis. The stability of a generalized quartic FE is also explored, with results obtained in *n-$\mathcal{BS}$s* using fixed-point techniques and sub-additive control functions. The discussion also includes non-Archimedean *n-$\mathcal{BS}$s*, offering a clear and unified approach to understanding the stability of these FEs across different mathematical contexts.

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Chapter 3

Some Fixed Point Results in C^*-Algebra Valued $\mathfrak{m}_{\mathcal{R}}$-Metric Space

3.1 Introduction

The aim of this chapter is to address and overcome certain limitations inherent in Banach's fixed-point theorem and its recent generalization. To address these limitations, we introduce the notion of C^*-algebra valued $\mathfrak{m}_{\mathcal{R}}$-metric space($C^*_{AV}$-$\mathfrak{m}_{\mathcal{R}}$-MS) which is a generalization of C^*_{AV}-$\mathfrak{m}$-MS and $\mathcal{R}$-MS . The first section deals the basic definition and intrinsic properties of C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-MS, including convergence of sequences and completeness. In the second section, we generalize some well-known contraction mappings and prove fixed point theorems on the $\mathcal{R}$-complete C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-MS (not necessarily complete in metric sense).
Our findings extend various fixed point results in the literature. Moreover, we provide examples where Banach-type contractions yield the desired results in this structure, which is not the case in various generalized metric spaces. Finally, we utilize our findings to establish the existence and uniqueness of solutions for an operator equation. The results of this chapter are presented in [1].

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[1]Yadav, K., & Kumar, D. C^*-algebra valued $\mathfrak{m}_{\mathcal{R}}$-metric space and fixed point results with application, *Asian-European Journal of Mathematics*, 2550109.

3.2 C^*-Algebra Valued $\mathfrak{m}_{\mathcal{R}}$-Metric Space

In this section, we introduced a new notion of C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$ -MS along with the intrinsic properties and some illustrative examples of this framework.

Definition 3.2.1. *$(\Omega, \mathbb{B}, \mathcal{R}, \varpi)$ is c.t.b. a C^*-algebra valued $\mathfrak{m}_{\mathcal{R}}$-metric space ($C^*_{AV}$-$\mathfrak{m}_{\mathcal{R}}$-MS) if it satisfies:*

(i) $(\Omega, \mathbb{B}, \varpi)$ is a C^*_{AV}-$\mathfrak{m}$-MS;

(ii) $\mathcal{R}$ is a reflexive binary relation on Ω.

Remark 3.2.2. *A C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-MS need not to be an C^*_{AV}-$\mathfrak{m}$-MS. For instance, consider $\Omega = \mathbb{R}$ and $I \in \mathbb{B}$ where $\mathbb{B} = M_2(\mathbb{R})$ with relation defined as $(\varrho, \vartheta) \in \mathcal{R}$ $\Leftrightarrow$ $\varrho = \vartheta$ or $\varrho.\vartheta > 0$. Then, $\varpi(\varrho, \vartheta) = \max \{|\varrho|, |\vartheta|\}I$ is an C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-metric but it is not C^*_{AV}-$\mathfrak{m}$-metric. This is because, for $\varrho = -2$ and $\vartheta = 2$, we have $\varpi(\varrho, \varrho) = \varpi(\varrho, \vartheta) = \varpi(\vartheta, \vartheta) = 2I$. But $\varrho \neq \vartheta$.*

Remark 3.2.3. *Every C^*_{AV}-$\mathcal{R}$-MS is C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-MS. But the converse may not hold. Consider*

Example 3.2.4. *Let $\Omega = [0, \infty)$ and $\mathbb{B} = M_2([0, \infty))$. Let involution on $\mathbb{B}$ be defined as $M^* = M^t$ $\forall M \in \mathbb{B}$, where M^t denotes transpose of M and zero element $\theta_{\mathbb{B}} = 0_{2\times 2}$. Norm on $\mathbb{B}$ is defined as $\|M\| = \max\limits_{1\leq i,j\leq 2}\Big|m_{ij}\Big|$, for $M = [m_{ij}]$. Define $\varpi : \Omega \times \Omega \to M_2([0, \infty))$ as:*

$$\varpi(\varrho, \vartheta) = \begin{bmatrix} \frac{\varrho+\vartheta}{2} & 0 \\ 0 & \frac{\varrho+\vartheta}{2} \end{bmatrix}.$$

For $P = [p_{ij}], Q = [q_{ij}] \in M_2([0, \infty))$, we define $P \preceq Q \Leftrightarrow p_{ij} \leq q_{ij}$, $\forall i, j = 1, 2$. One can easily verify that $(\Omega, \mathbb{B}, \varpi)$ is C^*_{AV}-$\mathfrak{m}$-MS. Let $\mathcal{R}$ be a binary relation on Ω defined as $\varrho\mathcal{R}\vartheta \Leftrightarrow \varrho = \vartheta$ or $\varrho.\vartheta = 0$. Then $(\Omega, \mathbb{B}, \mathcal{R}, \varpi)$ is C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-MS.

Example 3.2.5. *Let $\Omega = [0, \infty)$ and $\mathbb{B} = M_2([0, \infty))$. Let involution on $\mathbb{B}$ be defined as $M^* = M^t$ $\forall M \in \mathbb{B}$, where M^t denotes transpose of M. Norm on $\mathbb{B}$ is defined as $\|M\| = \max\limits_{1\leq i,j\leq 2}\Big|m_{ij}\Big|$, for $M = [m_{ij}]$. Define $\varpi : \Omega \times \Omega \to M_2([0, \infty))$ as:*

$$\varpi(\varrho, \vartheta) = \begin{bmatrix} [\frac{\varrho+\vartheta}{2}]^{\varrho} & 0 \\ 0 & [\frac{\varrho+\vartheta}{2}]^{\varrho} \end{bmatrix}, \text{where } \varrho \geq 1.$$

For $P = [p_{ij}], Q = [q_{ij}] \in M_2([0, \infty))$, we define $P \preceq Q \Leftrightarrow p_{ij} \leq q_{ij}$, $\forall i, j = 1, 2$. Let $\mathcal{R}$ be a binary relation on Ω defined as $\varrho\mathcal{R}\vartheta \Leftrightarrow \varrho = \vartheta$ or $\varrho.\vartheta = 0$. Here, $(\Omega, \mathbb{B}, \mathcal{R}, \varpi)$ is C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-MS..

Example 3.2.6. *Let Ω be a $\mathcal{BS}$ with norm $\|.\|$. Let $\mathbb{B} = M_2([0,\infty))$ and I denote the identity matrix. Then $(\Omega, \mathbb{B}, \mathcal{R}, \mathfrak{m})$ be a C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-MS, where $\mathfrak{m}$-metric is defined as:*

$$\varpi(\varrho, \vartheta) = \|\varrho - \vartheta\| I + \min\{\|\varrho\|, \|\vartheta\|\} I \ \forall \varrho, \vartheta \in \Omega,$$

and $\mathcal{R}$ be a reflexive relation defined on Ω.

Example 3.2.7. *Let H be a Hilbert space with inner product $<,>_H$ and $B(H)$ be the set of all bounded linear operator on H. Clearly, $B(H)$ with the usual norm is a $\mathcal{BS}$. Then for a positive operator P consider the metric defined as:*

$$\varpi(\Gamma_1, \Gamma_2) = \|\Gamma_1 - \Gamma_2\| P + \min\{\|\Gamma_1\|, \|\Gamma_2\|\} P,$$

where $\|.\|$ on $B(H)$ defined as $\|\Gamma\| = \sup\limits_{\varrho \in H, \varrho \neq 0} \dfrac{\|\Gamma(\varrho)\|}{\|\varrho\|}$. Let relation $\mathcal{R}$ on $B(H)$ be defined as $(\Gamma_1, \Gamma_2) \in \mathcal{R} \Leftrightarrow |< \Gamma_1(\varrho), \varrho >_H| \leq |< \Gamma_2(\varrho), \varrho >_H| \ \forall \varrho \in H$. Then, $B(H)$ is a C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-MS.

Definition 3.2.8. *Let $(\Omega, \mathbb{B}, \mathcal{R}, \varpi)$ is C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-MS. An $\mathcal{R}_{seq}$ in $(\Omega, \mathbb{B}, \mathcal{R}, \varpi)$ is convergent to some $\varrho \in \Omega$, if $\forall \epsilon > 0$, $\exists \hbar_0 \in \mathbb{N}$ s.t. $\|\varpi(\varrho_\hbar, \varrho) - \varpi_{\varrho_\hbar \varrho}\| \leq \epsilon \ \ \forall \hbar \geq \hbar_0$ i.e.,*

$$\varpi(\varrho_\hbar, \varrho) - \varpi_{\varrho_\hbar \varrho} \to \theta_{\mathbb{B}} \text{ as } \hbar \to \infty.$$

Definition 3.2.9. *Let $(\Omega, \mathbb{B}, \mathcal{R}, \varpi)$ is C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-MS . An $\mathcal{R}_{seq}$ in $(\Omega, \mathbb{B}, \mathcal{R}, \varpi)$ is c.t.b. $\mathfrak{m}_{\mathcal{R}}$-Cauchy if*

$$\lim_{\hbar, m \to \infty} \varpi(\varrho_\hbar, \varrho_m) - \varpi_{\varrho_\hbar \varrho_m} \text{ and } \lim_{\hbar, m \to \infty} M_{\varrho_\hbar \varrho_m} - \varpi_{\varrho_\hbar \varrho_m} \text{ exists finitely.}$$

Definition 3.2.10. *$(\Omega, \mathbb{B}, \mathcal{R}, \varpi)$ is c.t.b. a $\mathcal{R}$-complete C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-MS if all $\mathfrak{m}_{\mathcal{R}}$-$C_{seq}$ in $(\Omega, \mathbb{B}, \mathcal{R}, \varpi)$ is convergent in Ω i.e., $\exists \varrho \in \Omega$ s.t.*

$$\varpi(\varrho_\hbar, \varrho) - \varpi_{\varrho_\hbar \varrho} \to \theta_{\mathbb{B}} \text{ and } M_{\varrho_\hbar \varrho} - \varpi_{\varrho_\hbar \varrho} \to \theta_{\mathbb{B}} \text{ as } \hbar \to \infty.$$

Remark 3.2.11. *Let $(\Omega, \mathbb{B}, \mathcal{R}, \varpi)$ is C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-MS . Then*

(i) $\varpi^w(\varrho, \vartheta) = \varpi(\varrho, \vartheta) + M_{\varrho\vartheta} - 2\varpi_{\varrho\vartheta}$;

(ii) $\varpi_s(\varrho, \vartheta) = \begin{cases} \varpi(\varrho, \vartheta) - \varpi_{\varrho\vartheta}, & \text{if } \varrho \neq \vartheta; \\ \theta_{\mathbb{B}}, & \text{if } \varrho = \vartheta, \end{cases}$

are C^*_{AV}-$\mathcal{R}$-metric and $(\Omega, \mathbb{B}, \mathcal{R}, \mathfrak{m}^w)$ and $(\Omega, \mathbb{B}, \mathcal{R}, \varpi_s)$ are C^*_{AV}-$\mathcal{R}$-MSs.

Lemma 3.2.12. *Let $(\Omega, \mathbb{B}, \mathcal{R}, \varpi)$ be a C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-MS . Let $\{\varrho_\hbar\}$ be a sequence in Ω. Then,*

(i) $\{\varrho_\hbar\}$ *is $\mathfrak{m}_{\mathcal{R}}$-Cauchy in* $(\Omega, \mathbb{B}, \mathcal{R}, \varpi) \Leftrightarrow \{\varrho_\hbar\}$ *is $\mathcal{R}$-Cauchy in* $(\Omega, \mathbb{B}, \mathcal{R}, \mathfrak{m}^w)$*;*

(ii) $(\Omega, \mathbb{B}, \mathcal{R}, \varpi)$ *is complete* $\Leftrightarrow (\Omega, \mathbb{B}, \mathcal{R}, \mathfrak{m}^w)$ *is complete;*

(iii) $\{\varrho_\hbar\}$ *is $\mathfrak{m}_{\mathcal{R}}$-Cauchy in* $(\Omega, \mathbb{B}, \mathcal{R}, \varpi) \Leftrightarrow \{\varrho_\hbar\}$ *is $\mathcal{R}$-Cauchy in* $(\Omega, \mathbb{B}, \mathcal{R}, \varpi_s)$*;*

(iv) $(\Omega, \mathbb{B}, \mathcal{R}, \varpi)$ *is complete* $\Leftrightarrow (\Omega, \mathbb{B}, \mathcal{R}, \varpi_s)$ *is complete.*

Lemma 3.2.13. *Let $(\Omega, \mathbb{B}, \mathcal{R}, d)$ be a C^*_{AV}-$\mathcal{R}$-MS and $a, b \in \mathbb{B}$ with $a, b \succ \theta_{\mathbb{B}}$. Then $(\Omega, \mathbb{B}, \mathcal{R}, \varpi)$ is a C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-MS, where $\varpi(\varrho, \vartheta) = ad(\varrho, \vartheta) + b \ \forall (\varrho, \vartheta) \in \mathcal{R}$.*

Definition 3.2.14. *Let $(\Omega, \mathbb{B}, \mathcal{R}, \varpi)$ be a C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-MS . Let $\Gamma : \Omega \to \Omega$. Γ is c.t.b. $\mathcal{R}$-preserving if $\forall \varrho, \vartheta \in \Omega$, $(\varrho, \vartheta) \in \mathcal{R}$ implies $(\Gamma\varrho, \Gamma\vartheta) \in \mathcal{R}$.*

Remark 3.2.15. *Let $(\Omega, \mathbb{B}, \mathcal{R}, \varpi)$ be a C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-MS . For $\varrho, \vartheta, z \in \Omega$, s.t. $(\varrho, \vartheta) \in \mathcal{R}$, $(\varrho, z) \in \mathcal{R}$ and $(z, \vartheta) \in \mathcal{R}$, we have*

(i) $\theta_{\mathbb{B}} \preceq M_{\varrho\vartheta} + \varpi_{\varrho\vartheta} = \varpi(\varrho, \varrho) + \varpi(\vartheta, \vartheta)$*;*

(ii) $\theta_{\mathbb{B}} \preceq M_{\varrho\vartheta} - \varpi_{\varrho\vartheta} = \bigg(\varpi(\varrho, \varrho) - \varpi(\vartheta, \vartheta)\bigg) \vee \bigg(\varpi(\vartheta, \vartheta) - \varpi(\varrho, \varrho)\bigg)$*;*

(iii) $M_{\varrho\vartheta} - \varpi_{\varrho\vartheta} \preceq M_{\varrho z} - \varpi_{\varrho z} + M_{z\vartheta} - \varpi_{z\vartheta}$.

3.3 Some Fixed Point Results in C^*-Algebra Valued $\mathfrak{m}_{\mathcal{R}}$-Metric Space

In this section, some fixed point results are established using some well known contraction mapping in the framework of C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-MS.

Definition 3.3.1. *Let $(\Omega, \mathbb{B}, \mathcal{R}, \varpi)$ be a C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-MS and $\Gamma : \Omega \to \Omega$ be a mapping on Ω. Then Γ is c.t.b. C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-contraction if $\exists$ an $\mathcal{A} \in \mathbb{B}$ with $\|\mathcal{A}\| < 1$, s.t.*

$$\varpi(\Gamma\varrho, \Gamma\vartheta) \preceq \mathcal{A}^* \varpi(\varrho, \vartheta) \mathcal{A}, \tag{3.3.1}$$

$\forall \varrho, \vartheta \in \Omega$, s.t. $(\varrho, \vartheta) \in \mathcal{R}$.

Theorem 3.3.2. *Let $(\Omega, \mathbb{B}, \mathcal{R}, \varpi)$ be an $\mathcal{R}$-complete C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-MS, and let $\Gamma : \Omega \to \Omega$ satisfy:*

(i) Γ *is* $\mathcal{R}$*-preserving;*

(ii) $\exists$ *an* $\varrho_0 \in \Omega$ *s.t.* $(\varrho_0, \vartheta) \in \mathcal{R}$, $\forall \vartheta \in \Omega$*;*

(iii) Γ *is a* C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$*-contraction.*

Then, Γ *possesses exactly one fixed point.*

Proof. For $\varrho_0 \in \Omega$ satisfying condition (ii), define the iterative sequence $\{\varrho_\hbar\}$ as $\varrho_\hbar = \Gamma \varrho_{\hbar-1}$ $\forall \hbar \in \mathbb{N}$.
If $\varrho_\hbar = \varrho_{\hbar-1}$ for some $\hbar \in \mathbb{N}$. Then $\Gamma \varrho_{\hbar-1} = \varrho_\hbar = \varrho_{\hbar-1}$ implies $\varrho_{\hbar-1}$ is a fixed point of Γ. Hence, the result holds.
Now, assume that $\varrho_\hbar \neq \varrho_{\hbar-1}$ $\forall \hbar \in \mathbb{N}$.
As, $\varrho_0 \in \Omega$ satisfying condition (ii) and Γ is $\mathcal{R}$-preserving. Hence,

$$\begin{aligned} (\varrho_0, \vartheta) \in \mathcal{R} \ \ \forall \vartheta \in \Omega \ &\Rightarrow \ (\varrho_0, \Gamma \varrho_0) \in \mathcal{R} \\ &\Rightarrow \ (\varrho_0, \varrho_1) \in \mathcal{R}. \end{aligned}$$

On repeated use of $\mathcal{R}$-preserving property, we have

$$(\varrho_\hbar, \varrho_{\hbar+1}) \in \mathcal{R}.$$

Also, by assumption (ii), we have

$$(\varrho_0, \Gamma^k(\varrho_0)) \in \mathcal{R}, \text{ where } k \in \mathbb{N}.$$

Therefore, using $\mathcal{R}$-preserving property, we can easily prove that $\{\varrho_\hbar\}$ is an $\mathcal{R}_{seq}$ i.e., $(\varrho_\hbar, \varrho_{\hbar+k}) \in \mathcal{R}$ $\forall \hbar, k \in \mathbb{N}$.
Now, using (3.3.1), we have

$$\begin{aligned} \varpi(\varrho_{\hbar+1}, \varrho_\hbar) &\preceq \mathcal{A}^* \varpi(\varrho_\hbar, \varrho_{\hbar-1}) \mathcal{A} \\ &\preceq (\mathcal{A}^*)^2 \varpi(\varrho_{\hbar-1}, \varrho_{\hbar-2}) \mathcal{A}^2 \\ &\ldots \\ &\preceq (\mathcal{A}^*)^\hbar \varpi(\varrho_1, \varrho_0) \mathcal{A}^\hbar. \end{aligned}$$

Let $\varpi(\varrho_1, \varrho_0) = \beta$. Then,

$$\begin{aligned} \varpi(\varrho_{\hbar+1}, \varrho_\hbar) &\preceq (\mathcal{A}^*)^\hbar \beta \mathcal{A}^\hbar = (\mathcal{A}^*)^\hbar \beta^{\frac{1}{2}}.\beta^{\frac{1}{2}} \mathcal{A}^\hbar \\ &= [\beta^{\frac{1}{2}} \mathcal{A}^\hbar]^*.[\beta^{\frac{1}{2}} \mathcal{A}^\hbar] \preceq \|\beta^{\frac{1}{2}} \mathcal{A}^\hbar\|^2 I_{\mathbb{B}} \\ &\preceq \|\beta^{\frac{1}{2}}\|^2 \|\mathcal{A}^\hbar\|^2 I_{\mathbb{B}}. \end{aligned}$$

As $\|\mathcal{A}\| < 1$ and $\|\mathcal{A}^\hbar\| \leq \|\mathcal{A}\|^\hbar \Rightarrow \|\mathcal{A}^\hbar\| \to 0$.
Hence,

$$\lim_{\hbar\to\infty} \varpi(\varrho_{\hbar+1}, \varrho_\hbar) = \theta_{\mathbb{B}}. \tag{3.3.2}$$

Also, $\varpi_{\varrho_\hbar\varrho_{\hbar+1}} = \min\left\{\varpi(\varrho_\hbar, \varrho_\hbar), \varpi(\varrho_{\hbar+1}, \varrho_{\hbar+1})\right\} \preceq \varpi(\varrho_{\hbar+1}, \varrho_\hbar) \to \theta_{\mathbb{B}}$ as $\hbar \to \infty$.
Hence,

$$\lim_{\hbar\to\infty} \varpi(\varrho_{\hbar,}, \varrho_\hbar) \to \theta_{\mathbb{B}}. \tag{3.3.3}$$

Using (3.3.3), we have

$$\lim_{\hbar,m\to\infty} \varpi_{\varrho_\hbar\varrho_m} = \theta_{\mathbb{B}}. \tag{3.3.4}$$

For $m \geq \hbar$, consider

$$\begin{aligned}
\varpi(\varrho_m, \varrho_\hbar) - \varpi_{\varrho_m\varrho_\hbar} \;&\leq\; \varpi(\varrho_m, \varrho_{m-1}) - \varpi_{\varrho_m\varrho_{m-1}} \\
&+\; \varpi(\varrho_{m-1}, \varrho_{m-2}) - \varpi_{\varrho_{m-1}\varrho_{m-2}} \\
&\ldots\; +\varpi(\varrho_{\hbar+1}, \varrho_\hbar) - \varpi_{\varrho_{\hbar+1}\varrho_\hbar} \\
&\preceq\; \varpi(\varrho_m, \varrho_{m-1}) + \varpi(\varrho_{m-1}, \varrho_{m-2})\ldots + \varpi(\varrho_{\hbar+1}, \varrho_\hbar) \\
&\preceq\; [\mathcal{A}^*]^{m-1}\beta\mathcal{A}^{m-1} + [\mathcal{A}^*]^{m-2}\beta\mathcal{A}^{m-2} + \ldots + [\mathcal{A}^*]^{\hbar}\beta\mathcal{A}^{\hbar} \\
&=\; \sum_{k=\hbar}^{\mathfrak{m}-1}[\mathcal{A}^*]^k\beta\mathcal{A}^k \;= \sum_{k=\hbar}^{\mathfrak{m}-1}[\beta^{\frac{1}{2}}\mathcal{A}^k]^*.[\beta^{\frac{1}{2}}\mathcal{A}^k] \\
&\preceq\; \sum_{k=\hbar}^{\mathfrak{m}-1}\|\beta^{\frac{1}{2}}\mathcal{A}^k\|^2 I_{\mathbb{B}} \preceq \sum_{k=\hbar}^{\mathfrak{m}-1}\|\beta^{\frac{1}{2}}\|^2\|\mathcal{A}^k\|^2 I_{\mathbb{B}} \\
&\preceq\; \|\beta^{\frac{1}{2}}\|^2\sum_{k=\hbar}^{\infty}\|\mathcal{A}^k\|^2 I_{\mathbb{B}} = \|\beta^{\frac{1}{2}}\|^2\frac{\|\mathcal{A}^\hbar\|^2}{1-\|\mathcal{A}\|^2} I_{\mathbb{B}}.
\end{aligned}$$

Now, since $\|\mathcal{A}\| < 1 \Rightarrow \|\mathcal{A}^\hbar\| \to 0$, then we have

$$\lim_{\hbar,m\to\infty} \varpi(\varrho_m, \varrho_\hbar) - \varpi_{\varrho_m\varrho_\hbar} = \theta_{\mathbb{B}}.$$

Using (3.3.3) and Remark 3.2.15, we have

$$\lim_{\hbar,m\to\infty} M_{\varrho_m\varrho_\hbar} - \varpi_{\varrho_m\varrho_\hbar} = \theta_{\mathbb{B}}.$$

Hence, $\{\varrho_\hbar\}$ is an $\mathfrak{m}_{\mathcal{R}}$-$C_{seq}$.
As, $(\Omega, \mathbb{B}, \mathcal{R}, \varpi)$ is an $\mathcal{R}$-complete C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-MS and $\{\varrho_\hbar\}$ is $\mathfrak{m}_{\mathcal{R}}$ -C_{seq}. Therefore, $\exists\varrho \in \Omega$ s.t.

$$\varpi(\varrho_\hbar, \varrho) - \varpi_{\varrho_\hbar\varrho} \to \theta_{\mathbb{B}} \quad \text{and} \quad M_{\varrho_\hbar\varrho} - \varpi_{\varrho_\hbar\varrho} \to \theta_{\mathbb{B}} \;\text{ as } \hbar \to \infty. \tag{3.3.5}$$

Using (3.3.3), we have

$$\varpi_{\varrho_\hbar \varrho} = \min\left\{\varpi(\varrho_\hbar, \varrho_\hbar), \varpi(\varrho, \varrho)\right\} \preceq \varpi(\varrho_\hbar, \varrho_\hbar) \to \theta_{\mathbb{B}} \text{ as } \hbar \to \infty. \tag{3.3.6}$$

Using (3.3.6) and Remark 3.2.15, we have

$$\varpi(\varrho_\hbar, \varrho) \to \theta_{\mathbb{B}},\ M_{\varrho_\hbar \varrho} \to \theta_{\mathbb{B}} \text{ and } \varpi(\varrho, \varrho) = \theta_{\mathbb{B}}. \tag{3.3.7}$$

$$\varpi_{\varrho\Gamma\varrho} = \min\left\{\varpi(\varrho, \varrho), \varpi(\Gamma\varrho, \Gamma\varrho)\right\} \preceq \varpi(\varrho, \varrho) = \theta_{\mathbb{B}} \Rightarrow \varpi_{\varrho\Gamma\varrho} = \theta_{\mathbb{B}}. \tag{3.3.8}$$

Using (3.3.1), (3.3.8) and the triangle inequality, we have

$$\begin{aligned} \varpi(\varrho, \Gamma\varrho) &= \varpi(\varrho, \Gamma\varrho) - \varpi_{\varrho\Gamma\varrho} \preceq \varpi(\varrho, \varrho_\hbar) - \varpi_{\varrho_\hbar\varrho} + \varpi(\varrho_\hbar, \Gamma\varrho) - \varpi_{\varrho_\hbar\Gamma\varrho} \\ &\preceq \limsup_{\hbar\to\infty} \varpi(\varrho, \varrho_\hbar) - \varpi_{\varrho_\hbar\varrho} + \varpi(\varrho_\hbar, \Gamma\varrho) - \varpi_{\varrho_\hbar\Gamma\varrho} \\ &= \limsup_{\hbar\to\infty} \varpi(\varrho_\hbar, \Gamma\varrho) - \varpi_{\varrho_\hbar\Gamma\varrho}, \end{aligned}$$

or

$$\varpi(\varrho, \Gamma\varrho) \preceq \limsup_{\hbar\to\infty} \varpi(\varrho_\hbar, \Gamma\varrho) - \varpi_{\varrho_\hbar\Gamma\varrho}. \tag{3.3.9}$$

Using (3.3.1), (3.3.6) and (3.3.7), in (3.3.9), we have

$$\begin{aligned} \theta_{\mathbb{B}} \leq \varpi(\varrho, \Gamma\varrho) &\preceq \limsup_{\hbar\to\infty} \varpi(\varrho_\hbar, \Gamma\varrho) - \varpi_{\varrho_\hbar\Gamma\varrho} \\ &\preceq \limsup_{\hbar\to\infty} \varpi(\varrho_\hbar, \Gamma\varrho) = \limsup_{\hbar\to\infty} \varpi(\Gamma\varrho_{\hbar-1}, \Gamma\varrho) \\ &\preceq \limsup_{\hbar\to\infty} \mathcal{A}^*\varpi(\varrho_{\hbar-1}, \varrho)\mathcal{A} = \theta_{\mathbb{B}}. \end{aligned} \tag{3.3.10}$$

Using (3.3.1) and (3.3.7), we have

$$\theta_{\mathbb{B}} \prec \varpi(\Gamma\varrho, \Gamma\varrho) \preceq \mathcal{A}^*\varpi(\varrho, \varrho)\mathcal{A} = \theta_{\mathbb{B}}. \tag{3.3.11}$$

By (3.3.3), (3.3.10) and (3.3.11), we have

$$\varpi(\varrho, \varrho) = \varpi(\varrho, \Gamma\varrho) = \varpi(\Gamma\varrho, \Gamma\varrho) \Rightarrow \Gamma\varrho = \varrho,$$

i.e, ϱ is fixed point of Γ.

Uniqueness: let $\vartheta \neq \varrho \in \Omega$ be another fixed point of Γ with $\varpi(\vartheta, \vartheta) = \theta_{\mathbb{B}}$. From assumption (ii), $(\varrho_0, \vartheta) \in \mathcal{R}$. Using $\mathcal{R}$-preserving property of Γ, we have $(\Gamma^\hbar(\varrho_0), \Gamma^\hbar(\vartheta)) \in \mathcal{R}\ \forall \hbar \in N$. Then, using (3.3.1), we have

$$\begin{aligned} \varpi(\varrho_\hbar, \vartheta) = \varpi(\Gamma^\hbar(\varrho_0), \Gamma^\hbar(\vartheta)) &\preceq \mathcal{A}^*\varpi(\Gamma^{\hbar-1}(\varrho), \Gamma^{\hbar-1}(\vartheta))\mathcal{A} \\ &\dots \\ &\preceq \mathcal{A}^{*^\hbar}\varpi(\varrho_0, \vartheta)\mathcal{A}^\hbar \end{aligned} \tag{3.3.12}$$

Taking limit as $\hbar \to \infty$ on both side of (3.3.12), we have

$$\varpi(\varrho, \vartheta) = \theta_{\mathbb{B}}$$

i.e., $\varrho = \vartheta$. Hence, Γ possesses exactly one fixed point. $\square$

Theorem 3.3.3. *Let $(\Omega, \mathbb{B}, \mathcal{R}, \varpi)$ be an $\mathcal{R}$-complete C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-MS, and let $\Gamma : \Omega \to \Omega$ satisfy:*

(i) Γ *is $\mathfrak{m}_{\mathcal{R}}$-preserving;*

(ii) $\exists$ *an $\varrho_0 \in \Omega$ s.t. $(\varrho_0, \vartheta) \in \mathcal{R}$, $\forall \vartheta \in \Omega$;*

(iii) $\exists$ *an $\mathcal{A} \in \mathbb{A}'_+$ with $\|\mathcal{A}\| < \frac{1}{2}$, satisfying*

$$\varpi(\Gamma\varrho, \Gamma\vartheta) \preceq \mathcal{A}\Big(\varpi(\varrho, \Gamma\varrho) + \varpi(\vartheta, \Gamma\vartheta)\Big), \tag{3.3.13}$$

$\forall \varrho, \vartheta \in \Omega$ *s.t.* $(\varrho, \vartheta) \in \mathcal{R}$.

Then, Γ possesses exactly one fixed point.

Proof. For $\varrho_0 \in \Omega$ satisfying condition (ii), define the iterative sequence, $\{\varrho_\hbar\}$ s.t. $\varrho_\hbar = \Gamma\varrho_{\hbar-1}$, $\forall \hbar \in \mathbb{N}$.
If $\varrho_\hbar = \varrho_{\hbar-1}$ for some $\hbar \in \mathbb{N}$, then $\varrho_\hbar = \Gamma\varrho_{\hbar-1} = \varrho_{\hbar-1}$ implies $\varrho_{\hbar-1}$ is a fixed point of Γ. Hence, the result holds.
Now, consider $\varrho_\hbar \neq \varrho_{\hbar-1}$, $\forall \hbar \in \mathbb{N}$.
As, $\varrho_0 \in \Omega$ satisfies condition (ii) and Γ is $\mathfrak{m}_{\mathcal{R}}$-preserving, we have

$$\begin{aligned}(\varrho_0, \vartheta) \in \mathcal{R} \ \ \forall \vartheta \in \Omega \ \ &\Rightarrow \ \ (\varrho_0, \Gamma\varrho_0) \in \mathcal{R} \\ &\Rightarrow \ \ (\varrho_0, \varrho_1) \in \mathcal{R}.\end{aligned}$$

On repeated use of $\mathcal{R}$-preserving property, we have

$$(\varrho_\hbar, \varrho_{\hbar+1}) \in \mathcal{R}.$$

Also, by assumption (ii), we have

$$(\varrho_0, \Gamma^k(\varrho_0)) \in \mathcal{R}, \text{ where } k \in \mathbb{N}.$$

Therefore, using $\mathcal{R}$-preserving property, we can easily prove that $\{\varrho_\hbar\}$ is an $\mathcal{R}_{seq}$ i.e.,$(\varrho_\hbar, \varrho_{\hbar+k}) \in \mathcal{R}$, for each $\hbar, k \in \mathbb{N}$.
Using (3.3.13), we have

$$\begin{aligned}\varpi(\varrho_{\hbar+1}, \varrho_\hbar) &= \varpi(\Gamma\varrho_\hbar, \Gamma\varrho_{\hbar-1}) \preceq \mathcal{A}\Bigg(\varpi(\varrho_\hbar, \Gamma\varrho_\hbar) + \varpi(\varrho_{\hbar-1}, \Gamma\varrho_{\hbar-1})\Bigg)\\ &= \mathcal{A}\Bigg(\varpi(\varrho_\hbar, \varrho_{\hbar+1}) + \varpi(\varrho_{\hbar-1}, \varrho_\hbar)\Bigg).\end{aligned}$$

As, $\|\mathcal{A}\| < \frac{1}{2}$, by Lemma 1.2.34, $(I_\mathbb{B} - \mathcal{A})^{-1}$ exists. Also, $\|(I_\mathbb{B} - \mathcal{A})^{-1}\mathcal{A}\| < 1$. Therefore,

$$(I_\mathbb{B} - \mathcal{A})\varpi(\varrho_{\hbar+1}, \varrho_\hbar) \preceq \mathcal{A}(\varpi(\varrho_\hbar, \varrho_{\hbar-1})).$$

implies

$$\begin{aligned}\varpi(\varrho_{\hbar+1}, \varrho_\hbar) &\preceq (I_\mathbb{B} - \mathcal{A})^{-1}\mathcal{A}(\varpi(\varrho_\hbar, \varrho_{\hbar-1}))\\ &\preceq \Bigg((I_\mathbb{B} - \mathcal{A})^{-1}\mathcal{A}\Bigg)^2 \varpi(\varrho_{\hbar-1}, \varrho_{\hbar-2})\\ &\cdots\\ &= \Bigg((I_\mathbb{B} - \mathcal{A})^{-1}\mathcal{A}\Bigg)^\hbar \varpi(\varrho_1, \varrho_0).\end{aligned}$$

Let $\varpi(\varrho_1, \varrho_0) = \beta$ and $(I_\mathbb{B} - \mathcal{A})^{-1}\mathcal{A} = t$. Then,

$$\varpi(\varrho_{\hbar+1}, \varrho_\hbar) \preceq t^\hbar\beta \preceq \|t^\hbar\beta\| I_\mathbb{B} \preceq \|\beta\|\|t\|^\hbar I_\mathbb{B}.$$

As, $\|t\| < 1 \Rightarrow \|t\|^\hbar \to 0$. Therefore,

$$\varpi(\varrho_{\hbar+1}, \varrho_\hbar) \to \theta_\mathbb{B}. \tag{3.3.14}$$

Also, $\varpi_{\varrho_\hbar\varrho_{\hbar+1}} = \min\Bigg\{\varpi(\varrho_\hbar, \varrho_\hbar), \varpi(\varrho_{\hbar+1}, \varrho_{\hbar+1})\Bigg\} \preceq \varpi(\varrho_{\hbar+1}, \varrho_\hbar) \to \theta_\mathbb{B}$, as $\hbar \to \infty$. Hence,

$$\lim_{\hbar\to\infty} \varpi(\varrho_{\hbar,}, \varrho_\hbar) \to \theta_\mathbb{B}. \tag{3.3.15}$$

Using (3.3.15), we have

$$\lim_{\hbar,m\to\infty} \varpi_{\varrho_\hbar\varrho_m} = \theta_\mathbb{B}. \tag{3.3.16}$$

For $\mathfrak{m} \geq \hbar$, consider

$$\begin{aligned}
\varpi(\varrho_m, \varrho_\hbar) - \varpi_{\varrho_m \varrho_\hbar} &= \varpi(\varrho_m, \varrho_{m-1}) - \varpi_{\varrho_m \varrho_{m-1}} \\
&+ \varpi(\varrho_{m-1}, \varrho_{m-2}) - \varpi_{\varrho_{m-1} \varrho_{m-2}} \\
&... +\varpi(\varrho_{\hbar+1}, \varrho_\hbar) - \varpi_{\varrho_{\hbar+1} \varrho_\hbar} \\
&\preceq \varpi(\varrho_m, \varrho_{m-1}) + \varpi(\varrho_{m-1}, \varrho_{m-2}) ... + \varpi(\varrho_{\hbar+1}, \varrho_\hbar) \\
&\preceq \left((I_{\mathbb{B}} - \mathcal{A})^{-1}\mathcal{A}\right)^{m-1} \beta + \left((I_{\mathbb{B}} - \mathcal{A})^{-1}\mathcal{A}\right)^{m-2} \beta + \\
&... + \left((I_{\mathbb{B}} - \mathcal{A})^{-1}\mathcal{A}\right)^{\hbar} \beta \\
&= t^{m-1}\beta + t^{m-2}\beta + ... + t^\hbar \beta = \sum_{k=\hbar}^{m-1} t^k \beta \\
&\preceq \sum_{k=\hbar}^{\infty} t^k \beta \preceq \sum_{k=\hbar}^{\infty} \|t^k \beta\| I_{\mathbb{B}} \preceq \sum_{k=\hbar}^{\infty} \|\beta\| \|t^k\| I_{\mathbb{B}} \\
&\preceq \|\beta\| \sum_{k=\hbar}^{\infty} \|t\|^k I_{\mathbb{B}} = \|\beta\| \frac{\|t\|^\hbar}{1 - \|t\|} I_{\mathbb{B}} \to \theta_{\mathbb{B}}.
\end{aligned}$$

Hence,

$$\varpi(\varrho_m, \varrho_\hbar) - \varpi_{\varrho_m \varrho_\hbar} \to \theta_{\mathbb{B}}.$$

Using (3.3.15), we have

$$M_{\varrho_m \varrho_\hbar} - \varpi_{\varrho_m \varrho_\hbar} \to \theta_{\mathbb{B}}.$$

Hence, $\{\varrho_\hbar\}$ is $\mathfrak{m}_{\mathcal{R}}$-$C_{seq}$. As, $(\Omega, \mathbb{B}, \mathcal{R}, \varpi)$ be a $\mathcal{R}$-complete C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-MS and $\{\varrho_\hbar\}$ is $\mathfrak{m}_{\mathcal{R}}$ -C_{seq}. Hence, $\exists \varrho \in \Omega$ s.t.

$$\varpi(\varrho_\hbar, \varrho) - \varpi_{\varrho_\hbar \varrho} \to \theta_{\mathbb{B}} \text{ and } M_{\varrho_\hbar \varrho} - \varpi_{\varrho_\hbar \varrho} \to \theta_{\mathbb{B}} \text{ as } \hbar \to \infty. \tag{3.3.17}$$

Also, using (3.3.15),

$$\varpi_{\varrho_\hbar \varrho} = \min \left\{ \varpi(\varrho_\hbar, \varrho_\hbar), \varpi(\varrho, \varrho) \right\} \preceq \varpi(\varrho_\hbar, \varrho_\hbar) \to \theta_{\mathbb{B}}. \tag{3.3.18}$$

Using (3.3.18) and Remark 3.2.15, we have

$$\varpi(\varrho_\hbar, \varrho) \to \theta_{\mathbb{B}}, M_{\varrho_\hbar \varrho} \to \theta_{\mathbb{B}} \text{ and } \varpi(\varrho, \varrho) = \theta_{\mathbb{B}}. \tag{3.3.19}$$

Also,

$$\varpi_{\varrho \Gamma \varrho} = \min \left\{ \varpi(\varrho, \varrho), \varpi(\Gamma\varrho, \Gamma\varrho) \right\} \preceq \varpi(\varrho, \varrho) = \theta_{\mathbb{B}} \Rightarrow \varpi_{\varrho \Gamma \varrho} = \theta_{\mathbb{B}}. \tag{3.3.20}$$

Using (3.3.13), (3.3.20) and the triangle inequality, we have

$$\begin{aligned}
\varpi(\varrho,\Gamma\varrho) &= \varpi(\varrho,\Gamma\varrho)-\varpi_{\varrho\Gamma\varrho}\preceq \varpi(\varrho,\varrho_\hbar)-\varpi_{\varrho_\hbar\varrho}+\varpi(\varrho_\hbar,\Gamma\varrho)-\varpi_{\varrho_\hbar\Gamma\varrho}\\
&\preceq \limsup_{\hbar\to\infty}\varpi(\varrho,\varrho_\hbar)-\varpi_{\varrho_\hbar\varrho}+\varpi(\varrho_\hbar,\Gamma\varrho)-\varpi_{\varrho_\hbar\Gamma\varrho}\\
&= \limsup_{\hbar\to\infty}\varpi(\varrho_\hbar,\Gamma\varrho)-\varpi_{\varrho_\hbar\Gamma\varrho},
\end{aligned}$$

or

$$\varpi(\varrho,\Gamma\varrho)\preceq \limsup_{\hbar\to\infty}\varpi(\varrho_\hbar,\Gamma\varrho)-\varpi_{\varrho_\hbar\Gamma\varrho}. \tag{3.3.21}$$

Using (3.3.13), (3.3.18) and (3.3.19) in (3.3.21), we have

$$\begin{aligned}
\varpi(\varrho,\Gamma\varrho) &\preceq \limsup_{\hbar\to\infty}\varpi(\varrho_\hbar,\Gamma\varrho)-\varpi_{\varrho_\hbar\Gamma\varrho}\\
&\preceq \limsup_{\hbar\to\infty}\varpi(\varrho_\hbar,\Gamma\varrho)\\
&= \limsup_{\hbar\to\infty}\varpi(\Gamma\varrho_{\hbar-1},\Gamma\varrho)\\
&\preceq \limsup_{\hbar\to\infty}\mathcal{A}\Big(\varpi(\varrho_{\hbar-1},\Gamma\varrho_{\hbar-1})+\varpi(\varrho,\Gamma\varrho)\Big)\\
&= \limsup_{\hbar\to\infty}\mathcal{A}\Big(\varpi(\varrho_{\hbar-1},\varrho_\hbar)+\varpi(\varrho,\Gamma\varrho)\Big)=\mathcal{A}\,\varpi(\varrho,\Gamma\varrho)
\end{aligned}$$

This implies

$$\|\varpi(\varrho,\Gamma\varrho)\| \;\leq\; \|\mathcal{A}\varpi(\varrho,\Gamma\varrho)\|\leq\|\mathcal{A}\|\|\varpi(\varrho,\Gamma\varrho)\|.$$

Since $\|\mathcal{A}\|<\frac{1}{2}$ implies $\|\varpi(\varrho,\Gamma\varrho)\|=0$, therefore

$$\varpi(\varrho,\Gamma\varrho)=\theta_{\mathbb{B}}. \tag{3.3.22}$$

Now, consider

$$\theta_{\mathbb{B}}\preceq\varpi(\Gamma\varrho,\Gamma\varrho)\;\preceq\;\mathcal{A}\Big(\varpi(\varrho,\Gamma\varrho)+\varpi(\varrho,\Gamma\varrho)\Big)=2\mathcal{A}\,\varpi(\varrho,\Gamma\varrho)=\theta_{\mathbb{B}}.$$

Hence,

$$\varpi(\Gamma\varrho,\Gamma\varrho)=\theta_{\mathbb{B}}. \tag{3.3.23}$$

Using (3.3.19), (3.3.22) and (3.3.23), we have $\varpi(\varrho,\Gamma\varrho)=\varpi(\varrho,\varrho)=\varpi(\Gamma\varrho,\Gamma\varrho)$ implies $\varrho=\Gamma\varrho$, i.e., ϱ is a fixed point of Γ.
Uniqueness: let $\vartheta\neq\varrho\in\Omega$ be another fixed point of Γ with $\varpi(\vartheta,\vartheta)=\theta_{\mathbb{B}}$.

From assumption (ii), $(\varrho_0, \vartheta) \in \mathcal{R}$. Using $\mathcal{R}$-preserving property of Γ, we have $(\Gamma^{\hbar}(\varrho_0), \Gamma^{\hbar}(\vartheta)) \in \mathcal{R}\ \forall \hbar \in N$. Then, using (3.3.13), we have

$$\begin{aligned}
\varpi(\varrho_{\hbar}, \vartheta) &= \varpi(\Gamma^{\hbar}(\varrho_0), \Gamma^{\hbar}(\vartheta)) = \varpi\left(\Gamma\left(\Gamma^{\hbar-1}(\varrho_0)\right), \Gamma\left(\Gamma^{\hbar-1}(\vartheta)\right)\right) \\
&\preceq \mathcal{A}\left(\varpi(\varrho_0, \Gamma\left(\Gamma^{\hbar-1}(\varrho_0)\right) + \varpi(\vartheta, \Gamma\left(\Gamma^{\hbar-1}(\vartheta)\right)\right) \\
&= \mathcal{A}\left(\varpi(\varrho_0), \Gamma^{\hbar}(\varrho_0)\right) = \mathcal{A}\left(\varpi(\varrho_0, \varrho_{\hbar})\right) \qquad (3.3.24)
\end{aligned}$$

Taking limit as $\hbar \to \infty$ on both side of (3.3.24), we have

$$\varpi(\varrho, \vartheta) = \theta_{\mathbb{B}}.$$

Therefore, $\varpi(\varrho, \vartheta) = \varpi(\varrho, \varrho) = \varpi(\vartheta, \vartheta) = \theta_{\mathbb{B}}$. This implies $\varrho = \vartheta$. Hence, Γ possesses exactly one fixed point. $\square$

Theorem 3.3.4. *Let $(\Omega, \mathbb{B}, \mathcal{R}, \varpi)$ be an $\mathcal{R}$-complete C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-MS and $\Gamma : \Omega \to \Omega$ satisfying:*

(i) Γ is $\mathcal{R}$-preserving;

(ii) $\exists$ an $\varrho_0 \in \Omega$ s.t. $(\varrho_0, \vartheta) \in \mathcal{R}$, $\forall \vartheta \in \Omega$;

(iii) for $\varrho_0 \in \Omega$, $\varpi(\Gamma^{\hbar+1}\varrho_0, \Gamma^{\hbar+1}\varrho_0) \preceq \varpi(\Gamma^{\hbar}\varrho_0, \Gamma^{\hbar}\varrho_0)$, $\forall \hbar \in \mathbb{N}$;

(iv) $\exists$ an $\mathcal{A} \in \mathbb{A}'_+$ with $\|\mathcal{A}\| < \frac{1}{2}$, s.t.

$$\varpi(\Gamma\varrho, \Gamma\vartheta) \preceq \mathcal{A}\Big(\varpi(\varrho, \Gamma\vartheta) + \varpi(\vartheta, \Gamma\varrho)\Big), \qquad (3.3.25)$$

$\forall \varrho, \vartheta \in \Omega$ s.t. $(\varrho, \vartheta) \in \mathcal{R}$. Then, Γ possesses unique fixed point.

Proof. For $\varrho_0 \in \Omega$ satisfying condition (ii), define the iterative sequence $\{\varrho_{\hbar}\}$ s.t. $\varrho_{\hbar} = \Gamma\varrho_{\hbar-1}\ \forall \hbar \in \mathbb{N}$. If $\varrho_{\hbar} = \varrho_{\hbar-1}$ for some $\hbar$, then $\Gamma\varrho_{\hbar-1} = \varrho_{\hbar} = \varrho_{\hbar-1}$ implies $\varrho_{\hbar-1}$ is a fixed point of Γ. Hence, the result holds. Now, consider $\varrho_{\hbar} \neq \varrho_{\hbar-1}$, $\forall \hbar \in \mathbb{N}$. As, $\varrho_0 \in \Omega$ satisfying condition (ii) and Γ is $\mathcal{R}$-preserving. Hence,

$$\begin{aligned}
(\varrho_0, \vartheta) \in \mathcal{R}\ \ \forall \vartheta \in \Omega &\Rightarrow (\varrho_0, \Gamma\varrho_0) \in \mathcal{R} \\
&\Rightarrow (\varrho_0, \varrho_1) \in \mathcal{R}.
\end{aligned}$$

On repeated use of $\mathcal{R}$-preserving property, we have

$$(\varrho_{\hbar}, \varrho_{\hbar+1}) \in \mathcal{R}.$$

Also, by assumption (ii), we have

$$(\varrho_0, \Gamma^k(\varrho_0)) \in \mathcal{R}, \text{ where } k \in \mathbb{N}.$$

Therefore, using $\mathcal{R}$-preserving property, we can easily prove that $\{\varrho_\hbar\}$ is an $\mathcal{R}_{seq}$ i.e.,$(\varrho_\hbar, \varrho_{\hbar+k}) \in \mathcal{R}$, for each $\hbar, k \in \mathbb{N}$.
Now, using assumption (iii), we have

$$\varpi_{\varrho_{\hbar-1}\varrho_\hbar} = \varpi(\varrho_\hbar, \varrho_\hbar), \varpi_{\varrho_\hbar\varrho_{\hbar+1}} = \varpi(\varrho_{\hbar+1}, \varrho_{\hbar+1}) \text{ and } \varpi_{\varrho_{\hbar-1}\varrho_{\hbar+1}} = \varpi(\varrho_{\hbar+1}, \varrho_{\hbar+1}).$$

Using (3.3.25), we have

$$\begin{aligned}
\varpi(\varrho_{\hbar+1}, \varrho_\hbar) = \varpi(\Gamma\varrho_\hbar, \Gamma\varrho_{\hbar-1}) &\preceq \mathcal{A}\Big(\varpi(\varrho_\hbar, \Gamma\varrho_{\hbar-1}) + \varpi(\varrho_{\hbar-1}, \Gamma\varrho_\hbar)\Big) \\
&= \mathcal{A}\Big(\varpi(\varrho_\hbar, \varrho_\hbar) + \varpi(\varrho_{\hbar-1}, \varrho_{\hbar+1})\Big) \\
&\preceq \mathcal{A}\Big(\varpi(\varrho_\hbar, \varrho_\hbar) + \varpi(\varrho_{\hbar-1}, \varrho_\hbar) - \varpi_{\varrho_{\hbar-1}\varrho_\hbar} \\
&\quad +\varpi(\varrho_\hbar, \varrho_{\hbar+1}) - \varpi_{\varrho_\hbar\varrho_{\hbar+1}} + \varpi_{\varrho_{\hbar-1}\varrho_{\hbar+1}}\Big) \\
&\preceq \mathcal{A}\Big(\varpi(\varrho_{\hbar+1}, \varrho_\hbar) + \varpi(\varrho_\hbar, \varrho_{\hbar-1})\Big).
\end{aligned}$$

Since $\|\mathcal{A}\| < \frac{1}{2}$, therefore by Lemma 1.2.34, $(I_\mathbb{B} - \mathcal{A})^{-1}$ exists. Also, $\|(I_\mathbb{B} - \mathcal{A})^{-1}\mathcal{A}\| < 1$. Thus,

$$(I_\mathbb{B} - \mathcal{A})\varpi(\varrho_{\hbar+1}, \varrho_\hbar) \preceq Am(\varrho_\hbar, \varrho_{\hbar-1}).$$

implies

$$\begin{aligned}
\varpi(\varrho_{\hbar+1}, \varrho_\hbar) &\preceq (I_\mathbb{B} - \mathcal{A})^{-1}\mathcal{A}(\varpi(\varrho_\hbar, \varrho_{\hbar-1})) \\
&\preceq \Big((I_\mathbb{B} - \mathcal{A})^{-1}\mathcal{A}\Big)^2 \varpi(\varrho_{\hbar-1}, \varrho_{\hbar-2}) \\
&\cdots \\
&= \Big((I_\mathbb{B} - \mathcal{A})^{-1}\mathcal{A}\Big)^\hbar \varpi(\varrho_1, \varrho_0).
\end{aligned}$$

Let $\varpi(\varrho_1, \varrho_0) = \beta$ and $(I_\mathbb{B} - \mathcal{A})^{-1}\mathcal{A} = t$. Then,

$$\varpi(\varrho_{\hbar+1}, \varrho_\hbar) \preceq t^\hbar\beta \preceq \|t^\hbar\beta\| I_\mathbb{B} \preceq \|\beta\|\|t\|^\hbar I_\mathbb{B}.$$

Now, $\|t\| < 1 \Rightarrow \|t\|^{\hbar} \to 0$. Hence,

$$\varpi(\varrho_{\hbar+1}, \varrho_{\hbar}) \to \theta_{\mathbb{B}}. \tag{3.3.26}$$

Also, $\varpi_{\varrho_{\hbar}\varrho_{\hbar+1}} = \min\left\{\varpi(\varrho_{\hbar}, \varrho_{\hbar}), \varpi(\varrho_{\hbar+1}, \varrho_{\hbar+1})\right\} \preceq \varpi(\varrho_{\hbar+1}, \varrho_{\hbar}) \to \theta_{\mathbb{B}}$ as $\hbar \to \infty$. Hence,

$$\lim_{\hbar\to\infty} \varpi(\varrho_{\hbar,}, \varrho_{\hbar}) \to \theta_{\mathbb{B}} \tag{3.3.27}$$

and

$$\lim_{\hbar,\mathfrak{m}\to\infty} \varpi_{\varrho_{\hbar}\varrho_{m}} = \theta_{\mathbb{B}}. \tag{3.3.28}$$

For $m \geq \hbar$, consider

$$\begin{aligned}
\varpi(\varrho_m, \varrho_{\hbar}) - \varpi_{\varrho_m\varrho_{\hbar}} &= \varpi(\varrho_m, \varrho_{m-1}) - \varpi_{\varrho_m\varrho_{m-1}} \\
&+ \varpi(\varrho_{m-1}, \varrho_{m-2}) - \varpi_{\varrho_{m-1}\varrho_{m-2}} \\
&... +\varpi(\varrho_{\hbar+1}, \varrho_{\hbar}) - \varpi_{\varrho_{\hbar+1}\varrho_{\hbar}} \\
&\preceq \varpi(\varrho_m, \varrho_{m-1}) + \varpi(\varrho_{m-1}, \varrho_{m-2}) ... + \varpi(\varrho_{\hbar+1}, \varrho_{\hbar}) \\
&\preceq \left((I_{\mathbb{B}} - \mathcal{A})^{-1}\mathcal{A}\right)^{\mathfrak{m}-1}\beta + \left((I_{\mathbb{B}} - \mathcal{A})^{-1}\mathcal{A}\right)^{\mathfrak{m}-2}\beta + \\
&... +\left((I_{\mathbb{B}} - \mathcal{A})^{-1}\mathcal{A}\right)^{\hbar}\beta \\
&= t^{m-1}\beta + t^{m-2}\beta + ... + t^{\hbar}\beta = \sum_{k=\hbar}^{m-1} t^k\beta \\
&\preceq \sum_{k=\hbar}^{\infty} t^k\beta \preceq \sum_{k=\hbar}^{\infty} \|t^k\beta\| I_{\mathbb{B}} \preceq \sum_{k=\hbar}^{\infty} \|\beta\|\|t^k\| I_{\mathbb{B}} \preceq \|\beta\| \sum_{k=\hbar}^{\infty} \|t\|^k I_{\mathbb{B}} \\
&= \|\beta\| \frac{\|t\|^{\hbar}}{1 - \|t\|} I_{\mathbb{B}} \to \theta_{\mathbb{B}} \text{ as } \hbar \to \infty.
\end{aligned}$$

Hence,

$$\lim_{\hbar,m\to\infty} \varpi(\varrho_m, \varrho_{\hbar}) - \varpi_{\varrho_m\varrho_{\hbar}} = \theta_{\mathbb{B}}.$$

Similarly, using (3.3.27), we have

$$\lim_{\hbar,m\to\infty} M_{\varrho_m\varrho_{\hbar}} - \varpi_{\varrho_m\varrho_{\hbar}} = \theta_{\mathbb{B}}.$$

Hence, $\{\varrho_{\hbar}\}$ is $\mathfrak{m}_{\mathcal{R}}$-$C_{seq}$. As, $(\Omega, \mathbb{B}, \mathcal{R}, \varpi)$ be a $\mathcal{R}$-complete C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-MS and $\{\varrho_{\hbar}\}$ is $\mathfrak{m}_{\mathcal{R}}$ -C_{seq}. Hence, $\exists$ some $\varrho \in \Omega$ s.t.

$$\varpi(\varrho_{\hbar}, \varrho) - \varpi_{\varrho_{\hbar}\varrho} \to \theta_{\mathbb{B}} \text{ and } M_{\varrho_{\hbar}\varrho} - \varpi_{\varrho_{\hbar}\varrho} \to \theta_{\mathbb{B}} \text{ as } \hbar \to \infty. \tag{3.3.29}$$

Also, using (3.3.28),

$$\varpi_{\varrho_\hbar\varrho} = \min\left\{\varpi(\varrho_\hbar,\varrho_\hbar),\varpi(\varrho,\varrho)\right\} \preceq \varpi(\varrho_\hbar,\varrho_\hbar) \to \theta_{\mathbb{B}}. \tag{3.3.30}$$

Now, using (3.3.30) and Remark 3.2.15, we have

$$\varpi(\varrho_\hbar,\varrho) \to \theta_{\mathbb{B}},\ M_{\varrho_\hbar\varrho} \to \theta_{\mathbb{B}} \text{ and } \varpi(\varrho,\varrho) = \theta_{\mathbb{B}}. \tag{3.3.31}$$

Also,

$$\varpi_{\varrho\Gamma\varrho} = \min\left\{\varpi(\varrho,\varrho),\varpi(\Gamma\varrho,\Gamma\varrho)\right\} \preceq \varpi(\varrho,\varrho) = \theta_{\mathbb{B}} \Rightarrow \varpi_{\varrho\Gamma\varrho} = \theta_{\mathbb{B}}. \tag{3.3.32}$$

Using (3.3.25), (3.3.32) and the triangle inequality, we have

$$\begin{aligned}
\varpi(\varrho,\Gamma\varrho) &= \varpi(\varrho,\Gamma\varrho) - \varpi_{\varrho\Gamma\varrho} \preceq \varpi(\varrho,\varrho_\hbar) - \varpi_{\varrho_\hbar\varrho} + \varpi(\varrho_\hbar,\Gamma\varrho) - \varpi_{\varrho_\hbar\Gamma\varrho}\\
&\preceq \limsup_{\hbar\to\infty} \varpi(\varrho,\varrho_\hbar) - \varpi_{\varrho_\hbar\varrho} + \varpi(\varrho_\hbar,\Gamma\varrho) - \varpi_{\varrho_\hbar\Gamma\varrho}\\
&= \limsup_{\hbar\to\infty} \varpi(\varrho_\hbar,\Gamma\varrho) - \varpi_{\varrho_\hbar\Gamma\varrho}.
\end{aligned}$$

Using (3.3.25), (3.3.30) and (3.3.31), we have

$$\begin{aligned}
\varpi(\varrho,\Gamma\varrho) &\preceq \limsup_{\hbar\to\infty} \varpi(\varrho_\hbar,\Gamma\varrho) - \varpi_{\varrho_\hbar\Gamma\varrho} \preceq \limsup_{\hbar\to\infty} \varpi(\Gamma\varrho_{\hbar-1},\Gamma\varrho)\\
&\preceq \limsup_{\hbar\to\infty} \mathcal{A}\Big(\varpi(\varrho_{\hbar-1},\Gamma\varrho) + \varpi(\varrho,\Gamma\varrho_{\hbar-1})\Big)\\
&\preceq \limsup_{\hbar\to\infty} \mathcal{A}\Big(\varpi(\varrho_{\hbar-1},\Gamma\varrho) + \varpi(\varrho,\varrho_\hbar)\Big)\\
&\preceq \limsup_{\hbar\to\infty} \mathcal{A}\Big(\varpi(\varrho_{\hbar-1},\Gamma\varrho\Big)\\
&\preceq \limsup_{\hbar\to\infty} \mathcal{A}\Big(\varpi(\varrho_{\hbar-1},\varrho) - \varpi_{\varrho_{\hbar-1}\varrho} + \varpi(\varrho,\Gamma\varrho) - \varpi_{\varrho\Gamma\varrho} + \varpi_{\varrho_{\hbar-1}\Gamma\varrho}\Big),
\end{aligned} \tag{3.3.33}$$

or

$$\varpi(\varrho,\Gamma\varrho) \preceq \mathcal{A}\Big(\varpi(\varrho,\Gamma\varrho) + \varpi_{\varrho_{\hbar-1}\Gamma\varrho}\Big). \tag{3.3.34}$$

Using (3.3.27), we have

$$\varpi_{\varrho_{\hbar-1}\Gamma\varrho} = \min\left\{\varpi(\varrho_{\hbar-1},\varrho_{\hbar-1}),\varpi(\Gamma\varrho,\Gamma\varrho)\right\} \preceq \varpi(\varrho_{\hbar-1},\varrho_{\hbar-1}) \Rightarrow \varpi_{\varrho_{\hbar-1}\Gamma\varrho} = \theta_{\mathbb{B}}. \tag{3.3.35}$$

Hence, using (3.3.35) in (3.3.34), we have

$$\varpi(\varrho, \Gamma\varrho) \preceq \mathcal{A}\varpi(\varrho, \Gamma\varrho) \text{ i.e., } \|\varpi(\varrho, \Gamma\varrho)\| \leq \|\mathcal{A}\|\|\varpi(\varrho, \Gamma\varrho)\|.$$

Now, since $\|\mathcal{A}\| < \frac{1}{2}$ implies $\|\varpi(\varrho, \Gamma\varrho)\| = 0$, therefore

$$\varpi(\varrho, \Gamma\varrho) = \theta_{\mathbb{B}}. \tag{3.3.36}$$

Thus,

$$\theta_{\mathbb{B}} \preceq \varpi(\Gamma\varrho, \Gamma\varrho) \quad \preceq \quad \mathcal{A}(\varpi(\varrho, \Gamma\varrho) + \varpi(\varrho, \Gamma\varrho)) = 2Am(\varrho, \Gamma\varrho) = \theta_{\mathbb{B}}.$$

Hence,

$$\varpi(\Gamma\varrho, \Gamma\varrho) = \theta_{\mathbb{B}}. \tag{3.3.37}$$

Using (3.3.31), (3.3.36) and (3.3.37), we have $\varpi(\varrho, \Gamma\varrho) = \varpi(\varrho, \varrho) = \varpi(\Gamma\varrho, \Gamma\varrho)$ implies $\varrho = \Gamma\varrho$, i.e., ϱ is a fixed point of Γ.

