A STUDY OF COUPLED FIBONACCI AND LUCAS SEQUENCE, THEIR FUNDAMENTAL PROPERTIES AND APPLICATIONS

Thesis Submitted for the Award of the Degree of

DOCTOR OF PHILOSOPHY

in

(Mathematics)

By

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2025

Declaration

I, Vikas Ranga, declared that the presented work in the thesis entitled "A STUDY OF COUPLED FIBONACCI AND LUCAS SEQUENCE, THEIR FUNDAMENTAL PROPERTIES AND APPLICATIONS" in fulfillment of degree of **Doctor of Philosophy (Ph. D.)** is outcome of research work carried out by me under the supervision of Prof. (Dr.) A.K. Awasthi, working as Professor in the Mathematics Department, School of Chemical Engineering and Physical Sciences, Lovely Professional University, Phagwara. In keeping with general practice of reporting scientific observations, due acknowledgment have been made whenever work described here has been based on findings of other investigator. This work has not been submitted in part or full to any other University or Institute for the award of any degree.

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Certificate

This is to certify that the work reported in the Ph. D. thesis entitled "A STUDY OF COUPLED FIBONACCI AND LUCAS SEQUENCE, THEIR FUNDAMENTAL PROPERTIES AND APPLICATIONS" submitted in fulfillment of the requirement for the award of degree of **Doctor of Philosophy (Ph.D.)** in the Mathematics Department, School of Chemical Engineering and Physical Sciences, is a research work carried out by Vikas Ranga, 41800749, is bonafide record of his/her original work carried out under my supervision and that no part of thesis has been submitted for any other degree, diploma or equivalent course.

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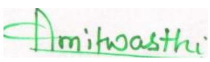

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Abstract

Leonardo of Pisa's extraordinary investigation into the Fibonacci numbers, one of God's best-gifted numbers, revealed how important they are to our daily life. Fibonacci numbers are a result of Leonardo of Pisa's famous Rabbit problem, which we will cover in more detail in the next chapters of this thesis. These numbers play an important role in our daily lives, but they also have a wide range of applications in things like music, nature, and other fields that are difficult to describe.

In Chapter 1, the entire thesis is centered on the idea of the stunning, divinely Coupled Fibonacci and Lucas sequence. This thesis is composed of six chapters.The Fibonacci numbers that make up history are discussed in general terms in the first chapter, along with some of the fields in which they are used. Additionally, we quickly review a few definitions and well-known outcomes of the Fibonacci numbers, which meet the minimal requirement for the succeeding chapters. Basic definitions of first, second, third, fourth, and fifth orders of the Multiplicative Coupled Fibonacci series as well as first, second, third, and fourth orders of the Multiplicative Triple Fibonacci sequence are discussed. This chapter also has a part on the literature review that highlights the study on connected Fibonacci sequences that has been done by various researchers. The review has indicated the area for additional investigation. The goals and methods to close these gaps have also been described in this chapter.

The subsequent chapters make an effort to explore the behavior and many characteristics of the coupled Lucas sequence and the multiplicative coupled and triple Fibonacci sequence. In this thesis, we focus primarily on triple and multiplicative coupled Fibonacci sequences. We also establish the Generalized Coupled Lucas sequence's determinantal identities. We use a variety of approaches to accomplish our goal.

Chapter 2 discusses the fifth order of Multiplicative coupled Fibonacci sequence and the results on some special Schemes under fifth order.

We worked on the Scheme

$$X_{m+5} = Y_{m+4}.Y_{m+3}.Y_{m+2}.Y_{m+1}.Y_{m}, \qquad m \geq 0$$

$$Y_{m+5} = X_{m+4}.X_{m+3}.X_{m+2}.X_{m+1}.X_{m}, \qquad m \geq 0$$

We have discovered certain identities through research into the various orders of the multiplicative coupled Fibonacci sequence, and we are currently applying mathematical induction and combinatorics to solve the theorems.

In Chapter 3, we have discovered the titan of a triple sequence of the first, second,

and third orders, and we have suggested various identities as a result.

We worked on the below scheme under 2nd order MTFS

First scheme

$Ӿ_{m+2} = Ӿ_{m+1}.Ӿ_{m}$

$Ɏ_{m+2} = Ɏ_{m+1}.Ɏ_{m}$

$Ƶ_{m+2} = Ƶ_{m+1}.Ƶ_{m}$

Second scheme

$Ӿ_{m+2} = Ɏ_{m+1}.Ɏ_{m}$

$Ɏ_{m+2} = Ƶ_{m+1}.Ƶ_{m}$

$Ƶ_{m+2} = Ӿ_{m+1}.Ӿ_{m}$

Third scheme

$Ӿ_{m+2} = Ƶ_{m+1}.Ƶ_{m}$

$Ɏ_{m+2} = Ӿ_{m+1}.Ӿ_{m}$

$Ƶ_{m+2} = Ɏ_{m+1}.Ɏ_{m}$

3rd order of MTFS

First scheme

$Ӿ_{n+3} = Ɏ_{m+2}.Ƶ_{m+1}.Ӿ_{m}$

$Ɏ_{m+3} = Ƶ_{m+2}.Ӿ_{m+1}.Ɏ_{m}$

$Ƶ_{m+3} = Ӿ_{m+2}.Ɏ_{m+1}.Ƶ_{m}$

Second scheme

$Ӿ_{m+3} = Ӿ_{m+2}.Ƶ_{m+1}.Ɏ_{m}$

$Ɏ_{m+3} = Ɏ_{m+2}.Ӿ_{m+1}.Ƶ_{m}$

$Ƶ_{m+3} = Ƶ_{m+2}.Ɏ_{m+1}.Ӿ_{m}$

Third scheme

$Ӿ_{m+3} = Ƶ_{m+2}.Ɏ_{m+1}.Ӿ_{m}$

$Ɏ_{m+3} = Ӿ_{m+2}.Ƶ_{m+1}.Ɏ_{m}$

$Ƶ_{m+3} = Ɏ_{m+2}.Ӿ_{m+1}.Ƶ_{m}$

Fourth scheme

$Ӿ_{m+3} = Ӿ_{m+2}.Ɏ_{m+1}.Ƶ_{m}$

$Ɏ_{m+3} = Ɏ_{m+2}.Ƶ_{m+1}.Ӿ_{m}$

$Ƶ_{m+3} = Ƶ_{m+2}.Ӿ_{m+1}.Ɏ_{m}$

Fifth scheme

$Ӿ_{m+3} = Ɏ_{m+2}.Ӿ_{m+1}.Ƶ_{m}$

$Ɏ_{m+3} = Ƶ_{m+2}.Ɏ_{m+1}.Ӿ_{m}$

$Ƶ_{m+3} = Ӿ_{m+2}.Ƶ_{m+1}.Ɏ_{m}$

In chapter 4, we explored the fourth-order triple sequence and proposed several related identities.

4th order of MTFS

First scheme

$Ӿ_{m+4} = Ӿ_{m+3}.Ӿ_{m+2}.Ӿ_{m+1}.Ӿ_{m}$

$Ɏ_{m+4} = Ɏ_{m+3}.Ɏ_{m+2}.Ɏ_{m+1}.Ɏ_{m}$

$Ƶ_{m+4} = Ƶ_{m+3}.Ƶ_{m+2}.Ƶ_{m+1}.Ƶ_{m}$

Second scheme

$Ӿ_{m+4} = Ɏ_{m+3}.Ɏ_{m+2}.Ɏ_{m+1}.Ɏ_{m}$

$Ɏ_{m+4} = Ƶ_{m+3}.Ƶ_{m+2}.Ƶ_{m+1}.Ƶ_{m}$

$Ƶ_{m+4} = Ӿ_{m+3}.Ӿ_{m+2}.Ӿ_{m+1}.Ӿ_{m}$

Third scheme

$Ӿ_{m+4} = Ƶ_{m+3}.Ƶ_{m+2}.Ƶ_{m+1}.Ƶ_{m}$

$Ɏ_{m+4} = Ӿ_{m+3}.Ӿ_{m+2}.Ӿ_{m+1}.Ӿ_{m}$

$Ƶ_{m+4} = Ɏ_{m+3}.Ɏ_{m+2}.Ɏ_{m+1}.Ɏ_{m}$

Fourth scheme

$Ӿ_{m+4} = Ӿ_{m+3}.Ɏ_{m+2}.Ƶ_{m+1}.Ӿ_{m}$

$Ɏ_{m+4} = Ɏ_{m+3}.Ƶ_{m+2}.Ӿ_{m+1}.Ɏ_{m}$

$Ƶ_{m+4} = Ƶ_{m+3}.Ӿ_{m+2}.Ɏ_{m+1}.Ƶ_{m}$

Fifth scheme

$Ӿ_{m+4} = Ƶ_{m+3}.Ӿ_{m+2}.Ɏ_{m+1}.Ƶ_{m}$

$Ɏ_{m+4} = Ӿ_{m+3}.Ɏ_{m+2}.Ƶ_{m+1}.Ӿ_{m}$

$Ƶ_{m+4} = Ɏ_{m+3}.Ƶ_{m+2}.Ӿ_{m+1}.Ɏ_{m}$

Sixth scheme

$Ӿ_{m+4} = Ɏ_{m+3}.Ƶ_{m+2}.Ӿ_{m+1}.Ɏ_{m}$

$Ɏ_{m+4} = Ƶ_{m+3}.Ӿ_{m+2}.Ɏ_{m+1}.Ƶ_{m}$

$Ƶ_{m+4} = Ӿ_{m+3}.Ɏ_{m+2}.Ƶ_{m+1}.Ӿ_{m}$

Seventh scheme

$Ӿ_{m+4} = Ӿ_{m+3}.Ƶ_{m+2}.Ɏ_{m+1}.Ӿ_{m}$

$Ɏ_{m+4} = Ɏ_{m+3}.Ӿ_{m+2}.Ƶ_{m+1}.Ɏ_{m}$

Eighth scheme

$Ӿ_{m+4} = Ɏ_{m+3}.Ӿ_{m+2}.Ƶ_{m+1}.Ɏ_{m}$

$Ɏ_{m+4} = Ƶ_{m+3}.Ɏ_{m+2}.Ӿ_{m+1}.Ƶ_{m}$

Ninth scheme

$Ӿ_{m+4} = Ƶ_{m+3}.Ɏ_{m+2}.Ӿ_{m+1}.Ƶ_{m}$

$Ɏ_{m+4} = Ӿ_{m+3}.Ƶ_{m+2}.Ɏ_{m+1}.Ӿ_{m}$

$$Ƶ_{m+4} = Ƶ_{m+3}.Ɏ_{m+2}.Ӿ_{m+1}.Ƶ_{m} \quad Ƶ_{m+4} = Ӿ_{m+3}.Ƶ_{m+2}.Ɏ_{m+1}.Ӿ_{m} \quad Ƶ_{m+4} = Ɏ_{m+3}.Ӿ_{m+2}.Ƶ_{m+1}.Ɏ_{m}$$

Through our research on different orders of the multiplicative triple Fibonacci sequence, we identified several identities and are now using mathematical induction and combinatorics to prove the theorems.

Chapter 5 discusses Coupled Lucas Sequence of Second order and Fibonacci Lucas Sequence's Determinantal Identities. We defined 2-L Sequences as coupled order recurrence relations for Lucas numbers and Lucas sequences.

$$Ł_{m+2} = Ɱ_{m+1} + 2Ɱ_{m},\ m \geq 0$$

$$Ɱ_{m+2} = Ł_{m+1} + 2Ł_{m},\ m \geq 0$$

$$Ł_0 = a, Ł_1 = b, Ɱ_0 = c, Ɱ_1 = d$$

The Lucas sequence is also thought to have a similar perception. The recurrence relation confirms that the Lucas sequence is genuine.

$$L_{m} = L_{m-1} + L_{m-2},\ m \geq 2 \text{ and } L_0 = 2, L_1 = 1$$

We use recurrence to illustrate the Generalized Fibonacci sequence $\{Ƀ_{m}\}_{m=0}^{\infty}$ in this area:

$$Ƀ_{m} = Ƀ_{m-1} + Ƀ_{m-2},\ m \geq 2 \text{ and } Ƀ_0 = 2b, Ƀ_1 = s$$

b and s must both be non-negative integers.

One of the key components of number theory, recurrence relations draw attention from researchers not just in Mathematics but also in other disciplines such as physics, economics, and a wide range of computer science applications. There are many different forms of recurrence relations sequences in higher Mathematics. The Fibonacci sequence of numbers, the Lucas numbers, the Chebyshev polynomial sequences, and the Pell numbers are some unique sorts of recurrence formula sequence with outlined in simple terms.According to renowned theorist Carl Friedrich Gauss, number theory is the queen of mathematical studies, and Mathematics as a field of study and a branch of science is the queen of all science. Studying numerology is based on looking at integer and rational number features that go beyond simple mathematical operations. Relationships between recurring subjects are used in both Mathematics and Economics. The convergence of the series of recurrences is significantly impacted by the recovery coefficient.

Acknowledgment

I want to start by thanking God and my parents, who are the source of all wisdom, for everything. They have served as my lifelong mentors and advocates, which has given me the courage and joy to successfully overcome every obstacle I have encountered.

I would like to express my sincere gratitude to my supervisor, Prof.(Dr.) A. K. Awasthi, for their constant support, valuable feedback, and expert guidance throughout this research journey. I sincerely appreciate the help with my research that the head of the department and the entire maths faculty at Lovely Professional University in Phagwara provided.

Special thanks to my dear friends Ms. Ekta Singal, Dr. Bhagwan Dass, Dr. Navneet Kaur, Dr. Navdeep Kaur, and Dr. Arun for their encouragement, late-night discussions, and unwavering motivation during challenging times. Your presence made this journey enjoyable and memorable.

This thesis would not have been possible without the support and inspiration of each one of you.

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$F_{\mathbb{m}}$	:	$\mathbb{m}^{th}$ Fibonacci number
$L_{\mathbb{m}}$	:	$\mathbb{m}^{th}$ Lucas number
Φ	:	Golden Ratio
$[.]$	:	Greatest integer
det $[A]$	:	determinant of a matrix A
$\sum .$	:	Summation
$\prod .$	:	Product
$\mathfrak{f}_{\mathbb{m}}(\mathfrak{x})$	:	$\mathbb{m}^{th}$term of Fibonacci polynomial
$\mathbb{L}_{\mathbb{m}}(\mathfrak{x})$	:	$\mathbb{m}^{th}$ term of Lucas polynomial
$\binom{\omega}{k}$	:	Binomial coefficient

Chapter - 1

General Introduction

1.1 Introduction

In the past, figures and numbers were the two things that sparked people's curiosity. Numerous mathematicians have been drawn to the field of number theory over the years because of its clarity, intellectual rigour, and beauty of presentation. Since the beginning of time, the study of number sequences has drawn the attention of numerous number theorists. Many ideas have uses in various Mathematical fields in the world of Mathematics. If these essential notions were missing, the various subfields of Mathematics would all appear disjointed and unrelated to the topics covered in other areas of Mathematics. An individual by the name of Leonardo Pisano made one of these findings back in the early part of the 13th century. According to O'Connor and Robertson [1], Leonardo Pisano was born into a family of merchants in the year 1175 A.D. in the city of Pisa, which is located in Italy. He is better known by the name Fibonacci, which was given to him. His father, Guglielmo Bonacci, was an ambassador, and he received the most of his education outside of Italy while he was stationed in North Africa. It is believed that Fibonacci's father served as a representative for the Pisan merchants in the city of Bugia, which is located in the northwestern region of Algeria on the Mediterranean coast. During his time in Bugia, Fibonacci acquired his formal education in the field of Mathematics. During his teenage years, Fibonacci spent a lot of time in the Mediterranean with his father, which broadened his worldview and increased his appreciation for the region's many diverse civilizations. The journeys that Fibonacci took around the Mediterranean fueled his passion for Mathematics by exposing him to innovative mathematical ideas and concepts that were prevalent in a number of different countries. In the Middle Ages, a renowned mathematician was Leonardo of Pisa (1170–1250), also known as Fibonacci. Fibonacci is best known for his "Fibonacci numbers," which bear his name. Fibonacci's exposure to the earliest works of algebra, arithmetic, and geometry occurred during his frequent trips to North Africa. He also visited Mediterranean nations and researched the mathematical practices that were being used there.

Mathematicians have long been fascinated by the Fibonacci sequence (FS). The FS is the name given to this particular idea in modern times. The FS has developed as one of the most exciting notions in all of Mathematics as a result of its astonishing features, practical applications in a number of different areas of Mathematics, including geometry, discrete Mathematics, and number theory, as well as indisputable evidence of the magnificent creation that God has made. The generalized FS is utilized in a wide variety of fields, some of which are computer algorithms, encryption, optical networks, probability theory, and many more. The general sequences of Second order are the subject of numerous literary studies. Falcon, Sergio, Angel Plaza, Posamentier Alfred S., Ingmar Lehmann and T. Koshy [2], [3], and [4]instance, the Lucas, Jacobsthal, and k-FS .The generalized FS is a generalization of the FS that is created by altering either the initial condition or the recurrence relation, or both.

1.2 The Book Liber Abaci

When Fibonacci [5] went back to his birthplace in 1200 A.D., his most renowned mathematical work, Liber Abaci, which literally translates to "The Book of Calculations, " was already a well-known publication. Many of the mathematical concepts that Fibonacci encountered while exploring the Mediterranean are found in his Liber Abaci.The first edition of Liber Abaci appeared in 1202 and was later revised in 1228". It's said that Fibonacci's work in Liber Abaci was influenced by that of Egyptian mathematician Abu Kamil.

