A STUDY ON BACK ORDERING FOR DIFFERENT INVENTORY MODELS UNDER FINITE PLANNING HORIZON

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By

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DECLARATION

I, hereby declared that the presented work in the thesis entitled "**A study on back ordering for different inventory models under finite planning horizon**" in fulfilment of degree of **Doctor of Philosophy (Ph. D.)** is outcome of research work carried out by me under the supervision **Dr. Nitin Kumar Mishra**, working as Professor, in the School of Chemical Engineering and Physical Sciences Lovely Professional University, Punjab, India. In keeping with general practice of reporting scientific observations, due acknowledgements have been made whenever work described here has been based on findings of another investigator. This work has not been submitted in part or full to any other University or Institute for the award of any degree.

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CERTIFICATE

This is to certify that the work reported in the Ph. D. thesis entitled "**A study on back ordering for different inventory models under finite planning horizon**" submitted in fulfillment of the requirement for the reward of degree of **Doctor of Philosophy (Ph.D.)** School of Chemical Engineering and Physical Sciences Lovely Professional University, Punjab, is a research work carried out by **Namwade Renuka Sheshrao**, Registration No.- 42000189, is bonafide record of his/her original work carried out under my supervision and that no part of thesis has been submitted for any other degree, diploma or equivalent course.

(Signature of Supervisor)

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Dedicated to my Dear Husband Shri. Santosh D. Kakade and my beloved Son Darsh

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Abstract

In this thesis the put forward work would study the variety of backordering, an analogy of those various types. This study will lighten up the new tracks for the firms when, why, and which method will be more helpful for managing inventory. Also, the proposed research work is relevant to environmental issues along with business issues and government issues. I have considered various types of backorders. To complete backorders most of the manufacturers may use more machinery, machinery which is under maintenance that causes more pollution, pollution leads to temperature increase. If we consider in the industrial area total companies are 100 and out of that 50 companies got the backorders and they are taking more manpower, more types of machinery for manufacturing and working for 24 hours then the average pollution will automatically increase which leads to global warming. We will try to find the solution for such back-ordering problems. To tackle the environmental issue.

In Chapter 3 of this thesis, in order to design a new replenishment policy, we introduced an improved inventory model for organizations that struggle with back orders regularly. The current research aims at enhancing this solution by including backorders and stressing the dynamics of the considered system as well as providing the best solutions over multiple iterations of the replenishment procedure. The model enhances the new parameters under a finite planning horizon and backordering involving totally back-ordered and linear and quadratic back-order functions to minimize the economic cost of ordering. We assessed both linear and quadratic backorders and optimized the cost through a positive definite Hessian matrix. To solve the model, we used numerical iterative methods in MATLAB; MATHEMATICA Version was 12. Tabular displays of the results are provided to demonstrate the findings when the parameters are adjusted. Furthermore, in sensitivity analysis, the results are shown in the tables and figures. At the end of the article, the author presents and discusses the practical implications of the study together with the insights obtained therein.

Based on these premises, chapter 4 This research pays attention to shortage control strategies. Integrated into an inventory model for price-sensitive demand

carrying an exponential complete backordering policy within a finite planning horizon for linear trends. There are mathematical parameters like demand fluctuation, and backorder expenses, as well as how demand is influenced by the price that is captured by exponential backordering terms. Calculations were performed in MATHEMATICA 12, while Python tools were used to create graphs and heatmaps for a more detailed result presentation. The main goal of the research is to find the least value of total variable cost with the view of establishing the appropriate replenishment and shortage periods. An example shows how the model can be applied for inventory control problems to avoid the stockout of particular types of items, to have better relationships between suppliers and retailers, and to minimize expenses. To highlight the importance of these parameters, the sensitivity analysis also underlines how the model can indeed be adjusted. This work offers a clear model for improving supply chain management, especially in conditions where demand fluctuation, price elasticity, and back orders are critical.

After that, chapter 5, this study is centered on the economic production quantity (EPQ) model with exponential partial backordering under a finite production time horizon where customers are committed to the price of the product. The optimality model for determining requisite re-order points is fixed that integrates cost as well as other parameters for replenishment. In a sensitivity analysis, the impact of these parameters on replenishment cycles, order quantity, and costs borne by the retail service is examined. The validity of the mathematical solutions is then confirmed by using Mathematica 12 to verify the results. Further, specific machine learning tools such as Python are used in the creation of more detailed graphics that must illustrate movements in inventory. The findings of the Research provide tips to the organizations on how they can adopt backordering strategies in order to minimize cost and improve the inventories of the companies. This is true because the findings show how companies have to balance holding and shortage costs through backordering and how accurate demand forecasts can limit holding costs. This research is relevant to the existing literature because it expands upon the EPQ model and provides actionable plans for inventory management that will increase an organization’s overall performance and customer satisfaction levels.

And lastly, chapter 6 This paper discusses an inventory management model appropriate to the contemporary business setting as businesses continue to adopt machine learning for demand-driven stocks management. The proposed model considers fuzzy and non-incompetent terms in that it allows for shortages and partial back ordering of an imperfect and degrading product. Defective rates and degradation rates are discussed as fuzzy variables because such characteristics are considered stochastic or depend on some uncertain condition. The goal is to identify the rightful stock replenishment cycles and order quantities and to minimize the overall cost which also encompasses carbon emissions for the given planning horizon. The defuzzification process is performed with the help of sign distance approximation. Machine learning is used to enable the implementation of a seasonal demand forecasting strategy to make the model more responsive to cyclical demand requirements. More accurately, a numerical example is provided, and the calculation and simulation results prove the applicability of the developed mathematical model in determining the demand for deteriorating products and its interaction with inventory management. A comparative analysis looks at the key benefits of implementing the AI forecasting system over demand scenarios. The sensitivity analysis allows for understanding the impact of key parameters on the optimal solution and provides significant managerial considerations for further strategies development.

Chapter 1

Introduction

1.1. INTRODUCTION TO OPERATIONS RESEARCH

Operations research (OR) is a multidisciplinary field dedicated to the application of advanced analytical methods to help make better decisions. By employing techniques from mathematics, statistics, and computer science, OR seeks to provide a rational basis for decision-making by seeking to understand and structure complex situations and to predict system behavior. OR is primarily focused on optimizing decision-making processes by either maximizing desired outcomes, such as profits, or minimizing undesired factors, such as costs, within a given set of constraints.

In the context of inventory management, (Veinott et al. 1963) provided one of the earliest systematic treatments of inventory systems, offering mathematical models to balance costs and resources efficiently, which laid a foundation for modern OR applications in supply chain and inventory theory. These constraints can include limited resources, environmental regulations, and operational limitations, among others.

1.2. HISTORICAL CONTEXT AND SIGNIFICANCE

The field of operations research originated during World War II out of the necessity for the British Army to address unprecedented strategic and tactical challenges. The British military, facing a war of immense scale and complexity, required innovative approaches to enhance their operational effectiveness. Recognizing the need for systematic and quantitative decision-making under uncertainty, they brought together interdisciplinary teams composed of scientists, engineers, and mathematicians. This collaborative effort aimed to develop new methodologies for optimizing resource allocation, logistics, and strategic planning.

These teams focused on applying mathematical and statistical techniques to military operations, leading to significant improvements in areas such as radar detection, convoy protection, and resource distribution. The success of these efforts was evident in the enhanced effectiveness of Allied operations, which played a crucial role in achieving victory. The legacy of OR in World War II not only underscored its critical impact on military strategy but also highlighted its potential for solving complex problems in various domains.

1.3. APPLICATION BEYOND WARFARE

After the war, the principles and techniques of operations research were recognized for their broader applicability beyond military contexts. One notable development was the creation of Linear Programming (LPP) by the US Army, a method that revolutionized resource allocation and decision-making processes. LPP allowed for the optimization of linear objective functions, subject to linear constraints, facilitating more efficient allocation of limited resources.

In the post-war era, industries began to adopt OR methodologies to improve their operational efficiencies. By the 1970s and 1980s, OR techniques had permeated diverse sectors such as agriculture, where it optimized crop production and distribution; manufacturing, where it enhanced production schedules and inventory management; transportation, where it improved routing and scheduling; and finance, where it optimized investment portfolios and risk management.

For example, in agriculture, OR techniques have been used to determine the optimal mix of crops to maximize yield and minimize costs, considering factors such as soil quality, weather patterns, and market demand. In manufacturing, OR has helped in streamlining production processes, reducing waste, and improving product quality through better scheduling and resource allocation. In transportation, OR has optimized logistics networks, reducing delivery times and costs by determining the most efficient routes and schedules for vehicles. In finance, OR has aided in developing strategies for asset allocation, risk assessment, and financial planning, enhancing profitability and stability.

As technology advanced, OR techniques evolved to address more complex and dynamic challenges. Today, operations research continues to be a vital tool in various fields, leveraging big data, artificial intelligence, and machine learning to solve contemporary problems and improve decision-making processes. In particular, (Edward A. Silver et al. 2016) demonstrated how modern OR integrates these advanced tools with supply chain and inventory management, offering strategies for demand forecasting, production planning, and backordering in dynamic business environments.

It remains a cornerstone of strategic planning and operational efficiency in numerous industries, driving innovation and competitive advantage.

1.4. ADVANTAGES AND LIMITATIONS OF OPERATIONS RESEARCH

Operations research (OR) is a pivotal discipline that offers organizations significant advantages in decision-making and systemic optimization. Despite its strengths, OR also confronts several challenges that require careful consideration for effective implementation and utilization.

1.4.1. ADVANTAGES

Scientific Rigor: OR employs rigorous mathematical techniques to model and solve complex problems, ensuring accuracy and reliability in decision-making processes. By basing decisions on quantitative data, organizations can achieve outcomes that are grounded in solid analytical foundations.

Quality Decision Making: By utilizing data-driven insights OR facilitates informed decision-making that reduces the risk of errors and improves overall decision quality. This approach allows organizations to evaluate multiple scenarios and assess potential outcomes systematically.

Systemic Approach: OR takes a holistic view of organizational systems, considering interdependence and relationships across various functions. This systemic perspective enables OR to offer integrated solutions that address underlying issues comprehensively, leading to sustainable improvements in efficiency and performance.

Optimization of Resources: OR enables organizations to optimize resource allocation, whether it's manpower, finances, or materials. By identifying the most efficient use of resources, OR helps minimize waste and maximize productivity, ultimately contributing to cost savings and enhanced operational efficiency.

Forecasting and Planning: OR methodologies include robust forecasting models that help organizations anticipate future trends and plan accordingly. This proactive approach to planning allows businesses to adapt to changing market conditions and make strategic decisions with greater confidence.

1.4.2. LIMITATIONS

i. **Quantitative Bias:** OR's emphasis on numerical data may overlook qualitative factors that influence decision-making. Human judgment, cultural considerations, and non-measurable variables can significantly impact outcomes but may not be fully captured by OR models, potentially leading to suboptimal decisions.

ii. **Complexity in Variable Handling:** Large-scale problems with numerous variables pose challenges in model formulation and analysis. OR may struggle to effectively manage and interpret vast amounts of data, increasing the risk of computational errors and limiting the scalability of solutions.

iii. **Resistance to Change:** Implementing OR initiatives often face resistance from stakeholders within organizations. Employees and decision-makers may be hesitant to adopt new methodologies or technologies introduced by OR, preferring familiar practices. This resistance can hinder the successful implementation and integration of OR solutions into existing workflows.

iv. **Cost and Time Intensive:** Developing and implementing OR solutions can be resource-intensive, requiring significant investments in specialized software, training, and skilled personnel. The complexity and time required to build and validate OR models can also pose practical challenges for organizations, particularly those with limited resources or tight deadlines.

v. **Ethical Considerations:** OR decisions and models may have ethical implications, especially when they involve optimizing outcomes that impact human lives, such as healthcare allocation or transportation logistics. Ensuring that OR solutions uphold ethical standards and consider social responsibilities is crucial for maintaining trust and credibility.

1.4.3. INVENTORY MANAGEMENT: ESSENTIAL FOR BUSINESS OPERATIONS

Inventory management is crucial for businesses seeking to optimize operations, meet customer demand efficiently, and control costs. It involves the strategic handling and optimization of stocks throughout their lifecycle, encompassing raw materials, work-in-progress items, and finished goods.

1.5. TYPES OF INVENTORIES

1.5.1. DIRECT INVENTORY

i. **Production Inventory:** These are raw materials essential for manufacturing processes. They undergo transformation through various stages of production, contributing directly to the creation of finished products. Managing production inventory involves ensuring consistent supply chains and timely procurement to support manufacturing operations effectively.

ii. **Work-in-Progress Inventory:** Items in various stages of production that require further processing before becoming finished goods. This inventory stage is critical for managing production workflows, ensuring that each production step proceeds smoothly without delays or interruptions.

iii. **Finished Goods Inventory:** Consists of completed products ready for distribution or sale. Managing finished goods inventory involves balancing stock levels to meet customer demand while minimizing storage costs and obsolescence risks. Efficient management ensures timely delivery and customer satisfaction.

iv. **Maintenance, Repair, and Operating (MRO) Inventory:** Supplies necessary for equipment maintenance and daily operations. MRO inventory includes spare parts, lubricants, and other consumables essential for keeping machinery and facilities operational. Effective management of MRO inventory minimizes downtime, enhances equipment reliability, and supports uninterrupted production schedules.

v. **Miscellaneous Inventory:** Includes items such as packaging materials and consumables necessary for production and logistics operations. While not directly part of the final product, these items are essential for maintaining operational efficiency and ensuring smooth logistics management.

1.5.2. INDIRECT INVENTORY

i. **Pipeline Inventory:** Goods in transit between locations or processes within the supply chain. Pipeline inventory management focuses on optimizing transportation and logistics to ensure timely delivery and minimize transit times. This inventory type is crucial for maintaining supply chain efficiency and meeting customer delivery expectations.

ii. **Safety Stock:** Buffer inventory maintained to safeguard against variability in demand or supply chain disruptions. Safety stock acts as insurance against unexpected fluctuations in customer orders or delays from suppliers, ensuring continuity in operations and mitigating risks associated with stockouts.

iii. **Decoupling Inventory:** Held to manage discrepancies between production rates and demand variability. This inventory buffer helps smooth out production schedules, allowing businesses to adjust output levels without disrupting overall operations. It plays a vital role in balancing inventory levels to meet fluctuating customer demands efficiently.

iv. **Seasonal Inventory:** Stocks maintained to accommodate seasonal fluctuations in customer demand. Businesses anticipate peak seasons or promotional periods and stockpile inventory accordingly to ensure product availability. Effective management of seasonal inventory supports sales forecasts, prevents stockouts during high-demand periods, and optimizes inventory turnover rates.

v. **Lot Size Inventory:** Strategy leveraging economies of scale through bulk purchasing. By buying larger quantities at discounted rates, businesses reduce procurement costs per unit and optimize inventory holding costs. This approach is beneficial for products with stable demand patterns and helps businesses manage inventory levels efficiently while maximizing cost savings.

vi. **Anticipation Inventory:** Accumulated in anticipation of future demand fluctuations or new product launches. This proactive inventory management approach involves forecasting market trends, customer preferences, and promotional activities to ensure adequate inventory levels. Anticipation inventory supports strategic business growth initiatives and minimizes the risks of understocking during demand surges.

1.6. CHARACTERISTICS OF EFFECTIVE INVENTORY SYSTEMS

Effective inventory management systems are characterized by several key attributes that are crucial for enhancing operational efficiency, minimizing costs, and ensuring seamless supply chain operations:

i. **Optimized Utilization:** Effective inventory systems prioritize maximizing the use of available resources while minimizing waste and obsolescence. By implementing just-in-time (JIT) inventory practices and using advanced forecasting models, businesses can ensure that materials are used efficiently throughout their lifecycle. This optimization helps in maintaining optimal stock levels without unnecessary accumulation or shortages, thus supporting cost-effective inventory management practices.

ii. **Timely Supply Management:** A critical aspect of effective inventory systems is maintaining timely supply management. This involves accurately forecasting demand, monitoring inventory levels in real-time, and ensuring that stock levels are sufficient to meet customer orders promptly. By leveraging automated inventory tracking systems and cloud-based inventory management software, businesses can prevent stockouts and minimize carrying costs, thereby improving overall operational efficiency.

iii. **Efficiency:** Efficient inventory systems leverage technology and data analytics to streamline procurement and production processes. By integrating demand planning software and inventory optimization tools, businesses can analyze historical data, market trends, and customer preferences to make informed decisions regarding inventory levels and reorder points. This proactive approach helps in optimizing inventory turnover rates, reducing lead times, and enhancing supply chain efficiency.

iv. **Loss Prevention:** Effective inventory management systems implement robust measures to prevent shrinkage, damage, and spoilage during handling and storage. This includes implementing strict inventory control procedures, conducting regular inventory audits, and investing in secure storage facilities. By minimizing inventory losses, businesses can protect the value of their inventory and reduce unnecessary costs associated with replacement or disposal.

v. **Cost Efficiency:** Cost efficiency is a fundamental characteristic of effective inventory systems. These systems strike a balance between inventory holding costs (such as storage, insurance, and obsolescence) and procurement costs (including ordering, transportation, and handling). By adopting lean inventory practices, implementing vendor-managed inventory (VMI) agreements, and negotiating favorable terms with suppliers, businesses can minimize total inventory expenditure while maintaining adequate stock levels to meet customer demand.

vi. **Operational Continuity:** Maintaining operational continuity is essential for effective inventory management systems. This involves developing contingency plans and maintaining safety stocks to mitigate potential disruptions in the supply chain. By collaborating closely with suppliers, diversifying sourcing strategies, and implementing supply chain visibility tools, businesses can anticipate and respond to unforeseen events, ensuring uninterrupted operations and fulfilling customer commitments.

vii. **Risk Management:** Effective inventory systems focus on identifying and managing risks associated with inventory shortages, overages, and operational inefficiencies. Proactive risk management strategies include scenario planning, establishing alternative sourcing options, and conducting comprehensive risk assessments. By integrating risk mitigation strategies into inventory management practices, businesses can minimize financial losses, enhance supply chain resilience, and maintain business continuity during challenging times.

viii. **Sustainability:** Increasingly, effective inventory management systems prioritize sustainability and environmental stewardship. By adopting ecofriendly packaging materials, optimizing transportation routes to reduce carbon emissions, and implementing recycling programs for packaging and returned products, businesses can align inventory practices with corporate sustainability goals. This not only reduces environmental impact but also enhances brand reputation and attracts environmentally conscious consumers.

1.7. INVENTORY MANAGEMENT STRATEGIES AND COSTS

Effective inventory management strategies encompass meticulous planning, organized execution, and rigorous control measures aimed at optimizing inventory levels while managing associated costs. These strategies are essential for businesses to maintain adequate stock levels to meet customer demand promptly without incurring unnecessary expenses.

i. **Comprehensive Planning:** Successful inventory management begins with thorough planning. Businesses must forecast demand accurately based on historical data, market trends, and seasonal fluctuations. By anticipating demand patterns and aligning procurement strategies accordingly, organizations can minimize stockouts and overstock situations, thereby optimizing inventory levels.

ii. **Organized Execution:** Organized execution involves implementing structured inventory control measures and operational procedures. This includes establishing clear inventory management policies, defining reorder points, and implementing inventory tracking systems. By streamlining processes and reducing manual handling errors, businesses can enhance operational efficiency and reduce labor costs associated with inventory management.

iii. **Cost Optimization:** A critical aspect of inventory management is controlling costs across various stages of the inventory lifecycle. Key cost components include procurement costs (such as ordering and transportation expenses), storage costs (including rent, insurance, and utilities), handling costs (related to receiving, picking, and packing), and maintenance costs (for equipment and facilities).

iv. **Procurement Costs:** Efficient procurement strategies involve negotiating favorable terms with suppliers, leveraging bulk purchasing discounts, and adopting just-in-time (JIT) inventory practices to minimize inventory holding costs. By optimizing procurement processes, businesses can reduce lead times and avoid over-ordering, thereby lowering overall procurement expenses.

v. **Storage Costs:** Managing storage costs requires optimizing warehouse space utilization, implementing efficient inventory layout designs, and utilizing advanced inventory management software to track inventory movements. Businesses should also consider adopting lean inventory principles to reduce excess stock levels and minimize storage-related expenses.

vi. **Handling Costs:** Efficient handling practices involve automating repetitive tasks, optimizing picking and packing processes, and investing in technology-driven solutions like barcode scanning and automated storage systems. By reducing manual handling errors and improving workflow efficiency, businesses can lower labor costs and enhance operational productivity.

vii. **Maintenance Costs:** Maintaining the inventory in good condition is crucial for minimizing maintenance expenses. Implementing preventive maintenance programs for storage equipment and investing in quality storage solutions can help businesses mitigate repair costs and extend the lifespan of inventory storage facilities.

1.8. ADVANTAGES OF INVENTORY MANAGEMENT

Effective inventory management offers numerous advantages for businesses across various industries, enhancing operational efficiency, customer satisfaction, and overall profitability.

i. **Enhanced Resource Utilization:** Inventory management maximizes the utilization of available resources by optimizing production schedules and minimizing waste. By maintaining optimal inventory levels, businesses can ensure that raw materials are used efficiently, production processes run smoothly, and operational costs are minimized. This efficient resource allocation contributes to streamlined operations and improved profitability.

ii. **Improved Customer Service:** A well-managed inventory system ensures timely order fulfillment and reduces lead times, thereby enhancing customer satisfaction. Meeting customer demand promptly increases reliability and trust, fostering long-term customer relationships. Satisfied customers are more likely to repeat purchases and recommend the business to others, contributing to business growth and competitiveness.

iii. **Cost Savings:** Effective inventory management achieves cost savings through various means. By leveraging economies of scale through bulk purchasing, businesses can negotiate better prices with suppliers and reduce procurement costs. Efficient inventory handling practices, such as just-in-time (JIT) inventory systems, minimizing storage costs and inventory holding expenses. These cost-saving measures contribute directly to improving the company's bottom line and overall profitability.

iv. **Risk Mitigation:** Proactive inventory planning and management mitigate risks associated with stockouts, overstocking, and inventory obsolescence. By accurately forecasting demand, businesses can maintain optimal inventory levels to meet customer needs without an excess inventory that may become obsolete. Proper risk management strategies reduce financial losses, enhance operational resilience, and enable businesses to adapt swiftly to market fluctuations and unforeseen challenges.

1.9. CHALLENGES IN INVENTORY MANAGEMENT

Inventory management presents several challenges that businesses must navigate to maintain operational efficiency and profitability.

i. **Space Constraints:** Effective inventory management requires adequate storage space, which becomes a challenge as inventory volumes increase. Larger storage facilities incur higher operational costs, including rent, utilities, and maintenance. Efficient space utilization strategies such as vertical storage systems and optimized warehouse layouts are essential to minimize storage expenses and maximize operational efficiency.

ii. **Cost Considerations:** Inventory management involves various costs, including storage facilities, insurance premiums to protect against losses, and expenses related to handling equipment and personnel. Balancing these costs is crucial for businesses to maintain profitability. Strategies such as implementing lean inventory practices, optimizing inventory turnover rates, and negotiating favorable terms with suppliers can help mitigate these costs while ensuring efficient inventory management.

iii. **Risk of Obsolescence:** One of the significant risks in inventory management is inventory obsolescence, where goods become outdated or unusable due to changes in consumer preferences, technological advancements, or market trends. Managing inventory turnover, closely monitoring product lifecycles, and implementing effective inventory forecasting techniques are essential to minimize the risk of obsolescence. Businesses must continuously assess market demands and adjust their inventory strategies accordingly to prevent losses associated with obsolete inventory.

iv. **Complex Logistics:** Managing diverse inventory types, supply chains, and global distribution networks presents logistical challenges. Effective logistics and supply chain management strategies are crucial to ensure timely delivery of goods while minimizing transportation costs and optimizing inventory levels across different locations. Leveraging technology such as inventory management software, automated tracking systems, and real-time data analytics helps streamline logistics operations and enhance overall supply chain efficiency.

1.10. FOCUS OF MY RESEARCH: UNDERSTANDING BACKORDERING

My research work primarily delves into the various aspects and types of back-ordering in supply chain management. Back ordering can signify a product currently in production or one yet to commence production. The essence of effective back ordering lies in clear and consistent communication. By informing customers about back orders, suppliers provide critical updates on the status of goods, including what is on its way and the expected arrival time. This transparency enables both suppliers and customers to maintain uninterrupted operations, even in the face of delays. However, it is important to note that backorders can influence inventory levels and associated holding costs. (Abad 1996) further emphasized this by analyzing optimal pricing and lot-sizing decisions under perishability and partial backordering, showing how backorders significantly affect total cost structures and customer service levels.

1.10.1. BACK-ORDERING: AN IN-DEPTH OVERVIEW

Introduction to Backordering: Backordering is a supply chain strategy where orders for out-of-stock products are accepted and fulfilled once the items become available. This approach enables businesses to continue sales even when inventory levels are temporarily insufficient to meet demand. Effective communication between suppliers and customers is crucial to ensure transparency regarding product availability and delivery timelines.

1.10.1.1. TYPES OF BACKORDERING

1. **Current Production Backorders:**

   These back orders represent goods that are currently in the production process. The supplier informs the customer that the product is being manufactured and will be available once production is complete.

2. **Future Production Backorders:**

   These back-orders involve goods that have not yet begun production. The supplier notifies the customer of the expected start date for production and provides an estimated delivery timeframe.

1.10.2. CHALLENGES IN BACKORDERING

1. **Customer Dissatisfaction:** If not managed properly, backorders can lead to customer frustration and dissatisfaction. Delays in product delivery may result in lost sales and damage to the company’s reputation.

2. **Supply Chain Disruptions:** Backorders can be affected by disruptions in the supply chain, such as supplier delays, transportation issues, or production .customer satisfaction.
3. **Forecasting Difficulties:** Accurate demand forecasting is essential for effective backorder management. Misjudging demand can lead to either excessive backorder or overproduction, both of which have financial implications.
4. **Increased Administrative Work:** Managing back orders requires additional administrative efforts, including tracking, communication, and coordination with suppliers and customers. This can increase the workload for supply chain and customer service teams.

1.10.3. BEST PRACTICES FOR BACKORDER MANAGEMENT

1. **Clear Communication:** Maintain transparent and proactive communication with customers about back-order status, expected delivery dates, and any changes to the timeline. This helps manage customer expectations and reduces frustration.
2. **Robust Inventory Tracking:** Implement advanced inventory management systems that provide real-time visibility into stock levels, production schedules, and order statuses. Accurate tracking helps in managing backorders efficiently.
3. **Supplier Collaboration:** Develop strong relationships with suppliers to ensure timely delivery of raw materials and finished products. Collaborative planning and communication with suppliers can mitigate the risk of supply chain disruptions.
4. **Demand Forecasting:** Use sophisticated forecasting techniques, such as predictive analytics and market trend analysis, to anticipate demand accurately. This helps in planning inventory levels and reducing the likelihood of backorders.
5. **Flexible Production Planning:** Adopt flexible production strategies that can quickly adapt to changes in demand. This may include scalable manufacturing processes and agile supply chain networks.

1.10.4. CAUSES OF BACKORDERS

1. Unusual Demand: Sudden spikes in customer demand can lead to backorders, as the existing inventory may not be sufficient to meet the unexpected surge. This situation often occurs during peak seasons, promotional periods, or in response to market trends that significantly increase product popularity.

2. Low Safety Stock: Inadequate safety stock levels can result in backorders when regular inventory runs out before replenishment. Safety stock acts as a buffer against uncertainties in demand and supply. Insufficient safety stock leaves the supply chain vulnerable to disruptions and delays, causing backorders.

3. Supplier Issues: Problems with the supplier's end, such as production delays, quality control issues, or logistical challenges, can lead to backorders. Reliable supplier performance is crucial for maintaining a steady supply of goods. Any disruption in the supplier's operations directly impacts the ability to fulfill orders on time.

4. Use of Multiple Suppliers: Relying on multiple suppliers can sometimes complicate the supply chain, especially if coordination and communication are lacking. Variations in lead times, order processing speeds, and delivery schedules among different suppliers can cause misalignments, resulting in backorders.

5. Variation in Order Patterns: Fluctuations in the ordering patterns of customers can create challenges in maintaining optimal inventory levels. Irregular or unpredictable ordering behavior makes it difficult to forecast demand accurately, leading to potential stockouts and subsequent backorders.

1.10.5. ADVANTAGES OF BACKORDERS

i. **Indicator of Healthy Demand:** One significant advantage of backorders is that they can signal a healthy increase in demand for a product. When managed properly and kept to a minimum, back orders reflect strong customer interest and consistent sales growth. This valuable insight can help companies plan for future growth, adjust production schedules, and optimize inventory management to meet the rising demand efficiently.

ii. **Lower Inventory Levels:** Backorders allow companies to maintain lower inventory levels, which in turn frees up cash flow for other operational needs and expansion opportunities. By not holding excessive stock, businesses can allocate resources more effectively, ensuring that capital is available for strategic investments, such as new product development, marketing campaigns, or facility upgrades.

iii. **Improved Cash Flow:** Operating with leaner inventory levels due to backorders enhances cash flow management. Instead of tying up funds in excess inventory, companies can use the available cash to support day-to-day operations, invest in growth initiatives, or navigate unexpected financial challenges. This improved liquidity strengthens the overall financial health of the business.

iv. **Tax Benefits:** Lower inventory levels can also result in reduced tax liabilities. Inventory is often subject to taxation based on its value, and by minimizing the amount of stock on hand, companies can decrease their taxable inventory value. This reduction in tax obligations can lead to significant cost savings, contributing to overall profitability.

v. **Reduced Labor and Holding Costs:** Backorders can help companies reduce labor and holding costs associated with managing and storing large inventories. With fewer products to handle, businesses can streamline warehouse operations, reduce staffing requirements, and minimize storage expenses. These savings directly impact the bottom line, making operations more cost-effective and efficient.

vi. **Enhanced Profitability:** By running leaner and managing back orders effectively, companies can drive greater profitability. The combination of improved cash flow, lower tax liabilities, and reduced operational costs creates a more efficient and financially robust business model. This lean approach not only boosts immediate profits but also supports sustainable long-term growth.

vii. **Demand Signaling:** Backorders can indicate a strong demand for a product. By tracking back-order levels, companies can gain insights into market trends and adjust production schedules accordingly. This helps in planning for future inventory needs and aligning supply with demand.

viii. **Inventory Management:** Backorders allow companies to maintain lower inventory levels, reducing the need for excessive stock. This lean inventory approach minimizes holding costs, including storage, insurance, and obsolescence, while freeing up capital for other business operations.

ix. **Cash Flow Optimization:** Operating with lower inventory levels improves cash flow management. Companies can allocate resources to other critical areas, such as research and development, marketing, or infrastructure improvements, enhancing overall financial stability.

x. **Cost Savings:** Reduced inventory levels lead to lower taxes, as inventory is often subject to valuation-based taxation. Additionally, companies save on labor and handling costs associated with managing large inventories, contributing to overall cost efficiency.

xi. **Customer Relationship Management:** Effective backorder management fosters strong customer relationships. By keeping customers informed about product availability and delivery schedules, companies can build trust and loyalty. Satisfied customers are more likely to return for future purchases and recommend the business to others.

xii. **Market Responsiveness:** Backorders provide flexibility in responding to market changes and fluctuations in demand. Companies can quickly adjust production and supply chain strategies to meet new customer needs without overcommitting resources to inventory that may not sell.

Chapter 2

Literature Review

2. LITERATURE REVIEW

In recent years, supply chain management has become increasingly important for businesses across various industries. Consequently, an increasing emphasis has been placed on different facets of supply chain management, with inventory management being a prominent area of interest. Despite the abundance of literature concerning supply chain inventory models, this topic continues to receive significant attention, there remains a research gap in understanding the most effective approach to managing inventory in complex and dynamic environments. This review article is designed to explore various models within the realm of supply chain inventory and pinpoint areas where existing research may be lacking or incomplete.

2.1. BACKGROUND AND SIGNIFICANCE OF THE RESEARCH GAP

This segment provides a synopsis of the difficulties encountered in managing inventory within the supply chain. and the importance of addressing the research gap. In today's highly competitive marketplace, the role of supply chain management has evolved to become a vital component in ensuring business success. Effective inventory management is key, optimizing inventory levels to ensure smooth operations and cost reduction. Despite extensive literature, there remains a research gap in understanding the best approach for inventory management in complex environments. This article aims to contribute to knowledge by analyzing existing literature on inventory models.

2.1.1. MOTIVATION AND OBJECTIVES OF THE LITERATURE REVIEW

Traditionally, back ordering has been an important aspect of inventory management and this directly influences cost, service levels, and decision making within supply chains. (Veinott et al. 1963) laid out a systematic treatment of inventory systems and generalized this to perishability and partial backordering by (Abad 1996). Despite these contributions, shortages, degeneration and price sensitive demand continue to challenge finite planning horizons. It is reported that recent studies have proposed (Sujit Kumar De and Matadial Ojha 2023) the development of the models by including refinement and exponential and partial backordering with fuzzy environments to suit the complexities of the real world more(Surendra Vikram Singh Padiyar et al. 2023). This gives the impetus of the present study, which will attempt to generalize the classical theory to dynamic scenarios. The objective of this review is to provide a thorough understanding and critique of assorted models associated with supply chain inventory while pinpointing existing research gaps and limitations within the current body of literature. The key aims of this article encompass:

i. To provide an overview of different types of inventory models, including those that consider backordering, shortages, and price-sensitive demand.
ii. To examine and compare linear and quadratic complete backordering models, identifying their strengths and limitations.
iii. To describe and evaluate the exponential complete backordering comparison inventory model, including its effectiveness in managing shortages for both linear trends and price-sensitive demand.
iv. To delve into the cost-efficient production quantity inventory framework with partial and complete backordering, evaluating its effectiveness and efficiency in managing inventory levels.
v. To investigate fuzzy backordering in inventory models with shortages, analyzing the impact of this concept on the planning horizon and comparing it to existing literature.
vi. To conclude the analysis of existing literature and identify overarching research gaps and limitations in supply chain inventory models.
vii. To provide a comprehensive summary of the reviewed literature and research findings, highlighting key insights and limitations in current inventory management strategies.
viii. Lastly, offer closing remarks on the importance of further research and advancements in logistics and inventory control to meet the demands of complex and dynamic environments.
ix. To successfully manage supply chain inventory, it is essential to have a thorough understanding of different inventory models and their implications.

However, the existing literature on logistics inventory models is vast and diverse, making it difficult to navigate and assess the various approaches. Therefore, this literature review aims to offer an all-encompassing synopsis of the different supply chain inventory models and identify the research gaps and limitations within them, and Figure 2.1 shows the keywords of backorder model.

Fig. 2. 1 Showing supply chain with different types of back order

2.2. OVERVIEW OF SUPPLY CHAIN INVENTORY MODELS

Inventory models refer to mathematical or analytical techniques used to optimize inventory levels and decision-making in supply chain management. They help businesses determine the most efficient strategies for ordering, stocking, and replenishing inventory. There are several types of inventory models, such as:

i. **Optimal Order Quantity (EOQ) Model:** the ideal order quantity to minimize overall inventory costs by factoring in ordering costs, holding costs, and demand patterns.

ii. **Just-in-Time (JIT) Model: A** model that minimizes inventory by delivering materials and goods precisely when required in the production process, focusing on waste reduction, efficiency improvement, and lean inventory maintenance.

iii. **Periodic Review Model:** In this model, inventory levels are reviewed at fixed intervals, and orders are placed to replenish inventory to a predetermined level. It helps balance inventory holding costs and stockouts.

iv. **Continuous Review Model:** Also known as the reorder point model, this approach continuously monitors inventory levels and triggers an order when the stock reaches a specified reorder point. It helps prevent stockouts while minimizing inventory carrying costs.

v. **ABC Analysis:** ABC analysis classifies inventory items according to their value or significance. This classification system allows companies to concentrate their efforts and distribute resources strategically, facilitating optimal inventory levels, financial effectiveness, and heightened customer satisfaction.