Uniqueness: let $\vartheta \neq \varrho \in \Omega$ be another fixed point of Γ with $\varpi(\vartheta, \vartheta) = \theta_{\mathbb{B}}$. From assumption (ii), $(\varrho_0, \vartheta) \in \mathcal{R}$. Using $\mathcal{R}$-preserving property of Γ, we have $(\Gamma^{\hbar}(\varrho_0), \Gamma^{\hbar}(\vartheta)) \in \mathcal{R}$, for $\hbar \in N$. Then, using (3.3.25), we have

$$\begin{aligned}
\varpi(\varrho_{\hbar}, \vartheta) &= \varpi(\Gamma^{\hbar}(\varrho_0), \Gamma^{\hbar}(\vartheta)) = \varpi\left(\Gamma\left(\Gamma^{\hbar-1}(\varrho_0)\right), \Gamma\left(\Gamma^{\hbar-1}(\vartheta)\right)\right) \\
&\preceq \mathcal{A}\left(\varpi\left(\Gamma^{\hbar-1}(\varrho_0), \Gamma^{\hbar}(\vartheta)\right) + \varpi\left(\Gamma^{\hbar-1}(\vartheta), \Gamma^{\hbar}(\varrho_0)\right)\right) \\
&= \mathcal{A}\left(\varpi(\varrho_{\hbar-1}, \vartheta) + \varpi(\varrho_{\hbar}, \vartheta)\right),
\end{aligned}$$

or

$$\begin{aligned}
(I_{\mathbb{B}} - \mathcal{A})\varpi(\varrho_{\hbar}, \vartheta) &\preceq \mathcal{A}\left(\varpi(\varrho_{\hbar} - 1, \vartheta)\right) \\
\Rightarrow \varpi(\varrho_{\hbar}, \vartheta) &\preceq \frac{\mathcal{A}}{I_{\mathbb{B}} - \mathcal{A}}\left(\varpi(\varrho_{\hbar} - 1, \vartheta)\right) \\
&\cdots \\
\Rightarrow \varpi(\varrho_{\hbar}, \vartheta) &\preceq \frac{\mathcal{A}^{\hbar}}{(I_{\mathbb{B}} - \mathcal{A})^{\hbar}}(\varpi(\varrho_0, \vartheta)).
\end{aligned} \tag{3.3.38}$$

Now, $\|\mathcal{A}\| < \frac{1}{2}$ implies $(I_{\mathbb{B}} - \mathcal{A})$ is invertible and $\|\frac{\mathcal{A}}{I_{\mathbb{B}} - \mathcal{A}}\| < 1$. Therefore, taking limit as $\hbar \to \infty$ on both side of (3.3.38), we have

$$\varpi(\varrho, \vartheta) = \theta_{\mathbb{B}}.$$

As, $\varpi(\varrho, \vartheta) = \varpi(\varrho, \varrho) = \varpi(\vartheta, \vartheta) = \theta_{\mathbb{B}}$. Hence, $\varrho = \vartheta$, i.e., Γ has exactly one fixed point. $\square$

Example 3.3.5. *Let $\Omega = [0, \infty)$ and $\mathbb{B} = M_2([0, \infty))$, with involution $\mathcal{A}^* = \mathcal{A}^t$ $\forall \mathcal{A} \in \mathbb{B}$, where $\mathcal{A}^t$ denotes the transpose of $\mathcal{A}$. For $\mathcal{A} = [a_{ij}]$, $\|\mathcal{A}\| = \max_{1 \leq i,j \leq 2} \left| a_{ij} \right|$. Let $\varpi : \Omega \times \Omega \to M_2(\mathbb{R})$ be defined as*

$$\varpi(\varrho, \vartheta) = \begin{bmatrix} \frac{\varrho+\vartheta}{2} & 0 \\ 0 & \frac{\varrho+\vartheta}{2} \end{bmatrix}.$$

For $P = [p_{ij}]$ and $Q = [q_{ij}]$, $P \preceq Q \Leftrightarrow p_{ij} \leq q_{ij}$, for $1 \leq i,\ j \leq 2$. $(\varrho, \vartheta) \in \mathcal{R} \Leftrightarrow \varrho = \vartheta$ or $\varrho\vartheta = 0$. Then $(\Omega, \mathbb{B}, \mathcal{R}, \varpi)$ is $\mathcal{R}$-complete C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-MS . Let Γ be

$$\Gamma(\varrho) = \begin{cases} \frac{\varrho}{7}, & \text{if } \varrho \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Now, we will prove that for $\mathcal{A} = \begin{bmatrix} \frac{1}{\sqrt{6}} & 0 \\ 0 & \frac{1}{\sqrt{6}} \end{bmatrix}$, Γ satisfies Theorem 3.3.2. As, relation $\mathcal{R}$ on Ω is defined as $(\varrho, \vartheta) \in \mathcal{R}$ if $\varrho = \vartheta$ or $\varrho.\vartheta = 0$. Then,

Case(i) For $\varrho = \vartheta \leq 1$, we have

$$\varpi(\Gamma\varrho, \Gamma\varrho) = \varpi\left(\frac{\varrho}{7}, \frac{\varrho}{7}\right) = \begin{bmatrix} \frac{1}{2}(\frac{\varrho}{7} + \frac{\varrho}{7}) & 0 \\ 0 & \frac{1}{2}(\frac{\varrho}{7} + \frac{\varrho}{7}) \end{bmatrix} = \begin{bmatrix} \frac{\varrho}{7} & 0 \\ 0 & \frac{\varrho}{7} \end{bmatrix},$$

$$\varpi(\varrho, \varrho) = \begin{bmatrix} \frac{\varrho+\varrho}{2} & 0 \\ 0 & \frac{\varrho+\varrho}{2} \end{bmatrix} = \begin{bmatrix} \varrho & 0 \\ 0 & \varrho \end{bmatrix}.$$

For $\mathcal{A} = \mathcal{A}^* = \begin{bmatrix} \frac{1}{\sqrt{6}} & 0 \\ 0 & \frac{1}{\sqrt{6}} \end{bmatrix}$, we have

$$\varpi(\Gamma\varrho, \Gamma\vartheta) \preceq \mathcal{A}^*\varpi(\varrho, \vartheta)\mathcal{A}.$$

Case(ii) For $\varrho = \vartheta > 1$, we have

$$\varpi(\Gamma\varrho, \Gamma\vartheta) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\varpi(\varrho, \vartheta) = \varpi(\varrho, \varrho) = \begin{bmatrix} \frac{\varrho+\varrho}{2} & 0 \\ 0 & \frac{\varrho+\varrho}{2} \end{bmatrix}.$$

Hence, for $\mathcal{A} = \mathcal{A}^* = \begin{bmatrix} \frac{1}{\sqrt{6}} & 0 \\ 0 & \frac{1}{\sqrt{6}} \end{bmatrix}$, we have

$$\varpi(\Gamma\varrho, \Gamma\vartheta) \preceq \mathcal{A}^*\varpi(\varrho, \vartheta)\mathcal{A}.$$

Let $\varrho \neq \vartheta$, then $\varrho.\vartheta = 0$ i.e., $\varrho = 0$ or $\vartheta = 0$. Without loss of generality, let $\vartheta = 0$.

Case(iii) For $\varrho > 1$ and $\vartheta = 0$, we have

$$\varpi(\Gamma\varrho, \Gamma\vartheta) = \varpi(0, 0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\varpi(\varrho,\vartheta) = \begin{bmatrix} \frac{\varrho}{2} & 0 \\ 0 & \frac{\varrho}{2} \end{bmatrix}.$$

Hence, for $\mathcal{A} = \mathcal{A}^* = \begin{bmatrix} \frac{1}{\sqrt{6}} & 0 \\ 0 & \frac{1}{\sqrt{6}} \end{bmatrix}$, we have

$$\varpi(\Gamma\varrho,\Gamma\vartheta) \preceq \mathcal{A}^*\varpi(\varrho,\vartheta)\mathcal{A}.$$

Case(iv) For $\varrho \leq 1$ and $\vartheta = 0$, we have

$$\varpi(\Gamma\varrho,\Gamma\vartheta) = \varpi\left(\frac{\varrho}{7}, 0\right) = \begin{bmatrix} \frac{1}{2}(\frac{\varrho}{7}) & 0 \\ 0 & \frac{1}{2}(\frac{\varrho}{7}) \end{bmatrix},$$

$$\varpi(\varrho,\vartheta) = \begin{bmatrix} \frac{\varrho}{2} & 0 \\ 0 & \frac{\varrho}{2} \end{bmatrix}.$$

Hence, for $\mathcal{A} = \mathcal{A}^* = \begin{bmatrix} \frac{1}{\sqrt{6}} & 0 \\ 0 & \frac{1}{\sqrt{6}} \end{bmatrix}$, we have

$$\varpi(\Gamma\varrho,\Gamma\vartheta) \preceq \mathcal{A}^*\varpi(\varrho,\vartheta)\mathcal{A}.$$

Thus, Γ satisfies Theorem (3.3.2). Hence, Γ possesses exactly one fixed point.

Example 3.3.6. *Let $\Omega = [0,\infty)$ and $\mathbb{B} = M_2([0,\infty))$, with involution $\mathcal{A}^* = \mathcal{A}^t$ $\forall \mathcal{A} \in \mathbb{B}$, where $\mathcal{A}^t$ denotes the transpose of $\mathcal{A}$. For $\mathcal{A} = [a_{ij}]$, $\|\mathcal{A}\| = \max\limits_{1\leq i,j\leq 2}\left|a_{ij}\right|$. Let $\varpi : \Omega \times \Omega \to M_2(\mathbb{R}_+)$ be defined as*

$$\varpi(\varrho,\vartheta) = \begin{bmatrix} \frac{\varrho+\vartheta}{2} & 0 \\ 0 & \frac{\varrho+\vartheta}{2} \end{bmatrix}.$$

Also, for $P = [p_{ij}]$ and $Q = [q_{ij}]$, $P \preceq Q \Leftrightarrow p_{ij} \leq q_{ij}$ $\forall 1 \leq i,j \leq 2$. Then $(\Omega,\mathbb{B},\mathcal{R},\varpi)$ is $\mathcal{R}$-complete C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-MS . $(\varrho,\vartheta) \in \mathcal{R} \Leftrightarrow \varrho = \vartheta$ or $\varrho\vartheta = 0$. Now Γ s.t.

$$\Gamma(\varrho) = \begin{cases} \frac{\varrho}{7}, & \textit{if } \varrho \leq 3 \\ \frac{\varrho}{\varrho+4}, & \textit{otherwise.} \end{cases}$$

Now, we will prove that for $\mathcal{A} = \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{6} \end{bmatrix}$, Γ satisfies Theorem 3.3.3. As, relation $\mathcal{R}$ defined as $(\varrho,\vartheta) \in \mathcal{R}$, if $\varrho = \vartheta$ or $\varrho\vartheta = 0$. Then

Case(i) For $\varrho = \vartheta \leq 3$, we have

$$\begin{aligned} \varpi(\Gamma\varrho,\Gamma\vartheta) = \varpi\left(\frac{\varrho}{7}, \frac{\varrho}{7}\right) &= \begin{bmatrix} \frac{1}{2}(\frac{\varrho}{7}+\frac{\varrho}{7}) & 0 \\ 0 & \frac{1}{2}(\frac{\varrho}{7}+\frac{\varrho}{7}) \end{bmatrix} \\ &\preceq \begin{bmatrix} \frac{\varrho}{7} & 0 \\ 0 & \frac{\varrho}{7} \end{bmatrix}, \end{aligned}$$

$$\varpi(\vartheta,\Gamma\vartheta)=\varpi(\varrho,\Gamma\varrho)=\begin{bmatrix}\frac{\varrho+\frac{\varrho}{7}}{2} & 0\\ 0 & \frac{\varrho+\frac{\varrho}{7}}{2}\end{bmatrix}.$$

Hence, for $\mathcal{A}=\begin{bmatrix}\frac{1}{6} & 0\\ 0 & \frac{1}{6}\end{bmatrix}$, $\varpi(\Gamma\varrho,\Gamma\vartheta)\preceq\mathcal{A}\Big(\varpi(\varrho,\Gamma\varrho)+\varpi(\vartheta,\Gamma\vartheta)\Big)$. Also, $\mathcal{A}\in\mathcal{A}'_+$.

Case(ii) For $\varrho=\vartheta>3$, we have

$$\begin{aligned}\varpi(\Gamma\varrho,\Gamma\vartheta)=\varpi\left(\frac{\varrho}{\varrho+4},\frac{\varrho}{\varrho+4}\right) &= \begin{bmatrix}\frac{1}{2}(\frac{\varrho}{\varrho+4}+\frac{\varrho}{\varrho+4}) & 0\\ 0 & \frac{1}{2}(\frac{\varrho}{\varrho+4}+\frac{\varrho}{\varrho+4})\end{bmatrix}\\ &\preceq \begin{bmatrix}\frac{1}{2}(\frac{\varrho}{7}+\frac{\varrho}{7}) & 0\\ 0 & \frac{1}{2}(\frac{\varrho}{7}+\frac{\varrho}{7})\end{bmatrix}\\ &= \begin{bmatrix}\frac{\varrho}{7} & 0\\ 0 & \frac{\varrho}{7}\end{bmatrix},\end{aligned}$$

$$\varpi(\vartheta,\Gamma\vartheta)=\varpi(\varrho,\Gamma\varrho)=\begin{bmatrix}\frac{\varrho+\frac{\varrho}{\varrho+4}}{2} & 0\\ 0 & \frac{\varrho+\frac{\varrho}{\varrho+4}}{2}\end{bmatrix}.$$

Hence, for $\mathcal{A}=\begin{bmatrix}\frac{1}{6} & 0\\ 0 & \frac{1}{6}\end{bmatrix}$, $\varpi(\Gamma\varrho,\Gamma\vartheta)\preceq\mathcal{A}\Big(\varpi(\varrho,\Gamma\varrho)+\varpi(\vartheta,\Gamma\vartheta)\Big)$. Also, $\mathcal{A}\in\mathcal{A}'_+$.

Case(iii) Without loss of generality, let $\vartheta=0$ and $\varrho\leq 3$. Then, we have

$$\varpi(\Gamma\varrho,\Gamma\vartheta)=\varpi\left(\frac{\varrho}{7},0\right) = \begin{bmatrix}\frac{1}{2}(\frac{\varrho}{7}) & 0\\ 0 & \frac{1}{2}(\frac{\varrho}{7})\end{bmatrix},$$

$$\varpi(\varrho,\Gamma\varrho)=\begin{bmatrix}\frac{\varrho+\frac{\varrho}{7}}{2} & 0\\ 0 & \frac{\varrho+\frac{\varrho}{7}}{2}\end{bmatrix},$$

$$\varpi(\vartheta,\Gamma\vartheta)=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}.$$

Hence, for $\mathcal{A}=\begin{bmatrix}\frac{1}{6} & 0\\ 0 & \frac{1}{6}\end{bmatrix}$, $\varpi(\Gamma\varrho,\Gamma\vartheta)\preceq\mathcal{A}\Big(\varpi(\varrho,\Gamma\varrho)+\varpi(\vartheta,\Gamma\vartheta)\Big)$. Also, $\mathcal{A}\in\mathcal{A}'_+$.

Case(iv) For $\vartheta=0$ and $\varrho>3$, we have

$$\varpi(\Gamma\varrho,\Gamma\vartheta)=\varpi\left(\frac{\varrho}{\varrho+4},0\right) = \begin{bmatrix}\frac{1}{2}(\frac{\varrho}{\varrho+4}) & 0\\ 0 & \frac{1}{2}(\frac{\varrho}{\varrho+4})\end{bmatrix},$$

$$\varpi(\varrho,\Gamma\varrho)=\begin{bmatrix}\frac{\varrho+\frac{\varrho}{\varrho+4}}{2} & 0\\ 0 & \frac{\varrho+\frac{\varrho}{\varrho+4}}{2}\end{bmatrix},$$

$$\varpi(\vartheta,\Gamma\vartheta)=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}.$$

Hence, for $\mathcal{A} = \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{6} \end{bmatrix}$, $\varpi(\Gamma\varrho, \Gamma\vartheta) \preceq \mathcal{A}\bigg(\varpi(\varrho, \Gamma\varrho) + \varpi(\vartheta, \Gamma\vartheta)\bigg)$. Also, $\mathcal{A} \in \mathcal{A}'_{+}$. Thus, Γ satisfies the conditions of Theorem (3.3.3). Hence, Γ possesses exactly one fixed point.Hence, by Theorem (3.3.3), Γ has exactly one fixed point.

Example 3.3.7. *Let $\Omega = [0, 1)$ and $\varpi : \Omega \times \Omega \to \mathbb{R}$ be C^*_{AV}-$\mathfrak{m}$- metric defined on Ω as $\varpi(\varrho, \vartheta) = \frac{\varrho+\vartheta}{2}$. Then for $\mathbb{B} = \mathbb{R}$ with $a^* = a$, $\|a\| = |a|$ and $a \preceq b \Leftrightarrow a \leq b$. $(\Omega, \mathbb{B}, \mathcal{R}, \varpi)$ is $\mathcal{R}$-complete C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-MS, where $a\mathcal{R}b$ iff $a = b$ or $a.b = 0$. Let Γ be a self mapping defined on Ω as:*

$$\Gamma(\varrho) = \begin{cases} 0, & \text{if } \varrho \leq \frac{3}{4} \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

$\varrho\mathcal{R}\vartheta$ implies $\varrho = \vartheta$ or $\varrho.\vartheta = 0$. Now $\varrho = \vartheta$ implies $\Gamma\varrho = \Gamma\vartheta$ i.e., $(\Gamma\varrho, \Gamma\vartheta) \in \mathcal{R}$. If $\varrho \neq \vartheta$ i.e., $\varrho = 0$ or $\vartheta = 0$ implies $\Gamma\varrho = 0$ or $\Gamma\vartheta = 0$. Hence, $(\Gamma\varrho, \Gamma\vartheta) \in \mathcal{R}$ $\forall(\varrho, \vartheta) \in \mathcal{R}$ i.e., Γ is $\mathcal{R}$-preserving.

Case(I) *Let $\varrho = \vartheta$, then*

(i) *For $\varrho = \vartheta \leq \frac{3}{4}$,*

$$\varpi(\Gamma\varrho, \Gamma\vartheta) = \varpi(\Gamma\varrho, \Gamma\varrho) = \varpi(0, 0) = 0,$$

$$\varpi(\varrho, \vartheta) = \varpi(\varrho, \varrho) = \varrho.$$

For $\mathcal{A}^* = \mathcal{A} = \sqrt{\frac{2}{3}}$

$$\varpi(\Gamma\varrho, \Gamma\vartheta) \preceq \mathcal{A}^*\varpi(\varrho, \vartheta)\mathcal{A}.$$

(ii) *For $\varrho = \vartheta > \frac{3}{4}$,*

$$\varpi(\Gamma\varrho, \Gamma\vartheta) = \varpi(\Gamma\varrho, \Gamma\varrho) = \varpi\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2},$$

$$\varpi(\varrho, \vartheta) = \varpi(\varrho, \varrho) = \varrho.$$

For $\mathcal{A}^* = \mathcal{A} = \sqrt{\frac{2}{3}}$, we have

$$\varpi(\Gamma\varrho, \Gamma\vartheta) \preceq \mathcal{A}^*\varpi(\varrho, \vartheta)\mathcal{A}.$$

Csae(II) *Let $\varrho \neq \vartheta$ implies $\varrho.\vartheta = 0$. If $\vartheta = 0$. Then,*

(i) *For $\varrho \leq \frac{3}{4}$ and $\vartheta = 0$, we have*

$$\varpi(\Gamma\varrho, \Gamma\vartheta) = \varpi(0, 0) = 0.$$

Hence, for $\mathcal{A}^* = \mathcal{A} = \sqrt{\frac{2}{3}}$, we have

$$\varpi(\Gamma\varrho, \Gamma\vartheta) \preceq \mathcal{A}^*\varpi(\varrho, \vartheta)\mathcal{A}.$$

(ii) *For $\varrho > \frac{3}{4}$ and $\vartheta = 0$, we have*

$$\varpi(\Gamma\varrho, \Gamma\vartheta) = \varpi(\frac{1}{2}, 0) = \frac{1}{4}$$

and

$$\varpi(\varrho, \vartheta) = \varpi(\varrho, 0) = \frac{\varrho}{2}.$$

Hence, for $\mathcal{A}^* = \mathcal{A} = \sqrt{\frac{2}{3}}$, we have

$$\varpi(\Gamma\varrho, \Gamma\vartheta) \preceq \mathcal{A}^*\varpi(\varrho, \vartheta)\mathcal{A}.$$

Hence by Theorem 3.3.2, Γ has exactly one fixed point.

Remark 3.3.8. *In the above example, $\Omega = [0, 1)$ is a $\mathcal{R}$-complete C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-MS. But it is not complete in usual $\mathfrak{m}$-MS as well as C^*-AV-$\mathfrak{m}$-MS. Hence Banach contraction in $\mathfrak{m}$-MS and C^*-AV-$\mathfrak{m}$-MS does not ensure the existence of the fixed point.*

Example 3.3.9. *Let $\Omega = [0, 2)$ and $\varpi : \Omega \times \Omega \to \mathbb{R}$ be C^*_{AV}-$\mathfrak{m}$- metric defined on Ω as $\varpi(\varrho, \vartheta) = \frac{\varrho+\vartheta}{2}$. Then for $\mathbb{B} = \mathbb{R}$ with $a^* = a$, $\|a\| = |a|$ and $a \preceq b \Leftrightarrow a \leq b$. $(\Omega, \mathbb{B}, \mathcal{R}, \varpi)$ is $\mathcal{R}$-complete C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-MS, where $a\mathcal{R}b$ iff $a = b$ or $a.b = 0$. Let Γ be a self mapping defined on Ω as:*

$$\Gamma(\varrho) = \begin{cases} \frac{\varrho}{5}, & \text{if } \varrho \leq 1 \\ \frac{1}{10}, & \text{if } 1 < \varrho < 2. \end{cases}$$

$\varrho\mathcal{R}\vartheta$ implies $\varrho = \vartheta$ or $\varrho.\vartheta = 0$. Now $\varrho = \vartheta$ implies $\Gamma\varrho = \Gamma\vartheta$ i.e., $(\Gamma\varrho, \Gamma\vartheta) \in \mathcal{R}$. If $\varrho \neq \vartheta$ i.e., $\varrho = 0$ or $\vartheta = 0$ implies $\Gamma\varrho = 0$ or $\Gamma\vartheta = 0$. Hence, $(\Gamma\varrho, \Gamma\vartheta) \in \mathcal{R}$ for every $(\varrho, \vartheta) \in \mathcal{R}$, i.e., Γ is $\mathcal{R}$-preserving.
Case(I) *Let $\varrho = \vartheta$, then we have*
(i) *For $\varrho = \vartheta \leq 1$, we have*

$$\begin{aligned}
\varpi(\Gamma\varrho, \Gamma\vartheta) = \varpi(\Gamma\varrho, \Gamma\varrho) &= \varpi\bigg(\frac{\varrho}{5}, \frac{\varrho}{5}\bigg) = \frac{\varrho}{5} \\
&= \frac{1}{5}\bigg(\frac{1}{2}(\varrho + \varrho)\bigg) \\
&\leq \frac{1}{5}\bigg(\frac{1}{2}(\varrho + \frac{\varrho}{5}) + \frac{1}{2}(\varrho + \frac{\varrho}{5})\bigg) \\
&= \frac{1}{5}\bigg(\varpi(\varrho, \Gamma\varrho) + \varpi(\varrho, \Gamma\varrho)\bigg) \\
&= \mathcal{A}\bigg(\varpi(\varrho, \Gamma\varrho) + \varpi(\vartheta, \Gamma\vartheta)\bigg).
\end{aligned}$$

For $\mathcal{A} = \frac{1}{5} < \frac{1}{2}$, we have $\varpi(\Gamma\varrho, \Gamma\vartheta) \leq \mathcal{A}(\varpi(\varrho, \Gamma\varrho) + \varpi(\vartheta, \Gamma\vartheta))$.
(ii) *For $1 < \varrho = \vartheta < 2$, we have*

$$
\begin{aligned}
\varpi(\Gamma\varrho, \Gamma\vartheta) = \varpi(\Gamma\varrho, \Gamma\varrho) &= \varpi\bigg(\frac{1}{10}, \frac{1}{10}\bigg) = \frac{1}{10} \\
&= \frac{1}{10}\bigg(\frac{1}{2}(1+1)\bigg) \\
&\leq \frac{1}{5}\bigg(\frac{1}{2}(\varrho+\varrho)\bigg) \\
&\leq \frac{1}{5}\bigg(\frac{1}{2}(\varrho+\frac{\varrho}{5}) + \frac{1}{2}(\varrho+\frac{\varrho}{5})\bigg) \\
&= \frac{1}{5}\bigg(\varpi(\varrho, \Gamma\varrho) + \varpi(\varrho, \Gamma\varrho)\bigg) \\
&= \mathcal{A}\bigg(\varpi(\varrho, \Gamma\varrho) + \varpi(\vartheta, \Gamma\vartheta)\bigg).
\end{aligned}
$$

Hence, For $\mathcal{A} = \mathcal{A}^* = \frac{1}{5} < \frac{1}{2}$, we have $\varpi(\Gamma\varrho, \Gamma\vartheta) \leq \mathcal{A}\bigg(\varpi(\varrho, \Gamma\varrho) + \varpi(\vartheta, \Gamma\vartheta)\bigg)$.
Case(II) *Let $\varrho \neq \vartheta$ implies $\varrho.\vartheta = 0$. Without loss of generality, let $\vartheta = 0$.*
(i) *For $\varrho \leq 1$ and $\vartheta = 0$, we have*

$$
\begin{aligned}
\varpi(\Gamma\varrho, \Gamma\vartheta) = \varpi\bigg(\frac{\varrho}{5}, 0\bigg) &= \frac{\varrho}{10} \\
&\leq \frac{1}{5}\bigg(\frac{1}{2}(\varrho+\frac{\varrho}{5})\bigg) \\
&= \mathcal{A}\bigg(\varpi(\varrho, \Gamma\varrho) + \varpi(\vartheta, \Gamma\vartheta)\bigg),
\end{aligned}
$$

where $\mathcal{A} = \frac{1}{5}$ and $\varpi(\vartheta, \Gamma\vartheta) = 0$.
Hence, For $\mathcal{A} = \mathcal{A}^* = \frac{1}{5} < \frac{1}{2}$, we have $\varpi(\Gamma\varrho, \Gamma\vartheta) \leq \mathcal{A}\bigg(\varpi(\varrho, \Gamma\varrho) + \varpi(\vartheta, \Gamma\vartheta)\bigg)$.
(ii) *For $1 < \varrho < 2$ and $\vartheta = 0$, we have*

$$
\begin{aligned}
\varpi(\Gamma\varrho, \Gamma\vartheta) = \varpi\bigg(\frac{1}{10}, 0\bigg) &= \frac{1}{20} \leq \frac{1}{10} \\
&\leq \frac{1}{5}\bigg(\frac{1}{2}(\varrho+\frac{1}{10})\bigg) \\
&= \mathcal{A}\bigg(\varpi(\varrho, \Gamma\varrho) + \varpi(\vartheta, \Gamma\vartheta)\bigg),
\end{aligned}
$$

where $\mathcal{A} = \mathcal{A}^* = \frac{1}{5}$ *and* $\varpi(\vartheta, \Gamma\vartheta) = 0$.
For $\mathcal{A} = \mathcal{A}^* = \frac{1}{5} < \frac{1}{2}$, *we have* $\varpi(\Gamma\varrho, \Gamma\vartheta) \leq \mathcal{A}\Bigg(\varpi(\varrho, \Gamma\varrho) + \varpi(\vartheta, \Gamma\vartheta)\Bigg)$.
Therefore, by Theorem 3.3.3, Γ *has exactly one fixed point.*

Remark 3.3.10. *In the above example,* $\Omega = [0, 2)$ *is a* $\mathcal{R}$*-complete* C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-*MS . But it is not complete in usual* $\mathfrak{m}$*-MS and* C^**-AV-*$\mathfrak{m}$*-MS. Hence, Kannan-contraction in* $\mathfrak{m}$*-MS and* C^**-AV-*$\mathfrak{m}$*-MS does not ensure the existence of the fixed point.*

Remark 3.3.11. *The completeness in the fixed point results of Theorem 3.3.2, 3.3.3 and 3.3.4 can be further relaxed with the following hypotheses*

(H) $\exists$ *an* $\mathcal{R}$*-complete subset* Υ *of* Ω *s.t.* $\Gamma(\Omega) \subseteq \Upsilon$.

Using the hypothesis (H) , one can prove the Theorems 3.3.2, 3.3.3 and 3.3.4. To verify the claim consider

Example 3.3.12. *Let* $\Omega = [0, 2)$ *and* $\varpi : \Omega \times \Omega \to \mathbb{R}$ *be* $\mathfrak{m}$*- metric defined on* Ω *as* $\varpi(\varrho, \vartheta) = \frac{\varrho+\vartheta}{2}$. *Then for* $\mathbb{B} = \mathbb{R}$ *with* $a^* = a$, $\|a\| = |a|$ *and* $a \preceq b \Leftrightarrow a \leq b$. *Then* $(\Omega, \mathbb{B}, \mathcal{R}, \varpi)$ *is complete* C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$*-MS, where* $\mathcal{R} = \Omega \times \Omega$. *Let* Γ *be a self mapping defined on* Ω *as following:*

$$\Gamma(\varrho) = \begin{cases} \frac{\varrho}{5}, & \text{if } \varrho \leq 1 \\ \frac{1}{10}, & \text{if } \varrho > 1. \end{cases}$$

Then, Γ *is* $\mathcal{R}$*-preserving. Also,* Γ *satisfies the additional assumption with* $\Upsilon = [0, 1/5]$. *Therefore,* Γ *meets all the criteria of Theorem 3.3.2 with* $\mathcal{A} = \mathcal{A}^* = \frac{1}{\sqrt{5}}$ *and Theorem 3.3.3 with* $\mathcal{A} = \frac{1}{5}$. *Consequently,* Γ *possesses exactly one fixed point. One can observe that in the above example*

- Ω *is not complete;*
- Γ *is not* $\mathcal{R}$*-continuous.*

But still Γ *satisfies the Banach type contraction in the* C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$*-MS, but not in MS.*

3.4 Existence of Solution to the Operator equation

In this section, a potential application of the derived fixed point results is discussed. The applicability of these results in operator equations demonstrates their significance.

Theorem 3.4.1. *Let H be a Hilbert space and $B(H)$ be the set of all bounded linear operators on H. Suppose $S_\hbar \in B(H)$ s.t. $\sum_{\hbar=1}^{\infty} \|S_\hbar\|^2 < 1$. Then the operator equation*

$$M - \sum_{\hbar=1}^{\infty} S_\hbar^* M S_\hbar = 0 \tag{3.4.1}$$

has exactly one solution.

Proof. Define a mapping $\varpi : B(H) \times B(H) \to B(H)_+$ as

$$\varpi(M_1, M_2) = \|M_1 - M_2\| P + \min\{\|M_1\|, \|M_2\|\} P,$$

for an positive operator P define on H. Consider a reflexive binary relation $\mathcal{R}$ defined as $(M_1, M_2) \in \mathcal{R} \Leftrightarrow M_1 + M_2 = M_1 \vee M_2$. Then, $B(H)$ is a C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-MS. Consider a self mapping Γ defined on $B(H)$ as:

$$\Gamma(M) = \sum_{\hbar=1}^{\infty} S_\hbar^* M S_\hbar.$$

Step (i) Γ is $\mathcal{R}$-preserving:

Let $(M_1, M_2) \in \mathcal{R}$ i.e., $M_1 + M_2 = M_1 \vee M_2$. Then,

$$\Gamma(M_1 + M_2) = \sum_{\hbar=1}^{\infty} S_\hbar^*(M_1 + M_2) S_\hbar = \sum_{\hbar=1}^{\infty} S_\hbar^* M_1 S_\hbar \vee \sum_{\hbar=1}^{\infty} S_\hbar^* M_2 S_\hbar = \Gamma(M_1) \vee \Gamma(M_2),$$

i.e., Γ is $\mathcal{R}$-preserving.

Step (ii) Γ is C^*-AV $\mathfrak{m}_{\mathcal{R}}$-contraction:

$$\begin{aligned}
\varpi(\Gamma(M_1), \Gamma(M_2)) &= \|\Gamma(M_1) - \Gamma(M_2)\| P + \min\{\|\Gamma(M_1)\|, \|\Gamma(M_2)\|\} P \\
&\leq \left\| \sum_{\hbar=1}^{\infty} S_\hbar^*(M_1 - M_2) S_\hbar \right\| P + \min\left\{ \left\| \sum_{\hbar=1}^{\infty} S_\hbar^* M_1 S_\hbar \right\|, \left\| \sum_{\hbar=1}^{\infty} S_\hbar^* M_2 S_\hbar \right\| \right\} P \\
&\leq \sum_{\hbar=1}^{\infty} \|S_\hbar\|^2 \|M_1 - M_2\| P + \min\left\{ \sum_{\hbar=1}^{\infty} \|S_\hbar\|^2 \|M_1\|, \sum_{\hbar=1}^{\infty} \|S_\hbar\|^2 \|M_2\| \right\} P \\
&\leq \kappa \left(\|M_1 - M_2\| P + \min\{\|M_1\|, \|M_2\|\} P \right) \\
&= \alpha^* \varpi(M_1, M_2) \alpha,
\end{aligned}$$

where $\alpha = \sqrt{\kappa} < 1$.
Clearly, $B(H)$ is complete and Γ satisfies all the conditions of Theorem 3.3.2. Hence the operator equation (3.4.1) has exactly one solution. $\square$

3.5 Conclusion

In the present chapter, we introduced a notion of C^*_{AV}-$\mathfrak{m}_{\mathcal{R}}$-MS as a generalization of C^*_{AV}-$\mathfrak{m}$-MS and $\mathcal{R}$-MS, and proved some fixed point results in this framework. To support our findings, some illustrative examples are discussed. At last, we ensure the existence and uniqueness of solutions to an operator equation using the results proved in the chapter. Our findings generalized the following results in the literature.

(i) By taking $\mathcal{R} = \Omega \times \Omega$ in Theorem 3.3.2, one can derive the fixed point result for Banach-type self mappings in C^*_{AV}-$\mathfrak{m}$-MS (Alsamir et al., 2019).

(ii) By taking $\mathcal{R} = \Omega \times \Omega$ in Theorem 3.3.3, one can derive the fixed point result for Kannan-type self mappings in C^*_{AV}-$\mathfrak{m}$-MS (Alsamir et al., 2019).

(iii) By considering $\mathbb{B} = \mathbb{R}$ with $\|a\| = |a|$ $\forall a \in \mathbb{B}$ in Theorem 3.3.2, one can derive the fixed point results for Banach-type self mappings in $\mathcal{R}_{\mathfrak{m}}$-MS (Khalil et al., 2021).

(iv) By considering $\mathbb{B} = \mathbb{R}$ with $\|a\| = |a|$ $\forall a \in \mathbb{B}$ in Theorem 3.3.3, one can derive the fixed point results for Kannan-type self mappings in $\mathcal{R}_{\mathfrak{m}}$-MS (Khalil et al., 2021).

(v) By taking $\mathcal{R} = \Omega \times \Omega$ and $\mathbb{B} = \mathbb{R}$ with $\|a\| = |a|$, $\forall a \in \mathbb{B}$ in Theorem 3.3.2, one can derive the fixed point results for Banach-type self mappings in $\mathfrak{m}$-MS (Asadi et al., 2014).

(vi) By taking $\mathcal{R} = \Omega \times \Omega$ and $\mathbb{B} = \mathbb{R}$ with $\|a\| = |a|$, $\forall a \in \mathbb{B}$ in Theorem 3.3.3, one can derive the fixed point results for Kannan-type self mappings in $\mathfrak{m}$-MS (Asadi et al., 2014).

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Chapter 4

Some Fixed Point and Common Fixed Point Results in Multiplicative $\mathfrak{m}$-Metric Space

4.1 Introduction

In the present chapter of the thesis, we introduce the notion of multiplicative $\mathfrak{m}$-metric space, inspired by the concepts of multiplicative metric space and $\mathfrak{m}$-metric space. The first section deals with the fundamental definitions for the multiplicative $\mathfrak{m}$-metric space, its intrinsic properties and illustrative examples. In the second section, we establish some fixed point results using contraction mappings in complete multiplicative $\mathfrak{m}$-metric space. To support our findings, we discuss some illustrative examples, where well known fixed point results in the literature does not ensure the existence of fixed point. In the third section, we discuss some common fixed point results for a pair of self mapping using various contraction. An illustrative example, involving discontinuous self mappings, is discussed, along with numerical iterations to approximate the common fixed point, supported by graphical representations. At the last, we prove the existence of solution to a system of multiplicative integral equation and multiplicative initial value problem using the fixed point results proved in the chapter.

The results of the chapter are presented in [1] [2] [3].

4.2 Multiplicative $\mathfrak{m}$-Metric Space

In this section, we have introduced a new notion of multiplicative $\mathfrak{m}$-MS along with the intrinsic properties and some illustrative examples of this framework.

Definition 4.2.1. *A mapping $\mu : \Omega \times \Omega \to [1, \infty)$ is c.t.b. multiplicative $\mathfrak{m}$-metric if it satisfies:*

(i) $\mu(\varrho, \vartheta) = \mu(\vartheta, \vartheta) = \mu(\varrho, \varrho) \Leftrightarrow \varrho = \vartheta$;

(ii) $\mu_{\varrho\vartheta} \leq \mu(\varrho, \vartheta)$;

(iii) $\mu(\varrho, \vartheta) = \mu(\vartheta, \varrho)$;

(iv) $\dfrac{\mu(\varrho, \vartheta)}{\mu_{\varrho\vartheta}} \leq \dfrac{\mu(\varrho, \xi)}{\mu_{\varrho\xi}} . \dfrac{\mu(\xi, \vartheta)}{\mu_{\xi\vartheta}}$,

where $\mu_{\varrho\vartheta} = \min\Big\{\mu(\varrho, \varrho), \mu(\vartheta, \vartheta)\Big\}$, *and* $\mu^*_{\varrho\vartheta} = \max\Big\{\mu(\varrho, \varrho), \mu(\vartheta, \vartheta)\Big\}$ $\forall \varrho, \vartheta, \xi \in \Omega$. *Also, (Ω, μ) is c.t.b. a multiplicative $\mathfrak{m}$-MS.*

Remark 4.2.2. *Every multiplicative metric is a multiplicative $\mathfrak{m}$-metric. But the converse is not true.*

Example 4.2.3. *Let $\Omega = [0, \infty)$ and $\mu(\varrho, \vartheta) = e^{\frac{\varrho+\vartheta}{2}}$, then (Ω, μ) is a multiplicative $\mathfrak{m}$-MS. But (Ω, μ) is not a multiplicative MS. Because, for $\varrho \neq 0$, $\mu(\varrho, \varrho) = e^{\varrho} \neq 1$.*

Remark 4.2.4. *Let (Ω, μ) be a multiplicative $\mathfrak{m}$-MS. Then, $\forall \varrho, \vartheta, \varrho \in \Omega$, we have*

(i) $1 \leq \mu^*_{\varrho\vartheta} . \mu_{\varrho\vartheta} = \mu(\varrho, \varrho) . \mu(\vartheta, \vartheta)$;

(ii) $1 \leq \dfrac{\mu^*_{\varrho\vartheta}}{\mu_{\varrho\vartheta}} = \left|\dfrac{\mu(\varrho, \varrho)}{\mu(\vartheta, \vartheta)}\right|_*$;

[1] Yadav, K., & Kumar, D. (2024). Multiplicative $\mathfrak{m}$-metric space, fixed point theorems with applications in multiplicative integral equations and numerical results. *Journal of Applied Analysis*, *30*(1), 173–186.

[2] Yadav, K., & Kumar, D. (2024). Common fixed point theorems in multiplicative $\mathfrak{m}$-metric space with applications to the system of multiplicative integral equations and numerical results. *Hacettepe Journal of Mathematics and Statistics*, 1–12.

[3] Yadav, K., & Kumar, D. Fixed Point Results Using a Three-Point Analogue of Contraction in Multiplicative $\mathfrak{m}$-Metric Spaces, (Communicated)

(iii) $\dfrac{\mu^*_{\varrho\vartheta}}{\mu_{\varrho\vartheta}} \leq \dfrac{\mu^*_{\varrho\xi}}{\mu_{\varrho\xi}}.\dfrac{\mu^*_{\xi\vartheta}}{\mu_{\xi\vartheta}}$,

where $\mu_{\varrho\vartheta} = \min\Big\{\mu(\varrho,\varrho), \mu(\vartheta,\vartheta)\Big\}$, $\mu^*_{\varrho\vartheta} = \max\Big\{\mu(\varrho,\varrho), \mu(\vartheta,\vartheta)\Big\}$ *and* $|a|_* = \begin{cases} a, a \geq 1; \\ \frac{1}{a}, a < 1. \end{cases}$ *for* $a \in \mathbb{R}_+$.

Remark 4.2.5. *Let* (Ω, μ) *be a multiplicative* $\mathfrak{m}$*-MS. Then,* $\forall \varrho, \vartheta \in \Omega$*, we have*

(i) *(a)* $\mu^w(\varrho,\vartheta) = \dfrac{\mu(\varrho,\vartheta).\mu^*_{\varrho\vartheta}}{\mu_{\varrho\vartheta}.\mu_{\varrho\vartheta}}$;

(b) $\mu_s(\varrho,\vartheta) = \begin{cases} \frac{\mu(\varrho,\vartheta)}{\mu_{\varrho\vartheta}}, & \textit{if } \varrho \neq \vartheta \\ 1, & \textit{if } \varrho = \vartheta, \end{cases}$

are multiplicative metric on Ω.

(ii) *(a)* $\dfrac{\mu(\varrho,\vartheta)}{\mu^*_{\varrho\vartheta}} \leq \mu^w(\varrho,\vartheta) \leq \mu(\varrho,\vartheta).\mu^*_{\varrho\vartheta}$;

(b) $\dfrac{\mu(\varrho,\vartheta)}{\mu^*_{\varrho\vartheta}} \leq \mu_s(\varrho,\vartheta) \leq \mu(\varrho,\vartheta)$.

Example 4.2.6. *Let* (Ω, u) *be a multiplicative MS. Let* $\phi : [1,\infty) \to [a,\infty)$*, where* $a = \phi(1)$ *be a one to one and non-decreasing function s.t.*

$$\phi(\varrho.\vartheta) \leq \frac{\phi(\varrho).\phi(\vartheta)}{\phi(1)}, \tag{4.2.1}$$

$\forall \varrho, \vartheta \geq 1$*. Then,* $\mu(\varrho,\vartheta) = \phi(u(\varrho,\vartheta))$ *is a multiplicative* $\mathfrak{m}$*-metric.*

Proof. It is given that (Ω, u) is a multiplicative MS. In order to prove $\mu(\varrho,\vartheta) = \phi(u(\varrho,\vartheta))$ is a multiplicative $\mathfrak{m}$-MS, we shall prove the following :
(i) Since, (Ω, u) is a multiplicative MS. Therefore,

$$\begin{aligned} \varrho = \vartheta &\Leftrightarrow u(\varrho,\varrho) = u(\vartheta,\vartheta) = u(\varrho,\vartheta) = 1 \\ &\Leftrightarrow \phi(u(\varrho,\varrho)) = \phi(u(\vartheta,\vartheta)) = \phi(u(\varrho,\vartheta)) = \phi(1) \quad (\text{As, } \phi \text{ is one to one }) \\ &\Leftrightarrow \mu(\varrho,\varrho) = \mu(\vartheta,\vartheta) = \mu(\varrho,\vartheta). \end{aligned}$$

(ii) Since, $\mu(\varrho,\varrho) = \phi(u(\varrho,\varrho)) = \phi(1)$ and $\mu(\vartheta,\vartheta) = \phi(\vartheta,\vartheta) = \phi(1)$. Therefore, $\mu_{\varrho\vartheta} = \min\Big\{\mu(\varrho,\varrho), \mu(\vartheta,\vartheta)\Big\} = \phi(1) = a \leq \phi(u(\varrho,\vartheta)) = \mu(\varrho,\vartheta)$. This is because of the fact that $\phi(\varrho) \geq a$, $\forall \varrho \in [1,\infty)$.

(iii) Also, $\mu(\varrho,\vartheta) = \phi(u(\varrho,\vartheta)) = \phi(u(\vartheta,\varrho)) = \mu(\vartheta,\varrho)$. This is because of the fact that (Ω, u) is a multiplicative MS.
(iv) For triangle inequality, we have

$$\begin{aligned} \frac{\mu(\varrho,\vartheta)}{\mu_{\varrho\vartheta}} = \frac{\phi(u(\varrho,\vartheta))}{\phi(1)} &\leq \frac{\phi(u(\varrho,\varrho).u(\varrho,\vartheta))}{\phi(1)} \\ &\leq \frac{\phi(u(\varrho,\xi))}{\phi(1)}.\frac{\phi(u(\xi,\vartheta))}{\phi(1)} \quad \text{(Using (4.2.1))} \\ &= \frac{\mu(\varrho,\xi)}{\mu_{\xi\varrho}}.\frac{\mu(\xi,\vartheta)}{\mu_{\xi\vartheta}}. \end{aligned}$$

Thus, $\mu(\varrho,\vartheta) = \phi(u(\varrho,\vartheta))$ is a multiplicative $\mathfrak{m}$-metric and (Ω,μ) is a multiplicative $\mathfrak{m}$-MS. □

Example 4.2.7. *Let (Ω, u) be a multiplicative MS. Then $\mu(\varrho,\vartheta) = b.u(\varrho,\vartheta)^a$, where $a, b > 1$ is a multiplicative $\mathfrak{m}$-metric.*

Example 4.2.8. *Let $(\Omega,\mathfrak{m})$ be an $\mathfrak{m}$-MS. Then $\mu(\varrho,\vartheta) = e^{\mathfrak{m}(\varrho,\vartheta)}$ is a multiplicative $\mathfrak{m}$-metric on Ω and (Ω,μ) is a multiplicative $\mathfrak{m}$-MS.*

Example 4.2.9. *Let (Ω, d) be a MS. Then, $\forall a, b > 0$, $\mu(\varrho,\vartheta) = e^{ad(\varrho,\vartheta)+b}$ is a multiplicative $\mathfrak{m}$-metric and (Ω,μ) is a multiplicative $\mathfrak{m}$-MS.*

Example 4.2.10. *Let (Ω,μ) be a multiplicative $\mathfrak{m}$-MS then $\mathfrak{m}(\varrho,\vartheta) = \ln\mu(\varrho,\vartheta)$ is an $\mathfrak{m}$-metric and $d(\varrho,\vartheta) = \ln\mu(\varrho,\vartheta) + \ln\mu^*_{\varrho\vartheta} - 2\ln(\mu_{\varrho\vartheta})$ is a usual metric.*

Definition 4.2.11. *Let $\{\varrho_\hbar\}$ be a sequence in (Ω,μ). Then $\{\varrho_\hbar\}$ is c.t.b. multiplicative*

(i) ***convergent*** *if $\exists\varrho$ in Ω s.t.*

$$\lim_{\hbar\to\infty} \frac{\mu(\varrho_\hbar,\varrho)}{\mu_{\varrho_\hbar\varrho}} = 1.$$

(ii) ***$\mathfrak{m}$-Cauchy*** *if*

$$\lim_{\hbar,m\to\infty} \frac{\mu(\varrho_\hbar,\varrho_m)}{\mu_{\varrho_\hbar\varrho_m}} \quad \text{and} \quad \lim_{\hbar,m\to\infty} \frac{\mu^*_{\varrho_\hbar\varrho_m}}{\mu_{\varrho_\hbar\varrho_m}} \text{ exist finitely.}$$

Also, if every multiplicative $\mathfrak{m}$-C_{seq} in Ω is convergent in Ω, i.e., $\exists\varrho\in\Omega$ s.t.

$$\lim_{\hbar\to\infty} \frac{\mu(\varrho_\hbar,\varrho)}{\mu_{\varrho_\hbar\varrho}} = 1 \quad \text{and} \quad \lim_{\hbar\to\infty} \frac{\mu^*_{\varrho_\hbar\varrho}}{\mu_{\varrho_\hbar\varrho}} = 1.$$

Then (Ω,μ) is a multiplicative $\mathfrak{m}$-complete MS.

Lemma 4.2.12. *Let (Ω, μ) be a multiplicative $\mathfrak{m}$-MS. Consider the $\mathfrak{m}$-metric and usual metric defined by $\mathfrak{m}(\varrho, \vartheta) = ln(\mu(\varrho, \vartheta))$ and $d(\varrho, \vartheta) = ln(\mu(\varrho, \vartheta)) + ln(\mu^*_{\varrho\vartheta}) - 2ln(\mu_{\varrho\vartheta})$ respectively. Then $\{\varrho_\hbar\}$ is a multiplicative $\mathfrak{m}$-C_{seq} iff $\{\varrho_\hbar\}$ is Cauchy w.r.t to either of the multiplicative metric 'μ^w', 'μ_s', $\mathfrak{m}$-metric '$\mathfrak{m}$' and usual metric 'd'. Also, the multiplicative $\mathfrak{m}$-MS (Ω, μ) is complete iff it is complete w.r.t. to either of the multiplicative metric 'μ^w', 'μ_s', $\mathfrak{m}$-metric '$\mathfrak{m}$' and usual metric 'd'.*

Proof. Using Definitions 4.2.11, one can easily verify the above result. □

Lemma 4.2.13. *Let $\{\varrho_\hbar\}$ and $\{\vartheta_\hbar\}$ be two sequences in (Ω, μ) s.t. $\varrho_\hbar \to \varrho$ and $\vartheta_\hbar \to \vartheta$. Then,*

$$\lim_{\hbar\to\infty} \frac{\mu(\varrho_\hbar, \vartheta_\hbar)}{\mu_{\varrho_\hbar\vartheta_\hbar}} = \frac{\mu(\varrho, \vartheta)}{\mu_{\varrho\vartheta}}.$$

Proof. Consider the sequences $\{\varrho_\hbar\}$ and $\{\vartheta_\hbar\}$ s.t. $\varrho_\hbar \to \varrho$ and $\vartheta_\hbar \to \vartheta$. Then,

$$\lim_{\hbar\to\infty} \frac{\mu(\varrho_\hbar, \varrho)}{\mu_{\varrho_\hbar\varrho}} = 1, \lim_{\hbar\to\infty} \frac{\mu(\vartheta_\hbar, \vartheta)}{\mu_{\vartheta_\hbar\vartheta}} = 1. \tag{4.2.2}$$

Now, consider

$$\frac{\mu(\varrho_\hbar, \vartheta_\hbar)}{\mu_{\varrho_\hbar\vartheta_\hbar}} \leq \frac{\mu(\varrho_\hbar, \varrho)}{\mu_{\varrho_\hbar\varrho}} . \frac{\mu(\varrho, \vartheta)}{\mu_{\varrho\vartheta}} . \frac{\mu(\vartheta_\hbar, \vartheta)}{\mu_{\vartheta_\hbar\vartheta}}.$$

Taking limit as $\hbar \to \infty$ on both sides and using (4.2.2), we have

$$\lim_{\hbar\to\infty} \frac{\mu(\varrho_\hbar, \vartheta_\hbar)}{\mu_{\varrho_\hbar\vartheta_\hbar}} \leq \frac{\mu(\varrho, \vartheta)}{\mu_{\varrho\vartheta}}. \tag{4.2.3}$$

Similarly,

$$\frac{\mu(\varrho, \vartheta)}{\mu_{\varrho\vartheta}} \leq \lim_{\hbar\to\infty} \frac{\mu(\varrho_\hbar, \vartheta_\hbar)}{\mu_{\varrho_\hbar\vartheta_\hbar}}. \tag{4.2.4}$$

From (4.2.3) and (4.2.4), we have

$$\lim_{\hbar\to\infty} \frac{\mu(\varrho_\hbar, \vartheta_\hbar)}{\mu_{\varrho_\hbar\vartheta_\hbar}} = \frac{\mu(\varrho, \vartheta)}{\mu_{\varrho\vartheta}}.$$

□

Lemma 4.2.14. *Let $\{\varrho_\hbar\}$ be a sequence in (Ω, μ) s.t. $\varrho_\hbar \to \varrho$ and $\varrho_\hbar \to \vartheta$. Then $\mu(\varrho, \vartheta) = \mu_{\varrho\vartheta}$. Also, in case $\mu(\varrho, \varrho) = \mu(\vartheta, \vartheta)$, then $\varrho = \vartheta$.*

Proof. Using Lemma 4.2.13, one can easily prove the required result. □

Lemma 4.2.15. *Let (Ω, μ) be a multiplicative $\mathfrak{m}$-MS and $\{\varrho_\hbar\}$ be a sequence in Ω s.t.*

$$\mu(\varrho_{\hbar+1}, \varrho_\hbar) \leq \mu(\varrho_\hbar, \varrho_{\hbar-1})^r, \quad \forall\ \hbar \in \mathbb{N},\ \textit{where}\ r \in [0, 1). \tag{4.2.5}$$

Then,

(i) $\lim\limits_{\hbar\to\infty} \mu(\varrho_{\hbar+1}, \varrho_\hbar) = 1.$

(ii) $\lim\limits_{\hbar\to\infty} \mu(\varrho_\hbar, \varrho_\hbar) = 1.$

(iii) $\lim\limits_{\hbar,m\to\infty} \mu_{\varrho_\hbar\varrho_m} = 1.$

(iv) $\{\varrho_\hbar\}$ *is a multiplicative* $\mathfrak{m}$-C_{seq}.

Proof. (i) Consider a sequence $\{\varrho_\hbar\}$ in multiplicative $\mathfrak{m}$-MS (Ω, μ). Then using (4.2.5), we have

$$\begin{aligned}
\mu(\varrho_{\hbar+1}, \varrho_\hbar) &\leq \mu(\varrho_\hbar, \varrho_{\hbar-1})^r \\
&\leq \mu(\varrho_{\hbar-1}, \varrho_{\hbar-2})^{r^2} \\
&\quad \ldots \\
&\leq \mu(\varrho_1, \varrho_0)^{r^\hbar}.
\end{aligned}$$

Taking limit as $\hbar \to \infty$, we have

$$\lim_{\hbar\to\infty} \mu(\varrho_{\hbar+1}, \varrho_\hbar) \leq \lim_{\hbar\to\infty} \mu(\varrho_1, \varrho_0)^{r^\hbar} \to 1. \quad (\text{As } r < 1) \tag{4.2.6}$$

Also, by Definition 4.2.1, we have

$$\mu(\varrho_{\hbar+1}, \varrho_\hbar) \geq 1. \tag{4.2.7}$$

From (4.2.6) and (4.2.7), we have

$$\lim_{\hbar\to\infty} \mu(\varrho_{\hbar+1}, \varrho_\hbar) = 1.$$

(ii) By using (i) and Definition 4.2.1, we have

$$\lim_{\hbar\to\infty} \mu_{\varrho_{\hbar+1}\varrho_\hbar} = \lim_{\hbar\to\infty} \min\left\{\mu(\varrho_{\hbar+1}, \varrho_{\hbar+1}), \mu(\varrho_\hbar, \varrho_\hbar)\right\} \leq \lim_{\hbar\to\infty} \mu(\varrho_{\hbar+1}, \varrho_\hbar) = 1,$$

or

$$\lim_{\hbar\to\infty} \mu(\varrho_\hbar, \varrho_\hbar) \leq 1.$$

Also, by the Definition of multiplicative $\mathfrak{m}$-metric, we have

$$\mu(\varrho_\hbar, \varrho_\hbar) \geq 1.$$

Thus, $\lim_{\hbar\to\infty} \mu(\varrho_\hbar, \varrho_\hbar) = 1$.
(iii) Using (ii), one can easily deduce the required result.
(iv) Let $\{\varrho_\hbar\}$ be the given sequence. Using triangle inequality, we have

$$\begin{aligned}
\frac{\mu(\varrho_m, \varrho_\hbar)}{\mu_{\varrho_m\varrho_\hbar}} &\leq \frac{\mu(\varrho_m, \varrho_{m-1})}{\mu_{\varrho_m\varrho_{m-1}}}.\frac{\mu(\varrho_{m-1}, \varrho_{m-2})}{\mu_{\varrho_{m-1}\varrho_{m-2}}}...\frac{\mu(\varrho_{\hbar+1}, \varrho_\hbar)}{\mu_{\varrho_{\hbar+1}\varrho_\hbar}} \\
&\leq \mu(\varrho_1, \varrho_0)^{r^{m-1}}.\mu(\varrho_1, \varrho_0)^{r^{m-2}}...\mu(\varrho_1, \varrho_0)^{r^\hbar} \\
&\leq \mu(\varrho_1, \varrho_0)^{r^{m-1}+r^{m-2}+...r^\hbar} \\
&\leq \mu(\varrho_1, \varrho_0)^{\frac{r^\hbar}{1-r}} \to 1 \quad \text{as } \hbar \to \infty \text{ (Because } r < 1).
\end{aligned}$$

Using (ii) and Remark 4.2.4, we have

$$\frac{\mu^*_{\varrho_m\varrho_\hbar}}{\mu_{\varrho_m\varrho_\hbar}} = \left|\frac{\mu(\varrho_m, \varrho_m)}{\mu(\varrho_\hbar, \varrho_\hbar)}\right|_* \to 1 \text{ as } \hbar, m \to \infty.$$

Hence, $\{\varrho_\hbar\}$ is a multiplicative $\mathfrak{m}$-C_{seq}. □

4.3 Some Fixed Point Results in Multiplicative $\mathfrak{m}$-Metric Space

In this section, we established some fixed point results using some well known contractions in the framework of multiplicative $\mathfrak{m}$-MS.

Theorem 4.3.1. *Let $\Gamma : \Omega \to \Omega$ be a mapping defined on a multiplicative $\mathfrak{m}$-complete MS (Ω, μ). Suppose $\exists a, b, c \geq 0$, with $a + b + c < 1$ s.t.*

$$\mu(\Gamma\varrho, \Gamma\vartheta) \leq \Big(\mu(\varrho, \vartheta)\Big)^a \Big(\mu(\varrho, \Gamma\varrho)\Big)^b \Big(\mu(\vartheta, \Gamma\vartheta)\Big)^c \ \forall \varrho, \vartheta \in \Omega. \tag{4.3.1}$$

Then, Γ possesses exactly one fixed point.