The first lines of Liber Abaci by Fibonacci begin, "The Indians' nine figures are as follows: 9 8 7 6 5 4 3 2 1. Any number can be written using these nine figures and the Zephyr, or zero in Arabic, as will be demonstrated below. For the first time, the book's puzzles were able to demonstrate the advantages of the new Hindu-Arabic numeral system. Liber Abaci was regarded as a complete source of mathematical knowledge during the time of Fibonacci. This book's publication sparked further study in algebra and Mathematics, and it remained a crucial resource for hundreds of years. A number is made up of units, and as they are added, the number grows indefinitely. The numbers, which range from one to ten, are first composed. Second, the numbers from ten to one hundred are created from the tens. Third, the numbers that range from 100 to 1000 are created from the hundreds. As a consequence of this, by following an infinite sequence of steps, any number can be produced by combining the numbers that came before it. The first spot in the written representation of the numbers is to the right. The Second one is the one that comes after the previous one to the left. The adoption of these Hindu-Arabic numerals brought about a permanent change in the Mathematics of the western world. "Now we turn our attention to Indian mathematicians and the part they played in the development of the Fibonacci numbers. Although Leonardo

Fibonacci, who was previously mentioned in detail, is the name-bearer of the Fibonacci numbers, it's intriguing to note that these numbers were known much before his time. Fibonacci numbers have their roots in ancient India. Singh asserts that the first person to be familiar with the Fibonacci numbers was the Indian mathematician Pingala. He is thought to have lived somewhere around 400 B.C. Gopala, who was born about 1135 A.D., is thought to have been the first Indian mathematician to record the Fibonacci numbers in writing. Acharya Virashanka, who flourished between 600 and 800 A.D. and plays a significant role in the Fibonacci numbers, is another notable figure in this field.

Figure 1.1: Leonardo Fibonacci [77]

In addition to the Book Liber Abaci, Fibonacci wrote three other important books.

(1) In 1220, Practica Geometriae (Practice of Geometry) was published. Fibonacci used algebra to solve geometric problems and geometry to solve algebraic problems in the eight chapters that make up this book.

(2) The 1225 publication Flos (Blossom or Flower) discusses number theory.

(3) Number theory is covered in the 1225 publication Liber Quadratorum (The Book of Square Numbers). It only addresses Second-degree Diophantine problems. Liber Quadratorum is thought to have made the greatest contribution to number theory during the Latin Middle Ages before the work of Bachet and Fermat. Fibonacci's status as a significant number theorist was established by Liber Quadratorum. Between the French mathematician Pierre de Fermat and the Greek mathematician Diophantine (circa 250 AD), he was ranked Third (1601-1665).

Al-Khwarizmi and Abu-Kamil, two Persian mathematicians, made significant contributions to algebra, which are covered in fifteen chapters in liber Abaci (ca.900). The majority of scholars of his time could not compare to the brilliance and originality of Fibonacci. The qualities of Fibonacci are demonstrated by the works floss and liber quadratorum. The Second edition of Liber Abaci, which Fibonacci revised in 1228 and dedicated to Michael Scott, the most well-known philosopher and astrologer at Frederick II's court, bears his name. During his time at Frederick II's court as the Roman emperor (1194–1250), Fibonacci engaged in scientific discussions with philosophers.

1.3 The FS with Rabbit Problem

Fibonacci defined the FS as the following made-up situation in his work Liber Abaci, which was first published in the year 1202.A man put one pair of rabbits in a space that was totally surrounded by a wall on all sides. If the characteristics of these rabbits are such that each pair gives birth to a new pair every month, and that new pair starts producing offspring from the Second month onward, then how many new pairs of rabbits can one pair of rabbits generate in a single year.

As Fibonacci began to look into this particular problem, he discovered a sequence that involved the number of rabbits that were paired together. The problem is being caused by a pair of juvenile bunnies. After the first month has passed, the first pair of newborn bunnies will have matured and be ready to breed after they have reached this point. Assuming that a rabbit has a gestation period of one month on average, the first pair of rabbits will have another litter of rabbits at the beginning of the Third month, making the total number of rabbits born three. At this point in time, there are four rabbits total: two adult rabbits, two baby bunnies, and a pair of adult rabbits. In his calculation, Fibonacci uses the premise that after a couple of rabbits reach adulthood, they reproduce once a month on average. This is the starting point for his equation. At the beginning of the fourth month, the present pair of baby rabbits in the problem are able to reproduce, and beyond that point, they conceive a pair of baby bunnies every month. This continues until the problem is resolved. In order to guarantee the continuity of his work and ensure that none of the rabbits die, Fibonacci sets the additional premise that none of them do.
Fibonacci (in the year 1202) presented the number of rabbit pairs that could occur under ideal conditions as the real problem.

- Start off with two neonate bunnies.

- before maturation, one month
- one month before giving birth
- imitate two newborn
- once more, intimate, and so on
- No rabbit perishes.

After completing each month of the inquiry, he eventually arrived at a series of numbers that contained the number of rabbit pairings as the terms and the corresponding month numbers as the subscripts for those terms. His results lead him to this sequence of numbers eventually. Fibonacci rabbit is an illustration of two bunnies. The images of the smaller rabbits are new born, while those of the larger rabbits are adults who have been around for at least a month.

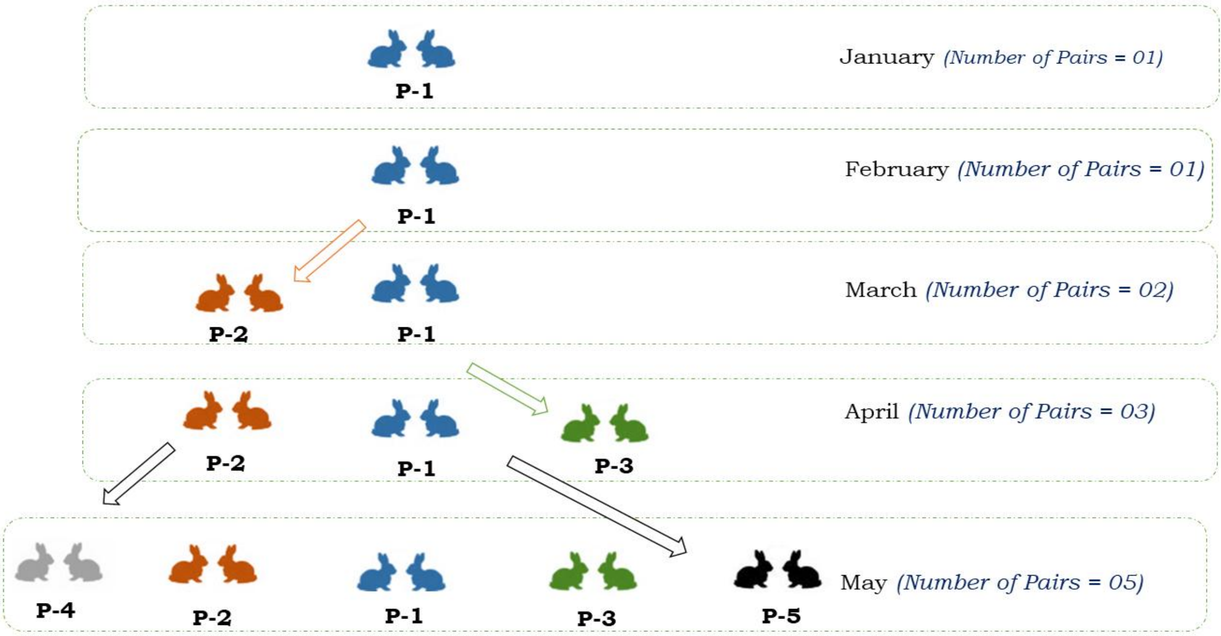


Figure 1.2: Fibonacci's Hypothetical Rabbit Problem [77]

In Figure 1.2, A pair of rabbits (one male and one female) is placed in a field. Rabbits reach reproductive maturity after one month, and each mature pair produces another pair (one male and one female) every month. Rabbits never die, and each new pair follows the same reproduction pattern.

Table 1.1: Growth of Rabbit Colony

Month	Youth Pair	Matured Pair	Total
1	1	0	1
2	0	1	1
3	1	1	2
4	1	2	3
5	2	3	5
6	3	5	8
7	5	8	13
8	8	13	21
9	13	21	34
10	21	34	55
11	34	55	89
12	55	89	144
13	89	144	233

1.4 The Fibonacci Sequence

Fibonacci made numerous contributions to Mathematics, but his most well-known achievement is the FS. Each number in the sequence, which is a recurrence relation, is the sum of the two numbers that came before it.

Fibonacci figures produced by,

$$F_{m+1} = F_m + F_{m-1} \text{ for } m = 1,2,3,... \quad (1.4.1)$$

starting with two seeds $F_0 = 0, F_1 = 1$

Figure 1.3 Fibonacci Spiral Aloe [77]

Grigas [6] highlights Fibonacci sequence's natural and historical significance. Burton's textbook [7] provides foundational concepts in number theory, including Fibonacci.Particular focus has been placed on the existence of Fibonacci numbers in pine cones by Cook [8]. Two sets of spirals one going clockwise and the other going counterclockwise can be seen from the top view of a pine cone, as seen in Figure 1.3. It has been observed that the patterns defined by the Fibonacci sequence Cook [8] are followed by the arrangements of some plant's leaves, some flower's petals, and other objects, as seen in figures 1.4 and 1.5. The Fibonacci sequence is produced exactly when the entries in Pascal's triangle are added together after a diagonal, as seen in figure 1.6. Chris [9] discusses factors influencing recurrence relations in mathematical sequences.

Blaise Pascal (1623–1662) is credited with creating this triangle by mathematicians. A ratio of two Fibonacci numbers usually invariably characterizes the ratio of the numbers in each pair. On the stems of many plants, the arrangement of their leaf's forms Fibonacci helices, which are based on minuscule Fibonacci numbers. The shape of some sea shells and snail shells is a natural example of the Fibonacci spiral, which is also connected to the Fibonacci sequence.

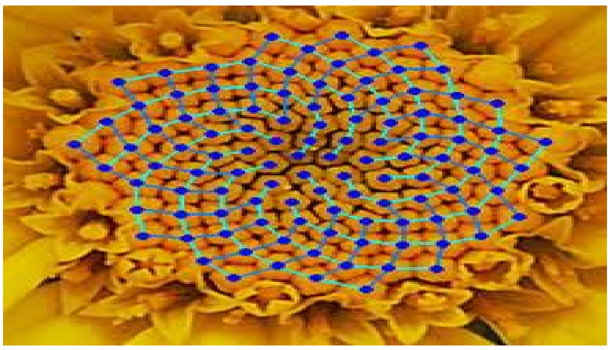


Figure 1.4 Fibonacci spiral pattern in a sunflower head [77]

Figure 1.5 Fibonacci Spiral Pattern in a Nature [77]

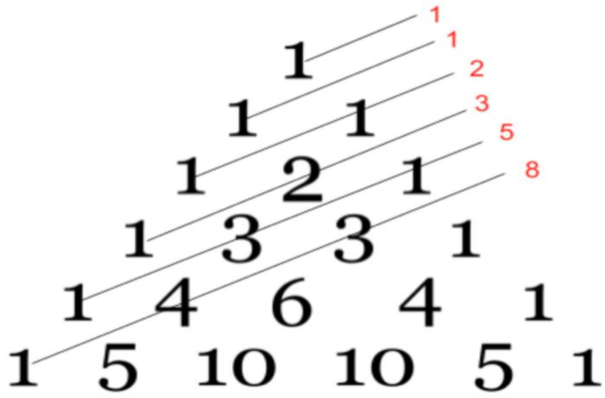


Figure 1.6 Fibonacci Triangle [77]

Other notable mathematicians who made significant contributions to the Fibonacci numbers include Jean- Dominique Cassini (1625-1712), Robert Simson (1687-1768), Jacques Binet (1786-1856), Gabriel Lame (1795-1870), Eugene Catalan (1814-1894), and Steven Vajda (1901-1995).

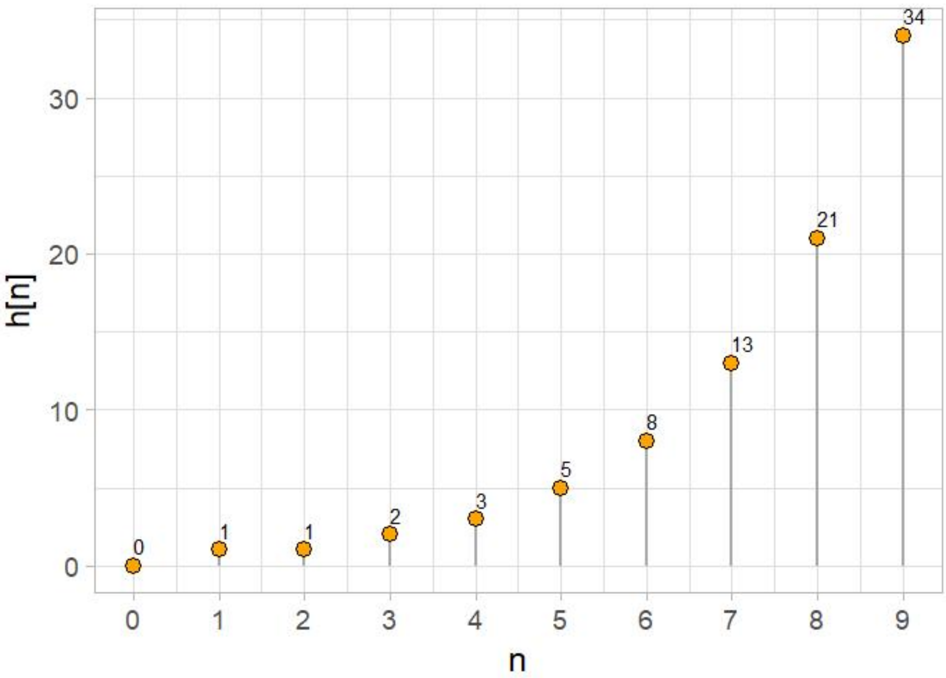


Figure 1.7: Fibonacci Numbers [77]

Additionally, the generating function $g_F(\mathfrak{x})$ for Fibonacci numbers is as follows:

$$\sum_{\mathfrak{m}=0}^{\infty} F_{\mathfrak{m}}\mathfrak{x}^{\mathfrak{m}} = g_F(\mathfrak{x}) = \frac{\mathfrak{x}}{1-\mathfrak{x}-\mathfrak{x}^2}$$

1.4.1 Binet's Formula for Fibonacci Numbers:

Jacques Philippe Marie Binet (1786–1856), a French mathematician, created a definition for the Fibonacci numbers in 1843. To figure out the $\mathbb{m}^{th}$ Fibonacci number, the Binet's formula [10] is used as:

$$F_{\mathbb{m}} = \frac{X^{\mathbb{m}} - Y^{\mathbb{m}}}{X - Y}\,;\; X = \frac{1+\sqrt{5}}{2} \text{ and } Y = \frac{1-\sqrt{5}}{2}$$

1.5 Fibonacci Number and Golden Ratio

If the ratio between two amounts is equal to the ratio between the larger of the two amounts, then the two amounts are in the golden ratio.

$$\frac{a+b}{a} = \frac{a}{b} = \Phi \tag{1.5.1}$$

Dunlap [11] explains connections between Fibonacci numbers and the golden ratio., denoted by the Greek letter phi (Φ). It's worth is:

$$\Phi = \frac{1+\sqrt{5}}{2} = 1.61803398$$

The left fraction can be used as a starting point to determine the value of Φ.

$$\frac{a}{b} = \Phi \text{ and } \frac{b}{a} = \frac{1}{\Phi}$$

Then,

$$\frac{a+b}{a} = 1 + \frac{b}{a} = 1 + \frac{1}{\Phi}$$

By equation 1.5.1, we get

$$1 + \frac{1}{\Phi} = \Phi$$

Multiplying both side by Φ

$$\Phi + 1 = \Phi^2$$

$$\Phi^2 - \Phi - 1 = 0$$

There are two answers that can be found using the quadratic formula:

$$\Phi = \frac{1+\sqrt{5}}{2} = 1.61803398$$

and

$$\Phi = \frac{1-\sqrt{5}}{2} =- 0.61803398$$

The most aesthetically pleasing rectangles are known as golden rectangles, which can be created using the golden ratio. These rectangles are unique because the length to width ratio is the golden ratio.

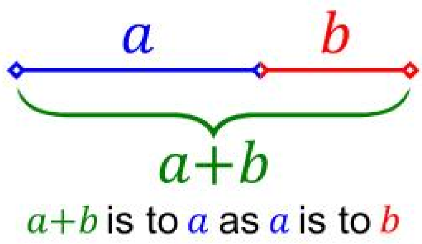


Figure 1.8: Golden Rectangle [79]

A golden rectangle is created by placing a square with side length b adjacent to a rectangle with longer side a and shorter side b. The resulting figure is another golden rectangle, where the new longer side is a + b and the shorter side remains a.

This illustrates the relationship.

$$\frac{a+b}{a} = \frac{a}{b} = \Phi$$

The sequence obtained approaches Φ by dividing the ratio of two consecutive Fibonacci numbers by their smaller counterparts.