Considering back-ordering, shortages, and price-sensitive demand in supply chain models is crucial for several reasons. Firstly, back-ordering helps maintain customer satisfaction by allowing orders to be fulfilled even when items are temporarily out of stock. Secondly, accounting for shortages enables businesses to proactively manage disruptions and minimize the impact on operations. Thirdly, incorporating price-sensitive demand allows for effective pricing strategies and revenue optimization. Lastly, addressing these factors enhances the accuracy and realism of supply chain models, leading to improved decision-making and overall supply chain performance.

Firstly the classical EOQ formula was discovered by (Harris 1990). The inaugural publication in inventory management was penned by (Raymond 1931), (Ouyang et al. 1996) proposed a model by taking shortages and solving the total shortages as a combined form of lost sales and backorder. (Scarf and Herbert Scarf 1959) estimated a stochastic model of multi-period with shortages and given a policy (s, S) for an optimal solution with backorders. (Veinott et al. 1963) think that finding the exact backorder cost was a hard task so they developed moa del which calculates the backorder cost. They considered back-ordering as a constant function, with shortages. (Zangwill 1969) proposed a multi-period model along with shortages and backorders. (Fogarty and Aucamp 1985) gave the model with shortages and back-ordering. (Aardal et al. 1989) proposed a model by taking the random demand (q, r) model given by Hadley and Within. But backorders are not considered but they assure that the yearly backorders cannot cross the upper boundaries. (Çetinkaya and Parlar 1998) established a generalized model by taking two different types of backorder costs.

(Liao and Shyu 1991) By assuming a fixed batch size model and normally distributed demand, the goal is to minimize the anticipated total cost that includes backorders, while simultaneously adjusting the processing time (Pan et al. 2004) an inventory model has been formulated considering negotiable lead time and backorder discounts. In this approach, the supplier factors in potential future losses and profits.

Consequently, this could enable the customer to acquire the necessary items promptly, allowing them to resume production without substantial delays(Bayındır et al. 2007) established an EPQ model taking general stock-dependent backordering. (San-José et al. 2014) The authors suggested an optimal purchase volume model for a singular product, considering partial backlogs and time-dependent shortages, with allowances for partial backordering. The model presumes a constant portion of the demand rate being backlogged at any particular time. Implementing this model led to the development of an ideal policy, thus minimizing inventory expenses. (Pan and Hsiao 2005) Expanded upon the research conducted by (Ouyang et al. 1996). Adopted a comprehensive inventory system in which shortages, backorders, and lead times are all subject to negotiation. They elaborated on two models: the first one considers normally distributed demand, and the second one deals with generally distributed demand. In these scenarios, a supplier might propose a backorder cost reduction to customers willing to wait. (Sazvar et al. 2013) Formulated an inventory model for goods subject to deterioration, factoring in instances of shortages and comprehensive back-ordering scenarios. (Naser Ghasemi et al. 2013) proposed two models taking holding cost as increasing continuous functions. The first model with no shortages & the second model is with shortages and complete backordering. (Kumar et al. 2021) proposed an economic policy by taking demand as power depending on time, with shortages and complete backordering.

2.2.1. LINEAR AND QUADRATIC COMPLETE BACKORDERING WITH SHORTAGES AND PRICE-SENSITIVE DEMAND

(Grubbström and Erdem 1999) Implemented an arithmetical approach to validate the equations for Optimal Purchase Quantity (Optimal Order Quantity) and Optimum Production Volume (Optimal Production Quantity), taking into account a singular cost related to backorders, which is solely linear in nature (depending on time). (Hu et al. 2009) proposed a model of backordering a as linear & quadratic function, partially backlogged. (Cárdenas-Barrón 2001) Provided an algebraic method to substantiate the equations for the EOQ and EPQ, incorporating a singular backordering cost that is linearly dependent on time. (Taleizadeh et al. 2012) Proposed an Optimal Purchase Quantity model by taking linear holding cost (depend on price), partially backlogged &

backorder is a linear function. (Taleizadeh et al. 2013b) proposed two EOQ models (a) by taking holding cost linear dependent on time, partially backlogged, backorders are linear function, lost sale cost as fixed and partially delayed payments. (b) by taking holding cost linear depends on time, partially backlogged, backorders are linear functions, lost sale cost is fixed & partially prepayments. (Yang 2014) established an Optimal Purchase Quantity model by taking non-linear stock-dependent holding cost, partial backlogging, backorders are linear, a lost sale is fixed, and the demand rate is stock dependent.

In the research paper (Luo 2019) Analytic and algebraic techniques have been employed by researchers to resolve inventory models that include fixed and linear backorder costs. This process led to the division of the viable domain and the proposal of a broader, generalized solution. However, further examination reveals questionable results and the need for future researchers to address this open problem in solving such inventory models. (Lee 1994) The research article introduces an inventory model with partial backorders, considering deterministic demand, normally distributed lead time, and a varying backorder ratio. A goal-oriented function is designed to reduce costs associated with backorders that are proportional to time, as well as fixed penalties incurred per unit lost. A repeating, step-by-step solution method has been devised to identify the best reorder point and the quantity to order. To illustrate this suggested technique, a numeric example has been given. In the research paper (Seliaman et al. 2020) Using algebraic methods, inventory models are broadened and generalized for the optimal decision-making in supply chain inventory, specifically focusing on orchestrating production-inventory decisions within a unified supply chain system that spans n stages. The research devises a mathematical model for total cost and creates a repeatable algebraic algorithm for making optimal decisions regarding inventory restocking. Numerical examples demonstrate the effectiveness of the proposed algorithm, The results demonstrate a decrease in total system cost when compared to the traditional cycle time mechanism. The research paper (Sphicas 2006) Examines the EOQ and EPQ models featuring fixed and linear costs associated with backorders, proving all formulas using algebra. Two cases are identified regarding the allowance of backorders. The results encompass various known special cases. (Kumar et al. 2021) This paper introduces a multi-product economic inventory model that accounts for items with demand that varies over time following a power law, incorporates shortages, and allows for comprehensive backordering.

The production rate is assumed proportional to the demand rate, while demand is assumed to decrease linearly with price. The paper aims to maximize overall profit and calculate optimal costs, prices, re-order points, and scheduling periods. The model permits shortages and full backordering, and its validity is established through a numeric example and sensitivity analysis using Mathematica software. It presents tables containing optimal values. (Chung and Cárdenas-Barrón 2012) This paper addresses the EOQ/EPQ inventory models with fixed and linear backorder costs, providing a comprehensive and analytical solution procedure for locating and ensuring optimal solutions. It establishes necessary and sufficient conditions for the existence of optimal solutions and identifies the optimal solutions even when these conditions are not satisfied. The research culminates with two lemmas and four beneficial theorems that assist in achieving optimal solutions for both inventory challenges. The research paper (Chung et al. 2009) centers on probing the convex nature of the total annual cost function within inventory models, particularly those that incorporate linear and fixed backorder expenses. The study explores the convex characteristics of total annual cost functions and introduces solution strategies for both inventory models based on these convex properties. The paper introduces the (M, R, T) inventory model incorporating quadratic backorder expenses and continuous processing times, extending the prior model that featured constant lead times. The inventory costs under constant lead time are averaged using a gamma distribution assumption, and The Bessel function with an imaginary argument is widely used in calculating the average across the various states of processing time. The results from the earlier model serve as the foundation for developing the model presented in this paper. The research article (M. O. Omorodion and J. Babalola 2014) This study takes into account a continuous processing time modelled as a gamma distribution, quadratic backorder expenses, and demand following a normal distribution. The expected backorder costs are computed and averaged over lead time states using the Bessel function. The inventory cost, which comprises anticipated backorder costs and existing inventory expenses, is assessed across various states of processing time.

2.2.2. EXPONENTIAL COMPLETE BACKORDERING COMPARISON INVENTORY MODEL

The exponential complete backordering model is crucial for optimizing inventory and reducing costs while ensuring accurate demand forecasting. It helps maintain customer satisfaction by minimizing back orders and enabling timely order fulfillment. Some literature on backorder as an exponential function,(Cohen 1977) gives the first model by assuming the deterioration as an exponential rate selling price is constant. He gave the optimal as well as ordering policy for no shortages & complete backordering cases. (Abad 1996) The initial model that incorporates a partial backordering rate, which escalates in relation to the delivery time, was created. (Pentico et al. 2015) gave the two approximation models taking partial backordering and backordering as a function of the exponential with deterioration process.

(Lee et al. 2006) This paper introduces an extended model based on Wu and Tsai's work, where the rate of backorders is treated as a regulating factor. The study investigates how the backorder rate is influenced by the processing time and eases presumptions regarding the shape of the lead time demand distribution. The problem is addressed using the minimax distribution-free procedure, which integrates a service level constraint in place of a stock-out term. The study showcases computational procedures to determine the optimal order quantity and processing time, supplemented by numerical examples that elucidate the results. The paper (Omorodion et al. 2014) examines the (Q, R) inventory model, taking the backorder cost as an exponential function of the backorder time. It derives expected backorder costs and introduces the assumption of demand following a normal distribution, along with the associated mathematical properties. The study also introduces the primary derivatives of the inventory backorder expenses. The research article (Omorodion 2014) The study draws out the inventory expenses for a model featuring exponentially increasing backorder costs, unpredictable supply, and continuous processing times, further extending the findings of the initial series. The model incorporates averaging the exponential backorder costs from series 1 over different lead time states, where lead time follows a gamma distribution. (Gholami-Qadikolaei et al. 2013) This research introduces inventory models with a mix of backorders and lost sales, featuring a manageable negative exponential backorder rate. It takes into account aspects such as processing time, storage space restrictions, and budgetary constraints. The study employs chance-constrained programming and minimax distribution-free methodologies, offering a numerical instance to depict the models and their respective solution strategies.

Accounting for shortages is essential when addressing inventory management for both linear trends and price-sensitive demand. Recognizing the impact shortages have on customer satisfaction, sales revenue, and overall business performance, incorporating their effects in inventory models enables informed decision-making regarding order quantities, pricing strategies, and inventory replenishment, ultimately minimizing stockouts and optimizing customer service levels. Linear trend and price-sensitive demand models have direct relevance to real-world supply chain modeling. The linear trend model enables businesses to project long-term demand patterns and align their operations accordingly. Price-sensitive demand models consider the impact of pricing on consumer behavior, aiding companies in optimizing pricing strategies to maximize revenue and competitiveness. By incorporating these models, businesses can make informed decisions about production, inventory, pricing, and supply chain optimization, resulting in improved operational efficiency, customer satisfaction, and profitability in dynamic market environments.

2.2.3. OPTIMAL PRODUCTION QUANTITY INVENTORY MODEL WITH PARTIAL AND COMPLETE BACKORDERING

The concepts of partial and complete backordering are important aspects of inventory management. Partial backordering refers to the scenario where only a portion of customer demand can be fulfilled immediately, while the remaining quantity is backordered. In contrast, complete backordering occurs when all unfulfilled demand is backordered, resulting in potential delays for customers. Recent research indicates that there has been a transition in the use of traditional mathematical modeling to AI/ML-based optimization, which enhances forecasting, inventory optimization, and cost effectiveness in dairy supply chains. (Rosario Huerta-Soto et al. 2023) The study on the improvement of operational excellence through modernization and focus on sustainability and environmental issues in DSC management is presented by PRISMA methodology. Understanding and analyzing these concepts is crucial for effective inventory control and customer satisfaction in supply chain operations. The research paper (Saithong et al. 2020) outlined a scenario wherein unpredictable supply interruptions occur, thereby necessitating the identification of optimal parameters for an effective inventory policy. This is particularly applicable when examining a two-tier system that includes a supplier and a retail element. The importance lies in appropriately identifying and adjusting these variables to effectively manage inventory in the face of uncertain supply disruptions, ensuring smooth operations within the supply chain. (Akbarzadeh et al. 2015) This paper examines Optimal Production Quantity (EPQ) models involving items of imperfect quality, both in situations with and without shortages, within a two-stage supply chain that includes a single producer and purchaser. The focus is on evaluating the impact of a vendor-managed inventory (VMI) policy by minimizing total costs, determining optimal production lot size, and allowable backorder levels. Numeric illustrations and sensitivity analysis are utilized to demonstrate the efficiency of the improved supply chain model of VMI implementation. The research article (Oktavia et al. 2017) explores the utilization of Optimal Production Quantity (EPQ) models for effective inventory management and policy implementation. It specifically addresses EPQ models applied to products with partial backorders and their corresponding components. The study adopts a coordinated planning approach to ensure synchronization between component inventory management and EPQ planning for the final product, which incorporates partial backorders. Numerical simulations, based on data from UD. Adi Mulya is undertaken to illustrate the real-world applicability of these models. The authors of the research article (Djuliah 2018) discussed the rapid growth of the industrial world necessitates efficient operations and effective inventory control for companies to remain competitive. This model focuses on optimizing production and minimizing inventory costs by considering factors such as defective product rework, backorders, and associated costs, ensuring high-quality outputs. The authors (Huang and Shyu 2006) studied the Optimal Production Quantity (EPQ) model of backorders. Which was assumed defective items were not repaired and given a novel study that proposes a straightforward algebraic method as a substitute for the differential calculus technique to identify the ideal production batch size and the maximum permissible level of backorders, With the goal of reducing the anticipated yearly expense. (Pratiwi et al. 2013) The study develops basic EPQ and partial backorder EPQ models for inventory systems with insufficient supply to meet consumer demand. The partial backorder EPQ model yields optimal total inventory costs compared to the basic EPQ model, considering the option for customers to wait for their orders.

The research article (Maghfiroh 2017) described inventory control as crucial for companies to optimize production outcomes and minimize total inventory costs. The EPQ model with a backorder and variable setup costs proves to be more optimal, resulting in approximately 12% savings in total inventory expenses. The research article (Jayanti and Purwanto 2014) examined that the EPQ model minimizes total inventory costs by determining the optimal production quantity. This article discusses the EPQ model with backorder, incorporating a normal distribution to estimate average defective products and analyzing sensitivity to determine the minimum total inventory cost. The research paper (E. Ebrahimi et al. 2022) introduces a multi-objective programming model, S-EPQ, based on the triple bottom line (TBL) strategy. It optimizes total profits, environmental emissions, and turn costs cost while considering uncertain factors and shortages. The Particle Swarm Optimization algorithm is used for optimal solutions. A case study in the Iranian dairy industry validates the model and provides managerial insights for manufacturers.

2.2.4. INTEGRATION OF BACK ORDERING IN AN OPTIMAL PRODUCTION QUANTITY MODEL

Back ordering is incorporated into the Optimal Production Quantity (EPQ) model by including the costs and implications of backorders within the model's calculations. The EPQ model is employed to ascertain the most favorable production quantity that reduces the overall costs to the lowest level possible, considering elements such as setup costs, storage expenses, and procurement costs. To incorporate back ordering, the model is adjusted to account for the costs associated with not meeting customer demand immediately and fulfilling backlogged orders. A review of clinical laboratories (Kemlall Ramdass et al. 2023) study revealed that inventory management is important in providing reliable patient results. The study utilizing these principles, e.g., safety stock adjustment, demand planning, and removing obsolete stock, resulted in serious efficiency gains in cost and service ratio and suggested that more of these techniques should be implemented in real-time inventory systems. This involves considering additional cost components such as the cost of lost sales, backorder holding costs, and any other expenses incurred due to backlogs in production or delivery. By integrating these costs into the comprehensive cost function, the model is capable of determining the ideal production quantity that reduces the total expenses to the lowest possible level, taking into account the trade-offs between production, inventory, and backorder costs. This integration allows businesses to make informed decisions about production and inventory management, ensuring a balance between meeting customer demand and minimizing costs associated with back orders.

Back ordering offers several benefits to businesses. Firstly, it allows companies to maintain customer satisfaction by fulfilling orders even when products are temporarily out of stock. This helps retain customers and reduce the risk of lost sales. Back ordering also provides a competitive advantage by enabling businesses to offer a wider range of products without the need for excessive inventory levels. It helps optimize production and inventory management by allowing for economies of scale, reducing setup costs, and minimizing holding costs. Additionally, back ordering can improve cash flow as it reduces the need for immediate production and inventory investment. By leveraging backordering effectively, businesses can enhance customer service, streamline operations, and achieve cost efficiencies in their supply chain.

2.2.5. FUZZY BACKORDERING IN INVENTORY MODELS WITH SHORTAGES

Fuzzy backordering is a concept that incorporates uncertainty and vagueness in backordering decisions, recognizing that traditional all-or-nothing approaches may not be suitable. Unlike binary systems, fuzzy back ordering allows for partial fulfillment of orders based on available inventory, considering factors such as demand uncertainty and lead times. This approach balances customer satisfaction and inventory management by quantifying the degree of fulfillment. The implications for inventory management are significant, as fuzzy backordering provides a more realistic representation of supply chain dynamics and enables better decision-making in situations with limited information. It helps manage customer expectations, optimize resource allocation, minimize revenue loss, and maintain an optimal inventory level. However, implementing fuzzy backordering requires careful modeling and analysis to determine appropriate fulfillment thresholds and ensure overall supply chain performance.

The research article (Soni and Joshi 2015) This study proposes a periodically reviewed inventory model within a fuzzy stochastic environment, taking into account processing time and the backorder rate as variables to control. They proposed a model that incorporates fuzzy random demand and protection interval demand, and a solution procedure using the Scan and zoom method is provided for determining the optimal policy. (Mahata 2015) This study explores a two-tier supply chain inventory model within a framework of uncertainty, or a 'fuzzy' context, emphasizing a single vendor and single buyer situation where backorders are permitted. The decision variables in this context are the uncertain order quantity and shortage quantity, which are represented as triangular fuzzy numbers. A mathematical model is created for the collective total relevant cost in an uncertain, or 'fuzzy', context. The membership function of this model is calculated and clarified using the centroid method. This proposed model is then solved and juxtaposed against its clear-cut equivalent, assessing them based on the joint total relevant cost. (De and Mahata 2017) This study investigates the Optimal Order Quantity inventory model associated with backorders within an environment marked by high uncertainty, referred to as a 'cloudy fuzzy' environment, emphasizing the removal of fuzziness over time. A comparative study is undertaken involving clear-cut, generally uncertain (fuzzy), and highly uncertain (cloudy fuzzy) solutions, backed by diagrammatic representations and numeric illustrations. (Chang et al. 1998) This study delves into the inventory issue related to backorders, focusing particularly on backorders characterized by uncertainty, or 'fuzzy' backorders, using triangular fuzzy numbers. The results show that the fuzzy model allows for better utilization of economic fuzzy quantities despite a slightly higher total cost compared to the crisp model. A recent study suggested a multi-item fuzzy EOQ model to include the level of greenness, imprecision, inflation, trade credit, and partial backlogging. The findings of (Ravendra Kumar et al. 2022) indicate that profits decrease as imprecision increases and increase with trade credit and retailers can optimize profit by balancing greenness and selling price, whilst considering deterioration and inflation.

The incorporation of fuzzy backordering in an inventory model involves considering backorder quantities as fuzzy numbers, typically represented by triangular or trapezoidal fuzzy sets. This allows for the representation of uncertainty and imprecision in backorder quantities, enabling more realistic modeling and decision-making in inventory management. The principles of fuzzy logic and fuzzy set theory are utilized to compute uncertain or fuzzy costs, optimize fuzzy order quantities, and analyze the impact of fuzzy backordering on inventory performance measures. (Wulan and Andyan 2013) This paper introduce the EOQ fuzzy model with trapezoidal membership functions for estimating costs and determining optimal inventory levels. The fuzzy model shows an increase in optimal Q and V values, allowing for better cost estimation and problem-solving in inventory management. (W. F. Khan and Oshmita Dey 2022) This study presents a recurrently assessed inventory model that includes both backorders and unfulfilled sales, using the rate of backorders as a regulating factor. Fuzziness and randomness are considered, and an algorithm is proposed to minimize the total annual inventory cost.

This investigation focuses on incorporating fuzzy backordering into an inventory model, with a particular emphasis on elucidating the mathematical framework and the decision-making process entailed in this integration. The examination extends to assessing how the inclusion of fuzzy backordering affects the planning horizon, considering the potential implications and consequences. Moreover, a comprehensive survey of existing studies is conducted to gather valuable insights and deepen our understanding of the field, providing a broader perspective on the subject matter.

Table 2. 1 Summary of literature on the supply chain from the year 1959

Author & year	Inventory Model	Shortage	Backorder	Partial Backorder	Linear Backorder	Time-Dependent Shortages	Lost Sale
(Aardal, et al. 1989)	Yes	Yes	No	No	No	No	No

(Bayındır et al. 2007)	Yes	Yes	No	No	No	No	No
(Çetinkaya and Parlar 1998)	Yes	No	Yes	No	No	No	No
(Fogarty and Aucamp 1985)	No	Yes	Yes	No	No	No	No
(Harris 1990)	Yes	No	No	No	No	No	No
(Kumar et al. 2021)	Yes	Yes	Yes	Yes	No	No	Yes
(Liao and Shyu 1991)	Yes	No	No	No	Yes	No	No
(Naser Ghasemi et al. 2013)	Yes	No	No	No	No	No	No
(Ouyang, et al. 1996)	Yes	Yes	Yes	No	No	No	No
(Pan and Hsiao 2005)	Yes	Yes	Yes	No	Yes	No	No
(Pan, et al. 2004)	Yes	Yes	No	No	No	No	No
(Raymond 1931)	No	No	No	No	No	No	No
(San-José, et al. 2014)	Yes	Yes	Yes	Yes	No	No	Yes
(Sazvar, et al. 2013)	Yes	Yes	Yes	Yes	No	Yes	No

(Scarf and H. Scarf 1959)	No	Yes	Yes	No	No	No	No
(Veinott et al. 1963)	No	No	Yes	No	No	No	No
(Zangwill 1969)	Yes	No	Yes	No	No	No	No

Table 2. 2 Summary of literature about Linear and Quadratic Backorder from the year 1973

Author and year	linear backorder	quadratic backorder	shortage	backorder	price-sensitive demand	fixed backorder	finite planning horizon
Cárdenas-Barrón 2001				✓			
Grubbström and Erdem 1999				✓			
Hu, Kim, and Banerjee 2009				✓			
K.-W. Lee 1994				✓			
Kumar et al. 2021							

Luo 2019	√		√				
Montgomery et al. 1973							
Montgomery et al. 1973				√			
Seliaman et al. 2020				√			
Sphicas 2006						√	
Taleizadeh et al. 2012	√						
Taleizadeh et al. 2013	√		√	√			
Taleizadeh et al. 2013	√			√			
Xu-Ren Luo 2019	√		√				
Yang 2014							√

Table 2. 3 Summary of literature about exponential backorder from the year 1973

Author & year	Exponential Backorder	Shortage	Backorder	Partial Backorder	Complete Backorder	Finite Planning Horizon
K.-W. Lee 1994	√		√			
Goyal 1977						
W.-C. Lee et al. 2006		√	√			
Omorodion, Ayinde, and K 2014	√		√			
Omorodion 2014	√				√	
Montgomery et al. 1973	√		√		√	
Gholami-Qadikolaei, Sobhanallahi, and Mirzazadeh 2013	√	√	√			
Pentico, Toews, and Drake 2015	√					
Cohen 1977						
Abad 1996		√		√		√

Table 2. 4 Summary of literature about fuzzy backorder from the year 1998

Keywords	(Soni and Joshi 2015)	(Mahata 2015)	(De and Mahata 2017)	(Chang, Yao, and Lee 1998)	(Wulan and Andyan 2013)	(W. F. Khan and Oshmita Dey 2022)	(Park 1987)	(S. H. Chen and Hsieh 1999)
Fuzzy backorder	√	√	√	√	√	√		√
Shortage	√	√	√	√	√	√		√
Back order	√	√		√			√	
Partial backorder	√				√			√
Complete backorder	√							√
Finite planning horizon						√		

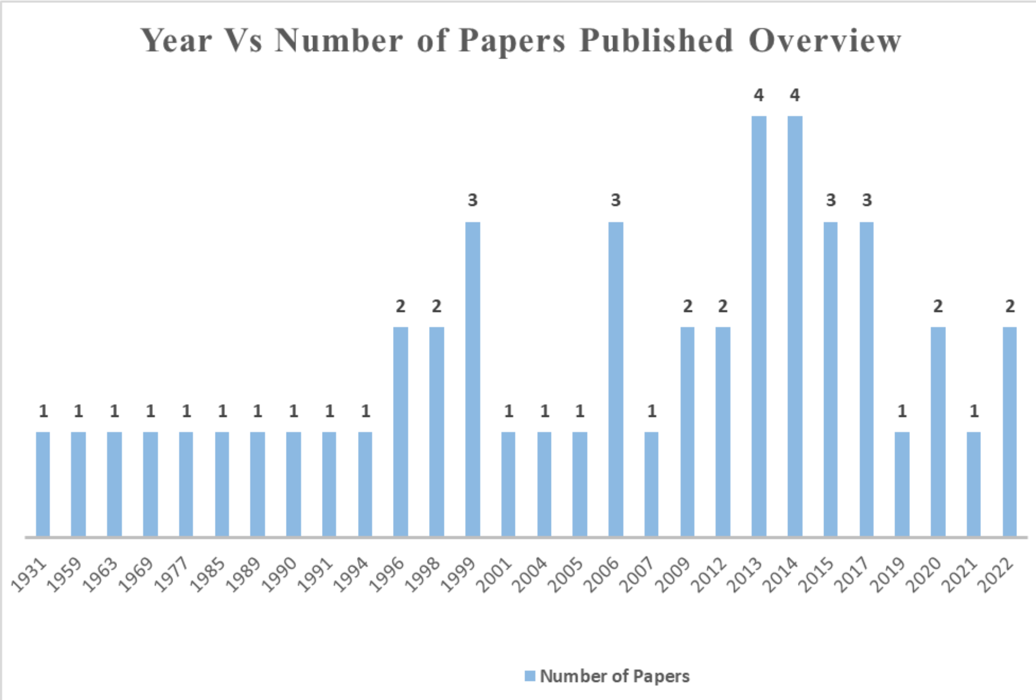

Fig. 2. 2 Graphical representation of year vs number of papers published Overview

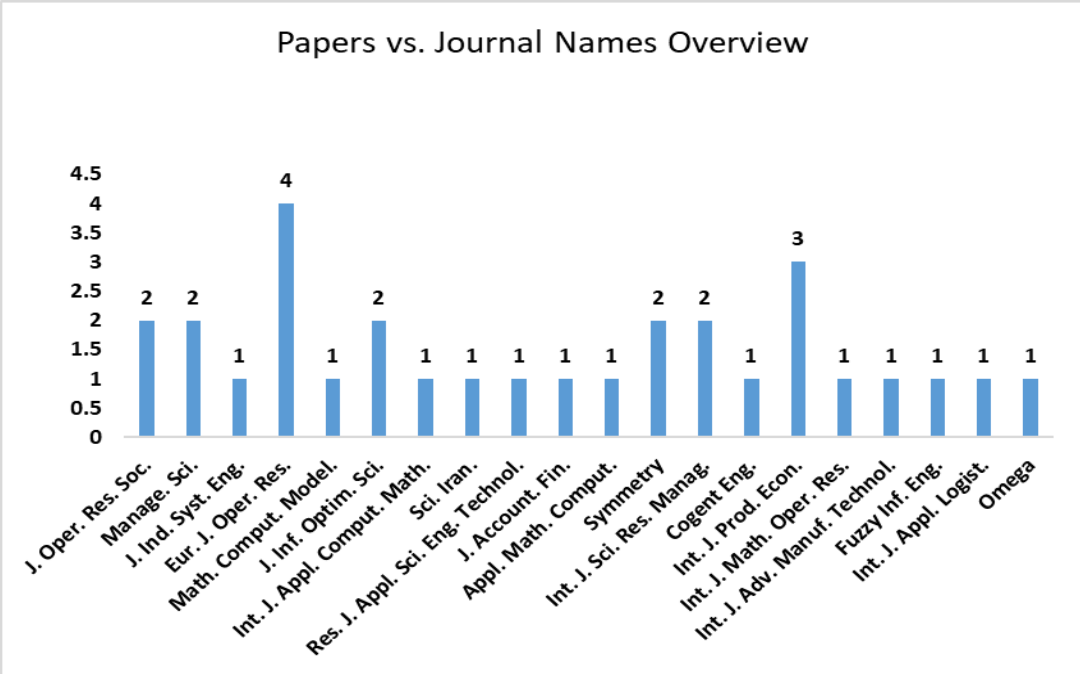

Fig. 2. 3 Graphical representations of names of the journals vs number of papers published Overview

2.3. DIFFICULTIES ENCOUNTERED

The research process of incorporating backordering into the Optimal Production Quantity (EPQ) model involves various challenges and trade-offs. One major challenge is the complexity introduced by incorporating additional cost components associated with backorders, such as the cost of lost sales and backorder holding costs. Determining the accurate values for these costs and their impact on the total cost function can be challenging due to the subjective nature of customer satisfaction and potential financial losses. Another challenge is the uncertainty in demand forecasting and lead time estimation, as backorders rely on accurate predictions. Variability in customer demand and behavior, as well as changes in market conditions, can make it difficult to estimate the optimal production quantity. Trade-offs also arise when considering the costs and benefits of backordering. While back ordering can enhance customer service and provide flexibility, there are associated costs, such as the risk of customer dissatisfaction or lost sales. Balancing these trade-offs requires careful analysis, understanding the specific context of the business, and finding the optimal balance between customer service, inventory management, and cost efficiency. The effectiveness and efficiency of incorporating back-ordering into the Optimal Production Quantity (EPQ) model depend on various factors. By considering the costs associated with unfulfilled demand and backlogged orders, the model can effectively minimize costs and meet customer needs. Accuracy in demand forecasting, lead time estimation, and the inclusion of backorder costs in the total cost function are essential for the model's effectiveness. Incorporating back ordering improves efficiency by reducing stockouts, optimizing setup costs, and utilizing economies of scale. It streamlines production and inventory management processes, resulting in cost savings and improved resource utilization. However, efficiency also hinges on accurate back-order fulfillment processes. Evaluating the model's performance through cost reduction, customer satisfaction, and inventory turnover metrics, along with continuous monitoring and refinement, helps assess its effectiveness and efficiency.

2.4. DISCUSSION AND CONCLUSION

Comprehensive analysis of inventory models with backordering: Although several research papers have addressed inventory models with variations of backorder costs and functions, there is a need for further investigation in critically examining and validating the obtained solutions. Table 2.1 Summary of literature on the supply chain from the year 1959, Table 2.2 Summary of literature about Linear and Quadratic Backorder from the year 1973, Table 2.3 Summary of literature about exponential backorder from the year 1973, Table 2. 4 Summary of literature about fuzzy backorder from the year 1998, Figure 2. 2 Graphical representation of year vs number of papers published Overview, Figure 2. 3 Graphical representations of names of the journals vs number of papers published Overview. This includes addressing questionable results and ensuring the accuracy and effectiveness of analytic and algebraic methods in solving inventory models with quadratic and linear backorder costs. The research gap can be identified as the need for further exploration and enhancement in resolving inventory models that involve fixed and progressively increasing backorder costs. While previous research has used analytic and algebraic methods, questionable results and open problems remain, indicating the necessity for future researchers to address these issues. Additionally, there exists a research void in the examination of inventory models featuring partial backorders, predictable demand, and fluctuating backorder ratios. More research is required to develop iterative solution methods and determine optimal reorder points and order quantities in such scenarios. Furthermore, the orchestration of decisions regarding production and inventory in unified supply chain systems signifies a potential field for research, focusing on developing mathematical models and algorithms for optimal inventory replenishment decisions.

While some studies have proposed solution procedures, such as minimax distribution-free procedures or computational algorithms, there is a research gap in developing more advanced and efficient solution methods for inventory models with exponential backorder costs. Exploring optimization techniques such as metaheuristics, dynamic programming, or simulation-based approaches can enhance the accuracy and computational efficiency of solving complex inventory optimization problems.

A research gap exists in the practical implementation of fuzzy backordering models in real-world supply chain scenarios, as the existing literature primarily discusses the benefits and implications without sufficient empirical research. Further studies are needed to explore the challenges, effectiveness, and best practices associated with implementing fuzzy backordering. Additionally, there is a need for more comprehensive comparative analyses to evaluate the performance of fuzzy backordering models against traditional inventory models, considering various performance measures. Exploring alternative uncertainty modeling techniques and determining optimal fulfillment thresholds are also areas that require further investigation. Integrating fuzzy backordering with advanced technologies and examining its sustainability implications are emerging research areas that can contribute to the advancement of knowledge in this field.

Chapter 3

A Replenishment Policy for an Inventory Model with Price-Sensitive Demand with Linear and Quadratic Back Order in a Finite Planning Horizon

A Replenishment Policy for an Inventory Model with Price-Sensitive Demand with Linear and Quadratic Back Order in a Finite Planning Horizon

3.1. INTRODUCTION

While dealing with the inventory problem the basic thing to be remember the developing technologies along with the inventory models. New technologies are growing due to recent research done on inventory. (Seliaman et al. 2020) are developing new techniques day by day to make inventory management easier. Back orders are for products that a firm cannot currently fill because demand exceeds supply. Back ordering can refer to items that are presently in production or those that have not yet started production. For handling the backorder communication is the key. By communicating the presence of back order, the supplier gets the information about the customer's actual demand for the product, what is inbound, and when the balance items will be there. This allows both the suppliers and the customers to continue the operations uninterrupted. As the backorder may impact inventory and other holding costs. In this article, we made an advanced model of inventory for the firms/companies who frequently deal with backorders. In this model we take the price-sensitive demand, backorder as a quadratic function with shortages in a finite planning horizon, and a case of linear backorder is discussed with lead time is zero.

3.2. LITERATURE REVIEW

Inventory is a very interesting topic for researchers. So much work has been done from the decade and still mostly work is going to be done. Firstly, the classical EOQ formula was discovered by (Harris 1990) which is also known as 'Square-root formula'. The first book on inventory management was written by (Raymond 1931). (Veinott et al. 1963) studied that in the real-life situations, the demand rate is dynamic. So, they developed the first dynamic economic order quantity model which is developed by modifying F. W. Harris's square-root model.

(Ouyang et al. 1996) provided a model by taking shortages and solving the total shortages as a combined form of lost sales and backorder. (Scarf and Herbert Scarf 1959) estimated a stochastic model of multi-period with shortages and given a policy (s, S) for an optimal solution with backorders. (Veinott et al. 1963) think that finding the exact backorder cost was a hard task so they developed the model which calculates the backorder cost. They considered back-ordering as a constant function, with shortages. (Zangwill 1969) proposed a multi-period model along with shortages and backorders. (Fogarty and Aucamp 1985) gave the model with shortages and back-ordering. (Aardal et al. 1989)proposed a model by taking the random demand (q, r) model given by (Veinott et al. 1963). Backorders are not considered but they assure that the yearly backorders cannot cross the upper boundaries.