Proof. Let $\varrho_0 \in \Omega$ and consider the sequence $\{\varrho_\hbar\}$ is s.t. $\varrho_1 = \Gamma\varrho_0$ and $\varrho_\hbar = \Gamma\varrho_{\hbar-1}$. If $\varrho_\hbar = \varrho_{\hbar-1}$, then, $\Gamma\varrho_{\hbar-1} = \varrho_\hbar = \varrho_{\hbar-1}$ implies $\varrho_{\hbar-1}$ is a fixed point of Γ. Hence, the result holds.

Now, consider $\varrho_\hbar \neq \varrho_{\hbar-1}$, $\forall \hbar \in \mathbb{N}$
Using (4.3.1), we have

$$\begin{aligned}\mu(\varrho_{\hbar+1}, \varrho_\hbar) &= \mu(\Gamma\varrho_\hbar, \Gamma\varrho_{\hbar-1}) \\ &\leq \Big(\mu(\varrho_\hbar, \varrho_{\hbar-1})\Big)^a \Big(\mu(\varrho_\hbar, \Gamma\varrho_\hbar)\Big)^b \Big(\mu(\varrho_{\hbar-1}, \Gamma\varrho_{\hbar-1})\Big)^c \\ &= \Big(\mu(\varrho_\hbar, \varrho_{\hbar-1})\Big)^a \Big(\mu(\varrho_\hbar, \varrho_{\hbar+1})\Big)^b \Big(\mu(\varrho_{\hbar-1}, \varrho_\hbar)\Big)^c\end{aligned}$$

or

$$\Big(\mu(\varrho_{\hbar+1}, \varrho_\hbar)\Big)^{1-b} \leq \Big(\mu(\varrho_{\hbar-1}, \varrho_\hbar)\Big)^{a+c} \quad \Rightarrow \mu(\varrho_{\hbar+1}, \varrho_\hbar) \leq \Big(\mu(\varrho_{\hbar-1}, \varrho_\hbar)\Big)^{\dfrac{a+c}{1-b}}.$$

Since $\frac{a+c}{1-b} < 1$, using Lemma 4.2.15, we have

$$\lim_{\hbar\to\infty} \mu(\varrho_{\hbar+1}, \varrho_\hbar) = 1, \tag{4.3.2}$$

$$\lim_{\hbar\to\infty} \mu(\varrho_\hbar, \varrho_\hbar) = 1, \tag{4.3.3}$$

$$\lim_{\hbar, m\to\infty} \mu_{\varrho_\hbar\varrho_m} = 1$$

and $\{\varrho_\hbar\}$ is a multiplicative $\mathfrak{m}$-C_{seq}. As (Ω, μ) is complete, hence $\exists \varrho$ in Ω s.t.

$$\lim_{\hbar\to\infty} \frac{\mu(\varrho_\hbar, \varrho)}{\mu_{\varrho_\hbar\varrho}} = 1 \quad \text{and} \quad \lim_{\hbar\to\infty} \frac{\mu^*_{\varrho_\hbar\varrho}}{\mu_{\varrho_\hbar\varrho}} = 1. \tag{4.3.4}$$

Also, using (4.3.3), we have

$$\lim_{\hbar\to\infty} \mu_{\varrho_\hbar\varrho} = \lim_{\hbar\to\infty} \min\Big\{\mu(\varrho_\hbar, \varrho_\hbar), \mu(\varrho, \varrho)\Big\} \leq \lim_{\hbar\to\infty} \mu(\varrho_\hbar, \varrho_\hbar) \to 1. \tag{4.3.5}$$

Using (4.3.4), (4.3.5) and Remark 4.2.4, we have

$$\lim_{\hbar\to\infty} \mu(\varrho_\hbar, \varrho) = 1, \lim_{\hbar\to\infty} \mu^*_{\varrho_\hbar\varrho} = 1 \text{ and } \mu(\varrho, \varrho) = 1. \tag{4.3.6}$$

Also,

$$1 \leq \mu_{\varrho\Gamma\varrho} = \min\Big\{\mu(\varrho, \varrho), \mu(\Gamma\varrho, \Gamma\varrho)\Big\} \leq \mu(\varrho, \varrho) = 1 \Rightarrow \mu_{\varrho\Gamma\varrho} = 1. \tag{4.3.7}$$

By triangle inequality, (4.3.2) and (4.3.7), we have

$$\mu(\varrho, \Gamma\varrho) = \frac{\mu(\varrho, \Gamma\varrho)}{\mu_{\varrho\Gamma\varrho}} \leq \limsup_{\hbar\to\infty} \frac{\mu(\varrho, \varrho_\hbar)}{\mu_{\varrho_\hbar\varrho}} . \frac{\mu(\varrho_\hbar, \Gamma\varrho)}{\mu_{\varrho_\hbar\Gamma\varrho}} \leq \limsup_{\hbar\to\infty} \mu(\varrho_\hbar, \Gamma\varrho),$$

or

$$\mu(\varrho, \Gamma\varrho) \leq \limsup_{\hbar\to\infty} \mu(\Gamma\varrho_{\hbar-1}, \Gamma\varrho). \tag{4.3.8}$$

Using (4.3.1) and (4.3.6) in (4.3.8), we have

$$\mu(\varrho, \Gamma\varrho) \leq \limsup_{\hbar\to\infty} \Big(\mu(\varrho_{\hbar-1}, \varrho)\Big)^a \Big(\mu(\varrho_{\hbar-1}, \Gamma\varrho_{\hbar-1})\Big)^b \Big(\mu(\varrho, \Gamma\varrho)\Big)^c = \mu(\varrho, \Gamma\varrho)^c. \tag{4.3.9}$$

Since $c < 1$, we have

$$\mu(\varrho, \Gamma\varrho) = 1. \tag{4.3.10}$$

Using (4.3.1), we have

$$\begin{aligned} 1 \leq \mu(\Gamma\varrho, \Gamma\varrho) &\leq \Big(\mu(\varrho, \varrho)\Big)^a \Big(\mu(\varrho, \Gamma\varrho)\Big)^b \Big(\mu(\varrho, \Gamma\varrho)\Big)^c \\ &= \Big(\mu(\varrho, \varrho)\Big)^a \Big(\mu(\varrho, \Gamma\varrho)\Big)^{b+c} = 1, \end{aligned}$$

or

$$\mu(\Gamma\varrho, \Gamma\varrho) = 1. \tag{4.3.11}$$

Thus, by (4.3.6), (4.3.10) and (4.3.11), we have

$$\mu(\varrho, \varrho) = \mu(\Gamma\varrho, \Gamma\varrho) = \mu(\varrho, \Gamma\varrho).$$

Hence, by Definition 4.2.1, $\Gamma\varrho = \varrho$.

Uniqueness: let $\vartheta \neq \varrho \in \Omega$ be an another fixed point of Γ. Using (4.3.1), we have

$$\begin{aligned} \mu(\varrho, \vartheta) &= \mu(\Gamma\varrho, \Gamma\vartheta) \\ &\leq \Big(\mu(\varrho, \vartheta)\Big)^a \Big(\mu(\varrho, \Gamma\varrho)\Big)^b \Big(\mu(\vartheta, \Gamma\vartheta)\Big)^c \\ &= \Big(\mu(\varrho, \vartheta)\Big)^a \Big(\mu(\varrho, \varrho)\Big)^b \Big(\mu(\vartheta, \vartheta)\Big)^c \\ &= \Big(\mu(\varrho, \vartheta)\Big)^a < \mu(\varrho, \vartheta), \end{aligned}$$

a contradiction. Hence, $\varrho = \vartheta$. $\square$

Corollary 4.3.2. *Let $\Gamma : \Omega \to \Omega$ be a mapping defined on a multiplicative $\mathfrak{m}$-complete MS (Ω, μ). Suppose $\exists a \in [0,1)$ s.t.*

$$\mu(\Gamma\varrho, \Gamma\vartheta) \leq \Big(\mu(\varrho, \vartheta)\Big)^a \ \forall \varrho, \vartheta \in \Omega. \tag{4.3.12}$$

Then, Γ possesses exactly one fixed point.

Corollary 4.3.3. *Let $\Gamma : \Omega \to \Omega$ be a mapping defined on a multiplicative $\mathfrak{m}$-complete MS (Ω, μ). Suppose $\exists a \in \Big[0, \frac{1}{2}\Big)$ s.t.*

$$\mu(\Gamma\varrho, \Gamma\vartheta) \leq \Big(\mu(\varrho, \Gamma\varrho).\mu(\vartheta, \Gamma\vartheta)\Big)^a \ \forall \varrho, \vartheta \in \Omega. \tag{4.3.13}$$

Then, Γ possesses exactly one fixed point.

Theorem 4.3.4. *Let $\Gamma : \Omega \to \Omega$ be a mapping defined on a multiplicative $\mathfrak{m}$-complete MS (Ω, μ). Suppose $\exists \varrho_0 \in \Omega$ and $a \in \Big[0, \frac{1}{2}\Big)$ s.t.*

$$\mu(\Gamma\varrho, \Gamma\vartheta) \leq \Big(\mu(\varrho, \Gamma\vartheta).\mu(\vartheta, \Gamma\varrho)\Big)^a, \tag{4.3.14}$$

and

$$\mu(\Gamma^{\hbar+1}\varrho_0, \Gamma^{\hbar+1}\varrho_0) \leq \mu(\Gamma^{\hbar}\varrho_0, \Gamma^{\hbar}\varrho_0) \ \forall \hbar \in \mathbb{N}, \tag{4.3.15}$$

$\forall \varrho, \vartheta \in \Omega$. Then, Γ possesses exactly one fixed point.

Proof. For $\varrho_0 \in \Omega$, consider the sequence $\{\varrho_\hbar\}$ s.t. $\varrho_1 = \Gamma\varrho_0$. On generalizing, $\varrho_\hbar = \Gamma\varrho_{\hbar-1} \ \forall \hbar \in \mathbb{N}$.
If, $\varrho_\hbar = \varrho_{\hbar-1}$ for some $\hbar \in \mathbb{N}$. Then, $\Gamma\varrho_{\hbar-1} = \varrho_\hbar = \varrho_{\hbar-1}$ implies $\varrho_{\hbar-1}$ is a fixed point of Γ. Hence, the result holds.
Now, consider $\varrho_\hbar \neq \varrho_{\hbar-1}$, $\forall \hbar \in \mathbb{N}$.
Using (4.3.14) and triangle inequality, we have

$$\begin{aligned}
\mu(\varrho_{\hbar+1}, \varrho_\hbar) &= \mu(\Gamma\varrho_\hbar, \Gamma\varrho_{\hbar-1}) \\
&\leq \Big(\mu(\varrho_\hbar, \Gamma\varrho_{\hbar-1}).\mu(\varrho_{\hbar-1}, \Gamma\varrho_\hbar)\Big)^a \\
&= \Big(\mu(\varrho_\hbar, \varrho_\hbar).\mu(\varrho_{\hbar-1}, \varrho_{\hbar+1})\Big)^a \\
&\leq \Big(\mu(\varrho_\hbar, \varrho_\hbar).\frac{\mu(\varrho_{\hbar-1}, \varrho_\hbar)}{\mu_{\varrho_{\hbar-1}\varrho_\hbar}}.\frac{\mu(\varrho_\hbar, \varrho_{\hbar+1})}{\mu_{\varrho_\hbar\varrho_{\hbar+1}}}.\mu_{\varrho_{\hbar-1}\varrho_{\hbar+1}}\Big)^a,
\end{aligned}$$

or

$$\mu(\varrho_{\hbar+1},\varrho_{\hbar}) \leq \left(\mu(\varrho_{\hbar},\varrho_{\hbar}).\frac{\mu(\varrho_{\hbar-1},\varrho_{\hbar})}{\mu_{\varrho_{\hbar-1}\varrho_{\hbar}}}.\frac{\mu(\varrho_{\hbar},\varrho_{\hbar+1})}{\mu_{\varrho_{\hbar}\varrho_{\hbar+1}}}.\mu_{\varrho_{\hbar-1}\varrho_{\hbar+1}}\right)^{a}. \tag{4.3.16}$$

Using (4.3.15), we have

$$\mu_{\varrho_{\hbar}\varrho_{\hbar-1}} = \mu(\varrho_{\hbar},\varrho_{\hbar}) \text{ and } \mu_{\varrho_{\hbar-1}\varrho_{\hbar+1}} = \mu(\varrho_{\hbar+1},\varrho_{\hbar+1}) = \mu_{\varrho_{\hbar}\varrho_{\hbar+1}}. \tag{4.3.17}$$

Using (4.3.17) in (4.3.16), we have

$$\begin{aligned}
\mu(\varrho_{\hbar+1},\varrho_{\hbar}) &\leq \left(\mu(\varrho_{\hbar},\varrho_{\hbar}).\frac{\mu(\varrho_{\hbar-1},\varrho_{\hbar})}{\mu_{\varrho_{\hbar-1}\varrho_{\hbar}}}.\frac{\mu(\varrho_{\hbar},\varrho_{\hbar+1})}{\mu_{\varrho_{\hbar}\varrho_{\hbar+1}}}.\,\mu_{\varrho_{\hbar-1}\varrho_{\hbar+1}}\right)^{a} \\
&= \left(\mu(\varrho_{\hbar-1},\varrho_{\hbar}).\mu(\varrho_{\hbar},\varrho_{\hbar+1})\right)^{a} \\
\Rightarrow \mu(\varrho_{\hbar+1},\varrho_{\hbar})^{1-a} &\leq \mu(\varrho_{\hbar-1},\varrho_{\hbar})^{a} \\
\Rightarrow \mu(\varrho_{\hbar+1},\varrho_{\hbar}) &\leq \mu(\varrho_{\hbar-1},\varrho_{\hbar})^{\frac{a}{1-a}}.
\end{aligned}$$

Since $a<\frac{1}{2}\Rightarrow\frac{a}{1-a}<1$, using Lemma 4.2.15, we have

$$\lim_{\hbar\to\infty}\mu(\varrho_{\hbar+1},\varrho_{\hbar}) = 1, \tag{4.3.18}$$

$$\lim_{\hbar\to\infty}\mu(\varrho_{\hbar},\varrho_{\hbar}) = 1, \tag{4.3.19}$$

$$\lim_{\hbar,m\to\infty}\mu_{\varrho_{\hbar}\varrho_{m}} = 1$$

and $\{\varrho_{\hbar}\}$ is a multiplicative $\mathfrak{m}$-C_{seq}. As (Ω,μ) is complete, hence $\exists\varrho\in\Omega$ s.t.

$$\lim_{\hbar\to\infty}\frac{\mu(\varrho_{\hbar},\varrho)}{\mu_{\varrho_{\hbar}\varrho}} = 1 \quad\text{and}\quad \lim_{\hbar\to\infty}\frac{\mu^{*}_{\varrho_{\hbar}\varrho}}{\mu_{\varrho_{\hbar}\varrho}} = 1. \tag{4.3.20}$$

Also, using (4.3.19), we have

$$\lim_{\hbar\to\infty}\mu_{\varrho_{\hbar}\varrho} = \lim_{\hbar\to\infty}\min\left\{\mu(\varrho_{\hbar},\varrho_{\hbar}),\mu(\varrho,\varrho)\right\} \leq \lim_{\hbar\to\infty}\mu(\varrho_{\hbar},\varrho_{\hbar}) \to 1. \tag{4.3.21}$$

Using (4.3.20), (4.3.21) and Remark 4.2.4, we have

$$\lim_{\hbar\to\infty}\mu(\varrho_{\hbar},\varrho) = 1, \lim_{\hbar\to\infty}\mu^{*}_{\varrho_{\hbar}\varrho} = 1 \text{ and } \mu(\varrho,\varrho) = 1. \tag{4.3.22}$$

Also,

$$1 \leq \mu_{\varrho\Gamma\varrho} = \min\left\{\mu(\varrho,\varrho),\mu(\Gamma\varrho,\Gamma\varrho)\right\} \leq \mu(\varrho,\varrho) = 1 \Rightarrow \mu_{\varrho\Gamma\varrho} = 1. \tag{4.3.23}$$

Using triangle inequality, (4.3.22) and (4.3.23), we have

$$\mu(\varrho,\Gamma\varrho) \;=\; \frac{\mu(\varrho,\Gamma\varrho)}{\mu_{\varrho\Gamma\varrho}} \leq \limsup_{\hbar\to\infty}\frac{\mu(\varrho,\varrho_{\hbar})}{\mu_{\varrho_{\hbar}\varrho}}.\frac{\mu(\varrho_{\hbar},\Gamma\varrho)}{\mu_{\varrho_{\hbar}\Gamma\varrho}} \leq \limsup_{\hbar\to\infty}\mu(\varrho_{\hbar},\Gamma\varrho),$$

or

$$\mu(\varrho, \Gamma\varrho) \leq \limsup_{\hbar\to\infty} \mu(\Gamma\varrho_{\hbar-1}, \Gamma\varrho). \tag{4.3.24}$$

Using, (4.3.14), (4.3.22), (4.3.23) and the triangle inequality in (4.3.24), we have

$$\begin{aligned}
\mu(\varrho, \Gamma\varrho) &\leq \limsup_{\hbar\to\infty}\Big(\mu(\varrho_{\hbar-1}, \Gamma\varrho).\mu(\varrho, \Gamma\varrho_{\hbar-1})\Big)^a \\
&\leq \limsup_{\hbar\to\infty}\Big(\frac{\mu(\varrho_{\hbar-1}, \varrho)}{\mu_{\varrho_{\hbar-1}\varrho}}.\frac{\mu(\varrho, \Gamma\varrho)}{\mu_{\varrho\Gamma\varrho}}.\mu_{\varrho_{\hbar-1}\Gamma\varrho}.\, \mu(\varrho, \varrho_\hbar)\Big)^a \\
&= \limsup_{\hbar\to\infty}\Big(\mu(\varrho, \Gamma\varrho).\mu_{\varrho_{\hbar-1}\Gamma\varrho}\Big)^a,
\end{aligned}$$

or

$$\mu(\varrho, \Gamma\varrho) \leq \limsup_{\hbar\to\infty}\Big(\mu(\varrho, \Gamma\varrho).\mu_{\varrho_{\hbar-1}\Gamma\varrho}\Big)^a. \tag{4.3.25}$$

Since, $\mu_{\varrho_{\hbar-1}\Gamma\varrho} = \min\Big\{\mu(\varrho_{\hbar-1}, \varrho_{\hbar-1}), \mu(\Gamma\varrho, \Gamma\varrho)\Big\} \leq \mu(\varrho_{\hbar-1}, \varrho_{\hbar-1}) \to 1$, by (4.3.25), we have

$$\mu(\varrho, \Gamma\varrho) \leq \Big(\mu(\varrho, \Gamma\varrho)\Big)^a.$$

Since, $a < \frac{1}{2}$, we have

$$\mu(\varrho, \Gamma\varrho) = 1. \tag{4.3.26}$$

Using (4.3.14), we have

$$1 \leq \mu(\Gamma\varrho, \Gamma\varrho) \leq \Big(\mu(\varrho, \Gamma\varrho).\mu(\varrho, \Gamma\varrho)\Big)^a = 1. \tag{4.3.27}$$

Thus, by (4.3.22), (4.3.26) and (4.3.27), we have

$$\mu(\varrho, \varrho) = \mu(\Gamma\varrho, \Gamma\varrho) = \mu(\varrho, \Gamma\varrho).$$

Hence, by Definition 4.2.1 $\Gamma\varrho = \varrho$.
Uniqueness: let $\vartheta \neq \varrho \in \Omega$ be an another fixed point of Γ. Using (4.3.14), we have

$$\mu(\varrho, \vartheta) = \mu(\Gamma\varrho, \Gamma\vartheta) \leq \Big(\mu(\varrho, \Gamma\vartheta).\mu(\vartheta, \Gamma\varrho)\Big)^a = \Big(\mu(\varrho, \vartheta).\mu(\varrho, \vartheta)\Big)^a < \mu(\varrho, \vartheta),$$

a contradiction. Hence, $\varrho = \vartheta$. □

Example 4.3.5. *Consider $\Omega = [0,\infty)$ along with $\mu(\varrho,\vartheta) = e^{\frac{\varrho+\vartheta}{2}}$, $\forall\ \varrho,\vartheta \in \Omega$. Here, (Ω,μ) is a multiplicative $\mathfrak{m}$-complete MS. Consider $\Gamma : \Omega \to \Omega$ s.t.*

$$\Gamma(\varrho) = \begin{cases} \dfrac{\varrho}{5}, & \text{if } \varrho \in [0,3) \\ \dfrac{\varrho}{\varrho+2}, & \text{if } \varrho \geq 3. \end{cases}$$

Now, we will prove that Γ satisfies all the conditions of Theorem 4.3.1 with $a = \frac{1}{5}$, $b = \frac{1}{3}$ and $c = \frac{1}{4}$.

Case (i) For $\varrho,\vartheta < 3$, we have

$$\begin{aligned} \mu(\Gamma\varrho,\Gamma\vartheta) &= e^{\frac{1}{2}\left(\frac{\varrho}{5}+\frac{\vartheta}{5}\right)} \\ &\leq e^{\frac{1}{5}\left(\frac{\varrho+\vartheta}{2}\right)+\frac{1}{3}\left(\frac{1}{2}(\varrho+\frac{\varrho}{5})\right)+\frac{1}{4}\left(\frac{1}{2}(\vartheta+\frac{\vartheta}{5})\right)} \\ &= \left(\mu(\varrho,\vartheta)\right)^{\frac{1}{5}}.\left(\mu(\varrho,\Gamma\varrho)\right)^{\frac{1}{3}}.\left(\mu(\vartheta,\Gamma\vartheta)\right)^{\frac{1}{4}}. \end{aligned}$$

Case (ii) For $\varrho,\vartheta \geq 3$, we have

$$\begin{aligned} \mu(\Gamma\varrho,\Gamma\vartheta) &= e^{\frac{1}{2}\left(\frac{\varrho}{\varrho+2}+\frac{\vartheta}{\vartheta+2}\right)} \\ &\leq e^{\frac{1}{2}\left(\frac{\varrho}{5}+\frac{\vartheta}{5}\right)} = e^{\frac{1}{5}\left(\frac{\varrho}{2}+\frac{\vartheta}{2}\right)} \\ &\leq e^{\frac{1}{5}\left(\frac{\varrho}{2}+\frac{\vartheta}{2}\right)+\frac{1}{3}\left(\frac{1}{2}(\varrho+\frac{\varrho}{\varrho+2})\right)+\frac{1}{4}\left(\frac{1}{2}(\vartheta+\frac{\vartheta}{\vartheta+2})\right)} \\ &= \left(\mu(\varrho,\vartheta)\right)^{\frac{1}{5}}.\left(\mu(\varrho,\Gamma\varrho)\right)^{\frac{1}{3}}.\left(\mu(\vartheta,\Gamma\vartheta)\right)^{\frac{1}{4}}. \end{aligned}$$

Case (iii) For $\varrho < 3$ and $\vartheta \geq 3$, we have

$$\begin{aligned} \mu(\Gamma\varrho,\Gamma\vartheta) &= e^{\frac{1}{2}\left(\frac{\varrho}{5}+\frac{\vartheta}{\vartheta+2}\right)} \leq e^{\frac{1}{2}\left(\frac{\varrho}{5}+\frac{\vartheta}{5}\right)} \\ &\leq e^{\frac{1}{5}\left(\frac{\varrho}{2}+\frac{\vartheta}{2}\right)+\frac{1}{3}\left(\frac{1}{2}(\varrho+\frac{\varrho}{\varrho+2})\right)+\frac{1}{4}\left(\frac{1}{2}(\vartheta+\frac{\vartheta}{\vartheta+2})\right)} \\ &= \left(\mu(\varrho,\vartheta)\right)^{\frac{1}{5}}.\left(\mu(\varrho,\Gamma\varrho)\right)^{\frac{1}{3}}.\left(\mu(\vartheta,\Gamma\vartheta)\right)^{\frac{1}{4}}. \end{aligned}$$

For $a = \frac{1}{5}$, $b = \frac{1}{3}$ and $c = \frac{1}{4}$, Γ satisfies all the conditions of Theorem 4.3.1. Hence, Γ possesses exactly one fixed point i.e., $\varrho = 0$.

Example 4.3.6. *Consider* $\Omega = [0, 1]$ *with multiplicative* $\mathfrak{m}$*-metric* $\mu(\varrho, \vartheta) = e^{\frac{\varrho+\vartheta}{2}}$. *Clearly,* (Ω, μ) *is multiplicative* $\mathfrak{m}$*-complete MS. Consider* $\Gamma : \Omega \to \Omega$

$$\Gamma(\varrho) = \begin{cases} \frac{\varrho}{4}, & if \;\; \varrho < 1 \\ \frac{1}{8}, & if \, \varrho = 1. \end{cases}$$

Now, we will prove that Γ satisfies all the conditions of Corollary 4.3.2 with $a = \frac{1}{4}$.
Case (i) For $\varrho, \vartheta < 1$, we have

$$\begin{aligned} \mu(\Gamma\varrho, \Gamma\vartheta) &= e^{\dfrac{1}{2}\left(\dfrac{\varrho}{4} + \dfrac{\vartheta}{4}\right)} \\ &\leq e^{\dfrac{1}{4}\left(\dfrac{\varrho + \vartheta}{2}\right)} \\ &= \Big(\mu(\varrho, \vartheta)\Big)^{a}. \end{aligned}$$

Case (ii) For $\varrho = \vartheta = 1$, we have

$$\begin{aligned} \mu(\Gamma\varrho, \Gamma\vartheta) &= e^{1/8} \leq e^{1/4} \\ &= \Big(\mu(\varrho, \vartheta)\Big)^{a}. \end{aligned}$$

Case (iii) For $\varrho < 1$ and $\vartheta = 1$, we have

$$\begin{aligned} \mu(\Gamma\varrho, \Gamma\vartheta) &= e^{\dfrac{1}{2}\left(\dfrac{\varrho}{4} + \dfrac{1}{8}\right)} \\ &\leq e^{\dfrac{1}{2}\left(\dfrac{\varrho}{4} + \dfrac{1}{4}\right)} \\ &= \Big(\mu(\varrho, \vartheta)\Big)^{a}. \end{aligned}$$

For $a = \frac{1}{4} \in [0, 1)$, Γ satisfies all the conditions of Corollary 4.3.2. Hence, Γ possesses exactly one fixed point i.e., $\varrho = 0$.

Remark 4.3.7. *In Example 4.3.6, self mapping Γ is not continuous (in multiplicative MS). Hence, multiplicative Banach contraction is not applicable. Therefore, the existence of fixed point can not be guaranteed in multiplicative MS. On other hand, Γ satisfies Corollary 4.3.2. Hence, the existence of a fixed point is guaranteed in multiplicative $\mathfrak{m}$-MS.*

Example 4.3.8. *Consider $\Omega = [0,1]$ with multiplicative $\mathfrak{m}$-metric $\mu(\varrho, \vartheta) = e^{\frac{\varrho+\vartheta}{2}}$. (Ω, μ) is multiplicative $\mathfrak{m}$-complete MS. Consider $\Gamma : \Omega \to \Omega$*

$$\Gamma(\varrho) = \begin{cases} \frac{\varrho}{3}, & \text{if } \varrho < 1 \\ \frac{1}{7}, & \text{if } \varrho = 1. \end{cases}$$

Now, we will prove that Γ satisfies all the conditions of Corollary 4.3.3 with $a = \frac{1}{3}$.
Case (i) For $\varrho, \vartheta < 1$, we have

$$\begin{aligned} \mu(\Gamma\varrho, \Gamma\vartheta) &= e^{\dfrac{1}{2}\left(\dfrac{\varrho}{3} + \dfrac{\vartheta}{3}\right)} \\ &\leq \left(e^{\dfrac{1}{2}(\varrho + \dfrac{\varrho}{3})}.e^{\dfrac{1}{2}(\vartheta + \dfrac{\vartheta}{3})}\right)^{\dfrac{1}{3}} \\ &= \left(\mu(\varrho, \Gamma\varrho).\mu(\vartheta, \Gamma\vartheta)\right)^{a}. \end{aligned}$$

Case (ii) For $\varrho = \vartheta = 1$, we have

$$\begin{aligned} \mu(\Gamma\varrho, \Gamma\vartheta) &= e^{1/7} = \left(e^{8/14}.e^{8/14}\right)^{1/8} \\ &\leq \left(e^{8/14}.e^{8/14}\right)^{1/3} \\ &= \left(\mu(\varrho, \Gamma\varrho).\mu(\vartheta, \Gamma\vartheta)\right)^{a}. \end{aligned}$$

Case (iii) For $\varrho < 1$ and $\vartheta = 1$, we have

$$\begin{aligned} \mu(\Gamma\varrho, \Gamma\vartheta) &= e^{\dfrac{1}{2}\left(\dfrac{\varrho}{3} + \dfrac{1}{7}\right)} = e^{\dfrac{1}{3}\left(\dfrac{1}{2}(\varrho + \dfrac{3}{7})\right)} \\ &\leq e^{\dfrac{1}{3}\left(\dfrac{1}{2}(\varrho + \dfrac{\varrho}{3} + 1 + \dfrac{1}{7})\right)} \\ &= \left(\mu(\varrho, \Gamma\varrho).\mu(\vartheta, \Gamma\vartheta)\right)^{a}. \end{aligned}$$

For, $a = \frac{1}{3} \in [0, \frac{1}{2})$, Γ satisfies all the conditions of Corollary 4.3.3. Hence, Γ possesses exactly one fixed point i.e., $\varrho = 0$.

Remark 4.3.9. *In Example 4.3.8, one can easily verify that the self mapping Γ with induced multiplicative metric $u(\varrho, \vartheta) = e^{\frac{|\varrho-\vartheta|}{2}}$ does not satisfy Kannan-contraction in multiplicative MS for $\varrho = 0$ and $\vartheta = \frac{1}{3}$. Hence, the existence of a fixed point in multiplicative MS can not be guaranteed. On the other hand, Γ satisfies the conditions of Corollary 4.3.3. Hence, existence of a fixed point is guaranteed in multiplicative $\mathfrak{m}$-MS.*

4.3.1 Numerical Approximation of Fixed Point

In this section, we have presented some iterations for the approximation of unique fixed point of Γ in Example 4.3.5, 4.3.6 and 4.3.8. Also, we established the convergence of Picard's iterative sequence graphically and concluded that the fixed point of the mapping does not depend on the initial point of the iterative procedure.

x_0	1.000000	2.000000	3.000000	4.000000	5.000000
x_1	0.200000	0.400000	0.600000	0.666667	0.714286
x_2	0.040000	0.080000	0.120000	0.133333	0.142857
x_3	0.008000	0.016000	0.024000	0.026667	0.028571
x_4	0.001600	0.003200	0.004800	0.005333	0.005714
x_5	0.000320	0.000640	0.000960	0.001067	0.001143
x_6	0.000064	0.000128	0.000192	0.000213	0.000229
x_7	0.000013	0.000026	0.000038	0.000043	0.000046
x_8	0.000003	0.000005	0.000008	0.000009	0.000009
x_9	0.000001	0.000001	0.000002	0.000002	0.000002
x_{10}	0.000000	0.000000	0.000000	0.000000	0.000000

Figure 4.1: Iteration for Picard's sequence of Example 4.3.5.

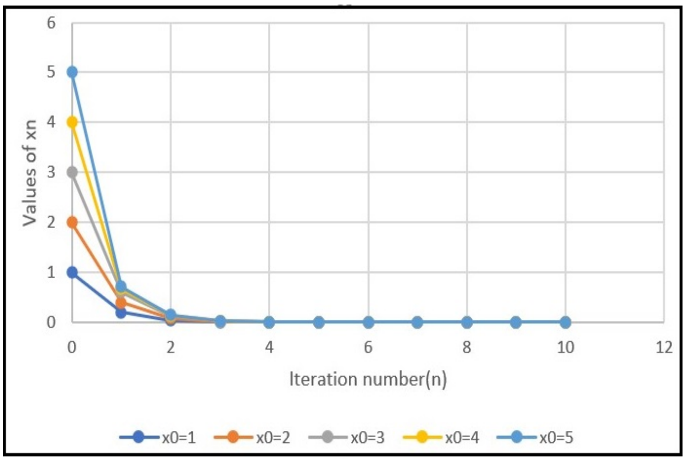

Figure 4.2: Convergence behaviour of Picard's sequence at different points for Example 4.3.5.

x_0	0.200000	0.400000	0.600000	0.800000	1.000000
x_1	0.050000	0.100000	0.150000	0.200000	0.125000
x_2	0.012500	0.025000	0.037500	0.050000	0.031250
x_3	0.003125	0.006250	0.009375	0.012500	0.007813
x_4	0.000781	0.001563	0.002344	0.003125	0.001953
x_5	0.000195	0.000391	0.000586	0.000781	0.000488
x_6	0.000049	0.000098	0.000147	0.000195	0.000122
x_7	0.000012	0.000025	0.000037	0.000049	0.000031
x_8	0.000003	0.000006	0.000009	0.000012	0.000008
x_9	0.000001	0.000002	0.000002	0.000003	0.000002
x_{10}	0.000000	0.000001	0.000001	0.000001	0.000001
x_{11}	0.000000	0.000000	0.000000	0.000000	0.000000

Figure 4.3: Iteration for Picard's sequence of Example 4.3.6.

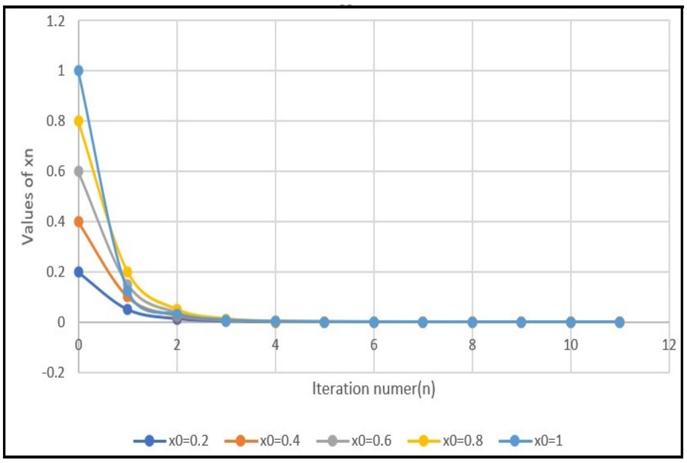

Figure 4.4: Convergence behaviour of Picard's sequence at different points for Example 4.3.6.

x_0	0.210000	0.410000	0.610000	0.810000	1.000000
x_1	0.070000	0.136667	0.203333	0.270000	0.142857
x_2	0.023333	0.045556	0.067778	0.090000	0.047619
x_3	0.007778	0.015185	0.022593	0.030000	0.015873
x_4	0.002593	0.005062	0.007531	0.010000	0.005291
x_5	0.000864	0.001687	0.002510	0.003333	0.001764
x_6	0.000288	0.000562	0.000837	0.001111	0.000588
x_7	0.000096	0.000187	0.000279	0.000370	0.000196
x_8	0.000032	0.000062	0.000093	0.000123	0.000065
x_9	0.000011	0.000021	0.000031	0.000041	0.000022
x_{10}	0.000004	0.000007	0.000010	0.000014	0.000007
x_{11}	0.000001	0.000002	0.000003	0.000005	0.000002
x_{12}	0.000000	0.000001	0.000001	0.000002	0.000001
x_{13}	0.000000	0.000000	0.000000	0.000001	0.000000

Figure 4.5: Iteration for Picard's sequence of Example 4.3.8.

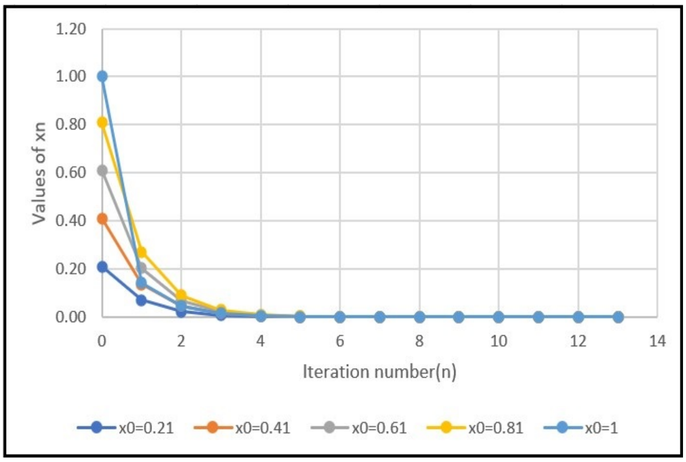

Figure 4.6: Convergence behaviour of Picard's sequence at different points for Example 4.3.8.

4.4 Some Common Fixed Point Results in Multiplicative $\mathfrak{m}$-Metric Space

In this section, we have established some common fixed point results for a pair of self mappings using well known contractions in the framework of multiplicative $\mathfrak{m}$-MS.

Theorem 4.4.1. *Let $\Gamma_1, \Gamma_2 : \Omega \to \Omega$ be mappings defined on multiplicative $\mathfrak{m}$-complete MS (Ω, μ). Suppose $\exists a_1, a_2, a_3 \in (0,1)$ with $a_1 + a_2 + a_3 < 1$ s.t.*

$$\mu(\Gamma_1\varrho, \Gamma_2\vartheta) \leq (\mu(\varrho, \vartheta))^{a_1} (\mu(\varrho, \Gamma_1\varrho))^{a_2} (\mu(\vartheta, \Gamma_2\vartheta))^{a_3} \ \forall \varrho, \vartheta \in \Omega. \tag{4.4.1}$$

Then, either Γ_1 or Γ_2 has a fixed point $\varrho_0 \in \Omega$ (say). Moreover, if $\mu^*_{\Gamma_1\varrho_0, \Gamma_2\varrho_0} \leq \mu(\Gamma_1\varrho_0, \Gamma_2\varrho_0)$, then Γ_1, Γ_2 have exactly one common fixed point.

Proof. For $\varrho_0 \in \Omega$, construct a sequence in Ω

$$\varrho_{2\hbar+1} = \Gamma_1\varrho_{2\hbar}, \text{ and } \varrho_{2\hbar+2} = \Gamma_2\varrho_{2\hbar+1} \text{ for } \hbar \in \mathbb{N}_0.$$

If for some $\hbar_0 \in \mathbb{N}_0$ we have $\varrho_{2\hbar_0+1} = \varrho_{2\hbar_0+2}$. Then,

$$\varrho_{2\hbar_0+1} = \Gamma_1\varrho_{2\hbar_0} = \varrho_{2\hbar_0+2} = \Gamma_2\varrho_{2\hbar_0+1}$$

implies $\varrho_{2\hbar_0+1} = \varrho_{2\hbar_0+2}$ is the fixed point of mapping Γ_2.
Now, consider $\varrho_{2\hbar+1} \neq \varrho_{2\hbar+2}$ for $\hbar \in \mathbb{N}_0$. Then,

$$\begin{aligned}
\mu(\varrho_{2\hbar+1}, \varrho_{2\hbar+2}) &= \mu(\Gamma_1\varrho_{2\hbar}, \Gamma_2\varrho_{2\hbar+1}) \\
&\leq (\mu(\varrho_{2\hbar}, \varrho_{2\hbar+1}))^{a_1}(\mu(\varrho_{2\hbar}, \Gamma_1\varrho_{2\hbar}))^{a_2}(\mu(\varrho_{2\hbar+1}, \Gamma_2\varrho_{2\hbar+1}))^{a_3} \\
&= (\mu(\varrho_{2\hbar}, \varrho_{2\hbar+1}))^{a_1}(\mu(\varrho_{2\hbar}, \varrho_{2\hbar+1}))^{a_2}(\mu(\varrho_{2\hbar+1}, \varrho_{2\hbar+2}))^{a_3}
\end{aligned}$$

or

$$\mu(\varrho_{2\hbar+1}, \varrho_{2\hbar+2})^{1-a_3} \leq (\mu(\varrho_{2\hbar}, \varrho_{2\hbar+1}))^{a_1+a_2} \Leftrightarrow \mu(\varrho_{2\hbar+1}, \varrho_{2\hbar+2}) \leq (\mu(\varrho_{2\hbar}, \varrho_{2\hbar+1}))^{\frac{a_1+a_2}{1-a_3}}.$$

Using similar arguments, we have

$$\mu(\varrho_{2\hbar+2}, \varrho_{2\hbar+3}) \leq (\mu(\varrho_{2\hbar+1}, \varrho_{2\hbar+2}))^{\frac{a_1+a_2}{1-a_3}}.$$

Therefore,

$$\mu(\varrho_{\hbar}, \varrho_{\hbar+1}) \leq \mu(\varrho_{\hbar-1}, \varrho_{\hbar})^{\frac{a_1+a_2}{1-a_3}} \text{ or } \mu(\varrho_{\hbar+1}, \varrho_{\hbar}) \leq \mu(\varrho_{\hbar}, \varrho_{\hbar-1})^{\frac{a_1+a_2}{1-a_3}} \quad \forall\, \hbar \in \mathbb{N}.$$

Since $\frac{a+b}{1-c} < 1$, using Lemma 4.2.15, we have

$$\lim_{\hbar\to\infty} \mu(\varrho_{\hbar+1}, \varrho_{\hbar}) = 1, \tag{4.4.2}$$

$$\lim_{\hbar\to\infty} \mu(\varrho_{\hbar}, \varrho_{\hbar}) = 1, \tag{4.4.3}$$

$$\lim_{\hbar,m\to\infty} \mu_{\varrho_{\hbar},\varrho_m} = 1 \tag{4.4.4}$$

and $\{\varrho_{\hbar}\}$ is a multiplicative $\mathfrak{m}$-C_{seq}. As, (Ω, μ) is multiplicative $\mathfrak{m}$-complete, therefore $\exists \varrho \in \Omega$ s.t.

$$\lim_{\hbar\to\infty} \frac{\mu(\varrho_{\hbar}, \varrho)}{\mu_{\varrho_{\hbar},\varrho}} = 1 \text{ and } \lim_{\hbar\to\infty} \frac{\mu^*(\varrho_{\hbar}, \varrho)}{\mu^*_{\varrho_{\hbar},\varrho}} = 1. \tag{4.4.5}$$

Moreover, using (4.4.3), we have

$$\lim_{\hbar\to\infty} \mu_{\varrho_{\hbar}\varrho} = \lim_{\hbar\to\infty} \min\{\mu(\varrho_{\hbar}, \varrho_{\hbar}), \mu(\varrho, \varrho)\} \leq \lim_{\hbar\to\infty} \mu(\varrho_{\hbar}, \varrho_{\hbar}) = 1. \tag{4.4.6}$$

Using (4.4.5), (4.4.6) and Remark 4.2.4 , we have

$$\lim_{\hbar\to\infty} \mu(\varrho_{\hbar}, \varrho) = 1, \lim_{\hbar\to\infty} \mu^*_{\varrho_{\hbar}\varrho} = 1 \text{ and } \mu(\varrho, \varrho) = 1. \tag{4.4.7}$$

Also,

$$\mu_{\varrho,\Gamma_1\varrho} = \min\{\mu(\varrho, \varrho), \mu(\Gamma_1\varrho, \Gamma_1\varrho)\} \leq \mu(\varrho, \varrho) = 1, \tag{4.4.8}$$

and

$$\mu_{\varrho,\Gamma_2\varrho} = \min\{\mu(\varrho, \varrho), \mu(\Gamma_2\varrho, \Gamma_2\varrho)\} \leq \mu(\varrho, \varrho) = 1. \tag{4.4.9}$$

Further, using (4.4.6), (4.4.9) and the triangle inequality, we have

$$\begin{aligned}
\mu(\varrho, \Gamma_2\varrho) = \frac{\mu(\varrho, \Gamma_2\varrho)}{\mu_{\varrho,\Gamma_2\varrho}} &\leq \frac{\mu(\varrho, \varrho_{2\hbar+2})}{\mu_{\varrho,\varrho_{2\hbar+1}}} . \frac{\mu(\varrho_{2\hbar+1}, \Gamma_2\varrho)}{\mu_{\varrho_{\varrho_{2\hbar+1}},\Gamma_2\varrho}} \\
&\leq \limsup_{\hbar\to\infty} \frac{\mu(\varrho, \varrho_{2\hbar+2})}{\mu_{\varrho,\varrho_{2\hbar+1}}} . \frac{\mu(\varrho_{2\hbar+1}, \Gamma_2\varrho)}{\mu_{\varrho_{2\hbar+1},\Gamma_2\varrho}} \\
&\leq \limsup_{\hbar\to\infty} \mu(\varrho_{2\hbar+1}, \Gamma_2\varrho),
\end{aligned}$$

or

$$\mu(\varrho, \Gamma_2\varrho) \leq \limsup_{\hbar\to\infty} \mu(\Gamma_1\varrho_{2\hbar}, \Gamma_2\varrho). \tag{4.4.10}$$

Using (4.4.1), (4.4.6), (4.4.9) and the triangle inequality in (4.4.10), we have

$$\begin{aligned}
\mu(\varrho, \Gamma_2\varrho) &\leq \limsup_{\hbar\to\infty} \mu(\Gamma_1\varrho_{2\hbar}, \Gamma_2\varrho) \\
&\leq \limsup_{\hbar\to\infty} (\mu(\varrho_{2\hbar}, \varrho))^{a_1} (\mu(\varrho_{2\hbar}, \Gamma_1\varrho_{2\hbar}))^{a_2} \mu(\varrho, \Gamma_2\varrho)^{a_3} \\
&\leq \limsup_{\hbar\to\infty} (\mu(\varrho_{2\hbar}, \varrho))^{a_1} (\mu(\varrho_{2\hbar}, \varrho_{2\hbar+1}))^{a_2} \mu(\varrho, \Gamma_2\varrho)^{a_3} \\
&= \mu(\varrho, \Gamma_2\varrho)^c.
\end{aligned}$$

Since, $a_3 < 1$. Hence,

$$\mu(\varrho, \Gamma_2\varrho) = 1. \tag{4.4.11}$$

On similar lines, we have

$$\mu(\varrho, \Gamma_1\varrho) = 1. \tag{4.4.12}$$

Using (4.4.1), (4.4.7), (4.4.11) and (4.4.12), we have

$$\mu(\Gamma_1\varrho, \Gamma_2\varrho) \leq (\mu(\varrho, \varrho))^{a_1}.(\mu(\varrho, \Gamma_1\varrho))^{a_2}.(\mu(\varrho, \Gamma_2\varrho))^{a_3} = 1,$$

or

$$\mu(\Gamma_1\varrho, \Gamma_2\varrho) = 1. \tag{4.4.13}$$

Also,

$$\mu_{\Gamma_1\varrho, \Gamma_2\varrho} = \min\{\mu(\Gamma_1\varrho, \Gamma_1\varrho), \mu(\Gamma_2\varrho, \Gamma_2\varrho)\} \leq \mu(\Gamma_1\varrho, \Gamma_2\varrho) = 1.$$

Suppose, $\mu(\Gamma_1\varrho, \Gamma_1\varrho) \leq \mu(\Gamma_2\varrho, \Gamma_2\varrho)$. Then $\mu(\Gamma_1\varrho, \Gamma_1\varrho) = 1$. Hence, $\mu(\Gamma_1\varrho, \Gamma_1\varrho) = 1 = \mu(\varrho, \varrho) = \mu(\varrho, \Gamma_1\varrho)$ implies $\Gamma_1\varrho = \varrho$, i.e., ϱ is the fixed point of Γ_1.
Further, suppose that $\mu^*_{\Gamma_1\varrho, \Gamma_2\varrho} \leq \mu(\Gamma_1\varrho, \Gamma_2\varrho)$. Then,

$$\mu(\Gamma_2\varrho, \Gamma_2\varrho) = \max\{\mu(\Gamma_1\varrho, \Gamma_1\varrho), \mu(\Gamma_2\varrho, \Gamma_2\varrho)\} = \mu^*_{\Gamma_1\varrho, \Gamma_2\varrho} \leq \mu(\Gamma_1\varrho, \Gamma_2\varrho) = 1,$$

or

$$\mu(\Gamma_2\varrho, \Gamma_2\varrho) = 1.$$

Therefore,

$$\mu(\Gamma_1\varrho, \Gamma_2\varrho) = \mu(\Gamma_1\varrho, \Gamma_1\varrho) = \mu(\Gamma_2\varrho, \Gamma_2\varrho) = 1,$$

implies $\Gamma_1\varrho = \Gamma_2\varrho = \varrho$.
Uniqueness: let $\vartheta \neq \varrho \in \Omega$ be an another common fixed point of Γ_1, Γ_2. Then, using (4.4.1), we have

$$\begin{aligned}
\mu(\varrho, \vartheta) &= \mu(\Gamma_1\varrho, \Gamma_2\vartheta) \\
&\leq \Big(\mu(\varrho, \vartheta)\Big)^{a_1} \Big(\mu(\varrho, \Gamma_1\varrho)\Big)^{a_2} \Big(\mu(\vartheta, \Gamma_2\vartheta)\Big)^{a_3} \\
&= \Big(\mu(\varrho, \vartheta)\Big)^{a_1} \Big(\mu(\varrho, \varrho)\Big)^{a_2} \Big(\mu(\vartheta, \vartheta)\Big)^{a_3} \\
&= \Big(\mu(\varrho, \vartheta)\Big)^{a_1} < \mu(\varrho, \vartheta),
\end{aligned}$$

a contradiction. Hence, $\vartheta = \varrho$. □

Example 4.4.2. *Consider $\Omega = [0, \infty)$ with multiplicative m-metric $\mu(\varrho, \vartheta) = e^{\max\{\varrho, \vartheta\}}$. Clearly, (Ω, μ) is multiplicative $\mathfrak{m}$-complete MS.Consider, $\Gamma_1, \Gamma_2 : \Omega \to \Omega$*

$$\Gamma_1\varrho = \begin{cases} \frac{\varrho}{5}, & \text{if } \varrho \in [0,1) \\ \frac{1}{10}, & \text{otherwise.} \end{cases}, \ \Gamma_2\varrho = \begin{cases} \frac{\varrho}{7}, & \text{if } \varrho \in [0,1) \\ \frac{1}{14}, & \text{otherwise.} \end{cases}.$$

Then,

(i) for $\varrho, \vartheta \in [0,1)$, we have

$$\begin{aligned} \mu(\Gamma_1\varrho, \Gamma_2\vartheta) &= e^{\max\{\Gamma_1\varrho, \Gamma_2\vartheta\}} \\ &= e^{\max\{\frac{\varrho}{5}, \frac{\varrho}{7}\}} = e^{\frac{\varrho}{5}} \\ &\le \mu(\varrho, \vartheta)^{\frac{1}{5}}. \end{aligned}$$

(ii) For $\varrho, \vartheta \ge 1$, we have

$$\begin{aligned} \mu(\Gamma_1\varrho, \Gamma_2\vartheta) &= e^{\max\{\Gamma_1\varrho, \Gamma_2\vartheta\}} \\ &= e^{\max\{\frac{1}{10}, \frac{1}{14}\}} = e^{\frac{1}{10}} \\ &\le (e^{\max\{\varrho, \vartheta\}})^{\frac{1}{5}} = \mu(\varrho, \vartheta)^{\frac{1}{5}}. \end{aligned}$$

(iii) For $\varrho > 1, \vartheta \le \frac{7}{10}$, we have

$$\begin{aligned} \mu(\Gamma_1\varrho, \Gamma_2\vartheta) &= e^{\max\{\Gamma_1\varrho, \Gamma_2\vartheta\}} \\ &= e^{\max\{\frac{1}{10}, \frac{\vartheta}{7}\}} = e^{\frac{1}{10}} \\ &\le (e^{\max\{\varrho, \vartheta\}})^{\frac{1}{5}} = \mu(\varrho, \vartheta)^{\frac{1}{5}}. \end{aligned}$$

(iv) For $\varrho > 1, \frac{7}{10} < \vartheta < 1$, we have

$$\begin{aligned} \mu(\Gamma_1\varrho, \Gamma_2\vartheta) &= e^{\max\{\Gamma_1\varrho, \Gamma_2\vartheta\}} \\ &= e^{\max\{\frac{1}{10}, \frac{\vartheta}{7}\}} = e^{\frac{\vartheta}{7}} \\ &\le (e^{\max\{\varrho, \vartheta\}})^{\frac{1}{7}} \le \mu(\varrho, \vartheta)^{\frac{1}{5}}. \end{aligned}$$

(v) For $\vartheta > 1, \varrho \le \frac{5}{14}$, we have

$$\begin{aligned} \mu(\Gamma_1\varrho, \Gamma_2\vartheta) &= e^{\max\{\Gamma_1\varrho, \Gamma_2\vartheta\}} \\ &= e^{\max\{\frac{\varrho}{5}, \frac{1}{14}\}} = e^{\frac{1}{14}} \le (e^{\vartheta})^{\frac{1}{7}} \\ &\le (e^{\max\{\varrho, \vartheta\}})^{\frac{1}{7}} \le \mu(\varrho, \vartheta)^{\frac{1}{5}}. \end{aligned}$$

(vi) For $\vartheta > 1, \frac{5}{14} < \varrho < 1$, we have

$$\begin{aligned}
\mu(\Gamma_1\varrho, \Gamma_2\vartheta) &= e^{\max\{\Gamma_1\varrho, \Gamma_2\vartheta\}} \\
&= e^{\max\{\frac{\varrho}{5}, \frac{1}{14}\}} = e^{\frac{\varrho}{5}} \leq (e^{\varrho})^{\frac{1}{5}} \\
&\leq (e^{\max\{\varrho, \vartheta\}})^{\frac{1}{5}},
\end{aligned}$$

and $\mu^*_{\Gamma_1\varrho, \Gamma_2\varrho} = \max\{\mu(\Gamma_1\varrho, \Gamma_1\varrho), \mu(\Gamma_2\varrho, \Gamma_2\varrho)\} = \max\left\{e^{\Gamma_1\varrho}, e^{\Gamma_2\varrho}\right\} \leq e^{\max\{\Gamma_1\varrho, \Gamma_2\varrho\}} = \mu(\Gamma_1\varrho, \Gamma_2\varrho)$. *Therefore, Γ_1, Γ_2 satisfy Theorem 4.4.1 with $a_1 = \frac{1}{5}, a_2 = 0 = a_3$. Hence, Γ_1, Γ_2 have exactly one common fixed point.*

4.4.1 Numerical Approximation of Common Fixed Point

In this section, we have presented some iterations for approximating the common fixed point of Γ_1, Γ_2 in Example 4.4.2. In addition, we graphically demonstrated the convergence of Iterative sequence and concluded that the fixed point of the mapping is independent of the iterative procedure's initial point (see Figure 4.4.1). The iteration scheme used for the approximation is given as

$$\text{For initial point } x_0,\ x_{2\hbar+1} = \Gamma_1 x_{2\hbar} \text{ and } x_{2\hbar+2} = \Gamma_2 x_{2\hbar+1}.$$

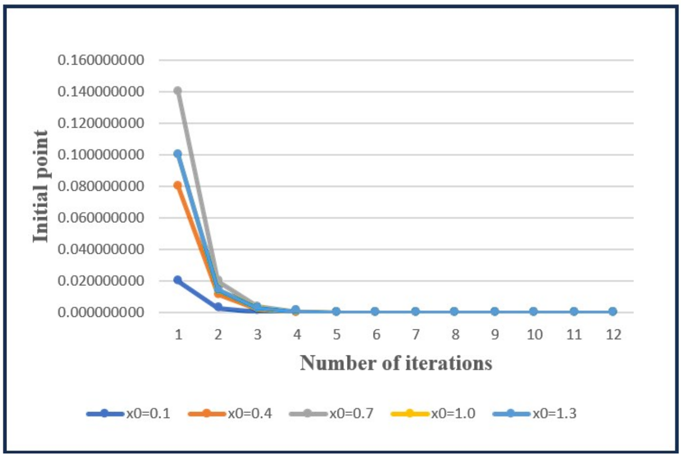

Figure 4.7: Convergence behaviour of iteration scheme at different initial points for Example 4.4.2.

x0	0.10	0.40	0.70	1.00	1.30
x1	0.020000000	0.080000000	0.140000000	0.100000000	0.100000000
x2	0.002857143	0.011428571	0.020000000	0.014285714	0.014285714
x3	0.000571429	0.002285714	0.004000000	0.002857143	0.002857143
x4	0.000081633	0.000326531	0.000571429	0.000408163	0.000408163
x5	0.000016327	0.000065306	0.000114286	0.000081633	0.000081633
x6	0.000002332	0.000009329	0.000016327	0.000011662	0.000011662
x7	0.000000466	0.000001866	0.000003265	0.000002332	0.000002332
x8	0.000000067	0.000000267	0.000000466	0.000000333	0.000000333
x9	0.000000013	0.000000053	0.000000093	0.000000067	0.000000067
x10	0.000000002	0.000000008	0.000000013	0.000000010	0.000000010
x11	0.000000000	0.000000002	0.000000003	0.000000002	0.000000002
x12	0.000000000	0.000000000	0.000000000	0.000000000	0.000000000

Figure 4.8: Numerical iteration for Example 4.4.2.

4.5 Some Fixed Point Results using Three Point Analogue of Contraction Mappings

In this section, we have established some fixed point results using three point analogue of contraction in multiplicative $\mathfrak{m}$-MS.

Definition 4.5.1. *Consider an multiplicative $\mathfrak{m}$-MS (Ω, μ). A self mapping Γ is c.t.b.* ***self distance contraction*** *on Ω if $\exists k_0 \in [0,1)$ s.t.*

$$\mu(\Gamma\varrho, \Gamma\varrho) \leq (\mu(\varrho, \varrho))^{k_0} \ \ \forall \varrho \in \Omega. \tag{4.5.1}$$

Definition 4.5.2. *Consider an multiplicative $\mathfrak{m}$-complete MS (Ω, μ). A self mapping Γ is c.t.b.* ***contracting perimeter of triangle*** *on Ω if $\exists k \in [0,1)$ s.t.*

$$\mu(\Gamma\varrho, \Gamma\vartheta).\mu(\Gamma\vartheta, \Gamma\xi).\mu(\Gamma\xi, \Gamma\varrho) \leq (\mu(\varrho, \vartheta).\mu(\vartheta, \xi).\mu(\xi, \varrho))^k, \tag{4.5.2}$$

$\forall$ pairwise distinct $\varrho, \vartheta, \xi \in \Omega$.