Thus,

$$\lim_{m \to \infty} \frac{F_{m+1}}{F_m} = \Phi$$

1 /1 = 1, 2/ 1 = 2, 3/ 2 = 1.5, 5 / 3 = 1.666,

8/ 5 = 1.6, 13 /8 = 1.625, 21/13 = 1.61538……

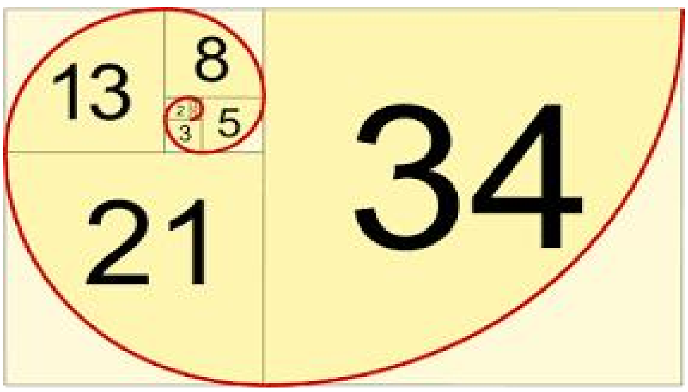


Figure 1.9: Fibonacci Spiral [77]

You can see from the graph of this information how they seem to be approaching a threshold, as illustrated in the picture below.

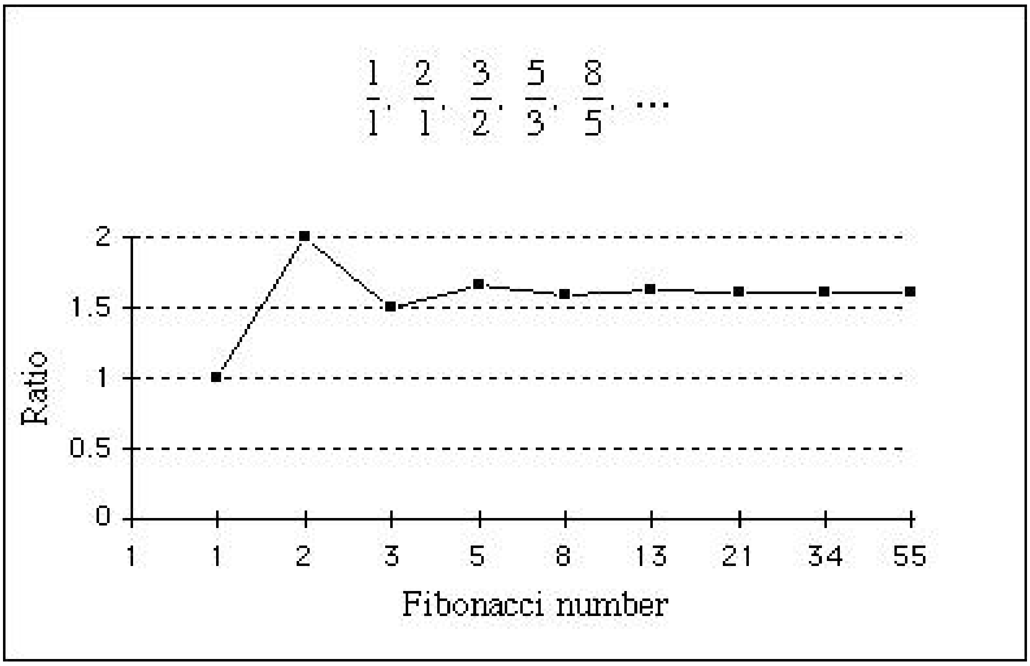


Figure 1.10: Fibonacci Numbers Approaching to Golden Ratio [77]

1.6 Application of Fibonacci Sequence

In addition to Mathematics, Fibonacci numbers play a significant role in nature, daily life, and a wide range of other fields. There are numerous flowers whose patella's display a Fibonacci number sequence. Lilies, for example, have three petals, and buttercups have five, while chicory has 34 petals, plantains, daisies, and asters have 8, 13, and 21 petals each, and delphiniums, daisies, and pyrethrum have five.

Additionally, some flowers have spiral patterns that, whether counted clockwise or anticlockwise, are Fibonacci numbers. Popular topics for mathematical enrichment and popularization include the Fibonacci numbers. They are well known for a variety of intriguing and unexpected qualities, and appear in textbooks, articles in magazines, and websites. Garland [12] explores Fibonacci numbers' patterns, mysteries, and mathematical magic. It would be simple to conclude that they are a singular and unique phenomenon based on all of this attention.

A wide range of numerical sequences identified by the Second-order linear recurrences communicate the majority of the qualities of the Fibonacci numbers. Some of the facts pertaining to Fibonacci numbers were found and described in the nineteenth century by Lucas and his contemporaries, who were well-aware of this. Numerous references analyze a number of remarkable Fibonacci features and analysis about the different types of sequences. Numerous mathematical puzzles contain the Fibonacci and Lucas numbers. A data structure known as a Fibonacci bunch serves as the foundation for many quick algorithms in the field of computer science that manage graphs. Both computer science and the counting of mathematical objects such as sets, permutations, and sequences are two areas in which the Fibonacci numbers are utilized.

They are studied as part of number theory. Regarding the repetition of these thin pastry leaves in the same alignment, see (filo pastry). Extensive research has been conducted on the Fibonacci series in three distinct spiral configurations, and it has been observed in phylotaxis. The Fibonacci numbers also show up in the natural world. Fibonacci numbers or patterns can be seen in a variety of things, including seashells, flower petals, sunflower seed heads, pine cones, palm trees, pineapples, and more. 90% of plants have different leaf/petal arrangements or bromeliads.

There are numerous other places where the Fibonacci numbers can be found. There are hints in the field of physics that the golden ratio and the Fibonacci numbers have something to do with both the arrangement of the planets in the solar system and the structure of atoms.

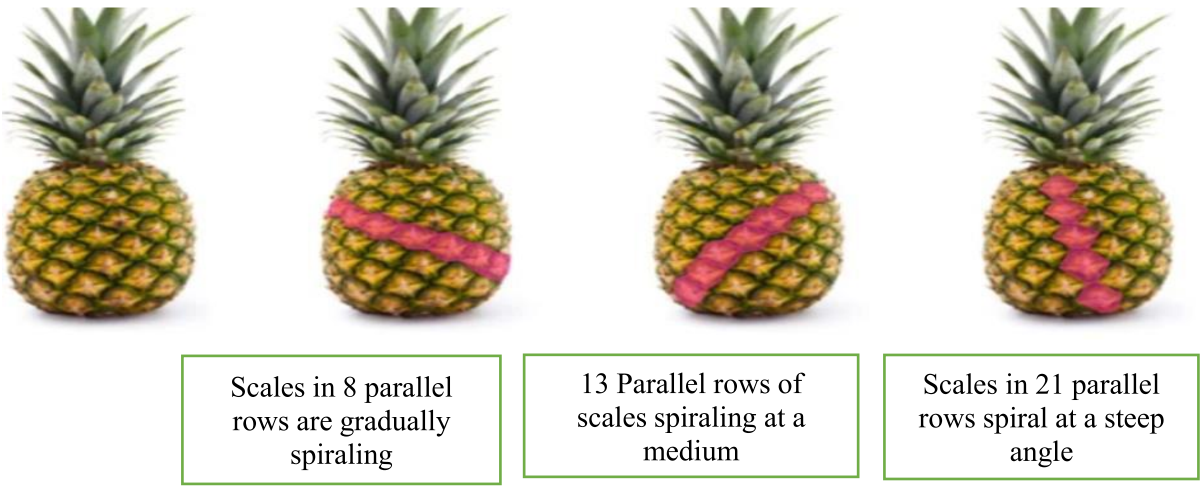


Figure 1.11: Fibonacci number in Pineapple [80]

Key moments in human ageing and development are denoted by Fibonacci numbers. The FS can be found at the core of all things beautiful, artistic, and meaningful in life. Even the music in the series has a basis. Timing in musical compositions frequently exhibits Fibonacci and phi (Φ) relationships. Fibonacci and phi (Φ) are used in the design of violins as well as in the manufacture of speaker wire of the highest calibre.

1.7 Fibonacci Polynomials

A set of Fibonacci polynomials are produced by the Q matrix, as demonstrated by Basin, S. L. [13]. He uses the matrix method to derive the explicit forms and the generating function. Richard A. Hayes [14] also uses the matrix method to derive a number of identities. M. N. S. Swami [15] and The Fibonacci polynomials were defined almost simultaneously by Hoggatt, V. E., jr. [16]. In 1883, Belgian mathematician Eugene Charles Catalan used the Fibonacci polynomials $\{F_\omega(\mathfrak{x})\}_{\omega\geq 0}$[20] to develop the idea of the Fibonacci numbers.

The recurrence relation serves as the basis for the Fibonacci polynomials.

$$F_{\omega+1}(\mathfrak{x}) = \mathfrak{x}F_\omega(\mathfrak{x}) + F_{\omega-1}(\mathfrak{x}) \tag{1.7.1}$$

with $F_1(\mathfrak{x}) = 1$ and $F_2(\mathfrak{x}) = \mathfrak{x}$for integral values $\omega \geq 2$.

For (1.7.1), the explicit sum formula is provided by

$$F_\omega(\mathfrak{x}) = \sum_{k=0}^{\left[\frac{\omega-1}{2}\right]} \binom{\omega-k-1}{k} \mathfrak{x}^{\omega-1-2k} \tag{1.7.2}$$

Where $[\mathfrak{x}]$ is defined as the largest integer, and $\binom{\omega}{k}$ is a binomial coefficient.

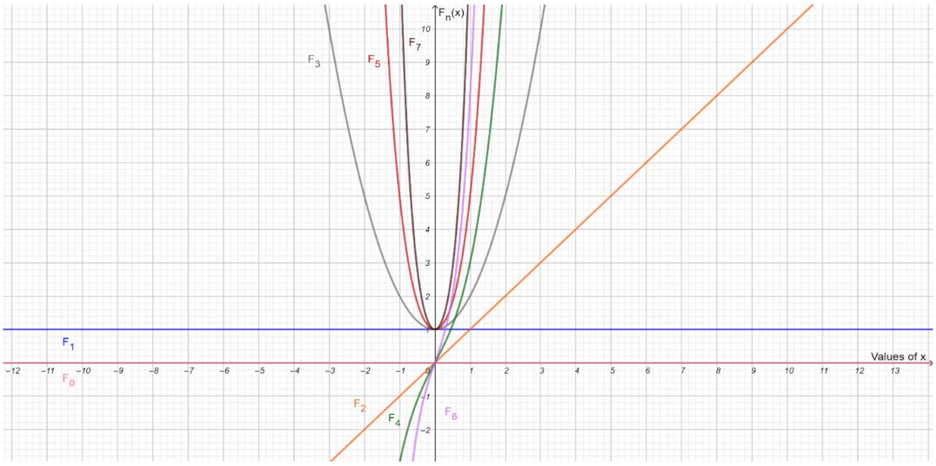


Figure 1.12: Fibonacci Polynomial

Koshy[17] presents Fibonacci and Lucas numbers with real-world applications.The recurrence relation defines the Lucas polynomials.

$$\mathbb{L}_{\omega+1}(\mathfrak{x}) = \mathfrak{x}\mathbb{L}_\omega(\mathfrak{x}) + \mathbb{L}_{\omega-1}(\mathfrak{x}) \tag{1.7.3}$$

with $\mathbb{L}_0(\mathfrak{x}) = 2$ and $\mathbb{L}_1(\mathfrak{x}) = \mathfrak{x}$for integral values $\omega \geq 2$.

For (1.7.3), the explicit sum formula is provided by

$$\mathbb{L}_\omega(\mathfrak{x}) = \sum_{k=0}^{\left[\frac{\omega}{2}\right]} \left(\frac{\omega}{\omega-k}\right) \binom{\omega-k}{k} \mathfrak{x}^{\omega-2k} \tag{1.7.4}$$

Where $[\mathfrak{x}]$ is defined as the largest integer, and $\binom{\omega}{k}$ is a binomial coefficient.

1.8 Generalized Fibonacci Polynomials

By extending the h(x) Fibonacci polynomials, Cohen and Niven [18] explore properties and applications of generalized Fibonacci polynomials. Researchers developed the following Generalized Fibonacci polynomials:

$$\mathfrak{f}_{\mathfrak{k},\yen,\mathfrak{m}+1}(\mathfrak{g}) = \mathfrak{k}(\mathfrak{g})\mathfrak{f}_{\mathfrak{k},\yen,\mathfrak{m}}(\mathfrak{g}) + \yen(\mathfrak{g})\mathfrak{f}_{\mathfrak{k},\yen,\mathfrak{m}-1}(\mathfrak{g}) \text{ for } \mathfrak{m} = 2,3, \ldots\ldots \quad (1.8.1)$$

$\mathfrak{k}(g)$ and $\yen(g)$ are real coefficient polynomials with $\mathfrak{f}_{\mathfrak{k},\yen,0}(\mathfrak{g}) = 0$ and $\mathfrak{f}_{\mathfrak{k},\yen,1}(\mathfrak{g}) = 1$. The sequence given by (1.8.1) becomes the Fibonacci number sequence if $\mathfrak{k}(\mathfrak{g}) = \yen(\mathfrak{g}) = 1$.
Further,

$$\sum_{\mathfrak{m}=0}^{\infty} \mathfrak{f}_{\mathfrak{k},\yen,\mathfrak{m}}(\mathfrak{g})\, \mathfrak{u}^{\mathfrak{m}} = \mathfrak{g}_{\mathfrak{f}}\ (\mathfrak{u}) = \frac{\mathfrak{u}}{1-\mathfrak{k}(\mathfrak{g})\mathfrak{u} - \yen(\mathfrak{g})\mathfrak{u}^2}$$

1.9 Fibonacci Polynomial's Properties

Liu and Zhang [19] discuss properties of Fibonacci polynomials in number theory. Here are some Fibonacci polynomial's well-known characteristics.

(i) Sum Formula

$$\sum_{\kappa=1}^{\mathfrak{m}+1} \mathfrak{x}^{\mathfrak{m}-\kappa+1}\, \mathfrak{f}_{\kappa}(\mathfrak{x}) = \mathfrak{x}^{\mathfrak{m}}\mathfrak{f}_1(\mathfrak{x}) + \mathfrak{x}^{\mathfrak{m}-1}\mathfrak{f}_2(\mathfrak{x}) + \ldots + \mathfrak{f}_{\mathfrak{m}+1}(\mathfrak{x}) = \mathfrak{f}_{\mathfrak{m}+3}(\mathfrak{x}) - \mathfrak{x}^{\mathfrak{m}+1}\mathfrak{f}_2(\mathfrak{x}) \quad (1.9.1)$$

(ii) Sum of odd terms

$$\sum_{\kappa=1}^{\mathfrak{m}} \mathfrak{f}_{2\kappa-1}(\mathfrak{x}) = \mathfrak{f}_1(\mathfrak{x}) + \mathfrak{f}_3(\mathfrak{x}) + \ldots + \mathfrak{f}_{2\mathfrak{m}-1}(\mathfrak{x}) = \frac{\mathfrak{f}_{2\mathfrak{m}}(\mathfrak{x})}{\mathfrak{x}} \quad (1.9.2)$$

(iii) Sum of even terms

$$\sum_{\kappa=1}^{\mathfrak{m}} \mathfrak{f}_{2\kappa}(\mathfrak{x}) = \mathfrak{f}_2(\mathfrak{x}) + \mathfrak{f}_4(\mathfrak{x}) + \ldots + \mathfrak{f}_{2\mathfrak{m}}(\mathfrak{x}) = \frac{\mathfrak{f}_{2\mathfrak{m}+1}(\mathfrak{x})-1}{\mathfrak{x}} \quad (1.9.3)$$

(iv) Two consecutive Fibonacci polynomial's sum of squares

$$\mathfrak{f}_{\mathfrak{m}}^2(\mathfrak{x}) + \mathfrak{f}_{\mathfrak{m}+1}^2(\mathfrak{x}) = \mathfrak{f}_{2\mathfrak{m}+1}(\mathfrak{x}) \quad (1.9.4)$$

(v) Squares of two different Fibonacci polynomial's differences

$$\mathfrak{f}_{\mathfrak{m}+2}^2(\mathfrak{x}) - \mathfrak{f}_{\mathfrak{m}}^2(\mathfrak{x}) = \mathfrak{x}\mathfrak{f}_{2\mathfrak{m}+2}(\mathfrak{x}) \quad (1.9.5)$$

(vi) Identity of Catalan

$$\mathfrak{f}_{\mathfrak{m}}^2(\mathfrak{x}) - \mathfrak{f}_{\mathfrak{m}+\mathfrak{r}}(\mathfrak{x})\mathfrak{f}_{\mathfrak{m}-\mathfrak{r}}(\mathfrak{x}) = (-1)^{\mathfrak{m}-\mathfrak{r}}\mathfrak{f}_{\mathfrak{r}}^2(\mathfrak{x}) \quad (1.9.6)$$

(vii) Who is D'Ocagne?

$$\mathfrak{f}_{\mathfrak{mm}+1}(\mathfrak{x})\mathfrak{f}_{\mathfrak{m}}(\mathfrak{x}) - \mathfrak{f}_{\mathfrak{mm}}(\mathfrak{x})\mathfrak{f}_{\mathfrak{m}+1}(\mathfrak{x}) = (-1)^{\mathfrak{mm}}\mathfrak{f}_{\mathfrak{m}-\mathfrak{mm}}(\mathfrak{x}) \quad (1.9.7)$$

(viii) Identity of Cassini

$$f_{m+1}(x)f_{m-1}(x) - f_m^2(x) = (-1)^m \tag{1.9.8}$$

1.10 Generalized Fibonacci Sequence

The FS has been studied and generalized by numerous authors.Horadam [20] first described and investigated the characteristics of a generalized Fibonacci sequence $\{\mathbb{H}_m\}$, which he defined as the recurrence relation:

$$\mathbb{H}_{m+2} = \mathbb{H}_{m+1} + \mathbb{H}_m, \mathbb{H}_0 = q \text{ and } \mathbb{H}_1 = p, m \geq 0 \tag{1.10.1}$$

where p and q are two arbitrary integers.