Various other models discussed in the literature are compared and contrasted to showcase the advancements made in inventory management research. (Çetinkaya and Parlar 1998) established a generalized model by taking two different types of backorder costs. The research of (Sarkar et al. 2012) concerned with optimal inventory replenishment for a degrading item with time-quadratic demand and time-dependent partial backlogging. The analytical model yields optimum solutions, which are demonstrated numerically.(Liao and Shyu 1991) Given a model of predefined lot size and demand is assumed to be regularly distributed, with lead time as the variable, the estimated total cost with the backorder is minimized. (Pan et al. 2004) established an inventory model by taking the lead time & backorder discounts are negotiable in the way that the supplier may take into account the future & present loss & profit. The buyer may be ready to obtain the item as quickly as it can be obtained to ensure production may restart. (Bayındır et al. 2007) established an EPQ model taking general stock dependent backordering. (San-José et al. 2014) Proposed an EOQ model for a single item with partially backlogging, shortages time-dependent, partial backordering, the demand rate is backlogged at any instant is a constant fraction with shortages & obtained an optimal policy & less inventory cost. (Pan and Hsiao 2005) Extended the work of (Ouyang et al. 1996). Taken an integrated inventory system with shortages and backorder as well as lead time are negotiable. A provider may provide waiting consumers with a backorder cost reduction in the first of two models they described, which had normally distributed demand, and widely dispersed demand in the second. (Sazvar et al. 2013) established an inventory model for deteriorating goods by taking shortages and complete backordering. (Naser Ghasemi et al. 2013) proposed two models taking holding cost as increasing continuous functions. The first model with no shortages & the second model is with shortages and complete backordering. (Kumar et al. 2021)proposed an economic policy by taking demand as power depending on time, with shortages and complete backordering. (Nitin Kumar Mishra and Ranu 2023) This article discusses the importance of supplier-retailer coordination in managing deteriorating inventory with decreasing demand, addressing a research gap in supply chain literature. It presents a numerical solution and conducts a sensitivity analysis to illustrate the concept further.

Backordering was studied over the decade and still the work is going on. Back ordering is a major problem for the business, organization that's why researchers readily study backorder taking different types of backordering like linear, non-linear, exponential, negative exponential, constant function and quadratic function, etc.) (Grubbström and Erdem 1999) applied algebraic approach to develop the equations for both the EOQ (Economic Order Quantity) and the Economic Production Quantity (EPQ), while taking into account a single backordering cost that is only linear with respect to time. (Cárdenas-Barrón 2001) developed an algebraic method to prove the mathematical equations for EOQ and EPQ with a single cost of backordering, only linear (depending on time). (Taleizadeh et al. 2012) Proposed an EOQ model by taking linear holding cost (depend on price), partially backlogged & backorder is a linear function. (Taleizadeh et al. 2013b) proposed two EOQ models (a) by taking holding cost linear dependent on time, partially backlogged, backorders are linear function, lost sale cost as fixed and partially delayed payments. (Taleizadeh et al. 2013a) by taking holding cost linear depends on time, partially backlogged, backorders are linear functions, lost sale cost is fixed & partially prepayments. (Yang 2014) established an EOQ model by taking non-linear stock dependent holding cost, partially backlogging, backorders are linear, a lost sale is fixed, the demand rate is stock dependent. By taking different types of backordering singly researchers were not satisfied with the output, so they started taking two types of backorders together, like linear plus fixed, linear, and quadratic Some of the literature surveys are as follows, (Montgomery et al. 1976) firstly took the linear plus fixed backorder and solved by calculus and solved the system of equations &they get the first-order condition. (Sphicas 2006) extended the study of (Grubbström and Erdem 1999) by taking two parameters combine i.e., Linear & fixed backorder cost for the EOQ & EPQ models. They discussed two conditions first is when fixed backorder cost is high then we can't get any optimal backorder & the second case if the back-order cost is very lesser than there should be optimally some of the backorders. The result reveals that linear backorder cost plays no role. (Chung and Cárdenas-Barrón 2012) given the complete solution procedure for the EOQ/EPQ models, and backorder cost is taken as fixed & linear. Most of the models are failed to give an argument & surety of the optimal situation but (Chung et al. 2009) given every aspect of the approved solution procedure, we ensure the most effective possible solution. They discussed two cases in their paper for the existence of optimal solution & if the conditions are not satisfied then how to identify the condition by which optimal solution is sure. And derives four theorems &two lemmas for an optimal solution. (Nitin Kumar Mishra and Renuka S. Namwad 2023) This discussion focuses on an inventory model that addresses items with minimal lead time and deterioration, utilizing cubic demand and deterioration functions. It emphasizes the advantages of employing cubic functions for practical applicability, numerical validation, and graphical representation. Additionally, it includes a numerical example and a comprehensive sensitivity analysis. (Wee et al. 2014) Proposed an EOQ model by taking linear holding cost (depend on price), partially backlogged & backorder is linear & fixed function. (Sphicas 2014) proposed an EOQ model holding cost is linear & dependent on time, completely backlogged, and backorders are fixed &linear. (Hu et al. 2009) proposed a model of backordering as linear & quadratic function, partially backlogged. In Table 3.1 a literature survey is carried out.

Table 3. 1 Survey of existing literature

Authors	**Demand Type**	**Shortages**	**Backorder Type**	**Finite Planning Horizon**
(Scarf and Herbert Scarf 1959)	Stochastic	Optimal with backorders	Optimal	-
(Veinott et al. 1963)	Dynamic EOQ	Yes (Dynamic)	Dynamic	-
(Zangwill 1969)	Multi-period	Yes	Yes	-
(Montgomery et al. 1973)	Linear plus fixed	-	Linear & Fixed	-
(Harris 1990)	Classical EOQ (Square-root)	-	-	-
(Liao and Shyu 1991)	Normally distributed	Expected Total Cost	Expected Total Cost	-

		with Backorder		
(Ouyang et al. 1996)	-	Lost Sales & Backorder	Lost Sales & Backorder	-
(Silver et al. 1998)	Time-dependent	Yes	-	-
(Pan and Hsiao 2005)	-	Yes	Linear	-
(Hu et al. 2009)	-	Partially backlogged	Linear & Quadratic	-
(Taleizadeh et al. 2012)	Linear holding cost (on price)	Partially backlogged	Partial	-
(Sazvar et al. 2013)	-	Complete Backordering	Complete	-
(Naser Ghasemi et al. 2013)	-	No & yes with complete backordering	No & Complete	-
(Taleizadeh et al. 2013a)	Time-dependent holding cost	Partially backlogged	Partial	-
(Taleizadeh et al. 2013b)	Time-dependent holding cost	Partially backlogged	Partial	-
(San-José et al. 2014)	Time-dependent	Partially backlogging	Partial	-
(Yang 2014)	Stock-dependent	Partially backlogging	Partial	-
(Wee et al. 2014)	Linear holding cost (on price)	Partially backlogged	Partial	-
(Sphicas 2014)	Time-dependent holding cost	Completely backlogged	Complete	-
(Kumar et al. 2021)	Power depending on time	Yes, with complete backordering	Complete	Yes

This paper	Price-sensitive	Yes	Complete Linear and Quadratic	Yes

3.2. RESEARCH GAP AND PROBLEM IDENTIFICATION

In this study, we aim to address the research gap related to price-dependent quadratic backorder. Despite the extensive research conducted in this field, there is still a lack of understanding regarding a very few research conducted in the finite planning horizon. Therefore, this study seeks to contribute to the existing literature by price-sensitive demand and price-dependent quadratic backorder. The research question addressed in this study is a replenishment policy for the quadratic backorder and linear backorder inventory models discussed in the finite planning horizon.

3.3. ASSUMPTIONS AND NOTATIONS

3.3.1. ASSUMPTIONS

i. The total stock level is initially zero.
ii. The cost of storing stays constant.
iii. The lagging time is zero.
iv. The cost of ordering is predetermined.
v. Under a finite planning horizon, shortages are acceptable and continuous ones.
vi. Back ordering is complete and described as a quadratic function and linear.

3.3.2. NOTATIONS

i. H is Fixed time horizon.
ii. a is the initial demand rate per year
iii. b is the increasing demand rate per year
iv. r is the amount that is carried per unit per order.
v. The demand rate is D and $D(t) = a - bp$, where p is the unit selling price.
vi. W_h is the cost of replenishing or purchasing per order.
vii. I_{j+1} is the total inventory carried out during the interval $[t_j, s_j]$
viii. S is the total amount of shortages in the interval $[s_j, t_{j+1}]$.
ix. S_j denotes the time at which the inventory level reaches zero in the j^{th} replenishment cycle j= 1,2, 3, …., n.
x. t_j is the j^{th} replenishment time j =1, 2, 3, ……n.
xi. n is the number of orders during the time horizon H.

xii. $D_1 = a - bp - cp^2$ is the price dependent quadratic backordering, a, b, c are all constants.

xiii. Q is the total optimal order quantity during the planning horizon H.

xiv. I_b instantons shortage during the shortage period.

xv. θ is an inventory dependent parameter

3.4. MATHEMATICAL SOLUTION OF THE MODEL

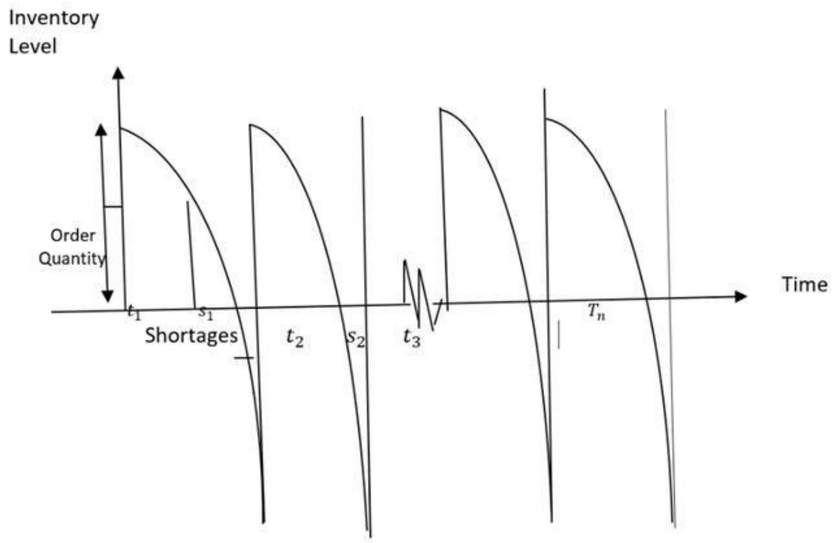

Fig. 3. 1 Inventory model diagram

The initial inventory equation is given by,

$$\frac{dI_{j+1}(t)}{dt} + (\theta_1)I_{j+1}(t) = -D(t) \qquad t_j < t < s_{j+1} \qquad \ldots (3.1)$$

where, j=1, 2, 3, …, n_1.

$$\frac{dI_{j+1}(t)}{dt} = -D(t) - \theta_1 I_{j+1}(t) \qquad t_j < t < s_{j+1} \qquad \ldots (3.2)$$

Considering the boundary condition $I_{i+1}(s_j) = 0$.

Solution of Eq. (3.2) is,

$$I_{j+1}(t) = e^{-\theta_1 * t} \int_{t}^{s_{j+1}} D(u) e^{u}\, du \qquad \ldots (3.3)$$

$$I_{j+1}(t) = \int_{t}^{s_{j+1}} D(u) e^{\theta_1 (u-t)}\, du \qquad \ldots (3.4)$$

$$I_{j+1}(t) = \frac{1}{\theta_1} [e^{\theta_1 (s_{j+1}-t)} - 1] D(t) \qquad \ldots (3.5)$$

During the shortage phase, the instantaneously arising shortage $I_b(t)$ is offered by,

$$I_b(t) = D_1(t_{j+1} - s_j) \qquad \ldots (3.6)$$

where, $D_1 = a - bp - cp^2$ is the price dependent quadratic backorder.

$$I_b(t) = a - bp - cp^2(t_j - s_j) \qquad \ldots (3.7)$$

Considering the boundary condition, $I_b(s_j) = 0$.

$$Q_{j+1} = I_{j+1}(t_j) = \frac{1}{\theta_1}[e^{\theta_1(s_{j+1}-t_j)} - 1]D(t) \qquad \text{... (3.8)}$$

where, $D(t) = a - b * p$.

Considering the reorganization of the ordering, S_{j+1} can be given as,

$$S_{j+1} = \int_{s_j}^{t_j} I_b(t)dt = \int_{s_j}^{t_j} (a - b * p - c * p^2)(t_j - s_j)dt \qquad \text{...(3.9)}$$

The entire purchase amount for a limited time frame of planning,

$$Q_{nt} = \sum_{j=1}^{n_1} Q_{j+1} = \sum_{j=1}^{n_1}\{I_{j+1} + S_{j+1}\} \qquad \text{... (3.10)}$$

$$Q_{j+1} = \frac{1}{\theta_1}[e^{\theta_1(s_{j+1}-t_j)} - 1]D(t) + \int_{s_j}^{t_j}(a - b * p - c * p^2)(t_j - s_j)dt \qquad \text{... (3.11)}$$

The total retailer cost over a specified time horizon is given by,

$$Total\ cost = Resupply\ expenses + Cost\ of\ retaining\ stocks + purchasing\ cost + Storage\ cost$$

$$T_R(t_j, s_j, n_1) = n_1 * O_r + \sum_{j=0}^{n_1-1} H \int_{t_j}^{s_{j+1}} I_{j+1}(t)dt + \sum_{j=0}^{n_1-1} W_h * Q_{j+1} + \sum_{j=0}^{n_1-1} s \int_{s_j}^{t_j} I_b(t)dt \quad \ldots (3.12)$$

$$T_R(t_j, s_j, n_1) = n_1 * O_r + \sum_{j=0}^{n_1-1} H \int_{t_j}^{s_{j+1}} I_{j+1}(t)dt + \sum_{j=0}^{n_1-1} W_h * Q_{j+1} + \sum_{j=0}^{n_1-1} s \int_{s_j}^{t_j} I_b(t)dt \quad \ldots (3.13)$$

$$T_R(t_j, s_j, n_1) = n_1 * O_r + \sum_{j=0}^{n_1-1} H \int_{t_j}^{s_{j+1}} \frac{1}{\theta_1}[e^{\theta_1(s_{j+1}-t)} - 1]D(t)dt + \sum_{j=0}^{n_1-1} W_h * (\frac{1}{\theta_1}[e^{\theta_1(s_{j+1}-t_j)} - 1]D(t) + \int_{s_j}^{t_j}(a - b * p - c * p^2)(t_j - s_j)dt) \quad \ldots (3.14)$$

$$T_R(t_j, s_j, n_1) = n_1 * O_r + H \int_{t_{j-1}}^{s_j} \frac{1}{\theta_1}[e^{\theta 1(s_j - t)} - 1]D(t)dt + \int_{t_j}^{s_{j+1}} \frac{1}{\theta_1}[e^{\theta 1(s_{j+1} - t)} - 1]D(t)dt + W_h c * (\frac{1}{\theta_1}[e^{\theta 1(s_j - t_{j-1})} - 1]D(t) + W_h * (\frac{1}{\theta_1}[e^{\theta 1(s_{j+1} - t_j)} - 1]D(t) + s(a - b * p - c * p^2)(t_{j-1} - s_{j-1})^2 + s(a - b * p - c * p^2)(t_j - s_i)^2 \qquad (3.15)$$

To achieve the lowest possible total cost in the inventory system, the essential conditions for minimizing the total cost are as follows,

$$\frac{\partial TC(t_j, s_j,\ n_1)}{\partial t_j} = 0, \qquad j = 1, 2, 3, \dots, n \qquad \dots (3.16)$$

$$\frac{\partial TC(t_j, s_j,\ / n)}{\partial s_j} = 0, \qquad j = 1, 2, 3, \dots, n \qquad \dots (3.17)$$

$$\frac{\partial T_R(t_j, s_j, n_1)}{\partial t_j} = \sum_{j=0}^{n_1-1} -H * \frac{1}{\theta_1}[e^{\theta 1(s_{j+1}-t_j)} - 1]D(t) - \sum_{j=0}^{n_1-1} W_h * ([e^{\theta 1(s_{j+1}-t_j)}]D(t) + \sum_{j=0}^{n_1-1} 2 * s(a - b * p - c * p^2)(t_j - s_j) \qquad \dots (3.18)$$

$$\frac{\partial T_R(t_j, s_j, n_1)}{\partial s_j} = \sum_{j=0}^{n_1-1} H \int_{t_{i-1}}^{s_j}[e^{\theta 1(s_j-t_j)}]D(t)dt - \sum_{j=0}^{n_1-1} W_h * ([e^{\theta 1(s_j-t_{j-1})}]D(t) - \sum_{j=0}^{n_1-1} 2 * s(a - b * p - c * p^2)(t_j - s_j) \qquad \dots (3.19)$$

The total cost's Hessian matrix must be positive definite for a fixed n in order for the total cost to be least (i.e. $\nabla^2 TC$).

$$\frac{\partial^2 T_R(t_j, s_j, n_1)}{\partial {t_j}^2} = \sum_{j=0}^{n_1-1} H * [e^{\theta 1(s_{j+1}-t_j)}]D(t) + \sum_{j=0}^{n_1-1} \theta_1 * W_h * ([e^{\theta 1(s_{j+1}-t_j)}]D(t) + \sum_{j=0}^{n_1-1} 2 * s * (a - b * p - c * p^2) \quad \ldots (3.20)$$

$$\frac{\partial^2 T_R(t_j, s_j, n_1)}{\partial {s_i}^2} = \sum_{j=0}^{n_1-1} H * \theta_1 \left(\int_{t_{j-1}}^{s_j} [e^{\theta 1(s_j-t_j)}]D(t)dt + 1\right) - \sum_{j=0}^{n_1-1} W_h * \theta_1([e^{\theta 1(s_j-t_{j-1})}]D(t) + \sum_{j=0}^{n_1-1} 2 * s(a - b * p - c * p^2) \quad \ldots (3.21)$$

$$\frac{\partial^2 T_R(t_j, s_j, n_1)}{\partial t_i \partial s_i} = \sum_{j=0}^{n_1-1} -2 * s(a - b * p - c * p^2) \quad \ldots (3.22)$$

3.4.1. TOTAL COST OF SUPPLIER

$$T_S(t_j, s_j, n_1) = n_1 * Ss + Cs * \sum_{j=0}^{n_1-1} \frac{1}{\theta_1}[e^{\theta 1(s_{j+1}-t_j)} - 1]D(t) + \int_{s_j}^{t_j} (a - b * p - c * p^2)(t_j - s_j)dt \quad \dots (3.23)$$

3.5. THEOREMS

I. If the following conditions are satisfied:

(i) $\frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_j^2} \geq 0,$

(ii) $\frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial s_j^2} \geq 0,$

(iii) $\frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_j^2} - |\frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_j \partial s_j}| \geq 0$ and

(iv) $\frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial s_j^2} - |\frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_j \partial s_j}| \geq 0$ for all j= 1, 2, ..., n

Then, $T_R(t_j, s_j, n_1)$ will be positive definite. This set of conditions is sufficient to ensure that $T_R(t_j, s_j, n_1)$ is at its minimum for a fixed value of n_1. The theorem establishes that $T_R(t_j, s_j, n_1)$ is indeed positive. Therefore, we can compute the optimal values of t_j and s_j for a given positive integer n_1 using iterative methods and Mathematica software based on Eq. (3.18) and Eq. (3.19).

II. When considering a convex set S ⊆ R^n, a cost function is deemed convex across S if it satisfies the condition that, for any x_1 and x_2 belonging to S, and for any λ within the interval [0, 1], the following inequality holds: $\lambda f(x_1) + (1 - \lambda) f(x_2) \geq f(\lambda x_1 + (1 - \lambda)x_2)$. Should this inequality always be held as a strict inequality, then the function f is denoted as a strictly convex cost function on S.

III. Consider an open convex subset S, which is non-empty, of R^n, and a cost function f: S → R that is twice differentiable on S. In this context, f is convex on S if and only if the Hessian matrix $\nabla^2 f(x)$ is positive semi-definite for all x in S.

IV. In the scenario where S is an open convex set in R^n and f: S → R is a cost function that is twice differentiable, if the Hessian matrix ∇^2 f(x) is positive definite for all x in S, then f is a strictly convex function on S.

$$\nabla T_R(t_j, s_j, n_1) = \begin{bmatrix} \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_1^2} & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_1\,\partial s_1} & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial s_2\,\partial t_1} & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial s_1^2} & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial s_1\,\partial t_2} & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_2\,\partial s_1} & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_2^2} & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_2\,\partial s_2} & \cdots & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_{n_1-1}\,\partial s_{n_1-1}} & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial s_{n_1-1}^2} & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial s_{n_1-1}\,\partial t_{n_1}} \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_{n_1}\,\partial s_{n_1-1}} & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_{n_1}^2} \end{bmatrix}$$

3.6. NUMERICAL ILLUSTRATION

A numerical example to validate our model, using specific parameter values $a = 1.25$, $b = 0.2$, $c = 18.4$, $r = 60$, $e = 2.7$, W_h=2, H=4, p=0.01, S=2, $s_1 = 0$, $\theta_1 = 0.03$ expressed in their appropriate units. For the solution of Eq. (3.18) and Eq. (3.19), Mathematica (version 12) was the computational program that we utilized. 'MATHEMATICA VERSION 12' efficiently handles the calculations and analysis required for the inventory model considering backorders. The associated total cost for various resupply cycles, i.e., for n = 1, 2, ... for quadratic backorder are given in Table 3.2, Figures 3.2, 3.3, 3.4, 3.5, we notice that for each resupply cycle, the most efficient number of replenishments time for the corresponding minimum total cost gets supplied in appropriate units. The optimal solutions for t_j and s_{j+1} for n = 4 are given in Tables 3.3 and 3.4, Figures 3.6 and 3.7 the x-axis is labeled as Cycle Number (n), and the y-axis is labeled as Replenishment Time (t_j) and replenishment time during shortage period (s_j) respectively. In Table 3.5 highlights various total cost values for retailers, suppliers and total optimal quantity for the different values of parameter a.

Table 3. 2 Total cost for the retailer for different replenishment cycle in quadratic back order

↓a →n	1	2	3	4	5	6
0.81675	28.8679	**28.494**	28.6444	29.2919	304364	32.078
0.9375	32.5612	31.5384	**31.1177**	31.2676	31.9882	33.2794
1.089	37.1951	35.3582	34.2208	**33.7465**	33.9352	34.7868
1.25	42.1195	39.4175	37.5185	36.3808	**36.0042**	36.3888

Table 3. 3 The optimal solutions for t_j (replenishment time) for Quadratic back order

↓a →t_j	t_0	t_1	t_2	t_3	t_4	t_5
0.81675	0	3.3374	4			
0.9375	0	2.95889	3.53907	4		
1.089	0	2.58054	3.13523	3.53925	4	
1.25	0	2.20229	2.73147	3.13543	3.5393	**4**

Table 3. 4 The optimal solutions for s_j (time of shortage) for quadratic back order

↓a →s_j	s_0	s_1	s_2	s_3	s_4	s_5
0.81675	0	3.59373	4			
0.9375	0	3.19009	3.59601	4		
1.089	0	2.78658	3.19216	3.59608	4	
1.25	0	2.38314	2.78839	3.19226	3.59613	4

Table 3. 5 Total cost for retailer, supplier and quantity is given for optimal value for quadratic back order

$\downarrow a$	T_R	T_S	Q_{nt}
0.81675	28.494	12.2224	9.40793
0.9375	31.1177	16.7166	8.72184
1.089	33.7465	21.2081	8.02711
1.25	36.0042	25.6457	7.15235

3.7. SENSITIVITY ANALYSIS

We will now talk about how the ideal solution responds to variations in the values of various parameters. The comparative study is carried out by altering all of the parameters' a, b, c, θ, W_h, r, and S by ±20% and ±10%, one at a time, while keeping the other parameters constant. The effect on total cost due to percentage changes in parameters a, b, c, θ, S, r, W_h and all parameters is shown in Figures 3.8, 3.9, 3.10, 3.11, 3.12, 3.13, 3.14, 3.15 respectively. A detailed analysis of the table acknowledges the following perceptions

Table 3. 6 Sensitivity analysis of the parameters in quadratic back order

Parameters	% Changes	Optimal Replenishment Cycle	Total Order Quantity Q_{nt}	Total Cost of Retailer T_R	Total Cost of Supplier T_S
a	$+20$	6	6.53586	38.8763	30.1608
	$+10$	5	7.87023	37.6106	25.8611
	0	5	7.15235	36.0042	25.6457
	-10	4	8.29353	34.3355	21.2881
	$\{-20$	4	7.36848	32.2903	21.0105

b	+20	3	8.2967	30.1857	3.69051
	+10	3	8.57701	30.8002	3.81515
	0	3	8.85732	31.4147	3.9398
	−10	4	7.23527	31.9957	4.13207
	{−20	4	7.45728	32.4866	4.25878
c	+20	5	7.16731	36.0382	5.57456
	+10	5	7.16731	36.0382	5.57456
	0	5	7.16731	36.0382	5.57456
	−10	5	7.16731	36.0382	5.57456
	{−20	5	7.16731	36.0382	5.57456
$\boldsymbol{\theta}$	+20	5	7.53094	36.8115	5.68158
	+10	5	7.34484	36.4147	5.62539
	0	5	7.15235	36.0042	5.56728
	−10	5	6.95334	35.5798	5.50721
	{−20	5	6.74774	35.1413	5.44518
$\boldsymbol{W_h}$	+20	5	5.87409	33.6968	5.44791
	+10	5	6.47731	34.7681	5.49681
	0	5	7.15235	36.0042	5.56728
	−10	5	7.8991	37.4047	5.65929
	{−20	5	8.71747	38.9696	5.77282
r	+20	5	7.83037	37.1705	5.23008
	+10	5	7.50244	36.6031	5.37882
	0	5	7.15235	36.0042	5.56728
	−10	5	6.7769	35.3707	5.80891
	{−20	5	6.37212	34.6989	6.12359
S	+20	6	6.11979	40.4676	6.73796
	+10	5	7.48108	38.3417	5.85373
	0	5	7.15235	36.0042	5.56728
	−10	5	6.78255	33.7416	5.26615
	{−20	4	8.55565	31.2096	4.7597

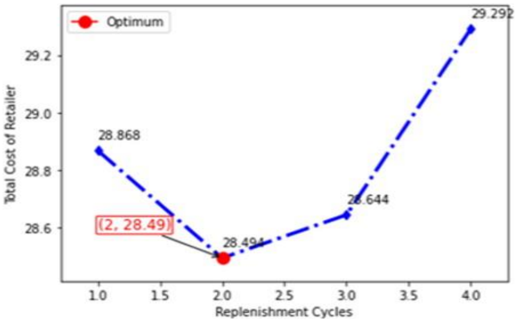

Fig. 3. 2 Convexity of total cost for retailer in 2nd replenishment cycle

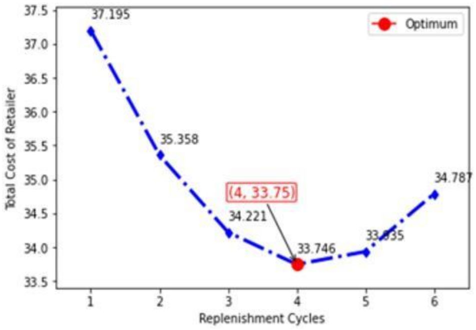

Fig. 3. 3 Convexity of total cost for retailer in 3rd replenishment cycle

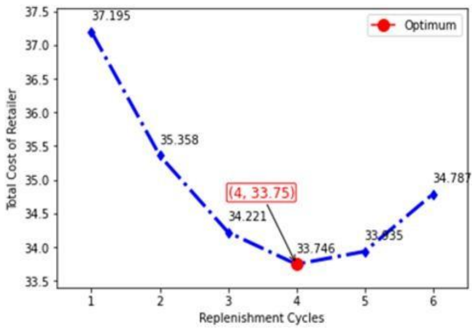

Fig. 3. 4 Convexity of total cost for retailer in 4th replenishment cycle

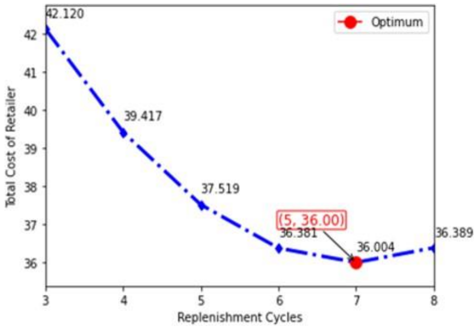

Fig. 3. 5 Convexity of total cost for retailer in 5th replenishment cycle

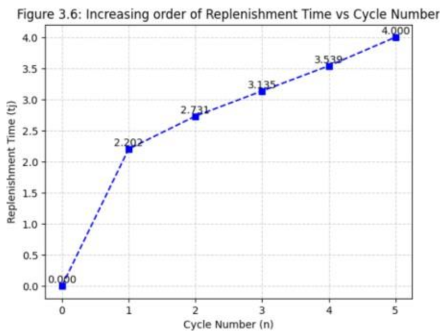

Fig. 3.6 Increasing order of replenishment time t_j

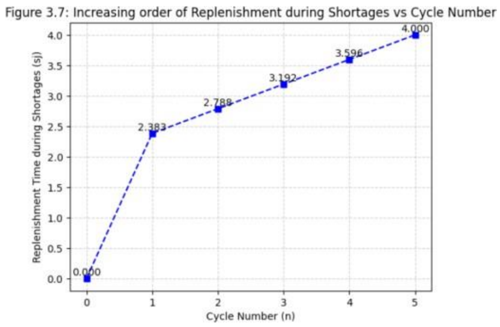

Fig. 3.7 Increasing order of replenishment time s_j

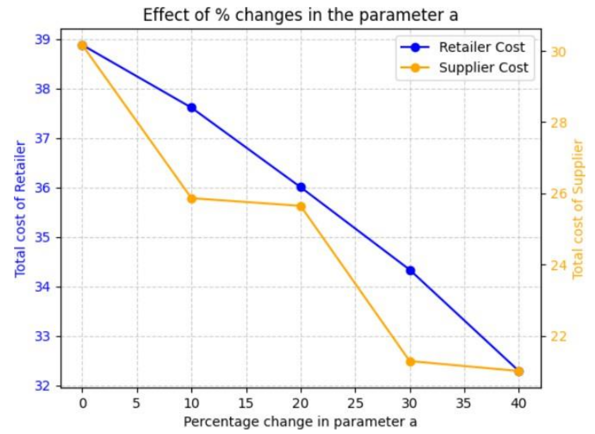

Fig. 3. 8 Effect on total cost of retailer and supplier due to parameter 'a'

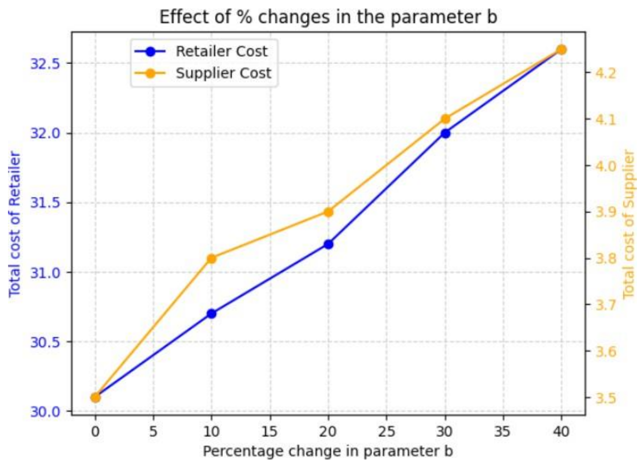

Fig. 3. 9 Effect on total cost of retailer and supplier due to parameter 'b'

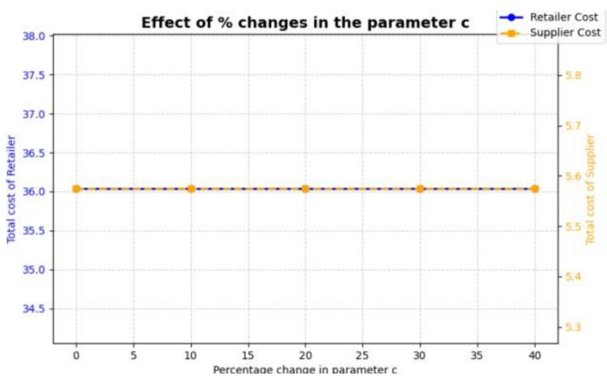

Fig. 3. 10 Effect on total cost of retailer and supplier due to parameter 'c'

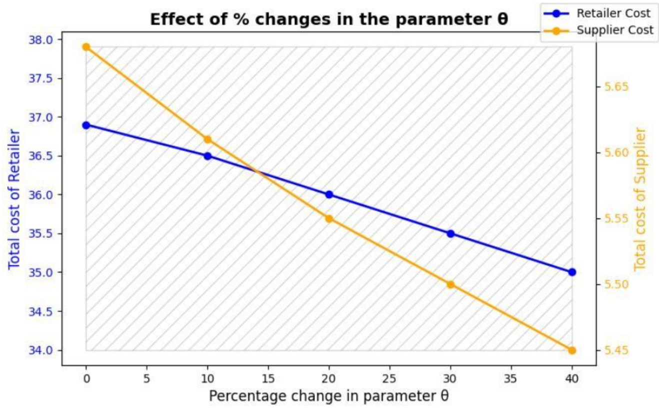

Fig. 3. 11 Effect on total cost of retailer and supplier due to parameter 'θ'

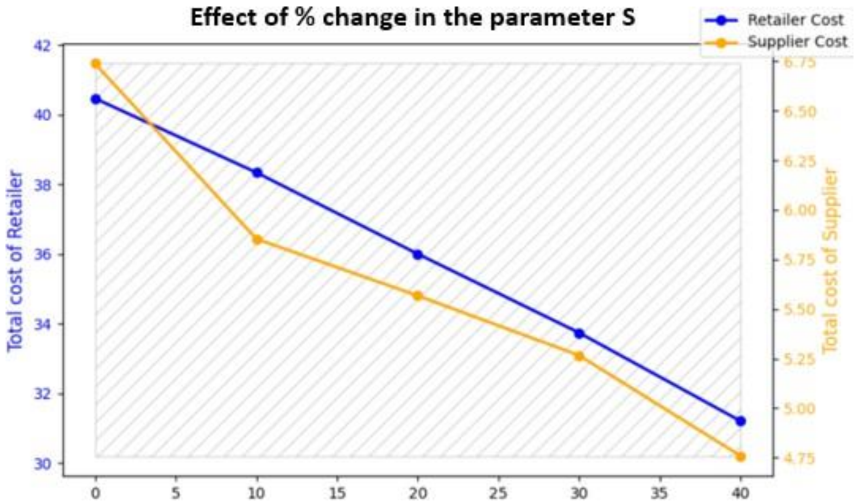

Fig. 3. 12 Effect on total cost of retailer and supplier due to parameter 'S'

The optimal replenishment cycle, n, is sensitive to varying in most parameters. It is extremely responsive to variations in the parameter 'a'. While decreasing 'a' by 20%, the optimal replenishment cycle, n, decreases from 5 to 4, shows a 20% decrease. On the other hand, with a 20% increase in 'a', the cycle increases to 6, a 20% increase. From this, we analyze that as 'a' expands or contracts, the optimal replenishment cycle moves in conjunction. Variations in parameters impact total cost and efficiency, providing a deeper understanding of the system's behavior under different scenarios.