Remark 4.5.3. *The condition (pairwise distinct) given in Definition 4.5.2 made the contraction substantially different from the Banach contraction principle. If ϱ, ϑ, ξ are not distinct, then the condition (4.5.2) is reduced to classic contraction in multiplicative $\mathfrak{m}$-MS.*

$$\mu(\Gamma\varrho, \Gamma\vartheta) \leq (\mu(\varrho, \vartheta))^k.$$

Remark 4.5.4. *A self mapping Γ contracting perimeter of triangle in multiplicative $\mathfrak{m}$-MS need not to be continuous. But in the usual MS the mapping must be continuous. For illustration see the Example 4.5.5.*

Example 4.5.5. *Consider* $\Omega = [0,2]$ *with multiplicative* $\mathfrak{m}$*-metric,* $\mu(\varrho,\vartheta) = e^{\frac{\varrho+\vartheta}{2}}$. *Clearly,* (Ω,μ) *is multiplicative* $\mathfrak{m}$*-complete MS. Consider,* $\Gamma : \Omega \to \Omega$

$$\Gamma\varrho = \begin{cases} \frac{\varrho}{3}, & \text{if } \varrho < 1 \\ \frac{1}{7}, & \text{otherwise.} \end{cases}$$

(i) If $\varrho,\vartheta,\xi < 1$*. Then,*

$$\begin{aligned} \mu(\Gamma\varrho,\Gamma\vartheta).\mu(\Gamma\vartheta,\Gamma\xi).\mu(\Gamma\xi,\Gamma\varrho) &= e^{\frac{\Gamma\varrho+\Gamma\vartheta}{2}}.e^{\frac{\Gamma\vartheta+\Gamma\xi}{2}}.e^{\frac{\Gamma\xi+\Gamma\varrho}{2}} \\ &= e^{\frac{\varrho+\vartheta}{6}}.e^{\frac{\vartheta+\xi}{6}}.e^{\frac{\xi+\varrho}{6}} \\ &\le \left(e^{\frac{\varrho+\vartheta}{2}}.e^{\frac{\vartheta+\xi}{2}}.e^{\frac{\xi+\varrho}{2}}\right)^{\frac{1}{3}} \\ &= (\mu(\varrho,\vartheta).\mu(\vartheta,\xi).\mu(\xi,\varrho))^{\frac{1}{3}}. \end{aligned}$$

(ii) For distinct ϱ,ϑ,ξ*, assume that* $\varrho,\vartheta < 1$ *&* $\xi = 1$*. Then,*

$$\begin{aligned} \mu(\Gamma\varrho,\Gamma\vartheta).\mu(\Gamma\vartheta,\Gamma\xi).\mu(\Gamma\xi,\Gamma\varrho) &= e^{\frac{\Gamma\varrho+\Gamma\vartheta}{2}}.e^{\frac{\Gamma\vartheta+\Gamma\xi}{2}}.e^{\frac{\Gamma\xi+\Gamma\varrho}{2}} \\ &= e^{\frac{\frac{\varrho}{3}+\frac{\vartheta}{3}}{2}}.e^{\frac{\frac{\vartheta}{3}+\frac{1}{7}}{2}}.e^{\frac{\frac{1}{7}+\frac{\varrho}{3}}{2}} \\ &= e^{\frac{\varrho+\vartheta}{6}}.e^{\frac{\vartheta+\frac{3}{7}}{6}}.e^{\frac{\frac{3}{7}+\varrho}{6}} \\ &\le e^{\frac{\varrho+\vartheta}{6}}.e^{\frac{\vartheta+1}{6}}.e^{\frac{1+\varrho}{6}} \\ &\le \left(e^{\frac{\varrho+\vartheta}{2}}.e^{\frac{\vartheta+\xi}{2}}.e^{\frac{\xi+\varrho}{2}}\right)^{\frac{1}{3}} \\ &= (\mu(\varrho,\vartheta).\mu(\vartheta,\xi).\mu(\xi,\varrho))^{\frac{1}{3}}. \end{aligned}$$

Then, Γ *is a contracting perimeter of triangle in multiplicative* $\mathfrak{m}$*-MS with* $k = \frac{1}{3}$*. Also,* Γ_1 *is a self distance contraction.*

Let $LHS = d(\Gamma\varrho,\Gamma\vartheta)+d(\Gamma\vartheta,\Gamma\xi)+d(\Gamma\xi,\Gamma\varrho)$ *and* $RHS = d(\varrho,\vartheta)+d(\vartheta,\xi)+d(\xi,\varrho)$*, where* $d(\varrho,\vartheta) = |\varrho-\vartheta|\ \forall \varrho,\vartheta \in [0,2]$*. The figures* **??** *shows that there is no such* $k \in [0,1)$ *that satisfies*

$$d(\Gamma\varrho,\Gamma\vartheta)+d(\Gamma\vartheta,\Gamma\xi)+d(\Gamma\xi,\Gamma\varrho) \le k\left(d(\varrho,\vartheta)+d(\vartheta,\xi)+d(\xi,\varrho)\right)\ \forall \varrho,\vartheta,\xi \in [0,2].$$

Therefore with usual metric $d(\varrho,\vartheta) = |\varrho-\vartheta|$*,* Γ *is not contracting perimeter of triangle in* $([0,2],d)$ *i.e.,* Γ *does not meet the requirement of Theorem* **??***.*

Remark 4.5.6. *Consider a self mapping* Γ *satisfying the Banach contraction in multiplicative* $\mathfrak{m}$*-MS with* $0 \leq \alpha < \frac{1}{3}$ *i.e.,*

$$\mu(\Gamma\varrho, \Gamma\vartheta) \leq (\mu(\varrho, \vartheta))^{\alpha}, \quad \forall\ \varrho, \vartheta \in \Omega.$$

Then Γ *is a mapping contracting the perimeter of triangle in multiplicative* $\mathfrak{m}$*-MS* (Ω, μ)*.*

Theorem 4.5.7. *Let* $\Gamma : \Omega \to \Omega$ *be a mapping defined on a multiplicative* $\mathfrak{m}$*-complete MS* (Ω, μ)*. Suppose* Γ *satisfies*

(i) Γ *is contracting perimeter of triangle on* Ω*;*

(ii) Γ *is a self distance contraction mapping;*

(iii) there is no periodic point of order 2 in Ω*.*

Then, Γ *possesses a fixed point. Moreover,* Γ *has atmost two fixed point.*

Proof. Let $\varrho_0 \in \Omega$ be any point. Consider the iterative sequence $\{\varrho_\hbar\}$ generated by mapping Γ with initial point ϱ_0 as

$$\varrho_1 = \Gamma\varrho_0,\ \varrho_2 = \Gamma\varrho_1,\ \ldots\ \varrho_{\hbar+1} = \Gamma\varrho_\hbar.$$

Suppose that $\varrho_\hbar$ is not a fixed point $\forall \hbar \in \mathbb{N}$. Then, $\varrho_{\hbar+1} = \Gamma\varrho_\hbar \neq \varrho_\hbar$. Also, there is no periodic point of order 2 implies $\varrho_{\hbar+2} = \Gamma(\varrho_{\hbar+1}) = \Gamma^2\varrho_\hbar \neq \varrho_\hbar$. Moreover, by assumption $\varrho_{\hbar+1}$ is not a fixed point of Γ i.e., $\varrho_{\hbar+2} = \Gamma\varrho_{\hbar+1} \neq \varrho_{\hbar+1}$, proves that $\varrho_\hbar, \varrho_{\hbar+1}, \varrho_{\hbar+2}$ are all pairwise distinct.
Now consider the sequence $\{\beta_\hbar\}$ generated by perimeter of triangle in multiplicative distance structure with vertices as the consecutive member of the sequence $\{\varrho_\hbar\}$ as

$$\beta_\hbar = \mu(\varrho_\hbar, \varrho_{\hbar+1}).\mu(\varrho_{\hbar+1}, \varrho_{\hbar+2}).\mu(\varrho_{\hbar+2}, \varrho_\hbar).$$

Now, as $\varrho_\hbar, \varrho_{\hbar+1}, \varrho_{\hbar+2}$ are all pairwise distinct and Γ is contracting perimeter of triangle in (Ω, μ). Therefore, we have

$$\begin{aligned}
\beta_\hbar &= \mu(\varrho_\hbar, \varrho_{\hbar+1}).\mu(\varrho_{\hbar+1}, \varrho_{\hbar+2}).\mu(\varrho_{\hbar+2}, \varrho_\hbar) \\
&= \mu(\Gamma\varrho_{\hbar-1}, \Gamma\varrho_\hbar).\mu(\Gamma\varrho_\hbar, \Gamma\varrho_{\hbar+1}).\mu(\Gamma\varrho_{\hbar+1}, \Gamma\varrho_{\hbar-1}) \\
&\leq (\mu(\varrho_{\hbar-1}, \varrho_\hbar).\mu(\varrho_\hbar, \varrho_{\hbar+1}).\mu(\varrho_{\hbar+1}, \varrho_{\hbar-1}))^k \\
&\leq (\beta_{\hbar-1})^k.
\end{aligned}$$

Moreover, as $k < 1$. Then,

$$\beta_0 > \beta_1 > \beta_2 > .... > \beta_{\hbar-1} > \beta_\hbar > ...$$

We claim that $\varrho_i, \varrho_{i+1}, \varrho_{i+1}$ are all distinct. Suppose $\exists j \geq 3$ s.t. $\varrho_j = \varrho_i$ for some $0 \leq i \leq j-2$. Then,

$$\varrho_i = \varrho_j \Rightarrow \varrho_{i+1} = \Gamma\varrho_i = \Gamma\varrho_j = \varrho_{j+1} \Rightarrow \varrho_{i+2} = \Gamma\varrho_{i+1} = \Gamma\varrho_{j+1} = \varrho_{j+2},$$

implies

$$\beta_i = \mu(\varrho_i, \varrho_{i+1}).\mu(\varrho_{i+1}, \varrho_{i+2}).\mu(\varrho_{i+2}, \varrho_i) = \mu(\varrho_j, \varrho_{j+1}).\mu(\varrho_{j+1}, \varrho_{j+2}).\mu(\varrho_{j+2}, \varrho_j) = \beta_j,$$

a contradiction.
Consider,

$$\begin{aligned}
\mu(\varrho_\hbar, \varrho_{\hbar+1}) &\leq \mu(\varrho_\hbar, \varrho_{\hbar+1}).\mu(\varrho_{\hbar+1}, \varrho_{\hbar+2}).\mu(\varrho_{\hbar+2}, \varrho_\hbar) = \beta_\hbar \\
&\leq (\beta_{\hbar-1})^k \leq \cdots \leq (\beta_0)^{k^\hbar}. \qquad (4.5.3)
\end{aligned}$$

As, $k < 1$. Then, taking limit as $\hbar$ tends to infinity in (4.5.3), we have

$$\lim_{\hbar\to\infty} \mu(\varrho_\hbar, \varrho_{\hbar+1}) = 1. \qquad (4.5.4)$$

Also,

$$\lim_{\hbar\to\infty} \mu_{\varrho_\hbar\ \varrho_{\hbar+1}} = \lim_{\hbar\to\infty} \min\{\mu(\varrho_\hbar, \varrho_\hbar), \mu(\varrho_{\hbar+1}, \varrho_{\hbar+1})\} \leq \lim_{\hbar\to\infty} \mu(\varrho_\hbar, \varrho_{\hbar+1}) = 1, \quad (4.5.5)$$

$$\begin{aligned}
\lim_{\hbar\to\infty} \mu(\varrho_\hbar, \varrho_\hbar) = \lim_{\hbar\to\infty} \mu(\varrho_{\hbar+1}, \varrho_{\hbar+1}) &= \lim_{\hbar\to\infty} \min\{\mu(\varrho_\hbar, \varrho_\hbar), \mu(\varrho_{\hbar+1}, \varrho_{\hbar+1})\} \\
&= \lim_{\hbar\to\infty} \mu_{\varrho_\hbar\ \varrho_{\hbar+1}} = 1, \qquad (4.5.6)
\end{aligned}$$

and

$$\lim_{\hbar\to\infty} \mu^*_{\varrho_\hbar\ \varrho_\hbar+1} = \lim_{\hbar\to\infty} \max\{\mu(\varrho_\hbar, \varrho_\hbar), \mu(\varrho_{\hbar+1}, \varrho_{\hbar+1})\} = 1. \qquad (4.5.7)$$

Next, we will show that $\{\varrho_\hbar\}$ is a multiplicative $\mathfrak{m}$-C_{seq}. Consider,

$$\begin{aligned}
\frac{\mu(\varrho_\hbar, \varrho_m)}{\mu_{\varrho_\hbar,\varrho_m}} &\leq \frac{\mu(\varrho_\hbar, \varrho_{\hbar+1})}{\mu_{\varrho_\hbar,\varrho_{\hbar+1}}}.\frac{\mu(\varrho_{\hbar+1}, \varrho_{\hbar+2})}{\mu_{\varrho_{\hbar+1},\varrho_{\hbar+2}}}.\cdots.\frac{\mu(\varrho_{m+1}, \varrho_m)}{\mu_{\varrho_{m+1},\varrho_m}} \\
&\leq \mu(\varrho_\hbar, \varrho_{\hbar+1}).\mu(\varrho_{\hbar+1}, \varrho_{\hbar+2}).\cdots.\mu(\varrho_{m+1}, \varrho_m) \\
&\leq \beta_\hbar.\beta_{\hbar+1}.\cdots.\beta_{m+1} \\
&\leq (\beta_0)^{k^\hbar}.(\beta_0)^{k^{\hbar+1}}.\cdots.(\beta_0)^{k^{m+1}} \\
&\leq (\beta_0)^{k^\hbar(1+k+k^2+...k^{m+1-\hbar})} = (\beta_0)^{k^\hbar(\frac{1-k^{m+1-\hbar}}{1-k})}. \qquad (4.5.8)
\end{aligned}$$

As $k < 1$. Then, by taking limit as $\hbar, m$ tends to infinity in (4.5.8), we have

$$\lim_{\hbar . m\to\infty} \frac{\mu(\varrho_\hbar, \varrho_m)}{\mu_{\varrho_\hbar\, \varrho_m}} = 1, \tag{4.5.9}$$

i.e., $\{\varrho_\hbar\}$ is multiplicative $\mathfrak{m}$-Cauchy. Further, as (Ω, μ) is multiplicative $\mathfrak{m}$-complete. Therefore, $\exists \varrho \in \Omega$ s.t.

$$\lim_{\hbar\to\infty} \frac{\mu(\varrho_\hbar, \varrho)}{\mu_{\varrho_\hbar \varrho}} = 1 = \lim_{\hbar\to\infty} \frac{\mu^*_{\varrho_\hbar \varrho}}{\mu_{\varrho_\hbar \varrho}}. \tag{4.5.10}$$

Using (4.5.6), we have

$$\lim_{\hbar\to\infty} \mu_{\varrho_\hbar \varrho} = \lim_{\hbar\to\infty} \min\{\mu(\varrho_\hbar, \varrho_\hbar), \mu(\varrho, \varrho)\} \leq \lim_{\hbar\to\infty} \mu(\varrho_\hbar, \varrho_\hbar) = 1. \tag{4.5.11}$$

Using (4.5.10), (4.5.11) and Remark 4.2.4 , we have

$$\lim_{\hbar\to\infty} \mu(\varrho_\hbar, \varrho) = 1, \ \lim_{\hbar\to\infty} \mu^*_{\varrho_\hbar \varrho} = 1 \text{ and } \mu(\varrho, \varrho) = 1. \tag{4.5.12}$$

Also,

$$\mu_{\varrho\Gamma\varrho} = \min\{\mu(\varrho, \varrho), \mu(\Gamma\varrho, \Gamma\varrho)\} = 1. \tag{4.5.13}$$

Using (4.5.2),(4.5.13) and the triangle inequality, we have

$$\begin{aligned} 1 \leq \mu(\varrho, \Gamma\varrho) &= \frac{\mu(\varrho, \Gamma\varrho)}{\mu_{\varrho\Gamma\varrho}} \\ &\leq \frac{\mu(\varrho, \varrho_\hbar)}{\mu_{\varrho_\hbar \varrho}} . \frac{\mu(\varrho_\hbar, \Gamma\varrho)}{\mu_{\varrho_\hbar \Gamma\varrho}} \\ &\leq \frac{\mu(\varrho, \varrho_\hbar)}{\mu_{\varrho_\hbar \varrho}} . \frac{\mu(\varrho_\hbar, \Gamma\varrho)}{\mu_{\varrho_\hbar \Gamma\varrho}} . \mu(\varrho_\hbar, \varrho_{\hbar+1}) . \mu(\varrho_{\hbar+1}, \Gamma\varrho) \\ &\leq \mu(\varrho, \varrho_\hbar) . \mu(\Gamma\varrho, \Gamma\varrho_{\hbar-1}) . \mu(\Gamma\varrho_{\hbar-1}, \Gamma\varrho_\hbar) . \mu(\Gamma\varrho_\hbar, \Gamma\varrho) \\ &\leq \mu(\varrho, \varrho_\hbar) \left(\mu(\varrho, \varrho_{\hbar-1}) . \mu(\varrho_{\hbar-1}, \varrho) . \mu(\varrho_\hbar, \varrho)\right)^k . \end{aligned} \tag{4.5.14}$$

Taking limit as $\hbar \to \infty$ in (4.5.14), we have

$$\mu(\varrho, \Gamma\varrho) = 1. \tag{4.5.15}$$

Also, Γ is a self distance contraction. Therefore, $\exists k_0 \in [0, 1)$ s.t.

$$\mu(\Gamma\varrho, \Gamma\varrho) \leq (\mu(\varrho, \varrho))^{k_0},$$

or

$$\mu(\Gamma\varrho, \Gamma\varrho) = 1. \tag{4.5.16}$$

Using (4.5.12), (4.5.13), (4.5.15) and (4.5.16), we have

$$\mu(\varrho,\varrho)=\mu(\varrho,\Gamma\varrho)=\mu(\Gamma\varrho,\Gamma\varrho)=1,$$

By using axiom (i) of Definition 4.2.1, we have $\varrho=\Gamma\varrho$. Hence ϱ is the fixed point of Γ.
In order to prove that Γ has atmost two fixed points, suppose that there are three distinct fixed points say ϱ,ϑ,ξ. Then, we have $\mu(\varrho,\varrho)=\mu(\vartheta,\vartheta)=\mu(\xi,\xi)=1$ and $\Gamma\varrho=\varrho,\Gamma\vartheta=\vartheta,\Gamma\xi=\xi$. Consider

$$\begin{aligned}\mu(\varrho,\vartheta).\mu(\vartheta,\xi).\mu(\xi,\varrho) &= \mu(\Gamma\varrho,\Gamma\vartheta).\mu(\Gamma\vartheta,\Gamma\xi).\mu(\Gamma\xi,\Gamma\varrho)\\ &\leq (\mu(\varrho,\vartheta).\mu(\vartheta,\xi).\mu(\xi,\varrho))^k\\ &< \mu(\varrho,\vartheta).\mu(\vartheta,\xi).\mu(\xi,\varrho),\end{aligned}$$

a contradiction. Hence, the mapping Γ has atmost two fixed points. □

Definition 4.5.8. *Let (Ω,μ) be an multiplicative $\mathfrak{m}$-MS. A triplet $(\Gamma_1,\Gamma_2,\Gamma_3)$ of self mapping is c.t.b. contracting perimeter of triangle on Ω if $\exists k\in[0,1)$ s.t.*

$$\mu(\Gamma_1\varrho,\Gamma_2\vartheta).\mu(\Gamma_2\vartheta,\Gamma_3\xi).\mu(\Gamma_3\xi,\Gamma_1\varrho)\leq(\mu(\varrho,\vartheta).\mu(\vartheta,\xi).\mu(\xi,\varrho))^k, \tag{4.5.17}$$

$\forall$ pairwise distinct $\varrho,\vartheta,\xi\in\Omega$.

We extend the concept of periodic point further for a pair of self mappings (Γ_1,Γ_2) as : ϱ has prime period 2 pairwise w.r.t. (Γ_1,Γ_2) if $\Gamma_1\varrho\neq\varrho$,$\Gamma_2\varrho\neq\varrho$, but either $FG(\varrho)=\varrho$ or $GF(\varrho)=\varrho$. Moreover, ϱ has prime period 2 pairwise w.r.t. $(\Gamma_1,\Gamma_2,\Gamma_3)$, if ϱ has prime period 2 pairwise w.r.t. each pair of self mappings.

Example 4.5.9. *Consider the mapping Γ_1 and Γ_2 defines on $\mathbb{R}_+$ as $\Gamma_1\varrho=e^{\varrho}$ and $\Gamma_2\varrho=ln(\varrho)$. Then, $\forall$ $\varrho\in\mathbb{R}_+$, $\Gamma_1\varrho\neq\varrho$ and $Gx\neq\varrho$ but $FG(\varrho)=\varrho$. Therefore, every point in $\mathbb{R}_+$ has a prime order 2 pairwise w.r.t. mappings (Γ_1,Γ_2).*

Theorem 4.5.10. *Let $\Gamma_1,\Gamma_2,\Gamma_3:\Omega\to\Omega$ are mappings defined on complete multiplicative $\mathfrak{m}$-MS (Ω,μ). Suppose the triplet $(\Gamma_1,\Gamma_2,\Gamma_3)$ satisfies*

(i) triplet $(\Gamma_1,\Gamma_2,\Gamma_3)$ is contracting perimeter of triangle on Ω;

(ii) $\Gamma_1,\Gamma_2,\Gamma_3$ are self distance contraction mappings;

(iii) there is no point in Ω that has a prime period of order 2 pairwise w.r.t. $(\Gamma_1,\Gamma_2,\Gamma_3)$.

Then, the triplet $(\Gamma_1, \Gamma_2, \Gamma_3)$ *has a common fixed point. Moreover, the triplet* $(\Gamma_1, \Gamma_2, \Gamma_3)$ *has atmost two common fixed points.*

Proof. For some $\varrho_0 \in \Omega$, define the iterative sequence as

$\varrho_1 = \Gamma_1\varrho_0,\ \varrho_2 = \Gamma_2\varrho_1,\ \varrho_3 = \Gamma_3\varrho_2, ..., \varrho_{3\hbar+1} = \Gamma_1\varrho_{3\hbar},\ \varrho_{3\hbar+2} = \Gamma_2\varrho_{3\hbar+1},\ \varrho_{3\hbar+3} = \Gamma_3\varrho_{3\hbar+2},$

$\forall\ \hbar \in \mathbb{N}$. Without loss of generality, rename the sequence as $\varrho_0 = \vartheta_0$ and $\vartheta_\hbar = \varrho_{3\hbar}, \vartheta_{\hbar+1} = \varrho_{3\hbar+1}, \vartheta_{\hbar+2} = \varrho_{3\hbar+2}$ and so on..., for $\hbar \in \mathbb{N}$. Suppose that $\vartheta_\hbar$ is not a common fixed point $\forall \hbar \in \mathbb{N}$. Then, $\vartheta_\hbar, \vartheta_{\hbar+1}, \vartheta_{\hbar+2}$ are pairwise distinct. Now, we can define the sequence $\{\beta_\hbar\}$ generated by perimeter of triangle in multiplicative distance structure with vertices as the consecutive member of the sequence $\{\vartheta_\hbar\}$ as

$$\beta_\hbar = \mu(\vartheta_\hbar, \vartheta_{\hbar+1}).\mu(\vartheta_{\hbar+1}, \vartheta_{\hbar+2}).\mu(\vartheta_{\hbar+2}, \vartheta_\hbar).$$

On similar line of Theorem4.5.7, it can be observe that $\{\vartheta_\hbar\}$ is a multiplicative $\mathfrak{m}$-C_{seq} in Ω and

$$\lim_{\hbar\to\infty} \mu(\vartheta_\hbar, \vartheta_{\hbar+1}) = 1. \tag{4.5.18}$$

Also,

$$\lim_{\hbar\to\infty} \mu_{\vartheta_\hbar\ \vartheta_{\hbar+1}} = \lim_{\hbar\to\infty} \min\{\mu(\vartheta_\hbar, \vartheta_\hbar), \mu(\vartheta_{\hbar+1}, \vartheta_{\hbar+1})\} \leq \lim_{\hbar\to\infty} \mu(\vartheta_\hbar, \vartheta_{\hbar+1}) = 1, \tag{4.5.19}$$

$$\begin{aligned}\lim_{\hbar\to\infty} \mu(\vartheta_\hbar, \vartheta_\hbar) = \lim_{\hbar\to\infty} \mu(\vartheta_{\hbar+1}, \vartheta_{\hbar+1}) = \lim_{\hbar\to\infty} \min\{\mu(\vartheta_\hbar, \vartheta_\hbar), \mu(\vartheta_{\hbar+1}, \vartheta_{\hbar+1})\}\\ = \lim_{\hbar\to\infty} \mu_{\vartheta_\hbar\ \vartheta_{\hbar+1}} = 1,\end{aligned}$$

and

$$\lim_{\hbar\to\infty} \mu^*_{\vartheta_\hbar\ \vartheta_\hbar+1} = \lim_{\hbar\to\infty} \max\{\mu(\vartheta_\hbar, \vartheta_\hbar), \mu(\vartheta_{\hbar+1}, \vartheta_{\hbar+1})\} = 1. \tag{4.5.20}$$

Also, (Ω, μ) is complete. Therefore, $\exists \vartheta \in \Omega$ s.t.

$$\lim_{\hbar\to\infty} \frac{\mu(\vartheta_\hbar, \vartheta)}{\mu_{\vartheta_\hbar\vartheta}} = 1 = \lim_{\hbar\to\infty} \frac{\mu^*_{\vartheta_\hbar\vartheta}}{\mu_{\vartheta_\hbar\vartheta}}. \tag{4.5.21}$$

Using (4.5.20), we have

$$\lim_{\hbar\to\infty} \mu_{\vartheta_\hbar\vartheta} = \lim_{\hbar\to\infty} \min\{\mu(\vartheta_\hbar, \vartheta_\hbar), \mu(\vartheta, \vartheta)\} \leq \lim_{\hbar\to\infty} \mu(\vartheta_\hbar, \vartheta_\hbar) = 1. \tag{4.5.22}$$

Using (4.5.21), (4.5.22) and Remark 4.2.4, we have

$$\lim_{\hbar\to\infty} \mu(\vartheta_\hbar, \vartheta) = 1,\ \lim_{\hbar\to\infty} \mu^*_{\vartheta_\hbar\vartheta} = 1 \text{ and } \mu(\vartheta, \vartheta) = 1. \tag{4.5.23}$$

Also,

$$\mu_{\vartheta\Gamma_2\vartheta} = \min\{\mu(\vartheta,\vartheta), \mu(\Gamma_2\vartheta, \Gamma_2\vartheta)\} = 1. \tag{4.5.24}$$

Using (4.5.17), (4.5.24) and the triangle inequality, we have

$$\begin{aligned}
1 \leq \mu(\vartheta, \Gamma_2\vartheta) &= \frac{\mu(\vartheta, \Gamma_2\vartheta)}{\mu_{\vartheta\Gamma_2\vartheta}} \\
&\leq \frac{\mu(\vartheta, \varrho_{3\hbar+1})}{\mu_{\varrho_{3\hbar+1}\vartheta}} \cdot \frac{\mu(\varrho_{3\hbar+1}, \Gamma_2\vartheta)}{\mu_{\varrho_{3\hbar+1}\Gamma_2\vartheta}} \\
&\leq \frac{\mu(\vartheta, \varrho_{3\hbar+1})}{\mu_{\varrho_{3\hbar+1}\ \vartheta}} \cdot \frac{\mu(\varrho_{3\hbar+1}, \Gamma_2\vartheta)}{\mu_{\varrho_{3\hbar+1}\Gamma_2\vartheta}} . \mu(\Gamma_2\vartheta, \varrho_{3\hbar+3}).\mu(\varrho_{3\hbar+3}, \varrho_{3\hbar+1}) \\
&\leq \mu(\vartheta, \varrho_{3\hbar+1}).\mu(\Gamma_1\varrho_{3\hbar}, \Gamma_2\vartheta).\mu(\Gamma_2\vartheta, \Gamma_3\varrho_{3\hbar+2}).\mu(\Gamma_3\varrho_{3\hbar+2}, \Gamma_1\varrho_{3\hbar}) \\
&\leq \mu(\vartheta, \varrho_{3\hbar+1}) \left(\mu(\varrho_{3\hbar}, \vartheta).\mu(\vartheta, \varrho_{3\hbar+2}).\mu(\varrho_{3\hbar+2}, \varrho_{3\hbar})\right)^k \\
&= \mu(\vartheta, \vartheta_{\hbar+1}) \left(\mu(\vartheta_\hbar, \vartheta).\mu(\vartheta, \vartheta_{\hbar+2}).\mu(\vartheta_{\hbar+2}, \vartheta_\hbar)\right)^k . \qquad (4.5.25)
\end{aligned}$$

Taking limit as $\hbar \to \infty$ in (4.5.25), we have

$$\mu(\vartheta, \Gamma_2\vartheta) = 1. \tag{4.5.26}$$

Also, Γ_2 is a self distance contraction. Therefore, $\exists k_0 \in [0, 1)$ s.t.

$$\mu(\Gamma_2\varrho, \Gamma_2\varrho) \leq (\mu(\varrho, \varrho))^{k_0},$$

or

$$\mu(\Gamma_2\varrho, \Gamma_2\varrho) = 1. \tag{4.5.27}$$

Using (4.5.23), (4.5.26) and (4.5.27), we have

$$\mu(\vartheta, \vartheta) = \mu(\vartheta, \Gamma_2\vartheta) = \mu(\Gamma_2\vartheta, \Gamma_2\vartheta) = 1,$$

i.e., $\vartheta = \Gamma_2\vartheta$. Thus, ϑ is the fixed point of Γ_2.
On the similar lines one can prove that $\vartheta = \Gamma_1\vartheta = \Gamma_2\vartheta = \Gamma_3\vartheta$ i.e., ϑ is the common fixed point of self mappings $\Gamma_1, \Gamma_2, \Gamma_3$.
The rest part of the theorem is analogues to Theorem 4.5.7. □

Example 4.5.11. *Consider* $\Omega = \{a_1, a_2, a_3\}$ *with distance function* $\mathfrak{m}$ *defined as* $\mu(a_1, a_1) = \mu(a_2, a_2) = 1$, $\mu(a_3, a_3) = 2$ *and* $\mu(a_1, a_2) = \mu(a_2, a_3) = \mu(a_1, a_3) = \mu(a_2, a_1) = \mu(a_3, a_2) = \mu(a_3, a_1) = 4$. *Then clearly* (Ω, μ) *is multiplicative* $\mathfrak{m}$*-MS. Let* Γ *be a self mapping defined on* Ω *as* $\Gamma(a_1) = a_1, \Gamma(a_2) = a_2, \Gamma(a_3) = a_2$. *Then,* Γ_1 *satisfies Theorem 4.5.7 and has two fixed point.*

Example 4.5.12. *Consider* $\Omega = \{a_1, a_2, a_3\}$ *with distance function* μ *defined as* $\mu(a_1, a_1) = \mu(a_2, a_2) = \mu(a_3, a_3) = 2$ *and* $\mu(a_1, a_2) = \mu(a_2, a_3) = \mu(a_1, a_3) = \mu(a_2, a_1) = \mu(a_3, a_2) = \mu(a_3, a_1) = 4$. *Then clearly* (Ω, μ) *is multiplicative* $\mathfrak{m}$*-MS. Let* Γ *be a self mapping defined on* Ω *as* $\Gamma(a_1) = a_2, \Gamma(a_2) = a_1, \Gamma(a_3) = a_1$. *Then,* Γ *satisfies the condition of contracting perimeter and* a, b *are periodic points of prime order 2. Also,* Γ *has no fixed point.*

Remark 4.5.13. *If under the assumption of Theorem 4.5.7, the mapping* Γ *has a fixed point* ϱ *and it is a limit point of the iterative scheme* $\varrho_\hbar = \Gamma\varrho_{\hbar-1}$*.Then,* Γ *possesses exactly one fixed point.*
Let if possible $\vartheta \neq \varrho$ *is another fixed point. Clearly* $\varrho_\hbar, \varrho, \vartheta$ *are pairwise distinct. Consider*

$$\begin{aligned} \mu(\varrho_{\hbar+1}, \varrho)\mu(\varrho, \vartheta)\mu(\vartheta, \varrho_{\hbar+1}) &= \mu(\Gamma\varrho_\hbar, \Gamma\varrho).\mu(\Gamma\varrho, \Gamma\vartheta).\mu(\Gamma\vartheta, \Gamma\varrho_\hbar) \\ &\leq (\mu(\varrho_\hbar, \varrho)\mu(\varrho, \vartheta)\mu(\vartheta, \varrho_\hbar))^k \\ &< \mu(\varrho_\hbar, \varrho)\mu(\varrho, \vartheta)\mu(\vartheta, \varrho_\hbar). \end{aligned}$$

As, $(\lim_{\hbar\to\infty} \mu(\varrho_\hbar, \varrho) = \lim_{\hbar\to\infty} \mu(\varrho_\hbar, \vartheta) = 1)$. *Therefore, by taking limit as* $\hbar \to \infty$*, we have*

$$\mu(\varrho, \vartheta) < \mu(\varrho, \vartheta),$$

a contradiction.

Remark 4.5.14. *If under the assumption of Theorem 4.5.10, the triplet of mappings* $\Gamma_1, \Gamma_2, \Gamma_3$ *has a common fixed point* ϱ *and it is a limit point of the iterative scheme*

$$\varrho_1 = \Gamma_1\varrho_0,\ \varrho_2 = \Gamma_2\varrho_1,\ \varrho_3 = \Gamma_3\varrho_2, ..., \varrho_{3\hbar+1} = \Gamma_1\varrho_{3\hbar},\ \varrho_{3\hbar+2} = \Gamma_2\varrho_{3\hbar+1},\ \varrho_{3\hbar+3} = \Gamma_3\varrho_{3\hbar+2},$$

$\forall\hbar \in \mathbb{N}$. *Then,* $(\Gamma_1, \Gamma_2, \Gamma_3)$ *possesses exactly one common fixed point.*
Let if possible $\vartheta \neq \varrho$ *is another common fixed point. Clearly* $\varrho_{3\hbar}, \varrho, \vartheta$ *are pairwise distinct. Consider*

$$\begin{aligned} \mu(\varrho_{3\hbar+1}, \varrho)\mu(\varrho, \vartheta)\mu(\vartheta, \varrho_{3\hbar+1}) &= \mu(\Gamma_1\varrho_{3\hbar}, \Gamma_2\varrho).\mu(\Gamma_2\varrho, \Gamma_3\vartheta).\mu(\Gamma_3\vartheta, \Gamma_1\varrho_{3\hbar}) \\ &\leq (\mu(\varrho_{3\hbar}, \varrho)\mu(\varrho, \vartheta)\mu(\vartheta, \varrho_{3\hbar}))^k \\ &< \mu(\varrho_{3\hbar}, \varrho)\mu(\varrho, \vartheta)\mu(\vartheta, \varrho_{3\hbar}). \end{aligned}$$

As $(\lim_{\hbar\to\infty} \mu(\varrho_{3\hbar}, \varrho) = \lim_{\hbar\to\infty} \mu(\varrho_{3\hbar}, \vartheta) = 1)$. *Therefore, by taking limit as* $\hbar \to \infty$*, we have*

$$\mu(\varrho, \vartheta) < \mu(\varrho, \vartheta),$$

a contradiction.

Example 4.5.15. *Consider $\Omega = [0,\infty)$ with the multiplicative $\mathfrak{m}$-metric $\mu(\varrho,\vartheta) = e^{\frac{\varrho+\vartheta}{2}}$. Clearly, (Ω,μ) is multiplicative $\mathfrak{m}$-complete MS. Consider $\Gamma_1,\Gamma_2,\Gamma_3 : \Omega \to \Omega$*

$$\Gamma_1\varrho = \begin{cases} \frac{\varrho}{4}, & \text{if } \varrho < 1 \\ \frac{1}{8}, & \text{otherwise.} \end{cases}, \Gamma_2\varrho = \begin{cases} \frac{\varrho}{5}, & \text{if } \varrho < 1 \\ \frac{\varrho}{\varrho+9}, & \text{otherwise.} \end{cases}, \Gamma_3\varrho = \begin{cases} \frac{\varrho}{9}, & \text{if } \varrho < 1 \\ \frac{\varrho}{8\varrho+10}, & \text{otherwise.} \end{cases},$$

Then,

(i) Suppose $\varrho,\vartheta,\xi < 1$, then

$$\begin{aligned} \mu(\Gamma_1\varrho,\Gamma_2\vartheta).\mu(\Gamma_2\vartheta,\Gamma_3\xi).\mu(\Gamma_3\xi,\Gamma_1\varrho) &= e^{\frac{1}{2}(\frac{\varrho}{4}+\frac{\vartheta}{5})}.e^{\frac{1}{2}(\frac{\vartheta}{5}+\frac{\xi}{9})}.e^{\frac{1}{2}(\frac{\xi}{9}+\frac{\varrho}{4})} \\ &\leq e^{\max\left\{\frac{1}{4},\frac{1}{5}\right\}(\frac{\varrho}{2}+\frac{\vartheta}{2})}.e^{\max\left\{\frac{1}{5},\frac{1}{9}\right\}(\frac{\vartheta}{2}+\frac{\xi}{2})} \\ &\quad .e^{\max\left\{\frac{1}{9},\frac{1}{4}\right\}(\frac{\xi}{2}+\frac{\varrho}{2})} \\ &\leq \left(\mu(\varrho,\vartheta).\mu(\vartheta,\xi).\mu(\xi,\varrho)\right)^{k_1}, \end{aligned}$$

where $k_1 = \max\{\frac{1}{4},\frac{1}{5},\frac{1}{9}\} = \frac{1}{4}$.

(ii) Suppose $\xi \geq 1$ and $\varrho,\vartheta < 1$, then

$$\begin{aligned} \mu(\Gamma_1\varrho,\Gamma_2\vartheta).\mu(\Gamma_2\vartheta,\Gamma_3\xi).\mu(\Gamma_3\xi,\Gamma_1\varrho) &= e^{\frac{1}{2}(\frac{\varrho}{4}+\frac{\vartheta}{5})}.e^{\frac{1}{2}(\frac{\vartheta}{5}+\frac{\xi}{8\xi+10})}.e^{\frac{1}{2}(\frac{\xi}{8\xi+10}+\frac{\varrho}{4})} \\ &\leq e^{\frac{1}{2}(\frac{\varrho}{4}+\frac{\vartheta}{5})}.e^{\frac{1}{2}(\frac{\vartheta}{5}+\frac{\xi}{18})}.e^{\frac{1}{2}(\frac{\xi}{18}+\frac{\varrho}{4})} \\ &\leq e^{\max\left\{\frac{1}{4},\frac{1}{5}\right\}(\frac{\varrho}{2}+\frac{\vartheta}{2})}.e^{\max\left\{\frac{1}{5},\frac{1}{18}\right\}(\frac{\vartheta}{2}+\frac{\xi}{2})} \\ &\quad .e^{\max\left\{\frac{1}{18},\frac{1}{4}\right\}(\frac{\xi}{2}+\frac{\varrho}{2})} \\ &\leq \left(\mu(\varrho,\vartheta).\mu(\vartheta,\xi).\mu(\xi,\varrho)\right)^{k_2}, \end{aligned}$$

where $k_2 = \max\{\frac{1}{4},\frac{1}{5},\frac{1}{18}\} = \frac{1}{4}$.

(iii) Suppose $\vartheta \geq 1$ and $\varrho,\xi < 1$, then

$$\begin{aligned} \mu(\Gamma_1\varrho,\Gamma_2\vartheta).\mu(\Gamma_2\vartheta,\Gamma_3\xi).\mu(\Gamma_3\xi,\Gamma_1\varrho) &= e^{\frac{1}{2}(\frac{\varrho}{4}+\frac{\vartheta}{\vartheta+9})}.e^{\frac{1}{2}(\frac{\vartheta}{\vartheta+9}+\frac{\xi}{9})}.e^{\frac{1}{2}(\frac{\xi}{9}+\frac{\varrho}{4})} \\ &\leq e^{\frac{1}{2}(\frac{\varrho}{4}+\frac{\vartheta}{10})}.e^{\frac{1}{2}(\frac{\vartheta}{10}+\frac{\xi}{9})}.e^{\frac{1}{2}(\frac{\xi}{9}+\frac{\varrho}{4})} \\ &\leq e^{\max\left\{\frac{1}{4},\frac{1}{10}\right\}(\frac{\varrho}{2}+\frac{\vartheta}{2})}.e^{\max\left\{\frac{1}{10},\frac{1}{9}\right\}(\frac{\vartheta}{2}+\frac{\xi}{2})} \\ &\quad .e^{\max\left\{\frac{1}{9},\frac{1}{4}\right\}(\frac{\xi}{2}+\frac{\varrho}{2})} \\ &\leq \left(\mu(\varrho,\vartheta).\mu(\vartheta,\xi).\mu(\xi,\varrho)\right)^{k_3}, \end{aligned}$$

where $k_3 = \max\{\frac{1}{4},\frac{1}{9},\frac{1}{10}\} = \frac{1}{4}$.

(iv) Suppose $\varrho \geq 1$ and $\xi, \vartheta < 1$, then

$$\begin{aligned}
\mu(\Gamma_1\varrho, \Gamma_2\vartheta).\mu(\Gamma_2\vartheta, \Gamma_3\xi).\mu(\Gamma_3\xi, \Gamma_1\varrho) &= e^{\frac{1}{2}(\frac{1}{8}+\frac{\vartheta}{5})}.e^{\frac{1}{2}(\frac{\vartheta}{5}+\frac{\xi}{9})}.e^{\frac{1}{2}(\frac{\xi}{9}+\frac{1}{8})} \\
&\leq e^{\max\left\{\frac{1}{8},\frac{1}{5}\right\}(\frac{\varrho}{2}+\frac{\vartheta}{2})}.e^{\max\left\{\frac{1}{5},\frac{1}{9}\right\}(\frac{\vartheta}{2}+\frac{\xi}{2})} \\
&\quad .e^{\max\left\{\frac{1}{9},\frac{1}{8}\right\}(\frac{\xi}{2}+\frac{\varrho}{2})} \\
&\leq \left(\mu(\varrho, \vartheta).\mu(\vartheta, \xi).\mu(\xi, \varrho)\right)^{k_4},
\end{aligned}$$

where $k_4 = \max\{\frac{1}{8}, \frac{1}{5}, \frac{1}{9}\} = \frac{1}{5}$.

Let $k = \max\{k_1, k_2, k_3, k_4\} = \frac{1}{4}$. Then $(\Gamma_1, \Gamma_2, \Gamma_3)$ is a contracting perimeter of triangle in (Ω, μ) with $k = \frac{1}{4}$ and there is no pairwise periodic point of prime order 2. Also, $\Gamma_1, \Gamma_2, \Gamma_3$ are self distance contractions. Hence, $(\Gamma_1, \Gamma_2, \Gamma_3)$ satisfies all the requirement of Theorem 4.5.10. Hence the triplet $(\Gamma_1, \Gamma_2, \Gamma_3)$ possesses a common fixed point.

4.5.1 Numerical Approximation of Common Fixed Point

In this section, we have introduced several iterations to approximate the common fixed point of the triplet $(\Gamma_1, \Gamma_2, \Gamma_3)$ as defined in Example 4.5.15. We also provided a graphical representation to illustrate the convergence behavior of the iterative sequence, leading to the conclusion that the common fixed point of the triplet is independent of the initial point chosen for the iterative procedure (see figure 4.10). The iterative scheme employed for the approximation of the common fixed point is as follows:
For some $x_0 \in \Omega$,

$$x_1 = \Gamma_1 x_0,\ x_2 = \Gamma_2 x_1,\ x_3 = \Gamma_3 x_2, ..., x_{3n+1} = \Gamma_1 x_{3n},\ x_{3n+2} = \Gamma_2 x_{3n+1},\ x_{3n+3} = \Gamma_3 x_{3n+2},$$

$\forall \hbar \in \mathbb{N}$.

4.6 Existence of Solution to First-Order Multiplicative Initial Value Problem

In this section, we have discussed the applicability of the proved results by establishing the existence of solution to a first order multiplicative initial value problem.

x_0	0.210000	0.310000	0.410000	0.510000	0.710000	0.910000
x_1	0.052500	0.077500	0.102500	0.127500	0.177500	0.227500
x_2	0.010500	0.015500	0.020500	0.025500	0.035500	0.045500
x_3	0.001167	0.001722	0.002278	0.002833	0.003944	0.005056
x_4	0.000292	0.000431	0.000569	0.000708	0.000986	0.001264
x_5	0.000058	0.000086	0.000114	0.000142	0.000197	0.000253
x_6	0.000006	0.000010	0.000013	0.000016	0.000022	0.000028
x_7	0.000002	0.000002	0.000003	0.000004	0.000005	0.000007
x_8	0.000000	0.000000	0.000001	0.000001	0.000001	0.000001
x_9	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000

Figure 4.9: Numerical iteration for Example 4.5.15.

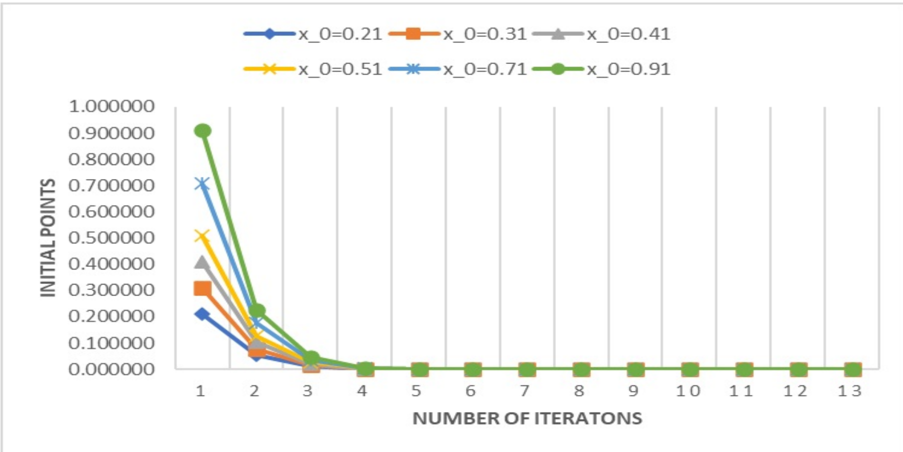

Figure 4.10: Convergence behaviour of iteration scheme at different initial points for Example 4.5.15.

Let $C^*[a,b]$ be the set of all real-valued multiplicative continuous function on $[a,b] \subset \mathbb{R}_+$ with multiplicative $\mathfrak{m}$-metric defined as

$$\mu(f,g) = \sup_{\varrho \in [a,b]} \left| \frac{f(\varrho)}{g(\varrho)} \right|_*,$$

where $|a|_* = \begin{cases} a, a \geq 1; \\ \frac{1}{a}, a < 1. \end{cases}$ for $a \in \mathbb{R}_+$. Then, $(C^*[a,b], \mu)$ is a multiplicative $\mathfrak{m}$-complete MS.

Consider the first order multiplicative initial value problem defined as

$$\begin{cases} \frac{d^*\vartheta}{d\varrho} = f\left(\varrho, \vartheta(\varrho)\right) \\ \vartheta(\varrho_0) = \vartheta_0 \end{cases}, \tag{4.6.1}$$

where $\varrho \in [1,\tau]$ for a sufficiently small $\tau > 1$. f is a multiplicative continuous function on $[1,\tau] \times C^*[1,\tau] \to \mathbb{R}_+$. Suppose f satisfies the multiplicative Lipschitz conditions with $\alpha \geq 1$ i.e.,

$$\left|\frac{f(s,\vartheta_1(s))}{f(s,\vartheta_2(s))}\right|_* \leq \alpha^{\left|\vartheta_1(s)/\vartheta_2(s)\right|_*},$$

$\forall \vartheta_1, \vartheta_2 \in C^*[1,\tau]$ and $s \in [1,\tau]$. Suppose $\varrho_0 \in [1,\tau]$, then multiplicative initial value problem given by equation (4.6.1) has a feasible solution on closed interval $[\varrho_0 - \kappa, \varrho_0 + \kappa]$, for a sufficiently small $\kappa > 0$ with $\kappa^\alpha < 1$.

Proof. Define a self mapping Γ_1 on $C^*[1,\tau]$ as below

$$\Gamma\vartheta(\varrho) = \vartheta_0 \int_{\varrho_0}^{\varrho} f(s,\vartheta)^{ds}.$$

Then, for distinct $\vartheta_1, \vartheta_2, \vartheta_3$, we have

$$\begin{aligned}
&\mu(\Gamma\vartheta_1,\Gamma\vartheta_2).\mu(\Gamma\vartheta_2,\Gamma\vartheta_3).\mu(\Gamma\vartheta_3,\Gamma\vartheta_1)\\
= &\sup_{\varrho\in[1,\tau]}\left|\frac{\Gamma(\vartheta_1(\varrho))}{\Gamma(\vartheta_2(\varrho))}\right|_* . \sup_{\varrho\in[1,\tau]}\left|\frac{\Gamma(\vartheta_2(\varrho))}{\Gamma(\vartheta_3(\varrho))}\right|_* . \sup_{\varrho\in[1,\tau]}\left|\frac{\Gamma(\vartheta_3(\varrho))}{\Gamma(\vartheta_1(\varrho))}\right|_*\\
= &\sup_{\varrho\in[1,\tau]}\left|\frac{\int_{\varrho_0}^{\varrho}(f(s,\vartheta_1(s)))^{ds}}{\int_{\varrho_0}^{\varrho}(f(s,\vartheta_1(s)))^{ds}}\right|_* . \sup_{\varrho\in[1,\tau]}\left|\frac{\int_{\varrho_0}^{\varrho}(f(s,\vartheta_2(s)))^{ds}}{\int_{\varrho_0}^{\varrho}(f(s,\vartheta_3(s)))^{ds}}\right|_* . \sup_{\varrho\in[1,\tau]}\left|\frac{\int_{\varrho_0}^{\varrho}(f(s,\vartheta_3(s)))^{ds}}{\int_{\varrho_0}^{\varrho}(f(s,\vartheta_1(s)))^{ds}}\right|_*\\
\leq &\sup_{\varrho\in[1,\tau]}\int_{\varrho_0}^{\varrho}\left|\frac{f(s,\vartheta_1)}{f(s,\vartheta_2)}\right|_*^{ds} . \sup_{\varrho\in[1,\tau]}\int_{\varrho_0}^{\varrho}\left|\frac{f(s,\vartheta_1)}{f(s,\vartheta_2)}\right|_*^{ds} . \sup_{\varrho\in[1,\tau]}\int_{\varrho_0}^{\varrho}\left|\frac{f(s,\vartheta_1)}{f(s,\vartheta_2)}\right|_*^{ds}\\
\leq &\sup_{\varrho\in[1,\tau]}\int_{\varrho_0}^{\varrho}(\alpha^{\left|\vartheta_1(s)/\vartheta_2(s)\right|_*})^{ds}. \sup_{\varrho\in[1,\tau]}\int_{\varrho_0}^{\varrho}(\alpha^{\left|\vartheta_2(s)/\vartheta_3(s)\right|_*})^{ds}. \sup_{\varrho\in[1,\tau]}\int_{\varrho_0}^{\varrho}(\alpha^{\left|\vartheta_2(s)/\vartheta_1(s)\right|_*})^{ds}\\
\leq &\sup_{\varrho\in[1,\tau]}\int_{\varrho_0}^{\varrho}(\alpha^{\mu(\vartheta_1,\vartheta_2)})^{ds}. \sup_{\varrho\in[1,\tau]}\int_{\varrho_0}^{\varrho}(\alpha^{\mu(\vartheta_2,\vartheta_3)})^{ds}. \sup_{\varrho\in[1,\tau]}\int_{\varrho_0}^{\varrho}(\alpha^{\mu(\vartheta_3,\vartheta_1)})^{ds}\\
\leq &\sup_{\varrho\in[1,\tau]}(|\varrho-\varrho_0|^\alpha)^{\mu(\vartheta_1,\vartheta_2)}.(|\varrho-\varrho_0|^\alpha)^{\mu(\vartheta_2,\vartheta_3)}.(|\varrho-\varrho_0|^\alpha)^{\mu(\vartheta_3,\vartheta_1)}\\
\leq &(\kappa^\alpha)^{\mu(\vartheta_1,\vartheta_2)}.(\kappa^\alpha)^{\mu(\vartheta_2,\vartheta_3)}.(\kappa^\alpha)^{\mu(\vartheta_3,\vartheta_1)} \leq (\mu(\vartheta_1,\vartheta_2).\mu(\vartheta_1,\vartheta_2).\mu(\vartheta_1,\vartheta_2))^{\kappa^\alpha}.
\end{aligned}$$

Here, $\kappa^\alpha < 1$. Therefore, Γ satisfies the conditions of Theorem 4.5.7 and has a fixed point i.e., initial value problem given by equation (4.6.1) has a feasible solution.

□

4.7 Existence of Solution to System of Multiplicative Fredholm Integral Equation

In this section, we have discussed the applicability of the proved results by establishing the existence of solution to a multiplicative Fredholm integral equations.

Theorem 4.7.1. *Consider the following system of multiplicative integral equation of Fredholm type*

$$\begin{cases} \vartheta(z) = \left[\int_1^2 \left(\vartheta(s)^{K_1(s,z)}\right)^{ds}\right]^{\alpha}, \text{ where } s, z \in I = [1,2] \\ \vartheta(z) = \left[\int_1^2 \left(\vartheta(s)^{K_2(s,z)}\right)^{ds}\right]^{\alpha}, \text{ where } s, z \in I = [1,2], \end{cases} \tag{4.7.1}$$

where $K_1(s,z), K_2(s,z)$ are continuous function defined on $I \times I$ s.t. $|K_i(s,z)| \leq \beta_i$ for $1 \leq i \leq 2$. If $\beta\alpha < 1$, where $\beta = \max\{\beta_1, \beta_2\}$, then we have exactly one solution to (4.7.1).

Proof. Consider the set of all multiplicative continuous positive function on [1,2] denoted as $C^*[1,2]$. Then the mapping $\mu : C^*[1,2] \times C^*[1,2] \to [1,\infty)$ defined as

$$\mu(\varrho,\vartheta) = \sup_{z\in[1,2]} \left|\frac{\varrho(z)}{\vartheta(z)}\right|_* . \min\left\{\sup_{z\in[1,2]} |\varrho(z)|_*, \sup_{z\in[1,2]} |\vartheta(z)|_*\right\},$$

where $|a|_* = \begin{cases} a, a \geq 1; \\ \frac{1}{a}, a < 1. \end{cases}$ is a multiplicative $\mathfrak{m}$-metric. Moreover, $C^*[1,2]$ is a complete multiplicative $\mathfrak{m}$-MS.

Define the self mappings Γ_1 and Γ_2 on $C^*[1,2]$ as

$$\Gamma_1(\vartheta(z)) = \left[\int_1^2 \left(\vartheta(s)^{K_1(s,z)}\right)^{ds}\right]^{\alpha},$$

$$\Gamma_2(\vartheta(z)) = \left[\int_1^2 \left(\vartheta(s)^{K_2(s,z)}\right)^{ds}\right]^{\alpha}.$$

Consider,

$$\mu(\Gamma_1(\vartheta_1), \Gamma_2(\vartheta_2)) = \sup_{z\in[1,2]} \left|\frac{\Gamma_1(\vartheta_1(z))}{\Gamma_2(\vartheta_2(z))}\right|_* . \min\left\{\sup_{z\in[1,2]} |\Gamma_1(\vartheta_1(z))|_*, \sup_{z\in[1,2]} |\Gamma_2(\vartheta_2(z))|_*\right\}$$

$$= \sup_{z\in[1,2]} \left|\left(\frac{\int_1^2 \left(\vartheta_1(s)^{K_1(s,z)}\right)^{ds}}{\int_1^2 \left(\vartheta_2(s)^{K_2(s,z)}\right)^{ds}}\right)^{\alpha}\right|_*$$

$$. \min\left\{\sup_{z\in[1,2]} \left|\left(\int_1^2 \left(\vartheta_1(s)^{K_1(s,z)}\right)^{ds}\right)^{\alpha}\right|_*, \sup_{z\in[1,2]} \left|\left(\int_1^2 \left(\vartheta_2(s)^{K_2(s,z)}\right)^{ds}\right)^{\alpha}\right|_*\right\}$$

$$
\begin{aligned}
&\leq \sup_{z\in[1,2]}\left(\int_1^2\left|\frac{\vartheta_1(s)}{\vartheta_2(s)}\right|_*^{ds}\right)^{\beta\alpha}.\min\left\{\sup_{z\in[1,2]}\left(\int_1^2|\vartheta_1(s)|_*^{ds}\right)^{\beta\alpha},\sup_{z\in[1,2]}\left(\int_1^2|\vartheta_2(s)|_*^{ds}\right)^{\beta\alpha}\right\}\\
&\leq \left(\sup_{z\in[1,2]}\left|\frac{\vartheta_1(s)}{\vartheta_2(s)}\right|_*.\min\left\{\sup_{z\in[1,2]}|\vartheta_1(z)|_*,\sup_{z\in[1,2]}|\vartheta_2(z)|_*\right\}\right)^{\beta\alpha}\\
&= \left(\mu\left(\vartheta_1,\vartheta_2\right)\right)^{\beta\alpha}.
\end{aligned}
$$

Also, $\mu_{\Gamma_1(\vartheta),\Gamma_2(\vartheta)} \leq \mu(\Gamma_1(\vartheta),\Gamma_2(\vartheta))$. Therefore, Γ_1,Γ_2 satisfies Theorem 4.4.1 with $a_1 = \beta\alpha < 1,\ a_2 = a_3 = 0$. Hence, Γ_1,Γ_2 have exactly one common fixed point i.e., system of equations (4.7.1) has exactly one solution. □

Theorem 4.7.2. *Consider the following multiplicative Fredholm integral equation*

$$
\vartheta(t) = \left[\int_1^2\left(\vartheta(s)^{K(s,t)}\right)^{ds}\right]^{\alpha},\ \textit{where } s,t\in I=[1,2] \tag{4.7.2}
$$

and $K(s,t)$ is real valued continuous function on $I\times I$ s.t. $|K(s,t)|\leq\beta$. If $\beta\alpha<1$, then we have exactly one solution to the equation (4.7.2).

Proof. The result follows as an direct consequence of Theorem 4.7.1. □

4.8 Conclusion

In this chapter, we established a generalized distance functions called multiplicative $\mathfrak{m}$-metric and proved some fixed point results. We provided various illustrations to support our results. Moreover, we present some common fixed point results of a pair of self mappings using a generalized contraction and common fixed point for a triplet of self mapping using three point analogue of contraction mapping in the context of multiplicative $\mathfrak{m}$-MS. We complemented our findings with numerical results and graphs to provide visual support for our conclusions. Furthermore, we explored the potential of utilizing the multiplicative $\mathfrak{m}$-metric to demonstrate the existence of a solution to a initial value problem and a system of multiplicative integral equation.

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Chapter 5

Some Common Fixed Point Results in Partial Metric Space

5.1 Introduction

Fractals, with their intricate and self-similar structures, have captivated researchers across various fields, including mathematics, physics, computer science, and art. Fixed point theory plays a crucial role in both the creation and characterization of fractals. In this chapter, we present several common fixed point results for self mappings in PMS using the $(\phi - \psi)$ Wardowski type contraction. Furthermore, some fixed point results are proven using generalized cyclic contractions, followed by illustrative examples. As an application, the existence of a fractal set for the Hutchinson-Barnsley operator is established using the established fixed point results. Finally, some iterations for generating fractal sets are presented, along with the resulting fractals.
The results of this chapter are presented in [1].

5.2 Some Common Fixed Point Results in Partial Metric Space

In this section, we present the coincidence point and common fixed point theorems for a pair of self mappings using a $(\phi - \psi)$ Wardowski contraction in partial metric space (PMS).

[1]Yadav, K., & Kumar, D. Fixed points of a generalized contraction in partial metric structure and application to fractal generation (Communicated)

Definition 5.2.1. *Consider a PMS* $(\Omega, \wp)$*. A pair* (Γ_1, Γ_2) *of self mappings on* Ω *is referred to as a* ***$(\phi - \psi)$-Wardowski type contraction pair*** *if, for some* $\phi \in \Phi$ *and* $\psi \in \Psi$*, we have*

$$\wp(\Gamma_1\varrho, \Gamma_1\vartheta) > 0 \Rightarrow \phi\left(\wp(\Gamma_1\varrho, \Gamma_1\vartheta)\right) \leq \psi\left(\phi\left(\mathcal{M}_{\Gamma_1,\Gamma_2}(\varrho, \vartheta)\right)\right), \tag{5.2.1}$$

where

$$\mathcal{M}_{\Gamma_1,\Gamma_2}(\varrho, \vartheta) = \max\left\{\wp(\Gamma_2\varrho, \Gamma_2\vartheta), \wp(\Gamma_2\varrho, \Gamma_1\varrho), \wp(\Gamma_2\vartheta, \Gamma_1\vartheta), \frac{\wp(\Gamma_2\varrho, \Gamma_1\vartheta) + \wp(\Gamma_2\vartheta, \Gamma_1\varrho)}{2}\right\},$$

$\forall \varrho, \vartheta \in \Omega$.