1.11 Generalized Fibonacci-Type Sequence

Many researchers have extended Fibonacci-type sequences and studied their various properties. Singh et al. [21] introduced a Fibonacci-like sequence $\{S_m\}$ defined by a specific recurrence relation

$$S_{m+2} = S_{m+1} + S_m, S_0 = 2 \text{ and } S_1 = 2, m \geq 0 \tag{1.11.1}$$

By using the recurrence relation, Badshah et al. [25] defined Generalized Fibonacci-Like sequence $\{M_m\}$.

$$M_{m+2} = M_{m+1} + M_m, M_0 = 2m \text{ and } M_1 = 1 + m, m \geq 0 \tag{1.11.2}$$

m is a constant positive integer.

1.12 Generalized Fibonacci Polynomials

The properties of Fibonacci polynomials have been studied by numerous authors who have generalized them. Lucas and Fibonacci polynomials were first introduced by Swamy [22]. The generalized Fibonacci polynomials are described as follows:

$$\mathbb{U}_m(x,y) = x\mathbb{U}_{m-1}(x,y) + y\mathbb{U}_{m-2}(x,y), m \geq 2, \mathbb{U}_0(x,y) = 0 \text{ and } \mathbb{U}_1(x,y) = 1 \tag{1.12.1}$$

Generalized Fibonacci-Type polynomials were defined by Singh, et al. [26].

$$\mathbb{V}_m(x) = px\mathbb{V}_{m-1}(x) + q\mathbb{V}_{m-2}(x), m \geq 2, \mathbb{V}_0(x) = a \text{ and } \mathbb{V}_1(x) = bx \tag{1.12.2}$$

Where p, q, a and b represent integers.

1.13 Coupled Fibonacci Sequence

In all of Mathematics, the FS is undoubtedly one of the most well-known and frequently discussed number sequences. Koken and Bozkurth [23] explore applications of coupled Fibonacci sequences in Mathematics.Using initial conditions $F_0 = 0$ &$F_1 = 1$, the FS has been described by the recurrence relation $F_m = F_{m-1} + F_{m-2}$, $m \geq 2$. wherein each succeeding filial generation is viewed as being composed of the two preceding generations. By using a pair of sequences, $\{X_i\}_{i=0}^{\infty}$ and $\{Y_i\}_{i=0}^{\infty}$ which can be generated by the well-known Fibonacci formula, Attanasov offered a fresh perspective on generalized FS in 1985. He offered four different strategies for producing connected FS.

$$X_0 = a, Y_0 = b, X_1 = c, Y_1 = d$$

First Scheme:

$$X_{m+2} = Y_{m+1} + Y_m, \qquad m \geq 0 \tag{1.13.1}$$
$$Y_{m+2} = X_{m+1} + X_m, \qquad m \geq 0$$

Second Scheme:

$$X_{m+2} = X_{m+1} + Y_m, \qquad m \geq 0 \tag{1.13.2}$$
$$Y_{m+2} = Y_{m+1} + X_m, \qquad m \geq 0$$

Third Scheme:

$$X_{m+2} = Y_{m+1} + X_m, \qquad m \geq 0 \tag{1.13.3}$$
$$Y_{m+2} = X_{m+1} + Y_m, \qquad m \geq 0$$

Fourth Scheme:

$$X_{m+2} = X_{m+1} + X_m, \qquad m \geq 0 \tag{1.13.4}$$
$$Y_{m+2} = Y_{m+1} + Y_m, \qquad m \geq 0$$

1.14 Generalized Coupled Fibonacci Sequences

A new class of generalized CFS was introduced by K. T. Atanassov.Ali and Kumar [24] discuss properties of generalized coupled Fibonacci sequences.Let there be two infinite sequences with initial conditions, $\{X_i\}_{i=0}^{\infty}$ and $\{Y_i\}_{i=0}^{\infty}$.

$$X_0 = a, Y_0 = b, X_1 = c, Y_1 = d$$

then definition of generalized coupled Fibonacci sequences is as follows:

$$X_m = pX_{m-1} + qX_{m-2}, \qquad m \geq 2 \tag{1.14.1}$$
$$Y_m = rY_{m-1} + sY_{m-2}, \qquad m \geq 2$$

1.15 Multiplicative Fibonacci Sequence

An intriguing twist on the Fibonacci sequence is that a new term is created by multiplying the two terms that came before it. The Multiplicative Fibonacci sequence, according to P. Glaister [25], is comprised of

$$F_{m+1} = F_m F_{m-1} \qquad \text{for } m \geq 0 \text{ and } F_0 = 1, F_1 = 2 \tag{1.15.1}$$

1, 2, 2, 4, 8, 32, and 256 are the sequence's few terms. This is identical to a series of powers of two, and the indexes are traditional Fibonacci numbers.

It is possible to write the recurrence relation (1.15.1) as

$$F_{m+2} = 2^{F_{m-1}} \text{ for } m \geq 2 \text{ and } F_0 = 1, F_1 = 1 \tag{1.15.2}$$

P. Hope generalized the multiplicative Fibonacci sequence [28] as

$$x_{m+2} = x_{m+1} x_m, \text{ for } m \geq 0 \text{ and } x_0 = a, x_1 = b. \tag{1.15.3}$$

with actual numbers a and b.One way to spell it is as

$$x_m = a^{F_{m-1}} b^{F_m} \text{ for } m \geq 1 \tag{1.15.4}$$

When there are multiple sequences, a multiplicative pattern might be employed.

1.16 Multiplicative Coupled Fibonacci Sequence of 2nd order:

Four distinct multiplicative techniques for connected Fibonacci sequences are announced by K. T. Atanassov [26, 27]. Let a, b, c and d be four randomly chosen real integers and $\{X_i\}_{i=0}^{\infty}$ and $\{Y_i\}_{i=0}^{\infty}$ be two infinite sequences.

The following are four various multiplicative Schemes for 2-FS:

$$X_0 = a, Y_0 = b, X_1 = c, Y_1 = d$$

First Scheme:

$$X_{m+2} = Y_{m+2}.Y_m, \qquad m \geq 0 \tag{1.16.1}$$

$$Y_{m+2} = X_{m+1}.X_m, \qquad m \geq 0$$

Second Scheme:

$$X_{m+2} = X_{m+1}.Y_m, \qquad m \geq 0 \tag{1.16.2}$$

$$Y_{m+2} = Y_{m+1}.X_m, \qquad m \geq 0$$

Third Scheme:

$$X_{m+2} = Y_{m+2}.X_m, \qquad m \geq 0 \tag{1.16.3}$$

$$Y_{m+2} = X_{m+1}.Y_m, \qquad m \geq 0$$

Fourth Scheme:

$$X_{m+2} = X_{m+1}.X_m, \qquad m \geq 0 \tag{1.16.4}$$

$$Y_{m+2} = Y_{m+1}.Y_m, \qquad m \geq 0$$

1.17 Multiplicative Coupled Fibonacci Sequence of 3rd order:

Ravi and Gupta [28] explore properties of multiplicative coupled Fibonacci sequences. Six random real numbersa, b, c, d, e and fare given, and let $\{Ӿ_i\}_{i=0}^{\infty}$ and $\{¥_i\}_{i=0}^{\infty}$ be two infinite sequences. There are eight possible approaches

First Scheme:

$$Ӿ_{m+3} = ¥_{m+2}.¥_{m+1}.¥_{m}, \qquad m \geq 0 \tag{1.17.1}$$

$$¥_{m+3} = Ӿ_{m+2}.Ӿ_{m+1}.Ӿ_{m}, \qquad m \geq 0$$

Second Scheme:

$$Ӿ_{m+3} = Ӿ_{m+2}.Ӿ_{m+1}.Ӿ_{m}, \qquad m \geq 0 \tag{1.17.2}$$

$$¥_{m+3} = ¥_{m+2}.¥_{m+1}.¥_{m}, \qquad m \geq 0$$

Third Scheme:

$$Ӿ_{m+3} = ¥_{m+2}.¥_{m+1}.Ӿ_{m}, \qquad m \geq 0 \tag{1.17.3}$$

$$¥_{m+3} = Ӿ_{m+2}.Ӿ_{m+1}.¥_{m}, \qquad m \geq 0$$

Fourth Scheme:

$$Ӿ_{m+3} = Ӿ_{m+2}.Ӿ_{m+1}.¥_{m}, \qquad m \geq 0 \tag{1.17.4}$$

$$¥_{m+3} = ¥_{m+2}.¥_{m+1}.¥_{m}, \qquad m \geq 0$$

Fifth Scheme:

$$Ӿ_{m+2} = ¥_{m+1}.Ӿ_{m+1}.¥_{m}, \qquad m \geq 0 \tag{1.17.5}$$

$$¥_{m+3} = Ӿ_{m+2}.¥_{m+1}.Ӿ_{m}, \qquad m \geq 0$$

Sixth Scheme:

$$Ӿ_{m+3} = Ӿ_{m+2}.¥_{m+1}.Ӿ_{m}, \qquad m \geq 0 \tag{1.17.6}$$

$$¥_{m+3} = ¥_{m+2}.Ӿ_{m+1}.¥_{m}, \qquad m \geq 0$$

Seventh Scheme:

$$Ӿ_{m+3} = Ӿ_{m+2}.¥_{m+1}.¥_{m}, \qquad m \geq 0 \tag{1.17.7}$$

$$¥_{m+3} = ¥_{m+2}.Ӿ_{m+1}.Ӿ_{m}, \qquad m \geq 0$$

Eighth Scheme:

$$Ӿ_{m+3} = ¥_{m+2}.Ӿ_{m+1}.Ӿ_{m}, \qquad m \geq 0 \tag{1.17.8}$$

$$¥_{m+3} = Ӿ_{m+2}.¥_{m+1}.¥_{m}, \qquad m \geq 0$$

1.18 Multiplicative Coupled Fibonacci Sequence of 4th order:

Sharma and Kumar [29] discuss applications of multiplicative coupled Fibonacci sequences.Eight arbitrary real numbers a, b, c, d, e, f, g, and h are given MCFS of fourth order. With $\{Ӿ_i\}_{i=0}^{\infty}$ and $\{¥_i\}_{i=0}^{\infty}$, two infinite sequences, the following 16 methods can be used to construct these sequences.

First Scheme:

$$Ӿ_{m+4} = Ӿ_{m+3}.Ӿ_{m+2}.Ӿ_{m+1}.Ӿ_{m}, \qquad m \geq 0 \tag{1.18.1}$$

$$¥_{m+4} = ¥_{m+3}.¥_{m+2}.¥_{m+1}.¥_{m}, \qquad m \geq 0$$

Second Scheme:

$$Ӿ_{m+4} = Ӿ_{m+3}.Ӿ_{m+2}.Ӿ_{m+1}.¥_{m}, \qquad m \geq 0 \tag{1.18.2}$$

$$¥_{m+4} = ¥_{m+3}.¥_{m+2}.¥_{m+1}.Ӿ_{m}, \qquad m \geq 0$$

Third Scheme:

$$Ӿ_{m+4} = Ӿ_{m+3}.Ӿ_{m+2}.¥_{m+1}.Ӿ_{m}, \qquad m \geq 0 \tag{1.18.3}$$

$$¥_{m+4} = ¥_{m+3}.¥_{m+2}.Ӿ_{m+1}.¥_{m}, \qquad m \geq 0$$

Fourth Scheme:

$$Ӿ_{m+4} = Ӿ_{m+3}.Ӿ_{m+2}.¥_{m+1}.¥_{m}, \qquad m \geq 0 \tag{1.18.4}$$

$$¥_{m+4} = ¥_{m+3}.¥_{m+2}.Ӿ_{m+1}.Ӿ_{m}, \qquad m \geq 0$$

Fifth Scheme:

$$Ӿ_{m+4} = Ӿ_{m+3}.¥_{m+2}.Ӿ_{m+1}.Ӿ_{m}, \qquad m \geq 0 \tag{1.18.5}$$

$$¥_{m+4} = ¥_{m+3}.Ӿ_{m+2}.¥_{m+1}.¥_{m}, \qquad m \geq 0$$

Sixth Scheme:

$$Ӿ_{m+4} = Ӿ_{m+3}.¥_{m+2}.Ӿ_{m+1}.¥_{m}, \qquad m \geq 0 \tag{1.18.6}$$

$$¥_{m+4} = ¥_{m+3}.Ӿ_{m+2}.¥_{m+1}.Ӿ_{m}, \qquad m \geq 0$$

Seventh Scheme:

$$Ӿ_{m+4} = ¥_{m+3}.Ӿ_{m+2}.Ӿ_{m+1}.Ӿ_{m}, \qquad m \geq 0 \tag{1.18.7}$$

$$¥_{m+4} = Ӿ_{m+3}.¥_{m+2}.¥_{m+1}.¥_{m}, \qquad m \geq 0$$

Eighth Scheme:

$$Ӿ_{m+4} = ¥_{m+3}.Ӿ_{m+2}.Ӿ_{m+1}.¥_{m}, \qquad m \geq 0 \tag{1.18.8}$$

$$¥_{m+4} = Ӿ_{m+3}.¥_{m+2}.¥_{m+1}.Ӿ_{m}, \qquad m \geq 0$$

Ninth Scheme

$$Ӿ_{m+4} = Ӿ_{m+3}.¥_{m+2}.¥_{m+1}.Ӿ_{m}, \qquad m \geq 0 \tag{1.18.9}$$

$$¥_{m+4} = ¥_{m+3}.Ӿ_{m+2}.Ӿ_{m+1}.¥_{m}, \qquad m \geq 0$$

Tenth Scheme:

$$Ӿ_{m+4} = Ӿ_{m+3}.¥_{m+2}.¥_{m+1}.¥_{m}, \qquad m \geq 0 \tag{1.18.10}$$

$$Ұ_{m+4} = Ұ_{m+3}.Ӿ_{m+2}.Ӿ_{m+1}.Ӿ_{m}, \qquad m \geq 0$$

Eleventh Scheme:

$$Ӿ_{m+4} = Ұ_{m+3}.Ұ_{m+2}.Ӿ_{m+1}.Ӿ_{m}, \qquad m \geq 0 \qquad (1.18.11)$$

$$Ұ_{m+4} = Ӿ_{m+3}.Ӿ_{m+2}.Ұ_{m+1}.Ұ_{m}, \qquad m \geq 0$$

Twelfth Scheme:

$$Ӿ_{m+4} = Ұ_{m+3}.Ұ_{m+2}.Ӿ_{m+1}.Ұ_{m}, \qquad m \geq 0 \qquad (1.18.12)$$

$$Ұ_{m+4} = Ӿ_{m+3}.Ӿ_{m+2}.Ұ_{m+1}.Ӿ_{m}, \qquad m \geq 0$$

Thirteenth Scheme:

$$Ӿ_{m+4} = Ұ_{m+3}.Ӿ_{m+2}.Ұ_{m+1}.Ӿ_{m}, \qquad m \geq 0 \qquad (1.18.13)$$

$$Ұ_{m+4} = Ӿ_{m+3}.Ұ_{m+2}.Ӿ_{m+1}.Ұ_{m}, \qquad m \geq 0$$

Fourteenth Scheme

$$Ӿ_{m+4} = Ұ_{m+3}.Ӿ_{m+2}.Ұ_{m+1}.Ұ_{m}, \qquad m \geq 0 \qquad (1.18.14)$$

$$Ұ_{m+4} = Ӿ_{m+3}.Ұ_{m+2}.Ӿ_{m+1}.Ӿ_{m}, \qquad m \geq 0$$

Fifteenth Scheme:

$$Ӿ_{m+4} = Ұ_{m+3}.Ұ_{m+2}.Ұ_{m+1}.Ӿ_{m}, \qquad m \geq 0 \qquad (1.18.15)$$

$$Ұ_{m+4} = Ӿ_{m+3}.Ӿ_{m+2}.Ӿ_{m+1}.Ұ_{m}, \qquad m \geq 0$$

Sixteenth Scheme:

$$Ӿ_{m+4} = Ұ_{m+3}.Ұ_{m+2}.Ұ_{m+1}.Ұ_{m}, \qquad m \geq 0 \qquad (1.18.16)$$

$$Ұ_{m+4} = Ӿ_{m+3}.Ӿ_{m+2}.Ӿ_{m+1}.Ӿ_{m}, \qquad m \geq 0$$

1.19 Multiplicative Coupled Fibonacci Sequence of 5th Order

Let $\{Ӿ_i\}_{i=0}^{\infty}$ and $\{Ұ_i\}_{i=0}^{\infty}$be two infinite sequences with initial value $a, b, c, d, e, f, g, h. i$ and j. MCFS of fifth order describes the following ways:

First Scheme:

$$Ӿ_{m+5} = Ұ_{m+4}.Ұ_{m+3}.Ұ_{m+2}.Ұ_{m+1}.Ұ_{m}, \qquad m \geq 0 \qquad (1.19.1)$$

$$Ұ_{m+5} = Ӿ_{m+4}.Ӿ_{m+3}.Ӿ_{m+2}.Ӿ_{m+1}.Ӿ_{m}, \qquad m \geq 0$$

Second Scheme:

$$Ӿ_{m+5} = Ӿ_{m+4}.Ӿ_{m+3}.Ӿ_{m+2}.Ӿ_{m+1}.Ӿ_{m}, \qquad m \geq 0 \qquad (1.19.2)$$

$$Ұ_{m+5} = Ұ_{m+4}.Ұ_{m+3}.Ұ_{m+2}.Ұ_{m+1}.Ұ_{m}, \qquad m \geq 0$$

Third Scheme:

$$Ӿ_{m+5} = Ұ_{m+4}.Ұ_{m+3}.Ұ_{m+2}.Ұ_{m+1}.Ӿ_{m}, \qquad m \geq 0 \qquad (1.19.3)$$

$$Ɏ_{𝕞+5} = Ӿ_{𝕞+4}.Ӿ_{𝕞+3}.Ӿ_{𝕞+2}.Ӿ_{𝕞+1}.Ɏ_{𝕞}, \qquad 𝕞 \geq 0$$

Fourth Scheme:

$$Ӿ_{𝕞+5} = Ɏ_{𝕞+4}.Ɏ_{𝕞+3}.Ɏ_{𝕞+2}.Ӿ_{𝕞+1}.Ɏ_{𝕞}, \qquad 𝕞 \geq 0 \tag{1.19.4}$$

$$Ɏ_{𝕞+5} = Ӿ_{𝕞+4}.Ӿ_{𝕞+3}.Ӿ_{𝕞+2}.Ɏ_{𝕞+1}.Ӿ_{𝕞}, \qquad 𝕞 \geq 0$$

Fifth Scheme:

$$Ӿ_{𝕞+5} = Ɏ_{𝕞+4}.Ɏ_{𝕞+3}.Ӿ_{𝕞+2}.Ɏ_{𝕞+1}.Ɏ_{𝕞}, \qquad 𝕞 \geq 0 \tag{1.19.5}$$

$$Ɏ_{𝕞+5} = Ӿ_{𝕞+4}.Ӿ_{𝕞+3}.Ɏ_{𝕞+2}.Ӿ_{𝕞+1}.Ӿ_{𝕞}, \qquad 𝕞 \geq 0$$

Sixth Scheme:

$$Ӿ_{𝕞+5} = Ɏ_{𝕞+4}.Ӿ_{𝕞+3}.Ɏ_{𝕞+2}.Ɏ_{𝕞+1}.Ɏ_{𝕞}, \qquad 𝕞 \geq 0 \tag{1.19.6}$$

$$Ɏ_{𝕞+5} = Ӿ_{𝕞+4}.Ɏ_{𝕞+3}.Ӿ_{𝕞+2}.Ӿ_{𝕞+1}.Ӿ_{𝕞}, \qquad 𝕞 \geq 0$$

Seventh Scheme:

$$Ӿ_{𝕞+5} = Ӿ_{𝕞+4}.Ɏ_{𝕞+3}.Ɏ_{𝕞+2}.Ɏ_{𝕞+1}.Ɏ_{𝕞}, \qquad 𝕞 \geq 0 \tag{1.19.7}$$

$$Ɏ_{𝕞+5} = Ɏ_{𝕞+4}.Ӿ_{𝕞+3}.Ӿ_{𝕞+2}.Ӿ_{𝕞+1}.Ӿ_{𝕞}, \qquad 𝕞 \geq 0$$

Eighth Scheme:

$$Ӿ_{𝕞+5} = Ӿ_{𝕞+4}.Ӿ_{𝕞+3}.Ӿ_{𝕞+2}.Ӿ_{𝕞+1}.Ɏ_{𝕞}, \qquad 𝕞 \geq 0 \tag{1.19.8}$$

$$Ɏ_{𝕞+5} = Ɏ_{𝕞+4}.Ɏ_{𝕞+3}.Ɏ_{𝕞+2}.Ɏ_{𝕞+1}.Ӿ_{𝕞}, \qquad 𝕞 \geq 0$$

Ninth Scheme:

$$Ӿ_{𝕞+5} = Ӿ_{𝕞+4}.Ӿ_{𝕞+3}.Ӿ_{𝕞+2}.Ɏ_{𝕞+1}.Ӿ_{𝕞}, \qquad 𝕞 \geq 0 \tag{1.19.9}$$

$$Ɏ_{𝕞+5} = Ɏ_{𝕞+4}.Ɏ_{𝕞+3}.Ɏ_{𝕞+2}.Ӿ_{𝕞+1}.Ɏ_{𝕞}, \qquad 𝕞 \geq 0$$

Tenth Scheme:

$$Ӿ_{𝕞+5} = Ӿ_{𝕞+4}.Ӿ_{𝕞+3}.Ɏ_{𝕞+2}.Ӿ_{𝕞+1}.Ӿ_{𝕞}, \qquad 𝕞 \geq 0 \tag{1.19.10}$$

$$Ɏ_{𝕞+5} = Ɏ_{𝕞+4}.Ɏ_{𝕞+3}.Ɏ_{𝕞+2}.Ӿ_{𝕞+1}.Ɏ_{𝕞}, \qquad 𝕞 \geq 0$$

Eleventh Scheme:

$$Ӿ_{𝕞+5} = Ӿ_{𝕞+4}.Ɏ_{𝕞+3}.Ӿ_{𝕞+2}.Ӿ_{𝕞+1}.Ӿ_{𝕞}, \qquad 𝕞 \geq 0 \tag{1.19.11}$$

$$Ɏ_{𝕞+5} = Ɏ_{𝕞+4}.Ӿ_{𝕞+3}.Ɏ_{𝕞+2}.Ɏ_{𝕞+1}.Ɏ_{𝕞}, \qquad 𝕞 \geq 0$$

Twelfth Scheme:

$$Ӿ_{𝕞+5} = Ɏ_{𝕞+4}.Ӿ_{𝕞+3}.Ӿ_{𝕞+2}.Ӿ_{𝕞+1}.Ӿ_{𝕞}, \qquad 𝕞 \geq 0 \tag{1.19.12}$$

$$Ɏ_{𝕞+5} = Ӿ_{𝕞+4}.Ɏ_{𝕞+3}.Ɏ_{𝕞+2}.Ɏ_{𝕞+1}.Ɏ_{𝕞}, \qquad 𝕞 \geq 0$$

Thirteen Scheme:

$$Ӿ_{𝕞+5} = Ӿ_{𝕞+4}.Ӿ_{𝕞+3}.Ӿ_{𝕞+2}.Ɏ_{𝕞+1}.Ɏ_{𝕞}, \qquad 𝕞 \geq 0 \tag{1.19.13}$$

$$Ɏ_{𝕞+5} = Ɏ_{𝕞+4}.Ɏ_{𝕞+3}.Ɏ_{𝕞+2}.Ӿ_{𝕞+1}.Ӿ_{𝕞}, \qquad 𝕞 \geq 0$$

Fourteenth Scheme:

$$Ӿ_{m+5} = Ӿ_{m+4}.Ӿ_{m+3}.Ɏ_{m+2}.Ӿ_{m+1}.Ɏ_{m}, \qquad m \geq 0 \tag{1.19.14}$$
$$Ɏ_{m+5} = Ɏ_{m+4}.Ɏ_{m+3}.Ӿ_{m+2}.Ɏ_{m+1}.Ӿ_{m}, \qquad m \geq 0$$

Fifteenth Scheme:

$$Ӿ_{m+5} = Ӿ_{m+4}.Ɏ_{m+3}.Ӿ_{m+2}.Ӿ_{m+1}.Ɏ_{m}, \qquad m \geq 0 \tag{1.19.15}$$
$$Ɏ_{m+5} = Ɏ_{m+4}.Ӿ_{m+3}.Ɏ_{m+2}.Ɏ_{m+1}.Ӿ_{m}, \qquad m \geq 0$$

Sixteenth Scheme:

$$Ӿ_{m+5} = Ɏ_{m+4}.Ӿ_{m+3}.Ӿ_{m+2}.Ӿ_{m+1}.Ɏ_{m}, \qquad m \geq 0 \tag{1.19.16}$$
$$Ɏ_{m+5} = Ӿ_{m+4}.Ɏ_{m+3}.Ɏ_{m+2}.Ɏ_{m+1}.Ӿ_{m}, \qquad m \geq 0$$

Seventeenth Scheme:

$$Ӿ_{m+5} = Ӿ_{m+4}.Ӿ_{m+3}.Ɏ_{m+2}.Ɏ_{m+1}.Ӿ_{m}, \qquad m \geq 0 \tag{1.19.17}$$
$$Ɏ_{m+5} = Ɏ_{m+4}.Ɏ_{m+3}.Ӿ_{m+2}.Ӿ_{m+1}.Ɏ_{m}, \qquad m \geq 0$$

Eighteenth Scheme:

$$Ӿ_{m+5} = Ӿ_{m+4}.Ɏ_{m+3}.Ӿ_{m+2}.Ɏ_{m+1}.Ӿ_{m}, \qquad m \geq 0 \tag{1.19.18}$$
$$Ɏ_{m+5} = Ɏ_{m+4}.Ӿ_{m+3}.Ɏ_{m+2}.Ӿ_{m+1}.Ɏ_{m}, \qquad m \geq 0$$

Nineteenth Scheme:

$$Ӿ_{m+5} = Ɏ_{m+4}.Ӿ_{m+3}.Ӿ_{m+2}.Ɏ_{m+1}.Ӿ_{m}, \qquad m \geq 0 \tag{1.19.19}$$
$$Ɏ_{m+5} = Ӿ_{m+4}.Ɏ_{m+3}.Ɏ_{m+2}.Ӿ_{m+1}.Ɏ_{m}, \qquad m \geq 0$$

Twentieth Scheme:

$$Ӿ_{m+5} = Ӿ_{m+4}.Ɏ_{m+3}.Ɏ_{m+2}.Ӿ_{m+1}.Ӿ_{m}, \qquad m \geq 0 \tag{1.19.20}$$
$$Ɏ_{m+5} = Ɏ_{m+4}.Ӿ_{m+3}.Ӿ_{m+2}.Ɏ_{m+1}.Ɏ_{m}, \qquad m \geq 0$$

Twenty-First Scheme:

$$Ӿ_{m+5} = Ɏ_{m+4}.Ӿ_{m+3}.Ɏ_{m+2}.Ӿ_{m+1}.Ӿ_{m}, \qquad m \geq 0 \tag{1.19.21}$$
$$Ɏ_{m+5} = Ӿ_{m+4}.Ɏ_{m+3}.Ӿ_{m+2}.Ɏ_{m+1}.Ɏ_{m}, \qquad m \geq 0$$

Twenty-Second Scheme:

$$Ӿ_{m+5} = Ɏ_{m+4}.Ɏ_{m+3}.Ӿ_{m+2}.Ӿ_{m+1}.Ӿ_{m}, \qquad m \geq 0 \tag{1.19.22}$$
$$Ɏ_{m+5} = Ӿ_{m+4}.Ӿ_{m+3}.Ɏ_{m+2}.Ɏ_{m+1}.Ɏ_{m}, \qquad m \geq 0$$

Twenty-Third Scheme:

$$Ӿ_{m+5} = Ɏ_{m+4}.Ɏ_{m+3}.Ɏ_{m+2}.Ӿ_{m+1}.Ӿ_{m}, \qquad m \geq 0 \tag{1.19.23}$$
$$Ɏ_{m+5} = Ӿ_{m+4}.Ӿ_{m+3}.Ӿ_{m+2}.Ɏ_{m+1}.Ɏ_{m}, \qquad m \geq 0$$

Twenty-Fourth Scheme:

$$Ӿ_{m+5} = Ɏ_{m+4}.Ɏ_{m+3}.Ӿ_{m+2}.Ɏ_{m+1}.Ӿ_{m}, \qquad m \geq 0 \tag{1.19.24}$$

$$Ɏ_{m+5} = Ӿ_{m+4}.Ӿ_{m+3}.Ɏ_{m+2}.Ӿ_{m+1}.Ɏ_{m}, \qquad m \geq 0$$

Twenty-Fifth Scheme:

$$Ӿ_{m+5} = Ɏ_{m+4}.Ӿ_{m+3}.Ɏ_{m+2}.Ɏ_{m+1}.Ӿ_{m}, \qquad m \geq 0 \tag{1.19.25}$$

$$Ɏ_{m+5} = Ӿ_{m+4}.Ɏ_{m+3}.Ӿ_{m+2}.Ӿ_{m+1}.Ɏ_{m}, \qquad m \geq 0$$

Twenty-Sixth Scheme:

$$Ӿ_{m+5} = Ӿ_{m+4}.Ɏ_{m+3}.Ɏ_{m+2}.Ɏ_{m+1}.Ӿ_{m}, \qquad m \geq 0 \tag{1.19.26}$$

$$Ɏ_{m+5} = Ɏ_{m+4}.Ӿ_{m+3}.Ӿ_{m+2}.Ӿ_{m+1}.Ɏ_{m}, \qquad m \geq 0$$

Twenty-Seventh Scheme:

$$Ӿ_{m+5} = Ɏ_{m+4}.Ɏ_{m+3}.Ӿ_{m+2}.Ӿ_{m+1}.Ɏ_{m}, \qquad m \geq 0 \tag{1.19.27}$$

$$Ɏ_{m+5} = Ӿ_{m+4}.Ӿ_{m+3}.Ɏ_{m+2}.Ɏ_{m+1}.Ӿ_{m}, \qquad m \geq 0$$

Twenty-Eighth Scheme:

$$Ӿ_{m+5} = Ɏ_{m+4}.Ӿ_{m+3}.Ɏ_{m+2}.Ӿ_{m+1}.Ɏ_{m}, \qquad m \geq 0 \tag{1.19.28}$$

$$Ɏ_{m+5} = Ӿ_{m+4}.Ɏ_{m+3}.Ӿ_{m+2}.Ɏ_{m+1}.Ӿ_{m}, \qquad m \geq 0$$

Twenty-Ninth Scheme:

$$Ӿ_{m+5} = Ӿ_{m+4}.Ɏ_{m+3}.Ɏ_{m+2}.Ӿ_{m+1}.Ɏ_{m}, \qquad m \geq 0 \tag{1.19.29}$$

$$Ɏ_{m+5} = Ɏ_{m+4}.Ӿ_{m+3}.Ӿ_{m+2}.Ɏ_{m+1}.Ӿ_{m}, \qquad m \geq 0$$

Thirtieth Scheme:

$$Ӿ_{m+5} = Ɏ_{m+4}.Ӿ_{m+3}.Ӿ_{m+2}.Ɏ_{m+1}.Ɏ_{m}, \qquad m \geq 0 \tag{1.19.30}$$

$$Ɏ_{m+5} = Ӿ_{m+4}.Ɏ_{m+3}.Ɏ_{m+2}.Ӿ_{m+1}.Ӿ_{m}, \qquad m \geq 0$$

Thirty-First Scheme:

$$Ӿ_{m+5} = Ӿ_{m+4}.Ɏ_{m+3}.Ӿ_{m+2}.Ɏ_{m+1}.Ɏ_{m}, \qquad m \geq 0 \tag{1.19.31}$$

$$Ɏ_{m+5} = Ɏ_{m+4}.Ӿ_{m+3}.Ɏ_{m+2}.Ӿ_{m+1}.Ӿ_{m}, \qquad m \geq 0$$

Thirty-Second Scheme:

$$Ӿ_{m+5} = Ӿ_{m+4}.Ӿ_{m+3}.Ɏ_{m+2}.Ɏ_{m+1}.Ɏ_{m}, \qquad m \geq 0 \tag{1.19.32}$$

$$Ɏ_{m+5} = Ɏ_{m+4}.Ɏ_{m+3}.Ӿ_{m+2}.Ӿ_{m+1}.Ӿ_{m}, \qquad m \geq 0$$

1.20 Multiplicative Triple Fibonacci Sequence of 2nd order

Singh and Sharma [30] explore the properties of the multiplicative triple Fibonacci sequence of 2nd order. Awasthi and Ranga [31] explore multiplicative triple Fibonacci sequences under specific Schemes in both Second and Third orders.