Similarly, changes in the parameter 'b' also show a significant impact on the replenishment cycle. An increase of 20% in 'b' remains the cycle at 3, but a decrease of 10% in 'b' moves it to 4, a 33.33% increase. This implies that as 'b' reduces, there is an impulse to have more extended cycles. The Total Cost for Retailer T_R is sensitive to changes in θ and W_h. For instance, when a 20% decrease in θ shows a decrease in T_R by approximately 3.37%. Meanwhile, a 20% increase in Wh shows an increase in T_R by about 3.78%. These changes indicate the parameter's direct effect on the retailer's total costs. The Total Cost for Supplier, T_S, on the other hand, reacts differently to changes in parameters. An evident observation is with 'a'. A 20% increase in 'a' decreases the T_S by approximately 27.72%.

The Total Order Quantity, Q_{nt}, shows significant changes with parameters 'b', 'θ', 'W_h', and 'S'. For 'b', a 20% increase results in an increase of approximately 7.53% in Q_{nt}. The same pattern for 'S'; a 20% increase in 'S' shows a decrease in Q_{nt} by approximately 14.42%.

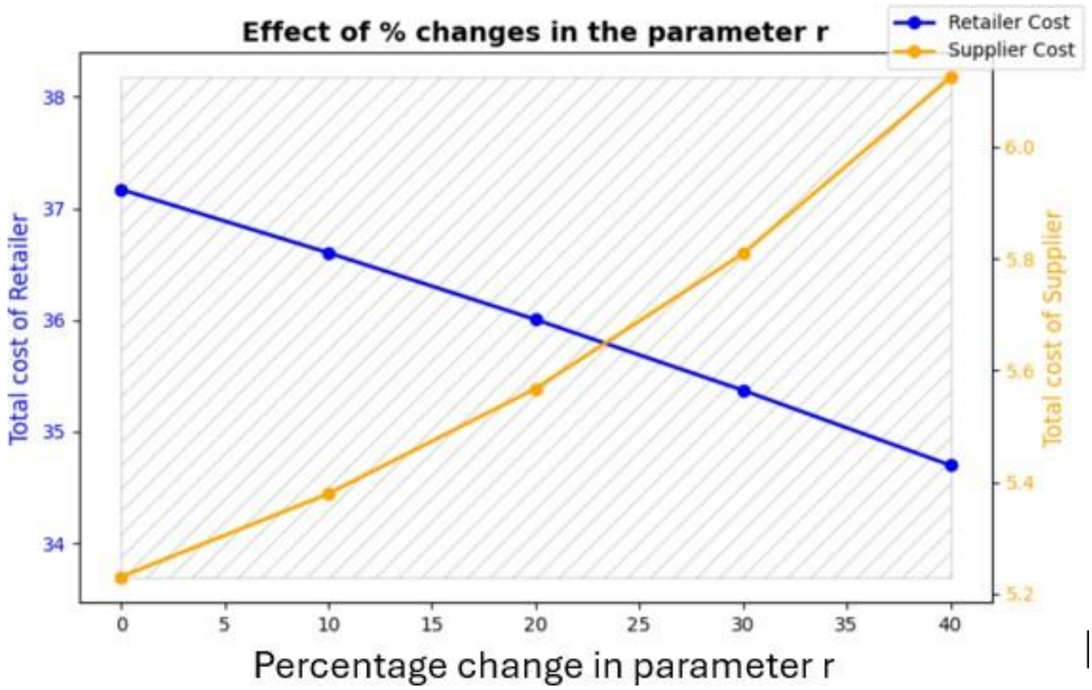

Fig. 3. 13 Effect on total cost of retailer and supplier due to parameter 'r'

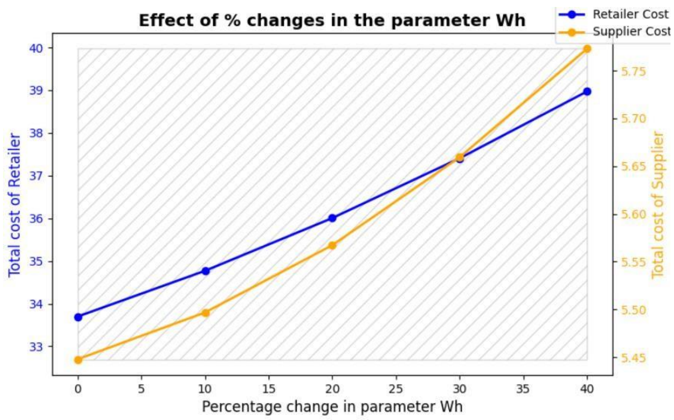

Fig. 3. 14 Effect on total cost of retailer and supplier due to parameter ‘W_h’

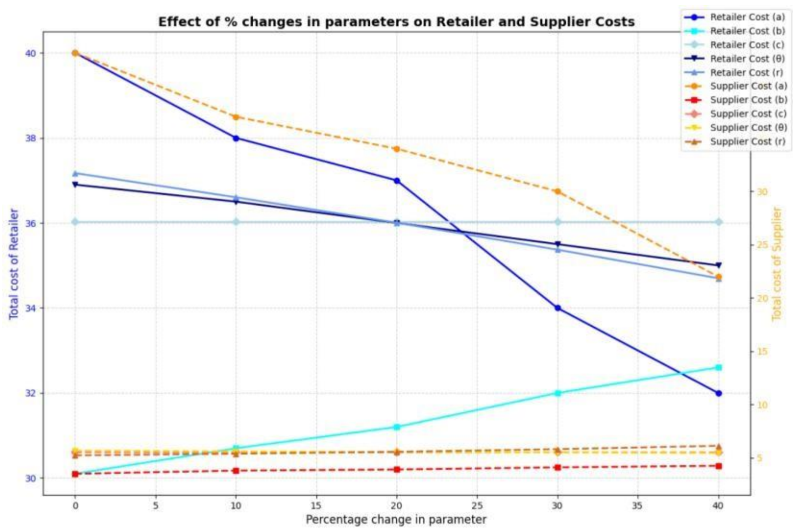

Fig. 3. 15 Effect on total cost of retailer and supplier due to all parameters

3.8. BACKORDERING AS A LINEAR FUNCTION

In the above solution, we considered quadratic back order $(a - bp - cp^2)$ if we put $c = 0$ then we form a linear backorder case for the model, Table 3.7 discusses the order quantity, the total cost of retailer and supplier for the linear back order.

Table 3. 7 Total cost of retailer and supplier for linear back order

Linear Back-Order Condition	Replenishment Cycle (n*}	Q_{nt} Order Quantity	Time Intervals (Years)		T_R Total Cost of Retailer	T_STotal Cost of Supplier
			t_j	s_j		
$c = 0$	2	9.42888	0.174335	0.980015	28.5422	12.2287
	3	8.74094	1.10189	1.76513	31.1614	16.7223
			1.85753	2.42678		
			2.50215	3.00577		
			3.07021	3.52538		
			3.58222	4.0000		
			1.1019	1.76504		
	4	8.04422	1.85749	2.4267	33.9692	21.2133
			2.50211	3.00571		
			3.07018	3.52535		
			3.5820	4.00000		

3.9. CONCLUSION

The optimization of replenishment policies outlined in this article is invaluable for businesses striving to enhance their supply chain management efficiency. By accurately modeling parameters such as price-sensitive demand and backordering, and minimizing total costs within a finite time horizon, companies can make informed decisions that lead to optimized inventory levels, reduced stockouts, and ultimately improved customer satisfaction. This approach provides a systematic framework for strategic planning, enabling businesses to allocate resources effectively, mitigate risks, and maximize profitability in a dynamic and competitive market environment.

An example of this model in practice may be observed in the electronics retail industry during festive sales of items such as smart home appliances. In such cases, demand is highly price elastic-lowing the prices leads to a drastic increase in the number of customers. Stockouts generate two types of backordering behavior: in the linear case, backorders increase step by step with each delay, whereas in the quadratic case, they rise more quickly as delays lengthen, reflecting growing urgency and demand buildup.

Because festive sales are limited to a short duration, this creates a finite planning horizon in which the retailer must determine replenishment, timing and size optimally to balance holding costs, backorder penalties, and revenue. Such a replenishment policy ensures better cost control, profit maximization, and efficient satisfaction of seasonal demand.

In this article, we tackled the optimization problem associated with a replenishment policy, focusing on various parameters that influence the cost and efficiency of the system. Specifically, we considered a scenario where demand is influenced by price, modeled as (a – bp), **p** is the unit selling price. and assumed complete backordering. Back ordering was modeled both as a quadratic function and a linear function, with shortages addressed within a finite time horizon H. Our model's primary objective was to lower the overall expense related to the replenishment procedure. We also examined how variations in parameters such as **'**a' is the initial demand rate per year, '**b'** is the increasing demand rate per year, '**r'** is the amount that is carried per unit per order, the demand rate is D, 'W_h' is the cost of replenishing or purchasing per order**,** **θ** is an inventory dependent parameter affect the total cost. The results revealed optimal values across cycles, highlighting the dynamic nature of the system. Future research may extend this model to multi-item scenarios and explore applications beyond finite horizons into infinite time settings.

Chapter 4

Exponential Backordering Inventory Model Addressing Shortages in Finite Planning Horizons

Exponential Backordering Inventory Model Addressing Shortages in Finite Planning Horizons

4.1. INTRODUCTION

Efficient inventory management is fundamental to sustaining competitive advantage in modern supply chain ecosystems. The complexity of demand patterns, coupled with uncertainties in supply, necessitates the development of sophisticated inventory models capable of addressing various challenges. Among these challenges, shortages of goods pose a significant concern, especially when considering both linear trend and price sensitive demand dynamics across the planning horizon. In response to this multifaceted problem, this research endeavours to investigate an exponential complete backordering comparison inventory model. This model aims to comprehensively analyse the implications of shortages while accommodating both linear trend and price sensitive demand throughout all cycles of the planning horizon. By integrating exponential backordering principles and considering the nuances of price sensitive demand, the proposed model seeks to provide insights into optimal inventory control strategies under such conditions.

The significance of this research is underscored by its potential to enhance decision making processes in inventory management, thereby minimizing costs and maximizing operational efficiency. The proposed model addresses critical factors such as demand variability, lead time, and backordering costs, offering a robust framework for managing inventory in environments characterized by fluctuating demand and supply uncertainties. Through a rigorous investigation of the proposed model, this study aims to contribute to the existing body of knowledge in inventory optimization, particularly in addressing shortages amidst dynamic demand and supply conditions. This research employs advanced numerical iterative methods and leverages the computational capabilities of MATHEMATICA (version 12) to validate the model. The results demonstrate optimized total costs, taking into account the exponential backorder rates and the price sensitive nature of demand within a finite planning horizon.

In light of these considerations, this paper outlines the objectives, methodology, results, implications, and future directions of the research. The overarching goal is to advance our understanding of inventory management in the face of evolving market dynamics, providing valuable insights for practitioners and decisionmakers aiming to improve their inventory control mechanisms and maintain a competitive edge in an ever-evolving business landscape.

4.2. LITERATURE REVIEW

The concept of exponential partial backordering has been widely studied in the inventory literature, providing a more realistic representation of customer behavior in situations where some customers are willing to wait for replenishment while others seek alternative sources. Researchers have explored various extensions and variations of this model, incorporating factors such as deteriorating items, time varying demand and production rates, pricing decisions, and different cost structures. Despite the extensive research in this area, there is still room for further exploration and refinement of these models to better capture the complexities of real-world inventory systems. (Steven Nahmias and Nahmias 2014) offers a comprehensive and contemporary examination of inventory management techniques. With a strong emphasis on the integration of demand forecasting and inventory control, the book provides valuable insights for professionals and academics seeking to optimize supply chain efficiency. ((David Simchi-Levi and Simchi-Levi 2005) discusses advanced inventory management strategies, including backordering and the impact of demand variability on inventory decisions. (Stephen C. Graves and Graves 2020) examines the interplay between pricing strategies and inventory control, proposing models that accommodate demand elasticity and price fluctuations. (Shen 2019) delves into the theoretical foundations of supply chain management, offering insights into inventory models that address shortages and backordering. (Arzum Akkaş et al. 2020) investigates inventory policies for perishable products with stochastic demand, providing insights into how inventory models can be adapted to manage products with varying shelf lives and demand patterns. (Gérard P. Cachon et al. 2000) discusses strategies for aligning supply and demand, emphasizing the need for flexible inventory models to manage shortages effectively.

(S.C. Graves et al. 1993) provides a comprehensive overview of production and inventory control, discussing strategies for managing backorders in complex supply chain networks. (Marshall L. Fisher et al. 2010) explores how analytics can improve inventory management, including the use of backordering policies to optimize supply chain performance. (Srinagesh Gavirneni et al. 1999) discusses the benefits of information sharing in supply chains, relevant to optimizing inventory levels and backordering strategies. (Renuka S. Namwad et al. 2023) The paper introduces a model on supply chain inventory replenishment addressing material degradation based on usage of green technologies, offering profits along with cost reduction as well as carbon emissions abatement through the use of an iterative numerical algorithm with theoretical insights and practical applications. (Sunil Chopra et al. 2003) provides a detailed examination of supply chain strategies, including inventory management and backordering policies to handle dynamic demand and supply conditions. (David A. Wuttke et al. 2016) explores how technology adoption, such as RFID, can improve inventory accuracy and management, including backordering. (Kumar 2020) investigates inventory control policies for products with price sensitive demand, providing insights into how pricing strategies influence inventory levels and back-ordering decisions. (Joseph Milner et al. 2002) examines how dynamic demand patterns influence inventory management strategies, including the role of backordering. (Vic Anand et al. 2023) discusses the role of contracts in capacity planning, including inventory management and back ordering strategies with limited resources.

(Akshay R. Rao et al. 2000) explores the impact of agency issues on supply chain coordination, focusing on price-sensitivity impact on demand. (Kris Johnson Ferreira et al. 2016) examines how demand forecasting and lead times affect inventory management, emphasizing the need for robust backordering models. (Hau L. Lee et al. 1997) explores the causes and consequences of demand variability in supply chains, emphasizing the need for effective inventory models to manage backorders. (Paul Zipkin and Zipkin 2000) explores the performance of various lot sizing rules in the presence of demand uncertainty, relevant to understanding the impact of backordering policies. (Khouja, M. 2024) provides a comprehensive review of single period inventory models, highlighting the challenges of managing backorders in such contexts. (Charles P. Schmidt et al. 1985) discusses the impact of setup costs on inventory decisions, relevant to the study of back-ordering models in contemporary supply chains. (Awi Federgruen et al. 1986) explores how production constraints impact inventory decisions, relevant to the study of backordering under capacity limitations. (A. J. Clark et al. 1960) provides foundational insights into multi-echelon inventory systems, highlighting the importance of coordinated inventory policies.

(Hau L. Lee 1992) discusses common challenges in supply chain inventory management and proposes strategies to address these issues, including backordering policies. (Evan L. Porteus and Porteus 1985) presents modifications to the Economic Order Quantity (EOQ) model, which are pertinent to understanding how setup cost reductions can affect back ordering strategies. (John A. Buzacott et al. 1994) provides insights into the role of stochastic processes in inventory management, highlighting the importance of probabilistic models in addressing supply chain uncertainties. (Song, J.S., & Zipkin, P. 2024) examines inventory models for systems where products are assembled to order, providing insights into the role of backordering in such environments. (Guillermo Gallego et al. 1994) explores how dynamic pricing strategies can influence inventory management and backordering decisions. (Marshall L. Fisher et al. 2010) discusses the benefits of information sharing in supply chains, relevant to optimizing inventory levels and backordering policies.

(Marshall L. Fisher et al. 1996) explores how analytics can improve inventory management, including the use of backordering policies to optimize supply chain performance. (Michael Ketzenberg et al. 2008) investigates inventory policies for perishable products with stochastic demand, providing insights into how inventory models can be adapted to manage products with varying shelf lives and demand patterns.

4.3. ASSUMPTIONS AND NOTATIONS

4.3.1. ASSUMPTIONS

a) The initial inventory level is zero.
b) The storage cost remains constant throughout.
c) There is no time delay in replenishment or supply.
d) The ordering cost is fixed and known in advance.
e) Shortages are allowed under a finite planning horizon and occur continuously.
f) Complete back ordering is applied, following an exponential function.

4.3.2. NOTATIONS

1) H is finite time horizon.

2) The demand is price-sensitive, and the demand rate is D and $D(t) = a - bp$.

3) a is the initial demand rate per year.

4) b is the increasing demand rate per year.

5) The amount that is carried per unit per order is denoted by r.

6) O_r is the cost of replenishing or purchasing per order.

7) I_{j+1} is the total inventory carried out during the interval $[t_j, s_{j+1}]$

8) S is the total amount of shortages in the interval $[s_j, t_j]$.

9) s_j denote the time at which the inventory level reaches zero in the j^{th} replenishment cycle j= 1,2, 3...n.

10) t_j is the j^{th} replenishment time, where j =1, 2, 3, ……n.

11) n_1 is the number of replenishment cycles during the time horizon H.

12) Q_{nt} is the total optimal order quantity during the planning horizon H.

13) I_b instaneous shortage during the shortage period.

14) α is inventory dependent parameter.

15) W_h is the whole sale price Per unit.

16) $T_R(t_j, s_j, n_1)$ is the total cost of retailer.

17) Q_{j+1} is order quantity in (j+1) th cycle in time t_j.

18) $B(t)$ is the back ordering rate taken as an exponential function, $B(t) = \rho e^{-ct}$

4.4. MODEL FORMULATION

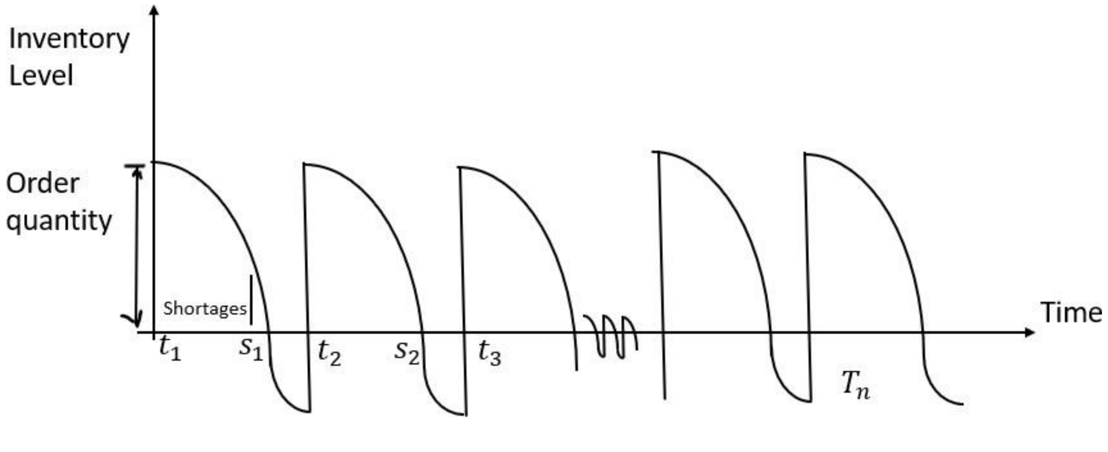

Fig. 4. 1. Pictorial representation of model

The initial equation is given by,

$$\frac{dI_{j+1}(t)}{dt} + I_{j+1}(t) = -D(t) \qquad t_j < t < s_{j+1}$$

Where, j=1, 2, 3, ……n_1 … (4.1)

$$\frac{dI_{j+1}(t)}{dt} = -D(t) - I_{j+1}(t) \qquad t_j < t < s_{j+1}$$

… (4.2)

Considering the boundary conditions $I_{j+1}(s_j) = 0$

The solution of the equation (4.2) is,

$$I_{j+1}(t) = e^{-\alpha t} \int_t^{s_{j+1}} D(u)\, e^u du$$

… (4.3)

$$I_{j+1}(t) = (a - bp)(e^{s_{j+1} - \alpha t} - e^{(1-\alpha)t})$$

... (4.4)

During the shortage phase, the instantaneously arising shortages $I_b(t)$ is offered by,

$$I_b(t) = B(t)(t_{j+1} - s_j)$$

… (4.5)

$$I_b(t) = \rho e^{-ct}(t_{j+1} - s_j)$$

… (4.6)

Considering the boundary conditions, $I_b(s_j) = 0$

$$Q_{j+1} = I_{j+1}(t) = (a - bp)(e^{sj+1-\alpha t_j} - e^{(1-\alpha)tj}) \quad \dots (4.7)$$

Considering the reorganization of the ordering s_{j+1} can be given as,

$$S_{j+1} = \int_{s_j}^{t_j} I_b(t)dt \quad \dots (4.8)$$

$$S_{j+1} = \int_{s_j}^{t_j} \rho e^{-ct}(t_{j+1} - s_j) \quad \dots (4.9)$$

$$S_{j+1} = \frac{-\rho}{c}(t_{j+1} - s_j)(e^{-ctj} - e^{-csj}) \quad \dots (4.10)$$

The entire purchase amount for a limited time frame of planning,

$$Q_{nt} = \sum_{j=1}^{n_1} Q_{j+1} = \sum_{j=1}^{n_1}(I_{j+1} + S_{j+1}) \quad \dots (4.11)$$

$$Q_{j+1} = \sum_{j=1}^{n_1}[(a - bp)(e^{(sj+1-\alpha t_j)} - e^{(1-\alpha)tj}) - \frac{\rho}{c}(t_{j+1} - s_j)(e^{-ctj} - e^{-csj})] \quad \dots (4.12)$$

The total retailer cost over a specified time horizon is given by,

$$Total\ Cost = Resupply\ expensess + Cost\ of\ retaining + Stocks + Purchacing\ Cost + Storage\ Cost \quad \dots (4.13)$$

$$T_R(t_j, s_j, n_1) = n_1.O_r + \sum_{j=0}^{n_1-1} H \int_{t_j}^{s_{j+1}} I_{j+1}(t)dt + \sum_{j=0}^{n_1-1} W_h.Q_{j+1} + \sum_{j=0}^{n_1-1} S \int_{s_j}^{t_j} I_b(t)dt \quad \ldots (4.14)$$

$$T_R(t_j, s_j, n_1) = n_1.O_r + \sum_{j=0}^{n_1-1} H \int_{t_j}^{s_{j+1}} (a-bp)(e^{s_{j+1}-\alpha t} - e^{(1-\alpha)t})dt + \sum_{j=0}^{n_1-1} W_h.[(a-bp)(e^{(s_{j+1}-\alpha t_j)} - e^{(1-\alpha)t_j})] + \sum_{j=0}^{n_1-1} S \int_{s_j}^{t_j} \rho e^{-at}(t_{j+1}-s_j)dt \quad \ldots (4.15)$$

$$T_R(t_j, s_j, n_1) = n_1.O_r \sum_{j=0}^{n_1-1} H.(a-bp)\left[\frac{-1}{\alpha}(1 - e^{s_{j+1}-\alpha t_j}) - \frac{1}{(1-\alpha)}(e^{(1-\alpha)s_{j+1}} - e^{(1-\alpha)t_j})\right] + \sum_{j=0}^{n_1-1} W_h.[(a-bp)(e^{(s_{j+1}-\alpha t_j)} - e^{(1-\alpha)t_j})] - \sum_{j=0}^{n_1-1} S\left[\frac{\rho}{c}(t_{j+1} - s_j)(e^{-ct_j} - e^{-cs_j})\right] \quad \ldots (4.16)$$

The primary goal is to determine the values of t_j and s_j, Fig 4.1. shows the pictorial representation, that minimize the total variable cost (T_R) in stock control and inventory management, to achieve the lowest possible total cost in the inventory system, the essential conditions for minimizing the total cost are as follows,

$$\frac{\partial T_R(t_j, s_j, n_1)}{\partial t_j} = 0 \ , \qquad\qquad j = 1, 2, 3, \ldots . n$$

… (4.17)

$$\frac{\partial T_R(t_j, s_j, n_1)}{\partial s_j} = 0 \ , \qquad\qquad j = 1, 2, 3, \ldots . n$$

… (4.18)

$$\frac{\partial T_R(t_j, s_j, n_1)}{\partial t_j}$$
$$= \sum_{j=0}^{n_1-1} H.(a - bp)[-e^{(sj+1-\alpha t_j)} + e^{(1-\alpha)tj}]$$
$$+ \sum_{j=0}^{n_1-1} W_h(a - bp)[-\alpha e^{(sj+1-\alpha t_j)} + (1 - \alpha)e^{(1-\alpha)tj}]$$
$$+ \sum_{j=0}^{n_1-1} S.\rho(t_{j+1} - s_j)e^{-\alpha tj}$$

… (4.19)

$$\frac{\partial T_R(t_j, s_j, n_1)}{\partial s_j} = S. \sum_{j=0}^{n_1-1} \frac{\rho}{c} [e^{-ctj} - e^{-csj} + c(t_{j+1} - s_j)e^{-sj}]$$

… (4.20)

The total cost of supplier is given by

$$T_S(t_j, s_j, n_1) = n_1. Ss$$
$$+ Cs \sum_{j=0}^{n_1-1} \int_{t_j}^{s_{j+1}} (a - bp)(e^{sj+1-\alpha t} - e^{(1-\alpha)t})dt$$
$$+ \int_{s_j}^{t_j} e^{-ct}(t_j - s_j)$$

… (4.21)

The total cost's Hessian matrix must be positive definite for a fixed n in order for the total cost to be least (i.e. $\nabla^2 TC$).

$$\nabla T_R(t_j, s_j, n_1)$$

$$= \begin{bmatrix} \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_1^2} & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_1\,\partial s_1} & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial s_2\,\partial t_1} & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial s_1^2} & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial s_1\,\partial t_2} & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_2\,\partial s_1} & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_2^2} & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_2\,\partial s_2} & \cdots & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_{n_1-1}\,\partial s_{n_1-1}} & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial s_{n_1-1}^2} & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial s_{n_1-1}\,\partial t_{n_1}} \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_{n_1}\,\partial s_{n_1-1}} & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_{n_1}^2} \end{bmatrix}$$

4.5. THEOREMS

I. First-Order Optimality Condition

Let f(x) be a differentiable function on $\mathbb{R}^n$ Then, necessarily, for $x^* \in \mathbb{R}^n$ to be a local minimum of f(x), the gradient of f(x) must be zero at x^* that is

$$\nabla f(x^*) = 0$$

This condition is now known as first-order optimality (Nocedal 2020)

Proof: The function f(x) is a local minimum at x^*, so no direction d satisfies:

$f(x^* + d) < f(x^*)$. From the first order Taylor expansion around x^*, we have

$$f(x^* + d) \approx f(x^*) + \nabla f(x^*)^T d$$

For that to be true for all directions d, it must be that $\nabla f(x^*) = 0$.

II. Second-order optimality condition

Let f(x) twice differentiable on $\mathbb{R}^n$ The point $x^*$is a local minimum of f(x) if

$\nabla f(x^*) = 0$ (first-order condition), and

The Hessian matrix $\mathrm{H}(f)(x^*)$ is positive semi-definite that is, $\mathrm{H}(f)(x^*) \geq 0$.

Proof: The second-order Taylor expansion of f(x) about x^* is

$$f(x^*d) \approx f(x^*) + \nabla f(x^*)^T d + \frac{1}{2} d^T H(f)(x^*)d$$

Because $\nabla f(x^*) = 0$ (by the first-order condition), this gives

$$f(x^*d) \approx f(x^*) + \frac{1}{2} d^T H(f)(x^*)d$$

The inequality $f(x^* + d) \geq f(x^*)$ for all directions dd holds only if $H(f)(x^*)$ is positive semi-definite.

III. KKT Conditions Karush-Kuhn-Tucker

For the nonlinear programming problem:

$$min\ f(x)\ subject\ to\quad g_i(x) \leq 0,\ h_j(x) = 0,$$

Karush-Kuhn-Tucker Conditions Suppose that f, g_i, h_j are differentiable. (Amir Beck 2023)Karush-Kuhn-Tucker conditions give necessary optimality conditions. Let x^*be a local minimum. Then there exist multipliers $\lambda_i \geq 0$ and μ_j such that

1. $\nabla f(x^*) + \sum i\lambda i\nabla gi(x^*) + \sum j\mu j\nabla hj(x^*) = 0$
2. $\lambda_i g_i(x^*) = 0 \quad for\ all\ i, (complementary\ slackness),$
3. $g_i(x^*) \leq 0,\ h_j(x^*) = 0\ for\ all\ i, j.$
4. $\lambda_i \geq 0\ for\ all\ i.$

4.6. NUMERICAL ILLUSTRATION

A numerical example is presented with particular values for a=0.65, b=0.01, c=0.35, rho(ρ)=0.26, e=2.7, Wh=0.3, H=0.1, p=1.5, r=2.6748, $s_1 = 0$, S=4 in appropriate units. To obtain the solutions of Eq. (4.19) and Eq. (4.20), we utilized Mathematica, version 12, that is the computational tool implemented in order to compute the results and analysis for inventory model with backordering. Fig 4.2. and Fig 4.3. shows the increasing values replenishment time t_j and s_j respectively.

4.7. RESULTING TABLES AND FIGURES

Table 4. 1 Replenishment time t_j for exponential back order

$\downarrow a$ $\rightarrow t_j$	t_0	t_1	t_2	t_3
0.1	0.01	0.122146		
0.2	0.01	2.37729	3.33812	
0.4	0.01	1.71775	3.33812	4
0.6	0.01	1.71775	3.33812	4

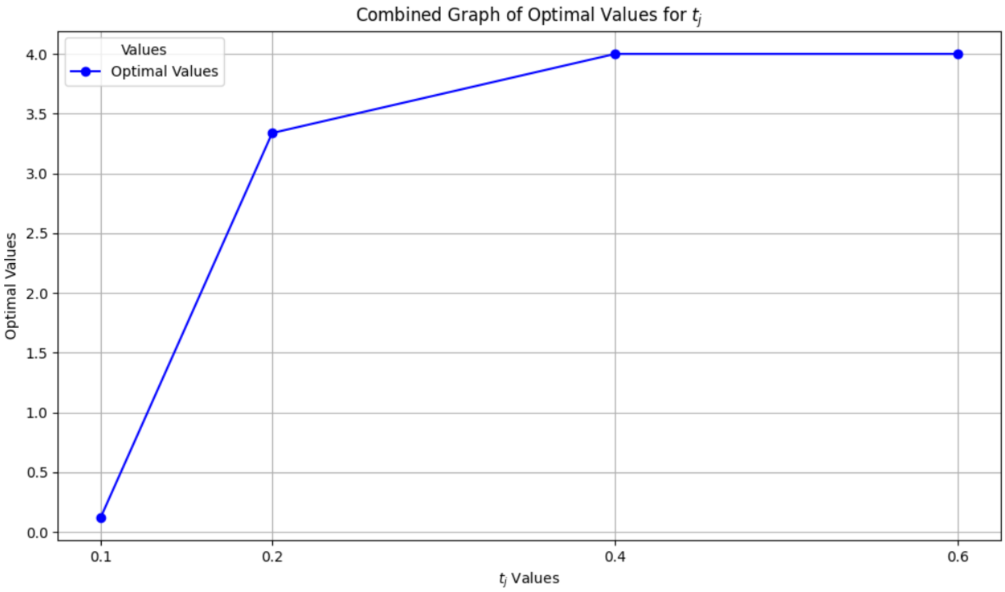

Fig. 4. 2 Combined graph of optimal values for t_j

Table 4. 2 Replenishment time s_j for exponential back order

↓ a	→ s_j	s_0	s_1	s_2	s_3
0.1		0	3.99881		
0.2		0	2.37743	3.9978	
0.3		0	1.71775	3.33812	4
0.4		0	1.71775	3.33812	4

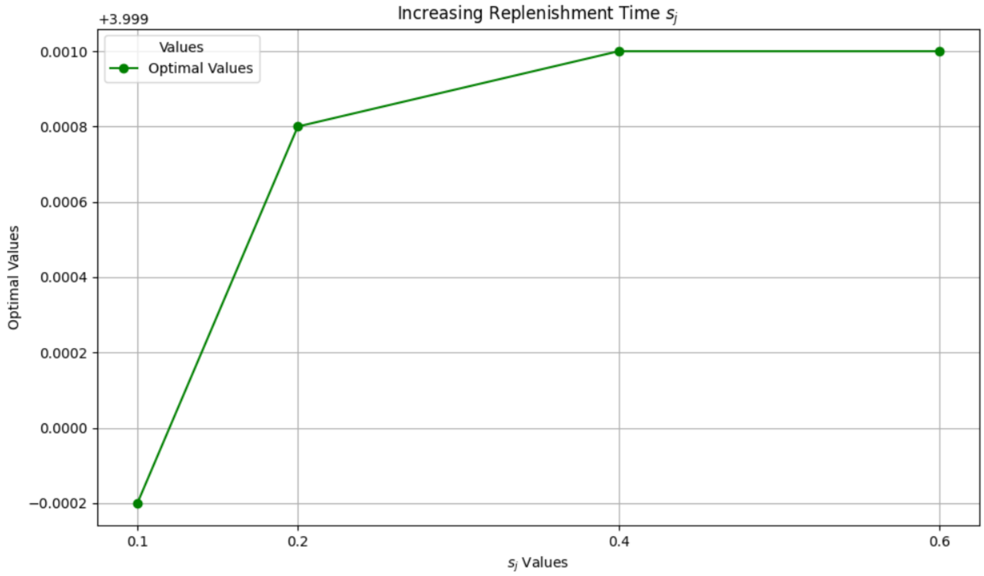

Fig. 4. 3 Increasing order of replenishment time s_j

Table 4. 3 Total cost for the retailer for different replenishment cycle for exponential back order

↓ *a* →n	1	2	3	4	5	6
0.1	**4.13456**	5.85722	9.00915	12.9165	18.3473	27.1847
0.2	**6.33814**	6.76708	10.272	15.9013	25.1825	42.6298
0.4	9.44631	**8.53743**	12.7736	21.8019	38.6834	73.1256
0.5	10.5904	**9.40242**	14.0225	24.7468	45.420 8	88.343
0.6	11.5592	**10.2558**	15.2711	27.6907	52.1556	103.554

Table 4. 4 Total cost for retailer, supplier and quantity for optimal values for exponential complete back order

$\downarrow$ a	T_R	T_S	Q_{nt}
0.1	4.13456	2.67411	2.67255
0.2	6.33814	2.67388	6.8361
0.4	8.53743	5.33857	7.86121
0.5	9.40242	5.33598	10.0195
0.6	10.2558	5.33346	12.1581

Table 4. 5 Days table for t_j

Days $\downarrow$ / n $\rightarrow$ **1**	2	3	4
148.156	313.488	609.206	904.924
19.7869	433.88	609.206	904.924
1.825	313.488	609.206	904.924
1.825	313.488	609.206	904.924

Table 4. 6 Days table for s_j for exponential complete back order

Days ↓ / n → 2	3	4
729.783	609.206	904.924
433.88	729.598	904.924
313.488	609.206	904.924
313.488	609.206	904.924

4.8. SENSITIVITY ANALYSIS

The sensitivity analysis elucidates the impact of changes in the crucial parameters on total costs for the retailer T_R, supplier T_S, and the total order quantity Q_{nt}. A 20% increase in parameter a causes T_R to shift up to 9.40242 from 8.53743 and T_S shifts down to its slightly decreased value from 5.33857 to 5.33598. The total order quantity Q_{nt} shifts up to 10.0195 from 7.86121. On the other hand, with the fall of 20% in a, T_R falls down to 6.33814 and T_S falls down to 2.67388, and Q_{nt} also falls down to 6.8361. This depicts that it is moderately sensitive to fluctuations in the cost and the order quantity of a. Changes have very minimal effects on parameter b. T_R varies within the range of 7.05339 to 7.04076 and T_S varies very slightly as it ranges from 0.00072145 to 0.000693299 whilst the total order quantity decreases slightly by the range of 4.0448 to 4.00439. This also shows that b has lesser effects on costs and order quantity Q_{nt}. The detailed sensitivity analysis is discussed in Table 4.7.