Theorem 5.2.2. *Let* $\Gamma_1, \Gamma_2 : \Omega \to \Omega$ *be mappings defined on a complete PMS* $(\Omega, \wp)$*. Suppose the pair* (Γ_1, Γ_2) *is a* $(\phi - \psi)$*-Wadowski type contraction pair, where* ϕ *and* ψ *are continuous functions. If* $\Gamma_1(\Omega) \subseteq \Gamma_2(\Omega)$ *and* $\Gamma_2(\Omega)$ *is complete, then* Γ_1 *possesses a coincidence point. Furthermore, if the pair* (Γ_1, Γ_2) *is weakly compatible, then it possesses exactly one common fixed point.*

Proof. Since $\Gamma_1(\Omega)$ be a subset of $\Gamma_2(\Omega)$, consequently for some $\varrho_0 \in \Omega$ $\exists$ an element ϱ_1 in Ω s.t. $\Gamma_1\varrho_0 = \Gamma_2\varrho_1$. Following the similar procedure, we can generate the sequence $\vartheta_\hbar \in \Omega$ s.t. $\vartheta_\hbar = \Gamma_1\varrho_\hbar = \Gamma_2\varrho_{\hbar+1}$ $\forall \hbar \in \mathbb{N}$.
We assume that $\vartheta_{\hbar+1} \neq \vartheta_\hbar$, $\forall \hbar \in \mathbb{N}$. Otherwise, suppose $\exists$ $\hbar_0 \in \mathbb{N}$ s.t. $\vartheta_{\hbar_0+1} = \vartheta_{\hbar_0}$ that suggests $\Gamma_1\varrho_{\hbar_0+1} = \Gamma_2\varrho_{\hbar_0+1}$, i.e., $\varrho_{\hbar_0+1}$ is a coincidence point of the mappings (Γ_1, Γ_2).
Next, we claim that $\wp(\vartheta_\hbar, \vartheta_\hbar + 1) > 0$. If possible, $\wp(\vartheta_{\hbar+1}, \vartheta_\hbar) = 0$, then $\wp(\vartheta_\hbar, \vartheta_\hbar)$ $= \wp(\vartheta_{\hbar+1}, \vartheta_{\hbar+1}) = 0$, i.e., $\vartheta_{\hbar+1} = \vartheta_\hbar$.
Therefore, $\wp(\vartheta_{\hbar+1}, \vartheta_\hbar) > 0$. Also, (Γ_1, Γ_2) be a $(\phi - \psi)$-Wadowski type contraction pair. Then, we have

$$\begin{aligned}\phi\left(\wp(\vartheta_{\hbar+1}, \vartheta_\hbar)\right) &= \phi\left(\wp(\Gamma_1\varrho_{\hbar+1}, \Gamma_1\varrho_\hbar)\right)\\ &\leq \psi\left(\phi\left(\mathcal{M}_{\Gamma_1,\Gamma_2}(\varrho_{\hbar+1}, \varrho_\hbar)\right)\right),\end{aligned}$$

where

$$\begin{aligned}\mathcal{M}_{\Gamma_1,\Gamma_2}(\varrho_{\hbar+1}, \varrho_\hbar) &= \max\Big\{\wp(\Gamma_2\varrho_{\hbar+1}, \Gamma_2\varrho_\hbar), \wp(\Gamma_2\varrho_{\hbar+1}, \Gamma_1\varrho_{\hbar+1}), \wp(\Gamma_2\varrho_\hbar, \Gamma_1\varrho_\hbar),\\ &\quad \frac{\wp(\Gamma_2\varrho_{\hbar+1}, \Gamma_1\varrho_\hbar) + \wp(\Gamma_1\varrho_{\hbar+1}, \Gamma_2\varrho_\hbar)}{2}\Big\}\\ &= \max\left\{\wp(\vartheta_\hbar, \vartheta_{\hbar-1}), \wp(\vartheta_\hbar, \vartheta_{\hbar+1}), \wp(\vartheta_{\hbar-1}, \vartheta_\hbar)\frac{\wp(\vartheta_\hbar, \vartheta_\hbar) + \wp(\vartheta_{\hbar+1}, \vartheta_{\hbar-1})}{2}\right\}\end{aligned}$$

$$\leq \max\left\{\wp(\vartheta_\hbar,\vartheta_{\hbar-1}),\wp(\vartheta_\hbar,\vartheta_{\hbar+1}),\frac{\wp(\vartheta_\hbar,\vartheta_{\hbar-1})+\wp(\vartheta_\hbar,\vartheta_{\hbar+1})}{2}\right\}$$
$$\leq \max\left\{\wp(\vartheta_\hbar,\vartheta_{\hbar-1}),\wp(\vartheta_\hbar,\vartheta_{\hbar+1})\right\}.$$

If $\mathcal{M}_{\Gamma_1,\Gamma_2}(\varrho_{\hbar+1},\varrho_\hbar)\leq\wp(\vartheta_\hbar,\vartheta_{\hbar+1})$. Then,

$$\phi\left(\wp(\vartheta_{\hbar+1},\vartheta_\hbar)\right)\leq\psi\Big(\phi\left(\wp(\vartheta_\hbar,\vartheta_{\hbar+1})\right)\Big)<\phi\left(\wp(\vartheta_\hbar,\vartheta_{\hbar+1})\right),$$

a contradiction. Hence, $\mathcal{M}_{\Gamma_1,\Gamma_2}(\varrho_\hbar,\varrho_{\hbar-1})\leq\wp(\vartheta_\hbar,\vartheta_{\hbar-1})$.
Consider

$$\phi\left(\wp(\vartheta_{\hbar+1},\vartheta_\hbar)\right)\leq\psi\Big(\phi\left(\wp(\vartheta_\hbar,\vartheta_{\hbar-1})\right)\Big)<\phi\left(\wp(\vartheta_\hbar,\vartheta_{\hbar-1})\right),$$

implies $\wp(\vartheta_{\hbar+1},\vartheta_\hbar)$ is a decreasing sequence. Also,

$$\begin{aligned}\phi\left(\wp(\vartheta_{\hbar+1},\vartheta_\hbar)\right) &\leq \psi\Big(\phi\left(\wp(\vartheta_\hbar,\vartheta_{\hbar-1})\right)\Big)\\ &\leq \psi^2\Big(\phi\left(\wp(\vartheta_{\hbar-1},\vartheta_{\hbar-2})\right)\Big)\\ &\ldots\\ &\leq \psi^\hbar\Big(\phi\left(\wp(\vartheta_1,\vartheta_0)\right)\Big).\end{aligned}$$

Taking the limit as $\hbar$ tends to ∞, we have

$$\lim_{\hbar\to\infty}\wp(\vartheta_{\hbar+1},\vartheta_\hbar)=0. \tag{5.2.2}$$

Also,

$$\lim_{\hbar\to\infty}\wp(\vartheta_\hbar,\vartheta_\hbar)\leq\lim_{\hbar\to\infty}\wp(\vartheta_{\hbar+1},\vartheta_\hbar)=0. \tag{5.2.3}$$

If possible, $\{\vartheta_\hbar\}$ is not a C_{seq}. Then, for $\epsilon>0$, $\exists$ two subsequences $\vartheta_{\hbar_\ell}\neq\vartheta_{m_\ell}$ s.t.

$$d_\wp(\vartheta_{\hbar_\ell},\vartheta_{m_\ell}) > \epsilon, \tag{5.2.4}$$

and

$$d_\wp(\vartheta_{\hbar_\ell-1},\vartheta_{m_\ell}) \leq \epsilon, \tag{5.2.5}$$

where $d_\wp(\varrho,\vartheta)=2\wp(\varrho,\vartheta)-\wp(\varrho,\varrho)-\wp(\vartheta,\vartheta)\ \forall\varrho,\vartheta\in\Omega$ is a metric.
Using (5.2.4) and triangle inequality, we have

$$\begin{aligned}\epsilon &< d_\wp(\vartheta_{\hbar_\ell},\vartheta_{m_\ell})\\ &\leq d_\wp(\vartheta_{\hbar_\ell},\vartheta_{\hbar_\ell-1})+d_\wp(\vartheta_{\hbar_\ell-1},\vartheta_{m_\ell}).\end{aligned}$$

Taking the limit as ℓ tends to ∞ and using (5.2.2) and (5.2.3), we have

$$\lim_{\ell\to\infty} d_{\wp}(\vartheta_{\hbar_\ell}, \vartheta_{m_\ell}) = \epsilon. \tag{5.2.6}$$

Also, using triangle inequality, we have

$$d_{\wp}(\vartheta_{\hbar_\ell}, \vartheta_{m_\ell}) \leq d_{\wp}(\vartheta_{\hbar_\ell}, \vartheta_{\hbar_\ell-1}) + d_{\wp}(\vartheta_{\hbar_\ell-1}, \vartheta_{m_\ell-1}) + d_{\wp}(\vartheta_{m_\ell-1}, \vartheta_{m_\ell}),$$

and

$$d_{\wp}(\vartheta_{\hbar_\ell-1}, \vartheta_{m_\ell-1}) \leq d_{\wp}(\vartheta_{\hbar_\ell-1}, \vartheta_{\hbar_\ell}) + d_{\wp}(\vartheta_{\hbar_\ell}, \vartheta_{m_\ell}) + d_{\wp}(\vartheta_{m_\ell}, \vartheta_{m_\ell-1}).$$

Taking the limit as ℓ tends to ∞ and using (5.2.6), we have

$$\lim_{\ell\to\infty} d_{\wp}(\vartheta_{\hbar_\ell-1}, \vartheta_{m_\ell-1}) = \epsilon.$$

$$\lim_{\ell\to\infty} 2\wp(\vartheta_{\hbar_\ell-1}, \vartheta_{m_\ell-1}) - \wp(\vartheta_{\hbar_\ell-1}, \vartheta_{\hbar_\ell-1}) - \wp(\vartheta_{m_\ell-1}, \vartheta_{m_\ell-1}) = \epsilon. \tag{5.2.7}$$

Using (5.2.2), (5.2.3) and (5.2.7), we have

$$\lim_{\ell\to\infty} \wp(\vartheta_{\hbar_\ell}, \vartheta_{m_\ell}) = \frac{\epsilon}{2} \text{ and } \lim_{\ell\to\infty} \wp(\vartheta_{\hbar_\ell-1}, \vartheta_{m_\ell-1}) = \frac{\epsilon}{2}. \tag{5.2.8}$$

Also, by (5.2.8), (5.3.2) and continuity of ϕ, we have

$$\begin{aligned}
\phi\left(\frac{\epsilon}{2}\right) &= \lim_{\ell\to\infty} \phi\left(\wp(\vartheta_{\hbar_\ell}, \vartheta_{m_\ell})\right) = \lim_{\ell\to\infty} \phi\left(\wp(\Gamma_1\varrho_{\hbar_\ell}, \Gamma_1\varrho_{m_\ell})\right) \\
&\leq \lim_{\ell\to\infty} \psi\left(\phi\left(\mathcal{M}_{\Gamma_1,\Gamma_2}(\varrho_{\hbar_\ell}, \varrho_{m_\ell})\right)\right) \\
&\leq \lim_{\ell\to\infty} \psi\Bigg(\phi\Bigg(\max\Bigg\{\wp(\Gamma_2\varrho_{\hbar_\ell}, \Gamma_2\varrho_{m_\ell}), \wp(\Gamma_2\varrho_{\hbar_\ell}, \Gamma_1\varrho_{\hbar_\ell}), \wp(\Gamma_2\varrho_{m_\ell}, \Gamma_1\varrho_{m_\ell}), \\
&\quad \frac{\wp(\Gamma_2\varrho_{\hbar_\ell}, \Gamma_1\varrho_{m_\ell}) + \wp(\Gamma_2\varrho_{m_\ell}, \Gamma_1\varrho_{\hbar_\ell})}{2}\Bigg\}\Bigg)\Bigg) \\
&= \lim_{\ell\to\infty} \psi\Bigg(\phi\Bigg(\max\Bigg\{\wp(\vartheta_{\hbar_\ell-1}, \vartheta_{m_\ell-1}), \wp(\vartheta_{\hbar_\ell-1}, \vartheta_{\hbar_\ell}), \wp(\vartheta_{m_\ell-1}, \vartheta_{m_\ell}), \\
&\quad \frac{\wp(\vartheta_{\hbar_\ell-1}, \vartheta_{m_\ell}) + \wp(\vartheta_{m_\ell-1}, \vartheta_{\hbar_\ell})}{2}\Bigg\}\Bigg)\Bigg) \\
&\leq \lim_{\ell\to\infty} \psi\Bigg(\phi\Bigg(\max\Bigg\{\wp(\vartheta_{\hbar_\ell-1}, \vartheta_{m_\ell-1}), \wp(\vartheta_{\hbar_\ell-1}, \vartheta_{\hbar_\ell}), \wp(\vartheta_{m_\ell-1}, \vartheta_{m_\ell}), \frac{\wp(\vartheta_{\hbar_\ell-1}, \vartheta_{m_\ell})}{2} + \\
&\quad \frac{\wp(\vartheta_{\hbar_\ell}, \vartheta_{\hbar_\ell-1}) + \wp(\vartheta_{\hbar_\ell-1}, \vartheta_{m_\ell-1}) - \wp(\vartheta_{\hbar_\ell-1}, \vartheta_{\hbar_\ell-1})}{2}\Bigg\}\Bigg)\Bigg) \\
&< \phi\left(\frac{\epsilon}{2}\right),
\end{aligned}$$

a contradiction. Therefore, $\{\vartheta_\hbar\}$ is a C_{seq} i.e.,

$$\lim_{\hbar,m\to\infty} d_{\wp}(\vartheta_\hbar, \vartheta_m) = 0,$$

or

$$\lim_{\hbar,m\to\infty} 2\wp(\vartheta_\hbar,\vartheta_m)-\wp(\vartheta_\hbar,\vartheta_\hbar)-\wp(\vartheta_m,\vartheta_m)=0.$$

Using (5.2.3), we have

$$\lim_{\hbar,m\to\infty} \wp(\vartheta_\hbar,\vartheta_m)=0,$$

i.e., $\{\vartheta_\hbar\}$ is 0-C_{seq} As, Ω is an 0-complete PMS, therefore for some $\vartheta\in\Omega$, we have

$$\lim_{\hbar\to\infty} \wp(\vartheta_\hbar,\vartheta)=\wp(\vartheta,\vartheta)=0. \tag{5.2.9}$$

As, $\Gamma_1(\Omega)\subseteq\Gamma_2(\Omega)$ and $\Gamma_2(\Omega)$ is complete, therefore $\exists$ some $\varrho\in\Omega$ s.t. $\Gamma_2\varrho=\vartheta$. Also, $\vartheta_\hbar\to\vartheta$ implies $\Gamma_1\varrho_\hbar\to\vartheta$ and $\Gamma_2\varrho_{\hbar+1}\to\vartheta$, i.e., $\wp(\Gamma_1\varrho_\hbar,\vartheta)\to\wp(\vartheta,\vartheta)=0$. Consider the following

$$\wp(\vartheta,\Gamma_1\vartheta)\leq\wp(\vartheta,\Gamma_1\varrho_\hbar)+\wp(\Gamma_1\varrho_\hbar,\Gamma_1\vartheta)-\wp(\Gamma_1\varrho_\hbar,\Gamma_1\varrho_\hbar)\leq\wp(\vartheta,\Gamma_1\varrho_\hbar)+\wp(\Gamma_1\varrho_\hbar,\Gamma_1\vartheta).$$

We claim that $\wp(\Gamma_1\varrho_\hbar,\Gamma_1\vartheta)>0$ or $\wp(\Gamma_1\varrho_{\hbar+1},\Gamma_1\vartheta)>0$. If not then, suppose $\wp(\Gamma_1\varrho_{\hbar_0},\Gamma_1\vartheta)=0$ and $\wp(\Gamma_1\varrho_{\hbar_0+1},\Gamma_1\vartheta)=0$, for some $\hbar_0\in\mathbb{N}$.
Now,

$$\begin{aligned}
\wp(\vartheta_{\hbar_o+1},\vartheta_{\hbar_o}) &= \wp(\Gamma_1\varrho_{\hbar_o+1},\Gamma_1\varrho_{\hbar_o})\\
&\leq \wp(\Gamma_1\varrho_{\hbar_o+1},\Gamma_1\varrho)+\wp(\Gamma_1\varrho_{\hbar_o},\Gamma_1\varrho)-\wp(\Gamma_1\varrho,\Gamma_1\varrho)\\
&\leq \wp(\Gamma_1\varrho_{\hbar_o+1},\Gamma_1\varrho)+\wp(\Gamma_1\varrho_{\hbar_o},\Gamma_1\varrho)=0,
\end{aligned}$$

a contradiction. Therefore, $\wp(\Gamma_1\varrho_\hbar,\Gamma_1\vartheta)>0$ or $\wp(\Gamma_1\varrho_{\hbar+1},\Gamma_1\vartheta)>0$. Without loss of generality suppose $\wp(\Gamma_1\varrho_\hbar,\Gamma_1\vartheta)>0$, then

$$\begin{aligned}
\phi\left(\wp(\Gamma_1\varrho,\vartheta)\right) &\leq \phi\left(\wp(\Gamma_1\varrho,\Gamma_1\varrho_\hbar)+\wp(\Gamma_1\varrho_\hbar,\vartheta)-\wp(\Gamma_1\varrho_\hbar,\Gamma_1\varrho_\hbar)\right)\\
&\leq \limsup_{\hbar\to\infty}\phi\left(\wp(\Gamma_1\varrho,\Gamma_1\varrho_\hbar)+\wp(\Gamma_1\varrho_\hbar,\vartheta)\right)\\
&\leq \limsup_{\hbar\to\infty}\phi\left(\wp(\Gamma_1\varrho_\hbar,\Gamma_1\varrho)\right)\\
&\leq \limsup_{\hbar\to\infty}\psi\left(\phi\left(\mathcal{M}_{\Gamma_1,\Gamma_2}(\varrho_\hbar,\varrho)\right)\right),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{M}_{\Gamma_1,\Gamma_2}(\varrho_\hbar,\varrho) &= \max\left\{\wp(\Gamma_2\varrho_\hbar,\Gamma_2\varrho),\wp(\Gamma_2\varrho,\Gamma_1\varrho),\wp(\Gamma_2\varrho_\hbar,\Gamma_1\varrho_\hbar),\frac{\wp(\Gamma_2\varrho,\Gamma_1\varrho_\hbar)+\wp(\Gamma_2\varrho_\hbar,\Gamma_1\varrho)}{2}\right\}\\
&= \max\left\{\wp(\vartheta_{\hbar-1},\Gamma_2\varrho),\wp(\Gamma_2\varrho,\Gamma_1\varrho),\wp(\vartheta_{\hbar-1},\vartheta_\hbar),\frac{\wp(\Gamma_2\varrho,\vartheta_\hbar)+\wp(\vartheta_{\hbar-1},\Gamma_1\varrho)}{2}\right\}\\
&= \max\left\{\wp(\vartheta_{\hbar-1},\vartheta),\wp(\vartheta,\Gamma_1\varrho),\wp(\vartheta_{\hbar-1},\vartheta_\hbar),\frac{\wp(\vartheta,\vartheta_\hbar)+\wp(\vartheta_{\hbar-1},\Gamma_1\varrho)}{2}\right\}\\
&\leq \max\Bigg\{\wp(\vartheta_{\hbar-1},\vartheta),\wp(\vartheta,\Gamma_1\varrho),\wp(\vartheta_{\hbar-1},\vartheta_\hbar),\\
&\quad \frac{\wp(\vartheta,\vartheta_\hbar)}{2}+\frac{\wp(\vartheta_{\hbar-1},\vartheta)+\wp(\vartheta,\Gamma_1\varrho)-\wp(\vartheta,\vartheta)}{2}\Bigg\}.
\end{aligned}$$

Here, $\wp(\vartheta_{\hbar-1},\vartheta)$, $\wp(\vartheta_{\hbar-1},\vartheta_\hbar)$ and $\wp(\vartheta,\vartheta_\hbar)\to 0$, as $\hbar\to\infty$. Let if possible $\wp(\vartheta,\Gamma_1\varrho)>0$, then $\mathcal{M}_{\Gamma_1,\Gamma_2}(\varrho_\hbar,\varrho)\leq\wp(\vartheta,\Gamma_1\varrho)$.

$$\begin{aligned}
\phi\left(\wp(\vartheta,\Gamma_1\varrho)\right) &\leq \psi\left(\phi\left(\mathcal{M}_{\Gamma_1,\Gamma_2}(\varrho_\hbar,\varrho)\right)\right)\\
&= \psi\left(\phi\left(\wp(\vartheta,\Gamma_1\varrho)\right)\right)\\
&< \phi\left(\wp(\vartheta,\Gamma_1\varrho)\right),
\end{aligned}$$

a contradiction. Therefore, $\wp(\vartheta,\Gamma_1\varrho)=0$ gives, $\wp(\vartheta,\vartheta)=\wp(\Gamma_1\varrho,\Gamma_1\varrho)=0=\wp(\varrho,\Gamma_1\varrho)$ i.e., $\Gamma_1\varrho=\vartheta$. Thus $\Gamma_1\varrho=\vartheta=\Gamma_2\varrho$ i.e., (Γ_1,Γ_2) has a coincidence point.
Uniqueness: let, ϑ_1 be another coincidence point of $\Gamma_1\&\Gamma_2$, then we have ϱ_1 s.t. $\Gamma_1\varrho_1=\Gamma_2\varrho_1=\vartheta_1$. If $\wp(\vartheta,\vartheta_1)>0$, then

$$\begin{aligned}
\phi\left(\wp(\vartheta,\vartheta_1)\right) &= \phi\left(\wp(\Gamma_1\varrho,\Gamma_1\varrho_1)\right)\\
&\leq \psi\left(\phi\left(\mathcal{M}_{\Gamma_1,\Gamma_2}(\varrho,\varrho_1)\right)\right),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{M}_{\Gamma_1,\Gamma_2}(\varrho,\varrho_1) &= \max\left\{\wp(\Gamma_2\varrho,\Gamma_2\varrho_1),\wp(\Gamma_2\varrho,\Gamma_1\varrho),\wp(\Gamma_2\varrho_1,\Gamma_1\varrho_1),\frac{\wp(\Gamma_2\varrho,\Gamma_1\varrho_1)+\wp(\Gamma_1\varrho,\Gamma_2\varrho_1)}{2}\right\}\\
&= \max\left\{\wp(\vartheta,\vartheta_1),\wp(\vartheta,\vartheta),\wp(\vartheta_1,\vartheta_1),\frac{\wp(\vartheta,\vartheta_1)+\wp(\vartheta,\vartheta_1)}{2}\right\}\\
&= \wp(\vartheta,\vartheta_1).
\end{aligned}$$

Therefore,

$$\begin{aligned}\phi\left(\wp(\vartheta,\vartheta_1)\right) &\leq \psi\left(\phi\left(\mathcal{M}_{\Gamma_1,\Gamma_2}(\varrho,\varrho_1)\right)\right)\\ &< \phi\left(\wp(\vartheta,\vartheta_1)\right),\end{aligned}$$

a contradiction. Therefore, $\vartheta = \vartheta_1$.
Also, the pair (Γ_1, Γ_2) is weakly compatible and $\Gamma_2\varrho = \Gamma_1\varrho$. Therefore, $\Gamma_1\Gamma_2\varrho = \Gamma_2\Gamma_1\varrho$. Then, $\Gamma_2\vartheta = \Gamma_2\Gamma_1\varrho = \Gamma_1\Gamma_2\varrho = \Gamma_1\vartheta$ implies ϑ is the point of coincidence for (Γ_1, Γ_2). By uniqueness, we have $\vartheta = \varrho$ i.e., $\Gamma_1\vartheta = \Gamma_2\vartheta = \vartheta$. Hence, the result holds.

□

Definition 5.2.3. *Consider a PMS $(\Omega, \wp)$. A self mapping $\Gamma : \Omega \to \Omega$ is considered a **$(\phi - \psi)$-Wardowski contraction**, if for some $\phi \in \Phi$ and $\psi \in \Psi$, we have*

$$\wp(\Gamma\varrho, \Gamma\vartheta) > 0 \Rightarrow \phi(\wp(\Gamma\varrho, \Gamma\vartheta)) \leq \psi(\phi(\mathcal{M}_\Gamma(\varrho, \vartheta))), \tag{5.2.10}$$

where

$$\mathcal{M}_\Gamma(\varrho, \vartheta) = \max\left\{\wp(\varrho, \vartheta), \wp(\varrho, \Gamma\varrho), \wp(\vartheta, \Gamma\vartheta), \frac{\wp(\varrho, \Gamma\vartheta) + \wp(\vartheta, \Gamma\varrho)}{2}\right\} \forall \varrho, \vartheta \in \Omega.$$

Theorem 5.2.4. *Let $\Gamma : \Omega \to \Omega$ be a mapping defined on a complete PMS $(\Omega, \wp)$. Suppose Γ is a $(\phi - \psi)$-Wadowski contraction, then Γ possesses exactly one fixed point.*

Proof. substituting $\Gamma_1 = \Gamma$ and $\Gamma_2 = I$, where I represents the identity map in Theorem 5.2.2, we obtain the required result. □

Corollary 5.2.5. *Let $\Gamma : \Omega \to \Omega$ be a mapping defined on a complete PMS $(\Omega, \wp)$. Suppose Γ satisfies:*

(i) $\exists \lambda \in (0, 1)$ *s.t.* $\wp(\Gamma\varrho, \Gamma\vartheta) \leq \lambda\wp(\varrho, \vartheta)$*;*

(ii) $\exists \lambda \in (0, \frac{1}{2})$ *s.t.* $\wp(\Gamma\varrho, \Gamma\vartheta) \leq \lambda\left(\wp(\varrho, \Gamma\varrho) + \wp(\vartheta, \Gamma\vartheta)\right)$*;*

(iii) $\exists a_0, a_1, a_2$ *with* $a_0 + a_1 + a_2 \in (0, 1)$ *s.t.* $\wp(\Gamma\varrho, \Gamma\vartheta) \leq a_o\wp(\varrho, \vartheta) + a_1\wp(\varrho, \Gamma\varrho) + a_2\wp(\vartheta, \Gamma\vartheta)$*;*

(iv) $\exists \digamma \in \boldsymbol{\digamma}_*$ *and* $\tau' > 0$ *s.t.* $\tau' + \digamma(\wp(\Gamma\varrho, \Gamma\vartheta)) \leq \digamma(\wp(\varrho, \vartheta))$*;*

(v) $\exists F \in \boldsymbol{F}_*$ *and* $\tau' > 0$ *s.t.* $\tau' + F(\wp(\Gamma\varrho, \Gamma\vartheta)) \leq F(\mathcal{M}(\varrho, \vartheta))$,
where $\mathcal{M}(\varrho, \vartheta) = \max\left\{\wp(\varrho, \vartheta), \wp(\varrho, \Gamma\varrho), \wp(\vartheta, \Gamma\vartheta), \frac{\wp(\varrho,\Gamma\vartheta)+\wp(\vartheta,\Gamma\varrho)}{2}\right\}$,

$\forall \varrho, \vartheta \in \Omega$. *Then,* Γ *possesses exactly one fixed point.*

Example 5.2.6. *Consider* $\Omega = [0, \infty]$ *along with* $\wp(\varrho, \vartheta) = \max\{\varrho, \vartheta\}$, *for every* $\varrho, \vartheta \in \Omega$. *Here,* $(\Omega, \wp)$ *is a complete PMS. Let* Γ_1, Γ_2 *be self mappings defined on* Ω *as*

$$\Gamma_1\varrho = \begin{cases} \frac{\varrho}{4}, & \text{if } \varrho \leq 1 \\ \frac{1}{8}, & \text{otherwise.} \end{cases}, \quad \Gamma_2\varrho = \begin{cases} \frac{\varrho}{3}, & \text{if } \varrho \leq 1 \\ \frac{1}{7}, & \text{otherwise.} \end{cases}$$

Observe that $\Gamma_1(\Omega) = [0, 1/4] \subseteq [0, \frac{1}{3}] = \Gamma_2(\Omega)$ *and* $\Gamma_2(\Omega)$ *is complete.*
Also, consider

(i) *for* $\varrho, \vartheta \in [0, 1]$, *we have*

$$\begin{aligned} p(\Gamma_1\varrho, \Gamma_1\vartheta) &= \max\left\{\frac{\varrho}{4}, \frac{\vartheta}{4}\right\} \\ &\leq \frac{1}{4}\max\{\varrho, \vartheta\} \\ &\leq \frac{1}{4}M_{\Gamma_1,\Gamma_2}(\varrho, \vartheta). \end{aligned}$$

(ii) *for* $\varrho, \vartheta > 1$, *we have*

$$\begin{aligned} p(\Gamma_1\varrho, \Gamma_1\vartheta) &= \max\{\Gamma\varrho, \Gamma\vartheta\} \\ &= \frac{1}{8} \leq \frac{1}{4}\max\{\varrho, \vartheta\} \\ &\leq \frac{1}{4}M_{\Gamma_1,\Gamma_2}(\varrho, \vartheta). \end{aligned}$$

(iii) *for* $\varrho \in [0, 1]$ *and* $\vartheta > 1$, *suppose* $\varrho \leq \frac{1}{2}$

$$\begin{aligned} p(\Gamma_1\varrho, \Gamma_1\vartheta) &= \max\left\{\frac{\varrho}{4}, \frac{1}{8}\right\} \\ &= \frac{1}{8} \leq \frac{1}{4}\max\{\varrho, \vartheta\} \\ &\leq \frac{1}{4}M_{\Gamma_1,\Gamma_2}(\varrho, \vartheta), \end{aligned}$$

and if $\varrho > \frac{1}{2}$, *then*

$$\begin{aligned} p(\Gamma_1\varrho, \Gamma_1\vartheta) &= \max\left\{\frac{\varrho}{4}, \frac{1}{8}\right\} \\ &= \frac{\varrho}{4} \leq \frac{1}{4}\max\{\varrho, \vartheta\} \\ &\leq \frac{1}{4}M_{\Gamma_1,\Gamma_2}(\varrho, \vartheta). \end{aligned}$$

Moreover, Γ_1, Γ_2 are weakly compatible. Therefore, Γ_1, Γ_2 satisfy all the hypothesis of Theorem 5.2.2. Hence, Γ_1 and Γ_2 have exactly one common fixed point.

For Example 5.2.6, we provided several iterations to approximate the common fixed point of S, Γ. The iterative scheme used is

For initial point x_0 : $x_1 = \Gamma_2 x_0,\ x_2 = \Gamma_1 x_1, \ldots\ldots, x_{2n+1} = \Gamma_2 x_{2n},\ x_{2n+2} = \Gamma_1 x_{2n+1}$.

Further graphically, we demonstrated the convergence of the iterative sequence and determined that the common fixed point of the mappings is independent of the initial point of the iterative process.

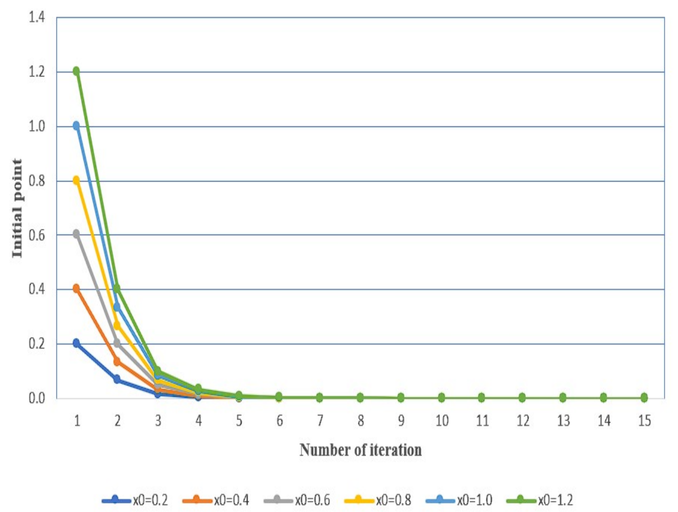

Figure 5.1: Convergence behaviour of iteration scheme at different initial points for Example 5.2.6

5.3 Fractal Generation via Fixed Point Approach using Generalized Cyclic Contraction

In this section, we present some fixed point results using generalized cyclic contraction. Later, the results are implemented to establish the existence of a fractal set for the Hutchinson-Barnsley operator of IFS.

Definition 5.3.1. *Consider a PMS $(\Omega, \wp)$, a positive integer m and non-empty subsets $A_i \subseteq \Omega$, for $1 \leq i \leq m$. A self mapping $\Gamma : \bigcup_{i=1}^{m} A_i \to \bigcup_{i=1}^{m} A_i$ is considered as* ***cyclic $(\phi - \psi)$-Wardowski contraction****, if for some $\phi \in \Phi$ and $\psi \in \Psi$, we have*

x0	0.2	0.4	0.6	0.8	1.0	1.2
x1	0.0666667	0.1333333	0.2000000	0.2666667	0.3333333	0.1428571
x2	0.0166667	0.0333333	0.0500000	0.0666667	0.0833333	0.0357143
x3	0.0055556	0.0111111	0.0166667	0.0222222	0.0277778	0.0119048
x4	0.0013889	0.0027778	0.0041667	0.0055556	0.0069444	0.0029762
x5	0.0004630	0.0009259	0.0013889	0.0018519	0.0023148	0.0009921
x6	0.0001157	0.0002315	0.0003472	0.0004630	0.0005787	0.0002480
x7	0.0000386	0.0000772	0.0001157	0.0001543	0.0001929	0.0000827
x8	0.0000096	0.0000193	0.0000289	0.0000386	0.0000482	0.0000207
x9	0.0000032	0.0000064	0.0000096	0.0000129	0.0000161	0.0000069
x10	0.0000008	0.0000016	0.0000024	0.0000032	0.0000040	0.0000017
x11	0.0000003	0.0000005	0.0000008	0.0000011	0.0000013	0.0000006
x12	0.0000001	0.0000001	0.0000002	0.0000003	0.0000003	0.0000001
x13	0.0000000	0.0000000	0.0000001	0.0000001	0.0000001	0.0000000
x14	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000

Figure 5.2: Numerical iteration for Example 5.2.6

(i) $\Gamma(A_i) \subseteq A_{i+1}\ \forall 1 \leq i \leq m$;

(ii) $\wp(\Gamma\varrho, \Gamma\vartheta) > 0 \Rightarrow \phi\left(\wp(\Gamma\varrho, \Gamma\vartheta)\right) \leq \psi\Big(\phi\left(\mathcal{M}_\Gamma(\varrho, \vartheta)\right)\Big)$,

where

$$\mathcal{M}_\Gamma(\varrho, \vartheta) = \max\left\{\wp(\varrho, \vartheta), \wp(\varrho, \Gamma\varrho), \wp(\vartheta, \Gamma\vartheta), \frac{\wp(\varrho, \Gamma\vartheta) + \wp(\vartheta, \Gamma\varrho)}{2}\right\},$$

and $A_{m+1} = A_1$, $\forall \varrho \in A_i, \vartheta \in A_{i+1}$.

Theorem 5.3.2. *Consider a complete PMS* $(\Omega, \wp)$, *a positive integer* m *and non-empty closed subsets* $A_i \subseteq \Omega$, *for* $1 \leq i \leq m$. *Suppose* $\Gamma : \bigcup_{i=1}^{m} A_i \to \bigcup_{i=1}^{m} A_i$ *be a cyclic* $(\phi - \psi)$*-Wardowski contraction w.r.t some* $\phi \in \Phi$ *and* $\psi \in \Psi$. *Then* $\bigcap_{i=1}^{m} A_i$ *is non-empty and* Γ *has exactly one fixed point. Moreover, the Picard sequence converges to* $\varrho \in \bigcap_{i=1}^{m} A_i$, *for any initial point* $\varrho_0 \in \bigcup_{i=1}^{m} A_i$.

Proof. For any $\varrho_0 \in \bigcup_{i=1}^{m} A_i$, consider the sequence $\varrho_{n+1} = \Gamma\varrho_n$, for $n \geq 0$. Observe that $\varrho_0 \in A_k$ for some k and $\Gamma(A_k) \subseteq A_{k+1}$ implies $\varrho_1 = \Gamma(\varrho_0) \in A_{k+1}$. On generalizing, for $n \geq 0$, we have some $i(l)$ s.t. $\varrho_n \in A_{i(l)}$ and $\varrho_{n+1} \in A_{i(l)+1}$.
In case, $\varrho_n = \varrho_{n+1}$, for some $n \in \mathbb{N}$. Then, ϱ_n is a fixed point of Γ.
Suppose $p(\varrho_{n+1}, \varrho_n) > 0$ and Γ is a cyclic $(\phi - \psi)$-Wardowski contraction. Then, on the outline of Theorem 6.2.3, we can verify that ϱ_n is a C_{seq}. Therefore, $\{\varrho_n\}$ is convergent to $\varrho \in\ (\Omega, \wp)$.

In addition, using the cyclic representation ($A_i : 1 \leq i \leq m$), it is feasible to determine subsequences $\{\varrho_{n_i}\} \in A_i$ that converge to ϱ. Also, A_i is closed for every $1 \leq i \leq m$, which implies $\varrho \in \bigcap_{i=1}^{m} A_i$.

Let $M = \bigcap_{i=1}^{m} A_i$ and $\Gamma' : M \to M$ be the restriction of Γ on Ω. Clearly, M is a complete subspace of Ω. Then, Γ' satisfies all the assumptions of Theorem 5.2.3. Therefore, Γ' has exactly one fixed point, i.e., $\Gamma|_\Omega$ has a fixed point, ϑ (say).

At last, suppose $\exists\ \varrho \in \bigcup_{i=1}^{m} A_i$ s.t. $\Gamma(\varrho) = \varrho$ and $\varrho \neq \vartheta$. Then,

$$
\begin{aligned}
\phi(\varrho, \vartheta) &= \phi\left(\wp(\Gamma\varrho, \Gamma\vartheta)\right) \leq \psi\Big(\phi\left(\mathcal{M}_\Gamma(\varrho, \vartheta)\right)\Big) \\
&\leq \psi\left(\phi\left(\max\left\{\wp(\varrho, \vartheta), \wp(\varrho, \Gamma\varrho), \wp(\vartheta, \Gamma\vartheta), \frac{\wp(\varrho, \Gamma\vartheta) + \wp(\vartheta, \Gamma\varrho)}{2}\right\}\right)\right) \\
&\leq \psi\Big(\phi\left(\wp(\varrho, \vartheta)\right)\Big) < \phi\left(\wp(\varrho, \vartheta)\right),
\end{aligned}
$$

a contradiction. Therefore, $\wp(\varrho, \vartheta) = 0 = \wp(\varrho, \varrho) = \wp(\vartheta, \vartheta)$ i.e., $\varrho = \vartheta$. □

Definition 5.3.3. *Consider a PMS $(\Omega, \wp)$, a positive integer m, and non-empty subsets $A_i \subseteq \Omega$, for $1 \leq i \leq m$. A self mapping $\Gamma : \bigcup_{i=1}^{m} A_i \to \bigcup_{i=1}^{m} A_i$ is considered a* ***cyclic $(\phi - \psi)$-Banach contraction****, if for some $\phi \in \Phi$ and $\psi \in \Psi$, we have*

(i) $\Gamma(A_i) \subseteq A_{i+1}\ \forall\ 1 \leq i \leq m$;

(ii) $\wp(\Gamma\varrho, \Gamma\vartheta) > 0 \Rightarrow \phi\left(\wp(\Gamma\varrho, \Gamma\vartheta)\right) \leq \psi\Big(\phi\left(\wp(\varrho, \vartheta)\right)\Big)$,

and $A_{m+1} = A_1$, $\forall \varrho \in A_i, \vartheta \in A_{i+1}$.

Remark 5.3.4. *Observe that if $\Gamma : \bigcup_{i=1}^{m} A_i \to \bigcup_{i=1}^{m} A_i$ is a cyclic $(\phi - \psi)$-Banach contraction w.r.t some $\phi \in \Phi$ and $\psi \in \Psi$. Then, $\Gamma : \bigcup_{i=1}^{m} A_i \to \bigcup_{i=1}^{m} A_i$ is a cyclic $(\phi - \psi)$-Wardowski contraction w.r.t $\phi \in \Phi$ and $\psi \in \Psi$.*

Corollary 5.3.5. *Consider a complete PMS $(\Omega, \wp)$, a positive integer m and non-empty closed subsets $A_i \subseteq \Omega$, for $1 \leq i \leq m$. Suppose $\Gamma : \bigcup_{i=1}^{m} A_i \to \bigcup_{i=1}^{m} A_i$ be a cyclic $(\phi - \psi)$-Banach contraction w.r.t some $\phi \in \Phi$ and $\psi \in \Psi$. Then $\bigcap_{i=1}^{m} A_i$ is non-empty and Γ has exactly one fixed point. Moreover, the Picard sequence converges to $\varrho \in \bigcap_{i=1}^{m} A_i$, for any initial point $\varrho_0 \in \bigcup_{i=1}^{m} A_i$.*

Proof. The result can be deduced using Theorem 5.3.2 and Remark 5.3.4. $\square$

Theorem 5.3.6. *Consider a complete PMS $(\Omega, \wp)$, a positive integer m and non-empty closed subsets $A_i \subseteq \Omega$, for $1 \leq i \leq m$. Suppose $\Gamma : \bigcup_{i=1}^{m} A_i \to \bigcup_{i=1}^{m} A_i$ be a continuous cyclic $(\phi-\psi)$-Banach contraction w.r.t some $\phi \in \Phi$ and $\psi \in \Psi$. Then, mapping $\Gamma' : \bigcup_{i=1}^{m} K(A_i) \to \bigcup_{i=1}^{m} K(A_i)$ defined as $\Gamma'(C) = \Gamma(C) = \{\Gamma(\varrho) : \varrho \in C\}$, $\forall C \in \bigcup_{i=1}^{m} K(A_i)$ is also a cyclic $(\phi-\psi)$-Banach contraction w.r.t $\phi \in \Phi$ and $\psi \in \Psi$ in PMS $(K(\Omega), H_\wp)$.*

Proof. Suppose that $\Gamma : \bigcup_{i=1}^{m} A_i \to \bigcup_{i=1}^{m} A_i$ be a continuous cyclic $(\phi - \psi)$-Banach contraction w.r.t some $\phi \in \Phi$ and $\psi \in \Psi$. Let $C \in K(A_i)$ for some $1 \leq i \leq m$. Then,

$$\begin{aligned} C \subseteq A_i &\Rightarrow \Gamma(C) \subseteq \Gamma(A_i) \subseteq A_{i+1} \\ &\Rightarrow \Gamma(C) \in K(A_{i+1}) \because \Gamma \text{ is continuous and cyclic.} \end{aligned}$$

This implies that $\Gamma'(C) \in K(A_{i+1})$ i.e., $\Gamma'(K(A_i)) \subseteq K(A_{i+1})$. Therefore, Γ' is cyclic.

Let $A \in K(A_i)$ and $B \in K(A_{i+1})$. Then, we have to show that

$$\phi\left(h_\wp\left(\Gamma'(A), \Gamma'(B)\right)\right) \leq \psi\left(\phi\left(h_\wp(A, B)\right)\right).$$

Let $\varrho_0 \in A$. Since $B \in K(A_{i+1})$, we have $\vartheta_0 \in B$ s.t. $\wp(\varrho_0, \vartheta_0) = \wp(\varrho_0, B) = \inf_{\vartheta \in B} \wp(\varrho_0, \vartheta)$. Then, for $\varrho_0 \in A \subseteq A_i$ and $\vartheta \in B \subseteq A_{i+1}$, we have

$$\begin{aligned} \phi\left(\wp\left(\Gamma\varrho_0, \Gamma(B)\right)\right) &= \phi\left(\inf_{\vartheta \in B} \wp\left(\varrho_0, \vartheta\right)\right) \leq \phi\left(\wp\left(\Gamma\varrho_0, \Gamma\vartheta_0\right)\right) \\ &\leq \psi\left(\phi\left(\wp\left(\varrho_0, \vartheta_0\right)\right)\right) = \psi\left(\phi\left(\wp\left(\varrho_0, B\right)\right)\right) \\ &\leq \psi\left(\phi\left(\sup_{\varrho \in A} \wp\left(\varrho, B\right)\right)\right) = \psi\left(\phi\left(h_\wp(A, B)\right)\right) \\ &\leq \psi\left(\phi\left(H_\wp(A, B)\right)\right). \end{aligned}$$

Also, ϱ_0 is arbitrary element of A and $\phi \in \Phi$. Therefore, we have

$$\phi\left(\wp\left(\Gamma\varrho, \Gamma(B)\right)\right) \leq \psi\left(\phi\left(H_\wp(A, B)\right)\right), \quad \forall\, \varrho \in A,$$

implies

$$\sup_{\varrho \in A} \phi \left(\wp \left(\Gamma \varrho, \Gamma(B) \right) \right) \leq \psi \left(\phi \left(H_{\wp}(A, B) \right) \right),$$

or

$$\phi \left(h_{\wp} \left(\Gamma'(A), \Gamma'(B) \right) \right) = \phi \left(h_{\wp} \left(\Gamma(A), \Gamma(B) \right) \right) \leq \psi \left(\phi \left(H_{\wp}(A, B) \right) \right).$$

On similar lines $\phi \left(h_{\wp} \left(\Gamma'(B), \Gamma'(A) \right) \right) \leq \psi \left(\phi \left(H_{\wp}(A, B) \right) \right)$.
Thus,

$$\begin{aligned} \phi \left(H_{\wp} \left(\Gamma'(A), \Gamma'(B) \right) \right) &= \phi \left(\max \left\{ h_{\wp} \left(\Gamma'(A), \Gamma'(B) \right), h_{\wp} \left(\Gamma'(B), \Gamma'(A) \right) \right\} \right) \\ &\leq \psi \left(\phi \left(H_{\wp}(A, B) \right) \right), \end{aligned}$$

i.e., Γ' is also a cyclic $(\phi - \psi)$-Banach contraction w.r.t some $\phi \in \Phi$ and $\psi \in \Psi$. $\square$

Theorem 5.3.7. *Consider a complete PMS $(\Omega, \wp)$, a positive integer m and non-empty closed subsets $A_i \subseteq \Omega$, for $1 \leq i \leq m$. Suppose $\Gamma_n : \bigcup_{i=1}^{m} A_i \to \bigcup_{i=1}^{m} A_i$ are continuous cyclic $(\phi - \psi_n)$-Banach contractions w.r.t some $\phi \in \Phi$ and $\psi_n \in \Psi$, where $n \in \mathbb{N}_{n_0}$ is a finite natural number. Then the Hutchinson operator $F : \bigcup_{i=1}^{m} K(A_i) \to \bigcup_{i=1}^{m} K(A_i)$ defined as $F(C) = \bigcup_{n=1}^{n_0} \Gamma'_n(C)$ has exactly one fixed point $A \in K(\Omega)$ and for any $B \in \bigcup_{i=1}^{m} K(A_i)$, the $\lim_{\hbar \to \infty} F^{\hbar}(B) = A$, which is the fractal generated by the IFS $\left\{ \bigcup_{i=1}^{m} A_i, \Gamma_n, n \in \mathbb{N}_{n_0} \right\}$.*

Proof. Suppose $\Gamma_n : \bigcup_{i=1}^{m} A_i \to \bigcup_{i=1}^{m} A_i$ are continuous cyclic $(\phi - \psi_n)$-Banach contractions w.r.t some $\phi \in \Phi$, $\psi_n \in \Psi$ and $C \in K(A_i)$ for some $1 \leq i \leq m$. By Theorem 5.3.6, $\forall n \in \mathbb{N}_{n_0}$, Γ'_n is a cyclic $(\phi - \psi_n)$-Banach contraction on $(K(\Omega), H_{\wp})$. As a result, $\Gamma'_n(C) \in K(A_{i+1})$ $\forall$ $1 \leq n \leq m$.
Also,

$$F(C) = \bigcup_{n=1}^{n_0} \Gamma'_n(C) \in K(A_i), i.e., F(K(A_i)) \subseteq K(A_{i+1}).$$

Further, for $A \in K(A_i)$ and $B \in K(A_{i+1})$, we have

$$\begin{aligned} H_\wp\left(F(A), F(B)\right) &= H_\wp\left(\bigcup_{n=1}^{n_0} \Gamma'_n(A), \bigcup_{n=1}^{n_0} \Gamma'_n(B)\right) \\ &\leq \max_{1\leq n\leq n_0} H_\wp\left(\Gamma'_n(A), \Gamma'_n(B)\right). \end{aligned}$$

Since ϕ, ψ are increasing functions. Therefore,

$$\begin{aligned} \phi(H_\wp(F(A), F(B))) &= \phi\left(\max_{1\leq n\leq n_0} H_\wp\left(\Gamma'_n(A), \Gamma'_n(B)\right)\right) \\ &= \max_{1\leq n\leq n_0} \phi\left(H_\wp\left(\Gamma'_n(A), \Gamma'_n(B)\right)\right) \\ &\leq \max_{1\leq n\leq n_0} \psi_i\left(\phi\left(H_\wp(A,B)\right)\right) \\ &= \psi\left(\phi\left(H_\wp(A,B)\right)\right). \end{aligned}$$

Hence, $F : \bigcup_{i=1}^{m} K(A_i) \to \bigcup_{i=1}^{m} K(A_i)$ is a cyclic $(\phi - \psi)$ Wardowski contraction on complete PMS $(K(\Omega), H_\wp)$, where $\psi(t) = \max_{1\leq n\leq n_0} \psi_n(t)$. Therefore, the result holds. $\square$

Example 5.3.8. *Consider $\Omega = \mathbb{R}$ with partial metric*

$$\wp(\varrho, \vartheta) = \begin{cases} |\varrho - \vartheta|, & \text{if } \varrho, \vartheta \in [1,3] \\ \max\{|\varrho|, |\vartheta|\}, & \text{otherwise.} \end{cases}$$

Then, $(X, \wp)$ is a complete PMS. Let $A_1 = [1,3]$ and $A_2 = [2,4]$. Define mapping $\Gamma_1 : A_1 \bigcup A_2 \to A_1 \bigcup A_2$ as

$$\Gamma_1(\varrho) = \begin{cases} \frac{17-\varrho}{7}, & \text{if } \varrho \in [1,3] \\ 2, & \text{if } \varrho \in [3,4]. \end{cases}$$

Hence,

$\Gamma_1(A_1) = \Gamma_1[1,3] = [2, 16/7] \subseteq [2,4] = A_2$ *and* $\Gamma_1(A_2) = \Gamma_1[2,4] = [2, 15/7] \subseteq [1,3] = A_1$.

Also, for $\varrho \in [1,3]$ and $\vartheta \in [2,3]$ implies $\Gamma_1\varrho, \Gamma_1\vartheta \in [1,3]$. Then,

$$\begin{aligned} \wp(\Gamma_1(\varrho), \Gamma_1\vartheta) &= \left|\frac{17-\varrho}{7} - \frac{17-\vartheta}{7}\right| \\ &= \left|\frac{\varrho-\vartheta}{7}\right| \\ &\leq \frac{1}{7}\wp(\varrho, \vartheta). \end{aligned}$$

Further, $\varrho \in [1,3]$ *and* $\vartheta \in [3,4]$ *implies* $\Gamma_1\varrho, \Gamma_1\vartheta \in [1,3]$. *Then,*

$$\begin{aligned}
\wp(\Gamma_1\varrho, \Gamma_1\vartheta) &= \left|\frac{17-\varrho}{7} - 2\right| \\
&= \left|\frac{3-\varrho}{7}\right| \leq \frac{2}{7} = \frac{2\times 3}{21}
\end{aligned}$$

$$\leq \frac{2}{21}\max\{\varrho, \vartheta\} = \frac{2}{21}\wp(\varrho, \vartheta).$$

Therefore, for $\phi(t) = t$ *and* $\psi(t) = \frac{1}{7}t$, Γ_1 *is a cyclic-*(ϕ,ψ)*-Banach contraction.*
Now, define $\Gamma_2 : A_1 \bigcup A_2 \to A_1 \bigcup A_2$ *as*

$$\Gamma_2(\varrho) = \begin{cases} \frac{23-\varrho}{7}, & \text{if } \varrho \in [1,3] \\ \frac{20}{7}, & \text{if } \varrho \in [3,4]. \end{cases}$$

Hence,

$\Gamma_2(A_1) = \Gamma_2[1,3] = [20/7, 22/7] \subseteq [2,4] = A_2$ *and* $\Gamma_2(A_2) = \Gamma_2[2,4] = [20/7, 21/7] \subseteq [1,3] = A_1$.

Also, $\varrho \in [1,3]$ *and* $\vartheta \in [2,3]$ *implies* $\Gamma_1\varrho, \Gamma_1\vartheta \in [1,3]$. *Then,*

$$\begin{aligned}
\wp(\Gamma_2(\varrho), \Gamma_2\vartheta) &= \left|\frac{23-\varrho}{7} - \frac{23-\vartheta}{7}\right| \\
&= \left|\frac{\varrho-\vartheta}{7}\right| \\
&\leq \frac{1}{7}\wp(\varrho, \vartheta).
\end{aligned}$$

Further, $\varrho \in [1,3]$ *and* $\vartheta \in [3,4]$ *implies* $\Gamma_1\varrho, \Gamma_1\vartheta \in [1,3]$. *Then,*

$$\begin{aligned}
\wp(\Gamma_2\varrho, \Gamma_2\vartheta) &= \left|\frac{23-\varrho}{7} - \frac{20}{7}\right| \\
&= \left|\frac{3-\varrho}{7}\right| \leq \frac{2}{7} = \frac{2\times 3}{21} \\
&\leq \frac{2}{21}\max\{\varrho, \vartheta\} = \frac{2}{21}\wp(\varrho, \vartheta).
\end{aligned}$$

Therefore, for $\phi(t) = t$ *and* $\psi(t) = \frac{1}{7}t$, Γ_2 *is a cyclic-*(ϕ,ψ)*-Banach contraction.*
Since both Γ_1 *and* Γ_2 *are continuous mappings, therefore by Theorem 5.3.7, the IFS* $\{A_1 \bigcup A_2; \Gamma_1, \Gamma_2\}$ *has exactly one fractal, i.e., the Hutchinson-Barnsley operator*

F has exactly one fixed point.
Moreover, few iteration of Hutchinson-Barnsley operator F with initial set $A_0 =$ $[2, 3]$ as follows

$$\Gamma_1(A_0) = \Gamma_1[2, 3] = [2, 15/7] \; and \; \Gamma_2(A_0) = \Gamma_2[2, 3] = [20/7, 21/7]$$

Then,

$$B_1 = F(A_0) = \Gamma_1(A_0) \bigcup \Gamma_2(A_0) = \left[2, \frac{15}{7}\right] \bigcup \left[\frac{20}{7}, \frac{21}{7}\right].$$

Similarly

$$B_2 = F^2(A_0) = F(F(A_0)) = F(B_1) = \Gamma_1(B_1) \bigcup \Gamma_2(B_2),$$

where

$$\Gamma_1(B_1) = \Gamma_1\left([2, 15/7] \bigcup [20/7, 21/7]\right) = \left[\frac{104}{49}, \frac{105}{49}\right] \bigcup \left[\frac{98}{49}, \frac{99}{49}\right]$$

and

$$\Gamma_2(B_1) = \Gamma_2\left([2, 15/7] \bigcup [20/7, 21/7]\right) = \left[\frac{146}{49}, \frac{147}{49}\right] \bigcup \left[\frac{140}{49}, \frac{141}{49}\right].$$

Therefore,

$$B_2 = \left[\frac{98}{49}, \frac{99}{49}\right] \bigcup \left[\frac{104}{49}, \frac{105}{49}\right] \bigcup \left[\frac{140}{49}, \frac{141}{49}\right] \bigcup \left[\frac{146}{49}, \frac{147}{49}\right]$$

and so on....

$$A = \lim_{\hbar \to \infty} F^{\hbar}(A_0) = \lim_{\hbar \to \infty} B_{\hbar}.$$

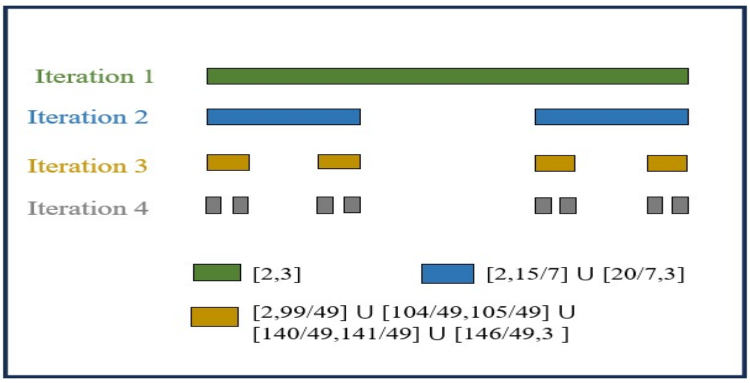

Figure 5.3: Fractals for Example 5.3.8 with different iterations

5.4 Conclusion

This chapter presents several common fixed-point theorems for self-mappings in partially metric spaces (PMS) using the $(\phi - \psi)$ Wardowski-type contraction. Moreover, it establishes fixed-point results using generalized cyclic contractions, supported by illustrative examples. As an application, the existence of a fractal set for the Hutchinson-Barnsley operator is demonstrated using the established fixed-point theorems. At the last, we present the iterative sequence for generating fractal sets and the resulting fractal.

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Chapter 6

Some Common Fixed Point Results in b-Metric Space

6.1 Introduction

In the present chapter of the thesis, we introduce the concept of $(\phi-\psi)$ generalized $\mathcal{R}$-contraction within a b-metric space, equipped with a binary relation $\mathcal{R}$ and proved some common fixed point results for a pair of self mappings. Also, we prove some fixed point results for self mapping using α-$(\phi-\psi)$ Wardowski contraction in the framework of b-MS. As an applications, we verify the existence and uniqueness of solution to an operator equation and a non-linear functional integral equation. The results of the chapter are presented in [1].

6.2 Some Common Fixed Point Results in b-Metric Space

In this section, we discuss some common fixed point results using generalized relation theoretic contraction in b-MS.

Definition 6.2.1. *Consider a b-MS (Ω, d_b) equipped with the binary relation $\mathcal{R}$. A pair of self mappings (Γ_1, Γ_2) defined on Ω is c.t.b. a* ***$(\phi-\psi)$ generalized $\mathcal{R}$-contraction pair*** *if $\exists$ functions $\phi \in \Phi$, and $\psi \in \Psi$ s.t.*

$$d_b(\Gamma_1\varrho, \Gamma_2\vartheta) > 0 \Rightarrow \phi\left(sd_b(\Gamma_1\varrho, \Gamma_2\vartheta)\right) \leq \psi\left(\phi\left(M_{\Gamma_1,\Gamma_2}(\varrho, \vartheta)\right)\right), \tag{6.2.1}$$

[1]Yadav, K., & Kumar, D. Existence of Solution for a Non-linear Functional Integral Equation and an Operator Equations via Fixed Point Approach (Communicated)

where

$$M_{\Gamma_1,\Gamma_2}(\varrho,\vartheta) = \max\left\{d_b(\varrho,\vartheta), d_b(\varrho,\Gamma_1\varrho), d_b(\vartheta,\Gamma_2\vartheta), \frac{d_b(\varrho,\Gamma_2\vartheta)+d_b(\vartheta,\Gamma_1\varrho)}{2s}\right\},$$

$\forall \varrho,\vartheta \in \Omega$ *satisfying* $(\varrho,\vartheta) \in \mathcal{R}$.

Example 6.2.2. *Consider the set* $\Omega = l_\infty$, *the set of all bounded sequence of real numbers. Then, for* $p \geq 1$, *the mapping* $d_b : l_\infty \times l_\infty \to [0,\infty)$ *defined as*

$$d_b(\xi,\varrho) = \sup_{\hbar\in\mathbb{N}} |\xi_\hbar - \varrho_\hbar|^p$$

is a b-metric. Also, consider a binary relation $\mathcal{R}$ *defined as* $(\xi,\varrho) \in \mathcal{R}$ *if* $\xi_j.\varrho_j = 0$, $\forall j \in \mathbb{N}$. *Then, the pair of* $\Gamma_1,\Gamma_2 : l_\infty \to l_\infty$ *defined as*

$$\Gamma_1(\xi_1,\xi_2,...,\xi_\hbar,...) = \left(0,\frac{\xi_1}{3},\frac{\xi_2}{3},...,\frac{\xi_{\hbar-1}}{3},...\right),$$

$$\Gamma_2(\xi_1,\xi_2,...,\xi_\hbar,...) = \left(0,\frac{\xi_1}{7},\frac{\xi_2}{7},...,\frac{\xi_{\hbar-1}}{7},...\right)$$

is a $(\phi-\psi)$ *generalized* $\mathcal{R}$*-contraction pair.*

Theorem 6.2.3. *Let* $\Gamma_1,\Gamma_2 : \Omega \to \Omega$ *be mappings defined on* $\mathcal{R}$*-complete b-MS* (Ω,d_b). *Suppose* Γ_1,Γ_2 *satisfy*

(i) (Γ_1,Γ_2) *is a* $(\phi$-$\psi)$ *generalized* $\mathcal{R}$*-contraction pair;*

(ii) $\exists$ *some* $\varrho_0 \in \Omega$ *s.t.* $(\varrho_0,\Gamma_1\varrho_0) \in \mathcal{R}$*;*

(iii) $\mathcal{R}$ *is* (Γ_1,Γ_2)*-regular closed;*

(iv) *(a)* Γ_1 *and* Γ_2 *are* $\mathcal{R}$*-continuous mappings;*
or
(b) $\mathcal{R}$ *is* d_b*-self closed on* Ω.