Let $\{Ӿ_i\}_{i=0}^{\infty}\{Ɏ_i\}_{i=0}^{\infty}$and $\{Ƶ_i\}_{i=0}^{\infty}$be three infinite sequences and called 3-F Sequence or TFS with initial value a, b, c, d, e and f.If $Ӿ_0 = a$, $Ɏ_0 = b$, $Ƶ_0 = c$, $Ӿ_1 = d$, $Ɏ_1 = e$, $Ƶ_1 = f$, then nine different Schemes of MTFS are as follows:

First Scheme:

$$\begin{aligned} X_{m+2} &= Y_{m+1}.Z_m \\ Y_{m+2} &= Z_{m+1}.X_m \\ Z_{m+2} &= X_{m+1}.Y_m \end{aligned} \tag{1.20.1}$$

Second Scheme:

$$\begin{aligned} X_{m+2} &= Z_{m+1}.Y_m \\ Y_{m+2} &= X_{m+1}.Z_m \\ Z_{m+2} &= Y_{m+1}.X_m \end{aligned} \tag{1.20.2}$$

Third Scheme:

$$\begin{aligned} X_{m+2} &= X_{m+1}.Y_m \\ Y_{m+2} &= Y_{m+1}.Z_m \\ Z_{m+2} &= Z_{m+1}.X_m \end{aligned} \tag{1.20.3}$$

Fourth Scheme:

$$\begin{aligned} X_{m+2} &= Y_{m+1}.X_m \\ Y_{m+2} &= Z_{m+1}.Y_m \\ Z_{m+2} &= X_{m+1}.Z_m \end{aligned} \tag{1.20.4}$$

Fifth Scheme:

$$\begin{aligned} X_{m+2} &= X_{m+1}.Z_m \\ Y_{m+2} &= Y_{m+1}.X_m \\ Z_{m+2} &= Z_{m+1}.Y_m \end{aligned} \tag{1.20.5}$$

Sixth Scheme:

$$\begin{aligned} X_{m+2} &= Z_{m+1}.X_m \\ Y_{m+2} &= X_{m+1}.Y_m \\ Z_{m+2} &= Y_{m+1}.Z_m \end{aligned} \tag{1.20.6}$$

Seventh Scheme:

$$\begin{aligned} X_{m+2} &= X_{m+1}.X_m \\ Y_{m+2} &= Y_{m+1}.Y_m \\ Z_{m+2} &= Z_{m+1}.Z_m \end{aligned} \tag{1.20.7}$$

Eighth Scheme:

$$X_{m+2} = Y_{m+1}.Y_m$$

$$¥_{\mathbb{m}+2} = Ƶ_{\mathbb{m}+1}.Ƶ_{\mathbb{m}} \quad (1.20.8)$$
$$Ƶ_{\mathbb{m}+2} = Ӿ_{\mathbb{m}+1}.Ӿ_{\mathbb{m}}$$

Ninth Scheme:

$$Ӿ_{\mathbb{m}+2} = Ƶ_{\mathbb{m}+1}.Ƶ_{\mathbb{m}}$$
$$¥_{\mathbb{m}+2} = Ӿ_{\mathbb{m}+1}.Ӿ_{\mathbb{m}} \quad (1.20.9)$$
$$Ƶ_{\mathbb{m}+2} = ¥_{\mathbb{m}+1}.¥_{\mathbb{m}}$$

1.21 Multiplicative Triple Fibonacci Sequence of 3rd order:

Awasthi and Ranga [31] investigate multiplicative triple Fibonacci sequences in the Second and Third orders using specific Schemes.Let $\{Ӿ_{i}\}_{i=0}^{\infty}\{¥_{i}\}_{i=0}^{\infty}$and $\{Ƶ_{i}\}_{i=0}^{\infty}$be three infinite sequences and called 3-F Sequence or TFS with initial value a, b, c, d, e, f, g, h and i be given.If $Ӿ_0 = a, ¥_0 = b, Ƶ_0 = c, Ӿ_1 = d, ¥_1 = e, Ƶ_1 = f, Ӿ_2 = g, ¥_2 = h, Ƶ_2 = i$ then twenty-seven different Schemes of MTFS are as follows:

First Scheme:

$$Ӿ_{\mathbb{m}+3} = ¥_{\mathbb{m}+2}.Ƶ_{\mathbb{m}+1}.Ӿ_{\mathbb{m}}$$
$$¥_{\mathbb{m}+3} = Ƶ_{\mathbb{m}+2}.Ӿ_{\mathbb{m}+1}.¥_{\mathbb{m}} \quad (1.21.1)$$
$$Ƶ_{\mathbb{m}+3} = Ӿ_{\mathbb{m}+2}.¥_{\mathbb{m}+1}.Ƶ_{\mathbb{m}}$$

Second Scheme:

$$Ӿ_{\mathbb{m}+3} = Ӿ_{\mathbb{m}+2}.Ӿ_{\mathbb{m}+1}.Ӿ_{\mathbb{m}}$$
$$¥_{\mathbb{m}+3} = ¥_{\mathbb{m}+2}.¥_{\mathbb{m}+1}.¥_{\mathbb{m}} \quad (1.21.2)$$
$$Ƶ_{\mathbb{m}+3} = Ƶ_{\mathbb{m}+2}.Ƶ_{\mathbb{m}+1}.Ƶ_{\mathbb{m}}$$

Third Scheme:

$$Ӿ_{\mathbb{m}+3} = Ӿ_{\mathbb{m}+2}.Ƶ_{\mathbb{m}+1}.¥_{\mathbb{m}}$$
$$¥_{\mathbb{m}+3} = ¥_{\mathbb{m}+2}.Ӿ_{\mathbb{m}+1}.Ƶ_{\mathbb{m}} \quad (1.21.3)$$
$$Ƶ_{\mathbb{m}+3} = Ƶ_{\mathbb{m}+2}.¥_{\mathbb{m}+1}.Ӿ_{\mathbb{m}}$$

Fourth Scheme:

$$Ӿ_{\mathbb{m}+3} = Ƶ_{\mathbb{m}+2}.¥_{\mathbb{m}+1}.Ӿ_{\mathbb{m}}$$
$$¥_{\mathbb{m}+3} = Ӿ_{\mathbb{m}+2}.Ƶ_{\mathbb{m}+1}.¥_{\mathbb{m}} \quad (1.21.4)$$
$$Ƶ_{\mathbb{m}+3} = ¥_{\mathbb{m}+2}.Ӿ_{\mathbb{m}+1}.Ƶ_{\mathbb{m}}$$

Fifth Scheme:

$$Ӿ_{\mathbb{m}+3} = Ӿ_{\mathbb{m}+2}.¥_{\mathbb{m}+1}.Ƶ_{\mathbb{m}}$$

$$Ɏ_{𝕞+3} = Ɏ_{𝕞+2}.Ƶ_{𝕞+1}.Ӿ_{𝕞} \tag{1.21.5}$$
$$Ƶ_{𝕞+3} = Ƶ_{𝕞+2}.Ӿ_{𝕞+1}.Ɏ_{𝕞}$$

Sixth Scheme:

$$\begin{aligned} Ӿ_{𝕞+3} &= Ӿ_{𝕞+2}.Ӿ_{𝕞+1}.Ɏ_{𝕞} \\ Ɏ_{𝕞+3} &= Ɏ_{𝕞+2}.Ɏ_{𝕞+1}.Ƶ_{𝕞} \\ Ƶ_{𝕞+3} &= Ƶ_{𝕞+2}.Ƶ_{𝕞+1}.Ӿ_{𝕞} \end{aligned} \tag{1.21.6}$$

Seventh Scheme:

$$\begin{aligned} Ӿ_{𝕞+3} &= Ӿ_{𝕞+2}.Ɏ_{𝕞+1}.Ӿ_{𝕞} \\ Ɏ_{𝕞+3} &= Ɏ_{𝕞+2}.Ƶ_{𝕞+1}.Ɏ_{𝕞} \\ Ƶ_{𝕞+3} &= Ƶ_{𝕞+2}.Ӿ_{𝕞+1}.Ƶ_{𝕞} \end{aligned} \tag{1.21.7}$$

Eighth Scheme:

$$\begin{aligned} Ӿ_{𝕞+3} &= Ɏ_{𝕞+2}.Ӿ_{𝕞+1}.Ӿ_{𝕞} \\ Ɏ_{𝕞+3} &= Ƶ_{𝕞+2}.Ɏ_{𝕞+1}.Ɏ_{𝕞} \\ Ƶ_{𝕞+3} &= Ӿ_{𝕞+2}.Ƶ_{𝕞+1}.Ƶ_{𝕞} \end{aligned} \tag{1.21.8}$$

Ninth Scheme:

$$\begin{aligned} Ӿ_{𝕞+3} &= Ӿ_{𝕞+2}.Ӿ_{𝕞+1}.Ƶ_{𝕞} \\ Ɏ_{𝕞+3} &= Ɏ_{𝕞+2}.Ɏ_{𝕞+1}.Ӿ_{𝕞} \\ Ƶ_{𝕞+3} &= Ƶ_{𝕞+2}.Ƶ_{𝕞+1}.Ɏ_{𝕞} \end{aligned} \tag{1.21.9}$$

Tenth Scheme:

$$\begin{aligned} Ӿ_{𝕞+3} &= Ӿ_{𝕞+2}.Ƶ_{𝕞+1}.Ӿ_{𝕞} \\ Ɏ_{𝕞+3} &= Ɏ_{𝕞+2}.Ӿ_{𝕞+1}.Ɏ_{𝕞} \\ Ƶ_{𝕞+3} &= Ƶ_{𝕞+2}.Ɏ_{𝕞+1}.Ƶ_{𝕞} \end{aligned} \tag{1.21.10}$$

Eleventh Scheme:

$$\begin{aligned} Ӿ_{𝕞+3} &= Ƶ_{𝕞+2}.Ӿ_{𝕞+1}.Ӿ_{𝕞} \\ Ɏ_{𝕞+3} &= Ӿ_{𝕞+2}.Ɏ_{𝕞+1}.Ɏ_{𝕞} \\ Ƶ_{𝕞+3} &= Ɏ_{𝕞+2}.Ƶ_{𝕞+1}.Ƶ_{𝕞} \end{aligned} \tag{1.21.11}$$

Twelfth Scheme:

$$\begin{aligned} Ӿ_{𝕞+3} &= Ɏ_{𝕞+2}.Ɏ_{𝕞+1}.Ƶ_{𝕞} \\ Ɏ_{𝕞+3} &= Ƶ_{𝕞+2}.Ƶ_{𝕞+1}.Ӿ_{𝕞} \\ Ƶ_{𝕞+3} &= Ӿ_{𝕞+2}.Ӿ_{𝕞+1}.Ɏ_{𝕞} \end{aligned} \tag{1.21.12}$$

Thirteenth Scheme:

$$Ꭓ_{m+3} = Ɏ_{m+2}.Ƶ_{m+1}.Ɏ_{m}$$
$$Ɏ_{m+3} = Ƶ_{m+2}.Ꭓ_{m+1}.Ƶ_{m} \quad (1.21.13)$$
$$Ƶ_{m+3} = Ꭓ_{m+2}.Ɏ_{m+1}.Ꭓ_{m}$$

Fourteenth Scheme:

$$Ꭓ_{m+3} = Ƶ_{m+2}.Ɏ_{m+1}.Ɏ_{m}$$
$$Ɏ_{m+3} = Ꭓ_{m+2}.Ƶ_{m+1}.Ƶ_{m} \quad (1.21.14)$$
$$Ƶ_{m+3} = Ɏ_{m+2}.Ꭓ_{m+1}.Ꭓ_{m}$$

Fifteenth Scheme:

$$Ꭓ_{m+3} = Ɏ_{m+2}.Ƶ_{m+1}.Ƶ_{m}$$
$$Ɏ_{m+3} = Ƶ_{m+2}.Ꭓ_{m+1}.Ꭓ_{m} \quad (1.21.15)$$
$$Ƶ_{m+3} = Ꭓ_{m+2}.Ɏ_{m+1}.Ɏ_{m}$$

Sixteenth Scheme:

$$Ꭓ_{m+3} = Ƶ_{m+2}.Ɏ_{m+1}.Ƶ_{m}$$
$$Ɏ_{m+3} = Ꭓ_{m+2}.Ƶ_{m+1}.Ꭓ_{m} \quad (1.21.16)$$
$$Ƶ_{m+3} = Ɏ_{m+2}.Ꭓ_{m+1}.Ɏ_{m}$$

Seventeenth Scheme:

$$Ꭓ_{m+3} = Ƶ_{m+2}.Ƶ_{m+1}.Ɏ_{m}$$
$$Ɏ_{m+3} = Ꭓ_{m+2}.Ꭓ_{m+1}.Ƶ_{m} \quad (1.21.17)$$
$$Ƶ_{m+3} = Ɏ_{m+2}.Ɏ_{m+1}.Ꭓ_{m}$$

Eighteenth Scheme:

$$Ꭓ_{m+3} = Ƶ_{m+2}.Ꭓ_{m+1}.Ɏ_{m}$$
$$Ɏ_{m+3} = Ꭓ_{m+2}.Ɏ_{m+1}.Ƶ_{m} \quad (1.21.18)$$
$$Ƶ_{m+3} = Ɏ_{m+2}.Ƶ_{m+1}.Ꭓ_{m}$$

Nineteenth Scheme:

$$Ꭓ_{m+3} = Ɏ_{m+2}.Ꭓ_{m+1}.Ɏ_{m}$$
$$Ɏ_{m+3} = Ƶ_{m+2}.Ɏ_{m+1}.Ƶ_{m} \quad (1.21.19)$$
$$Ƶ_{m+3} = Ꭓ_{m+2}.Ƶ_{m+1}.Ꭓ_{m}$$

Twentieth Scheme:

$$Ꭓ_{m+3} = Ꭓ_{m+2}.Ɏ_{m+1}.Ɏ_{m}$$
$$Ɏ_{m+3} = Ɏ_{m+2}.Ƶ_{m+1}.Ƶ_{m} \quad (1.21.20)$$
$$Ƶ_{m+3} = Ƶ_{m+2}.Ꭓ_{m+1}.Ꭓ_{m}$$

Twenty First Scheme:

$$\begin{aligned} \text{Ӿ}_{m+3} &= \text{¥}_{m+2}.\text{¥}_{m+1}.\text{Ӿ}_{m} \\ \text{¥}_{m+3} &= \text{Ƶ}_{m+2}.\text{Ƶ}_{m+1}.\text{¥}_{m} \\ \text{Ƶ}_{m+3} &= \text{Ӿ}_{m+2}.\text{Ӿ}_{m+1}.\text{Ƶ}_{m} \end{aligned} \tag{1.21.21}$$

Twenty Second Scheme:

$$\begin{aligned} \text{Ӿ}_{m+3} &= \text{Ӿ}_{m+2}.\text{Ƶ}_{m+1}.\text{Ƶ}_{m} \\ \text{¥}_{m+3} &= \text{¥}_{m+2}.\text{Ӿ}_{m+1}.\text{Ӿ}_{m} \\ \text{Ƶ}_{m+3} &= \text{Ƶ}_{m+2}.\text{¥}_{m+1}.\text{¥}_{m} \end{aligned} \tag{1.21.22}$$

Twenty Third Scheme:

$$\begin{aligned} \text{Ӿ}_{m+3} &= \text{Ƶ}_{m+2}.\text{Ӿ}_{m+1}.\text{Ƶ}_{m} \\ \text{¥}_{m+3} &= \text{Ӿ}_{m+2}.\text{¥}_{m+1}.\text{Ӿ}_{m} \\ \text{Ƶ}_{m+3} &= \text{¥}_{m+2}.\text{Ƶ}_{m+1}.\text{¥}_{m} \end{aligned} \tag{1.21.23}$$

Twenty Fourth Scheme:

$$\begin{aligned} \text{Ӿ}_{m+3} &= \text{Ƶ}_{m+2}.\text{Ƶ}_{m+1}.\text{Ӿ}_{m} \\ \text{¥}_{m+3} &= \text{Ӿ}_{m+2}.\text{Ӿ}_{m+1}.\text{¥}_{m} \\ \text{Ƶ}_{m+3} &= \text{¥}_{m+2}.\text{¥}_{m+1}.\text{Ƶ}_{m} \end{aligned} \tag{1.21.24}$$

Twenty Fifth Scheme:

$$\begin{aligned} \text{Ӿ}_{m+3} &= \text{¥}_{m+2}.\text{Ӿ}_{m+1}.\text{Ƶ}_{m} \\ \text{¥}_{m+3} &= \text{Ƶ}_{m+2}.\text{¥}_{m+1}.\text{Ӿ}_{m} \\ \text{Ƶ}_{m+3} &= \text{Ӿ}_{m+2}.\text{Ƶ}_{m+1}.\text{¥}_{m} \end{aligned} \tag{1.21.25}$$

Twenty Sixth Scheme:

$$\begin{aligned} \text{Ӿ}_{m+3} &= \text{¥}_{m+2}.\text{¥}_{m+1}.\text{¥}_{m} \\ \text{¥}_{m+3} &= \text{Ƶ}_{m+2}.\text{Ƶ}_{m+1}.\text{Ƶ}_{m} \\ \text{Ƶ}_{m+3} &= \text{Ӿ}_{m+2}.\text{Ӿ}_{m+1}.\text{Ӿ}_{m} \end{aligned} \tag{1.21.26}$$

Twenty Seventh Scheme:

$$\begin{aligned} \text{Ӿ}_{m+3} &= \text{Ƶ}_{m+2}.\text{Ƶ}_{m+1}.\text{Ƶ}_{m} \\ \text{¥}_{m+3} &= \text{Ӿ}_{m+2}.\text{Ӿ}_{m+1}.\text{Ӿ}_{m} \\ \text{Ƶ}_{m+3} &= \text{¥}_{m+2}.\text{¥}_{m+1}.\text{¥}_{m} \end{aligned} \tag{1.21.27}$$

1.22 Multiplicative Triple Fibonacci Sequence of 4^{th} order:

Ranga [32] examines the multiplicative triple Fibonacci sequence of the fourth order under nine specific Schemes in number theory.

Let $\{Ӿ_i\}_{i=0}^{\infty}\{Ɏ_i\}_{i=0}^{\infty}$and $\{Ƶ_i\}_{i=0}^{\infty}$ be three infinite sequences and called 3-F Sequence or Triple Fibonacci Sequence with initial value a, b, c, d, e, f, g, h, i, j, k and l be given.