This shows that parameter c is fairly sensitive. In fact, a 20% drop resulted in T_R sharply increasing from 6.87698 to 11.3531, T_S from 0.00101634 to 0.00129504, and quantity Q_{nt} from 3.7784 to 5.36078. α also turns out to be highly sensitive. Indeed, a 20% increase in it leads to a sharp increase in T_R from 5.37491 to 29.9734, and T_S

from 0.002066 to 0.0131941, while quantity Q_{nt} decreases from 3.03674 to 2. 44819.Therefore, c and α are actually important parameters, which largely dictate total costs, apart from the order quantities especially for the retailer. For graphical visualizations a Machine-Learning Tool-Python, with the help of it the heatmap are made. Fig 4.4 gives the visualization of the sensitivity of retailer's total cost; Fig 4.5 gives the graphical visualization of sensitivity of supplier's total cost and in Fig 4.6 gives the graphical visualization of optimal order quantity.

Table 4. 7 Sensitivity analysis of all the parameters for exponential complete back order

Parameters	%Changes	Optimal Replenishment cycle	Total order Quantity Q_{nt}	Total cost of Retailer T_R	Total cost of supplier T_S
a	+20	2	10.0195	9.40242	5.33598
	+10	2	12.1581	10.2558	5.33346
	0	2	7.86121	8.53743	5.33857
	−10	1	2.67255	4.13456	2.67411
	{−20	1	6.8361	6.33814	2.67388
b	+20	1	4.00439	7.04076	0.000693299
	+10	1	3.96288	7.02916	0.000662191
	0	1	4.0448	7.05339	0.00072145
	−10	1	4.16037	7.09689	0.000790488
	{−20	1	4.1227	7.08151	0.000769832
c	+20	1	3.32059	5.95992	0.00114326
	+10	1	2.92617	5.29343	0.00114757
	0	1	3.7784	6.87698	0.00101634
	−10	1	5.36078	11.3531	0.00129504
	{−20	1	4.82145	9.57865	0.000356569
α	+20	1	2.44819	4.21495	0.00286675
	+10	1	2.20027	4.04957	0.00278454
	0	1	3.03674	5.37491	0.002066
	−10	1	15.6442	29.9734	0.0131941
	{−20	1	8.74641	16.4004	0.00507976
W_h	+20	1	3.44331	6.64752	0.00139882
	+10	1	3.22745	6.49588	0.00165342
	0	1	3.7784	6.87698	0.00101634
	−10	1	10.5587	9.24053	0.00280211
	{−20	1	5.72052	7.96899	0.000725277

H	+20	1	3.92663	7.28774	0.000701578
	+10	1	4.08092	7.71854	0.00038045
	0	1	3.7784	6.87698	0.00101634
	−10	1	3.36164	5.73868	0.00192955
	{−20	1	3.49676	6.1048	0.00162995
Rho	+20	1	4.72833	8.0726	0.00100185
	+10	1	5.35694	9.13329	0.00114567
	0	1	4.0448	7.05339	0.00072145
	−10	1	1.69676	4.42553	0.0010419
	{−20	1	2.53281	5.2142	0.00027822
S	+20	1	4.0607	7.84175	0.00119037
	+10	1	4.35968	8.85483	0.00129419
	0	1	3.7784	6.87698	0.00101634
	−10	1	2.96604	4.39282	0.00233261
	{−20	1	3.22041	5.14019	0.000300246

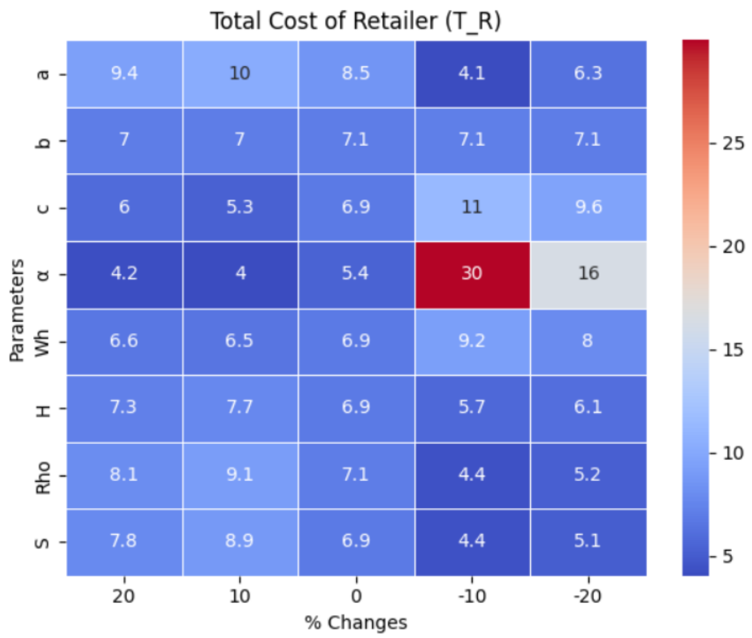

Fig. 4. 4 Heatmap of Retailer's Cost: Tracking the Ripple Effect of Parameter Changes

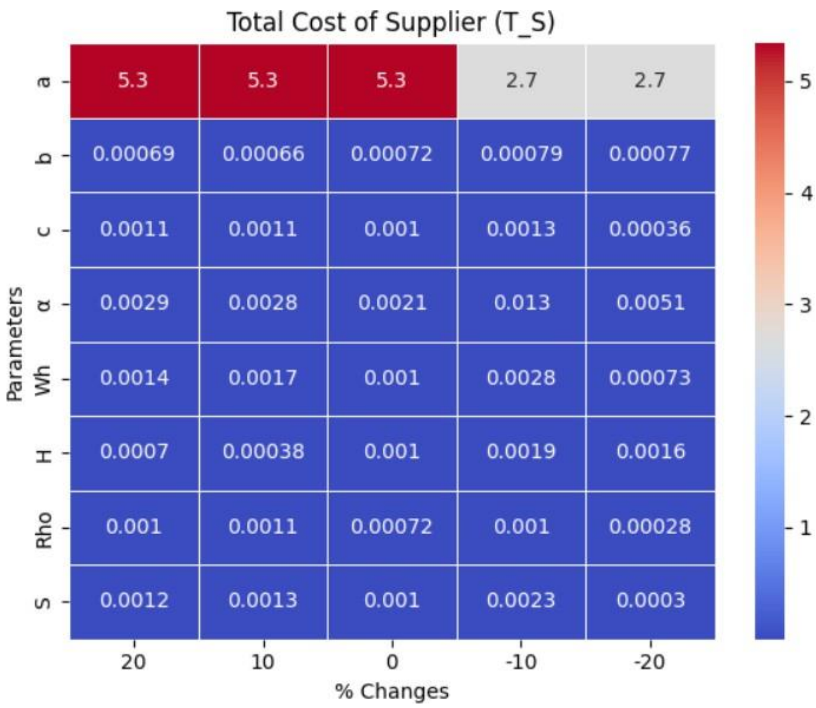

Fig. 4. 5 Heatmap of Supplier's Cost: Tracking the Ripple Effect of Parameter Changes

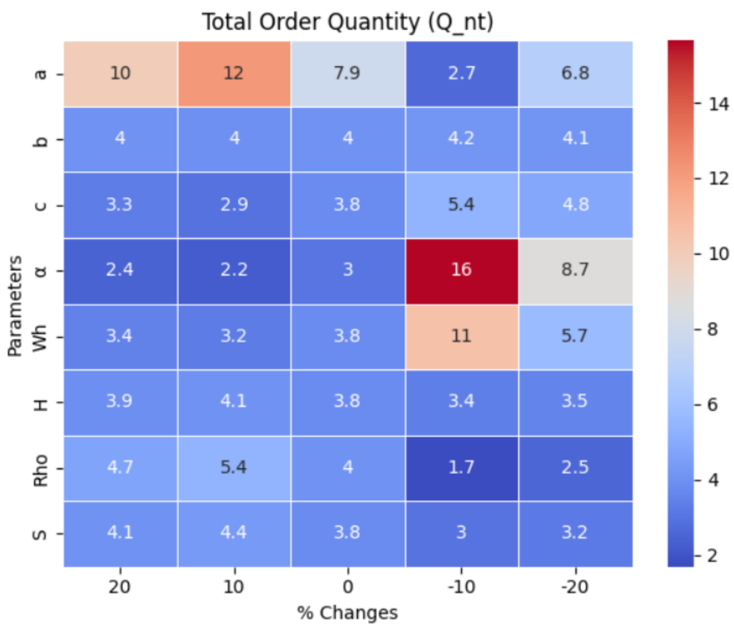

Fig. 4. 6 Total Order Quantity Trends with Parameter Variations

4.9. MANAGERIAL SUGGESTIONS

To benefit from back ordering to the fullest and to minimize total costs, the managers should initiate strategic moves, starting with optimum inventory levels based on accurate demand forecasting. Advanced techniques of forecasting to determine fluctuations in demand and when to allow back orders would indeed lead to both fulfilling customer orders on time and reducing excess holding costs. Various suppliers could be created to allow them to negotiate good terms for the back-ordered items, thus ensuring that the supplier stabilizes the cost during peak demand T_S, while such flexibility may enable managers to avoid misunderstandings or overreacting to changes in the marketplace. Again, providing discounts, loyalty points, or priority filling as an incentive to encourage customers to take backordered items can improve the satisfaction level of customers and optimize cash flows. This would enable customers to wait for the products and therefore minimize the loss of sales. The other thing of huge importance is conducting periodic backorders analysis. Analysis would permit managers to know the after-effects of back-ordering on efficiency in operations as well as customer satisfaction. This would make the necessary adjustments to the responsible strategies. With such controls, organizations can rightly monitor back ordering, lower their overall cost, and potentially increase profit without losing the relationship with the customer.

4.10. CONCLUSION

In this chapter, an exhaustive exponential complete backordering inventory model is developed for countering shortages at every cycle of a finite horizon with both linear trends and price-sensitive demands. An exhaustive, exponential complete backordering model will offer some significant improvement in the inventory management by considering exponential backordering, giving better applicability towards the real-world demand, where customers' demand is sensitive to the pricing and supply lags. Mathematica's mathematical verification, combined with computation support from the tools based on Python, makes this model even more versatile in giving accurate numerical solutions and intelligent graphical analysis.

A practical case may be seen in the pharmaceutical industry where the shortage of necessary medications is often caused because of the delays in production and sensitivity to price. There is often partial backorders in pharmacies and hospitals, and patients are ready to stand and wait as the stock is replenished, others change to an alternative. With exponential complete backordering model, suppliers can effectively plan to replenish the stocks and prices in a strategic manner to incur minimum costs and guarantee the availability of the drugs and to the satisfaction of the patients. Results of the numerical example will show that the model is optimizing the total cost while improving coordination between the suppliers and retailers. Results of the sensitivity analysis will provide an intuitive sense that total costs are sensitive to the backordering rate, α, and the cost coefficient, c; therefore, the importance of these parameters in the strategic decision-making for the inventory manager can be realized. This implies that the back ordering policies can be optimized by reducing stockouts in order to maximize customer satisfaction by a cost-effective supply chain.

Summarily, the developed model gives practitioners and decision-makers a critical tool for looking forward to improving the mechanisms of controlling inventory. Effective management of back orders under fluctuating demand and pricing conditions will ensure not only cost optimization but also improvements in efficiency in operations. This model may be extended into multi-echelon supply chains, with consideration for stochastic demand patterns and implications of dynamic pricing strategies.

Chapter 5

Managing Inventory Costs: A Study of Partial Backordering in the EPQ Model

Managing Inventory Costs: A Study of Partial Backordering in the EPQ Model

5.1. INTRODUCTION

Inventory management is central to business processes that are aimed at achieving efficiency in the use of resources and feeding the markets. Hence, it involves achieving an optimal flow between adequate supplies' availability and decreasing such costs. One of the key issues addressed in this field is back orders, which happen when demand exceeds the supply of a particular product, and the orders cannot be fulfilled on the same day, thus, potentially angering the customers. Order backlogs need to be managed well because they determine the service levels that clients get as well as organizational efficiency. Inventory theory has the Economic Production Quantity (EPQ) model as a basic model it uses, in order to determine with precision, the manufacturing amount, through minimizing the overall costs of inventory. The EPQ model does not allow for interruptions in production and demand, which is costly under circumstances that are unpredictable. Still the model is most appropriate where demand for the product is fixed. It offers important information concerning the place of holding costs and set-up costs and, therefore, helps in the efficient management of inventories.

Indeed, this paper's objective is to analyse partially and fully backordered EPQ models for products with exponential backordering characteristics. Therefore, this study aims at contributing to the managerial body of knowledge by establishing the efficacy of back ordering tactics while taking into account cost of inventory, service level, and operational performance.

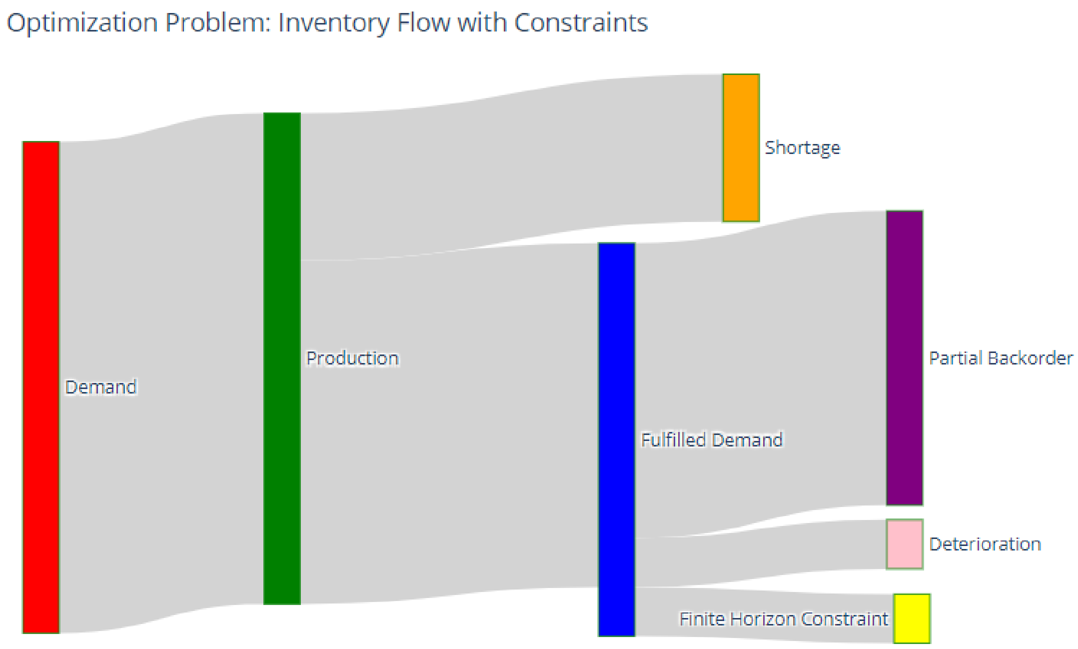

Fig. 5. 1 Inventory flow with constraints

This study aims to explore the effects of partial backordering within the EPQ model framework, specifically focusing on products with exponential backordering dynamics. Figure 5.1 shows the different constraints for inventory management, by analysing how various backordering strategies impact inventory costs, service levels, and operational efficiency, this research seeks to offer valuable insights that can the enhancements of decision-making performance of real-life environments by the developers, the optimization of the inventory performance and overall customer satisfaction.

5.2. LITERATURE REVIEW

It is necessary for a business to manage inventory as part of their general strategies towards resource use and client service. (Srinagesh Gavirneni et al. 1999) and (Harris 1990) Prediction of economic ordering point was suggested with Economic Order Quantity (EOQ) model which aims to find the best order quantity that minimizes the total inventory costs Searching for minimal total costs includes ordering costs and holding costs. These studies formed a basis for further development of the EOQ model into the Economic Production Quantity (EPQ) model developed by(Silver et al. 1998) for the cases of continuous production. These models have the capacity of creating a systematic way of determining the right amount of inventory to order or produce to minimize the costs of manufacturing and storing while at the same maintaining the right level of profitability and operation (Paul Zipkin and Zipkin 2000).

The management of back-orders or situations in which the orders are more than the stocks is another operational crucial since it ensures operational without compromising the service levels. These include the notes made by (John von Neumann et al. 1944) who identified that replacing the costs of holding inventory with ordering costs resulted in a trade-off. More on differentiation in backordering strategies were equally demonstrated by Lee and (Steven Nahmias and Nahmias 2014) indicating that at one extreme there is the complete backordering policy whereby all demands are met once there is restocking of inventories. On the other hand, (Gray 2012) focused on partial backordering policies with the objective of improving customers' service level by delivering orders based on their profitability or contract arrangements. These approaches provide information on the factors that affect the management of back orders with the aim of satisfying the customer's needs while at the same time minimizing costs. Literature on EPQ models incorporating back ordering policies has provided solutions to some of the primary problems in inventory control. A distinguishment review by (Edmund Terence Gomez et al. 2013) on the effect of partial backordering inside EPQ models in Malaysia indicated these vital tensions between holding costs and shortage costs. According to (Aris A. Syntetos et al. 2005) selected for enhancing on the demand forecasting accuracy especially in cases of intermittent demand essential in inventory management.

Further developments have been made to these basic topics. Multi-product inventory problems were investigated by(A. J. Clark et al. 1960) where he stressed the integration of the decisions of inventory for more than one product with minimal total system expense. (Evan L. Porteus and Porteus 2002) provided stochastic inventory models that put certain measures of variability into demand and lead times; this is useful in today's complex environment when making inventory management decisions. (Stephen C. Graves and Graves 2020) reviewed sophisticated active inventory management approaches such as the application of prognosis methods together with the provision of dynamic pricing in an effort to provide improved service to the customers. In the more recent past, new concepts and technologies for inventory management keep emerging with a view of refining decision-making tools.(Praveen Kumar et al. 2024) emphasized the steps used by companies in demand forecasting and inventory management through the use of data analytics and artificial intelligence resulting in enhanced efficiency in supply chain activities. Same as that (Fang Yin et al. 2023) have investigated the application of the ML method to forecast the demand flow and adjust the shops' inventory, which leads to lower holding costs and higher service levels. However, there has been a recent trend towards more specific studies of the application of the methods in industries and case studies. Analysing the sustainability aspects integrated with inventory management, (Chen and Wang 2010) focused on the application of environmental parameters along with the most popular cost and service factors.

Other relevant works include those carried out by (Chung et al. 2009) where the authors examined the effect of lead time variation on inventory control in production environments. (Goyal 1977) added a back-order factor to EPQ basic models; consequently, the EPQ model with back order costs was developed. (Saithong et al. 2020) have discussed about the use of queuing theory in the context of inventory management for aiming at the maximum service level and minimum cost of inventory in the service sector industries. Subsequent work by (Scott A. Wright et al. 2018) extended on the literature about inventory control with inventory management based on complex linear programming models where information on demand and supply variability was required. (Yongrok Choi et al. 2016) made a review on the various methods of stock management and their impact on the performance of supply chain, in a comparison with the different inventory management systems. (Christopher S. Tang et al. 2008) investigates the effects of SRM strategies to the performance on inventory, focusing particularly on the collaboration of inventory management. (Yuxin Liao et al. 2019, 201) focused on the discussion of using blockchain in managing inventories. They also established that blockchain improves the aspect of transparency and accountability since discrepancies are lowered and inventory exactness is boosted.

Similarly, Zhou et al. (2022) examined the IoT's applicability in inventory system where it presented that IoT in inventory system could greatly reduce the occurrence of stockout and overstock conditions.

Other recent investigations have also been directed towards the enhancement of backordering models. The dynamic backordering model that was recently developed by(Yajun Zhang et al. 2020) uses backorder rates that change regarding the current demand and supply circumstances. According to their model, the company offered better performance in as much as customer service level was maintained alongside cutting down on the costs of inventory. (Ching-Shih Tsou et al. 2011) proposed a partly backorder – completely backorder mixed policy whereby an organization can order a mixture of both to meet a higher level of service to customers. Xu and Huang, 2022 attempted to improve backordering policy, by using different demand scenarios for backordering policies using simulation-based optimization techniques. Some of the literature in this area has extended previous work on EPQ models with backordering. (Gaoke Wu et al. 2022) systemically reviewed the development of EPQ models and introduced the backordering development in the last decade, and they pointed out that there are many necessary research questions waiting to be explored, especially in the application of new technologies such as artificial intelligence and the internet of things for EPQ models. More specifically,(Li Xi et al. 2020) selected healthcare settings as the context of their study and showed that optimal backordering policies could cut costs and enhance the perceptions of services in healthcare supply systems. The study by (Malti Bansal et al. 2022) explored the effects of supply chain disruptions on EPQ models with backordering and recommended that firms should establish supply chains that can recover from such disruptions.

(Yuxin Liu et al. 2023) studied the fact that partnership in supply chain had a positive impact on inventory management decision where overall back ordering strategies were enhanced, information sharing was found to be efficient, and several other strategic aspects of supply chain were closer to perfection. (Shih-Yun Huang et al. 2022) studied the application of higher inventory optimization in EPQ models with backorder; the authors proved that the integration of metaheuristic algorithms, including genetic algorithms and particle swarm optimization, can enhance the usage of inventory management systems. (Yonggab Kim et al. 2022) established a new EPQ model which operated with the assistance of drone delivery to capture the inventory delivery system for minimizing lead time for back ordering and save more costs and enhance the service level for e-commerce supply chain. (Kris Johnson Ferreira et al. 2016) were interested in the use of digital twins in inventory management, proving that digital twins could assist in choosing the most favorable backordering policies and minimizing inventory costs by using digital copies of inventory systems for close monitoring and strategy testing. Another study by (TeronLemir et al. 2019) discussed the effect of climate change on inventory management and means that backordering technique should be modified to act as coping mechanism for the higher variability of demand as impacted by climate change. (Juana López-Martínez et al. 2002) studied the combined use of sustainability measures into EPQ models for improving the theoretical and practical measures for integrating environmental factors into inventory management for better stock control economic and environmental outcomes. In a study by (Jie Xu et al. 2017), they looked into the integration of advanced analytics and machine learning in identifying backorder probabilities whereby their model outperformed the general models in the prediction of backorders hence allowing for preliminary management and overall minimized costs of backordering.

5.3. ASSUMPTIONS AND NOTATIONS

5.3.1. ASSUMPTIONS

1) The initial inventory level is zero.
2) The storage cost remains constant throughout.
3) The ordering cost is fixed and known in advance.
4) Shortages are allowed under a finite planning horizon.
5) Partial back ordering is applied, following an exponential function.

5.3.2. NOTATIONS

1) H is finite time horizon.

2) The demand is price-sensitive, and the demand rate is D and $D(t) = a - bp,$ p is the unit selling price.

3) a is the initial demand rate per year.

4) b is the increasing demand rate per year.

5) The amount that is carried per unit per order is denoted by r.

6) O_r is the cost of replenishing or purchasing per order.

7) I_{j+1} is the total inventory carried out during the interval $[t_j, s_{j+1}]$

8) S is the total amount of shortages in the interval $[s_j, t_j]$.

9) s_j denote the time at which the inventory level reaches zero in the j^{th} replenishment cycle j= 1,2, 3...n.

10) t_j is the j^{th} replenishment time, where j =1, 2, 3, ……n.

11) n_1 is the number of replenishment cycles during the time horizon H.

12) Q_{nt} is the total optimal order quantity during the planning horizon H.

13) I_b instaneous shortage during the shortage period.

14) α is inventory dependent parameter.

15) W_h is the whole sale price Per unit.

16) $T_R(t_j, s_j, n_1)$ is the total cost of retailer.

17) Q_{j+1} is order quantity in (j+1) th cycle in time t_j.

18) $B(t)$ is the back ordering rate taken as an exponential function, $B(t) = \rho e^{-ct}$

5.4. MODEL FORMULATION

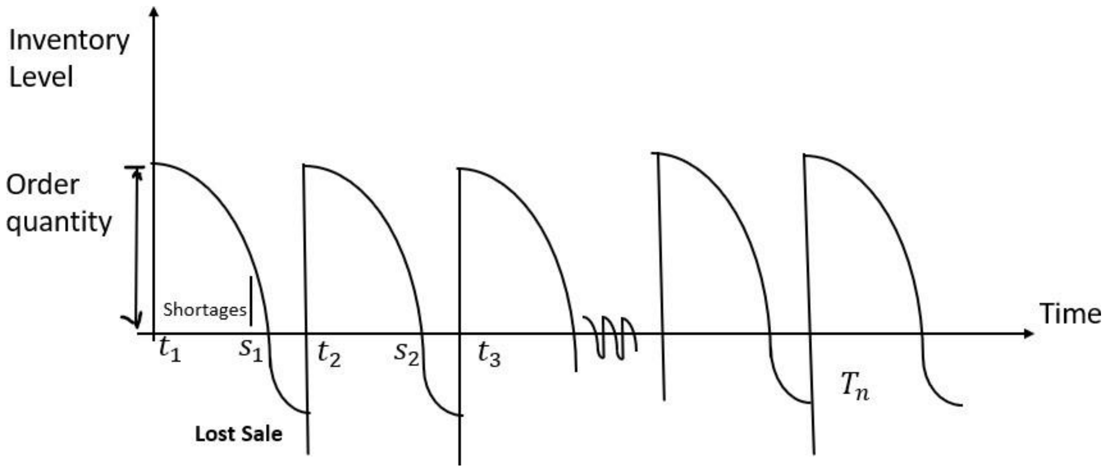

Fig. 5. 2. Model diagram

The initial equation is given by,

$$\frac{dI_{j+1}(t)}{dt} + I_{j+1}(t) = -D(t) \qquad t_j < t < s_{j+1} \qquad \ldots (5.1)$$

Where, j=1, 2, 3, ……n_1

$$\frac{dI_{j+1}(t)}{dt} = -D(t) - I_{j+1}(t) \qquad t_j < t < s_{j+1} \qquad \ldots (5.2)$$

Considering the boundary conditions $I_{j+1}(s_j) = 0$

The solution of the equation (5.2) is,

$$I_{j+1}(t) = e^{-\alpha t} \int_t^{s_{j+1}} D(u)\, e^u du \qquad \ldots (5.3)$$

$$I_{j+1}(t) = (a - bp)(e^{s_{j+1} - \alpha t} - e^{(1-\alpha)t}) \qquad \ldots (5.4)$$

During the shortage phase, the instantaneously arising shortages $I_b(t)$ is offered by,

$$I_b(t) = B(t)(t_{j+1} - s_j) \qquad \ldots (5.5)$$

$$I_b(t) = \rho e^{-ct}(t_{j+1} - s_j) \qquad \ldots (5.6)$$

Considering the boundary conditions, $I_b(s_j)=0$

$$Q_{j+1} = I_{j+1}(t) = (a - bp)(e^{s_{j+1} - \alpha t_j} - e^{(1-\alpha)t_j}) \qquad \ldots (5.7)$$

Considering the reorganization of the ordering s_{j+1} can be given as,

$$S = \int_{s_j}^{t_j} I_b(t)dt \qquad \ldots (5.8)$$

$$S = \int_{s_j}^{t_j} \rho e^{-ct}(t_{j+1} - s_j) \qquad \ldots (5.9)$$

$$S = \frac{-\rho}{c}(t_{j+1} - s_j)(e^{-ctj} - e^{-csj})$$
… (5.10)

The entire purchase amount for a limited time frame of planning,

$$Q_{nt} = \sum_{j=1}^{n_1} Q_{j+1} = \sum_{j=1}^{n_1}(I_{j+1} + S)$$
… (5.11)

$$Q_{j+1} = \sum_{j=1}^{n_1}[(a - bp)(e^{(sj+1-\alpha t_j)} - e^{(1-\alpha)tj}) - \frac{\rho}{c}(t_{j+1} - s_j)(e^{-ctj} - e^{-csj})]$$
… (5.12)

As the model consists of partial back ordering, there a lost sale is seen.

So, the total lost sale is given by,

$$L_i = \int_{s_j}^{t_j}[(a - bp) - \rho e^{-ct}(a - bp)]\, dt$$
… (5.13)

$$L_i = (a - bp)\int_{s_j}^{t_j}[(1 - \rho e^{-ct})]dt$$
… (5.14)

$$L_i = (a - bp)\{(t_j - s_j) - \rho\left(\frac{-1}{c}(e^{-ctj} - e^{-csj})\right)\}$$
… (5.15)

$$L_i = (a - bp)\{(t_j - s_j) + \left(\frac{\rho}{c}(e^{-ctj} - e^{-csj})\right)\}$$
… (5.16)

The total retailer cost over a specified time horizon is given by,

$$Total\ Cost = Resupply\ expensess + Cost\ of\ retaining + Stocks + Purchacing\ Cost + Lost\ Sale + Storage\ Cost \quad \ldots (5.17)$$

$$T_R(t_j, s_j, n_1) = n_1 . O_r + \sum_{j=0}^{n_1-1} H \int_{t_j}^{s_{j+1}} I_{j+1}(t)dt + \sum_{j=0}^{n_1-1} W_h . Q_{j+1} + L_i + \sum_{j=0}^{n_1-1} S \int_{s_j}^{t_j} I_b(t)dt \quad \ldots (5.18)$$

$$T_R(t_j, s_j, n_1) = n_1 . O_r + \sum_{j=0}^{n_1-1} H \int_{t_j}^{s_{j+1}} (a - bp)(e^{sj+1-\alpha t} - e^{(1-\alpha)t})dt + \sum_{j=0}^{n_1-1} W_h . [(a - bp)(e^{(sj+1-\alpha t_j)} - e^{(1-\alpha)tj})] + (a - bp)\{(t_j - s_j) + (\frac{\rho}{c}(e^{-ctj} - e^{-csj}))\} + \sum_{j=0}^{n_1-1} S \int_{s_j}^{t_j} \rho e^{-\alpha t}(t_{j+1} - s_j)dt \quad \ldots (5.19)$$

$$T_R(t_j, s_j, n_1) = n_1 . O_r \sum_{j=0}^{n_1-1} H.(a-bp) \left[\frac{-1}{\alpha}(1 - e^{s_{j+1}-\alpha t_j}) - \frac{1}{(1-\alpha)}(e^{(1-\alpha)s_{j+1}} - e^{(1-\alpha)t_j})\right]$$

$$+ \sum_{j=0}^{n_1-1} W_h.[(a-bp)(e^{(s_{j+1}-\alpha t_j)} - e^{(1-\alpha)t_j})]$$

$$+ (a-bp)\{(t_j - s_j) + (\frac{\rho}{c}(e^{-ct_j} - e^{-cs_j}))\}$$

$$- \sum_{j=0}^{n_1-1} S[\frac{\rho}{c}(t_{j+1} - s_j)(e^{-ct_j} - e^{-cs_j})$$

… (5.20)

The Fig 5.2. shows the pictorial representation of inventory model. The primary goal is to determine the values of t_j and s_j, that minimize the total variable cost (T_R) in stock control and inventory management, to achieve the lowest possible total cost in the inventory system, the essential conditions for minimizing the total cost are as follows,

$$\frac{\partial T_R(t_j, s_j, n_1)}{\partial t_j} = 0 \; , \qquad j = 1, 2, 3, \ldots . n$$

… (5.21)

$$\frac{\partial T_R(t_j, s_j, n_1)}{\partial s_j} = 0 \; , \qquad j = 1, 2, 3, \ldots . n$$

… (5.22)

$$\frac{\partial T_R(t_j, s_j, n_1)}{\partial t_j} = \sum_{j=0}^{n_1-1} H.(a-bp)\left[-e^{(s_{j+1}-\alpha t_j)} + e^{(1-\alpha)t_j}\right] + \sum_{j=0}^{n_1-1} W_h(a-bp)\left[-\alpha e^{(s_{j+1}-\alpha t_j)} + (1-\alpha)e^{(1-\alpha)t_j}\right] + (a-bp)(1-\rho e^{-ct_j}) + \sum_{j=0}^{n_1-1} S.\rho(t_{j+1}-s_j)e^{-at_j} \quad \ldots (5.23)$$

$$\frac{\partial T_R(t_j, s_j, n_1)}{\partial s_j} = S.\sum_{j=0}^{n_1-1} \frac{\rho}{c}\left[e^{-ct_j} - e^{-cs_j} + c(t_{j+1} - s_j)e^{-s_j}\right] - (a-bp)(1+\rho e^{-cs_j}) \quad \ldots (5.24)$$

The total cost's Hessian matrix must be positive definite for a fixed n in order for the total cost to be least (i.e. $\nabla^2 TC$).

$$\nabla T_R(t_j, s_j, n_1) = \begin{bmatrix}
\frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_1^2} & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_1\, \partial s_1} & 0 & 0 & \ldots & \ldots & 0 & 0 & 0 \\
\frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial s_2 \partial t_1} & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial s_1^2} & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial s_1 \partial t_2} & 0 & \ldots & \ldots & 0 & 0 & 0 \\
0 & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_2\, \partial s_1} & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_2^2} & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_2\, \partial s_2} & \ldots & \ldots & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & \ldots & \ldots & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_{n_1-1}\, \partial s_{n_1-1}} & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial s_{n_1-1}^2} & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial s_{n_1-1} \partial t_{n_1}} \\
0 & 0 & 0 & 0 & \ldots & \ldots & 0 & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_{n_1}\, \partial s_{n_1-1}} & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_{n_1}^2}
\end{bmatrix}$$

5.5. LEMMA

1. Convexity and Global Optimality

If f(x) is convex on $\mathbb{R}^n$, any local minimum x^* is automatically a global minimum.

Proof: By definition of convexity, for any two $x_1, x_2 \in \mathbb{R}^n$, we have

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2)$$

for all λ∈ [0,1]. Suppose that x^* is a local minimum, let $x \in \mathbb{R}^n$. Then, for any λ∈ [0,1], since f is convex,

$$f(x^* + \lambda)(x - x^*) \leq \lambda f(x) + (1-\lambda)f(x^*)$$

As it is bounded at a minimum of x^*since $f(x^*) \leq f(x)$, so it is global, hence $x^*$is also global.

2: Lagrange Multiplier in Constrained Optimization:

Let f(x) be the objective function and g(x)=0 be the constraint such that both are differentiable. Consider x^*as a local extremum of constrained optimization problem

$min\, f(x) \qquad subject\ to \qquad g(x) = 0,$

Then, of course, there is also a scalar $\lambda \in \mathbb{R}$ called the Lagrange multiplier-such that:

$$\nabla f(x^*) = \lambda \nabla g(x^*).$$

Proof: Consider the Lagrangian function

$$\mathcal{L}(x, \lambda) = f(x) + \lambda g(x).$$

At the optimum point x^*, the gradient of $\mathcal{L}$ with respect to both x and λ must be zero:

$$\nabla x\mathcal{L}(x^*, \lambda) = \nabla f(x^*) + \lambda \nabla g(x^*) = 0.$$

Thus, $\nabla f(x^*) = -\lambda \nabla g(x^*)$, that is, gradients of the objective function and the constraint are parallel.