Then pair (Γ_1,Γ_2) *has a common fixed point. Moreover, if* $(\varrho,\vartheta) \in \mathcal{R}$, $\forall \varrho,\vartheta \in CF(\Gamma_1,\Gamma_2)$, *where* $CF(\Gamma_1,\Gamma)$ *denotes the set of all the common fixed points of mappings* Γ_1,Γ_2, *then the pair* (Γ_1,Γ_2) *has exactly one common fixed point.*

Proof. From assumption (ii), we have an $\varrho_0 \in \Omega$ s.t. $(\varrho_0,\Gamma_1\varrho_0) \in R$. Consider the iterative sequence defined as

$$\varrho_{2\hbar+1} = \Gamma_1\varrho_{2\hbar} \text{ and } \varrho_{2\hbar+2} = \Gamma_2\varrho_{2\hbar+1} \;\; \forall\, \hbar \in \mathbb{N}\cup\{0\}.$$

Using (iii), we have

$$(\varrho_0, \varrho_1) = (\varrho_0, \Gamma_1 \varrho_0) \in \mathcal{R} \Leftrightarrow (\Gamma_1 \varrho_0, \Gamma_2 \varrho_1) \in \mathcal{R} \text{ and } (\Gamma_2 \varrho_1, \Gamma_1 \varrho_0) \in \mathcal{R},$$

or

$$(\varrho_2, \varrho_1) \in \mathcal{R} \text{ and } (\varrho_1, \varrho_2) \in \mathcal{R}.$$

Now, as $(\varrho_2, \varrho_1) \in \mathcal{R}$. Hence, using (iii), we have $(\varrho_3, \varrho_2) \in \mathcal{R}$. Repeating this process, we have

$$(\varrho_{\hbar+1}, \varrho_\hbar) \in \mathcal{R}, \quad \forall\, \hbar \in \mathbb{N},$$

i.e., $\{\varrho_\hbar\}$ is a $\mathcal{R}$-sequence.
Suppose that $\exists$ some $N \in \mathbb{N}$, s.t. $\varrho_{2N+1} = \varrho_{2N+2}$. Then,

$$\Gamma_1 \varrho_{2N} = \Gamma_2 \varrho_{2N+1} = \varrho_{2N+1} = \varrho_{2N+2}.$$

We claim that $\Gamma_2 \varrho_{2N+1} = \Gamma_1 \varrho_{2N+2}$. If possible, $d_b(\Gamma_2 \varrho_{2N+1}, \Gamma_1 \varrho_{2N+2}) > 0$, then by (6.2.1)

$$\begin{aligned}
\phi\left(d_b(\varrho_{2N+2}, \varrho_{2N+3})\right) &\leq \phi\left(s d_b(\varrho_{2N+2}, \varrho_{2N+3})\right) \\
&\leq \phi\left(s d_b(\Gamma_2 \varrho_{2N+1}, \Gamma_1 \varrho_{2N+2})\right) = \phi\left(s d_b(\Gamma_1 \varrho_{2N+2}, \Gamma_2 \varrho_{2N+1})\right) \\
&\leq \psi\Big(\phi\left(M_{\Gamma_2,\Gamma_1}(\varrho_{2N+2}, \varrho_{2N+1})\right)\Big),
\end{aligned}$$

where

$$\begin{aligned}
M_{\Gamma_1,\Gamma_2}(\varrho_{2N+2}, \varrho_{2N+1}) &= \max\Bigg\{ d_b(\varrho_{2N+2}, \varrho_{2N+1}), d_b(\varrho_{2N+2}, \Gamma_1 \varrho_{2N+2}), d_b(\varrho_{2N+1}, \Gamma_2 \varrho_{2N+1}), \\
&\qquad \frac{d_b(\varrho_{2N+2}, \Gamma_2 \varrho_{2N+1}) + d_b(\Gamma_1 \varrho_{2N+2}, \varrho_{2N+1})}{2s} \Bigg\} \\
&= \max\left\{0, d_b(\varrho_{2N+2}, \varrho_{2N+3}), 0, \frac{d_b(\varrho_{2N+1}, \varrho_{2N+2}) + d_b(\varrho_{2N+2}, \varrho_{2N+3})}{2}\right\} \\
&= d_b(\varrho_{2N+2}, \varrho_{2N+3}).
\end{aligned}$$

This implies

$$\begin{aligned}
\phi\left(d_b(\varrho_{2N+2}, \varrho_{2N+3})\right) &\leq \psi\Big(\phi\left(d_b(\varrho_{2N+2}, \varrho_{2N+3})\right)\Big) \\
&< \phi\left(d_b(\varrho_{2N+2}, \varrho_{2N+3})\right),
\end{aligned}$$

a contradiction. Therefore, $d_b(\Gamma_2\varrho_{2N+1}, \Gamma_1\varrho_{2N+2}) = 0$ which implies Γ_1 and Γ_2 have a common fixed point. Indeed, $\varrho_{2N+1} = \varrho_{2N+2}$ is a common fixed point.
Now, suppose that $\varrho_{2\hbar+1} \neq \varrho_{2\hbar+2}$ $\forall \hbar \in \mathbb{N}$. Then, using (6.2.1), we have

$$\begin{aligned}
\phi\left(d_b(\varrho_{2\hbar+1}, \varrho_{2\hbar+2})\right) &\leq \phi\left(sd_b(\varrho_{2\hbar+1}, \varrho_{2\hbar+2})\right) = \phi\left(sd_b(\Gamma_1\varrho_{2\hbar}, \Gamma_2\varrho_{2\hbar+1})\right) \\
&\leq \psi\left(\phi\left(M_{\Gamma_2,\Gamma_1}(\varrho_{2\hbar}, \varrho_{2\hbar+1})\right)\right),
\end{aligned}$$

where

$$\begin{aligned}
M_{\Gamma_1,\Gamma_2}(\varrho_{2\hbar}, \varrho_{2\hbar+1}) &= \max\Big\{d_b(\varrho_{2\hbar}, \varrho_{2\hbar+1}), d_b(\varrho_{2\hbar}, \Gamma_1\varrho_{2\hbar}), d_b(\varrho_{2\hbar+1}, \Gamma_2\varrho_{2\hbar+1}), \\
&\qquad \frac{d_b(\varrho_{2\hbar}, \Gamma_2\varrho_{2\hbar+1}) + d_b(\varrho_{2\hbar+1}, \Gamma_1\varrho_{2\hbar})}{2s}\Big\} \\
&= \max\left\{d_b(\varrho_{2\hbar}, \varrho_{2\hbar+1}), d_b(\varrho_{2\hbar+1}, \varrho_{2\hbar+2}), \frac{d_b(\varrho_{2\hbar}, \varrho_{2\hbar+2})}{2s}\right\} \\
&\leq \max\left\{d_b(\varrho_{2\hbar}, \varrho_{2\hbar+1}), d_b(\varrho_{2\hbar+1}, \varrho_{2\hbar+2}), \frac{d_b(\varrho_{2\hbar}, \varrho_{2\hbar+1}) + d_b(\varrho_{2\hbar+1}, \varrho_{2\hbar+2})}{2}\right\} \\
&= \max\left\{d_b(\varrho_{2\hbar}, \varrho_{2\hbar+1}), d_b(\varrho_{2\hbar+1}, \varrho_{2\hbar+2})\right\}.
\end{aligned}$$

If $\max\left\{d_b(\varrho_{2\hbar}, \varrho_{2\hbar+1}), d_b(\varrho_{2\hbar+1}, \varrho_{2\hbar+2})\right\} = d_b(\varrho_{2\hbar+1}, \varrho_{2\hbar+2})$. Then,

$$\begin{aligned}
\phi\left(d_b(\varrho_{2\hbar+1}, \varrho_{2\hbar+2})\right) &\leq \psi\left(\phi\left(d_b(\varrho_{2\hbar+1}, \varrho_{2\hbar+2})\right)\right) \\
&< \phi\left(d_b(\varrho_{2\hbar+1}, \varrho_{2\hbar+2})\right),
\end{aligned}$$

a contradiction. Therefore, $\max\left\{d_b(\varrho_{2\hbar}, \varrho_{2\hbar+1}), d_b(\varrho_{2\hbar+1}, \varrho_{2\hbar+2})\right\} = d_b(\varrho_{2\hbar}, \varrho_{2\hbar+1})$, implies

$$\begin{aligned}
\phi\left(sd_b(\varrho_{2\hbar+1}, \varrho_{2\hbar+2})\right) &\leq \psi\left(\phi\left(d_b(\varrho_{2\hbar}, \varrho_{2\hbar+1})\right)\right) \\
&< \phi\left(d_b(\varrho_{2\hbar}, \varrho_{2\hbar+1})\right).
\end{aligned}$$

As, ϕ is a increasing function. Therefore,

$$d_b(\varrho_{2\hbar+1}, \varrho_{2\hbar+2}) \leq \frac{1}{s} d_b(\varrho_{2\hbar}, \varrho_{2\hbar+1}).$$

Also,

$$\begin{aligned}
\phi\left(d_b(\varrho_{2\hbar+2}, \varrho_{2\hbar+3})\right) &= \phi\left(d_b(\varrho_{2\hbar+3}, \varrho_{2\hbar+2})\right) \\
&= \phi\left(d_b(\Gamma_1\varrho_{2\hbar+2}, \Gamma_{2\hbar+1})\right) \leq \phi\left(sd_b(\Gamma_1\varrho_{2\hbar+2}, \Gamma_2\varrho_{2\hbar+1})\right) \\
&\leq \psi\left(\phi\left(M_{\Gamma_2,\Gamma_1}(\varrho_{2\hbar+2}, \varrho_{2\hbar+1})\right)\right),
\end{aligned}$$

where

$$
\begin{aligned}
M_{\Gamma_1,\Gamma_2}(\varrho_{2\hbar+2},\varrho_{2\hbar+1}) &= \max\bigg\{ d_b(\varrho_{2\hbar+2},\varrho_{2\hbar+1}), d_b(\varrho_{2\hbar+2},\Gamma_1\varrho_{2\hbar+2}), d_b(\varrho_{2\hbar+1},\Gamma_2\varrho_{2\hbar+1}), \\
&\quad \frac{d_b(\varrho_{2\hbar+2},\Gamma_2\varrho_{2\hbar+1}) + d_b(\varrho_{2\hbar+1},\Gamma_1\varrho_{2\hbar+2})}{2s} \bigg\} \\
&= \max\left\{ d_b(\varrho_{2\hbar+2},\varrho_{2\hbar+1}), d_b(\varrho_{2\hbar+2},\varrho_{2\hbar+3}), \frac{d_b(\varrho_{2\hbar+1},\varrho_{2\hbar+3})}{2s} \right\}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \max\left\{ d_b(\varrho_{2\hbar+2},\varrho_{2\hbar+1}), d_b(\varrho_{2\hbar+2},\varrho_{2\hbar+3}), \frac{d_b(\varrho_{2\hbar+1},\varrho_{2\hbar+2}) + d_b(\varrho_{2\hbar+2},\varrho_{2\hbar+3})}{2} \right\} \\
&= \max\left\{ d_b(\varrho_{2\hbar+1},\varrho_{2\hbar+2}), d_b(\varrho_{2\hbar+2},\varrho_{2\hbar+3}) \right\}.
\end{aligned}
$$

If $\max\left\{ d_b(\varrho_{2\hbar+1},\varrho_{2\hbar+2}), d_b(\varrho_{2\hbar+2},\varrho_{2\hbar+3}) \right\} = d_b(\varrho_{2\hbar+2},\varrho_{2\hbar+3})$. Then,

$$
\begin{aligned}
\phi\left(d_b(\varrho_{2\hbar+2},\varrho_{2\hbar+3})\right) &\leq \psi\Big(\phi\left(d_b(\varrho_{2\hbar+2},\varrho_{2\hbar+3})\right)\Big) \\
&< \phi\left(d_b(\varrho_{2\hbar+2},\varrho_{2\hbar+3})\right),
\end{aligned}
$$

a contradiction. Therefore, $\max\left\{ d_b(\varrho_{2\hbar+1},\varrho_{2\hbar+2}), d_b(\varrho_{2\hbar+2},\varrho_{2\hbar+3}) \right\} = d_b(\varrho_{2\hbar+2},\varrho_{2\hbar+1})$. Now,

$$
\begin{aligned}
\phi\left(s d_b(\varrho_{2\hbar+2},\varrho_{2\hbar+3})\right) &\leq \psi\Big(\phi\left(d_b(\varrho_{2\hbar+2},\varrho_{2\hbar+1})\right)\Big) \\
&< \phi\left(d_b(\varrho_{2\hbar+2},\varrho_{2\hbar+1})\right),
\end{aligned}
$$

or

$$d_b(\varrho_{2\hbar+2},\varrho_{2\hbar+3}) \leq \frac{1}{s} d_b(\varrho_{2\hbar+1},\varrho_{2\hbar+2}).$$

Therefore, $\forall \hbar \in \mathbb{N}$, we have

$$d_b(\varrho_{\hbar+1},\varrho_{\hbar}) \leq \frac{1}{s} d_b(\varrho_{\hbar},\varrho_{\hbar-1}).$$

Using Lemma 1.2.27, we can conclude that $\{\varrho_\hbar\}$ is a C_{seq}, for $s > 1$.
In case $s = 1$, $d_b(\varrho_{\hbar+1},\varrho_\hbar)$ is a decreasing sequence. Then,

$$
\begin{aligned}
\phi\left(d_b(\varrho_{\hbar+1},\varrho_{\hbar})\right) &\leq \psi\Big(\phi\left(d_b(\varrho_{\hbar},\varrho_{\hbar-1})\right)\Big) \\
&\leq \psi^2\Big(\phi\left(d_b(\varrho_{\hbar-1},\varrho_{\hbar-2})\right)\Big) \\
&\quad \ldots \\
&\leq \psi^{\hbar}\Big(\phi\left(d_b(\varrho_{1},\varrho_{0})\right)\Big).
\end{aligned}
$$

Taking the limit as $\hbar$ tends to ∞, we have

$$\lim_{\hbar \to \infty} d_b(\varrho_{\hbar+1}, \varrho_{\hbar}) = 0. \tag{6.2.2}$$

Suppose that $\{\varrho_{\hbar}\}$ is not a C_{seq}. Then, for $\epsilon > 0$, $\exists$ two subsequences $\{\varrho_{2\hbar_k}\} \neq \{\varrho_{2l_k}\}$ s.t.

$$d_b(\varrho_{2\hbar_k+1}, \varrho_{2l_k}) \;\; > \;\; \epsilon, \tag{6.2.3}$$

and

$$d_b(\varrho_{2\hbar_k}, \varrho_{2l_k}) \;\; \leq \;\; \epsilon. \tag{6.2.4}$$

Using the triangle inequality and (6.2.3), we have

$$\begin{aligned} \epsilon &< d_b(\varrho_{2\hbar_k+1}, \varrho_{2l_k}) \\ &\leq d_b(\varrho_{2\hbar_k+1}, \varrho_{2\hbar_k}) + d_b(\varrho_{2\hbar_k}, \varrho_{2l_k}). \end{aligned}$$

Using (6.2.2), (6.2.4) and taking limit as k tends to ∞, we have

$$\lim_{k \to \infty} d_b(\varrho_{2\hbar_k+1}, \varrho_{2l_k}) = \epsilon. \tag{6.2.5}$$

Also, using the triangle inequality, we have

$$d_b(\varrho_{2\hbar_k+1}, \varrho_{2l_k}) \leq d_b(\varrho_{2\hbar_k+1}, \varrho_{2\hbar_k}) + d_b(\varrho_{2\hbar_k}, \varrho_{2l_k-1}) + d_b(\varrho_{2l_k-1}, \varrho_{2l_k}),$$

and

$$d_b(\varrho_{2\hbar_k}, \varrho_{2l_k-1}) \leq d_b(\varrho_{2\hbar_k}, \varrho_{2\hbar_k+1}) + d_b(\varrho_{2\hbar_k+1}, \varrho_{2l_k}) + d_b(\varrho_{2l_k}, \varrho_{2l_k-1}).$$

Taking the limit as k tends to ∞ and using (6.2.5), we have

$$\lim_{k \to \infty} d_b(\varrho_{2\hbar_k}, \varrho_{2l_k-1}) = \epsilon.$$

As, $d_b(\varrho_{2\hbar_{k+1}}, \varrho_{2l_k}) > \epsilon > 0$. Therefore, using (6.2.1), we have

$$\begin{aligned} \phi(\epsilon) &< \phi\left(d_b(\varrho_{2\hbar_{k+1}}, \varrho_{2l_k})\right) \\ &= \phi\left(d_b(\Gamma_1 \varrho_{2\hbar_k}, \Gamma_2 \varrho_{2l_k-1})\right) \\ &\leq \phi\left(s d_b(\Gamma_1 \varrho_{2\hbar_k}, \Gamma_2 \varrho_{2l_k-1})\right) \\ &\leq \psi\left(\phi\left(M_{\Gamma_1,\Gamma_2}(\varrho_{2\hbar_k}, \varrho_{2l_k-1})\right)\right), \end{aligned}$$

where

$$\begin{aligned}
M_{\Gamma_1,\Gamma_2}(\varrho_{2\hbar_k},\varrho_{2l_k-1}) &= \max\bigg\{d_b(\varrho_{2\hbar_k},\varrho_{2l_k-1}), d_b(\varrho_{2\hbar_k},\Gamma_1\varrho_{2\hbar_k}), d_b(\varrho_{2\hbar_k},\Gamma_2\varrho_{2l_k-1}),\\
&\quad \frac{d_b(\varrho_{2\hbar_k},\Gamma_2\varrho_{2l_k-1}) + d_b(\varrho_{2l_k-1},\Gamma_1\varrho_{2\hbar_k})}{2s}\bigg\}\\
&= \max\bigg\{d_b(\varrho_{2\hbar_k},\varrho_{2l_k-1}), d_b(\varrho_{2\hbar_k},\varrho_{2\hbar_k+1}), d_b(\varrho_{2\hbar_k},\varrho_{2l_k}),\\
&\quad \frac{d_b(\varrho_{2\hbar_k},\varrho_{2l_k}) + d_b(\varrho_{2l_k-1},\varrho_{2\hbar_k+1})}{2s}\bigg\}\\
&\leq \max\bigg\{d_b(\varrho_{2\hbar_k},\varrho_{2l_k-1}), d_b(\varrho_{2\hbar_k},\varrho_{2\hbar_k+1}), d_b(\varrho_{2\hbar_k},\varrho_{2l_k}),\\
&\quad \frac{d_b(\varrho_{2\hbar_k},\varrho_{2l_k})}{2s} + \frac{sd_b(\varrho_{2\hbar_k+1},\varrho_{2l_k}) + sd_b(\varrho_{2l_k},\varrho_{2l_k-1})}{2s}\bigg\}.
\end{aligned}$$

Taking limitsup as $k \to \infty$, we have

$$\begin{aligned}
\phi(\epsilon) &\leq \limsup_{k\to\infty} \psi\left(\phi\left(M_{\Gamma_1,\Gamma_2}(\varrho_{2\hbar_k},\varrho_{2l_k-1})\right)\right)\\
&= \psi\left(\phi(\epsilon)\right) < \phi(\epsilon),
\end{aligned}$$

a contradiction. Therefore, $\{\varrho_\hbar\}$ is a C_{seq}. Also, (Ω, d_b) is a $\mathcal{R}$-complete b-MS. Hence, $\exists$ an $\varrho \in \Omega$ s.t. $\varrho_\hbar$ converges to ϱ.
The following cases arises:

Case (i) If assumption (iv)(a) holds, i.e., if Γ_1, Γ_2 are $\mathcal{R}$-continuous mappings and $\{\varrho_\hbar\}$ is $\mathcal{R}$-preserving sequence converges to ϱ, we have

$$\{\Gamma_2\varrho_\hbar\} \to \Gamma_2\varrho \text{ and } \{\Gamma_1\varrho_\hbar\} \to \Gamma_1\varrho,$$

or

$$\{\Gamma_2\varrho_{2\hbar+1}\} \to \Gamma_2\varrho \text{ and } \{\Gamma_1\varrho_{2\hbar}\} \to \Gamma_1\varrho,$$

or

$$\{\varrho_{2\hbar+2}\} \to \Gamma_2\varrho \text{ and } \{\varrho_{2\hbar+1}\} \to \Gamma_1\varrho.$$

Also, we have

$$\{\varrho_{2\hbar+2}\} \to \varrho \text{ and } \{\varrho_{2\hbar+1}\} \to \varrho.$$

Therefore, $\Gamma_2\varrho = \varrho = \Gamma_1\varrho$, i.e., (Γ_1, Γ_2) has a common fixed point.

Case (ii) If $\mathcal{R}$ is d_b-self closed on Ω and $\{\varrho_\hbar\} \to \varrho$. Therefore, $\exists$ a subsequence $\{\varrho_{2\hbar_k}\}$ of $\{\varrho_\hbar\}$ with $(\varrho_{2\hbar_k}, \varrho) \in \mathcal{R}$. Thus using (6.2.1), we have

$$\begin{aligned} \phi\left(d_b(\Gamma_1\varrho_{2\hbar_k}, \Gamma_2\varrho)\right) &\leq \phi\left(sd_b(\Gamma_1\varrho_{2\hbar_k}, \Gamma_2\varrho)\right) \\ &\leq \psi\left(\phi\left(M_{\Gamma_2,\Gamma_1}(\varrho_{2\hbar_k}, \varrho)\right)\right), \end{aligned}$$

where

$$\begin{aligned} M_{\Gamma_1,\Gamma_2}(\varrho_{2\hbar_k}, \varrho) &= \max\Bigg\{d_b(\varrho_{2\hbar_k}, \varrho), d_b(\varrho_{2\hbar_k}, \Gamma_1\varrho_{2\hbar_k}), d_b(\varrho, \Gamma_2\varrho), \\ &\qquad \frac{d_b(\varrho_{2\hbar_k}, \Gamma_2\varrho) + d_b(\varrho, \Gamma_1\varrho_{2\hbar_k})}{2s}\Bigg\} \\ &\leq \max\Bigg\{d_b(\varrho_{2\hbar_k}, \varrho), d_b(\varrho_{2\hbar_k}, \Gamma_1\varrho_{2\hbar_k}), d_b(\varrho, \Gamma_2\varrho), \\ &\qquad \frac{d_b(\varrho_{2\hbar_k}, \varrho_{2\hbar_k+1})}{2} + \frac{d_b(\varrho_{2\hbar_k+1}, \Gamma_2\varrho)}{2} + \frac{d_b(\varrho, \Gamma_1\varrho_{2\hbar_k})}{2s}\Bigg\}, \end{aligned}$$

implies

$$\lim_{k\to\infty} M_{\Gamma_2,\Gamma_1}(\varrho_{2\hbar_k}, \varrho) \leq d_b(\varrho, \Gamma_2\varrho).$$

Also, using continuity of ϕ, we have

$$\begin{aligned} \phi\left(d_b(\varrho, \Gamma_2\varrho)\right) &= \lim_{k\to\infty} \phi\left(d_b(\Gamma_1\varrho_{2\hbar_k}, \Gamma_2\varrho)\right) \\ &\leq \lim_{k\to\infty} \psi\left(\phi\left(M_{\Gamma_2,\Gamma_1}(\varrho_{2\hbar_k}, \varrho)\right)\right) \\ &\leq \lim_{k\to\infty} \psi\left(\phi\left(d_b(\varrho, \Gamma_2\varrho)\right)\right) \\ &< \phi\left(d_b(\varrho, \Gamma_2\varrho)\right), \end{aligned}$$

a contradiction. Therefore, $\lim_{k\to\infty} d_b(\Gamma_1\varrho_{2\hbar_k}, \Gamma_2\varrho) = 0$, i.e.,

$$\{\Gamma_1\varrho_{2\hbar_k}\} \to \Gamma_2\varrho \Rightarrow \{\varrho_{2\hbar_k+1}\} \to \Gamma_2\varrho.$$

Also, $\{\varrho_{2\hbar_k+1}\} \to \varrho$. Hence, $\Gamma_2\varrho = \varrho$.
On the similar lines, $\Gamma_1\varrho = \varrho$. Therefore, $\Gamma_1\varrho = \varrho = \Gamma_2\varrho$ i.e., ϱ is a common fixed point of the pair (Γ_1, Γ_2) .

Uniqueness: If possible, let $\vartheta \neq \varrho$ with $(\varrho, \vartheta) \in \mathcal{R}$ be another common fixed point of the pair (Γ_1, Γ_2). Therefore, $d_b(\varrho, \vartheta) = d_b(\Gamma_1\varrho, \Gamma_2\vartheta) > 0$.
Using (6.2.1), we have

$$\begin{aligned}
\phi\left(d_b(\varrho, \vartheta)\right) &= \phi\left(d_b(Sx, \Gamma_2\vartheta)\right) \\
&\leq \phi\left(s d_b(\Gamma_1\varrho, \Gamma_2\vartheta)\right) \\
&\leq \psi\left(\phi\left(M_{\Gamma_2,\Gamma_1}(\varrho, \vartheta)\right)\right),
\end{aligned}$$

where

$$\begin{aligned}
M_{\Gamma_1,\Gamma_2}(\varrho, \vartheta) &= \max\left\{d_b(\varrho, \vartheta), d_b(\varrho, \Gamma_1\varrho), d_b(\vartheta, \Gamma_2\vartheta), \frac{d_b(\varrho, \Gamma_2\vartheta) + d_b(\vartheta, \Gamma_1\varrho)}{2s}\right\} \\
&= \max\left\{d_b(\varrho, \vartheta), 0, 0, \frac{d_b(\varrho, \vartheta)}{s}\right\} = d_b(\varrho, \vartheta).
\end{aligned}$$

Then,

$$\begin{aligned}
\phi\left(d_b(\varrho, \vartheta)\right) &= \psi\left(\phi\left(M_{\Gamma_2,\Gamma_1}(\varrho, \vartheta)\right)\right) \\
&= \psi\left(\phi\left(d_b(\varrho, \vartheta)\right)\right) \\
&< \phi\left(d_b(\varrho, \vartheta)\right),
\end{aligned}$$

a contradiction. Therefore, the pair (Γ_1, Γ_2) has exactly one common fixed point.

□

Corollary 6.2.4. *Let $\Gamma_1, \Gamma_2 : \Omega \to \Omega$ be mappings defined on $\mathcal{R}$-complete b-MS $(\Omega, d_b, \mathcal{R})$. Suppose Γ_1, Γ_2 satisfy*

(i) (a) $\exists z \in (0, \frac{1}{s})$ s.t. $d_b(\Gamma_1\varrho, \Gamma_2\vartheta) \leq z M_{\Gamma_2,\Gamma_1}(\varrho, \vartheta)$;
or

(b) $\exists F \in \boldsymbol{F}$ and $\tau > 0$ s.t. $\tau + F\left(s d_b(\Gamma_1\varrho, \Gamma_2\vartheta)\right) \leq F\left(d_b(\varrho, \vartheta)\right)$;
or

(c) $\exists F \in \boldsymbol{F}$ and $\tau > 0$ s.t. $\tau + F\left(s d_b(\Gamma_1\varrho, \Gamma_2\vartheta)\right) \leq F\left(M_{\Gamma_2,\Gamma_1}(\varrho, \vartheta)\right)$;

(ii) $\exists \varrho_0 \in \Omega$ s.t. $(\varrho_0, \Gamma_1\varrho_0) \in \mathcal{R}$;

(iii) $\mathcal{R}$ is (Γ_1, Γ_2)-regular closed;

(iv) (a) Γ_1 and Γ_2 are $\mathcal{R}$-continuous mappings;
or

(b) $\mathcal{R}$ *is* d_b*-self closed on* Ω*,*

where,

$$M_{\Gamma_1,\Gamma_2}(\varrho,\vartheta) = \max\left\{d_b(\varrho,\vartheta), d_b(\varrho,\Gamma_1\varrho), d_b(\vartheta,\Gamma_2\vartheta), \frac{d_b(\varrho,\Gamma_2\vartheta)+d_b(\vartheta,\Gamma_1\varrho)}{2s}\right\},$$

for all $\varrho,\vartheta \in \Omega$ *satisfying* $(\varrho,\vartheta) \in \mathcal{R}$*. Then, the pair* (Γ_1,Γ_2) *has a common fixed point. Moreover, if* $(\varrho,\vartheta) \in \mathcal{R}$*,* $\forall \varrho,\vartheta \in CF(\Gamma_1,\Gamma_2)$*, where* $CF(\Gamma_1,\Gamma_2)$ *denotes all the common fixed points of mappings* Γ_1,Γ_2*, then the pair* (Γ_1,Γ_2) *has exactly one common fixed point.*

Definition 6.2.5. *Let* (Ω, d_b) *be a b-MS equipped with the binary relation* $\mathcal{R}$*. A self mapping* Γ *defined on* Ω *is c.t.b. a* ***$(\phi-\psi)$ generalized $\mathcal{R}$-contraction*** *if* $\exists \phi \in \Phi,\ \psi \in \Psi$ *and* $\alpha : \Omega \times \Omega \to (0,\infty)$ *s.t.*

$$d_b(\Gamma\varrho,\Gamma\vartheta) > 0 \Rightarrow \phi\left(sd_b(\Gamma\varrho,\Gamma\vartheta)\right) \leq \psi\left(\phi\left(M_\Gamma(\varrho,\vartheta)\right)\right), \tag{6.2.6}$$

where,

$$M_{\Gamma}(\varrho,\vartheta) = \max\left\{d_b(\varrho,\vartheta), d_b(\varrho,\Gamma\varrho), d_b(\vartheta,\Gamma\vartheta), \frac{d_b(\varrho,\Gamma\vartheta)+d_b(\vartheta,\Gamma\varrho)}{2s}\right\},$$

$\forall \varrho,\vartheta \in \Omega$ *satisfying* $(\varrho,\vartheta) \in \mathcal{R}$*.*

Theorem 6.2.6. *Let* $\Gamma : \Omega \to \Omega$ *be a mapping defined on* $\mathcal{R}$*-complete b-MS. Suppose* Γ *satisfies*

(i) Γ *is a* $(\phi-\psi)$ *generalized* $\mathcal{R}$*-contraction;*

(ii) $\exists \varrho_0 \in \Omega$ *s.t.* $(\varrho_0, \Gamma\varrho_0) \in \mathcal{R}$*;*

(iii) $\mathcal{R}$ *is* Γ*-closed;*

(iv) (a) Γ *is* $\mathcal{R}$*-continuous mapping;*

(b) $\mathcal{R}$ *is* d_b*-self closed on* Ω*.*

Then, Γ *possesses a fixed point. Moreover, if* $(\varrho,\vartheta) \in \mathcal{R}$ $\forall \varrho,\vartheta \in Fix(\Gamma)$*, where* $Fix(\Gamma)$ *denotes the set of all the fixed points of mapping* Γ*, then* Γ *has exactly one fixed point.*

Proof. Considering $\Gamma_1 = \Gamma_2 = \Gamma$ in Theorem 6.2.3, one can easily deduce the result. $\square$

Corollary 6.2.7. *Let $\Gamma : \Omega \to \Omega$ be a mapping defined on $\mathcal{R}$-complete b-MS. Suppose Γ satisfies*

(i) *(a)* $\exists z \in (0, \frac{1}{s})$ *s.t.* $d_b(\Gamma\varrho, \Gamma\vartheta) \leq zM_\Gamma(\varrho, \vartheta)$*;*
or
(b) $\exists F \in \boldsymbol{F}$ *and* $\tau > 0$ *s.t.* $\tau + F(sd_b(\Gamma\varrho, \Gamma\vartheta)) \leq F(d_b(\varrho, \vartheta))$*;*
or
(c) $\exists F \in \boldsymbol{F}$ *and* $\tau > 0$ *s.t.* $\tau + F(sd_b(\Gamma\varrho, \Gamma\vartheta)) \leq F(M_\Gamma(\varrho, \vartheta))$*;*

(ii) $\exists \varrho_0 \in \Omega$ *s.t.* $(\varrho_0, \Gamma\varrho_0) \in \mathcal{R}$*;*

(iii) $\mathcal{R}$ *is* Γ*-closed;*

(iv) *(a)* Γ *is* $\mathcal{R}$*-continuous mappings;*
or
(b) $\mathcal{R}$ *is* d_b*-self closed on* Ω*,*

where,

$$M_\Gamma(\varrho, \vartheta) = \max\left\{d_b(\varrho, \vartheta), d_b(\varrho, \Gamma\varrho), d_b(\vartheta, \Gamma\vartheta), \frac{d_b(\varrho, \Gamma\vartheta) + d_b(\vartheta, \Gamma\varrho)}{2s}\right\},$$

$\forall \varrho, \vartheta \in \Omega$ *satisfying* $(\varrho, \vartheta) \in \mathcal{R}$*.*
Then, Γ *possesses a fixed point. Moreover, if* $(\varrho, \vartheta) \in \mathcal{R}$*,* $\forall \varrho, \vartheta \in Fix(\Gamma)$*, where* $Fix(\Gamma)$ *denotes the set of all the fixed points of mapping* Γ*, then* Γ *has exactly one fixed point.*

Proof. The results can be deduced from Theorem6.2.6, by substituting values of ϕ and ψ:

(i) for case i (a), consider $\phi(t) = t$ and $\psi(t) = zt$;

(ii) for case i (b) and i (c), consider $\psi(t) = e^{-\tau t}$ and $\phi(t) = e^{F(t)}$, where $F \in \mathbf{F}$.

□

Example 6.2.8. *Consider* $\Omega = (0, 2]$ *equipped with b-metric* $d_b(\varrho, \vartheta) = |\varrho - \vartheta|^2$*. Consider a binary relation* $\mathcal{R}$ *defined as*

$$(\varrho, \vartheta) \in \mathcal{R}, \text{ if either } \frac{1}{5} \leq \varrho, \vartheta \leq \frac{1}{4} \text{ or } \frac{1}{3} \leq \varrho, \vartheta \leq 2,$$

and a self mapping Γ *defined as*

$$\Gamma(\varrho) = \begin{cases} \frac{9}{40}, & if\, 0 \leq \varrho \leq \frac{1}{3}; \\ \frac{2}{3}, & if\, \frac{1}{3} \leq \varrho \leq 2. \end{cases}$$

One can easily verify that

- Γ *is not continuous but it is* $\mathcal{R}$ *continuous;*
- Ω *is not complete but it is* $\mathcal{R}$ *complete.*

Moreover, Γ *satisfies all the requirements of Theorem 6.2.6. Therefore,* Γ *possesses a fixed point. Also, fixed point obtained is not unique. Indeed,* $\frac{9}{40}\&\frac{2}{3}$ *are fixed points of* Γ*. Note that* $\left(\frac{9}{40}, \frac{2}{3}\right) \notin \mathcal{R}$*.*

Example 6.2.9. *Consider the space* l^2 *of all real valued sequences* $\{\varrho_\hbar\}$ *s.t.* $\sum\limits_{\hbar=1}^{\infty} |\varrho_\hbar|^2 < \infty$*. Then, the mapping* $d_b : l^2 \times l^2 \to [0, \infty)$ *defined as* $d_b(\varrho, \vartheta) = \sum\limits_{\hbar=1}^{\infty} |\varrho_\hbar - \vartheta_\hbar|^2$ *is a b-metric with* $s = 2$*. Define a relation on* l^2 *as* $\varrho\mathcal{R}\vartheta$ *if* $\varrho_i\vartheta_i = 0$*, for* $i \in \mathbb{N}$*. Then,* (l^2, d_b) *is* $\mathcal{R}$*-complete b-MS. Consider the self mappings* $\Gamma_1, \Gamma_2 : l^2 \to l^2$ *defined as*

$$\Gamma_1(\varrho) = \begin{cases} \left(0, \frac{\varrho_1}{7}, \frac{\varrho_2}{7}, \frac{\varrho_3}{7}, ...\right), & if\ \varrho \neq (1, 0, 0, 0, ...); \\ \left(\frac{1}{7}, 0, 0, ...\right), & otherwise. \end{cases}$$

and

$$\Gamma_2(\vartheta) = \left(0, \frac{\vartheta_1}{7}, \frac{\vartheta_2}{7}, \frac{\vartheta_3}{7}, ...\right).$$

Then, Γ_1, Γ_2 *are* $\mathcal{R}$ *continuous and satisfy the following conditions:*

(i) $\mathcal{R}$ ***is*** (Γ_1, Γ_2)***-regular closed****:*
Let $(\varrho, \vartheta) \in \mathcal{R}$ *i.e.,* $\varrho_i.\vartheta_i = 0$ *for* $i \in \mathbb{N}$*.*

(a) For $\varrho = (1, 0, 0, 0, ...)$*,* $\Gamma_1\varrho.\Gamma_2\vartheta = 0$ *i.e.,* $(\Gamma_1\varrho, \Gamma_2\vartheta) \in \mathcal{R}$*.*

(b) $\varrho \neq (1, 0, 0, 0, ...)$ *and* $(\varrho, \vartheta) \in \mathcal{R}$*. Then,*

$$\begin{aligned} \Gamma_1\varrho.\Gamma_2\vartheta &= \left(0, \frac{\varrho_1}{7}, \frac{\varrho_2}{7}, \frac{\varrho_3}{7}, ...\right).\left(0, \frac{\vartheta_1}{7}, \frac{\vartheta_2}{7}, \frac{\vartheta_3}{7}, ...\right) \\ &= \left(0, \frac{\varrho_1\vartheta_1}{7}, \frac{\varrho_2\vartheta_2}{7}, \frac{\varrho_3\vartheta_3}{7}, ...\right) \\ &= 0, \left(\quad as\ (\varrho, \vartheta) \in \mathcal{R}\right). \end{aligned}$$

(ii) (Γ_1, Γ_2) ***is*** $(\phi - \psi)$ ***generalized*** $\mathcal{R}$***-contraction***:
Let $(\varrho, \vartheta) \in \mathcal{R}$ *i.e.,* $\varrho_i.\vartheta_i = 0$ *for* $i \in \mathbb{N}$.

(a) If $\varrho \neq (1, 0, 0, 0, ...)$. *Then,*

$$\begin{aligned}
\phi(2d_b(\Gamma_1\varrho, \Gamma_2\vartheta)) &= 2d_b\left(\left(0, \frac{\varrho_1}{7}, \frac{\varrho_2}{7}, \frac{\varrho_3}{7}, ...\right), \left(0, \frac{\vartheta_1}{7}, \frac{\vartheta_2}{7}, \frac{\vartheta_3}{7}, ...\right)\right)\\
&= \sum_{i=1}^{\infty} 2\left|\frac{\varrho_i}{7} - \frac{\vartheta_i}{7}\right|^2\\
&\leq \frac{2}{49}d_b(\varrho, \vartheta) \leq \psi\left(\phi\left(M_{\Gamma_1,\Gamma_2}(\varrho, \vartheta)\right)\right).
\end{aligned}$$

(b) If $\varrho = (1, 0, 0, 0, ...)$, *i.e.,* $\vartheta_1 = 0$. *Then,*

$$\begin{aligned}
\phi(2d_b(\Gamma_1\varrho, \Gamma_2\vartheta)) &= 2d_b\left(\left(\frac{1}{7}, 0, 0, ...\right), \left(0, \frac{\vartheta_1}{7}, \frac{\vartheta_2}{7}, \frac{\vartheta_3}{7}, ...\right)\right)\\
&= 2\left(\left|\frac{1}{7}\right|^2 + \left|\frac{\vartheta_1}{7}\right|^2 + ...\right)\\
&= \frac{2}{49}\left(1 + |\vartheta_2|^2 + |\vartheta_3|^2 + ...\right)\\
&= \frac{2}{49}\left(|1 - \vartheta_1|^2 + |\vartheta_2|^2 + |\vartheta_3|^2 + ...\right)\\
&= \frac{2}{49}d_b(\varrho, \vartheta) \leq \psi\left(\phi\left(M_{\Gamma_1,\Gamma_2}(\varrho, \vartheta)\right)\right).
\end{aligned}$$

Therefore, (Γ_1, Γ_2) *meet all the requirements of Theorem (6.2.3), for* $\phi(t) = t$ *and* $\psi(t) = \frac{2}{49}t$. *Hence,* Γ_1, Γ_2 *have a common fixed point.*

Definition 6.2.10. *A pair of self mappings* (Γ_1, Γ_2) *defined on a non-empty b-MS* (Ω, d_b) *is c.t.b. a* ***generalized*** $\boldsymbol{\alpha-}$ $\boldsymbol{(\phi - \psi)}$ ***contraction pair*** *if* $\exists\ \phi \in \Phi,\ \psi \in \Psi$ *and* $\alpha : \Omega \times \Omega \to (0, \infty)$ *s.t.*

$$d_b(\Gamma_1\varrho, \Gamma_2\vartheta) > 0 \Rightarrow \phi(\alpha(\varrho, \vartheta)d_b(\Gamma_1\varrho, \Gamma_2\vartheta)) \leq \psi\left(\phi\left(M_{\Gamma_1,\Gamma_2}(\varrho, \vartheta)\right)\right), \quad (6.2.7)$$

where,

$$M_{\Gamma_1,\Gamma_2}(\varrho, \vartheta) = \max\left\{d_b(\varrho, \vartheta), d_b(\varrho, \Gamma_1\varrho), d_b(\vartheta, \Gamma_2\vartheta), \frac{d_b(\varrho, \Gamma_2\vartheta) + d_b(\vartheta, \Gamma_1\varrho)}{2s}\right\},$$

$\forall \varrho, \vartheta \in \Omega$ *satisfying* $\alpha(\varrho, \vartheta) \geq s$.

Theorem 6.2.11. *Let* $\Gamma_1, \Gamma_2 : \Omega \to \Omega$ *be mappings defined on complete b-MS* (Ω, d_b). *Suppose the pair* (Γ_1, Γ_2) *satisfies* *(i)* (Γ_1, Γ_2) *is a generalized* α-$(\phi - \psi)$ *contraction pair;*

(ii) $\exists \varrho_0 \in \Omega$ *s.t.* $\alpha(\varrho_0, \Gamma_1 \varrho_0) \geq s$ *and* $\alpha(\Gamma_1 \varrho_0, \varrho_0) \geq s$*;*

(iii) (Γ_1, Γ_2) *is generalized* α*-admissible of type* s*;*

(iv) *(a)* Γ_1 *and* Γ_2 *are continuous mappings;*

or

(b) if $\{\varrho_\hbar\}$ *is a sequence in* Ω *with* $\alpha(\varrho_\hbar, \varrho_{\hbar+1}) \geq s$ $\forall \hbar \in \mathbb{N} \cup \{0\}$ *s.t.* $\varrho_\hbar \to \varrho \in \Omega$ *as* $\hbar \to \infty$*, then* $\exists$ *a subsequence* $\{\varrho_{\hbar_k}\}$ *of* $\{\varrho_\hbar\}$ *s.t.* $\alpha(\varrho_{\hbar_k}, \varrho) \geq s$ $\forall k \in \mathbb{N} \cup \{0\}$.

Then, the pair (Γ_1, Γ_2) *has a common fixed point. Moreover, if* $\alpha(\varrho, \vartheta) \geq s$*,* $\forall \varrho, \vartheta \in CF(\Gamma_1, \Gamma_2)$*, where* $CF(\Gamma_1, \Gamma_2)$ *denotes the set of all the common fixed points of mappings* Γ_1, Γ_2*, then the pair* (Γ_1, Γ_2) *has exactly one common fixed point.*

Proof. Using assumption (ii), we have an $\varrho_0 \in \Omega$ s.t. $\alpha(\varrho_0, \Gamma_1 \varrho_0) \geq s$ and $\alpha(\Gamma_1 \varrho_0, \varrho_0) \geq s$. Consider the iterative sequence with initial point ϱ_0 and

$$\varrho_{2\hbar+1} = \Gamma_1 \varrho_{2\hbar} \text{ and } \varrho_{2\hbar+2} = \Gamma_2 \varrho_{2\hbar+1} \quad \forall\, \hbar \in \mathbb{N} \cup \{0\}.$$

Now, $\alpha(\varrho_0, \Gamma_1 \varrho_0) \geq s$ and (Γ_1, Γ_2) is generalized α-admissible. Therefore,

$$\alpha(\varrho_0, \varrho_1) = \alpha(\varrho_0, \Gamma_1 \varrho_0) \geq s \Rightarrow \alpha(\Gamma_1 \varrho_0, \Gamma_2 \varrho_1) = \alpha(\varrho_1, \varrho_2) \geq s \text{ and } \alpha(\varrho_2, \varrho_1) \geq s.$$

Repeating this process, we have

$$\alpha(\varrho_\hbar, \varrho_{\hbar+1}) \geq s \text{ and } \alpha(\varrho_{\hbar+1}, \varrho_\hbar) \geq s.$$

On the similar lines of Theorem 6.2.3, we can easily prove that $\{\varrho_\hbar\}$ is a C_{seq}. Also, (Ω, d_b) is a complete b-MS. Thus, $\exists \varrho \in \Omega$ s.t. $\{\varrho_\hbar\}$ converges to ϱ. The following cases arises:

Case (i) If Γ_1 and Γ_2 are continuous mappings and $\{\varrho_\hbar\}$ converges to ϱ. Then,

$$\{\Gamma_2 \varrho_\hbar\} \to \Gamma_2 \varrho \text{ and } \{\Gamma_1 \varrho_\hbar\} \to \Gamma_1 \varrho,$$

or

$$\{\Gamma_2 \varrho_{2\hbar+1}\} \to \Gamma_2 \varrho \text{ and } \{\Gamma_1 \varrho_{2\hbar}\} \to \Gamma_1 \varrho,$$

or

$$\{\varrho_{2\hbar+2}\} \to \Gamma_2\varrho \text{ and } \{\varrho_{2\hbar+1}\} \to \Gamma_1\varrho.$$

Also,

$$\{\varrho_{2\hbar+2}\} \to \varrho \text{ and } \{\varrho_{2\hbar+1}\} \to \varrho.$$

Therefore, $\Gamma_2\varrho = \varrho = \Gamma_1\varrho$, i.e., (Γ_1, Γ_2) has a common fixed point.

Case (ii) In case assumption (iv)(b) holds. As, $\alpha(\varrho_\hbar, \varrho_{\hbar+1}) \geq s$ and $\{\varrho_\hbar\} \to \varrho$. Therefore, $\exists$ a subsequence $\{\varrho_{\hbar_k}\}$ s.t. $\alpha(\varrho_{\hbar_k}, \varrho) \geq s$. Now, suppose that $d_b(\varrho, \Gamma_2\varrho) > 0$. Then,

$$\begin{aligned} \phi\left(d_b(\varrho_{2\hbar_k+1}, \Gamma_2\varrho)\right) &= \phi\left(\alpha(\varrho_{2\hbar_k}, \varrho)d_b(\Gamma_1\varrho_{2\hbar_k}, \Gamma_2\varrho)\right) \\ &\leq \psi\left(\phi\left(M_{\Gamma_1,\Gamma_2}(\varrho_{2\hbar_k}, \varrho)\right)\right), \end{aligned}$$

where

$$\begin{aligned} M_{\Gamma_1,\Gamma_2}(\varrho_{2\hbar_k}, \varrho)) &= \max\left\{ d_b(\varrho_{2\hbar_k}, \varrho), d_b(\varrho_{2\hbar_k}, Sx_{2\hbar_k}), d_b(\varrho, \Gamma_2\varrho), \right. \\ &\quad \left. \frac{d_b(\varrho_{2\hbar_k}, \Gamma_2\varrho) + d_b(\Gamma_1\varrho_{2\hbar_k}, \varrho)}{2s} \right\} \\ &= \max\left\{ d_b(\varrho_{2\hbar_k}, \varrho), d_b(\varrho_{2\hbar_k}, \varrho_{2\hbar_k+1}), d_b(\varrho, \Gamma_2\varrho), \right. \\ &\quad \left. \frac{d_b(\varrho_{2\hbar_k}, \Gamma_2\varrho) + d_b(\varrho_{2\hbar_k+1}, \varrho)}{2s} \right\} \\ &\leq \max\left\{ d_b(\varrho_{2\hbar_k}, \varrho), d_b(\varrho_{2\hbar_k}, \varrho_{2\hbar_k+1}), d_b(\varrho, \Gamma_2\varrho), \right. \\ &\quad \left. \frac{d_b(\varrho_{2\hbar_k+1}, \varrho)}{2s} + \frac{d_b(\varrho_{2\hbar_k}, \varrho)}{2s} + \frac{d_b(\varrho, \Gamma_2\varrho)}{2s} \right\}. \end{aligned}$$

Here, $\lim_{k\to\infty} d_b(\varrho_{2\hbar_k}, \varrho) = \lim_{k\to\infty} d_b(\varrho_{2\hbar_k}, \varrho_{2\hbar_k+1}) = 0$. As, $s \geq 1$. Hence,

$$\lim_{k\to\infty} M_{\Gamma_1,\Gamma_2}(\varrho_{2\hbar_k}, \varrho) \leq d_b(\varrho, \Gamma_2\varrho). \tag{6.2.8}$$

Using (6.2.8), and continuity of ϕ, we have

$$\begin{aligned} \phi\left(d_b(\varrho, \Gamma_2\varrho)\right) &= \lim_{k\to\infty} \phi\left(d_b(\varrho_{2\hbar_k+1}, \Gamma_2\varrho)\right) \\ &\leq \psi\left(\phi\left(d_b(\varrho, \Gamma_2\varrho)\right)\right) < \phi\left(d_b(\varrho, \Gamma_2\varrho)\right), \end{aligned}$$

a contradiction. Similarly, we can show that $d_b(\varrho, \Gamma_1\varrho) = 0$. Therefore, $\Gamma_1\varrho = \varrho = \Gamma_2\varrho$ i.e., ϱ is the common fixed point of the pair (Γ_1, Γ_2).

Uniqueness: let $\varrho \neq \vartheta$ be another common fixed points of mapping Γ_1, Γ_2 with $\alpha(\varrho, \vartheta) \geq s$. Then, using (6.2.7), we have

$$\begin{aligned}
\phi(d_b(\varrho,\vartheta)) &= \phi\left(d_b(\Gamma_1\varrho, \Gamma_2\vartheta)\right) \\
&\leq \phi\left(\alpha(\varrho,\vartheta)d_b(\Gamma_1\varrho, \Gamma_2\vartheta)\right) \\
&\leq \psi\left(\phi\left(M_{\Gamma_1,\Gamma_2}(\varrho,\vartheta)\right)\right),
\end{aligned}$$

where

$$\begin{aligned}
M_{\Gamma_1,\Gamma_2}(\varrho,\vartheta) &= \max\left\{d_b(\varrho,\vartheta), d_b(\varrho,\Gamma_1\varrho), d_b(\vartheta,\Gamma_2\vartheta), \frac{d_b(\varrho,\Gamma_2\vartheta)+d_b(\vartheta,\Gamma_1\varrho)}{2s}\right\} \\
&= \max\left\{d_b(\varrho,\vartheta), 0, 0, \frac{d_b(\varrho,\vartheta)}{s}\right\} \\
&= d_b(\varrho,\vartheta),
\end{aligned}$$

implies

$$\phi\left(d_b(\varrho,\vartheta)\right) \leq \psi\left(\phi\left(d_b(\varrho,\vartheta)\right)\right) < \phi\left(d_b(\varrho,\vartheta)\right),$$

a contradiction. Therefore, (Γ_1, Γ_2) has exactly one common fixed point.

$\square$

Definition 6.2.12. *A self mapping Γ defined on a non-empty b-MS (Ω, d_b) is c.t.b. a generalized* ***$\alpha-(\phi-\psi)$ contraction*** *if $\exists\phi \in \Phi,\ \psi \in \Psi$ and $\alpha : \Omega\times\Omega \to (0,\infty)$ s.t.*

$$d_b(\Gamma\varrho, \Gamma\vartheta) > 0 \Rightarrow \phi\left(\alpha(\varrho,\vartheta)d_b(\Gamma\varrho,\Gamma\vartheta)\right) \leq \psi\left(\phi\left(M_{\Gamma_1,\Gamma}(\varrho,\vartheta)\right)\right), \tag{6.2.9}$$

$\forall\varrho, \vartheta \in \Omega$ satisfying $\alpha(\varrho,\vartheta) \geq s$, where

$$M_{\Gamma_1,\Gamma}(\varrho,\vartheta) = \max\left\{d_b(\varrho,\vartheta), d_b(\varrho,\Gamma\varrho), d_b(\vartheta,\Gamma\vartheta), \frac{d_b(\varrho,\Gamma\vartheta)+d_b(\vartheta,\Gamma\varrho)}{2s}\right\}.$$

Theorem 6.2.13. *Let $\Gamma : \Omega \to \Omega$ be a mapping defined on a complete b-MS (Ω, d_b). Suppose Γ satisfies*

(i) *Γ is a generalized $\alpha-(\phi-\psi)$ contraction;*

(ii) *$\exists\varrho_0 \in \Omega$ s.t. $\alpha(\varrho_0, \Gamma\varrho_0) \geq s$ and $\alpha(\Gamma\varrho_0, \varrho_0) \geq s$;*

(iii) *Γ is α-admissible of type s;*

(iv) *(a)* Γ *is a continuous mapping;*

or

(b) *if* $\{\varrho_\hbar\}$ *is a sequence in* Ω *with* $\alpha(\varrho_\hbar, \varrho_{\hbar+1}) \geq s$ $\forall \hbar \in \mathbb{N} \cup \{0\}$ *s.t.* $\varrho_\hbar \to \varrho \in \Omega$ *as* $\hbar \to \infty$*, then* $\exists$ *a subsequence* $\{\varrho_{\hbar_k}\}$ *of* $\{\varrho_\hbar\}$ *s.t.* $\alpha(\varrho_{\hbar_k}, \varrho) \geq s$ $\forall k \in \mathbb{N} \cup \{0\}$.

Then Γ *has a fixed point. Moreover, if* $\alpha(\varrho, \vartheta) \geq s$, $\forall \varrho, \vartheta \in Fix(\Gamma)$*, where* $Fix(\Gamma)$ *denotes the set of all the fixed points of mapping* Γ*, then* Γ *has exactly one fixed point.*

Proof. Considering $\Gamma_1 = \Gamma$ in Theorem 6.2.11, one can easily deduce the result. $\square$

Corollary 6.2.14. *Let* $\Gamma : \Omega \to \Omega$ *be a mapping defined on a complete b-MS* (Ω, d_b)*. Suppose* Γ *satisfies*

(i) *(a)* $\exists F \in \boldsymbol{F}$ *and* $\tau > 0$ *s.t.* $\tau + F\left(\alpha(\varrho, \vartheta) d_b(\Gamma\varrho, \Gamma\vartheta)\right) \leq F\left(d_b(\varrho, \vartheta)\right)$*;*

or

(b) $\exists F \in \boldsymbol{F}$ *and* $\tau > 0$ *s.t.* $\tau + F\left(\alpha(\varrho, \vartheta) d_b(\Gamma\varrho, \Gamma\vartheta)\right) \leq F\left(M_\Gamma(\varrho, \vartheta)\right)$*;*

or

(c) $\exists \psi \in \Psi$ *s.t.* $\alpha(\varrho, \vartheta) d(\Gamma\varrho, \Gamma\vartheta) \leq \psi(d(\varrho, \vartheta))$*;*

(ii) $\exists \varrho_0 \in \Omega$ *s.t.* $\alpha(\varrho_0, \Gamma\varrho_0) \geq s$ *and* $\alpha(\Gamma\varrho_0, \varrho_0) \geq s$*;*

(iii) Γ *is* α*-admissible of type s;*

(iv) *(a)* Γ *is a continuous mapping;*

or

(b) *if* $\{\varrho_\hbar\}$ *is a sequence in* Ω *with* $\alpha(\varrho_\hbar, \varrho_{\hbar+1}) \geq s$ $\forall \hbar \in \mathbb{N} \cup \{0\}$ *s.t.* $\varrho_\hbar \to \varrho \in \Omega$ *as* $\hbar \to \infty$*, then* $\exists$ *a subsequence* $\{\varrho_{\hbar_k}\}$ *of* $\{\varrho_\hbar\}$ *s.t.* $\alpha(\varrho_{\hbar_k}, \varrho) \geq s$ $\forall k \in \mathbb{N} \cup \{0\}$,

where,

$$M_\Gamma(\varrho, \vartheta) = \max\left\{d_b(\varrho, \vartheta), d_b(\varrho, \Gamma\varrho), d_b(\vartheta, \Gamma\vartheta), \frac{d_b(\varrho, \Gamma\vartheta) + d_b(\vartheta, \Gamma\varrho)}{2s}\right\},$$

$\forall \varrho, \vartheta \in \Omega$.

Then, Γ has a fixed point. Moreover, if $\alpha(\varrho, \vartheta) \geq s$, $\forall \varrho, \vartheta \in Fix(\Gamma)$, where $Fix(\Gamma)$ denotes the set of all the fixed points of mapping Γ, then Γ has exactly one fixed point.

Proof. On substituting specific values of ϕ and ψ in Theorem 6.2.13, we can verify the above results.

(i) for case i (a) and i (b), consider $\psi(t) = e^{-\tau t}$ and $\phi(t) = e^{F(t)}$, where $F \in \mathbf{F}$;

(ii) for case i (c), consider $\phi(t) = t$.

$\square$

Example 6.2.15. *Let $\Omega = [0, \infty)$ and $d_b(\varrho, \vartheta) = |\varrho - \vartheta|^2$. Define $\Gamma_1, \Gamma_2 : \Omega \to \Omega$*

$$\Gamma_1(\varrho) = \frac{\varrho}{4}, \quad \Gamma_2(\varrho) = \frac{\varrho}{8}.$$

If $\alpha : \Omega \times \Omega \to [0, \infty)$ is defined as

$$\alpha(\varrho, \vartheta) = \begin{cases} 2 + cos(\varrho^2 + \vartheta), & \text{if } \varrho \in [0, 1]; \\ 0, & \text{otherwise.} \end{cases}$$

Then,

(i) (Ω, d_b) is a complete b-MS with $s = 2$;

(ii) (Γ_1, Γ_2) is generalized α-admissible;

(iii) For $\phi(t) = t$ and $\psi(t) = \frac{6t}{9}$, the pair (Γ_1, Γ_2) is a generalized α-$(\phi - \psi)$ contraction pair.

For $\varrho, \vartheta \in [0, 1]$

$$\begin{aligned} d_b(\Gamma_1\varrho, \Gamma_2\vartheta) &= \left|\frac{\varrho}{4} - \frac{\vartheta}{8}\right|^2 \\ &= \frac{1}{16}\left|\varrho - \frac{\vartheta}{2}\right|^2. \end{aligned}$$

(i) Let $\varrho > \frac{\vartheta}{2}$, then

$$\begin{aligned}
d_b(\Gamma_1\varrho, \Gamma_2\vartheta) &= \left|\frac{\varrho}{4} - \frac{\vartheta}{8}\right|^2 \\
&= \frac{1}{16}\left(\varrho^2 + \frac{\vartheta^2}{4} - \varrho\vartheta\right) \\
&\leq \frac{1}{16}\left(\varrho^2 + \varrho^2 - \varrho\vartheta\right) \\
&\leq \frac{2\varrho^2}{16} \\
&\leq \frac{1}{3} \times \frac{6}{9} \times \frac{9\varrho^2}{16} \\
&\leq \frac{1}{3}\left(\frac{6}{9} d_b(\varrho, \Gamma_1\varrho)\right) \\
&\leq \frac{1}{3}\left(\frac{6}{9} M_{\Gamma_1,\Gamma_2}(\varrho, \vartheta)\right).
\end{aligned}$$

(ii) Let $\varrho < \frac{\vartheta}{2}$, then

$$\begin{aligned}
d_b(\Gamma_1\varrho, \Gamma_2\vartheta) &= \left|\frac{\varrho}{4} - \frac{\vartheta}{8}\right|^2 \\
&= \frac{1}{16}\left(\frac{\vartheta^2}{4} + \frac{\vartheta^2}{4} - \varrho\vartheta\right) \\
&= \frac{1}{16}\left(\frac{\vartheta^2}{2}\right) \\
&\leq \frac{1}{3}\left(\frac{6}{9}\left(\frac{9}{64}\vartheta^2\right)\right) \\
&= \frac{1}{3}\left(\frac{6}{9}\left(d_b(\vartheta, \Gamma_2\vartheta)\right)\right) \\
&\leq \frac{1}{3}\left(\frac{6}{9} M_{\Gamma_1,\Gamma_2}(\varrho, \vartheta)\right).
\end{aligned}$$

(iii) Let $\varrho = \frac{\vartheta}{2}$, then

$$d_b(\Gamma_1\varrho, \Gamma_2\vartheta) = \left|\frac{\varrho}{4} - \frac{\vartheta}{8}\right|^2 = 0 \leq \frac{1}{3}\left(\frac{6}{9}\left(d_b(\varrho, \vartheta)\right)\right).$$

Therefore, for $\phi(t) = t$, $\psi(t) = 6t/9$, we have

$$\phi\left(\alpha(\varrho, \vartheta) d_b(\Gamma_1\varrho, \Gamma_2\vartheta)\right) \leq \phi\left(3 d_b(\Gamma_1\varrho, \Gamma_2\vartheta)\right) \leq \psi\left(\phi\left(M_{\Gamma_1,\Gamma_2}(\varrho, \vartheta)\right)\right).$$

Hence, all the requirements of Theorem 6.2.11 are satisfied. Indeed, $\varrho = 0$ is a common fixed point.