If $Ӿ_0 = a, Ɏ_0 = b, Ƶ_0 = c, Ӿ_1 = d, Ɏ_1 = e, Ƶ_1 = f, Ӿ_2 = g, Ɏ_2 = h, Ƶ_2 = i, Ӿ_3 = j, Ɏ_3 = k$ and $Ƶ_3 = l$ then twenty-seven different Schemes of MTFS. There are 81 Schemes of MTFS of fourth order. We are presenting some identities of fourth order under 3 specific Schemes and these Schemes are as follows:

First Scheme:

$$\begin{aligned} Ӿ_{m+4} &= Ӿ_{m+3}.Ӿ_{m+2}.Ӿ_{m+1}.Ӿ_m \\ Ɏ_{m+4} &= Ɏ_{m+3}.Ɏ_{m+2}.Ɏ_{m+1}.Ɏ_m \\ Ƶ_{m+4} &= Ƶ_{m+3}.Ƶ_{m+2}.Ƶ_{m+1}.Ƶ_m \end{aligned} \tag{1.22.1}$$

Second Scheme:

$$\begin{aligned} Ӿ_{m+4} &= Ɏ_{m+3}.Ɏ_{m+2}.Ɏ_{m+1}.Ɏ_m \\ Ɏ_{m+4} &= Ƶ_{m+3}.Ƶ_{m+2}.Ƶ_{m+1}.Ƶ_m \\ Ƶ_{m+4} &= Ӿ_{m+3}.Ӿ_{m+2}.Ӿ_{m+1}.Ӿ_m \end{aligned} \tag{1.22.2}$$

Third Scheme:

$$\begin{aligned} Ӿ_{m+4} &= Ƶ_{m+3}.Ƶ_{m+2}.Ƶ_{m+1}.Ƶ_m \\ Ɏ_{m+4} &= Ӿ_{m+3}.Ӿ_{m+2}.Ӿ_{m+1}.Ӿ_m \\ Ƶ_{m+4} &= Ɏ_{m+3}.Ɏ_{m+2}.Ɏ_{m+1}.Ɏ_m \end{aligned} \tag{1.22.3}$$

Fourth Scheme:

$$\begin{aligned} Ӿ_{m+4} &= Ӿ_{m+3}.Ɏ_{m+2}.Ƶ_{m+1}.Ӿ_m \\ Ɏ_{m+4} &= Ɏ_{m+3}.Ƶ_{m+2}.Ӿ_{m+1}.Ɏ_m \\ Ƶ_{m+4} &= Ƶ_{m+3}.Ӿ_{m+2}.Ɏ_{m+1}.Ƶ_m \end{aligned} \tag{1.22.4}$$

Fifth Scheme:

$$\begin{aligned} Ӿ_{m+4} &= Ƶ_{m+3}.Ӿ_{m+2}.Ɏ_{m+1}.Ƶ_m \\ Ɏ_{m+4} &= Ӿ_{m+3}.Ɏ_{m+2}.Ƶ_{m+1}.Ӿ_m \\ Ƶ_{m+4} &= Ɏ_{m+3}.Ƶ_{m+2}.Ӿ_{m+1}.Ɏ_m \end{aligned} \tag{1.22.5}$$

Sixth Scheme:

$$\begin{aligned} Ӿ_{m+4} &= Ɏ_{m+3}.Ƶ_{m+2}.Ӿ_{m+1}.Ɏ_m \\ Ɏ_{m+4} &= Ƶ_{m+3}.Ӿ_{m+2}.Ɏ_{m+1}.Ƶ_m \end{aligned} \tag{1.22.6}$$

$$Ƶ_{m+4} = Ӿ_{m+3}.Ɏ_{m+2}.Ƶ_{m+1}.Ӿ_{m}$$

Seventh Scheme:

$$\begin{aligned} Ӿ_{m+4} &= Ӿ_{m+3}.Ƶ_{m+2}.Ɏ_{m+1}.Ӿ_{m} \\ Ɏ_{m+4} &= Ɏ_{m+3}.Ӿ_{m+2}.Ƶ_{m+1}.Ɏ_{m} \\ Ƶ_{m+4} &= Ƶ_{m+3}.Ɏ_{m+2}.Ӿ_{m+1}.Ƶ_{m} \end{aligned} \quad (1.22.7)$$

Eighth Scheme:

$$\begin{aligned} Ӿ_{m+4} &= Ɏ_{m+3}.Ӿ_{m+2}.Ƶ_{m+1}.Ɏ_{m} \\ Ɏ_{m+4} &= Ƶ_{m+3}.Ɏ_{m+2}.Ӿ_{m+1}.Ƶ_{m} \\ Ƶ_{m+4} &= Ӿ_{m+3}.Ƶ_{m+2}.Ɏ_{m+1}.Ӿ_{m} \end{aligned} \quad (1.22.8)$$

Ninth Scheme:

$$\begin{aligned} Ӿ_{m+4} &= Ƶ_{m+3}.Ɏ_{m+2}.Ӿ_{m+1}.Ƶ_{m} \\ Ɏ_{m+4} &= Ӿ_{m+3}.Ƶ_{m+2}.Ɏ_{m+1}.Ӿ_{m} \\ Ƶ_{m+4} &= Ɏ_{m+3}.Ӿ_{m+2}.Ƶ_{m+1}.Ɏ_{m} \end{aligned} \quad (1.22.9)$$

1.23 Lucas Sequence

Currently, the nth term of the Fibonacci numbers, often known as Binet's Formula [10], can be expressed as

$$F_{m} = \frac{Ӿ^{m} - Ɏ^{m}}{Ӿ - Ɏ}$$

Where $Ӿ = \frac{1+\sqrt{5}}{2}$ and $Ɏ = \frac{1-\sqrt{5}}{2}$

$$Ӿ^2 = \frac{3+\sqrt{5}}{2}$$

$$Ӿ^3 = \frac{4+2\sqrt{5}}{2}$$

$$Ӿ^4 = \frac{7+3\sqrt{5}}{2}$$

As a result, it is clear that the coefficient of $\sqrt{5}$ in $Ӿ^{m}$ creates a Fibonacci number series, while the other terms create a new sequence denoted by

$$1,3,4,7,\ldots\ldots\ldots$$

Lucas (1878) [33] discusses Lucas sequences, while Smith and Doe (2019) [34] explore their modern applications in number theory. By providing the beginning term as 2, these sequences inspired the concept of Lucas numbers. Because of this, the Lucas numbers

[32], which take their name from the mathematician François Édouard Anatole Lucas, follow a recursive relation similar to that of the Fibonacci numbers, with the exception of their seed, which is different. Lucas numbers thus have the following relation:

$$\mathrm{L}_{\mathbb{m}+2} = \mathrm{L}_{\mathbb{m}+1} + \mathrm{L}_{\mathbb{m}} \text{ for } \mathbb{m} = 0,1,2,3, \ldots\ldots \qquad (1.23.1)$$

With $\mathrm{L}_0 = 2$ and $\mathrm{L}_1 = 1$

Consequently, the Lucas sequence is

$$2,1,3,4,7, \ldots\ldots\ldots$$

Binet's equation can be used to create Lucas sequences [28]:

$$\mathrm{L}_{\mathbb{m}} = \text{Ӿ}^{\mathbb{m}} + \text{¥}^{\mathbb{m}}, \mathbb{m} \geq 0$$

where Ӿ and ¥ satisfies

$$\mathrm{v}^2 - \mathrm{v} - 1 = 0$$

1.24 $\mathbb{m}^{\text{th}}$ Generalized Lucas Numbers

Smith and Johnson [35] explore applications of Nth Generalized Lucas Numbers in number theory. $\mathbb{m}^{\text{th}}$ Generalized Lucas numbers are listed in the relation as:

$$j_{\mathbb{m}}(\mathfrak{x}) = j_{\mathbb{m}-1}(\mathfrak{x}) + \mathfrak{x} j_{\mathbb{m}-2}(\mathfrak{x})$$

With $\mathbb{m} = 2,3,4, \ldots\ldots$ and $\mathfrak{x}$ is any positive integer with $j_0(\mathfrak{x}) = 2$ and $j_1(\mathfrak{x}) = 1$

1.25 Lucas Polynomials

Jones, Alice [36]defines the Applications of Lucas Polynomials in Modern Number Theory. The Lucas polynomials employ the identical recurrence but with various starting points:

$$\mathbb{L}_{\mathbb{m}}(\mathfrak{x}) = \begin{cases} 2, & \text{if } \mathbb{m} = 0 \\ \mathfrak{x}, & \text{if } \mathbb{m} = 1 \\ \mathfrak{x}\mathbb{L}_{\mathbb{m}-1}(\mathfrak{x}) + \mathbb{L}_{\mathbb{m}-2}(\mathfrak{x}), & \text{if } \mathbb{m} \geq 2 \end{cases}$$

$$\mathbb{L}_0(\mathfrak{x}) = 2$$

$$\mathbb{L}_1(\mathfrak{x}) = \mathfrak{x}$$

$$\mathbb{L}_2(\mathfrak{x}) = \mathfrak{x}^2 + 2$$

$$\mathbb{L}_3(\mathfrak{x}) = \mathfrak{x}^3 + 3\mathfrak{x}$$

$$\mathbb{L}_4(\mathfrak{x}) = \mathfrak{x}^4 + 4\mathfrak{x}^2 + 2$$

$$\mathbb{L}_5(\mathrm{x}) = \mathfrak{x}^5 + 5\mathfrak{x}^3 + 5\mathfrak{x}$$

$$\mathbb{L}_6(\mathrm{x}) = \mathfrak{x}^6 + 6\mathfrak{x}^4 + 9\mathfrak{x}^2 + 2$$

By evaluating the polynomials at $\mathfrak{x} = 1$, the Fibonacci, Lucas, and Pell numbers may be found. By evaluating F_n at $\mathfrak{x} = 2$, the Pell numbers can be found. F_n has degrees of $\mathfrak{m}$-1 and L_n has degrees of $\mathfrak{m}$. The sequences' standard generating function is:

$$\sum_{\mathfrak{m}=0}^{\infty} F_{\mathfrak{m}}(\mathfrak{x})t^{\mathfrak{m}} = \frac{t}{1-\mathfrak{x}t-t^2}$$

$$\sum_{\mathfrak{m}=0}^{\infty} L_{\mathfrak{m}}(\mathfrak{x})t^{\mathfrak{m}} = \frac{2-\mathfrak{x}t}{1-\mathfrak{x}t-t^2}$$

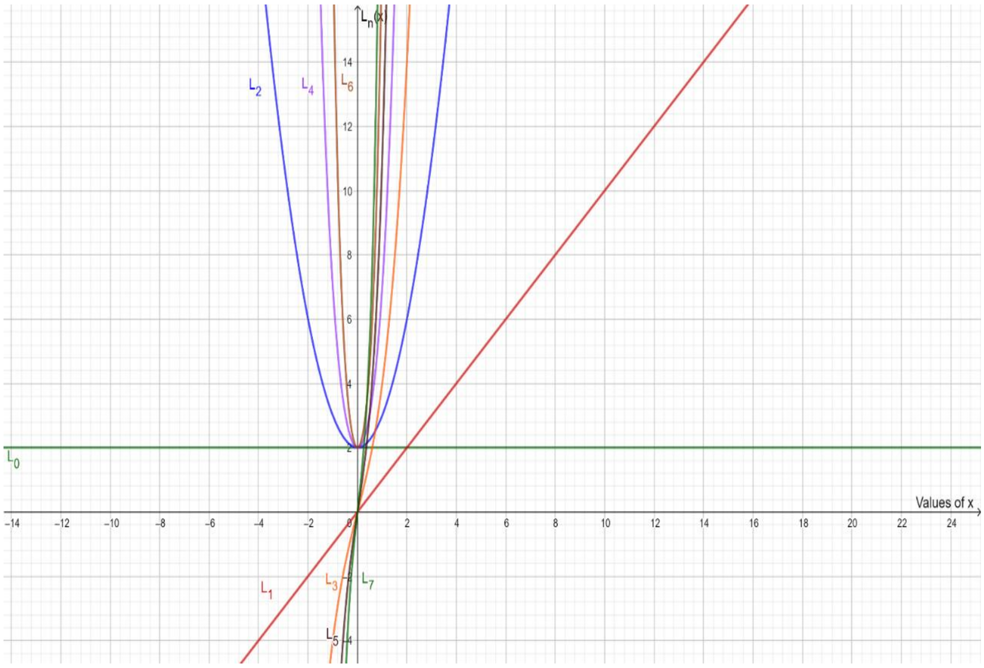


Figure 1.13: Lucas Polynomial [78]

1.26 Generalized Lucas Polynomials

The generalization of Lucas polynomials, known as generalized Lucas polynomials [36], is defined by

$$L_{\kappa,q,\mathfrak{m}+1}(\mathfrak{g}) = \kappa(\mathfrak{g})L_{\kappa,q,\mathfrak{m}}(\mathfrak{g}) + q(\mathfrak{g})L_{\kappa,q,\mathfrak{m}-1}(\mathfrak{g})$$

For $\mathfrak{n} = 1,2,3\ldots\ldots$

Where $L_{\kappa,q,0}(\mathfrak{g}) = 2$ and $L_{\kappa,q,1}(\mathfrak{g}) = \kappa(\mathfrak{g})$

Here, the real-coefficient polynomials $\kappa(\mathfrak{g})$ and $q(\mathfrak{g})$ are used.

The series given by (1.12) becomes the Lucas number sequence for $\kappa(\mathfrak{g}) = q(\mathfrak{g}) = 1$.

Also,

$$\sum_{n=0}^{\infty} L_{\kappa,q,m}(g)u^{m} = g_F(u) = \frac{2-\kappa(g)u}{1-\kappa(g)u-q(g)u^2}$$

1.27 Literature Review

The field of the Fibonacci numbers has been the focus of numerous researchers.Various summing equations for the "Coupled Fibonacci sequence and Multiplicative Coupled Fibonacci sequence" have been created in [37-46]. Similar characteristics of the "Fibonacci, Tribonacci, Coupled Fibonacci, Coupled Lucas sequence" have each been described in detail for the numbers [47-65].

Pain, Jean-Christophe [66] provided the various summing formulae for Generalized Fibonacci numbers, which are defined as

$$F_m = zF_{m-1} + sF_{m-2};\ F_0 = a, F_1 = b, \text{ for } m = 2,3,4, \ldots\ldots$$

The same type of work has been done for several sequences by Oduol, Fidel Ochieng, and Isaac Owino Okoth [67]. In [68], Sikhwal investigated a number of 2-Fibonacci sequences' qualities.

While some of the characteristics of Fibonacci numbers are straightforward and well-known, others have a wide range of application in scientific inquiry. Modern Mathematics has a wide range of applications for the Fibonacci and Lucas numbers. Fundamental characteristics of MCFS of Second order are presented by B. Singh and O. Sikhwal [39].