5.6. NUMERICAL ILLUSTRATION

A numerical example is presented with particular values for a=0.4, $b = 0.03, c = 0.5, \alpha = 0.4, \rho = 1.05, e = 2.7$, $W_h = 1.1, H = 0.1, p = 0.01, S = 4, s_1 = 0$ in appropriate units. To obtain the solutions of Eq. (5.23) and Eq. (5.24), we utilized Mathematica, version 12, that is the computational tool implemented in order to compute the results and analysis for inventory model with backordering. While Table 5.1 discusses the optimal solution for tj (replenishment time) and shown in Figure 5.3, Table 5.2 discusses the optimal solution for replenishment time s_j and graphical representation is given in Figure 5.4.

5.7. RESULTING TABLES AND FIGURES

Table 5. 1 The optimal solution for t_j (replenishment time) for exponential partial back order

↓*a*	→ t_j	t_0	t_1	t_2	t_3	t_4
0.1		0.01	1.8575			
0.2		0.01	2.9116	4		
0.4		0.01	2.9116	3.43628	3.93212	
0.6		0.01	2.9116	3.29334	3.60108	3.39965

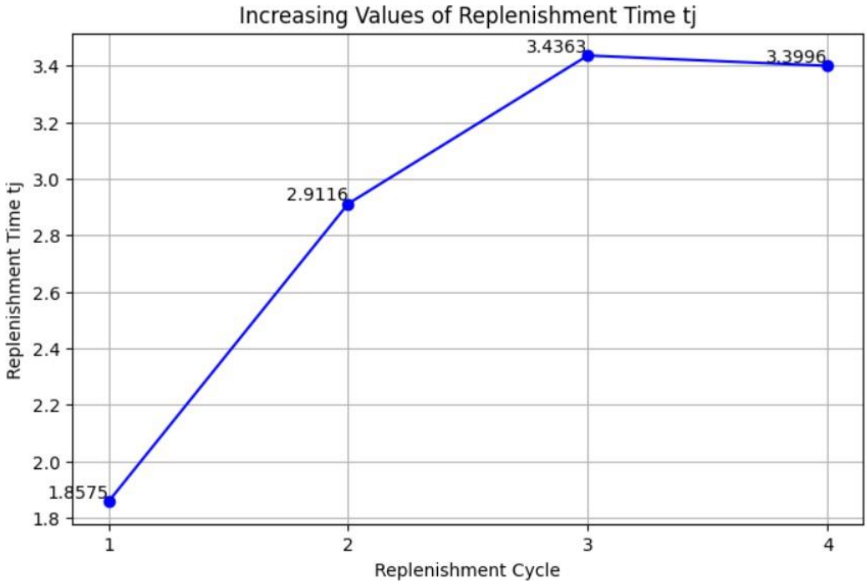

Fig. 5. 3 Stepping up values of replenishment time t_j

Table 5. 2 The optimal solution for s_j (time of shortage) for exponential partial back order

↓ a / → s_j	s_0	s_1	s_2	s_3	s_4
0.1	0	3.99976			
0.2	0	0.852092	4		
0.4	0	0.852092	0.963673	4	
0.6	0	0.852092	0.963673	1.18195	4

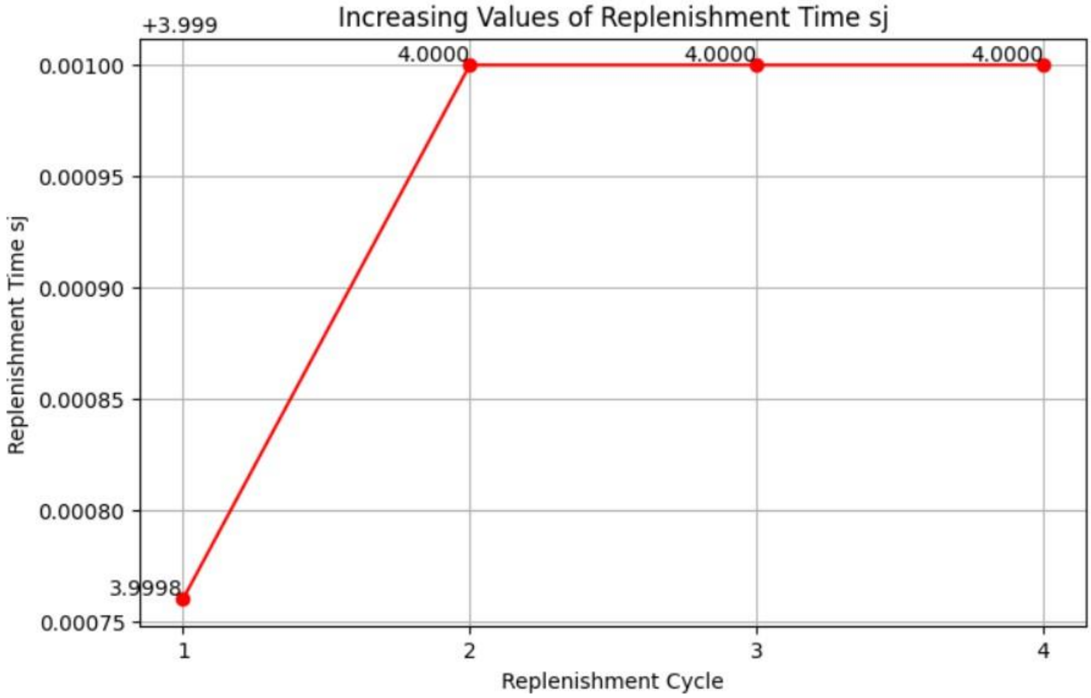

Fig. 5. 4 Stepping up value of replenishment time s_j

Table 5. 3 Total cost of retailer and quantity for optimal value

$\downarrow$ a	T_R	Q_{nt}
0.1	8.50505	4.92936
0.2	13.8734	10.0671
0.4	15.8044	12.0721
0.5	15.3763	12.0705
0.6	18.9012	15.344

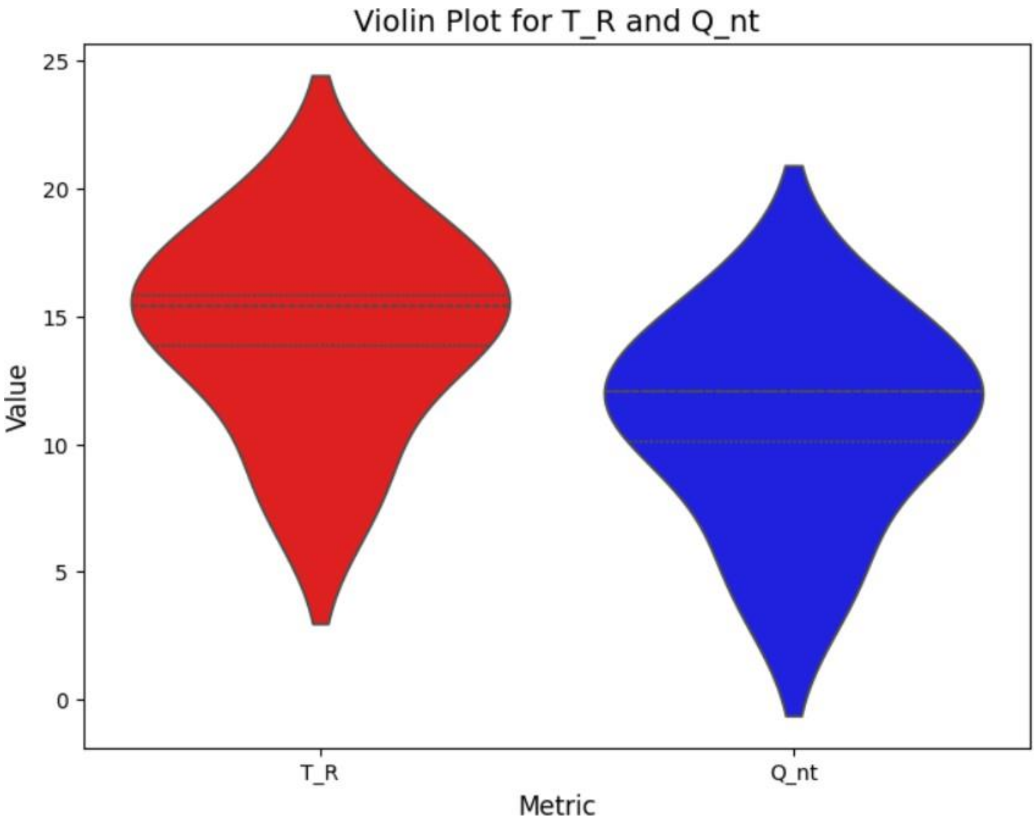

Fig. 5. 5 Violin Plot for total cost of retailer and quantity for optimal values

Table 5.3 discusses the values for total cost for retailer and quantity for optimal solution and the graphical representation is given in Figure 5.5 which displays a violin plot comparing the distributions of two metrics: Two groups and their representatives TR (in red) and Q_{nt} (in blue). The width of each violin is proportional to the data density, greater width meaning high data density or values concentrations. The horizontal line running through the interior of each violin corresponds to the median.

The plot also indicates that T_R has a wider range and a higher median than Q_{nt}, indicating higher variance and in essence, higher T_R values in general. Thus, this type of visualization is helpful in estimating the statistical properties and comparing the given measurements, which enhances the results of the study. Table 5.4 discusses the total cost for retailers in different replenishment 3D Plot is drawn using PYTHON programming which is shown in Figure 5.6.

Table 5. 4 Total cost for retailers in various replenishment cycle for Partial back order

↓ *a* →n	1	2	3	4	5
0.1	9.50507	**8.50505**	12.374	15.3953	18.4659
0.2	14.2953	**13.8734**	21.7559	28.2505	35.1411
0.4	18.9549	**15.8044**	26.054	32.4191	38.7843
0.5	18.4011	**15.3763**	26.0631	30.8818	37.6099
0.6	22.9665	**18.9012**	32.3932	39.4168	46.3726

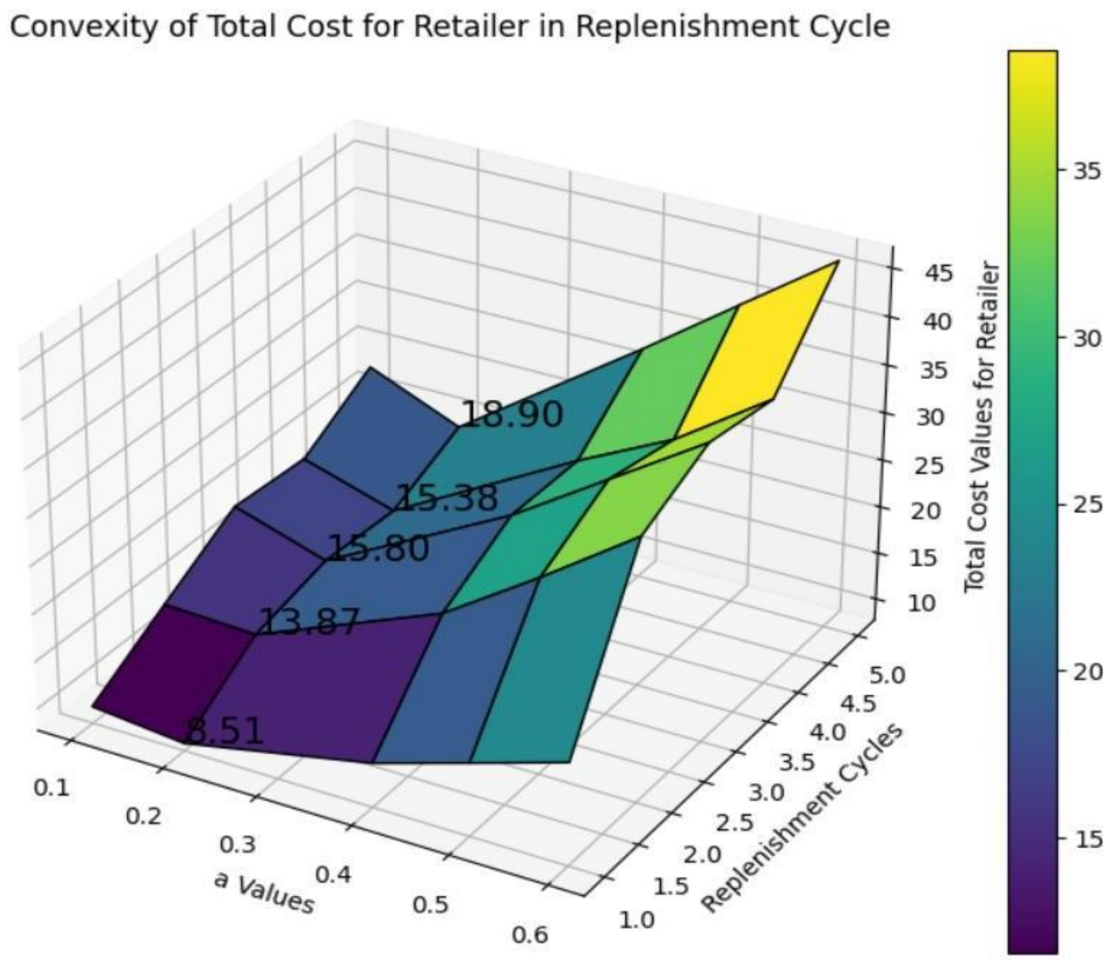

Fig. 5. 6 3D plot of Convexity of total cost for retailer in replenishment cycle

5.8. SENSITIVITY ANALYSIS

The sensitivity of all parameters is defined in Table 5.5. Based on the findings, the sensitivity analysis tests how the various parameters affecting the decision change the optimal replenishment cycle, the total order quantity (Q_{nt}), as well as the total cost of the retailer (T_R) within a fixed planning horizon when key parameters are altered as follows; +20%, +10%, 0%, -10% and -20% respectively is shown in Table 5.5. The demand has been assumed to be price sensitive according to the equation D(t)=a−bp which shows that variations in the demand rate (a) have far reaching consequences for the total cost even if the replenishment cycle has relatively small variations. The ordering cost (b) is virtually immune to change, producing consistent cycle and cost

Table 5. 5 Sensitivity Analysis of all parameters for exponential partial back order numbers.

Parameters	%Changes	Optimal Replenishment cycle	Total order Quantity Q_{nt}	Total cost of Retailer T_R
a	+20	2	12.0705	15.3763
	+10	2	15.344	18.9012
	0	2	12.0721	15.8044
	−10	2	4.92936	8.50505
	{−20	2	10.0671	13.8734
b	+20	2	12.0726	15.8052
	+10	2	12.0695	15.8018
	0	2	12.0721	15.8044
	−10	2	12.0767	15.809
	{−20	2	12.0636	15.8057
c	+20	1	13.4819	23.9274
	+10	1	13.6142	23.2977
	0	4	8.67743	16.3666
	−10	4	8.43607	16.0092
	{−20	4	8.24861	15.8551
p	+20	2	12.0726	15.8052
	+10	2	12.0695	15.8018
	0	2	12.0721	15.8044
	−10	2	12.0767	15.809
	{−20	2	12.0736	15.8057

α	+20 +10 0 −10 {−20	1 1 1 2 2	20.2052 20.5102 44.1533 20.4769 19.9426	29.5062 29.8362 63.9116 31.0092 27.2726
W_h	+20 +10 0 −10 {−20	2 2 2 2 1	7.89839 9.35408 12.0721 17.5657 16.8792	13.5127 17.6195 15.8044 7.98009 15.9912
H	+20 +10 0 −10 {−20	2 2 2 2 2	17.0713 14.1563 12.0721 10.4705 9.14794	19.486 17.6015 15.8044 14.3057 13.7399
Rho	+20 +10 0 −10 {−20	2 2 2 1 1	15.9766 11.0513 12.0721 10.1838 10.0998	18.3225 18.2849 15.8044 14.6801 13.2722
S	+20 +10 0 −10 {−20	2 2 1 1 1	8.39934 8.1668 9.0356 9.58188 9.43898	9.62728 9.30325 8.28784 7.0869 6.66815

The parameter c which stands for one of the costs of backordering affects the cycle for the replenishment and order quantity; higher value of c giving larger cycles indicate the link between backordering cost and inventory cost. On the other hand, α which is inventory dependent results in total cost and order cycle, the back ordering rates leading to shorter cycles but higher total cost. The warehouse holding cost (W_h) exhibits moderate degree of sensitivity, but an optimal order quantity decreases with an increase in maximum inventory level, and therefore yields shorter overall costs. Moreover, the deterioration rate also plays an important function with a negative impact because when ρ is greater, the replenishment cycle will decrease besides causing costs to go up. Finally, shortage cost or cost of stockouts (S) is used to estimate both the cycle and the total cost and notable shortages effects were found to have a direct and proportional impact on the total expense by the retailer. This analysis strengthens the argument that back ordering costs and other inventory-sensitive factors should be managed efficiently to reduce the overall ultimate costs for replenishment strategies.

5.9. MANAGERIAL RECOMMENDATIONS

Based on the sensitivity analysis, the following strategic recommendations would be drawn with respect to managing backorders: the inventories need to be optimized since the inventory dependent parameters ‘α’ have a huge effect on the replenishment cycle and, consequently, backorders, which are minimized through regular reviews and adjustments on the inventory. This also needs to be advanced in demand forecasting. Since demand is inversely proportional to price, organizations need to have advanced analytics or machine learning models for better predictions to minimize stockouts and align their inventory with their expected sales.

Furthermore, the careful observation of back ordering rates is crucial, given that the parameter ‘c’ significantly influences overall expenses. The adoption of real-time tracking mechanisms enables organizations to recognize patterns and modify their replenishment approaches in a timely manner. Additionally, participating in discussions with suppliers to enhance lead times and foster robust partnerships may further mitigate the effects of backorders on total costs.

Exploration of flexible pricing strategies may also encourage customer buying at a time of greater availability. The development of a responsive supply chain using diversified suppliers and in just-in-time practices will improve responsiveness and flexibility to changes in demand. Effective communication with customers on potential backorders is the critical factor in maintaining satisfaction even during shortages. Proactive management of customer expectations can create loyalty and strengthen relationships.

In this regard, effective and balanced scheduling is thus needed in making decisions about the inventory management. This balance of holding costs and shortage costs permits organizations to develop strategies that can enhance the level of inventories without raising costs overall. Thus, at the end, these can help managers face problems about back orders and eventually with inventory management at large with better operational efficiencies and customer satisfaction.

5.10. CONCLUSION

This study aims to provide a detailed analysis of the economic production quantity (EPQ) model where a partial backordering policy follows an exponential distribution with a finite replenishment cycle and price-sensitive demand. Other parameters such as the back ordering cost, represented here by the inventory-dependent parameter c and factor α, were examined in sensitivity analysis to determine effects on the replenishment cycle and total costs. The advantages of back ordering are highlighted, especially as they are related to positioning stocks and overcoming deficit problems.

To support the mathematical outcomes, software Mathematica-12 is used to check the mathematical answers hence confirming accuracy of complicated solutions. Also, one of the most effective tools for machine Learning and data analysis: Python is used for graphical representation and data visualization to make the results more interpretable. Some managerial implications were offered to assist managers to select the right mixture that reduces costs when managing back orders.

This work emphasizes the necessity of combining modern computational approaches and provides indicated proposals to managers on the cost, regeneration, and shortage approaches using backordering policies.

Chapter 6

Optimizing Inventory Management in a Fuzzy Environment: A Machine Learning-Based Approach with Seasonal Demand Forecasting

Optimizing Inventory Management in a Fuzzy Environment: A Machine Learning-Based Approach with Seasonal Demand Forecasting

6.1. INTRODUCTION

In the ever-evolving global market landscape, the intricate dance between seasonal and weather conditions exerts a profound influence on consumer demand, a cornerstone variable that presents multifaceted challenges to efficient inventory management across diverse industries (Joacim Rocklöv et al. 2020). The ebb and flow of seasonal demand, shaped by events such as festivals and climatic factors, introduces uncertainties and complexities into consumer purchasing behaviours, necessitating a sophisticated approach to inventory control (Jens Roehrich and Roehrich 2008). While conventional inventory models often hinge on deterministic demand assumptions, the real-world scenario unfolds with variations in product demand that adhere to distinct seasonal patterns. So, in our study, we have applied the time series algorithm to forecast seasonal demand.

The strategic imperative of effective demand prediction emerges as a key solution, offering the potential to refine inventory management strategies, curtail superfluous costs, and elevate overall customer service (Ron Kohavi et al. 2002). Leveraging Machine Learning (ML), with its advanced predictive capabilities, particularly through Decision Tree-based Algorithms, stands out as a transformative tool in achieving precise and accurate seasonal demand forecasts (Divyakant Agrawal et al. 2011). This paper delves into the convergence of seasonal demand dynamics, imperfect deteriorating products, and the contemporary imperative of considering carbon emissions in inventory systems. The intrinsic deterioration of physical products over time, be it during transit or storage, is a ubiquitous challenge across various industries (B. Hillier and O. Sahbaz 2005). Items such as fruits, medicines, flowers, foodstuffs, and vegetables are susceptible to decay during their holding and in-transit periods. This study acknowledges the deterministic approach traditionally applied to deterioration rates in inventory models but contends that real-world uncertainty demands a more sophisticated treatment (Silver et al. 1998). To address this, the model introduces a fuzzy variable for deterioration rates, acknowledging the uncertainty in their precise estimation.

Furthermore, the quantity of defective products, a critical consideration in inventory management, is recognized as another fuzzy variable due to unpredictable factors such as manufacturing defects, man-handling issues, and in-transit damage (Taleizadeh et al. 2013b). The study emphasizes the pressing concern of escalating carbon emissions in the modern era, driven by industrialization and contributing significantly to climate change. This prompts a paradigm shift in inventory system design, where scholars and organizations now focus on reducing the total cost, integrating considerations for carbon emissions. This research not only acknowledges permissible shortages but also accounts for partial backlogging, recognizing that not every consumer accepts delayed deliveries. By addressing these multifaceted challenges, the study endeavours to bridge gaps in existing literature concerning the impact of demand Predictions on inadequate decaying products. Two primary research questions guide this exploration: (a) How do AI based demand prediction techniques improve the certainty and predictability of seasonal predictions for demand for deteriorating products? (b) What are the benefits of using artificial intelligence-driven monthly projected demand versus constant demand in inventory management?

To resolve such issues, this article presents an AI-driven fuzzy inventory model that incorporates defective, deteriorating products and carbon emissions. The decision tree classifier, a powerful ML technique, is employed for demand forecasting, aiming to determine accurate seasonal demand (Lior Rokach et al. 2005). The goal is to optimize purchasing amount and replenishment periods, thereby minimizing the total average cost, while considering carbon emissions. The ensuing sections delve into a comprehensive review of the literature, outline notations and assumptions, articulate the mathematical model, detail the ML-based methodology, present validation through a numerical example, conduct sensitivity analysis, and culminate in conclusions and avenues for future research. In navigating this exploration, we aim to contribute insights that advance both the theoretical and practical dimensions of inventory management in the context of seasonal demand, imperfect deteriorating products, and the imperative of environmental sustainability.

6.2. LITERATURE REVIEW

Demand forecasting stands as a pivotal element in shaping business strategies, offering organizations the means to optimize operations, cut costs, and meet consumer expectations(Jens Roehrich and Roehrich 2008). (N. K. Mishra et al. 2024) puts forward a decentralized supply chain optimization model that incorporates blockchain and uses an iterative strategy to calculate overall costs for retailers and suppliers. By establishing its uniqueness and optimal results through theoretical analysis, the model efficiently determines optimal replenishment cycles using Wolfram Mathematica 13.0., guiding decision-makers with managerial insights for enhanced supply chain management efficiency and resilience. The continuous expansion of machine learning (ML) techniques provides a fertile ground for researchers aiming to enhance the accuracy of demand forecasting (Ron Kohavi et al. 2002) Early research by (Michael A. Persinger et al. 1983) delved into the relationship between weather conditions and individuals' moods, establishing that weather significantly influences consumer behaviour. (Scott A. Wright et al. 2018) emphasized the critical role of demand forecasting models in predicting overstocking and understocking situations, particularly during fluctuations in consumer demand. Notable festive seasons, such as Christmas and Diwali, witness a surge in demand for e-commerce giants like Amazon and Flipkart as consumers actively seek gifts and festive supplies. Accurate demand forecasting becomes imperative during such peak periods, emphasizing the need for dynamic models that surpass fixed demand predictions (Réal A. Carbonneau et al. 2008). Decision trees, recognized as powerful data mining techniques, have been extensively employed in various sectors for demand forecasting (T. Hastie et al. 2001). Researchers have explored the application of decision trees in joint analyses of continuous and discrete data, emphasizing their versatility (Kirshners et al., 2010). In the field of defective damaging goods, (P. M. Ghare and Ghare 1963) first developed research on damaging items in inventory systems, with a focus on con stant deterioration ratios. (Wee et al. 2014) conducted subsequent research and built a production inventory model particularly for damaging seasonal goods. The consideration of partial payment and trade credit policies in a non-instantaneous disintegrating concept of inventory was introduced by (Mohsen Lashgari et al. 2018). The reality of imperfect products, either due to manufacturing errors or deterioration, led to economic production quantity (EPQ) models for imperfect quality items (Salameh and Jaber 2000) Further investigations delved into imperfect quality inventory systems, integrating stochastic processes to model defective products (S. Papachristos et al. 2006).

Carbon emissions have become a significant concern in recent years, prompting researchers to incorporate environmental considerations into inventory models. (Nitin Kumar Mishra et al. 2022) provides a supply chain inventory model for degrading items that takes into account carbon emission-dependent demand, advanced payment methods, and the impact of carbon taxes and limits on a constrained planning horizon. It seeks to balance economic and environmental considerations, providing insights for businesses and potential relevance for government policies. (Guowei Hua et al. 2011) built an economic order quantity (EOQ) model which incorporates carbon costs associated with storage and shipment. Carbon taxes and emissions reduction policies have been explored to understand their impact on inventory costs (Saif Benjaafar et al. 2013). Green inventory models were studied under settings of carbon emission penalty fees, the cap-and-trade scheme systems, and severe emission limit laws (Md. Bokhtiar Hasan et al. 2021).

Fuzzy methods have gained attention in inventory management problems due to their ability to generate more relevant solutions. Early applications of fuzzy set theory in inventory models include analyses of economic order quantity (Park 1987; Chang et al. 1998). Extensions into fuzzy overall cost functions, considering fuzzy demand rates and proportions of defective products, have been explored (Thomas Philip Rúnarsson et al. 2000; Fwu-Ranq Chang and Chang 2004). Fuzzy inventory estimation methods for degrading items with time-dependent demand and backlog rates considered as fuzzy numbers have also been developed (Sushil Kumar et al. 2015). The integration of fuzzy concepts into inventory models, considering imperfect products, payment delays, variable demand and partial backlog, has been investigated (Chandra K. Jaggi et al. 2016) Shaikh et al., 2018). (Shivam Mishra et al. 2019) present a fuzzified supply chain finite planning horizon model, addressing deteriorating materials. The model uses fuzzy parameters, such as deterioration cost, and applies defuzzification methods with finite planning horizon. ("Lean Labour in AEC Industry: From Theory to

Implementation" 2020) provided research on the fuzziness of supplier-retailer supply coordination. The research explores credit terms in the context of managing items are degrading due to time-quadratic demand and partial backlog throughout all cycles across the finite planning horizon. (Ranu Singh and Vinod Kumar Mishra 2023) provides an inventory model that uses artificial intelligence to estimate demand, with a focus on imperfect deteriorating items and partial backlog concerns. The model also incorporates the impact of carbon emissions.

Table 6. 1 Literature Survey and Research Gap for fuzzy back order

Authors	Focus Area	Contribution
(Michael A. Persinger et al. 1983)	Weather's Impact on Moods	Demonstrated weather's influence on consumer moods.
(Scott A. Wright et al. 2018)	Demand Forecasting Models	Emphasized the role of demand forecasting in predicting overstocking and understocking.
(Réal A. Carbonneau et al. 2008)	Festive Demand	Simple time series algorithms were classified as conventional, and several ML approaches were examined.
(Ben Kirshner et al. 2010)	Decision Trees	Inductive decision trees were used to analyze both continuous and discrete data simultaneously.
(Wee et al. 2014)	Deteriorating Seasonal Items	Created a production supply model for decaying seasonal products.
(Salameh and Jaber 2000)	EPQ for Imperfect Quality Items	Proposed an EPQ model for imperfect quality items.
(Guowei Hua et al. 2011)	EOQ with Carbon Costs	Established an EOQ model considering carbon costs in transportation and stockkeeping.
(Md. Bokhtiar Hasan et al. 2021)	Green Inventory Model	Examined a green model forinventory under carbon emission penalties, cap-and-trade, and regulatory constraints.
Current Study	Deteriorating Inventory with Partial Backlogging	Created a decaying inventory estimation model with quadratic demand and partly backlog.

This literature review outlines the evolution of research in demand forecasting, imperfect deteriorating products, and environmental considerations. Literature surveys and research gaps are discussed in Table 6.1, While existing studies have addressed individual facets, there is a notable gap in integrating artificial intelligence for demand forecasts, considering partial backlogging, imperfect products and carbon emissions.

The current study endeavours to contribute to this intersection by providing a comprehensive model that extends previous frameworks and incorporates machine learning concepts in a fuzzy environment over the finite planning horizon.

6.3. SCHOLASTIC ACHIEVEMENT

The main innovation of this study is in crafting a distinctive inventory system supported by machine learning and fuzzy logic. This system is custom designed to confront the difficulties arising from defective degrading goods amid carbon emissions within a limited planning timeframe. Unlike traditional inventory models, this study addresses uncertainties surrounding defective percentages and deterioration rates by treating them as fuzzy variables. Through the application of a machine learning technique, specifically the time series prediction or (production), the model aims to enhance the precision of variable demand forecasts for degrading items over a limited planning horizon.

The importance of this research work is evident in its potential to revolutionize demand forecasting in businesses. By moving beyond fixed demand assumptions and incorporating machine learning-based monthly predicted demand, organizations can achieve more precise and reliable inventory management. The numerical experiment conducted in the study demonstrates a substantial reduction in overall costs when utilizing seasonal forecasted demand. Moreover, the model's incorporation of carbon emissions costs underscores a commitment to sustainability and environmental responsibility.

The research's overarching contribution is its holistic approach, fuzzy variables, integrating machine learning techniques and considerations for carbon emissions. This comprehensive framework not only improves demand forecasting accuracy but also provides businesses with a strategic means to minimize their ecological footprint. Ultimately, the research aims to identify optimal policies that minimize overall costs while simultaneously addressing the complexities associated with deteriorating products, thereby offering a valuable contribution to the field of inventory management over the finite planning horizon.

6.4. ASSUMPTIONS AND NOTATIONS

6.4.1. ASSUMPTIONS

1. Only one sort of deteriorating goods is evaluated, with an unlimited replacement rate.
2. The timing for replenishment orders will be limited.
3. The production pattern is based on expected demand.
4. β is back ordering cost.
5. The lead time will not be zero but rather nearly negligible.
6. Defective products result from imperfect manufacturing and worker handling, with the defective percentage (k) considered as an interval trapezoidal fuzzy number.
7. The decomposition rate is considered a trapezoidal fuzzy number.
8. Carbon emissions due to shipping, godown/storage, and decomposition are considered.
9. Shortages are permitted and partially backlogged.
10. The fixed transportation cost is incurred when the retailer initiates an order.

6.4.2. NOTATIONS

1. K: Defective percentage
2. D: Demand rate Unit/year
3. θ Degradation rate
4. θ_j: Fuzzy deterioration rate j=1,2,3,4
5. $\hat{h}_c$: The amount of CO_2 associated with each unit inventory holding cost (refrigeration cost). \$/unit/year
6. P_r: The cost of each acquired unit.
7. $\hat{P}_r$: The amount of CO2 emitted following the purchase costs per unit. c: Steady carbon emission cost per dollar. \$/unit
8. $Q_{j+1}=R_{j+1}+S_{j+1}$ is the ordered quantity at time t_j for j^{th} cycles where j=1,2,3… ....n.
9. τ: Carbon emissions cost per unit of carbon emitted.
10. C_e: The carrying amount of CO2 released during a replenishing process.

11. S_h: For each cycle, the shortage cost per unit time. \$/unit

12. $D_{t:}$ is the number of items that deteriorated in the j^{th} replenishment cycle. \$/unit

13. B Backlogging rate

14. $L_{j+1:}$ Total level of the stock that lost sale throughout the duration [sj, tj].

15. l: For each cycle, per unit time's lost sale cost \$/unit

16. D_{j+1}: The total number of items that deteriorated throughout the duration [tj, sj+1].

6.4.3. TRANSPORTATION VARIABLES

1. v_c- The cost of fuel consumed by the merchant during shipment is determined by the usage of fuel (\$/litre)
2. C_1-Fuel consumption of a retailer's vehicle when empty. (litres per kilometre)
3. C_2- Extra energy consumption for each ton of payload due to refrigeration and vehicle services during transportation. (litre/km/ton)
4. e_2 - Additional (refrigeration) carbon emission cost incurred by the retailer per unit item during transportation. (\$/unit/km)
5. e_1 - Fee for carbon dioxide emissions related to retail transportation. (\$/km)
6. d - The distance traveled from the supplier to the retailer. (km)
7. F_c - Fixed transportation costs are incurred when the retailer places an order. (\$)

6.5. METHODOLOGY FOR DEMAND FORECAST

The suggested approach in this work focuses on accurate demand forecasting using artificial intelligence (machine learning). While many academics believe that demand is predictable, the presence of unpredictable swings implies that an approach based on machine learning is more suited for demand prediction. For this study, time series method is chosen due to its simplicity and effectiveness in ML techniques. The primary objective is to ascertain the precise seasonal demand for deteriorating products. The flowchart illustrating the methodology for demand forecasting is presented above.

```python
import pandas as pd
import matplotlib.pyplot as plt
from statsmodels.tsa.arima.model import ARIMA

# Load the data
data_location = r'C:\Users\Ranu\OneDrive\Desktop\Daily_Demand_Forecasting_Orders.xlsx'
data = pd.read_excel(data_location)

# Display basic information about the dataset
print("Dataset Info:")
print(data.info())

# Display descriptive statistics
print("\nDescriptive Statistics:")
print(data.describe())

# Check column names
print("\nColumn Names:")
print(data.columns)
```

Fig. 6. 1 Python programming of Demand forecasting

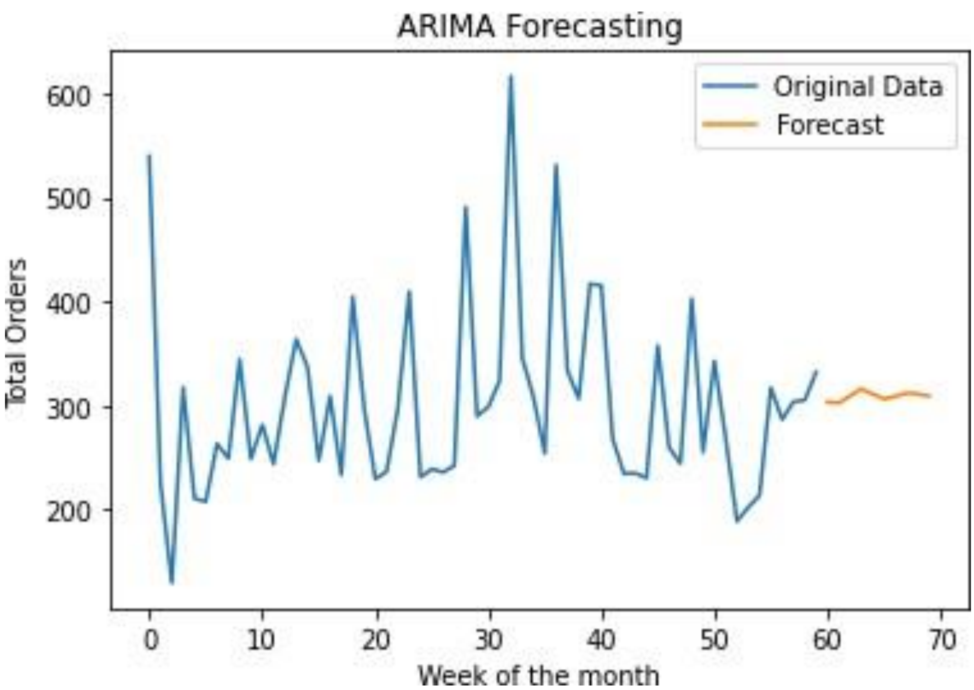

Fig. 6. 2 Demand forecasting

PYTHON code (version 3.10.0) is used to validate the forecasted inventory system by predicting seasonal demand for deteriorating products. Prior to running the PYTHON code, necessary packages such as Pandas for data manipulation and consuming time series data from the sklearn library are installed. Ultimately, by inputting the parameter (month), the seasonally demand for deteriorating products is obtained. As we can see the Figure 6.1 and Figure 6.2 respectively.