6.3 Existence of Solution to Operator Equation

In this section, we established the existence and uniqueness of the solution to an operator equation using Theorem 6.2.11.

Let $L(H)$ represents the set of bounded linear operators defined on the Hilbert space H. Let A be a bounded linear operator on H. Then, $L(H)$ is a normed space with norm on H defined as

$$\|A\| = \sup_{\varrho\neq 0\in H} \frac{\|A\varrho\|}{\|\varrho\|} = \sup_{\varrho\in H, \|\varrho\|=1} \|A\varrho\|.$$

Then, for $A, B, C \in L(H)$ and $d_b(A, B) = \|A - B\|^q$, we have

- $d_b(A, B) = 0$ implies $\|A - B\|^q = 0$ iff $\|A - B\| = 0$, i.e., $A = B$;
- $d_b(A, B) = \|A - B\|^q = \|B - A\|^q = d_b(B, A)$;
- for triangle inequality, consider

$$\begin{aligned} \|A\varrho - B\varrho\|^q &\leq 2^{q-1}(\|A\varrho - C\varrho\|^q + \|C\varrho - B\varrho\|^q) \\ &= s(\|A\varrho - C\varrho\|^q + \|C\varrho - B\varrho\|^q), \ \ \forall\ \varrho \in H. \end{aligned}$$

  i.e., $\|A - B\|^q \leq s(\|A - C\|^q + \|C - B\|^q)$ implies $d_b(A, B) \leq s(d_b(A, C) + d_b(C, B))$.

Therefore, space $(L(H), d_b)$ is a complete-b MS, where $d_b(A, B) = \|A - B\|^q$ with $s = 2^{q-1}$, $\forall A, B \in L(H)$ and $q > 1$.

Consider the operator equation

$$Y - \sum_{n=1}^{\infty} B_n^* F(Y) B_n = Q; \tag{6.3.1}$$

where $Y \in L(H)$, $Q \in L(H)_+$, B_n be a sequence of bounded linear operator on $L(H)$ and F is operator valued functions on $L(H)$. We will utilized the Theorem 6.2.11 to show the existence of the unique solution to the operator equation.

Theorem 6.3.1. *Let B_n be a sequence of non-zero bounded linear operators with $\sum_{n=1}^{\infty} \|B_n\|^2 = \eta$ is finite. Consider the assumption $\exists \kappa \in \mathbb{R}_+$ s.t. $\|F(Y) - F(\Omega)\| \leq \frac{\kappa}{\sqrt{2}}\|Y - \Omega\|$, $\forall Y \neq \Omega \in L(H)$. Then, the operator equation (6.3.1) has exactly one solution if $\eta\kappa < 1$.*

Proof. Define the self mappings S on $L(H)$ as

$$S(Y) = Q + \sum_{n=1}^{\infty} B_n^* F(Y) B_n$$

For $\alpha : L(H) \times L(H) \to (0, \infty)$ defined as $\alpha(Y, \Omega) = 2$. Clearly S is α-admissible. To prove S is α-admissible $(\phi - \psi)$ generalized contraction, consider the following

$$\begin{aligned}
d_b(S(Y), S(\Omega)) &= \|S(Y) - S(\Omega)\|^2 \\
&= \|\sum_{n=1}^{\infty} B_n^* F(Y) B_n - \sum_{n=1}^{\infty} B_n^* F(\Omega) B_n\|^2 \\
&\leq \|\sum_{n=1}^{\infty} B_n^* \left(F(Y) - F(\Omega)\right) B_n\|^2 \\
&\leq \left(\sum_{n=1}^{\infty} \|B_n^*\| \|F(Y) - F(\Omega)\| \|B_n\|\right)^2 \\
&\leq \left(\sum_{n=1}^{\infty} \|B_n\|^2 \frac{\kappa}{\sqrt{2}} \|Y - \Omega\|\right)^2 \\
&= \left(\eta \frac{\kappa}{\sqrt{2}} \|Y - \Omega\|\right)^2,
\end{aligned}$$

or

$$\alpha(Y, \Omega) d_b(S(Y), S(\Omega)) = (\varrho\varrho\|Y - \Omega\|)^2 = kd(Y, \Omega) \leq kM_{S,T}(Y, \Omega),$$

i.e., $\phi\left(\alpha(Y, \Omega) d_b(S(Y), T(\Omega))\right) \leq \psi(\phi(M_{S,T}(Y, \Omega)))$, where $\phi(t) = t$ and $\psi(t) = kt$, where $k = \sqrt{\eta\kappa} \in [0, 1)$.
As, S fulfills all the requirements of Theorem 6.2.13. Therefore, S possesses exactly one fixed point. i.e., the operator equations (6.3.1) has exactly one solution. □

6.4 Existence of Solution to Non-Linear Functional Integral Equation

In this section, we discussed the existence of the solution for the non-linear functional integral equation using Theorem 6.2.6. Additionally, some illustrative examples are discussed at the end of this section to support our findings.

$$\begin{cases} \varrho(z) = g(z, \varrho(z)) + f\left(z, \int_0^z u\left(z, s, \varrho(s)\right) ds, \varrho\left(\alpha(z)\right)\right) & \text{for } z \in I \\ g(z, \varrho(z)) \geq 1 & \text{for } z \in I. \end{cases} \tag{6.4.1}$$

Considering the following:

H1) The function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and $\exists a_1 : [0,1] \to [0,1]$ s.t.

$$|g(z,\varrho_1) - g(z,\varrho_2)| \leq a_1(z)|\varrho_1 - \varrho_2|;$$

H2) The function $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and $\exists a_2, a_3 : [0,1] \to [0,1]$ s.t.

$$\begin{aligned} |f(z,\vartheta_1,\varrho) - f(z,\vartheta_2,\varrho)| &\leq a_2(z)|\vartheta_1 - \vartheta_2| \\ |f(z,\vartheta,\varrho_1) - f(z,\vartheta,\varrho_2)| &\leq a_3(z)|\varrho_1 - \varrho_2|; \end{aligned}$$

H3) The function $u : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and $\exists\ a_4 : [0,1] \to [0,1]$ s.t.

$$|u(z,s,\varrho_1) - u(z,s,\varrho_2)| \leq a_4(z)|\varrho_1 - \varrho_2|;$$

H4) $2k + k^2 < 1$, where $k = \max\limits_{1 \leq i \leq 4} \left\{ \sup\limits_{z \in [0,1]} |a_i(z)| \right\}$.

Theorem 6.4.1. *Under the above mentioned assumptions the non-linear functional integral equation (6.4.1) has a solution.*

Proof. We consider the set $Y = \{\varrho \in C[0,1]\ :\ \varrho(z) > 0)\ \forall z \in [0,1]\}$ and define the relation $\mathcal{R}$ on Y as

$$(\varrho, \vartheta) \in \mathcal{R} \text{ whenever } \varrho(z).\vartheta(z) \geq \varrho(z) \vee \vartheta(z), \forall z \in [0,1].$$

Then, (Y, d_b) is a b-MS with $d_b(\varrho,\vartheta) = \sup\limits_{z \in [0,1]} |\varrho(z) - \vartheta(z)|^p$. Suppose $\varrho_0 = 1$, then $(\varrho_0, \varrho) \in \mathcal{R}\ \forall \varrho \in Y$. Let $\{\varrho_n\}$ be an $\mathcal{R}$-preserving C_{seq} and $\varrho_n \in C[0,1]$. It can be easily shown that $\{\varrho_n\}$ converges uniformly to some $\varrho \in C[0,1]$. Then, for $t \in [0,1]$ using definition of $\mathcal{R}$-preserving $\varrho_n(t).\varrho_{n+1}(t) \geq \varrho_n(t) \vee \varrho_{n+1}(t)$. As, $\{\varrho_n(t) > 0\}$, therefore $\exists$ an subsequence $\{\varrho_{n_k}\}$ s.t. $\varrho_{n_k} \geq 1$. Also, $\{\varrho_{n_k}\}$ converges to ϱ, i.e., $\varrho(t) \geq 1$, $\forall t \in [0,1]$. Hence, $\varrho \in Y$.
Now, we define the self mapping $T : Y \to Y$ as

$$T(\varrho(z)) = g(z, \varrho(z)) + f\left(z, \int_0^z u(z,s,\varrho(s))\, ds, \varrho(\alpha(z))\right).$$

Clearly, the fixed points of the self mapping T are the solutions of (6.4.1). To prove T satisfies all the conditions of Theorem 6.2.6, we shall prove that

(i) **$\mathcal{R}$ is T-closed**

Let $(\varrho(z), \vartheta(z)) \in \mathcal{R}$ i.e., $\varrho(z).\vartheta(z) \geq \varrho(z) \vee \vartheta(z)$. Then,

$$T\varrho(z) = g(z, \varrho(z)) + f\left(z, \int_0^z u\left(z, s, \varrho(s)\right) ds, \varrho(\alpha(z))\right) \geq 1,$$

or

$$T\varrho(z).T\vartheta(z) \geq T\vartheta(z),$$

i.e., $(T\varrho(z), T\vartheta(z)) \in \mathcal{R}$.

(ii) **T is $(\phi - \psi)$ generalized $\mathcal{R}$-contraction**

$$\begin{aligned}
|T\varrho(z) - T\vartheta(z)| &= \Bigg|g\left(z, \varrho(z)\right) + f\left(z, \int_0^z u\left(z, s, \varrho(s)\right) ds, \varrho\left(\alpha(z)\right)\right) - g\left(z, \vartheta(z)\right) \\
&\quad - f\left(z, \int_0^z u\left(z, s, \vartheta(s)\right) ds, \vartheta\left(\alpha(z)\right)\right)\Bigg| \\
&\leq \Big|g\left(z, \varrho(z)\right) - g\left(z, \vartheta(z)\right)\Big| + \\
&\quad \Bigg|f\left(z, \int_0^z u\left(z, s, \varrho(s)\right) ds, \varrho\left(\alpha(z)\right)\right) \\
&\quad - f\left(z, \int_0^z u\left(z, s, \varrho(s)\right) ds, \vartheta\left(\alpha(z)\right)\right)\Bigg| + \\
&\quad \Bigg|f\left(z, \int_0^z u\left(z, s, \varrho(s)\right) ds, \vartheta\left(\alpha(z)\right)\right) \\
&\quad - f\left(z, \int_0^z u\left(z, s, \vartheta(s)\right) ds, \vartheta\left(\alpha(z)\right)\right)\Bigg| \\
&\leq a_1(z)|\varrho(z) - \vartheta(z)| + a_2(z)\left|\int_0^z u(z, s, \varrho(s)) - \int_0^z u\left(z, s, \vartheta(s)\right)\right| \\
&\quad + a_3(z)|\varrho\left(\alpha(z)\right) - \vartheta\left(\alpha(z)\right)| \\
&\leq \left(a_1(z) + a_3(z)\right)|\varrho(z) - \vartheta(z)| \\
&\quad + a_2(z)\int_0^z |u\left(z, s, \varrho(s)\right) - u\left(z, s, \vartheta(s)\right)| ds \\
&\leq \left(a_1(z) + a_3(z)\right)|\varrho(z) - \vartheta(z)| + a_2(z)a_4(z)|\varrho(z) - \vartheta(z)|.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
\phi(d_b(T\varrho, T\vartheta)) &= \sup_{t\in[0,1]} |T\varrho(z) - T\vartheta(z)|^p \\
&\leq \sup_{z\in[0,1]} \Bigg[(a_1(z) + a_3(z)) \, |\varrho(z) - \vartheta(z)| \\
&\quad + a_2(z) \int_0^t |u\,(z, s, \varrho(s)) - u\,(z, s, \vartheta(s))\,|ds \Bigg]^p \\
&\leq \sup_{z\in[0,1]} \Big[(a_1(z) + a_3(z) + a_2(z)a_4(z)) \, |\varrho(z) - \vartheta(z)| \Big]^p \\
&\leq \sup_{z\in[0,1]} \Big[\left(2k + k^2\right) |\varrho(z) - \vartheta(z)| \Big]^p = \kappa^p d_b(\varrho, \vartheta) \\
&\leq \psi\left(\phi\left(M_T(\varrho, \vartheta)\right)\right),
\end{aligned}
$$

for $\phi(z) = z$ and $\psi(z) = \kappa^p z < z$, where $\kappa = 2k + k^2$.

(iii) **T is $\mathcal{R}$-continuous**

Let $\{\varrho_n\}$ be an $\mathcal{R}$-sequence converging to $\varrho \in Y$. Using the first part of the proof $\varrho(z) \geq 1 \; \forall z \in I$, hence $\varrho_n(z)\varrho(z) \geq \varrho_n(z) \; \forall n \in \mathbb{N}$ and all $z \in I$, therefore $\varrho_n \mathcal{R} \varrho$.

From the above part of the proof

$$
\begin{aligned}
d_b(T\varrho_n, T\varrho) &= \sup_{z\in[0,1]} |T\varrho_n(z) - T\varrho(z)|^p \\
&\leq k^p \sup_{z\in[0,1]} |\varrho_n(z) - \varrho(z)|^p \\
&= k^p d_b\left(\varrho_n, \varrho\right),
\end{aligned}
$$

implies $\{T\varrho_n\}$ be an $\mathcal{R}$-sequence converging to $T\varrho \in Y$.

Therefore, T satisfies all the requirements of Theorem 6.2.6. Hence, T has a fixed point i.e., non-linear functional integral equation (6.4.1) has a solution.

Example 6.4.2. *Consider the following fractional integral equation of the type*

$$
\begin{cases}
\varrho(z) = g(z) + \frac{a(z)}{\Gamma(\alpha)} \int_0^z b(s)(z-s)^{\alpha-1}\varrho(s)ds & \text{for } z \in I \\
g(z) \geq 1 & \text{for } z \in I.
\end{cases}
$$

The existence of the solution of this integral equation can be observed by Theorem 6.4.1 with $g(z, \varrho(z)) = g(z)$, $f(z, \vartheta(z), \varrho(\alpha(z))) = \frac{a(z)}{\Gamma(\alpha)}\vartheta(z)$, $u(z, s, \varrho(s)) = b(s)(z-s)^{\alpha-1}\varrho(s)$.

Example 6.4.3. *Volterra-Urysohn type integral equation*

$$\begin{cases} \varrho(z) = g(z) + \int_0^z u(z,s,\varrho(s))ds & \text{for } z \in I \\ g(z) \geq 1 & \text{for } z \in I. \end{cases}$$

has a solution under the requirements of Theorem 6.4.1.

Example 6.4.4. *Consider the following equation*

$$\varrho(z) = 3\cos\left(\frac{1}{1+2z}\right) + \frac{z}{7}\int_0^z s\cos(\varrho(s))ds.$$

Here, $g(z,\varrho) = 3\cos\left(\frac{1}{1+2z}\right)$, $f(z,\vartheta,\varrho) = \frac{\vartheta}{7}$, $u(z,s,\varrho(s)) = zs\cos\varrho(s)$ with $a_1(z) = 0$, $a_2(z) = \frac{1}{\sqrt{7}}$, $a_3(z) = 0$ and $a_4(z) = \frac{z}{\sqrt{7}}$. Therefore, for $k = \frac{1}{\sqrt{7}}$ all the conditions of Theorem 6.4.1 are fulfilled. □

6.5 Conclusion

In this chapter, we have introduced a novel approach for finding solutions to a class of non-linear functional integral equations within a b-MS equipped with a binary relation. Some common fixed point results are established in b-MS, which were then reduced to fixed point results for single mappings. The results presented in this chapter extend and generalize various existing fixed point results from the literature.

- If $s = 1$, Theorem 3.1 of Alam and Imdad (2015), Theorem 3.8, Theorem 3.9 of Gopal et al. (2016), Theorem 2.2 of Samet et al. (2012) can be deduced as the consequences of Corollary 6.2.7 and Corollary 6.2.14.
- If $\mathcal{R} = \Omega \times \Omega$, Theorem 1 and Theorem 3 of Czerwik (1993), Theorem 1, Theorem 2 and Theorem 3 of Kir and Kiziltunc (2013), Theorem 2 of Dubey et al. (2014), Theorem 3.5 of Pant and Panicker (2016) can be deduced as the consequences of Corollary 6.2.7.
- If $s = 1$ and $\mathcal{R} = \Omega \times \Omega$, Theorem 2.1 of Wardowski (2012), Theorem 2.4 of Wardowski and Dung (2014) and Theorem 2.1 of Piri and Kumam (2014) can be deduced as the consequences of Corollary 6.2.7.

As applications of the proved results, the existence of solution to a class of non-linear functional integral equation and an operator equation are established.

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Chapter 7

Some Fixed Point Results in $\mathfrak{m}$-Metric Space

7.1 Introduction

The present chapter delves into the study of fixed point results within the recently developed framework of $\mathfrak{m}$-metric spaces, providing both theoretical extension accompanied by some illustrations. This chapter has two main sections. In the first section, we explore various fixed point results for self mappings in $\mathfrak{m}$-metric spaces using interpolative-type contractions and $(\phi - \psi)$ Wardowski contractions. It has been highlighted that many in the existing literature are special cases of our findings. Several illustrative examples are provided to validate and elucidate these theoretical results.

The second section introduces a novel perspective on contraction mappings in $\mathfrak{m}$-metric spaces, particularly highlighting the cases where traditional metric space results are not applicable but their extended versions in $\mathfrak{m}$-metric spaces succeed. Unlike classical Banach contraction mappings, the requirement for continuity is relaxed in $\mathfrak{m}$-metric space, broadening its applicability. Also, we have compared the behaviour of contraction mappings for metric spaces and $\mathfrak{m}$-metric spaces, graphically. At last, we extend the scope by establishing results for the existence of common fixed points for pairs and triples of self mappings in incomplete spaces. These findings are supported by detailed numerical iterations, examples, and graphical representations to approximate the common fixed points effectively.

The results of the chapter are presented in [1] [2].

7.2 Comparison of Various Contraction in Metric Spaces and $\mathfrak{m}$-Metric Space

In this section, we have explored how fixed point theorems in $\mathfrak{m}$-MS represent a true generalization of those in MS. The procedure for determining the respective contraction constant within an abstract metric space is discussed. Some illustrative examples where traditional fixed point theorems fail, but the contractions in $\mathfrak{m}$-MS guarantee the existence of a fixed point are presented. Also, the algorithms to determine the Banach and Kannan contraction constants are presented.

Algorithm 1 How to find Banach Contraction constant with an distance function (δ)?

Initialize max_k = 0 {To track the maximum contraction constant}
for each pair of distinct points (ζ, υ) in the set X **do**
  Calculate the distance between the points:
    RHS $= \delta(\zeta, \upsilon)$
  Calculate the distance between their images under F:
    LHS $= \delta(F(\zeta), F(\upsilon))$
  Calculate the contraction ratio:
    $k_{\zeta,\upsilon} = \frac{LHS}{RHS}$
  if $k_{\zeta,\upsilon} >$ max_k **then**
    Update max_k $= k_{\zeta,\upsilon}$
  end if
  if max_k ≥ 1 **then**
    return "Not a Banach contraction" {Early exit if contraction condition fails}
  end if
end for
return max_k {Return Banach contraction constant if all $k_{\zeta,\upsilon} < 1$}

Example 7.2.1. *Consider the* $\mathfrak{m}$-*MS* $([0,2], \mathfrak{m})$ *, where* $\mathfrak{m} : [0,\infty) \times [0,2] \to [0,\infty)$

[1] Yadav, K., & Kumar, D. Some fixed point result in $\mathfrak{m}$-metric space using different contractions, In: Tomar, A., Jain, M. (eds) Banach Contraction Principle. Industrial and Applied Mathematics. Springer, Singapore.*Banach Contraction Principle: A Centurial Journey, Springer (Accepted)*

[2] Yadav, K., & Kumar, D. (2025). Common Fixed Point Theorems for Discontinuous Mappings in m-Metric Space and Numerical Approximations, *Journal of Computational and Applied Mathematics*, 470, 116720

Algorithm 2 How to find Kannan Contraction constant with an distance function (δ)?

Initialize max_k = 0 {To track the maximum contraction constant}
for each pair of distinct points (ζ, υ) in the set X **do**
  Calculate the distance between the points:
    RHS $= \delta(\zeta, \upsilon)$
  Calculate the distance between their images under F:
    LHS $= \delta(\zeta, F(\zeta)) + \delta(\upsilon, F(\upsilon))$
  Calculate the contraction ratio:
    $k_{\zeta,\upsilon} = \frac{LHS}{RHS}$
  if $k_{\zeta,\upsilon} >$ max_k **then**
    Update max_k $= k_{\zeta,\upsilon}$
  end if
  if max_k $\geq \frac{1}{2}$ **then**
    return "Not a Kannan contraction" {Early exit if contraction condition fails}
  end if
end for
return max_k {Return Kannan contraction constant if all $k_{\zeta,\upsilon} < \frac{1}{2}$}

defined as $\mathfrak{m}(\zeta, \upsilon) = \frac{\zeta+\upsilon}{2}$*. Let the self mapping* F *be defined as*

$$F\zeta = \begin{cases} \frac{\zeta^2}{5}, & \text{if } \zeta < 1 \\ \frac{1}{\zeta+9}, & \text{otherwise.} \end{cases},$$

Then, F *is a contraction w.r.t the* $\mathfrak{m}$*-metric* $\mathfrak{m}$ *i.e.,* $\mathfrak{m}(F\zeta, F\upsilon) \leq k\mathfrak{m}(\zeta, \upsilon)$, $\forall \zeta \in [0, 2)$*. One can verify the following cases:*

Case (i) *Let* $\zeta, \upsilon < 1$*. Then,*

$$\begin{aligned} \mathfrak{m}(F\zeta, F\upsilon) &= \frac{1}{2}\left(\frac{\zeta^2}{5} + \frac{\upsilon^2}{5}\right) \\ &\leq \frac{1}{2}\left(\frac{\zeta}{5} + \frac{\upsilon}{5}\right) \\ &\leq \frac{1}{5}\mathfrak{m}(\zeta, \upsilon). \end{aligned}$$

Case (ii) *Let* $\zeta < 1$ *and* $\upsilon \geq 1$. *Then,*

$$\begin{aligned} \mathfrak{m}(F\zeta, F\upsilon) &= \frac{1}{2}\left(\frac{\zeta^2}{5} + \frac{1}{\upsilon+9}\right) \\ &\leq \frac{1}{2}\left(\frac{\zeta}{5} + \frac{1}{10}\right) \\ &\leq \frac{1}{2}\left(\frac{\zeta}{5} + \frac{1}{5}\right) \\ &\leq \frac{1}{5}\mathfrak{m}(\zeta, \upsilon). \end{aligned}$$

Case (iii) *Let* $\zeta, \upsilon \geq 1$. *Then,*

$$\begin{aligned} \mathfrak{m}(F\zeta, F\upsilon) &= \frac{1}{2}\left(\frac{1}{\zeta+9} + \frac{1}{\upsilon+9}\right) \\ &\leq \frac{1}{2}\left(\frac{1}{10} + \frac{1}{10}\right) \\ &\leq \frac{1}{2}\left(\frac{\zeta}{5} + \frac{\upsilon}{5}\right) \\ &\leq \frac{1}{5}\mathfrak{m}(\zeta, \upsilon). \end{aligned}$$

Clearly, F *satisfies Banach contraction in* $\mathfrak{m}$*-MS. But one can easily verify that for usual metric* $d(\zeta, \upsilon) = |\zeta - \upsilon|$, F *does not satisfies the Banach contraction Principle. From figures 7.2.1 and 7.2.1, one can verify that there is no such* $k \in [0, 1)$ *for which* $d(F\zeta, F\upsilon) \leq kd(\zeta, \upsilon)$ $\forall \zeta, \upsilon \in [0, 2]$ *holds.*

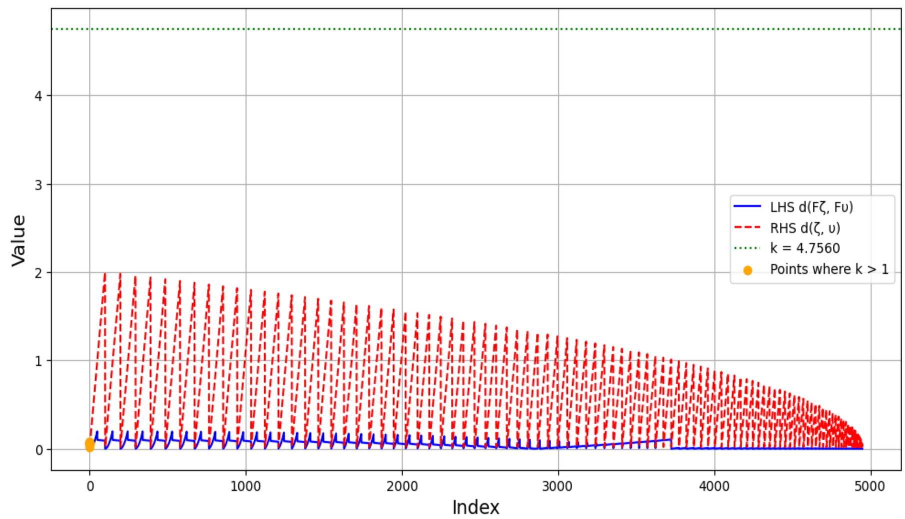

Figure 7.1: Banach contraction w.r.t. the usual metric

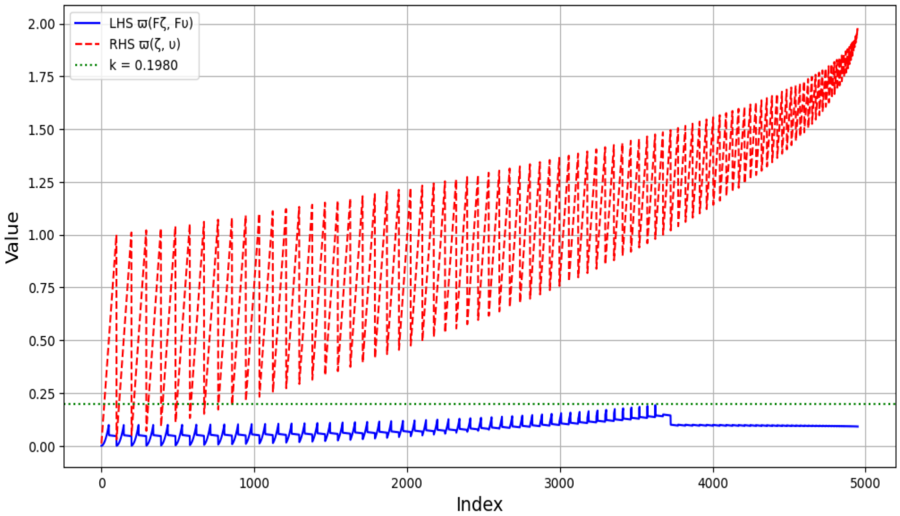

Figure 7.2: Banach contraction w.r.t. $\mathfrak{m}$-metric

Example 7.2.2. *Consider the $\mathfrak{m}$-MS $([0,2], \mathfrak{m})$, where $\mathfrak{m} : [0,2] \times [0,2] \to [0, \infty)$ defined as $\mathfrak{m}(\zeta, \upsilon) = \frac{\zeta+\upsilon}{2}$ and the self mapping F be defined as*

$$F\zeta = \begin{cases} \frac{\zeta}{3}, & \text{if } \zeta < 1 \\ \frac{1}{7}, & \text{otherwise.} \end{cases},$$

Then, F is a Kannan contraction w.r.t the $\mathfrak{m}$-metric $\mathfrak{m}$ i.e., $\mathfrak{m}(F\zeta, F\upsilon) \leq k(\mathfrak{m}(\zeta, F\zeta) + \mathfrak{m}(F\upsilon, \upsilon))$, $\forall \zeta \in [0, 2)$. One can verify the following cases:

Case (i) *Let $\zeta, \upsilon < 1$. Then,*

$$\begin{aligned} \mathfrak{m}(F\zeta, F\upsilon) &= \frac{1}{2}\left(\frac{\zeta}{3} + \frac{\upsilon}{3}\right) \\ &\leq \frac{1}{3}\left(\frac{1}{2}\left(\zeta + \frac{\zeta}{3}\right) + \frac{1}{2}\left(\upsilon + \frac{\upsilon}{3}\right)\right) \\ &\leq \frac{1}{3}\left(\mathfrak{m}(\zeta, F\zeta) + \mathfrak{m}(\upsilon, F\upsilon)\right). \end{aligned}$$

Case (ii) *Let $\zeta < 1$ and $\upsilon \geq 1$. Then,*

$$\begin{aligned} \mathfrak{m}(F\zeta, F\upsilon) &= \frac{1}{2}\left(\frac{\zeta}{3} + \frac{1}{7}\right) = \frac{1}{3}\left(\frac{1}{2}\left(\zeta + \frac{3}{7}\right)\right) \\ &\leq \frac{1}{3}\left(\frac{1}{2}\left(\zeta + \frac{\zeta}{3}\right) + \frac{1}{2}\left(1 + \frac{1}{7}\right)\right) \\ &\leq \frac{1}{3}\left(\mathfrak{m}(\zeta, F\zeta) + \mathfrak{m}(\upsilon, F\upsilon)\right). \end{aligned}$$

Case (iii) *Let $\zeta, \upsilon \geq 1$. Then,*

$$\begin{aligned}\mathfrak{m}(F\zeta, F\upsilon) &= \frac{1}{2}\left(\frac{1}{7}+\frac{1}{7}\right) = \frac{1}{3}\left(\frac{1}{2}\left(\frac{3}{7}+\frac{3}{7}\right)\right) \\ &\leq \frac{1}{3}\left(\frac{1}{2}\left(1+\frac{1}{7}\right)+\frac{1}{2}\left(1+\frac{1}{7}\right)\right) \\ &\leq \frac{1}{3}\left(\mathfrak{m}(\zeta, F\zeta)+\mathfrak{m}(\upsilon, F\upsilon)\right).\end{aligned}$$

Clearly, F satisfies Kannan contraction in $\mathfrak{m}$-MS. But one can easily verify that for usual metric $d(\zeta, \upsilon) = |\zeta - \upsilon|$, F does not satisfies the Kannan contraction. From figures 7.2.2 and 7.2.2, one can graphically visualize that there is no such $k \in [0, 1/2)$ for which $d(F\zeta, F\upsilon) \leq k(d(\zeta, F\zeta) + d(\upsilon, F\upsilon))$ $\forall \zeta, \upsilon \in [0, 2]$ holds.

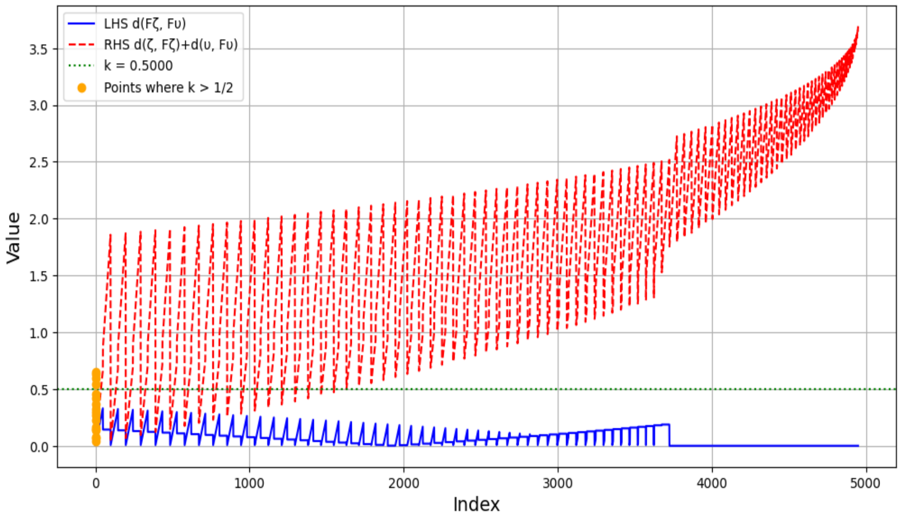

Figure 7.3: Kannan contraction w.r.t. the usual metric

7.3 Some Fixed Point Results in $\mathfrak{m}$-Metric Space

In this section, we gave the definition of interpolative $\mathfrak{m}$-contraction of Riech-Rus-Ćirić type and $(\phi - \psi)$ Wardowski contraction and presented some fixed point results using these contraction in $\mathfrak{m}$-MS.

Definition 7.3.1. *A self mapping $\Gamma : \Omega \to \Omega$ defined on a $\mathfrak{m}$-MS (Ω, ϖ) is c.t.b. a Riech-Rus-Ćirić type interpolative $\mathfrak{m}$-contraction if $\exists$ constants $\lambda \in [0, 1)$ and $a, b \in (0, 1)$ s.t.*

$$\varpi(\Gamma\varrho, \Gamma\vartheta) \leq \lambda\left(\varpi(\varrho, \vartheta)^{a}\varpi(\varrho, \Gamma\varrho)^{b}\varpi(\vartheta, \Gamma\vartheta)^{1-a-b}\right), \tag{7.3.1}$$

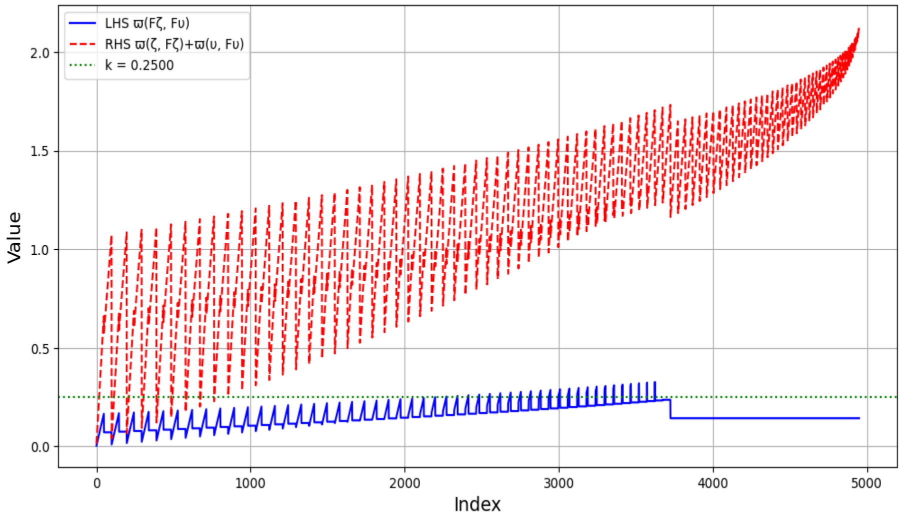

Figure 7.4: Kannan contraction w.r.t. the $\mathfrak{m}$-metric

$\forall \varrho, \vartheta \in \Omega$ *with* $\varrho \neq \Gamma\varrho, \vartheta \neq \Gamma\vartheta, \varpi(\varrho, \Gamma\varrho), \varpi(\vartheta, \Gamma\vartheta) \neq 0$.

Theorem 7.3.2. *Let* (Ω, ϖ) *be a complete* $\mathfrak{m}$*-MS, and* $\Gamma : \Omega \to \Omega$ *is a Riech-Rus-Ćirić type interpolative* $\mathfrak{m}$*-contraction, then* Γ *has a fixed point.*

Proof. For $\varrho_0 \in \Omega$, define the Picard sequence defined as $\varrho_\hbar = \Gamma\varrho_{\hbar-1}$, $\forall \hbar \in \mathbb{N}$. If $\exists$ a non negative integer $\hbar_0$ s.t. $\varrho_{\hbar_0} = \varrho_{\hbar_0-1}$, then $\Gamma\varrho_{\hbar_0-1} = \varrho_{\hbar_0} = \varrho_{\hbar_0-1}$ implies $\varrho_{\hbar_0-1}$ is a fixed point of the mapping Γ.
As, $\Gamma : \Omega \to \Omega$ is a Riech-Rus-Ćirić-type interpolative $\mathfrak{m}$-contraction, therefore, from Definition (7.3.1), we have

$$\begin{aligned}
\varpi(\varrho_{\hbar+1}, \varrho_\hbar) &= \varpi(\Gamma\varrho_\hbar, \Gamma\varrho_{\hbar-1}) \\
&\leq \lambda\Big((\varpi(\varrho_\hbar, \varrho_{\hbar-1}))^a(\varpi(\varrho_\hbar, \Gamma\varrho_\hbar))^b(\varpi(\varrho_{\hbar-1}, \Gamma\varrho_{\hbar-1}))^{1-a-b}\Big) \\
&= \lambda\Big((\varpi(\varrho_\hbar, \varrho_{\hbar-1}))^a(\varpi(\varrho_\hbar, \varrho_{\hbar+1}))^b(\varpi(\varrho_{\hbar-1}, \varrho_\hbar))^{1-a-b}\Big) \\
\Leftrightarrow (\varpi(\varrho_{\hbar+1}, \varrho_\hbar))^{1-b} &= \lambda\,(\varpi(\varrho_\hbar, \varrho_{\hbar-1}))^{1-b} \\
\Leftrightarrow \varpi(\varrho_{\hbar+1}, \varrho_\hbar) &= \lambda^{\frac{1}{1-b}}\varpi(\varrho_\hbar, \varrho_{\hbar-1}).
\end{aligned}$$

Since, $\lambda^{\frac{1}{1-b}} < 1$, therefore, from Lemma 4.2.15, we have

$$\lim_{\hbar\to\infty} \varpi(\varrho_{\hbar+1}, \varrho_\hbar) = 0, \tag{7.3.2}$$

$$\lim_{\ell,\hbar\to\infty} \varpi_{\varrho_\hbar, \varrho_\ell} = 0, \tag{7.3.3}$$

$$\lim_{\hbar\to\infty} \varpi(\varrho_\hbar, \varrho_\hbar) = 0 \tag{7.3.4}$$

and $\varrho_\hbar$ is an $\mathfrak{m}$-C_{seq}. Since, (Ω, ϖ) is complete, therefore $\exists \varrho \in \Omega$ s.t. $\varrho_\hbar \to \varrho$. Which implies

$$\lim_{\hbar\to\infty} \varpi(\varrho_\hbar, \varrho) - \varpi_{\varrho_\hbar \varrho} = 0 \text{ and } \lim_{\hbar\to\infty} M_{\varrho_\hbar,\varrho} - \varpi_{\varrho_\hbar,\varrho} = 0. \tag{7.3.5}$$

Also, using (7.3.4), we have

$$\lim_{\hbar\to\infty} \varpi_{\varrho_\hbar,\varrho} \leq \lim_{\hbar\to\infty} \varpi(\varrho_\hbar, \varrho_\hbar) = 0. \tag{7.3.6}$$

Using, (7.3.5), (7.3.6) and Lemma 1.2.10 , we have

$$\lim_{\hbar\to\infty} M_{\varrho_\hbar,\varrho} = 0, \ \lim_{\hbar\to\infty} \varpi(\varrho_\hbar, \varrho) = 0 \text{ and } \varpi(\varrho, \varrho) = 0. \tag{7.3.7}$$

and

$$\varpi_{\varrho,\Gamma\varrho} = \min\{\varpi(\varrho, \varrho), \varpi(\Gamma\varrho, \Gamma\varrho)\} \leq \varpi(\varrho, \varrho) = 0. \tag{7.3.8}$$

$$\begin{aligned}
\varpi(\Gamma\varrho_\hbar, \Gamma\varrho) &\leq \lambda\left((\varpi(\varrho_\hbar, \varrho))^a\,(\varpi(\varrho_\hbar, \Gamma\varrho_\hbar))^b\,(\varpi(\varrho, \Gamma\varrho))^{1-a-b}\right) \\
&= \lambda\left((\varpi(\varrho_\hbar, \varrho))^a\,(\varpi(\varrho_\hbar, \varrho_{\hbar+1}))^b\,(\varpi(\varrho, \Gamma\varrho))^{1-a-b}\right).
\end{aligned}$$

Taking limit as $\hbar \to \infty$ and using (7.3.2), we have

$$\lim_{\hbar\to\infty} \varpi(\Gamma\varrho_\hbar, \Gamma\varrho) = 0. \tag{7.3.9}$$

As a consequence of (7.3.9), we have

$$\lim_{\hbar\to\infty} \varpi_{\Gamma\varrho_\hbar,\Gamma\varrho} = \lim_{\hbar\to\infty} \min\{\varpi(\Gamma\varrho_\hbar, \Gamma\varrho_\hbar), \varpi(\Gamma\varrho, \Gamma\varrho)\} \leq \lim_{\hbar\to\infty} \varpi(\Gamma\varrho_\hbar, \Gamma\varrho) = 0. \tag{7.3.10}$$

or

$$\lim_{\hbar\to\infty} \varpi(\Gamma\varrho_\hbar, \Gamma\varrho) - \varpi_{\Gamma\varrho_\hbar,\Gamma\varrho} = 0,$$

which implies$\Gamma\varrho_\hbar$ converges to $\Gamma\varrho$.
Similarly, we have

$$\lim_{\hbar\to\infty} \varpi(\varrho_\hbar, \varrho_{\hbar+1}) = \lim_{\hbar\to\infty} \varpi(\varrho_\hbar, \Gamma\varrho_\hbar) = 0$$

and

$$\lim_{\hbar\to\infty} \varpi_{\varrho_\hbar,\Gamma\varrho_\hbar} \leq \lim_{\hbar\to\infty} \varpi(\varrho_\hbar, \Gamma\varrho_\hbar) = 0.$$

Now, $\varrho_\hbar$ and $\Gamma\varrho_\hbar$ converges to ϱ and $\Gamma\varrho$. From Lemma 1.2.8, we have

$$\lim_{\hbar\to\infty} \varpi(\varrho_\hbar, \Gamma\varrho_\hbar) - \varpi_{\varrho_\hbar,\Gamma\varrho_\hbar} = \varpi(\varrho, \Gamma\varrho) - \varpi_{\varrho,\Gamma\varrho} = 0. \tag{7.3.11}$$

Consider

$$\varpi(\Gamma\varrho, \Gamma\varrho) \leq \lambda\left((\varpi(\varrho, \varrho))^a (\varpi(\varrho, \Gamma\varrho))^b (\varpi(\varrho, \Gamma\varrho))^{1-a-b}\right) = 0. \tag{7.3.12}$$

Using (7.3.8), (7.3.11) and (7.3.12), we have

$$\varpi(\varrho, \varrho) = \varpi(\varrho, \Gamma\varrho) = \varpi_{\varrho,\Gamma\varrho}.$$

Hence, $\Gamma\varrho = \varrho$ i.e., ϱ is a fixed point of Γ.

$\square$

Definition 7.3.3. *A self mapping $\Gamma : \Omega \to \Omega$ defined on a non- empty $\mathfrak{m}$-MS (Ω, ϖ) is c.t.b. a $(\lambda - a - b - c)$ type interpolative $\mathfrak{m}$-contraction if $\exists$ constants $\lambda \in [0, 1)$ and $a, b, c \in (0, 1)$ with $a + b + c < 1$ s.t. Γ satisfies the following*

$$\varpi(\Gamma\varrho, \Gamma\vartheta) \leq \lambda\left((\varpi(\varrho, \vartheta))^a (\varpi(\varrho, \Gamma\varrho))^b (\varpi(\vartheta, \Gamma\vartheta))^c\right), \tag{7.3.13}$$

$\forall \varrho, \vartheta \in \Omega$ *with* $\varrho \neq \Gamma\varrho, \vartheta \neq \Gamma\vartheta, \varpi(\varrho, \Gamma\varrho), \varpi(\vartheta, \Gamma\vartheta) \neq 0$.

Theorem 7.3.4. *Let (Ω, ϖ) be a complete $\mathfrak{m}$-MS, and $\Gamma : \Omega \to \Omega$ be a $(\lambda - a - b - c)$ type interpolative $\mathfrak{m}$ contraction, then Γ has a fixed point.*

Proof. For $\varrho_0 \in \Omega$, define the Picard sequence defined as $\varrho_\hbar = \Gamma\varrho_{\hbar-1}\ \forall \hbar \in \mathbb{N}$. If $\exists$ a non negative integer $\hbar_0$ s.t. $\varrho_{\hbar_0} = \varrho_{\hbar_0-1}$, then $\Gamma\varrho_{\hbar_0-1} = \varrho_{\hbar_0} = \varrho_{\hbar_0-1}$ implies $\varrho_{\hbar_0-1}$ is a fixed point of the mapping Γ.
As, $\Gamma : \Omega \to Y$ is a of $(\lambda - a - b - c)$ type interpolative $\mathfrak{m}$-contraction, therefore, from Definition (7.3.3), we have

$$\begin{aligned}
\varpi(\varrho_{\hbar+1}, \varrho_\hbar) &= \varpi(\Gamma\varrho_\hbar, \Gamma\varrho_{\hbar-1}) \\
&\leq \lambda\left((\varpi(\varrho_\hbar, \varrho_{\hbar-1}))^a (\varpi(\varrho_\hbar, \Gamma\varrho_\hbar))^b (\varpi(\varrho_{\hbar-1}, \Gamma\varrho_{\hbar-1}))^c\right) \\
&= \lambda\left((\varpi(\varrho_\hbar, \varrho_{\hbar-1}))^a (\varpi(\varrho_\hbar, \varrho_{\hbar+1}))^b (\varpi(\varrho_{\hbar-1}, \varrho_\hbar))^c\right),
\end{aligned}$$

or

$$(\varpi(\varrho_{\hbar+1}, \varrho_\hbar))^{1-b} \leq \lambda\,(\varpi(\varrho_\hbar, \varrho_{\hbar-1}))^{a+c},$$

or

$$\begin{aligned}
\varpi(\varrho_{\hbar+1}, \varrho_\hbar) &= \lambda^{\frac{1}{1-b}} \varpi(\varrho_\hbar, \varrho_{\hbar-1})^{\frac{a+c}{1-b}} \\
&\leq \lambda^{\frac{1}{1-b}} \varpi(\varrho_\hbar, \varrho_{\hbar-1}).
\end{aligned}$$

Since $\lambda^{\frac{1}{1-b}} < 1$, therefore by Lemma 1.2.10, we have

$$\lim_{\hbar\to\infty} \varpi(\varrho_{\hbar+1}, \varrho_{\hbar}) = 0,$$
$$\lim_{\ell,\hbar\to\infty} \varpi_{\varrho_{\hbar},\varrho_{\ell}} = 0,$$
$$\lim_{\hbar\to\infty} \varpi(\varrho_{\hbar}, \varrho_{\hbar}) = 0$$

and $\varrho_{\hbar}$ is an $\mathfrak{m}$-C_{seq}.
The remaining proof can be done on the similar lines of Theorem 7.3.2. $\square$

Example 7.3.5. *Consider a set* $\Omega = \{0, 1, 2, 3\}$ *with* $\varpi(\varrho, \vartheta) = |\varrho - \vartheta| + 1$. *Then, clearly* (Ω, ϖ) *be an* $\mathfrak{m}$*-MS. A self mapping* Γ *is defined on* Ω *as* $\Gamma(0) = 0, \Gamma(1) = 1, \Gamma(2) = \Gamma(3) = 1$. *Then* Γ *satisfies all the conditions of Theorem 7.3.2 with* $\lambda = \frac{3}{4}, a = \frac{1}{3}, b = \frac{1}{3}$.
Case(i) *For* $\varrho = \vartheta = 2$, *we have*

$$\varpi(\Gamma\varrho, \Gamma\vartheta) = \varpi(\Gamma(2), \Gamma(2)) = \varpi(1, 1) = 1.$$

$$\varpi(\varrho, \vartheta) = \varpi(2, 2) = 1; \varpi(\varrho, \Gamma\varrho) = \varpi(2, \Gamma(2)) = \varpi(2, 1) = 2 = \varpi(\vartheta, \Gamma\vartheta).$$

Then,

$$\lambda(\varpi(\varrho, \vartheta))^a(\varpi(\varrho, \Gamma\varrho))^b(\varpi(\vartheta, \Gamma\vartheta))^{1-a-b} = 1.19 \geq 1 = \varpi(\Gamma\varrho, \Gamma\vartheta).$$

Case(ii) *For* $\varrho = \vartheta = 3$, *we have*

$$\varpi(\Gamma\varrho, \Gamma\vartheta) = \varpi(\Gamma(3), \Gamma(3)) = \varpi(1, 1) = 1.$$

$$\varpi(\varrho, \vartheta) = \varpi(3, 3) = 1; \varpi(\varrho, \Gamma\varrho) = \varpi(3, \Gamma(3)) = \varpi(3, 1) = 3 = \varpi(\vartheta, \Gamma\vartheta)$$

Then,

$$\lambda(\varpi(\varrho, \vartheta))^a(\varpi(\varrho, \Gamma\varrho))^b(\varpi(\vartheta, \Gamma\vartheta))^{1-a-b} = 1.56 \geq 1 = \varpi(\Gamma\varrho, \Gamma\vartheta)$$

Case(iii) *For* $\varrho = 2$ *and* $\vartheta = 3$, *we have*

$$\varpi(\Gamma\varrho, \Gamma\vartheta) = \varpi(\Gamma(2), \Gamma(3)) = \varpi(1, 1) = 1.$$

$$\varpi(\varrho, \vartheta) = \varpi(2, 3) = 2; \varpi(\varrho, \Gamma\varrho) = \varpi(2, \Gamma(2)) = \varpi(2, 1) = 2; \varpi(\vartheta, \Gamma\vartheta) = \varpi(3, \Gamma(3)) = \varpi(3, 1) = 3.$$

Then,

$$\lambda(\varpi(\varrho, \vartheta))^a(\varpi(\varrho, \Gamma\varrho))^b(\varpi(\vartheta, \Gamma\vartheta))^{1-a-b} = 1.36 \geq 1 = \varpi(\Gamma\varrho, \Gamma\vartheta)$$

Therefore, Γ *satisfies the hypotheses of Theorem 7.3.2. Hence,* Γ *has a fixed point. Indeed,* $\varrho = 1, 2$ *are two fixed points of* Γ.

Definition 7.3.6. *Let (Ω, ϖ) be an $\mathfrak{m}$-MS. A self mapping $\Gamma : \Omega \to \Omega$ is c.t.b. $(\phi - \psi)$-Wardowski type contraction, if*

$$\varpi(\Gamma\varrho, \Gamma\vartheta) > 0 \Rightarrow \phi(\varpi(\Gamma\varrho, \Gamma\vartheta)) \leq \psi\left(\phi\left(\mathcal{M}_\Gamma(\varrho, \vartheta)\right)\right), \tag{7.3.14}$$

where

$$\mathcal{M}_\Gamma(\varrho, \vartheta) = \max\{\varpi(\varrho, \vartheta), \varpi(\varrho, \Gamma\varrho), \varpi(\vartheta, \Gamma\vartheta)\},$$

$\forall \varrho, \vartheta \in \Omega$.

Theorem 7.3.7. *Let (Ω, ϖ) be a complete $\mathfrak{m}$-MS, and $\Gamma : \Omega \to \Omega$ be a $(\phi - \psi)$-Wadowski contraction with continuous function ψ and $\phi(0) = 0$, then Γ possesses exactly one fixed point.*

Proof. For $\varrho_0 \in \Omega$, define the Picard sequence defined as $\varrho_\hbar = \Gamma\varrho_{\hbar-1}$ $\forall \hbar \in \mathbb{N}$. If $\exists$ a non negative integer $\hbar_0$ s.t. $\varrho_{\hbar_0} = \varrho_{\hbar_0-1}$, then $\Gamma\varrho_{\hbar_0-1} = \varrho_{\hbar_0} = \varrho_{\hbar_0-1}$ implies $\varrho_{\hbar_0-1}$ is a fixed point of the mapping Γ.
As, $\Gamma : \Omega \to \Omega$ is $(\phi - \psi)$-Wardowski contraction, therefore, we have

$$\begin{aligned}
\phi(\varpi(\varrho_{\hbar+1}, \varrho_\hbar)) &= \phi(\varpi(\Gamma\varrho_\hbar, \Gamma\varrho_{\hbar-1})) \\
&\leq \psi\left(\phi\left(\mathcal{M}_\Gamma(\varrho_\hbar, \varrho_{\hbar-1})\right)\right),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{M}_\Gamma(\varrho_\hbar, \varrho_{\hbar-1}) &= \max\{\varpi(\varrho_\hbar, \varrho_{\hbar-1}), \varpi(\varrho_\hbar, \varrho_{\hbar+1}), \varpi(\varrho_{\hbar-1}, \varrho_\hbar)\} \\
&= \max\{\varpi(\varrho_\hbar, \varrho_{\hbar-1}), \varpi(\varrho_\hbar, \varrho_{\hbar+1})\}.
\end{aligned}$$

If possible $\mathcal{M}_\Gamma(\varrho_\hbar, \varrho_{\hbar-1}) = \varpi(\varrho_\hbar, \varrho_{\hbar+1})$. Then

$$\phi(\varpi(\varrho_{\hbar+1}, \varrho_\hbar)) \leq \psi(\phi(\varpi(\varrho_\hbar, \varrho_{\hbar+1}))) < \phi(\varpi(\varrho_\hbar, \varrho_{\hbar+1})),$$

a contradiction. Hence, $\mathcal{M}_\Gamma(\varrho_\hbar, \varrho_{\hbar-1}) = \varpi(\varrho_\hbar, \varrho_{\hbar-1})$.
Consider

$$\phi(\varpi(\varrho_{\hbar+1}, \varrho_\hbar)) \leq \psi(\phi(\varpi(\varrho_\hbar, \varrho_{\hbar-1}))) < \phi(\varpi(\varrho_\hbar, \varrho_{\hbar-1})),$$

implies $\varpi(\varrho_{\hbar+1}, \varrho_{\hbar})$ is a decreasing sequence.
Also,

$$\begin{aligned}
\phi\left(\varpi(\varrho_{\hbar+1}, \varrho_{\hbar})\right) &\leq \psi\Big(\phi\left(\varpi(\varrho_{\hbar}, \varrho_{\hbar-1})\right)\Big) \\
&\leq \psi^2\Big(\phi\left(\varpi(\varrho_{\hbar-1}, \varrho_{\hbar-2})\right)\Big) \\
&\dots \\
&\leq \psi^{\hbar}\Big(\phi\left(\varpi(\varrho_1, \varrho_0)\right)\Big).
\end{aligned}$$

Taking the limit as $\hbar$ tends to ∞, we have

$$\lim_{\hbar\to\infty} \varpi(\varrho_{\hbar+1}, \varrho_{\hbar}) = 0. \tag{7.3.15}$$

Also,

$$\varpi_{\varrho_{\hbar+1},\varrho_{\hbar}} = \min\{\varpi(\varrho_{\hbar+1}, \varrho_{\hbar+1}), \varpi(\varrho_{\hbar}, \varrho_{\hbar})\} \tag{7.3.16}$$

$$M_{\varrho_{\hbar+1},\varrho_{\hbar}} = \max\{\varpi(\varrho_{\hbar+1}, \varrho_{\hbar+1}), \varpi(\varrho_{\hbar}, \varrho_{\hbar})\}. \tag{7.3.17}$$

Taking the limit as $\hbar$ tends to ∞, we have

$$\lim_{\hbar\to\infty} \varpi_{\varrho_{\hbar+1},\varrho_{\hbar}} = 0,\ \lim_{\hbar\to\infty} \varpi(\varrho_{\hbar}, \varrho_{\hbar}) = 0 \text{ and } \lim_{\hbar\to\infty} M_{\varrho_{\hbar+1},\varrho_{\hbar}} = 0. \tag{7.3.18}$$

If possible $\{\varrho_{\hbar}\}$ is not $\mathfrak{m}$-Cauchy. Then, for $\epsilon > 0$ $\exists$ two subsequences $\varrho_{\hbar_\kappa} \neq \varrho_{\ell_\kappa}$ s.t.

$$d(\varrho_{\hbar_\kappa}, \varrho_{\ell_\kappa}) > \epsilon, \tag{7.3.19}$$

and

$$d(\varrho_{\hbar_{\kappa-1}}, \varrho_{\ell_\kappa}) \leq \epsilon, \tag{7.3.20}$$

where, $d(\varrho, \vartheta) = \varpi(\varrho, \vartheta) - 2\varpi_{\varrho,\vartheta} + M_{\varrho,\vartheta}$ $\forall \varrho, \vartheta \in \Omega$ is a metric.
Using triangle inequality and (7.3.19), we have

$$\begin{aligned}
\epsilon &< d(\varrho_{\hbar_\kappa}, \varrho_{\ell_\kappa}) \\
&\leq d(\varrho_{\hbar_\kappa}, \varrho_{\hbar_{\kappa-1}}) + d(\varrho_{\hbar_{\kappa-1}}, \varrho_{\ell_\kappa}).
\end{aligned}$$

Taking the limit as κ tends to ∞ and using (7.3.15), we have

$$\lim_{\kappa\to\infty} d(\varrho_{\hbar_\kappa}, \varrho_{\ell_\kappa}) = \epsilon. \tag{7.3.21}$$

Also, using triangle inequality, we have

$$d(\varrho_{\hbar_\kappa}, \varrho_{\ell_\kappa}) \leq d(\varrho_{\hbar_\kappa}, \varrho_{\hbar_{\kappa-1}}) + d(\varrho_{\hbar_{\kappa-1}}, \varrho_{\ell_{\kappa-1}}) + d(\varrho_{\ell_{\kappa-1}}, \varrho_{\ell_\kappa}),$$

and

$$d(\varrho_{\hbar_{\kappa-1}}, \varrho_{\ell_{\kappa-1}}) \leq d(\varrho_{\hbar_{\kappa-1}}, \varrho_{\hbar_\kappa}) + d(\varrho_{\hbar_\kappa}, \varrho_{\ell_\kappa}) + d(\varrho_{\ell_\kappa}, \varrho_{\ell_\kappa}).$$

Taking the limit as κ tends to ∞ and using (7.3.15) and (7.3.21), we have

$$\lim_{\kappa\to\infty} d(\varrho_{\hbar_{\kappa-1}}, \varrho_{\ell_{\kappa-1}}) = \epsilon. \tag{7.3.22}$$

Using (7.3.18) in (7.3.21) and (7.3.22), we have

$$\lim_{\kappa\to\infty} \varpi(\varrho_{\hbar_\kappa}, \varrho_{\ell_\kappa}) = \epsilon \text{ and } \lim_{\kappa\to\infty} \varpi(\varrho_{\hbar_{\kappa-1}}, \varrho_{\ell_{\kappa-1}}) = \epsilon. \tag{7.3.23}$$

Also, by continuity of ϕ, (7.3.14) and (7.3.23), we have

$$\begin{aligned}
\phi(\epsilon) &= \lim_{\kappa\to\infty} \phi\left(\varpi(\varrho_{\hbar_\kappa}, \varrho_{\ell_\kappa})\right) \\
&= \lim_{\kappa\to\infty} \phi\left(\varpi(\Gamma\varrho_{\hbar_{\kappa-1}}, \Gamma\varrho_{\ell_{\kappa-1}})\right) \\
&\leq \lim_{\kappa\to\infty} \psi\left(\phi\left(\mathcal{M}_\Gamma(\varrho_{\hbar_{\kappa-1}}, \varrho_{\ell_{\kappa-1}})\right)\right) \\
&\leq \lim_{\kappa\to\infty} \psi\left(\phi\left(\max\left\{\varpi(\varrho_{\hbar_{\kappa-1}}, \varrho_{\ell_{\kappa-1}}), \varpi(\varrho_{\hbar_{\kappa-1}}, \varrho_{\hbar_\kappa}), \varpi(\varrho_{\ell_{\kappa-1}}, \varrho_{\ell_\kappa})\right\}\right)\right) \\
&< \phi(\epsilon),
\end{aligned}$$

which contradicts itself. Therefore, $\{\varrho_\hbar\}$ is a $\mathfrak{m}$-C_{seq}. As, Ω is an $\mathfrak{m}$-complete MS, for some $\varrho \in \Omega$, we have

$$\lim_{\hbar\to\infty} \varpi(\varrho_\hbar, \varrho) - 2\varpi_{\varrho_\hbar,\varrho} + M_{\varrho_\hbar,\varrho} = 0. \tag{7.3.24}$$

Also, from (7.3.18), we have

$$\lim_{\hbar\to\infty} \varpi_{\varrho_\hbar,\varrho} = \lim_{\hbar\to\infty} \min\{\varpi(\varrho_\hbar, \varrho_\hbar), \varpi(\varrho, \varrho)\} \leq \lim_{\hbar\to\infty} \varpi(\varrho_\hbar, \varrho_\hbar) = 0. \tag{7.3.25}$$