6.6. MATHEMATICAL MODEL

This article explores a sustainable supply model designed for imperfect diminishing products within a retail setting. At the on set, the retailer acquires a quantity (Q_{j+1}) of products. The stock level experiences a reduction due to both demand and degradation throughout the period. A partial backlogging shortage occurs at a rate of β and persists until time T. By the conclusion of the cycle, the supply has exceeded its maximum deficit level. To rectify this backlog, the retailer initiates the replenishment of products. The dynamics of the stock level during the time intervals [0, t_0], [t_0, t_1] are governed by the following set of differential equations:

Inventory level with boundary condition $IL_{i+1}\,(s_{i+1}) = 0$

$$\frac{dI_{j+1}(t)}{dt} + (\theta)I_{j+1}(t) = -(1-\mathrm{k})\mathrm{D} \qquad t_j < \mathrm{t} < s_{j+1} \qquad \ldots (6.1)$$

Where {1,2,3 .... n_1}

$$\frac{dI_{j+1}(t)}{dt} = -(1-\mathrm{k})\mathrm{D} - (\theta)\, I_{j+1}(t) \qquad t_j < \mathrm{t} < s_{j+1} \qquad \ldots (6.2)$$

$$I_{j+1}(t) = \int_t^{s_{j+1}} -(1-\mathrm{k})\mathrm{D}\ \ e^{\theta(u-t)}\,\mathrm{du} \qquad \ldots (6.3)$$

$$I_{j+1}(t) = \theta(1-\mathrm{k})\mathrm{D}[1-e^{\theta(s_{j+1}-t)}] \qquad \ldots (6.4)$$

The current shortage level, denoted as $I_{j+1}(t)$ under the boundary condition $I_{j+1}(s_j) = 0$, is defined by the subsequent differential equation:

$$\frac{dI_{j+1}(t)}{dt} = D\ B(t) \qquad where \qquad s_j < \mathrm{t} < t_j \qquad \dots (6.5)$$

$$I_b(t) = \int_{s_j}^{t} D\,\beta\ dt = D\,\beta(s_j - t) \qquad \dots (6.6)$$

Therefore, the overall inventory quantity maintained throughout the interval $[t_j, s_{j+1}]$

$$Q_{j+1} = \int_{t_j}^{s_{j+1}} \{\int_{t}^{s_{j+1}} -(1-\mathrm{k})\mathrm{D}\ e^{\theta\,(u-t)}\,\mathrm{du}\}\,\mathrm{dt} \qquad \dots (6.7)$$

Equation (6.7) can be reformulated as follows by adjusting the integration position and omitting the higher-order terms of α^2.

$$Q_{j+1} = \frac{-(1-\mathrm{k})}{\theta} \int_{t_j}^{s_{j+1}} \{e^{\theta\,(s_{j+1}-t)} - 1\}\ D\ dt \qquad \dots (6.8)$$

Customers are waiting for the complete amount of that quantity i.e., the quantity of deficit throughout the timeframe $[s_j, t_j]$

After rearranging the ordering, S_{j+1} can be given as

$$S_{j+1} = \int_{s_j}^{t_j} I_b(t)\ dt = \int_{s_j}^{t_j} \{\int_{s_j}^{t} D\,\beta(s_j - u)du\}\ dt$$

$$= \frac{-1}{2}\int_{s_j}^{t_j} (D\ \beta(s_j - t))^2 dt$$

$$\dots (6.9)$$

$The\ total\ order\ quantity\ for$ a finite planning horizon

$$\mathrm{Q} = \sum_{j=1}^{n} Q_{j+1} = \sum_{j=1}^{n} \{ Q_{j+1} + S_{j+1} \} \qquad \dots (6.10)$$

$$Q_{j+1} = \frac{-(1-\mathrm{k})}{\theta} \int_{t_j}^{s_{j+1}} \{e^{\theta\,(sj+1-t)} - 1\}\ D\ dt\ +\ \frac{1}{3}(D\ \beta(s_j - t_j))^3 \qquad \dots (6.11)$$

The overall number of degraded components at each refill is as follows,

$$D_{j+1} = \int_{s_j}^{t_j} \theta\, I_{j+1}\ (t)\ dt\ = \{\theta^2 \int_{t_j}^{s_j} (1-\mathrm{k})\mathrm{D}[1 - e^{\theta(sj+1-t)}]\ dt\} \quad \dots (6.12)$$

In the case of Fast-Moving Consumer Goods (FMCG) or basic necessities (Nitin Kumar Mishra and Sakshi Sharma, 2023), consumers cannot delay their purchases, resulting in only a portion, β, of the demand being held back during stockouts. Consequently, the remaining fraction (1-β) is lost. (Sarkar et al. 2012) and (Pushpinder Singh et al. 2017)

The quantity that was lost throughout the interval $[s_j, t_j]$ is given as:

$$L_{j+1} = \int_{s_j}^{t_j} \{D - D\,\beta\}\,dt = \int_{s_j}^{t_j} \{(1-\beta)\ D\}\,dt = (1-\beta)\ D\ (t_j - s_j) \qquad \dots (6.13)$$

Cost of carbon emissions during the period $[t_j\ , s_{j+1}]$ can be expressed as:

$$Ce = \sum_{i=0}^{n1-1} \hat{c}\ +\ P_r\ *\ Q_{j+1}\ +\ \hbar_c \int_{t_j}^{s_{j+1}} I_{j+1}\ (t)\ dt \qquad \dots (6.14)$$

$$Ce = \sum_{j=0}^{n_1-1} \hat{c}\ +\ P_r\ *\ Q_{j+1}$$

$$+\ \hbar_c \int_{t_j}^{s_{j+1}} \{ \int_{t}^{s_{j+1}} -(1-\mathrm{k})\mathrm{D}\ \ e^{\theta(u-t)}\ \mathrm{du}\}\,dt \qquad \dots (6.15)$$

According to (Mishra and Ranu 2022), the overall cost of carbon emissions throughout the interval $[t_i, s_{i+1}]$ can be expressed as:

$$Ce = \tau\{\sum_{i=0}^{n_1-1} \hat{c} + P_r \{\theta \int_{t_j}^{s_j} (1-\mathrm{k})\mathrm{D}[1 - e^{\theta(s_{j+1}-t)}]\, dt\} - (1-\mathrm{k})\mathrm{D} * \hat{h}_c * \theta \int_{t_j}^{s_{j+1}} \{e^{\theta(s_{j+1}-t)} - 1)\mathrm{dt}\}\} \quad \ldots (6.16)$$

The transportation expenses for the retailer take into account both variable transportation costs and fixed costs, as well as carbon emissions resulting from FEC during refrigeration. Equation (6.17) consequently outlines the transportation cost as follows. (Namwad R.S., Mishra N.K., and Ranu., 2023) and (Mishra and Ranu 2023(a and b).

$$c = F_c + 2\mathrm{d}v_c C_1 + \mathrm{d} * v_c C_2 \int_{t_j}^{t_{j+1}} D\, e^{\theta(\mathrm{t}-\mathrm{t}_j)}\, \mathrm{d}t + 2\mathrm{d}\, e_1 + \mathrm{d}\, e_2 \int_{t_j}^{t_{j+1}} D\, e^{\theta(\mathrm{t}-\mathrm{t}_j)}\, \mathrm{d}t \quad \ldots (6.17)$$

$$\begin{aligned} Total\ cost\ =\ & Replenishment\ Cost\ +\ Inventory\ Holding\ Cost \\ & +\ Acquisition\ Cost\ +\ Depreciation\ Cost\ +\ Storage\ Expense \\ & +\ Lost\ Sales\ Cost\ +\ Carbon\ Emission\ Cost \\ & +\ Transportation\ Expense \end{aligned} \quad \ldots (6.18)$$

$$
\begin{aligned}
\mathrm{TC}(t_j, s_j, n_1) &= n_1 * O_r \\
&+ \sum_{j=0}^{n_1-1} H \int_{t_j}^{s_{j+1}} I_{j+1}(t)\, dt + \sum_{j=0}^{n_1-1} W_h * Q_{j+1} \\
&+ \sum_{j=0}^{n_1-1} D_t * \theta \int_{t_j}^{s_{j+1}} I_{j+1}(t)\ \ dt + \sum_{j=0}^{n_1-1} s \int_{s_j}^{t_j} I_b(t)\ \ dt \\
&+ \sum_{j=0}^{n_1-1} l \int_{s_j}^{t_j} (1-\beta)\, D\,(t_j - s_j)\ dt + \sum_{j=0}^{n_1-1} \hat{c} + \hat{P}_r\, Q_{j+1} \\
&+ \hbar_c \int_{t_j}^{s_{j+1}} I_{j+1}(t)\, dt + F_c + 2\mathrm{d}\, v_c C_1 + \mathrm{d} \\
&* v_c C_2 \int_{t_j}^{t_{j+1}} D\, e^{\theta(\mathrm{t}-\mathrm{t}_j)}\, \mathrm{d}t + 2\mathrm{d}\, e_1 \\
&+ \mathrm{d}\, e_2 \int_{t_j}^{t_{j+1}} D\ e^{\theta(\mathrm{t}-\mathrm{t}_j)}\, \mathrm{d}t
\end{aligned}
\qquad \ldots (6.19)
$$

$$
\begin{aligned}
\mathrm{TC}(t_j, s_j, n_1) &= n_1 * O_r \\
&+ \sum_{j=0}^{n_1-1} H \int_{t_j}^{s_{j+1}} I_{j+1}(t)\, dt + \sum_{j=0}^{n_1-1} W_h * Q_{j+1} \\
&+ \sum_{j=0}^{n_1-1} D_t * \theta \int_{t_j}^{s_{j+1}} I_{j+1}(t)\ dt + \sum_{i=0}^{n_1-1} s \int_{s_j}^{t_j} I_b(t)\quad dt \\
&+ \sum_{j=0}^{n_1-1} l \int_{s_j}^{t_j} (1-\beta)\, D\, (t_j - s_j)\ dt + \sum_{j=0}^{n_1-1} \tau \hat{c} + \tau\, \hat{P}_r\ Q_{j+1} \\
&+ \tau \hat{h}_c \int_{t_i}^{s_{i+1}} I_{j+1}(t)\, dt + F_c + 2\mathrm{d}\, v_c C_1 + \mathrm{d} \\
&* v_c C_2 \int_{t_j}^{t_{j+1}} D\, e^{\theta(t-t_j)}\, \mathrm{d}t + 2\mathrm{d}\, e_1 \\
&+ \mathrm{d}\, e_2 \int_{t_j}^{t_{j+1}} D\ e^{\theta(t-t_j)}\, \mathrm{d}t
\end{aligned}
\quad \text{... (6.20)}
$$

$$
\begin{aligned}
\mathrm{TC}(t_j, s_j, n_1) = n_1 * O_r &+ \sum_{j=0}^{n_1-1} \{H + \tau \hat{h}_c + D_t * \theta\} \int_{t_j}^{s_{j+1}} I_{j+1}(t)\, dt \\
&+ \{W_h + \tau \hat{P}_r\} \sum_{j=0}^{n_1-1} Q_{j+1} + \sum_{j=0}^{n_1-1} s \int_{s_j}^{t_j} D\, \beta(s_j - t)\quad dt \\
&+ \sum_{j=0}^{n_1-1} l \int_{s_j}^{t_j} (1-\beta)\, D\, (t_j - s_j)\ dt + \tau \hat{c} + F_c + 2\mathrm{d}\, v_c C_1 + \mathrm{d} \\
&* v_c C_2 \int_{t_j}^{t_{j+1}} D\, e^{\theta(t-t_j)}\, \mathrm{d}t + 2\mathrm{d}\, e_1 + \mathrm{d}\, e_2 \int_{t_j}^{t_{j+1}} D\ e^{\theta(t-t_j)}\, \mathrm{d}t
\end{aligned}
\quad \text{... (6.21)}
$$

$$\mathrm{TC}(t_j, s_j, n_1) = n_1 * O_r + \sum_{j=0}^{n_1-1} \{H + \tau \hat{h}_c + D_t * \theta\} \int_{t_j}^{s_{j+1}} I_{j+1}(t)\, dt$$
$$+ \{W_h + \tau \hat{P}_r\} \sum_{j=0}^{n_1-1} \left(\frac{-(1-\mathrm{k})}{\theta} \int_{t_j}^{s_{j+1}} \{e^{\theta(s_{j+1}-t)} - 1\}\, D\, dt \right.$$
$$+ \frac{1}{3}(D\,\beta(s_j - t_j))^3) + \sum_{j=0}^{n_1-1} s \int_{s_j}^{t_j} D\,\beta(s_j - t)\; dt$$
$$+ \sum_{j=0}^{n_1-1} l \int_{s_j}^{t_j} (1-\beta)\, D\,(t_j - s_j)\; dt + \tau \hat{\mathrm{c}} + F_c + 2\mathrm{d} * v_c C_1 + \mathrm{d}$$
$$* v_c C_2 \int_{t_j}^{t_{j+1}} D(t) e^{\theta(t-t_j)}\, \mathrm{d}t + 2\mathrm{d}\, e_1 + \mathrm{d}\, e_2 \int_{t_j}^{t_{j+1}} D(t) e^{\theta(t-t_j)}\, \mathrm{d}t \qquad \ldots (6.22)$$

$$\mathrm{TC}(t_j, s_j, n_1) = n_1 * O_r + \sum_{j=0}^{n_1-1} (1-\mathrm{k}) \{H + \tau \hat{h}_c + D_t * \theta\} \int_{t_j}^{s_{j+1}} [D\,\beta(s_j - t)]\, dt + \{W_h + \tau \hat{P}_r\} \sum_{j=0}^{n_1-1} \left(\frac{-(1-\mathrm{k})}{\theta} \int_{t_j}^{s_{j+1}} \{e^{\theta(s_{j+1}-t)} - 1\}\, D\, dt + \frac{1}{3}(D\,\beta(s_j - t_j))^3 \right) + \sum_{j=0}^{n_1-1} s \int_{s_j}^{t_j} D\,\beta(s_j - t)\; dt + \sum_{j=0}^{n_1-1} l \int_{s_j}^{t_j} (1-\beta)\, D\,(t_j - s_j)\; dt + \tau \hat{\mathrm{c}} + F_c + 2\mathrm{d}\, v_c C_1 + (\mathrm{d}\, e_2 + \mathrm{d} * v_c C_2) \int_{t_j}^{t_{j+1}} D e^{\theta(t-t_j)}\, \mathrm{d}t \qquad \ldots (6.23)$$

6.7. FUZZIFICATION OF MODEL

6.7.1 FUZZY MODEL

In inventory systems, pinpointing precise values for known parameters poses a challenge for decision-makers, introducing uncertainty into key parameters. Consequently, the defective percentage in quantity (k) and deterioration rate (θ) are treated as trapezoidal fuzzy interval types. Fuzzy arithmetic operations for trapezoidal fuzzy numbers are concisely explained based on the work of (Chen and Hsieh 1999) and (De and Mahata 2017) and (Ranu Singh and Vinod Kumar Mishra., 2023). Building upon these basic definitions and results, the proposed model is then fuzzified. Let $(\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3, \tilde{\theta}_4)$ and $(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3, \tilde{k}_4)$ represent trapezoidal fuzzy numbers, as depicted in Fig. 6.3. Consequently, the succinct total average expense function is converted into a fuzzy cost function. Since the deterioration rate (θ) and the defective quantity percentage (k) are both represented as trapezoidal fuzzy figures, the total cost (TC) is also treated as a trapezoidal fuzzy figure.

$$TC = (\tilde{T}C_1, \tilde{T}C_2, \tilde{T}C_3, \tilde{T}C_4) \quad \ldots (6.24)$$

$$TC_d = \frac{1}{4}(\tilde{T}C_1 + \tilde{T}C_2 + \tilde{T}C_3 + \tilde{T}C_4) \quad \ldots (6.25)$$

$$\begin{aligned}
\tilde{T}CR_j(t_j, s_j, n_1) &= n_1 * O_r + \sum_{j=0}^{n_1-1} (1-\tilde{k}_\iota)\{H + \tau\hat{h}_c + D_t * \tilde{\theta}_\iota\} \int_{t_j}^{s_{j+1}} [D\,\beta(s_j - t)]\,dt \\
&+ \{W_h \\
&+ \tau\hat{P}_r\} \sum_{j=0}^{n_1-1} \Big(\frac{-(1-\tilde{k})_\iota}{\tilde{\theta}_\iota} \int_{t_j}^{s_{j+1}} \{e^{\theta_i (s_{j+1}-t)} - 1\}\, D\, dt \\
&+ \frac{1}{3}(D\,\beta(s_j - t_j))^3\Big) + \sum_{j=0}^{n_1-1} s \int_{s_j}^{t_j} D\,\beta(s_j - t)\; dt \\
&+ \sum_{j=0}^{n_1-1} l \int_{s_j}^{t_j} (1-\beta)\, D\,(t_j - s_j)\; dt + \tau\hat{c} + F_c + 2\mathrm{d}v_c C_1(\mathrm{d}\, e_2 + \mathrm{d} \\
&* v_c C_2) \int_{t_j}^{t_{j+1}} D(t) e^{\theta_i(\mathrm{t}-\mathrm{t}_j)}\,\mathrm{d}t
\end{aligned}$$

$$\ldots(6.26)$$

The goal is to discover the basic values of t_i and s_i in order to lower the total variable cost (TC) of stock control and management. Figure 6.3 Exhibits the trapezoidal fuzzy numbers denoting the deterioration rate (θ) and the defective quantity percentage (k). The requirements to shows the TC_d to be minimum are given below:

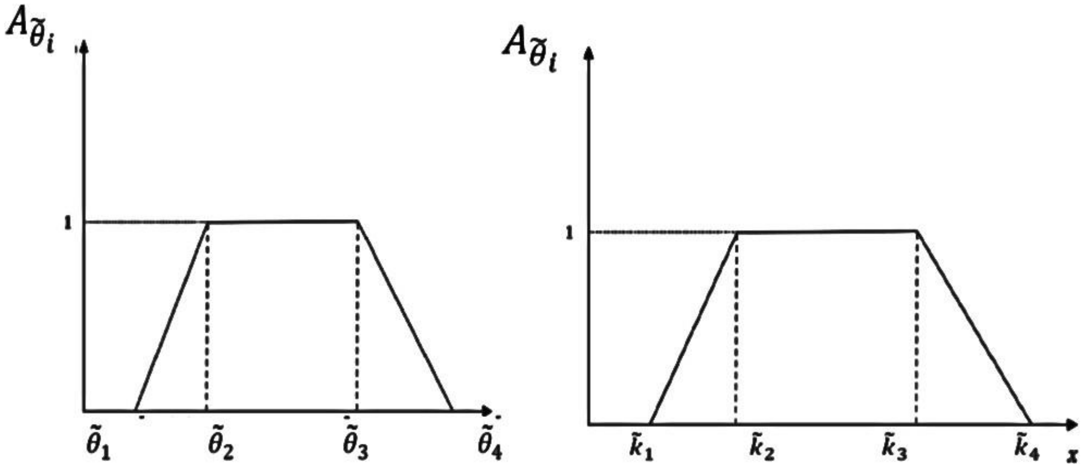

Fig. 6. 3 Displays the trapezoidal fuzzy

$$\frac{\partial \tilde{T}R_d(t_j, s_j, n_1)}{\partial t_j} = 0 \qquad \ldots(6.27)$$

$$\frac{\partial \tilde{T}R_d(t_j, s_j, n_1)}{\partial s_j} = 0 \qquad \ldots(6.28)$$

$$\frac{\tilde{\partial} \mathbf{C} R_d(t_j, s_j, n_1)}{\partial s_j}$$

$$= \{H + \tau \tilde{h}_c + D_t * \theta_\iota\} \int_{t_j}^{s_{j+1}} [D\,\beta]\,dt$$

$$+ \{W_h + \tau \tilde{P}_r\} \sum_{j=0}^{n_1-1} \left((D\,\beta(s_j - t_j))^2 \right) + \sum_{j=0}^{n_1-1} s \int_{s_j}^{t_j} D\,\beta\,dt$$

$$+ \sum_{j=0}^{n_1-1} l \int_{s_j}^{t_j} (\beta - 1)\,D\,dt$$

…(6.29)

$$\frac{\tilde{\partial} \mathbf{C} R_d(t_j, s_j n_1)}{\partial t_j}$$

$$= \{H + \tau \tilde{h}_c + D_t * \theta_\iota\} [D\,\beta(t_j - s_j)]$$

$$+ \{W_h$$

$$+ \tau \tilde{P}_r\} \sum_{j=0}^{n_1-1} \left(\frac{-(1 - \tilde{k}_\iota)}{\theta_\iota} (1 - e^{\theta_i (s_{j+1} - t_j)}) \right.$$

$$\left. - (D\,\beta(s_j - t_j))^2 \right) + \sum_{j=0}^{n_1-1} l \int_{s_j}^{t_j} (1 - \beta)\,D\,dt$$

$$+ l(1 - \beta)\,D(t_j - s_j) - (\mathrm{d}\,e_2 + \mathrm{d} * v_c C_2)$$

$$* \theta_\iota \int_{t_j}^{t_{j+1}} D\,e^{\theta_i(\mathrm{t} - \mathrm{t}_j)}\,\mathrm{d}t - (\mathrm{d}\,e_2 + \mathrm{d} * v_c C_2)\,D$$

… (6.30)

$$\frac{\partial^2 \tilde{T}CR_d(t_j, s_j, n_1)}{\partial s_j^{\,2}}$$

$$= \{W_h + \tau P_r\} \sum_{j=0}^{n_1-1} \left((D\,\beta(s_j - t_j)) \right) - sD\,\beta + l(\beta - 1)\,D$$

… (6.31)

$$\frac{\partial^2 \tilde{T}CR_d(t_j, s_j, n_1)}{\partial t_j^{2}}$$

$$= \{H + \tau h_c + D_t * \theta_\iota\}\,[D\,\beta]$$

$$+ \{W_h + \tau P_r\} \sum_{j=0}^{n_1-1} \left(-(1 - \mathrm{k}\backslash\tilde{\mathrm{k}}_\iota)(e^{\theta_i (s_{j+1} - t_j)}) + 2D\,\beta(s_j - t_j)\right)$$

$$+ \sum_{j=0}^{n_1-1} l(1-\beta)\,D + l(1-\beta)\,D + (\mathrm{d}\,e_2 + \mathrm{d}$$

$$* v_c C_2)\theta_\iota^{\,2} \int_{t_j}^{t_{j+1}} D e^{\tilde{\theta}_i(\mathrm{t} - \mathrm{t}_j)}\,\mathrm{d}t \;+ D(\mathrm{d}\,e_2 + \mathrm{d} * v_c C_2)\theta_\iota$$

… (6.32

The essential condition is that the Hessian matrix with TC_d must be positive definite for $\tilde{T}CR_d$ to achieve its minimum, with n_1 held constant. Additionally, the theorem establishes the positivity of TC_d. Consequently, through an iterative process

$$\nabla T_R(t_j, s_j, n_1) = \begin{bmatrix} \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_1^2} & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_1\,\partial s_1} & 0 & 0 & \cdots\cdots & 0 & 0 & 0 \\ \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial s_1\,\partial t_1} & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial s_1^{\,2}} & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial s_1\,\partial t_2} & 0 & \cdots\cdots & 0 & 0 & 0 \\ 0 & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_2\,\partial s_1} & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_2^{\,2}} & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_2\,\partial s_2} & \cdots\cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots\cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots\cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots\cdots & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_{n_1-1}\,\partial s_{n_1-1}} & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial s_{n_1-1}^{\,2}} & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial s_{n_1-1}\,\partial t_{n_1}} \\ 0 & 0 & 0 & 0 & \cdots\cdots & 0 & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_{n_1}\,\partial s_{n_1-1}} & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_{n_1}^{\,2}} \end{bmatrix}$$

for a specified positive integer using the Equations 6.29 and 6.30. Hessian matrix (Sarkar et al. 2012)

Theorem: If t_j and s_j satisfy the inequality,(i) $\frac{\partial^2 TCR_d(t_j,s_j,n_1)}{\partial t_j^2} \geq 0$,

(ii) $\frac{\partial^2 TCR_d(t_j,s_j,n_1)}{\partial s_j^2} \geq 0, (iii)\ \frac{\partial^2 TCR_d(t_j,s_j,n_1)}{\partial t_j^2} - \left|\frac{\partial TCR_d(t_j,s_j,n_1)}{\partial t_j\ \partial s_j}\right| \geq 0$ and (iv) $\frac{\partial^2 TCR_d(t_j,s_j,n_1)}{\partial s_j^2} - \left|\frac{\partial TCR_d(t_j,s_j,n_1)}{\partial s_j\ \partial t_j}\right| \geq 0$ for all j= 1, 2, . . ., n then TC will be positive definite.

$$\tilde{Q}_{1d} = \frac{-(1-\tilde{k}_1)}{\theta_1}\int_{t_j}^{s_{j+1}} \{e^{\theta_1 (s_{j+1}-t)} - 1\}\ D\ dt$$
$$+ \frac{-(1-\tilde{k}_2)}{\theta_2}\int_{t_j}^{s_{j+1}} \{e^{\theta_2 (s_{j+1}-t)} - 1\}\ D\ dt$$
$$+ \frac{-(1-\tilde{k}_3)}{\theta_3}\int_{t_j}^{s_{j+1}} \{e^{\theta_3 (s_{j+1}-t)} - 1\}\ D\ dt$$
$$+ \frac{-(1-\tilde{k}_4)}{\theta_4}\int_{t_j}^{s_{j+1}} \{e^{\theta_4 (s_{j+1}-t)} - 1\}\ D\ dt\ +\ (D\,\beta(s_j - t_j))^3$$

…(6.33)

$TCs_j(t_j, s_j, n_1)$

$$= n_1{}^{*}{}_1 * S_s + P_s \Big(\frac{-(1-\tilde{k}_1)}{\theta_1} \int_{t_j}^{s_{j+1}} \{e^{\theta_1 (s_{j+1}-t)} - 1\}\, D\, dt$$

$$+ \frac{-(1-\tilde{k}_2)}{\theta_2} \int_{t_j}^{s_{j+1}} \{e^{\theta_2 (s_{j+1}-t)} - 1\}\, D\, dt$$

$$+ \frac{-(1-\tilde{k}_3)}{\theta_3} \int_{t_j}^{s_{j+1}} \{e^{\theta_3 (s_{j+1}-t)} - 1\}\, D\, dt$$

$$+ \frac{-(1-\tilde{k}_4)}{\theta_4} \int_{t_j}^{s_{j+1}} \{e^{\theta_4 (s_{j+1}-t)} - 1\}\, D\, dt + (D\,\beta(s_j - t_j))^3 \Big)$$

…(6.34)

6.8. ILLUSTRATIVE SCENARIO

$D = 40$, $O_r = 0.5$, $\tilde{k}_1 = 0.01$, $\tilde{k}_2 = 0.02$ $\tilde{k}_3 = 0.03$ $\tilde{k}_4 = 0.04$, $H = 420$, $\tau = 0.2$, $\hbar_c = 0.1$, $D_t = 1$, $\theta_1 = 0.001$ $\theta_2 = 0.002$, $\theta_3 = 0.003$, $\theta_4 = 0.004$, $\beta = 0.1$, $W_h = 0.3$, $P_r = 0.2$, $\hat{c} = 0.1$, $F_c = 1.2$, d=2, $e_1 = 2$, $C_1 = 0.04$, $e_2 = 2$, $v_c = 3$, $C_2 = 0.2$, $S_s = 430$, $P_s = 0.001$, $l = 0.02$, $s = 2$. Equations 6.29 and 6.30, depicting nonlinear systems, were resolved utilizing Mathematica version 12 mathematical software, employing a numerical iterative technique to solve the nonlinear differential equation. The optimal condition of the overall system cost and replenishment cycles may be noticed in Tables 6.2, Table 6.3, Table 6.4, Figure 6.4, and Figure 6.5, respectively, for all the values specified in example 6.1.

The most optimal solution occurs at node 6, where $t_1 = 1.54979$, $t_2 = 2{,}03524$, $t_3 = 2.52702$, $t_4 = 3.01879$, $t_5 = 3.51056$, $t_6 = 4.00234$ and $s_1 = 0$, $s_2 = 1.54113$, $s_3 = 2.03291$, $s_4 = 2.52468$, $s_5 = 3.01645$, $s_6 = 3.50822$, and the total cost is 35.0379.

Table 6. 2 Optimal replenishment time interval, total Cost Table of retailer, supplier, and total quantity for fuzzy

↓ D	→ tj	t_0	t_1	t_2	t_3	t_4	t_5	t_6	$\tilde{C}R_j$	$\tilde{Q}_{\alpha+1d}$	TCs_j
50		0	1.54979	2.03524	2.52702	3.01879	3.51056	4.00234	35.0379	**21.255**	**54.8378**

Table 6. 3 Optimal time interval for shortage, total Cost Table of retailer, supplier, and total quantity for Fuzzy model

↓D	→ s_j	S_0	S_1	S_2	S_3	S_4	S_5	$\tilde{C}R_j$	$\tilde{Q}_{\alpha+1d}$	TCs_j
50		0	1.54113	2.03291	2.52468	301645	3.50822	35.0379	**21.255**	**54.8378**

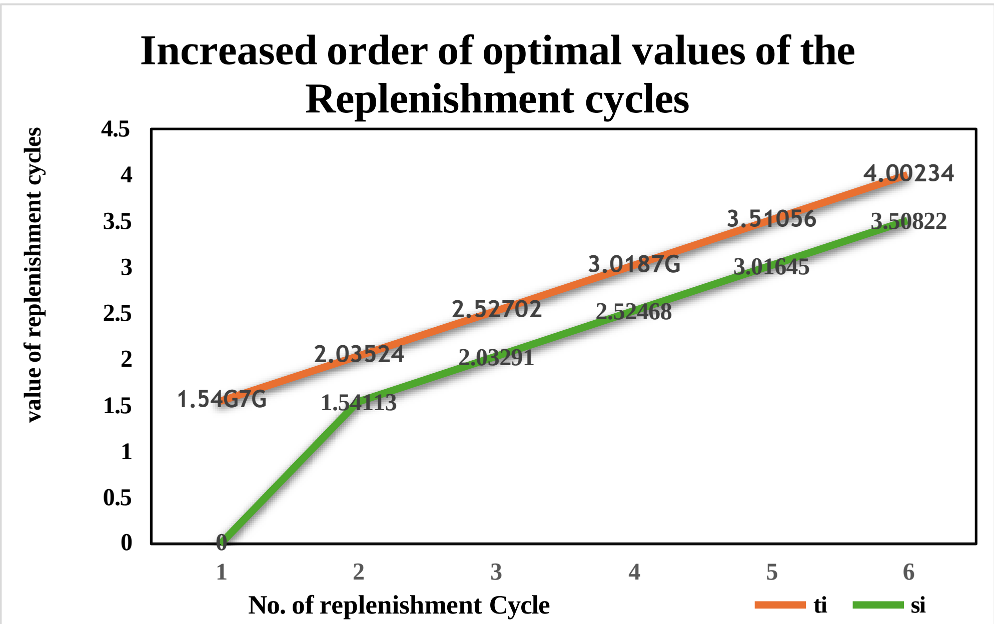

Fig. 6. 4 Graphical representation of the replenishment time and shortage time

Table 6. 4 Total Cost Table of retailer for Fuzzy model

↓ D	→n 1	1	2	3	4	5	6	7	$\tilde{T}CR_j$
50		539.29	540.79	542.29	543.79	504.85	35.037 9	548.29	35.037 9

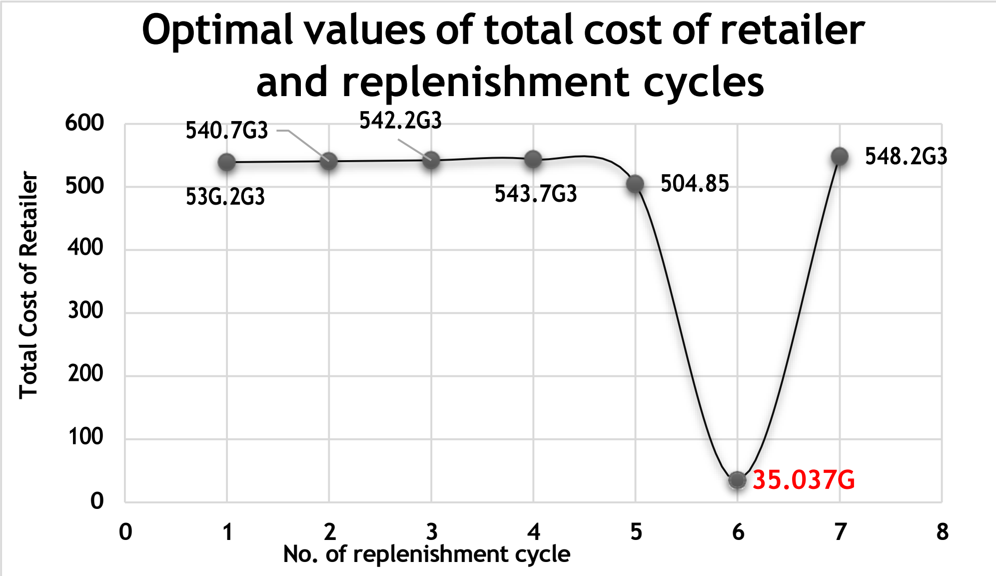

Fig. 6. 5 Graphical representation of the optimal values of total cost of retailer

6.9. SENSITIVITY INVESTIGATION

A sensitivity analysis is a crucial aspect of research papers, particularly in decision-making models or optimization problems. It helps understand how changes in parameters affect the outcomes or solutions. The sensitivity analysis conducted on the research model reveals insights into how variations in different parameters impact the optimal replenishment cycle and total order quantity. Across the parameters β, $\hat{h}_C$, L, V_c, O_r, T, z, s, and tau, the optimal replenishment cycle consistently remains at 6, showcasing its robustness to changes in these factors. For total order quantity, the analysis indicates minimal fluctuations in response to alterations in most parameters. Parameters like β, $\hat{h}_C$, L, V_c, O_r, T, and z exhibit marginal influences on total order quantity, with variations within a narrow range around the approximate value of 30.85. However, parameters s and tau demonstrate slightly more discernible effects on both the optimal replenishment cycle and total order quantity. Positive changes in s and tau lead to minor decreases in the optimal replenishment cycle and slight increases in total order quantity, while negative changes result in the opposite trends. As we can see, Table 6.5 and Figure 6.6.

Table 6. 5 Sensitivity analysis for all parameters in Fuzzy model

Parameters	**%Changes**	**Optimal Replenishment cycle**	**Total order Quantity Q_{nt}**	**Total cost of Retailer T_R**	**Total cost of supplier T_S**
β	+20	6	37.4136	12.54336	96.52709
	+10	6	26.1835	30.49022	67.5536
	0	6	17.0039	30.8503	43.8702
	−10	6	10.2267	31.2684	26.3850
	{−20	6	6.1971	31.7572	15.9883
$\tilde{h}_c$	+20	6	17.00816	12.2251	43.8810
	+10	6	17.00756	30.8491	43.8795
	0	6	17.00696	30.8493	43.8779
	−10	6	17.00636	30.84955	43.87643
	{−20	6	17.00576	30.8497	43.87488
H	+20	6	31.06859	12.8146	80.1569
	+10	6	23.7566	29.00219	61.2921
	0	6	17.00396	30.85033	43.870237
	−10	6	11.38452	33.17159	29.37207
	{−20	6	7.74971	36.15723	19.9942
l	+20	6	17.017007	12.25913	43.9038
	+10	6	17.01048	30.84763	43.88705
	0	6	17.00396	30.85033	43.87023
	−10	6	16.99744	30.85303	43.8534
	{−20	6	16.990929	30.85573	43.83659

O_r	+20 +10 0 −10 {−20	6	17.00396	14.02484	43.87023
		6	17.00396	31.75033	43.87023
		6	17.0039	30.85033	43.870237
		6	17.0039	29.95033	43.8702
		6	17.00396	29.0503	43.87023
S	+20 +10 0 −10 {−20	6	17.06070	11.8642	44.0166
		6	17.0322	32.61924	43.94327
		6	17.0039	30.8503	43.870237
		6	16.97577	29.0811	43.79749
		6	16.94769	27.31168	43.72504
D_t	+20 +10 0 −10 {−20	6	17.00426	12.22485	43.87101
		6	17.004118	30.8502	43.87062
		6	17.0039	30.85033	43.87023
		6	17.00381	30.85038	43.8698
		6	17.0036	30.8504	43.8694
τ	+20 +10 0 −10 {−20	6	17.0060	12.2332	43.8754
		6	17.0049	30.8619	43.8728
		6	17.0039	30.85033	43.87023
		6	17.0029	30.8387	43.8676
		6	17.0019	30.8271	43.865
v_c	+20 +10 0 −10 {−20	6	14.3255	12.32495	36.9598
		6	15.60286	31.4724	40.2553
		6	17.0039	30.8503	43.8702
		6	18.5331	30.2313	47.81556
		6	20.1947	29.61547	52.1025
d	+20 +10 0 −10 {−20	6	42.8818	13.40037	110.6350
		6	52.04806	24.65992	134.2840
		6	62.5602	23.3896	161.4053

		6 6	74.5001 87.95177	22.1549 20.9569	192.210 226.9155

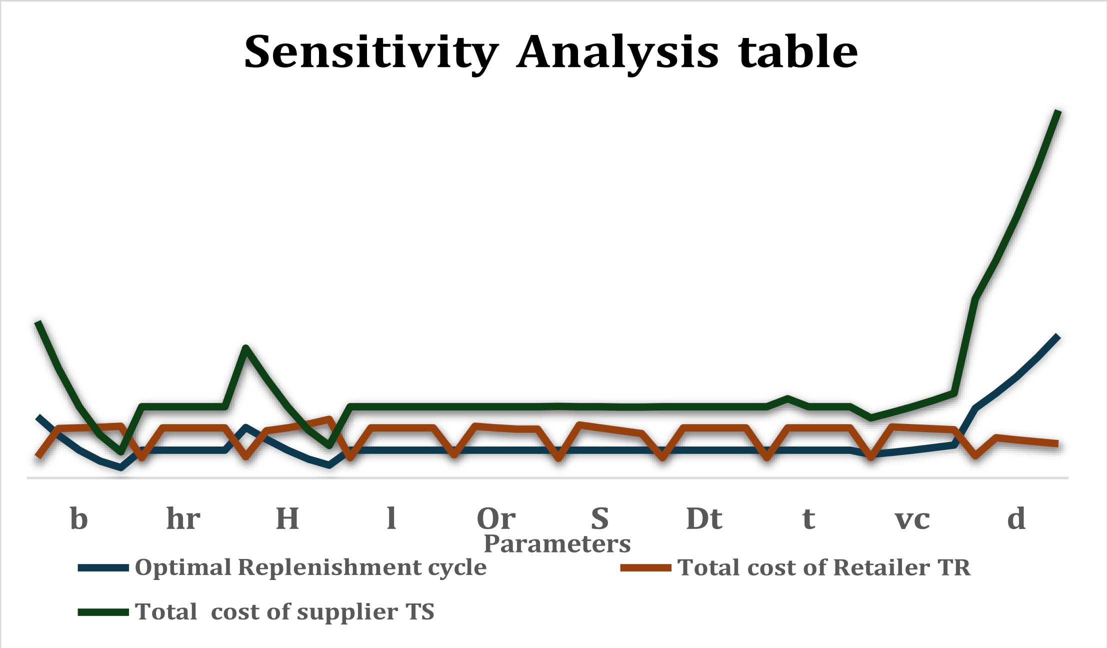

Fig. 6. 6 Sensitivity analysis of different parameters

6.10. MANAGERIAL INSIGHTS

This study offers valuable insights for inventory managers, providing effective strategies to address challenges posed by seasonal demand fluctuations. The emphasis on precisely forecasting seasonal demand highlights its pivotal role in optimizing inventory management and enhancing cost efficiency across various industries. A significant managerial insight arises from the integration of machine learning techniques, particularly decision tree classifiers, which markedly improve the accuracy of demand forecasts.

A crucial lesson from this research underscores the substantial costs associated with relying on fixed demand assumptions. Opting for seasonal forecasted demand over static predictions presents businesses with the opportunity to make noteworthy reductions in overall costs. For example, envision a retail company specializing in vegetables and fruits. Traditional approaches, maintaining a fixed inventory quantity year-round, may result in surplus during low-demand periods and shortages during peak seasons. However, employing machine learning for precise seasonal demand forecasting empowers managers to strategically adjust ordering quantities and replenishment periods. This adaptability allows companies to optimize inventory levels, reduce expenses linked to prevent stock-outs, surplus inventory and elevate the overall customer experience.

The study's findings also shed light on optimal policies for organizations grappling with imperfect deteriorating products in their inventory systems. Policymakers aiming to minimize overall costs can benefit by exercising greater control over sensitive parameters related to total cost. Businesses, especially in industries like retail and pharmaceuticals, stand to gain tangible advantages by incorporating these findings into their practical operations. The implementation of improved management policies, driven by seasonal demand estimates and optimized ordering practices, has the potential to enhance operational efficiency and boost profitability.

6.11. CONCLUSION

In summary, the present paper introduces a more practical inventory model designed for imperfectly decaying items, which incorporates a Machine Learning technique for seasonal demand prediction. The incorporation of deterioration rate and defective percentage quantity as fuzzy variables addresses inherent uncertainties, allowing for permitted shortages that are partially backlogged over the finite planning horizon.

Recognizing the pivotal role of product demand in business operations, particularly its seasonal variations, the study utilizes a time series predicting method to analyze seasonal forecasted demand. The results highlight the generation of direct month-wise predicted demand, offering managers valuable insights to enhance the management of inventory based on expected demand. The Sign Distance method is used for defuzzification, aiding in the determination of optimal replenishment periods and ordering quantities that minimize total average cost, including emission costs over the finite planning horizon.

The numerical example confirms the robustness of the mathematical model, unveiling a significant decrease in overall costs when employing projected seasonal demand as opposed to fixed demand. Sensitivity analysis pinpoints critical parameters demanding heightened managerial attention. Graphical representation illustrates the curvature of the total cost function, emphasizing its highly non-linear nature.

To enhance the model's versatility, future extensions could explore alternative demand forecasting approaches, such as decision tree methods forecasting. Additionally, a comparative study across different forecasting methods could provide valuable insights. Further extensions may involve incorporating different fuzzy variables to account for increased uncertainty in parameter estimation.

CHAPTER 7

CONCLUSION

7.1. CONCLUSION

During the research work of the "Study of different types of back ordering in Finite Planning Horizon," we consider our model to help in the implementation of supply chain management with the help of backordering in fixed planning horizon. The study presented will have huge benefits for inventory management as back ordering offers a lot of benefits like low inventory management and cash flow in business.

In Chapter 1: We presented a short historical overview of operations research and highlighted the role of inventory in the company. Further, the types of inventories are discussed, and considerable attention is paid to backordering as to its advantages and drawbacks.

In Chapter 2: To highlight the background and the importance of the research a literature survey was carried out. First, linear and quadratic back-ordering inventory models were briefly described. Secondly, we reviewed exponential backordering models taking into account the price sensitive demand and discussed both partial and complete backordering policies. We further talked about EOQ and EPQ models with backordering. Last but not least, we discussed the existing literature on fuzzy backordering models.

In chapter 3: As highlighted in this chapter, the enhancement of replenishment policies is of essence to the success of any business with the intention of enhancing supply chain efficiency. Through modeling decision factors like fluctuating, sensitive demand, and back ordering and achieving a minimum total cost within a finite time frame, then they are in a position to make efficient decisions in their respective fields that will de magnetically lead to better inventory management, reduced stock-outs, and high customers satisfaction. Analyzing it in detail, we can define that this approach helps take the structure of strategic planning, which in its turn creates proper ways to allocate resources in order to minimize risks and maximize profitability in the constantly growing and competitive environment.

Lastly, in a study based on this chapter, we applied the framework to enlighten the optimization problem pertaining to replenishment policies based on manipulation of influential cost and system parameters. More precisely, carrying cost-sensitive demand function is a−bp, and back ordering is fully allowed. The back ordering function was examined using quadratic and linear equations, with the amount of shortages controlled in finite planning horizon H.

The objective of our model was to determine the optimal total replenishment cost. For brevity's sake, the following are changes in parameters, a, b, c, p, S, H, θ, Wh : Our results demonstrate how each of the specified changes influences total costs. Most importantly, we noted that the best solutions occurred in the second set, a fine-tuning in the third, fourth and lastly in the fifth set of results indicating the dynamic nature of the system in place.

In Chapter 4: In this Chapter, solved a model for shortages at every cycle of a finite planning horizon, an exhaustive exponential complete backordering inventory model is also developed here, catering for linear trends and price sensitive demand. This model introduces enhancements in the inventory control by applying exponential backordering and better matches the situations where price, demand, and supply vary in the actual world. The use of Mathematica software accompanied by python-based computation tools improves the model to provide accurate numerical results and qualitative graphical results. The findings of the numerical example also indicate the extent of improvement in total costs achievable by the model and at the same time the extent of improved co-ordination between the suppliers and the retailers. Furthermore, acknowledging the results of the sensitivity analysis that showed that the total costs are essentially a function of the backordering rate α and the cost coefficient c, it is evident that the two ratios are strategic levers. This goes to confirm that there is opportunity in improving backordering policies by avoiding situations where supplies are out which compromise customer satisfaction, yet the supply systems are costly. Thus, the developed model enables practitioners and decision-makers to obtain a useful tool to improve inventory control decisions. Controlling backorders which are inevitable due to high variability in demand and pricing conditions is not only necessary for cost control but also necessary for control of supply chain operations.

In Chapter 5: This chapter presents an appropriate study of the Economic Production Quantity (EPQ) model for partially back-ordered items wherein demand is sensitive to price, and the lead time is defined by an exponential distribution in a finite production cycle. Important factors like backordering cost incorporating the inventory-dependent parameter cc and factor α were tested through sensitivity analysis to analyze the impact of backordering cost on the replenishment cycle and overall costs. The advantages of the back ordering strategy are highlighted especially in issues to do with stock placements and stock scarcities respectively.

In order to confirm mathematical findings, Mathematica 12 was used, which helped to achieve precise solutions in multifarious equations. Furthermore, as part of the analytical tools, Python, an efficient language in a machine learning context and in data analysis, was used to build graphical representations and visuals to enhance data comprehension and simplify analysis results.

Practical recommendations are offered to assist decision-makers in choosing the best strategies in how to reduce costs, given backorders. This work stresses the need to incorporate modern computational tools and provides measures on how to optimally determine the ordering cost, restocking cost, and backorder cost.

In Chapter 6: In conclusion, this research presents an efficient inventory model for perishable products with degradation aspects and incorporates a machine learning algorithm for demand variability of seasonal products. Owing to this, uncertainties in the model are in the deteriorating rate of inventory and the percentage of defective products where permitted shortages are partially backlogged over the planning horizon.

Since demand occupies a strategic position in business operations or management, especially its volatile nature due to seasonal variations, a time series approach of forecasting monthly demand is followed. These predictions provide useful information to managers in managing inventory in relation to expected demand profiles. For defuzzification, the Sign Distance method is used, allowing to identification of the most optimal replenishment cycles and order quantities that will minimize the total average cost including emissions costs through the planning horizon.

The numerical example supports the model and illustrates that total costs are much lower when utilizing seasonal demand forecasts compared to fixed demand estimates. Besides providing confidence intervals, sensitivity analysis indicates those variables that deserve keen attention from the manager. Moreover, graphical representations show how the total cost function behaves stepping up with its strong non-linear characteristics.

7.2. Limitations

The current research has critical contributions as it developed and confirmed an exponential complete backordering inventory model with price-sensitive demand in finite planning horizon. Although the model is sound and proves its great applicability to the real-world situations like retailing and pharmaceuticals, there are limits. One-item framework is chosen in the work to ensure clarity and mathematical tractability when many industries have to deal with numerous products at the same time. Demand has been assumed price-sensitive, with linear tendencies that may accurately reflect realistic customer behavior but are not exhaustive of the range of stochastic or seasonal tendencies. In addition, trapezoidal fuzzy numbers which are commonly used to represent deterioration and defective items were also used but more complex treatments of uncertainty could be used further to enrich the analysis. Lastly, validation was conducted based on numerical experiments instead of large-scale industry data, which was purposeful to highlight theory rigor and computational knowledge.

7.3. FUTURE SCOPE

Future studies might develop this study by expanding the model to the multi-item inventory systems, which will allow evaluating the different product classes and their interactions under the finite planning horizons. It was also possible to generalize the framework to infinite horizons to provide more information about long-term planning of inventory. The other opportunity is the inclusion of multi-echelon supply chains, which explicitly represent supplier-warehouse-retailer relationship in stochastic demand environments, in which pricing policies add an additional level of complexity to the inventory dynamics. In order to increase flexibility, future research can combine more complex demand forecasting tools, e.g., decision tree-based models, deep learning or hybrid machine learning, and perform comparative analyses to assess the performance of forecasting in different market environments. It was also observed that the model would be improved by including other fuzzy variables to better manage uncertain situations as well as an approximation of actual conditions in the real world.

Outside the classical supply chain extensions, there are new areas of optimization that offer new opportunities. These are green and sustainable optimization, in which emission and circular economy cost considerations are incorporated into inventory choices; robust and stochastic optimization, which has been used to manage such volatile market conditions, and quantum optimization, currently under investigation to solve complex inventory and logistics at scale. Moreover, combining optimization and Industry 4.0 technologies, including digital twins, smart warehouses with IoT, and blockchain-based transparent supply chains are also a groundbreaking move. Using the model in industry-specific case studies would further prove its practicability and will give practical guidance specific to organization environment. Together these extensions would help to build resiliency, efficient, and adaptive inventory management architectures, which could satisfy the needs of the current unpredictable and rapidly changing business environment.

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Appendix-A: Conference, workshop or seminar participation and presentations

A. 1. Conference presentations are as follows

- Presented a research article titled " An Economic Ordering Policy for Interest Earned on Sales Till the Permissible Period Without Paying Interest for the Items Kept in Advance Stock” at International Conference on “Recent Advances in Fundamental and Applied Sciences” (RAFAS 2021) held on June 25-26, 2021, organized by School of Chemical Engineering and Physical Sciences, Lovely Faculty of Technology and Sciences, Lovely Professional University, Punjab, India.
- Presented a research article titled " A replenishment policy for an inventory model with price-sensitive demand and quadratic back order in a finite planning horizon” at 2nd International Conference on Computational Applied Sciences and Its Applications -2023" from 13th to 14th July 2023 conducted by Department of Basic Sciences, University of Engineering & Management, Jaipur, India.
- Presented a research article titled " An Exponential Complete Backordering Comparison Inventory Model for Considering Shortages of Goods in All Cycles of the Planning Horizon” at International Conference on Basic Sciences: Global Challenges And Solutions For Sustainable Development” “(GCSSD)” held on October 13-14, 2023, organized by Chintamani Education Society, Ballarpur, in Collaboration with Gondwana University, Gadchiroli.
- Participated at One Day National Level Inter Disciplinary E-Conference on “Contemporary Dynamics in Humanities, Science & Technology” held on April 22, 2024, organized by Shri. R. L. T. College of Science, Akola, India.
- Namwad, R. S., N. K. Mishra, Ranu, (2024) “An Exponential Complete Backordering Comparison Inventory Model for Considering Shortages of

Goods in All Cycles of the Planning Horizon", Paper presented at the International Conference on Emerging Trends in Business Analytics and Management Science (BAMS-ORSI 2024), IIT Bombay, India. Submission ID: 711.

Certificate No. 225382

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Certificate of Participation

This is to certify that **Ms. Renuka Sheshrao Namwad**

of **Lovely Professional University**

has given poster presentation on **An economic ordering policy earn interest on sales till the permissible period and without paying interest for the items kept in advance stock.**

in the International Conference on "Recent Advances in Fundamental and Applied Sciences" (RAFAS 2021) held on June 25-26, 2021, organized by School of Chemical Engineering and Physical Sciences, Lovely Faculty of Technology and Sciences, Lovely Professional University, Punjab.

Date of Issue : 15-07-2021
Place of Issue: Phagwara (India)

Prepared by (Administrative Officer-Records)	Organizing Secretary (RAFAS 2021)	Convener (RAFAS 2021

University of Engineering & Management (UEM), Jaipur

2nd International Conference on Computational Applied Sciences and Its Applications (ICCASA-2023)

July 13-14, 2023

CERTIFICATE

This is to certify that

Ms./Mr./Dr./Prof. **Renuka Namwad**

of Lovely Professional University, Paghwara, Punjab

has presented a paper entitled

A replenishment policy for an inventory model with price-sensitive demand and quadratic back order in a finite planning horizon

in "2nd International Conference on Computational Applied Sciences and Its Applications -2023" from 13th to 14th July, 2023 conducted by Department of Basic Sciences, University of Engineering & Management, Jaipur.

Ms. Pallavi Malik	Dr. Priyanka Chhaparwal	Prof. Aniruddha Mukherjee
Convener, ICCASA - 2023	Convener, ICCASA - 2023	Dean Academics, UEM Jaipur

INTERNATIONAL CONFERENCE ON BASIC SCIENCES: GLOBAL CHALLENGES AND SOLUTIONS FOR SUSTAINABLE DEVELOPMENT

CERTIFICATE

ISBN: 978-93-91535-91-9

This is to certify that

Renuka Sheshrao Namwad

has participated and presented his/her paper titled

An Exponential Complete Backordering Comparison Inventory Model for Considering Shortages of Goods in All Cycles of the Planning Horizon

during the INTERNATIONAL CONFERENCE GCSSD-2K23 held on 13-14 October 2023

Organised by

Chintamani Education Society, Ballarpur's

CHINTAMANI COLLEGE OF SCIENCE, POMBHURNA, DIST. CHANDRAPUR, M.S

In Collaboration With

Gondwana University, Gadchiroli

Venue: VASANT-VAIBHAV HALL

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Dr. Rajiv G. Weginwar	Dr. Anil Chitade	Dr. Sudhir S. Hunge	Dr. Anant S. Deshpande
Principal and convener	Convener	Co-Convener	Organizing Secretary (IQAC - Coordinator)

Sahakar Maharshi Late Bhaskarrao Shingne Arts College

Khamgaon, Dist.-Buldhana (NAAC Re-accredited A+)

(Affiliated to Sant Gadge Baba Amravati University, Amravati, M. S.)

Shri R. L. T. College of Science, Akola (NAAC Re-accredited A)

(Affiliated to Sant Gadge Baba Amravati University, Amravati, M. S.)

&

Shri Sant Gajanan Maharaj College of Engineering, Shegaon

(Affiliated to Sant Gadge Baba Amravati University, Amravati, M. S.)

Jointly Organize

One Day National Level Inter-Disciplinary E-Conference on

Contemporary Dynamics in Humanities, Science & Technology

Monday, 22 April 2024

CERTIFICATE

This is to certify that Mr./Mrs./Miss/Dr. ***Renuka Namwad*** participated and presented a research paper in the One Day National Level Inter-Disciplinary E-Conference on ***"Contemporary Dynamics in Humanities, Science & Technology"*** on 22 April 2024.

Dr.Saurabh Jadhao	Dr.Pradip Deohate	Dr.Dayanand Raut	Dr. S. B. Somani	Dr. V. D. Nanoty	Dr. Sanjay Patil
Convener	Convener	Convener	Principal	Principal	Principal
S.S.G. M.C.E., Shegaon	Shri R. L. T. College of Science, Akola	S. M. B. S. College, Khamgaon	S.S.G.M.C.E., Shegaon	Shri R. L. T. College of Science, Akola	S. M. B. S. College, Khamgaon

SJMSOM

IIT Bombay

Indian Institute of Technology, Bombay

CERTIFICATE OF PRESENTATION

This is to certify that Mr./Ms./Prof./Dr. **Renuka S Namwad**

from **Lovely Professional University, Phagwara, Punjab**

presented their paper titled **An Exponential Complete Backordering Comparison Inventory Model for Considering Shortages of Goods in All Cycles of the Planning Horizon**

authored by Dr. Nitin Kumar Mishra, Dr. Ranu, Renuka S Namwad

at the International Conference on Emerging Trends in Business Analytics & Management Sciences, held as part of the 57th Annual Convention of the Operational Research Society of India (BAMS-ORSI 2024), organized by the Shailesh J. Mehta School of Management, IIT Bombay, from December 12 to 14, 2024.

Prof. Pankaj Dutta	Prof. M.D. Agrawal
Conference Chair	Chief Advisor

A. 2. Workshop and webinar participation are as follows:

- ❖ Five-Days Hands-on Workshop on Research Methodologies Based on SPSS, Python, JAMOVI, R, and COMSOL Five-Day Workshop organized by the Department of Mathematics, School of Advanced Sciences, and Department of MBA, VIT Business School, Vellore Institute of Technology, Chennai from 10th to 14th April 2024.
- ❖ Completed a "Python for Beginners" Course of 15 hours on May 7, 2024, organized by Udemy learning platform.

udemy

Certificate no: UC-360e87ad-80a5-4498-8e1b-81ec1ce62c4b
Certificate url: ude.my/UC-360e87ad-80a5-4498-8e1b-81ec1ce62c4b
Reference Number: 0004

CERTIFICATE OF COMPLETION

Python for beginners

Instructors **Bharath Thippireddy**

Renuka Sheshrao Namwad

Date **May 7, 2024**
Length **15 total hours**

VIT

Vellore Institute of Technology

(Deemed to be University under section 3 of UGC Act, 1956)

CHENNAI

Hands-on Workshop on Research Methodologies Based on SPSS, Python, JAMOVI, R, and COMSOL

Certificate of Participation

This is to certify that **Renuka Sheshrao Namwad** of **Lovely Professional University Jalendhar Punjab** participated in a Five-Day Workshop organized by the **Department of Mathematics, School of Advanced Sciences, and Department of MBA, VIT Business School, Vellore Institute of Technology, Chennai** from 10th to 14th April 2024.

Sandip Dalui

Dr. P. Konar	**Dr. S. Dalui**	**Dr. M. Bhattacharjee**	**Dr. Mahalakshmi S**	**Dr. Harikrishnan K**
Convener	**Convener**	**Convener**	**Dean, SAS**	**Dean, VITBS**

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Appendix-B: Published Papers

B. 1. The list of articles published is as follows (are part of the thesis):

1. Dr. Nitin Kumar Mishra, Namwad R. S., A book chapter titled “Supply Chain Inventory Models Literature Review” published in “Engineering the Future: Cutting-Edge Technologies and Sustainable Solutions” ISBN No.-978-81-979410-0-9

2. Mishra, N.K. and Namwad R. S., 2024. A Replenishment Policy for an Inventory Model with Price-Sensitive Demand with Linear and Quadratic Back Order in a Finite Planning Horizon. *Mathematical Modelling of Engineering Problems*, *11*(6). https://doi.org/10.18280/mmep.110614

3. Namwad R. S., Mishra, N.K. and Jain, P., 2024. Optimizing Inventory Management with Seasonal Demand Forecasting in a Fuzzy Environment. *Journal Européen des Systèmes Automatisés*, *57*(4). https://doi.org/10.18280/jesa.570416

4. Namwad R. S., Mishra, N. K., Jain, P. and Ranu., Exponential back-ordering inventory model addressing shortages in finite planning horizons. Industrial Management Advances, (2025), Volume No. 3 Issue No. 1.
doi: 10.59429/ima.v3i1.9928

IIETA International Information and Engineering Technology Association

Mathematical Modelling of Engineering Problems
Vol. 11, No. 6, June, 2024, pp. 1537-1546
Journal homepage: http://iieta.org/journals/mmep

A Replenishment Policy for an Inventory Model with Price-Sensitive Demand with Linear and Quadratic Back Order in a Finite Planning Horizon

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price-sensitive demand, shortage, linear backorder, quadratic backorder, supply chain, finite planning horizon

ABSTRACT

In this research article, we proposed an inventory model for the replenishment policy. The focus of our research article is on the companies that frequently deal with backorders. An advanced inventory model considering back order has been proposed the results highlight the dynamic nature of the system, with optimal values achieved in different cycles. In this model, replenishment policy is given and to lower the economic ordering cost, we used new parameters such as price-sensitive demand, complete back ordering, and backorder is taken as a quadratic function as well as linear backorder with shortages in a finite planning horizon. The result is discussed for both backorders (linear and quadratic), to minimize the total cost obtained using the Hessian matrix to be positive definite. Software 'MATHEMATICA VERSION 12' has been used for the solution of the proposed model by using numerical iterative method. For different parameters, different tables are provided. The outcomes of the sensitivity analysis with the help of tables and graphs are depicted. Finally, we have discussed the conclusion and practical implications.

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Optimizing Inventory Management with Seasonal Demand Forecasting in a Fuzzy Environment

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supply model, shortages, forecasting demand, artificial intelligence, machine learning, deterioration, carbon pollution policy, finite planning horizon

ABSTRACT

This study explores an inventory management model in today's business landscape, where organizations increasingly rely on Machine Learning for demand-driven stock control. The proposed model accounts for imperfect and deteriorating products within a fuzzy environment, allowing for shortages and partial backlogging. Degradation rates and faulty percentages are classified as fuzzy variables since they are unpredictable and impacted by undefined conditions. The goal is to calculate the appropriate replenishment cycle and ordering quantity while reducing the optimal overall cost, including carbon pollution costs, within a constrained planning horizon. The defuzzification technique uses the sign distance approximation technique. Leveraging Machine Learning, the study utilizes a seasonal demand forecasting methodology. A numerical illustration supports the mathematical approach by demonstrating its capacity to estimate demand for deteriorating products. This facilitates optimized inventory management aligned with forecasted demand. A comparative examination emphasizes the positive aspects of AI learning-based forecasting systems over determined demand circumstances. Sensitivity analysis provides insights into the impact of various parameters on optimal solutions, contributing valuable managerial perspectives.

Chapter 9

Supply Chain Inventory Models Literature Review

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I. Abstract:

This review article offers a comprehensive exploration of the scholarly literature surrounding inventory models within the supply chain context. We delve into the evolution of these models over the years, focusing on those involving backordering practices. From the Optimal Order Quantity (EOQ) to the Optimal Production Quantity (EPQ) models, this paper examines a variety of inventory models under diverse conditions and constraints, such as linear and fixed backorder costs and fuzzy backorder quantities. We also discuss the complexities posed by service-level constraints, perishability of items, vendor-managed inventory policies, and sustainability considerations.

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RESEARCH ARTICLE

Exponential backordering inventory model addressing shortages in finite planning horizons

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ABSTRACT

In today's highly dynamic and price-sensitive market environment, inventory management faces increasing challenges due to fluctuating demand and the need for efficient coordination between suppliers and retailers. Based on these premises, this paper develops an exponential complete backordering model of the inventory system that considers the shortages within a finite planning horizon and price-sensitive demand in the presence of linear trends. Included in considerations are variability of demand, backordering costs, and a demand-plus-price relationship modeled through exponential backordering functions taken into consideration. Supportive of MATHEMATICA 12, iterative calculations

B. 2. Other relevant articles communicated and accepted during research work are as follows (are not part of the thesis):

1. Renuka S. Namwad; Nitin Kumar Mishra (2023). "An Economic Ordering Policy For Iterest Earned On Sales Till The Permissible Period Without Paying Interest For The Items Kept In Advance Stock." in the International 284 AIP Conference on Materials for Emerging Technologies-2021 (ICMET 21). https://doi.org/10.1063/5.0163138
2. Namwad, R.S. and Mishra, N.K., 2023. Trade Credit and Preservation Technologies: An Inventory Replenishment Model for a Sustainable Supply Chain. *Journal Européen des Systèmes Automatisés*, *56*(6), p.1027 (Not a part of thesis). https://doi.org/10.18280/jesa.560613
3. Vishakha P. Totare, Balvinder Singh Sandhu, Namwad, R. S., A book chapter titled "Advancing Sustainable and Sludge treatment Innovations in Energy Conversion and Resources Recovery" published in "Engineering the Future: Cutting-Edge Technologies and Sustainable Solutions" ISBN No.-978-81-979410-0-9.
4. Ranu, Namwad, R. S., Mishra, N. K., Heeraman, J., & Jain, P. (2024). *Optimizing supply chain efficiency with Mathematica: Mathematical and statistical methods for finite planning horizons.* Integrative Approaches to Quality, Data Analysis, and Interdisciplinary Research.
5. 9. Jain, P., Mishra, N. K., Ranu, & Namwad, R. S., (2024) *Optimizing inventory management with a Stackelberg game approach: A retailer-manufacturer model*. Paper presented at the International Conference on Emerging Trends in Business Analytics and Management Science (BAMS-ORSI 2024), IIT Bombay, India. Submission ID: 65.

6. Nitin Kumar Mishra, Anushka Sharma, Ranu, and Renuka S Namwad, "Formulating a finite planning horizon model for Inventory Management, incorporating Linear Demand, Exponential Deterioration, and Trade Credit Policy." European Chemical doi: 10.48047/ecb/2023.12.4.186

7. Namwad, R. S Dr. H. S. Tomar, The Rise of Python in Mathematical Computing: A Comparative Analysis with Mathematica, International Journal of Science and Research (IJSR) ISSN: 2319-7064, https://dx.doi.org/10.21275/SR25430132447

An Economic Ordering Policy for Interest Earned on Sales Till the Permissible Period Without Paying Interest for the Items Kept in Advance Stock

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Abstract. An inventory model without shortages is discussed. Considering negligible lead time. In this model, we have discussed inventory management for deteriorating items. We have considered the demand as a cubic function as a polynomial of degree three. The rate of deterioration is considered a cubic polynomial as a function of time. We have also discussed inventory dependency on the cubic demand function. There are lots of advantages of using the cubic function in the model. Cubic functions are most helpful to relate the different models to real-life situations. That helps the researcher elaborate on the changes taking place and gives a clearer idea. The cubic function is not only useful for numerical justification but also gives a clear picture of the model with the help of a graph. In this paper, we try to give a clearer idea about better values so that we solve the model with the help of MATHEMATICA. Also given the graph to make everyone understand. To elaborate the model a numerical example is provided. An in-depth sensitivity analysis has been undertaken.

INTRODUCTION

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Trade Credit and Preservation Technologies: An Inventory Replenishment Model for a Sustainable Supply Chain

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supply chain replenishment, deterioration of materials, carbon emission, preservation technology, and trade credit

ABSTRACT

Achieving sustainability within today's competitive environment is highly challenging. Therefore, we proposed a supply chain inventory replenishment model incorporating a finite planning horizon. So, to enhance their profit and lessen the total cost and carbon, this research study examines the investment in green (carbon offset) and preservation technologies. Additionally, we analyzed the trade credit duration granted by suppliers to the retailers. Carbon offsets/green technology represent a prevalent and significant measure to reduce carbon emissions. Time becomes a critical factor influencing demand rates in this context, while the degradation of materials affects a vast number of business sectors. Therefore, the cost of investing in preservation or green technology to control the deterioration of the materials, and reduce environmental emissions, the cost for ordering, the holding cost, and the replenishment cycle duration are all calculated. Consequently, a numerical iterative algorithm is prepared to identify the optimized solution for the supply chain approach for inventory control and management challenges. The optimality and uniqueness of the parameters of the proposed research study are furnished with a theoretical, mathematical, tabular, and pictorial analysis. Also, proposed research studies are provided with managerial implications that provide practical insights for industry practitioners. In conclusion, this research not only contributes valuable theoretical insights but also offers a tangible framework applicable to real-world scenarios.

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The Rise of Python in Mathematical Computing: A Comparative Analysis with Mathematica

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Abstract: ***Python has become a fundamental tool in applied mathematics, offering extensive capabilities in numerical computation, optimization, and statistical analysis. Its growing adoption in mathematical research and education has positioned it as a strong alternative to traditional software like Mathematica. This paper explores Python's applications in solving differential equations, optimization problems, and statistical modeling, demonstrating its efficiency and versatility. Additionally, it examines Python's role in mathematics education, highlighting its impact on interactive learning, problem-solving, and accessibility. A comparative analysis with Mathematica is presented, emphasizing Python's advantages in terms of cost, flexibility, and integration with emerging technologies. Through this study, we aim to showcase Python's significance in modern mathematical applications and its transformative potential in both research and education. It highlights the importance in mathematics by focusing on how Python is becoming more popular in mathematical modelling, simulation and education than similar tools like Mathematica. This study gives research-based explanations along with practical insights to people involved in computational and applied mathematics.***

Keywords: Python in Mathematics, Computational Modeling, Mathematical Computing, Interactive Learning, Educational Tools, Open-Source Alternatives, Python vs Mathematica