Using (7.3.25) and Lemma 1.2.9, we have

$$\lim_{\hbar\to\infty} M_{\varrho_\hbar,\varrho} = 0, \varpi(\varrho, \varrho) = 0 \text{ and } \varpi_{\varrho,\Gamma\varrho} = 0. \tag{7.3.26}$$

We claim that $\varpi(\varrho, \Gamma\varrho) = 0$. If feasible, let $\varpi(\varrho, \Gamma\varrho) > 0$, then consider the following

$$\begin{aligned}
\phi(\varpi(\varrho, \Gamma\varrho) - \varpi_{\varrho,\Gamma\varrho}) &\leq \phi\left(\varpi(\varrho, \varrho_\hbar) - \varpi_{\varrho_\hbar,\varrho} + \varpi(\varrho_\hbar, \Gamma\varrho) - \varpi_{\varrho_\hbar,\Gamma\varrho}\right) \\
&= \limsup_{\hbar\to\infty} \phi\left(\varpi(\varrho, \varrho_\hbar) - \varpi_{\varrho_\hbar,\varrho} + \varpi(\varrho_\hbar, \Gamma\varrho) - \varpi_{\varrho_\hbar,\Gamma\varrho}\right) \\
&= \limsup_{\hbar\to\infty} \phi\left(\varpi(\varrho_\hbar, \Gamma\varrho) - \varpi_{\varrho_\hbar,\Gamma\varrho}\right) \\
&\leq \limsup_{\hbar\to\infty} \phi\left(\varpi(\varrho_\hbar, \Gamma\varrho)\right) = \phi\left(\varpi(\Gamma\varrho_\hbar, \Gamma\varrho)\right) \\
&\leq \limsup_{\hbar\to\infty} \psi\left(\phi(\mathcal{M}_\Gamma(\varrho_\hbar, \varrho))\right) \\
&\leq \limsup_{\hbar\to\infty} \psi\left(\phi\left(\max\{\varpi(\varrho_\hbar, \varrho), \varpi(\varrho_\hbar, \Gamma\varrho_\hbar), \varpi(\varrho, \Gamma\varrho)\}\right)\right) \\
&\leq \psi\left(\phi\left(\varpi(\varrho, \Gamma\varrho)\right)\right) \\
&< \phi\left(\varpi(\varrho, \Gamma\varrho)\right),
\end{aligned}$$

a contradiction. Hence, $\varpi(\varrho, \Gamma\varrho) = 0$.
Next, we claim that $\varpi(\Gamma\varrho, \Gamma\varrho) = 0$. If possible, let $\varpi(\varrho, \Gamma\varrho) > 0$, then

$$\begin{aligned}
\phi(\varpi(\Gamma\varrho, \Gamma\varrho)) &\leq \psi\left(\phi\left(\mathcal{M}_\Gamma(\varrho, \varrho)\right)\right) \\
&\leq \psi\left(\phi\left(\max\{\varpi(\varrho, \varrho), \varpi(\varrho, \Gamma\varrho)\}\right)\right) \\
&\leq \psi\left(\phi(0)\right) < \phi(0),
\end{aligned}$$

a contradiction. Hence, $\varpi(\Gamma\varrho, \Gamma\varrho) = 0$.
As we have already shown that $\varpi(\varrho, \varrho) = \varpi(\varrho, \Gamma\varrho) = \varpi(\Gamma\varrho, \Gamma\varrho) = 0$. Therefore, $\varrho = \Gamma\varrho$.
Uniqueness: If feasible, let $\vartheta \neq \varrho$ be another fixed point with $\varpi(\varrho, \varrho) = 0 = \varpi(\vartheta, \vartheta)$. If feasible, let $\varpi(\varrho, \vartheta) > 0$, then

$$\begin{aligned}
\phi(\varpi(\Gamma\varrho, \Gamma\vartheta)) &\leq \psi\left(\phi\left(\mathcal{M}_\Gamma(\varrho, \vartheta)\right)\right) \\
&\leq \psi\left(\phi\left(\max\{\varpi(\varrho, \vartheta), \varpi(\varrho, \Gamma\varrho), \varpi(\vartheta, \Gamma\vartheta)\}\right)\right) \\
&\leq \psi\left(\phi(0)\right) < \phi(0),
\end{aligned}$$

a contradiction. Therefore, $\varpi(\varrho, \vartheta) = 0 = \varpi(\varrho, \varrho) = \varpi(\vartheta, \vartheta)$, which implies $\varrho = \vartheta$. □

Corollary 7.3.8. *Let $\Gamma : \Omega \to \Omega$ be a mapping defined on complete $\mathfrak{m}$-MS (Ω, ϖ). If $\exists\ \lambda \in (0, 1)$ s.t.*

$$\varpi(\Gamma\varrho, \Gamma\vartheta) \leq \lambda\varpi(\varrho, \vartheta)\ \forall \varrho, \vartheta \in \Omega.$$

Then, Γ *possesses exactly one fixed point.*

Proof. Using $\phi(t) = t$ and $\psi(t) = kt$, for $k \in (0, 1)$, we have

$$\varpi(\Gamma\varrho, \Gamma\vartheta) \leq \lambda(\varpi(\varrho, \vartheta) \leq \lambda \max\{\varpi(\varrho, \vartheta), \varpi(\varrho, \Gamma\varrho), \varpi(\vartheta, \Gamma\vartheta)\} = \lambda\mathcal{M}_\Gamma(\varrho, \vartheta).$$

Thus, the result holds as a consequence of Theorem 7.3.7. □

Corollary 7.3.9. *Let* $\Gamma : \Omega \to \Omega$ *be a mapping defined on complete* $\mathfrak{m}$-*MS* (Ω, ϖ). *If* $\exists\lambda \in (0, \frac{1}{2})$ *s.t.*

$$\varpi(\Gamma\varrho, \Gamma\vartheta) \leq \lambda\left(\varpi(\varrho, \Gamma\varrho) + \varpi(\vartheta, \Gamma\vartheta)\right) \ \forall\varrho, \vartheta \in \Omega.$$

Then, Γ *possesses exactly one fixed point.*

Proof. Using $\phi(t) = t$ and $\psi(t) = 2\lambda t$ for $\lambda \in (0, \frac{1}{2})$, we have

$$\varpi(\Gamma\varrho, \Gamma\vartheta) \leq \lambda(\varpi(\varrho, \Gamma\varrho) + \varpi(\vartheta, \Gamma\vartheta)) \leq 2\lambda \max\{\varpi(\varrho, \vartheta), \varpi(\varrho, \Gamma\varrho), \varpi(\vartheta, \Gamma\vartheta)\} = 2\lambda M_\Gamma(\varrho, \vartheta).$$

Thus, the result holds as a consequence of Theorem 7.3.7. □

Corollary 7.3.10. *Let* $\Gamma : \Omega \to \Omega$ *be a mapping defined on complete* $\mathfrak{m}$-*MS* (Ω, ϖ). *If* $\exists a_i \geq 0$ *with* $a_0 + a_1 + a_2 \in (0, 1)$ *s.t.*

$$\varpi(\Gamma\varrho, \Gamma\vartheta) \leq a_o\varpi(\varrho, \vartheta) + a_1\varpi(\varrho, \Gamma\varrho) + a_2\varpi(\vartheta, \Gamma\vartheta) \ \forall\varrho, \vartheta \in \Omega.$$

Then, Γ *possesses exactly one fixed point.*

Proof. Using $\phi(t) = t$ and $\psi(t) = \lambda t$ for $\lambda = a_0 + a_1 + a_2 \in [0, 1)$, we have

$$\begin{aligned}
\varpi(\Gamma\varrho, \Gamma\vartheta) &\leq a_o\varpi(\varrho, \vartheta) + a_1\varpi(\varrho, \Gamma\varrho) + a_2\varpi(\vartheta, \Gamma\vartheta) \\
&\leq (a_0 + a_1 + a_2) \max\{\varpi(\varrho, \vartheta), \varpi(\varrho, \Gamma\varrho), \varpi(\vartheta, \Gamma\vartheta)\} \\
&= \lambda\mathcal{M}_\Gamma(\varrho, \vartheta).
\end{aligned}$$

Thus, the result holds as a consequence of Theorem 7.3.7. □

Corollary 7.3.11. *Let* $\Gamma : \Omega \to \Omega$ *be a mapping defined on complete* $\mathfrak{m}$-*MS* (Ω, ϖ). *If* $\exists F \in \boldsymbol{F}$ *and* $\tau > 0$ *s.t.*

$$\varpi(\Gamma\varrho, \Gamma\vartheta) > 0 \Rightarrow \tau + F(\varpi(\Gamma\varrho, \Gamma\vartheta)) \leq F(\varpi(\varrho, \vartheta)) \ \forall\varrho, \vartheta \in \Omega.$$

Then, Γ *possesses exactly one fixed point.*

Proof. Using $\phi(z) = e^{F(z)}$ and $\psi(z) = e^{-\tau}z$ for $\tau > 0$ and $F \in \mathbf{F}$, we have

$$\begin{aligned} \tau + F(\varpi(\Gamma\varrho, \Gamma\vartheta)) &\leq F(\varpi(\varrho, \vartheta)) \\ \Leftrightarrow e^{F(\varpi(\Gamma\varrho,\Gamma\vartheta))} &\leq e^{-\tau}e^{F(\varpi(\varrho,\vartheta))} \\ &\leq e^{-\tau}e^{F\max\{\varpi(\varrho,\vartheta),\varpi(\varrho,\Gamma\varrho),\varpi(\vartheta,\Gamma\vartheta)\}}. \end{aligned}$$

Thus, the result holds as a consequence of Theorem 7.3.7. □

Corollary 7.3.12. *Let $\Gamma : \Omega \to \Omega$ be a mapping defined on complete $\mathfrak{m}$-MS (Ω, ϖ). If $\exists F \in \boldsymbol{F}$ and $\tau > 0$ s.t.*

$$\varpi(\Gamma\varrho, \Gamma\vartheta) > 0 \Rightarrow \tau + F(\varpi(\Gamma\varrho, \Gamma\vartheta)) \leq F(\mathcal{M}_\Gamma(\varrho, \vartheta)) \;\forall \varrho, \vartheta \in \Omega,$$

where $\mathcal{M}_\Gamma(\varrho, \vartheta) = \max\{\varpi(\varrho, \vartheta), \varpi(\varrho, \Gamma\varrho), \varpi(\vartheta, \Gamma\vartheta)\}$. Then, Γ possesses exactly one fixed point.

Proof. Using $\phi(z) = e^{F(z)}$ and $\psi(z) = e^{-\tau}z$ for $\tau > 0$ and $f \in \mathbf{F}$ in Theorem 7.3.7, we have the required result. □

Example 7.3.13. *Consider $\Omega = [0, \infty)$ along with $\varpi(\varrho, \vartheta) = \frac{\varrho+\vartheta}{2}$, $\forall\ \varrho, \vartheta \in \Omega$. Here, (Ω, ϖ) is a complete $\mathfrak{m}$-MS. Consider the mapping $\Gamma : \Omega \to \Omega$ s.t.*

$$\Gamma(\varrho) = \begin{cases} \dfrac{\varrho}{7}, & \text{if } \varrho \in [0, 4) \\ \dfrac{\varrho}{\varrho + 3}, & \text{if } \varrho \geq 4. \end{cases}$$

Now, we will prove that Γ satisfies Theorem 7.3.7 with $\phi(t) = t$ and $\psi(t) = kt$ where $k \in (0, 1)$.

Case (i) For $\varrho, \vartheta < 4$, we have

$$\begin{aligned} \varpi(\Gamma\varrho, \Gamma\vartheta) &= \frac{1}{2}\left(\frac{\varrho}{7} + \frac{\vartheta}{7}\right) \\ &\leq \frac{1}{7}\left(\frac{\varrho + \vartheta}{2}\right) + \frac{1}{3}\left(\frac{1}{2}(\varrho + \frac{\varrho}{7})\right) + \frac{1}{4}\left(\frac{1}{2}(\varrho + \frac{\varrho}{7})\right) \\ &\leq \left(\frac{1}{7} + \frac{1}{3} + \frac{1}{4}\right)\max\left\{\frac{\varrho + \vartheta}{2}, \frac{1}{2}(\varrho + \frac{\varrho}{7}), \frac{1}{2}(\vartheta + \frac{\vartheta}{7})\right\} \\ &\leq k\mathcal{M}_\Gamma(\varrho, \vartheta), \end{aligned}$$

for $k = \frac{1}{7} + \frac{1}{3} + \frac{1}{4} < 1$.
Case (ii) For $\varrho, \vartheta \geq 4$, we have

$$\begin{aligned}
\varpi(\Gamma\varrho, \Gamma\vartheta) &= \frac{1}{2}\left(\frac{\varrho}{\varrho+3} + \frac{\vartheta}{\vartheta+3}\right) \\
&\leq \frac{1}{2}\left(\frac{\varrho}{7} + \frac{\vartheta}{7}\right) = \frac{1}{7}\left(\frac{\varrho}{2} + \frac{\vartheta}{2}\right) \\
&\leq \frac{1}{7}\left(\frac{\varrho}{2} + \frac{\vartheta}{2}\right) + \frac{1}{3}\left(\frac{1}{2}(\varrho + \frac{\varrho}{\varrho+3})\right) + \frac{1}{4}\left(\frac{1}{2}(\vartheta + \frac{\vartheta}{\vartheta+3})\right) \\
&\leq \left(\frac{1}{7} + \frac{1}{3} + \frac{1}{4}\right) \max\left\{\frac{\varrho+\vartheta}{2}, \frac{1}{2}(\varrho + \frac{\varrho}{\varrho+3}), \frac{1}{2}(\vartheta + \frac{\vartheta}{\vartheta+3})\right\} \\
&\leq k\mathcal{M}_\Gamma(\varrho, \vartheta),
\end{aligned}$$

for $k = \frac{1}{7} + \frac{1}{3} + \frac{1}{4} < 1$.
Case (iii) For $\varrho < 4$ and $\vartheta \geq 4$, we have

$$\begin{aligned}
\varpi(\Gamma\varrho, \Gamma\vartheta) &= \frac{1}{2}\left(\frac{\varrho}{7} + \frac{\vartheta}{\vartheta+4}\right) \leq \frac{1}{2}\left(\frac{\varrho}{7} + \frac{\vartheta}{7}\right) \\
&\leq \frac{1}{7}\left(\frac{\varrho}{2} + \frac{\vartheta}{2}\right) + \frac{1}{3}\left(\frac{1}{2}(\varrho + \frac{\varrho}{\varrho+3})\right) + \frac{1}{4}\left(\frac{1}{2}(\vartheta + \frac{\vartheta}{\vartheta+3})\right) \\
&\leq \left(\frac{1}{7} + \frac{1}{3} + \frac{1}{4}\right) \max\left\{\frac{\varrho+\vartheta}{2}, \frac{1}{2}(\varrho + \frac{\varrho}{\varrho+3}), \frac{1}{2}(\vartheta + \frac{\vartheta}{\vartheta+3})\right\} \\
&\leq k\mathcal{M}_\Gamma(\varrho, \vartheta),
\end{aligned}$$

for $k = \frac{1}{7} + \frac{1}{3} + \frac{1}{4} < 1$.
Therefore, Γ satisfies Theorem 7.3.7. Hence, Γ possesses exactly one fixed point. Indeed, $\varrho = 0$ is the unique fixed point of Γ.

7.4 Some Common Fixed Point Results in $\mathfrak{m}$-Metric Space

In this section, we present various fixed point results within the framework of $\mathfrak{m}$-MS, employing different generalized contraction conditions. From these proven results, we can derive several additional fixed point theorems in the existing literature.

Definition 7.4.1. *Let (Ω, ϖ) be an $\mathfrak{m}$-MS. A self mapping Γ_1 is c.t.b.* ***self distance contraction*** *on Ω if $\exists k_0 \in [0, 1)$ s.t.*

$$\varpi(\Gamma_1\varrho, \Gamma_1\varrho) \leq k_0\left(\varpi(\varrho, \varrho)\right), \tag{7.4.1}$$

$\forall \varrho \in \Omega$.

Theorem 7.4.2. *Let $\Gamma_1, \Gamma_2 : \Omega \to \Omega$ be mappings defined on complete $\mathfrak{m}$-MS (Ω, ϖ) and Y be a complete subspace of Ω s.t. $\Gamma_1(\Omega) \cup \Gamma_2(\Omega) \subseteq Y$. Suppose Γ_1, Γ_2 satisfy*

(i) $\exists \alpha, \beta, \gamma \geq 0$ *with* $\alpha + \beta + \gamma < 1$ *s.t.*

$$\varpi(\Gamma_1\varrho, \Gamma_2\vartheta) \leq \alpha\varpi(\varrho, \vartheta) + \beta\varpi(\varrho, \Gamma_1\varrho) + \gamma\varpi(\vartheta, \Gamma_2\vartheta), \tag{7.4.2}$$

$\forall \varrho, \vartheta \in \Omega$;

(ii) Γ_1, Γ_2 *are self distance contraction mappings.*

Then, the pair (Γ_1, Γ_2) possesses exactly one common fixed point.

Proof. Let $\varrho_0 \in \Omega$ be any point. Consider the sequence $\{\varrho_\hbar\}$ generated by mappings Γ_1, Γ_2 with initial point ϱ_0 as

$$\varrho_1 = \Gamma_1\varrho_0,\ \varrho_2 = \Gamma_2\varrho_1,\ \ldots\ \varrho_{2\hbar+1} = \Gamma_1\varrho_\hbar, \varrho_{2\hbar+2} = \Gamma_2\varrho_{2\hbar+1}.$$

Now, consider

$$\begin{aligned} \varpi(\varrho_{2\hbar+1,\varrho_{2\hbar}}) &= \varpi(\Gamma_1\varrho_{2\hbar}, \Gamma_2\varrho_{2\hbar-1}) \\ &\leq \alpha\varpi(\varrho_{2\hbar}, \varrho_{2\hbar-1}) + \beta\varpi(\varrho_{2\hbar}, \varrho_{2\hbar+1}) + \gamma\varpi(\varrho_{2\hbar-1}, \varrho_{2\hbar}), \\ \Leftrightarrow (1-\beta)\varpi(\varrho_{2\hbar+1}, \varrho_{2\hbar}) &\leq (\alpha+\gamma)\varpi(\varrho_{2\hbar}, \varrho_{2\hbar-1}), \\ \Leftrightarrow \varpi(\varrho_{2\hbar+1}, \varrho_{2\hbar}) &\leq \frac{\alpha+\gamma}{1-\beta}\varpi(\varrho_{2\hbar}, \varrho_{2\hbar-1}), \end{aligned}$$

or

$$\varpi(\varrho_{2\hbar+1}, \varrho_{2\hbar}) \leq \kappa\varpi(\varrho_{2\hbar}, \varrho_{2\hbar-1}),$$

where $\kappa = \frac{\alpha+\gamma}{1-\beta} < 1$. Without loss of generality, we have

$$\varpi(\varrho_{\hbar+1}, \varrho_\hbar) \leq \kappa\varpi(\varrho_\hbar, \varrho_{\hbar-1}).$$

Therefore, by Lemma 1.2.10, $\{\varrho_\hbar\}$ is $\mathfrak{m}$-Cauchy. Further, as Y is $\mathfrak{m}$-complete. Therefore, $\exists \varrho \in Y$ s.t.

$$\lim_{\hbar\to\infty} \varpi(\varrho_\hbar, \varrho) - \varpi_{\varrho_\hbar\varrho} = 0 = \lim_{\hbar\to\infty} M_{\varrho_\hbar\varrho} - \varpi_{\varrho_\hbar\varrho}. \tag{7.4.3}$$

Using Lemma 1.2.10, we have

$$\lim_{\hbar\to\infty} \varpi_{\varrho_\hbar\varrho} = \lim_{\hbar\to\infty} \min\{\varpi(\varrho_\hbar,\varrho_\hbar),\varpi(\varrho,\varrho)\} \le \lim_{\hbar\to\infty} \varpi(\varrho_\hbar,\varrho_\hbar) = 0. \tag{7.4.4}$$

Using (7.4.3), (7.4.4) and Lemma 1.2.9 , we have

$$\lim_{\hbar\to\infty} \varpi(\varrho_\hbar,\varrho) = 0,\ \lim_{\hbar\to\infty} M_{\varrho_\hbar\varrho} = 0 \text{ and } \varpi(\varrho,\varrho) = 0. \tag{7.4.5}$$

Also,

$$\varpi_{\varrho\Gamma_1\varrho} = \min\{\varpi(\varrho,\varrho),\varpi(\Gamma_1\varrho,\Gamma_1\varrho)\} = 0. \tag{7.4.6}$$

Using (7.4.2), (7.4.6) and the triangle inequality, we have

$$\begin{aligned}
0 \le \varpi(\varrho,\Gamma_1\varrho) &= \varpi(\varrho,\Gamma_1\varrho) - \varpi_{\varrho\Gamma_1\varrho} \\
&\le \varpi(\varrho,\varrho_{2\hbar}) - \varpi_{\varrho_{2\hbar}\varrho} + \varpi(\varrho_{2\hbar},\Gamma_1\varrho) - \varpi_{\varrho_{2\hbar}\Gamma_1\varrho} \\
&\le \varpi(\varrho,\varrho_{2\hbar}) + \varpi(\Gamma_1\varrho,\Gamma_2\varrho_{2\hbar-1}) \\
&\le \varpi(\varrho,\varrho_\hbar) + \alpha\left(\varpi(\varrho,\varrho_{2\hbar-1})\right) + \beta\left(\varpi(\varrho_{2\hbar-1},\varrho_{2\hbar})\right) \\
&\quad +\gamma\left(\varpi(\varrho,\Gamma_1\varrho)\right).
\end{aligned} \tag{7.4.7}$$

Taking limit as $\hbar\to\infty$ in (7.4.7), we have

$$\varpi(\varrho,\Gamma_1\varrho) = 0. \tag{7.4.8}$$

Also, Γ_1 is a self distance contraction. Therefore, $\exists\ k_0 \in [0,1)$ s.t.

$$\varpi(\Gamma_1\varrho,\Gamma_1\varrho) \le k_0(\varpi(\varrho,\varrho)),$$

or

$$\varpi(\Gamma_1\varrho,\Gamma_1\varrho) = 0. \tag{7.4.9}$$

Using (7.4.5), (7.4.6), (7.4.8) and (7.4.9), we have

$$\varpi(\varrho,\varrho) = \varpi(\varrho,\Gamma_1\varrho) = \varpi(\Gamma_1\varrho,\Gamma_1\varrho) = 0,$$

Using axiom (i) of Definition 1.2.5, we have $\varrho = \Gamma_1\varrho$. On similar lines, one can prove that $\varrho = \Gamma_2\varrho$. Therefore, ϱ is a common fixed point of Γ_1 and Γ_2.

Uniqueness: let $\vartheta \ne \varrho$ be another common fixed point with $\varpi(\varrho,\varrho) = \varpi(\vartheta,\vartheta) = 0$. Then,

$$\begin{aligned}
\varpi(\varrho,\vartheta) &= \varpi(\Gamma_1\varrho,\Gamma_2\vartheta) \\
&\le \alpha\left(\varpi(\varrho,\vartheta)\right) + \beta\left(\varpi(\varrho,\Gamma_1\varrho)\right) + \gamma\left(\varpi(\vartheta,\Gamma_2\vartheta)\right) \\
&= \alpha\left(\varpi(\varrho,\vartheta)\right) + \beta\left(\varpi(\varrho,\varrho)\right) + \gamma\left(\varpi(\vartheta,\vartheta)\right) \\
&< \varpi(\varrho,\vartheta),
\end{aligned}$$

a contradiction. Therefore, Γ_1, Γ_2 have exactly one common fixed point. $\square$

Theorem 7.4.3. *Let $\Gamma_1, \Gamma_2 : \Omega \to \Omega$ be mappings defined on complete $\mathfrak{m}$-MS (Ω, ϖ) and Y be a complete subspace of Ω s.t. $\Gamma_1(\Omega) \cup \Gamma_2(\Omega) \subseteq Y$. Suppose Γ_1, Γ_2 satisfy*

(i) $\exists \alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + \gamma < 1$ s.t.

$$\varpi(\Gamma_1\varrho, \Gamma_2\vartheta) \leq \alpha\varpi(\varrho, \vartheta) + \beta\varpi(\varrho, \Gamma_1\varrho) + \gamma\varpi(\vartheta, \Gamma_2\vartheta), \tag{7.4.10}$$

$\forall \varrho, \vartheta \in \Omega$;

(ii) $M_{\Gamma_1\varrho, \Gamma_2\varrho} \leq \varpi(\Gamma_1\varrho, \Gamma_2\varrho)$, $\forall \varrho \in \Omega$.

Then, the pair (Γ_1, Γ_2) possesses exactly one common fixed point.

Proof. The result is analogues to Theorem 7.4.2. $\square$

Definition 7.4.4. *Let (Ω, m) be an $\mathfrak{m}$-MS. A triplet $(\Gamma_1, \Gamma_2, \Gamma_3)$ of self mappings is c.t.b. contracting perimeter of triangle on Ω if $\exists$ some $k \in [0, 1)$ s.t.*

$$\varpi(\Gamma_1\varrho, \Gamma_2\vartheta) + \varpi(\Gamma_2\vartheta, \Gamma_3\xi) + \varpi(\Gamma_3\xi, \Gamma_1\varrho) \leq k\left(\varpi(\varrho, \vartheta) + \varpi(\vartheta, \xi) + \varpi(\xi, \varrho)\right), \tag{7.4.11}$$

$\forall \varrho, \vartheta, \xi \in \Omega$ that are pairwise distinct.

We extend the concept of periodic point further for a pair of self mappings (Γ_1, Γ_2) as : ϱ has pairwise prime period '2' w.r.t. (Γ_1, Γ_2) if $\Gamma_1\varrho \neq \varrho$,$\Gamma_2\varrho \neq \varrho$, but either $FG(\varrho) = \varrho$ or $GF(\varrho) = \varrho$. Moreover, ϱ has pairwise prime period 2 w.r.t. $(\Gamma_1, \Gamma_2, \Gamma_3)$, if ϱ has pairwise prime period '2' w.r.t. each pair of self mappings.

Example 7.4.5. *Consider the mapping Γ_1 and Γ_2 defines on $\mathbb{R}_+$ as $\Gamma_1\varrho = e^{\varrho}$ and $\Gamma_2\varrho = ln(\varrho)$. Then, $\forall\ \varrho \in \mathbb{R}_+$, $\Gamma_1\varrho \neq \varrho$ and $\Gamma_2\varrho \neq \varrho$ but $FG(\varrho) = \varrho$. Therefore, every point in $\mathbb{R}_+$ has a pairwise prime period '2' w.r.t. mappings (Γ_1, Γ_2).*

Theorem 7.4.6. *Let $\Gamma_1, \Gamma_2, \Gamma_3 : \Omega \to \Omega$ be mappings defined on complete $\mathfrak{m}$-MS (Ω, ϖ) and Y be a complete subspace of Ω s.t. $\Gamma_1(\Omega) \cup \Gamma_2(\Omega) \cup \Gamma_3(\Omega) \subseteq Y$. Suppose triplet $(\Gamma_1, \Gamma_2, \Gamma_3)$ satisfies*

(i) triplet $(\Gamma_1, \Gamma_2, \Gamma_3)$ is contracting perimeter of triangle on Ω;

(ii) $\Gamma_1, \Gamma_2, \Gamma_3$ *are self distance contraction mappings;*

(iii) *there is no point in* Ω *that has a pairwise prime period '2' w.r.t.* $(\Gamma_1, \Gamma_2, \Gamma_3)$.

Then, the triplet $(\Gamma_1, \Gamma_2, \Gamma_3)$ *possesses a common fixed point. Moreover, the triplet* $(\Gamma_1, \Gamma_2, \Gamma_3)$ *has atmost two common fixed points.*

Proof. For some $\varrho_0 \in \Omega$, define the iterative sequence as

$$\varrho_1 = \Gamma_1\varrho_0,\ \varrho_2 = \Gamma_2\varrho_1,\ \varrho_3 = \Gamma_3\varrho_2, ..., \varrho_{3\hbar+1} = \Gamma_1\varrho_{3\hbar},\ \varrho_{3\hbar+2} = \Gamma_2\varrho_{3\hbar+1},\ \text{and}\ \varrho_{3\hbar+3} = \Gamma_3\varrho_{3\hbar+2},$$

$\forall\ \hbar \in \mathbb{N}$. Suppose that $\varrho_{\hbar+1}$ is not a common fixed point $\forall \hbar \in \mathbb{N}$ and there is no point in Ω that has a prime order 2 pairwise w.r.t. $(\Gamma_1, \Gamma_2, \Gamma_3)$. Then, $\varrho_{3\hbar+1}, \varrho_{3\hbar+2}, \varrho_{3\hbar+3}$ are pairwise distinct. Now consider the sequence $\{\beta_\hbar\}$ generated by perimeter of triangle in multiplicative distance structure with vertices as the consecutive member of the sequence $\{\varrho_\hbar\}$ as

$$\beta_\hbar = \varpi(\varrho_{3\hbar+1}, \varrho_{3\hbar+2}) + \varpi(\varrho_{3\hbar+2}, \varrho_{3\hbar+3}) + \varpi(\varrho_{3\hbar+3}, \varrho_{3\hbar+1}).$$

Now, as $\varrho_{3\hbar}, \varrho_{3\hbar+1}, \varrho_{3\hbar+2}$ are all pairwise distinct and $(\Gamma_1, \Gamma_2, \Gamma_3)$ is contracting perimeter of triangle in (Ω, ϖ). Therefore, we have

$$\begin{aligned}
\beta_\hbar &= \varpi(\varrho_{3\hbar+1}, \varrho_{3\hbar+2}) + \varpi(\varrho_{3\hbar+2}, \varrho_{3\hbar+3}) + \varpi(\varrho_{3\hbar+3}, \varrho_{3\hbar+1}) \\
&= \varpi(\Gamma_1\varrho_{3\hbar}, \Gamma_2\varrho_{3\hbar+1}) + \varpi(\Gamma_2\varrho_{3\hbar+1}, \Gamma_3\varrho_{3\hbar+2}) + \varpi(\Gamma_3\varrho_{3\hbar+2}, \Gamma_1\varrho_{3\hbar}) \\
&\leq k\left(\varpi(\varrho_{3\hbar}, \varrho_{3\hbar+1}) + \varpi(\varrho_{3\hbar+1}, \varrho_{3\hbar+2}) + \varpi(\varrho_{3\hbar+2}, \varrho_{3\hbar})\right) \\
&\leq k(\beta_{\hbar-1}).
\end{aligned}$$

Moreover, since $k < 1$, therefore

$$\beta_0 > \beta_1 > \beta_2 > \cdots > \beta_{\hbar-1} > \beta_\hbar > \cdots .$$

We claim that $\varrho_{3i+1}, \varrho_{3i+2}, \varrho_{3i+3}$ are all distinct. Suppose $\exists\ j \geq 3$ s.t. $\varrho_{3j+1} = \varrho_{3i+1}$ for some $0 \leq i \leq j-2$. Then,

$$\varrho_{3i+1} = \varrho_{3j+1} \Rightarrow \varrho_{3i+2} = \Gamma_2\varrho_{3i+1} = \Gamma_2\varrho_{3j+1} = \varrho_{3j+2} \Rightarrow \varrho_{3i+3} = \Gamma_3\varrho_{3i+2} = \Gamma_3\varrho_{3j+2} = \varrho_{3j+3},$$

implies

$$\begin{aligned}
\beta_{3i} &= \varpi(\varrho_{3i+1}, \varrho_{3i+2}) + \varpi(\varrho_{3i+2}, \varrho_{3i+3}) + \varpi(\varrho_{3i+3}, \varrho_{3i+1}) \\
&= \varpi(\varrho_{3j+1}, \varrho_{3j+2}) + \varpi(\varrho_{3j+2}, \varrho_{3j+3}) + \varpi(\varrho_{3j+3}, \varrho_{3j+1}) \\
&= \beta_{3j},
\end{aligned}$$

a contradiction.
Consider,

$$\begin{aligned}\varpi(\varrho_{3\hbar+1},\varrho_{3\hbar+2}) &\leq \varpi(\varrho_{3\hbar+1},\varrho_{3\hbar+2})+\varpi(\varrho_{3\hbar+2},\varrho_{3\hbar+3})+\varpi(\varrho_{3\hbar+3},\varrho_{3\hbar+1})=\beta_\hbar\\ &\leq k(\beta_{\hbar-1})\leq\cdots\leq k^{\hbar}(\beta_0) \qquad (7.4.12)\end{aligned}$$

Since $k<1$, taking limit as $\hbar\to\infty$ in (7.4.12), we have

$$\lim_{\hbar\to\infty}\varpi(\varrho_{3\hbar+1},\varrho_{3\hbar+2})=0.$$

Without loss of generality, we have

$$\lim_{\hbar\to\infty}\varpi(\varrho_{\hbar},\varrho_{\hbar+1})=0. \qquad (7.4.13)$$

Also,

$$\lim_{\hbar\to\infty}\varpi_{\varrho_\hbar\ \varrho_{\hbar+1}}=\lim_{\hbar\to\infty}\min\{\varpi(\varrho_\hbar,\varrho_\hbar),\varpi(\varrho_{\hbar+1},\varrho_{\hbar+1})\}\leq\lim_{\hbar\to\infty}\varpi(\varrho_\hbar,\varrho_{\hbar+1})=0, \qquad (7.4.14)$$

$$\begin{aligned}\lim_{\hbar\to\infty}\varpi(\varrho_\hbar,\varrho_\hbar) &= \lim_{\hbar\to\infty}\varpi(\varrho_{\hbar+1},\varrho_{\hbar+1})=\lim_{\hbar\to\infty}\min\{\varpi(\varrho_\hbar,\varrho_\hbar),\varpi(\varrho_{\hbar+1},\varrho_{\hbar+1})\}\\ &= \lim_{\hbar\to\infty}\varpi_{\varrho_\hbar\ \varrho_{\hbar+1}}=0, \qquad (7.4.15)\end{aligned}$$

and

$$\lim_{\hbar\to\infty}M_{\varrho_\hbar\ \varrho_\hbar+1}=\lim_{\hbar\to\infty}\max\{\varpi(\varrho_\hbar,\varrho_\hbar),\varpi(\varrho_{\hbar+1},\varrho_{\hbar+1})\}=0. \qquad (7.4.16)$$

Next, we will show that $\{\varrho_\hbar\}$ is a $\mathfrak{m}$-C_{seq}. Consider,

$$\begin{aligned}\varpi(\varrho_{3\hbar+1},\varrho_{3m+1}) &- \varpi_{\varrho_{3\hbar+1},\varrho_{3m+1}}\\ &\leq \varpi(\varrho_{3\hbar+1},\varrho_{3\hbar+2})-\varpi_{\varrho_{3\hbar+1},\varrho_{3\hbar+2}}+\varpi(\varrho_{3\hbar+2},\varrho_{3\hbar+3})\\ &\quad -\varpi_{\varrho_{3\hbar+2},\varrho_{3\hbar+3}}+\cdots\\ &\quad +\varpi(\varrho_{3m+2},\varrho_{3m+1})-\varpi_{\varrho_{3m+2},\varrho_{3m+1}}\\ &\leq \varpi(\varrho_{3\hbar+1},\varrho_{3\hbar+2})+\varpi(\varrho_{3\hbar+2},\varrho_{3\hbar+3})+\cdots+\varpi(\varrho_{3m+2},\varrho_{3m+1})\\ &\leq \beta_\hbar+\beta_{\hbar+1}+\cdots+\beta_{m+1}\\ &\leq k^{\hbar}(\beta_0)+k^{\hbar+1}(\beta_0)+\cdots+k^{m+1}(\beta_0)\\ &\leq k^{\hbar}(1+k+k^2+...k^{m+1-\hbar})(\beta_0)\\ &= k^{\hbar}(\frac{1-k^{m+1-\hbar}}{1-k})(\beta_0). \qquad (7.4.17)\end{aligned}$$

Since $k < 1$, taking limit as $\hbar, m \to \infty$ in (7.4.17), we have

$$\lim_{\hbar, m\to\infty} \varpi(\varrho_{3\hbar+1}, \varrho_{3m+1}) - \varpi_{\varrho_{3\hbar+1}\ \varrho_{3m+1}} = 0.$$

Without loss of generality, we have

$$\lim_{\hbar, m\to\infty} \varpi(\varrho_{\hbar}, \varrho_{m}) - \varpi_{\varrho_{\hbar}\ \varrho_{m}} = 0, \tag{7.4.18}$$

i.e., $\{\varrho_\hbar\}$ is $\mathfrak{m}$-Cauchy.
Further, as Y is $\mathfrak{m}$-complete. Therefore, $\exists \varrho \in Y$ s.t.

$$\lim_{\hbar\to\infty} \varpi(\varrho_{\hbar}, \varrho) - \varpi_{\varrho_{\hbar}\varrho} = 0 = \lim_{\hbar\to\infty} M_{\varrho_{\hbar}\varrho} - \varpi_{\varrho_{\hbar}\varrho}. \tag{7.4.19}$$

Using (7.4.15), we have

$$\lim_{\hbar\to\infty} \varpi_{\varrho_{\hbar}\varrho} = \lim_{\hbar\to\infty} \min\{\varpi(\varrho_\hbar, \varrho_\hbar), \varpi(\varrho, \varrho)\} \leq \lim_{\hbar\to\infty} \varpi(\varrho_\hbar, \varrho_\hbar) = 0. \tag{7.4.20}$$

Using (7.4.19), (7.4.20) and Lemma 1.2.9, we have

$$\lim_{\hbar\to\infty} \varpi(\varrho_{\hbar}, \varrho) = 0,\ \lim_{\hbar\to\infty} M_{\varrho_{\hbar}\varrho} = 0 \text{ and } \varpi(\varrho, \varrho) = 0. \tag{7.4.21}$$

Also,

$$\varpi_{\varrho\Gamma_2\varrho} = \min\{\varpi(\varrho, \varrho), \varpi(\Gamma_2\varrho, \Gamma_2\varrho)\} = 0. \tag{7.4.22}$$

Using (7.4.11), (7.4.22) and the triangle inequality, we have

$$\begin{aligned}
0 \leq \varpi(\varrho, \Gamma_2\varrho) &= \varpi(\varrho, \Gamma_2\varrho) - \varpi_{\varrho\Gamma_2\varrho} \\
&\leq \varpi(\varrho, \varrho_{3\hbar+1}) - \varpi_{\varrho_{3\hbar+1}\varrho} + \varpi(\varrho_{3\hbar+1}, \Gamma_2\varrho) - \varpi_{\varrho_{3\hbar+1}\Gamma_2\varrho} \\
&\leq \varpi(\varrho, \varrho_{3\hbar+1}) - \varpi_{\varrho_{3\hbar+1}\ \varrho} + \varpi(\varrho_{3\hbar+1}, \Gamma_2\varrho) - \varpi_{\varrho_{3\hbar+1}\Gamma_2\varrho} \\
&\quad + \varpi(\Gamma_2\varrho, \varrho_{3\hbar+3}) + \varpi(\varrho_{3\hbar+3}, \varrho_{3\hbar+1}) \\
&\leq \varpi(\varrho, \varrho_{3\hbar+1}) + \varpi(\Gamma_1\varrho_{3\hbar}, \Gamma_2\varrho) + \varpi(\Gamma_2\varrho, \Gamma_3\varrho_{3\hbar+2}) \\
&\quad + \varpi(\Gamma_3\varrho_{3\hbar+2}, \Gamma_1\varrho_{3\hbar}) \\
&\leq \varpi(\varrho, \varrho_{3\hbar+1}) + k\Big(\varpi(\varrho_{3\hbar}, \varrho) + \varpi(\varrho, \varrho_{3\hbar+2}) \\
&\quad + \varpi(\varrho_{3\hbar+2}, \varrho_{3\hbar})\Big)
\end{aligned} \tag{7.4.23}$$

Taking limit as $\hbar \to \infty$ in (7.4.23), we have

$$\varpi(\varrho, \Gamma_2\varrho) = 0. \tag{7.4.24}$$

Also, Γ_2 is a self distance contraction. Therefore, $\exists\ k_0 \in [0, 1)$ s.t.

$$\varpi(\Gamma_2\varrho, \Gamma_2\varrho) \leq k_0(\varpi(\varrho, \varrho)),$$

or

$$\varpi(\Gamma_2\varrho, \Gamma_2\varrho) = 0. \tag{7.4.25}$$

Using (7.4.21), (7.4.24) and (7.4.25), we have

$$\varpi(\varrho, \varrho) = \varpi(\varrho, \Gamma_2\varrho) = \varpi(\Gamma_2\varrho, \Gamma_2\varrho) = 0,$$

i.e., $\varrho = \Gamma_2\varrho$. Thus, ϱ is the fixed point of Γ_2.
On the similar lines, one can prove that $\varrho = \Gamma_1\varrho = \Gamma_2\varrho = \Gamma_3\varrho$ i.e., ϱ is the common fixed point of self mappings $\Gamma_1, \Gamma_2, \Gamma_3$.
In order to prove that $\Gamma_1, \Gamma_2, \Gamma_3$ have atmost two common fixed points, suppose that $\exists$ three distinct common fixed points say ϱ, ϑ, ξ. Then, we have $\varpi(\varrho, \varrho) = \varpi(\vartheta, \vartheta) = \varpi(\xi, \xi) = 0$.
Consider,

$$\begin{aligned} \varpi(\varrho, \vartheta) + \varpi(\vartheta, \xi) + \varpi(\xi, \varrho) &= \varpi(\Gamma_1\varrho, \Gamma_2\vartheta) + \varpi(\Gamma_2\vartheta, \Gamma_3\xi) + \varpi(\Gamma_3\xi, \Gamma_1\varrho) \\ &\leq k\left(\varpi(\varrho, \vartheta) + \varpi(\vartheta, \xi) + \varpi(\xi, \varrho)\right) \\ &< \varpi(\varrho, \vartheta) + \varpi(\vartheta, \xi) + \varpi(\xi, \varrho), \end{aligned}$$

a contradiction. Hence, $\Gamma_1, \Gamma_2, \Gamma_3$ have atmost two common fixed points. □

Remark 7.4.7. *If under the assumption of Theorem 7.4.6, the triplet of mappings* $\Gamma_1, \Gamma_2, \Gamma_3$ *has a common fixed point* ϱ *and it is a limit point of the iterative scheme*

$$\varrho_1 = \Gamma_1\varrho_0,\ \varrho_2 = \Gamma_2\varrho_1,\ \varrho_3 = \Gamma_3\varrho_2, ...,\ \varrho_{3\hbar+1} = \Gamma_1\varrho_{3\hbar},\ \varrho_{3\hbar+2} = \Gamma_2\varrho_{3\hbar+1},\ \varrho_{3\hbar+3} = \Gamma_3\varrho_{3\hbar+2},$$

$\forall \hbar \in \mathbb{N}$. *Then,* $(\Gamma_1, \Gamma_2, \Gamma_3)$ *possesses exactly one common fixed point.*
If possible $\vartheta \neq \varrho$ *is another common fixed point. Clearly* $\varrho_{3\hbar}, \varrho, \vartheta$ *are pairwise distinct.*
Consider,

$$\begin{aligned} \varpi(\varrho_{3\hbar+1}, \varrho) + \varpi(\varrho, \vartheta) + \varpi(\vartheta, \varrho_{3\hbar+1}) &= \varpi(\Gamma_1\varrho_{3\hbar}, \Gamma_2\varrho) + \varpi(\Gamma_2\varrho, \Gamma_3\vartheta) + \varpi(\Gamma_3\vartheta, \Gamma_1\varrho_{3\hbar}) \\ &\leq k\left(\varpi(\varrho_{3\hbar}, \varrho) + \varpi(\varrho, \vartheta) + \varpi(\vartheta, \varrho_{3\hbar})\right) \\ &< \varpi(\varrho_{3\hbar}, \varrho) + \varpi(\varrho, \vartheta) + \varpi(\vartheta, \varrho_{3\hbar}). \end{aligned}$$

As $(\lim_{\hbar\to\infty} \varpi(\varrho_{3\hbar}, \varrho) = \lim_{\hbar\to\infty} \varpi(\varrho_{3\hbar}, \vartheta) = 0)$. *Therefore, by taking limit as* $\hbar \to \infty$, *we have*

$$\varpi(\varrho, \vartheta) < \varpi(\varrho, \vartheta),$$

a contradiction.

Definition 7.4.8. *Let $(\Omega, \mathfrak{m})$ be an $\mathfrak{m}$-MS . A self mapping Γ_1 is c.t.b.* ***contracting perimeter of triangle*** *on Ω if $\exists$ some $k \in [0,1)$ s.t.*

$$\varpi(\Gamma_1\varrho, \Gamma_1\vartheta).\varpi(\Gamma_1\vartheta, \Gamma_1\xi).\varpi(\Gamma_1\xi, \Gamma_1\varrho) \leq (\varpi(\varrho, \vartheta).\varpi(\vartheta, \xi).\varpi(\xi, \varrho))^k, \quad (7.4.26)$$

$\forall \varrho, \vartheta, \xi \in \Omega$ that are pairwise distinct.

Theorem 7.4.9. *Let $\Gamma_1 : \Omega \to \Omega$ be a mapping defined on complete $\mathfrak{m}$-MS (Ω, ϖ) and Y be a complete subspace of Ω s.t. $\Gamma_1(\Omega) \subseteq Y$. Suppose Γ_1 satisfies*

(i) Γ_1 is contracting perimeter of triangle on Ω;

(ii) Γ_1 is a self distance contraction mapping;

(iii) there is no periodic point of prime order 2 in Ω;

Then, Γ possesses a fixed point. Moreover, Γ_1 has atmost two fixed point.

Proof. The result holds as a consequence of Theorem 7.4.6, by substituting $\Gamma_2 = \Gamma_3 = \Gamma_1$. □

Example 7.4.10. *Consider a set $\Omega = \{p, q, s\}$ with distance function $\mathfrak{m}$ defined as $\varpi(p,p) = \varpi(q,q) = 0$, $\varpi(s,s) = 2$ and $\varpi(p,q) = \varpi(q,s) = \varpi(p,s) = \varpi(q,p) = \varpi(s,q) = \varpi(s,p) = 4$. Then, clearly (Ω, ϖ) is $\mathfrak{m}$-MS. Let $\Gamma : \Omega \to \Omega$ defined as $\Gamma(p) = p, \Gamma(q) = q, \Gamma(s) = q$. Then, Γ satisfies Theorem 7.4.9 and has two fixed point (say p & q).*

Example 7.4.11. *Consider a set $\Omega = \{p, q, s\}$ with distance function $\mathfrak{m}$ defined as $\varpi(p,p) = \varpi(q,q) = \varpi(s,s) = 2$ and $\varpi(p,q) = \varpi(q,s) = \varpi(p,s) = \varpi(q,p) = \varpi(s,q) = \varpi(s,p) = 4$. Then, clearly (Ω, ϖ) is $\mathfrak{m}$-MS. Let $\Gamma : \Omega \to \Omega$ defined as $\Gamma(p) = q, \Gamma(q) = p, \Gamma(s) = p$. Then, Γ satisfies the condition of contracting perimeter and p, q are periodic points of prime order '2' and has no fixed point.*

Remark 7.4.12. *If under the assumptions of Theorem 7.4.9, the mapping Γ_1 has a fixed point ϱ and it is a limit point of the iterative scheme $\varrho_\hbar = \Gamma_1\varrho_{\hbar-1}$. Then, Γ_1 possesses exactly one fixed point.*
If possible $\vartheta \neq \varrho$ is another fixed point. Clearly, $\varrho_\hbar, \varrho, \vartheta$ are pairwise distinct. Consider,

$$\begin{aligned}\varpi(\varrho_{\hbar+1}, \varrho) &+ \varpi(\varrho, \vartheta) + \varpi(\vartheta, \varrho_{\hbar+1}) \\ &= \varpi(\Gamma_1\varrho_\hbar, \Gamma_1\varrho) + \varpi(\Gamma_1\varrho, \Gamma_1\vartheta) + \varpi(\Gamma_1\vartheta, \Gamma_1\varrho_\hbar) \\ &\leq k\left(\varpi(\varrho_\hbar, \varrho) + \varpi(\varrho, \vartheta) + \varpi(\vartheta, \varrho_\hbar)\right) \\ &< \varpi(\varrho_\hbar, \varrho) + \varpi(\varrho, \vartheta) + \varpi(\vartheta, \varrho_\hbar).\end{aligned}$$

As, $\left(\lim_{\hbar\to\infty}\varpi(\varrho_\hbar,\varrho)=\lim_{\hbar\to\infty}\varpi(\varrho_\hbar,\vartheta)=0\right)$. *Therefore, by taking limit as* $\hbar\to\infty$, *we have*

$$\varpi(\varrho,\vartheta)<\varpi(\varrho,\vartheta),$$

a contradiction.

Example 7.4.13. *Consider the* $\mathfrak{m}$*-MS* $[0,2)$, *where* $\mathfrak{m}:[0,2)\times[0,2)\to[0,\infty)$ *defined as* $\varpi(\varrho,\vartheta)=\frac{\varrho+\vartheta}{2}$. *Let the self mappings* $\Gamma_1,\Gamma_2,\Gamma_3$ *are defined as*

$$\Gamma_1\varrho=\begin{cases}\frac{\varrho^2}{5}, & \textit{if } \varrho<1\\ \frac{1}{\varrho+9}, & \textit{otherwise.}\end{cases},\ \Gamma_2\varrho=\begin{cases}\frac{\varrho}{5}, & \textit{if } \varrho<1\\ \frac{\varrho}{\varrho+9}, & \textit{otherwise.}\end{cases},\ \Gamma_3\varrho=\begin{cases}\frac{\varrho}{3}, & \textit{if } \varrho<1\\ \frac{1}{7}, & \textit{otherwise.}\end{cases}.$$

Then,

(i) *Suppose* $\varrho,\vartheta,\xi<1$, *then*

$$\begin{aligned}\varpi(\Gamma_1\varrho,\Gamma_2\vartheta) &+ \varpi(\Gamma_2\vartheta,\Gamma_3\xi)+\varpi(\Gamma_3\xi,\Gamma_1\varrho)\\ &= \frac{1}{2}\left(\frac{\varrho^2}{5}+\frac{\vartheta}{5}\right)+\frac{1}{2}\left(\frac{\vartheta}{5}+\frac{\xi}{3}\right)+\frac{1}{2}\left(\frac{\xi}{3}+\frac{\varrho^2}{5}\right)\\ &\leq \frac{1}{5}\left(\frac{\varrho}{2}+\frac{\vartheta}{2}\right)+\max\left\{\frac{1}{5},\frac{1}{3}\right\}\left(\frac{\vartheta}{2}+\frac{\xi}{2}\right)+\max\left\{\frac{1}{5},\frac{1}{3}\right\}\left(\frac{\xi}{2}+\frac{\varrho}{2}\right)\\ &\leq k_1\left(\varpi(\varrho,\vartheta).\varpi(\vartheta,\xi).\varpi(\xi,\varrho)\right),\end{aligned}$$

where $k_1=\max\left\{\frac{1}{5},\frac{1}{3}\right\}=\frac{1}{3}$.

(ii) *Suppose* $\varrho\geq 1$ *and* $\xi,\vartheta<1$, *then*

$$\begin{aligned}\varpi(\Gamma_1\varrho,\Gamma_2\vartheta) &+ \varpi(\Gamma_2\vartheta,\Gamma_3\xi)+\varpi(\Gamma_3\xi,\Gamma_1\varrho)\\ &= \frac{1}{2}\left(\frac{1}{\varrho+9}+\frac{\vartheta}{5}\right)+\frac{1}{2}\left(\frac{\vartheta}{5}+\frac{\xi}{3}\right)+\frac{1}{2}\left(\frac{\xi}{3}+\frac{1}{\varrho+9}\right)\\ &\leq \frac{1}{2}\left(\frac{\varrho}{10}+\frac{\vartheta}{5}\right)+\frac{1}{2}\left(\frac{\vartheta}{5}+\frac{\xi}{3}\right)+\frac{1}{2}\left(\frac{\xi}{3}+\frac{\varrho}{10}\right)\\ &\leq \max\left\{\frac{1}{10},\frac{1}{5}\right\}\left(\frac{\varrho}{2}+\frac{\vartheta}{2}\right)+\max\left\{\frac{1}{5},\frac{1}{3}\right\}\left(\frac{\vartheta}{2}+\frac{\xi}{2}\right)\\ &\quad+\max\left\{\frac{1}{3},\frac{1}{10}\right\}\left(\frac{\xi}{2}+\frac{\varrho}{2}\right)\\ &\leq k_2\left(\varpi(\varrho,\vartheta)+\varpi(\vartheta,\xi).\varpi(\xi,\varrho)\right),\end{aligned}$$

where $k_2=\max\left\{\frac{1}{5},\frac{1}{3},\frac{1}{10}\right\}=\frac{1}{3}$.

(iii) Suppose $\vartheta \geq 1$ and $\varrho, \xi < 1$, then

$$\begin{aligned}
\varpi(\Gamma_1\varrho, \Gamma_2\vartheta) &+ \varpi(\Gamma_2\vartheta, \Gamma_3\xi) + \varpi(\Gamma_3\xi, \Gamma_1\varrho) \\
&= \frac{1}{2}\left(\frac{\varrho^2}{5} + \frac{\vartheta}{\vartheta+9}\right) + \frac{1}{2}\left(\frac{\vartheta}{\vartheta+9} + \frac{\xi}{3}\right) + \frac{1}{2}\left(\frac{\xi}{3} + \frac{\varrho^2}{5}\right) \\
&\leq \frac{1}{2}\left(\frac{\varrho}{5} + \frac{\vartheta}{10}\right) + \frac{1}{2}\left(\frac{\vartheta}{10} + \frac{\xi}{3}\right) + \frac{1}{2}\left(\frac{\xi}{3} + \frac{\varrho}{5}\right) \\
&\leq \max\left\{\frac{1}{5}, \frac{1}{10}\right\}\left(\frac{\varrho}{2} + \frac{\vartheta}{2}\right) + \max\left\{\frac{1}{10}, \frac{1}{3}\right\}\left(\frac{\vartheta}{2} + \frac{\xi}{2}\right) \\
&\quad + \max\left\{\frac{1}{3}, \frac{1}{5}\right\}\left(\frac{\xi}{2} + \frac{\varrho}{2}\right) \\
&\leq k_3\left(\varpi(\varrho, \vartheta).\varpi(\vartheta, \xi).\varpi(\xi, \varrho)\right),
\end{aligned}$$

where $k_3 = \max\left\{\frac{1}{3}, \frac{1}{5}, \frac{1}{10}\right\} = \frac{1}{3}$.

(iv) Suppose $\xi \geq 1$ and $\varrho, \vartheta < 1$, then

$$\begin{aligned}
\varpi(\Gamma_1\varrho, \Gamma_2\vartheta) &+ \varpi(\Gamma_2\vartheta, \Gamma_3\xi) + \varpi(\Gamma_3\xi, \Gamma_1\varrho) \\
&= \frac{1}{2}\left(\frac{\varrho^2}{5} + \frac{\vartheta}{5}\right) + \frac{1}{2}\left(\frac{\vartheta}{5} + \frac{1}{7}\right) + \frac{1}{2}\left(\frac{1}{7} + \frac{\varrho^2}{5}\right) \\
&\leq \frac{1}{5}\left(\frac{\varrho}{2} + \frac{\vartheta}{2}\right) + \max\left\{\frac{1}{5}, \frac{1}{7}\right\}\left(\frac{\vartheta}{2} + \frac{\xi}{2}\right) \\
&\quad + \max\left\{\frac{1}{7}, \frac{1}{5}\right\}\left(\frac{\xi}{2} + \frac{\varrho}{2}\right) \\
&\leq k_4\left(\varpi(\varrho, \vartheta).\varpi(\vartheta, \xi).\varpi(\xi, \varrho)\right),
\end{aligned}$$

where $k_4 = \max\left\{\frac{1}{5}, \frac{1}{7}\right\} = \frac{1}{5}$.

Let $k = \max\{k_1, k_2, k_3, k_4\} = \frac{1}{3}$. Then, $(\Gamma_1, \Gamma_2, \Gamma_3)$ is a contracting perimeter of triangle in (Ω, ϖ) with $k = \frac{1}{3}$ and there is no pairwise periodic point of prime order 2. Also, $\Gamma_1, \Gamma_2, \Gamma_3$ are self distance contractions. Hence, $(\Gamma_1, \Gamma_2, \Gamma_3)$ satisfies all the requirements of Theorem 7.4.6. Hence, the triplet $(\Gamma_1, \Gamma_2, \Gamma_3)$ has a common fixed point.

7.5 Conclusion

This chapter advances the study of fixed point theory by exploring novel results within the framework of $\mathfrak{m}$-MSs. Utilizing interpolative-type contractions and

$(\phi - \psi)$ Wardowski type contractions, we generalized several classical results and demonstrated their applicability through illustrative examples. The relaxation of continuity requirements in $\mathfrak{m}$-MSs broadens the scope of contraction mappings, addressing cases where traditional MS results are not applicable. Also, we analyzed the behavior of these mappings both theoretically as well as graphically and highlighted the significance of the $\mathfrak{m}$-MSs in fixed point theory. The results presented herein not only extend the existing work but also pave the way for future applications in more complex and incomplete spaces.

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List of Publications

1. **Yadav, K.**, & Kumar, D. (2024). Multiplicative $\mathfrak{m}$-metric space, fixed point theorems with applications in multiplicative integral equations and numerical results. *Journal of Applied Analysis*, *30*(1), 173–186. https://doi.org/10.1515/jaa-2023-0056. (ESCI/Scopus, SJR-0.243)

2. **Yadav, K.**, & Kumar, D. (2024). Some hyperstability results for quadratic type functional equations. *Applied Mathematics E-Notes*, *24*, 212–227. (ESCI/Scopus, SJR-0.221)

3. **Yadav, K.**, & Kumar, D. (2024). Common fixed point theorems in multiplicative $\mathfrak{m}$-metric space with applications to the system of multiplicative integral equations and numerical results. *Hacettepe Journal of Mathematics and Statistics*, 1–12. https://doi.org/10.15672/hujms.1336232. (SCIE/Scopus, SJR-0.288)

4. **Yadav, K.**, & Kumar, D. (2025). Common fixed point theorems for discontinuous mappings in M-metric space and numerical approximations, *Journal of Computational and Applied Mathematics*, 470, 116720, https://doi.org/10.1016/j.cam.2025.116720. (SCIE/Scopus, SJR-0.688)

5. **Yadav, K.**, & Kumar, D. (2025). Some Fixed Point Result in $\mathfrak{m}$-Metric Space Using Different Contractions. In: Tomar, A., Jain, M. (eds) Banach Contraction Principle. Industrial and Applied Mathematics. Springer, Singapore. https://doi.org/10.1007/978-981-96-4847-4_3

6. **Yadav, K.**, & Kumar, D. C^*-algebra valued $\mathfrak{m}_{\mathcal{R}}$ metric space and fixed point results with application, *Asian-European Journal of Mathematics*, 2550109. https://doi.org/10.1142/S1793557125501098. (ESCI/Scopus, SJR-0.310)

7. **Yadav, K.**, & Kumar, D. Stability analysis of a generalized quartic functional equation in n-Banach space, (Communicated).

8. **Yadav, K.**, & Kumar, D. Existence of solution for a non-linear functional integral equation and an operator equations via fixed point approach, (Communicated).

9. **Yadav, K.**, & Kumar, D. Fixed points of a generalized contraction in partial metric structure and application to fractal generation, (Communicated).

10. **Yadav, K.**, & Kumar, D. Fixed point results using a three-point analogue of contraction in multiplicative $\mathfrak{m}$-metric spaces, (Communicated) .

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Paper Presented in Conferences

1. Kapil Yadav and Deepak Kumar, Some Fixed-Point Results Using $(\phi - \psi)$ Wardowski Contraction in Partial-Metric Space, *International Symposium on Mathematical Analysis of Fractals and Dynamical Systems-2023* held from 24th to 25th Aug 2023 at Vellore Institute of Technology.

2. Kapil Yadav and Deepak Kumar, A Generalized Notion of Metric Space and Fixed Point Results, *Meeting on Analysis, Dimensions, and PDEs* on 27th July 2024 held at IIIT Allahabad

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Workshops Attended

1. Participated in *Advanced Training School on Functional Analysis and its Applications for Research Scholars and Teachers* from 11th to 22th Dec 2023 at Indian Institute of Technology, Tirupati.

2. Attend *Faculty Developent Program- Mathematics in Engineering and Science-Importance and Applications (MES 2024)* from (02nd to 6th Jan 2024) organised by Vellore Institute of Technology.

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