For every integer $m \geq 0$

1. $Y_0 . X_{3m+3} = X_0 . Y_{3m+3}$
2. $Y_1 . X_{3m+4} = X_1 . Y_{3m+4}$
3. $Y_2 . X_{3m+5} = X_2 . Y_{3m+5}$

For every integer $m \geq 0$

1. $X_{3m+3} = Y_1 . \prod_{i=0}^{3m+1} X_i$
2. $Y_{3m+3} = X_1 . \prod_{i=0}^{3m+1} Y_i$
3. $X_{3m+4} = X_1 . \prod_{i=0}^{3m+2} X_i$
4. $Y_{3m+4} = Y_1 . \prod_{i=0}^{3m+2} Y_i$
5. $X_{3m+5} = \frac{X_1}{X_0} . \prod_{i=0}^{3m+3} X_i$
6. $Y_{3m+5} = \frac{Y_1}{Y_0} . \prod_{i=0}^{3m+3} Y_i$

For every integer $\mathbb{m} \geq 0$

1. $\frac{Ӿ_{3\mathbb{m}+7}}{Ӿ_{3\mathbb{m}+4}} = Ӿ_0^{F_{3\mathbb{m}+4}}.¥_0^{F_{3\mathbb{m}+4}}.Ӿ_1^{F_{3\mathbb{m}+5}}.¥_1^{F_{3\mathbb{m}+5}}$
2. $\frac{¥_{3\mathbb{m}+7}}{¥_{3\mathbb{m}+4}} = Ӿ_0^{F_{3\mathbb{m}+4}}.¥_0^{F_{3\mathbb{m}+4}}.Ӿ_1^{F_{3\mathbb{m}+5}}.¥_1^{F_{3\mathbb{m}+5}}$
3. $\frac{Ӿ_{3\mathbb{m}+6}}{Ӿ_{3\mathbb{m}+3}} = Ӿ_0^{F_{3\mathbb{m}+3}}.¥_0^{F_{3\mathbb{m}+3}}.Ӿ_1^{F_{3\mathbb{m}+4}}.¥_1^{F_{3\mathbb{m}+4}}$
4. $\frac{¥_{3\mathbb{m}+6}}{¥_{3\mathbb{m}+3}} = Ӿ_0^{F_{3\mathbb{m}+3}}.¥_0^{F_{3\mathbb{m}+3}}.Ӿ_1^{F_{3\mathbb{m}+4}}.¥_1^{F_{3\mathbb{m}+4}}$
5. $\frac{Ӿ_{3\mathbb{m}+5}}{Ӿ_{3\mathbb{m}+2}} = Ӿ_0^{F_{3\mathbb{m}+2}}.¥_0^{F_{3\mathbb{m}+2}}.Ӿ_1^{F_{3\mathbb{m}+3}}.¥_1^{F_{3\mathbb{m}+3}}$
6. $\frac{¥_{3\mathbb{m}+5}}{¥_{3\mathbb{m}+2}} = Ӿ_0^{F_{3\mathbb{m}+2}}.¥_0^{F_{3\mathbb{m}+2}}.Ӿ_1^{F_{3\mathbb{m}+1}}.¥_1^{F_{3\mathbb{m}+1}}$

For every integer $\mathbb{m} \geq 0$

1. $Ӿ_{\mathbb{m}}.Ӿ_{\mathbb{m}+1}.Ӿ_{\mathbb{m}+2} = (Ӿ_0.¥_0)^{F_{\mathbb{m}+1}}(Ӿ_1.¥_1)^{F_{\mathbb{m}+2}}$
2. $¥_{\mathbb{m}}.¥_{\mathbb{m}+1}.¥_{\mathbb{m}+2} = (Ӿ_0.¥_0)^{F_{\mathbb{m}+1}}(Ӿ_1.¥_1)^{F_{\mathbb{m}+2}}$

For every integer $\mathbb{m} \geq 0$

1. $\frac{Ӿ_{\mathbb{m}+3}}{Ӿ_{\mathbb{m}}} = (Ӿ_0.¥_0)^{F_{\mathbb{m}}}(Ӿ_1.¥_1)^{F_{\mathbb{m}+1}}$
2. $\frac{¥_{\mathbb{m}+3}}{¥_{\mathbb{m}}} = (Ӿ_0.¥_0)^{F_{\mathbb{m}}}(Ӿ_1.¥_1)^{F_{\mathbb{m}+1}}$

For every integer $n \geq 0$

1. $$Ӿ_{\mathbb{m}+2} = \left\{\begin{matrix} Ӿ_0^{\frac{1}{2}\left(F_{\mathbb{m}+1}+3.\left[\frac{\mathbb{m}+2}{3}\right]-\mathbb{m}-1\right)}.¥_0^{\frac{1}{2}\left(F_{\mathbb{m}+1}-3.\left[\frac{\mathbb{m}+2}{3}\right]+\mathbb{m}-1\right)} \\ \alpha_1^{\frac{1}{2}(F_{\mathbb{m}+2}-3.\left[\frac{\mathbb{m}}{3}\right]+\mathbb{m}-1)}.¥_1^{\frac{1}{2}(F_{\mathbb{m}+2}+3.\left[\frac{\mathbb{m}}{3}\right]-\mathbb{m}+1)F_{3\mathbb{m}+5}} \end{matrix}\right\}$$

2. $$¥_{\mathbb{m}+2} = \left\{\begin{matrix} Ӿ_0^{\frac{1}{2}\left(F_{\mathbb{m}+1}-3.\left[\frac{\mathbb{m}+2}{3}\right]+\mathbb{m}+1\right)}.¥_0^{\frac{1}{2}\left(F_{\mathbb{m}+1}+3.\left[\frac{\mathbb{m}+2}{3}\right]-\mathbb{m}-1\right)} \\ Ӿ_1^{\frac{1}{2}(F_{\mathbb{m}+2}+3.\left[\frac{\mathbb{m}}{3}\right]-\mathbb{m}+1)}.¥_1^{\frac{1}{2}(F_{\mathbb{m}+2}+3.\left[\frac{\mathbb{m}}{3}\right]+\mathbb{m}-1)} \end{matrix}\right\}$$

where [] denote for greatest integer function.By utilizing Binet's formula, Cassini's identities, and Catalan's Identity, Gupta, Panwar, and Sikhwal [54] presented several features of the Generalized Fibonacci sequence.

(1) $V_{\mathbb{m}-2}V_{\mathbb{m}} - V_{\mathbb{m}-1}^2 = (-1)^{\mathbb{m}}2^{\mathbb{m}+1}$

(2) $\mathbb{U}_{\mathbb{m}+2}\mathbb{U} - \mathbb{U}_{\mathbb{m}-1}^{2} = (-1)^{\mathbb{m}}2^{\mathbb{m}+3}$

(3) $\mathbb{V}_{\mathbb{m}-r-1}\mathbb{V}_{\mathbb{m}+r-1} - \mathbb{V}_{\mathbb{m}-1}^{2} = (-1)^{\mathbb{m}-r+1}2^{\mathbb{m}-r}\mathbb{V}_{r-1}^{2}$

(4) $\mathbb{U}_{\mathbb{m}-r-1}\mathbb{U}_{\mathbb{m}+r-1} - \mathbb{U}_{\mathbb{m}-1}^{2} = (-1)^{\mathbb{m}-r+1}2^{\mathbb{m}-r}\mathbb{U}_{r-1}^{2}$

The following are some standard and determinant identities of generalized Fibonacci-Lucas sequences that M. Singh, Y. K. Gupta, and O. Sikhwal [55] proposed using Binet's formula and other straightforward techniques.

Sum of First $\mathbb{m}$ terms:

If B_n is the nth integer in the generalized Fibonacci - Lucas sequence, then the sum of the first n terms is

$$(B_1 + B_2 + B_3 + \dots + B_{\mathbb{m}}) = \sum_{k=1}^{\mathbb{m}} B_k = B_{n+2} - s$$

Sum of First $\mathbb{m}$ terms with even indices:

The sum of the first n terms with even indices, given that B_n is the nth element of the Fibonacci-Lucas sequence, is

$$(B_2 + B_4 + B_6 + \dots + B_{2\mathbb{m}}) = \sum_{k=1}^{\mathbb{m}} B_{2k} = B_{2\mathbb{m}+1} - s$$

Some standard identities and determinant identities of generalized Fibonacci-Lucas sequences were defined as Explicit Sum Formula by M. Singh, O. Sikhwal, and Y. K. Gupta [55].$G_{\mathbb{m}}$ should represent the $\mathbb{m}^{th}$ term in the generalized Fibonacci-Lucas sequence.Then

$$B_{\mathbb{m}} = 2b\sum_{k=0}^{\left[\frac{\mathbb{m}}{2}\right]}\binom{\mathbb{m}-k}{k} + (s-2b)\sum_{k=0}^{\left[\frac{\mathbb{m}-1}{2}\right]}\binom{\mathbb{m}-k-1}{k}$$

The authors C. Kzlate, B. Ekim, N. Tuglu, and T. Kim [69] provide the definitions of the families of three-variable polynomials along with the newly generalized polynomials that are connected to the generating functions of the well-known polynomials and literary numbers. These definitions may be found in [28]. The following definition describes what is known as a generating function for a novel and varied family of polynomials with three variables: by $S_j = S_j(x, y, z; k, m, \mathbb{m}, c)$:

$$T = M(x, y, z; k, m, \mathbb{m}, c) = \sum_{j=0}^{\infty} S_j t^j = \frac{1}{1 - x^k t - y^m t^{m+\mathbb{m}} - Z^c t^{m+\mathbb{m}+c}}$$

Following that, the partial differential equations for brand-new polynomials are derived as

$$\frac{\partial}{\partial x} S_j = k x^{k-1} \sum_{l=0}^{j-1} S_{j-l-1} S_l$$

$$\frac{\partial}{\partial y} S_j = \sum_{l=0}^{j-m-\mathbb{m}} m y^{m-1} S_{j-m-\mathbb{m}-l} S_l$$

$$\frac{\partial}{\partial z} S_j = c Z^{c-1} \sum_{l=0}^{j-m-\mathbb{m}-c} S_{j-m-\mathbb{m}-c-l} S_l$$

1.28 Objectives defined of the Thesis

Considering the previously completed research in the field of the Coupled Fibonacci and Lucas sequence, the objectives of the research work are:

- To obtain new identities and some special representations of the Coupled Lucas sequence of numbers and polynomials.
- To find the application of Coupled Fibonacci and Lucas sequences
- To obtain new generalizations and extensions of the Coupled Fibonacci sequence of numbers and polynomials.

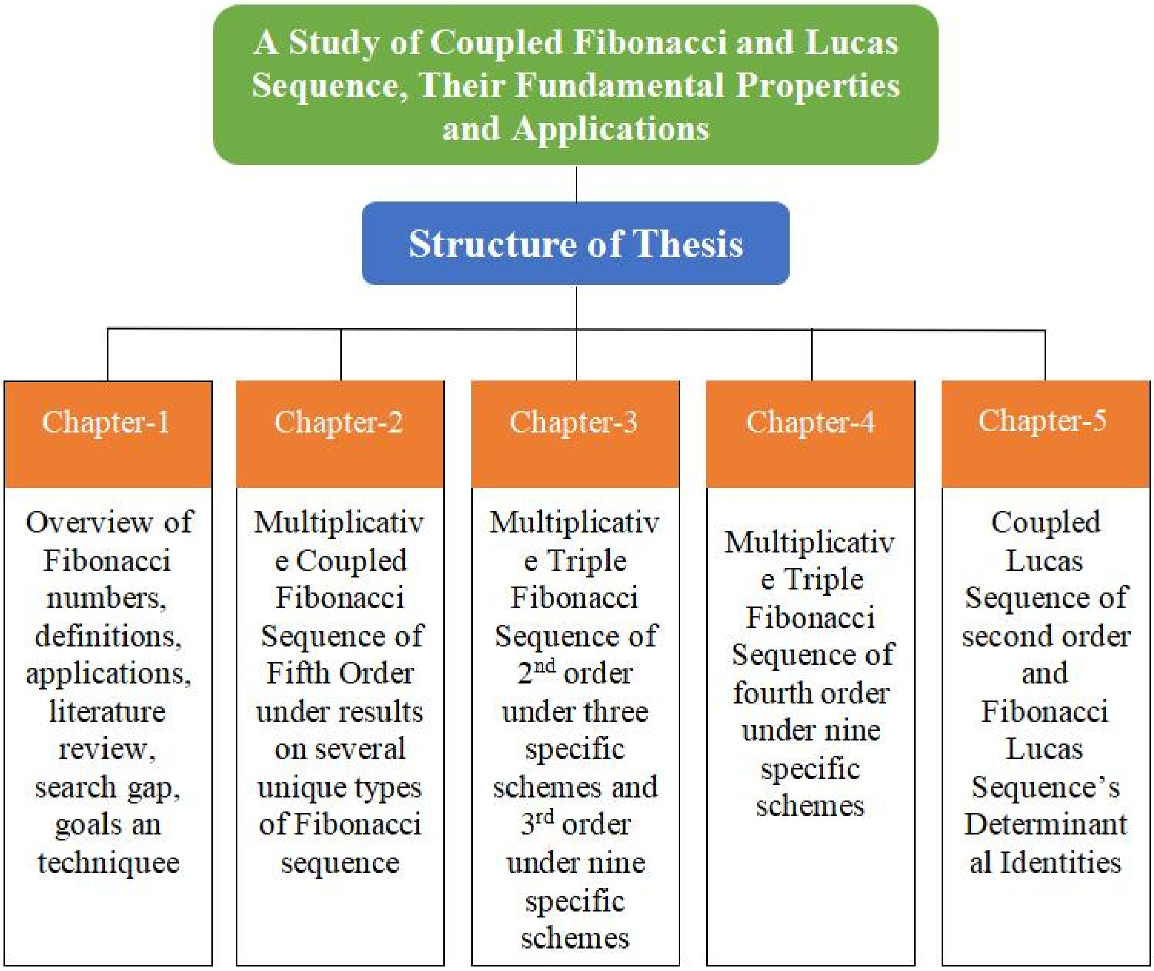
1.29 Methodology used in the Research Work

The chosen methodology is described below in order to accomplish the stated goal:

- Fifth order Multiplicative Coupled Fibonacci sequence attributes will be derived using the principle of mathematical induction. Additionally, the outcomes will be confirmed by those that have already been demonstrated for the Fibonacci numbers by the application of some specific Schemes.
- The procedure for finding the identities of the Multiplicative Coupled Fibonacci sequence and the Multiplicative Triple Fibonacci sequence will serve the outcomes of these sequences in a different order than the order in which they appear in the sequences themselves.

- Work on Coupled Lucas sequence and Generalized Lucas sequence with their identities
  and determinantal identities will be done utilizing a few properties of Lucas sequence and determinantal.

1.30 Structure of Thesis



Chapter-2

Multiplicative Coupled Fibonacci Sequence of Fifth Order

The work from this chapter has been published in the form of research paper entitled “Multiplicative Coupled Fibonacci Sequence of Fifth Order” AIP Conference Proceedings RAFAS-2021 (Scopus Indexed).

2.1 Introduction

Recent years have seen a significant investment of time into the Fibonacci Sequence (FS). It is not as well recognized for its Multiplicative FS. Atanassov K.T. [26] first investigated the generality and assets of the FS [27], [49] and [50]. K.T. Atanassov's [26] article describes the utilization of the Multiplicative Coupled Fibonacci Sequence (MCFS) from four distinct vantage points. Additionally, P. Glaister [25] and P. Hope [59] did research on MFS.

2.2 MCFS of Second Order

Let $\{Ӿ_i\}_{i=0}^{\infty}$ and $\{¥_i\}_{i=0}^{\infty}$be two infinite sequences and called 2-F Sequence or Coupled Fibonacci Sequence (CFS) with basic value a, b, c and d. then all the distinct schemes of MCFS are as follows:

$$Ӿ_0 = a, ¥_0 = b, Ӿ_1 = c, ¥_1 = \mathrm{d}$$

First Scheme:

$$Ӿ_{m+2} = ¥_{m+2}.¥_{m}, \qquad m \geq 0 \tag{2.1}$$
$$¥_{m+2} = Ӿ_{m+1}.Ӿ_{m}, \qquad m \geq 0$$

Second Scheme:

$$Ӿ_{m+2} = Ӿ_{m+1}.¥_{m}, \qquad m \geq 0 \tag{2.2}$$
$$¥_{m+2} = ¥_{m+1}.Ӿ_{m}, \qquad m \geq 0$$

Third Scheme:

$$Ӿ_{m+2} = ¥_{m+2}.Ӿ_{m}, \qquad m \geq 0 \tag{2.3}$$
$$¥_{m+2} = Ӿ_{m+1}.¥_{m}, \qquad m \geq 0$$

Fourth Scheme:

$$Ӿ_{m+2} = Ӿ_{m+1}.Ӿ_{m}, \qquad m \geq 0 \tag{2.4}$$
$$¥_{m+2} = ¥_{m+1}.¥_{m}, \qquad m \geq 0$$

Table 2.1: First ten terms of Scheme 2.1

m	$Ӿ_m$	$¥_m$
0	a	b
1	c	d

2	bd	ac
3	acd	bcd
4	abc^2d	$abcd^2$
5	$ab^2c^2d^3$	$a^2bc^3d^2$
6	$a^4b^4c^7d^6$	$a^4b^4c^6d^7$
7	$a^6b^7c^{10}d^{11}$	$a^7b^6c^{11}d^{10}$
8	$a^6b^7c^{10}d^{11}$	$a^7b^6c^{11}d^{10}$
9	$a^{11}b^{10}c^{17}d^{17}$	$a^{10}b^{11}c^{17}d^{17}$

2.3 MCFS of Third Order

Let $\{Ӿ_i\}_{\mathrm{i}=0}^{\infty}$ and $\{¥_i\}_{\mathrm{i}=0}^{\infty}$ be two infinite sequences and called 2-F Sequence or CFS with initial value a, b, c, d, e and f. There are eight specific schemes. G.P.S. Rathore, Shweta Jain and O.P. Sikhwal [37] studied various results of 3rd order MCFS.

2.4 MCFS of Fourth Order:

Let $\{Ӿ_i\}_{\mathrm{i}=0}^{\infty}$ and $\{¥_i\}_{\mathrm{i}=0}^{\infty}$ be two infinite sequences with initial value a, b, c, d, e, f, g and h. A. D. Godase [40] studied many results of fourth order MCFS.Here, we present some different identities on MCFS of Fifth order under two specific schemes.

2.5 MCFS of Fifth Order:

Let $\{Ӿ_i\}_{i=0}^{\infty}$ and $\{¥_i\}_{i=0}^{\infty}$be two infinite sequences with basic value $a, b, c, d, e, f, g, h.i$ and j. Fifth order MCFS describes the following ways:

$$Ӿ_{\mathbb{m}+5} = ¥_{\mathbb{m}+4}.¥_{\mathbb{m}+3}.¥_{\mathbb{m}+2}.¥_{\mathbb{m}+1}.¥_{\mathbb{m}}, \qquad \mathbb{m} \geq 0 \tag{2.5}$$

$$¥_{\mathbb{m}+5} = Ӿ_{\mathbb{m}+4}.Ӿ_{\mathbb{m}+3}.Ӿ_{\mathbb{m}+2}.Ӿ_{\mathbb{m}+1}.Ӿ_{\mathbb{m}}, \qquad \mathbb{m} \geq 0$$

There are thirty-two schemes under fifth order Multiplicative Coupled Fibonacci Sequence. We worked on Scheme no. 2.5.

Table 2.2: Some terms of Scheme 2.5

$\mathbb{m}$	$Ӿ_{\mathbb{m}}$	$¥_{\mathbb{m}}$
0	a	b

1	c	d
2	e	f
3	g	h
4	i	j
5	$bdfhj$	$acegi$
6	$acdefghij$	$bcdefghij$
7	$ac^2de^2f^2g^2hi^2j^2$	$abcd^2e^2f^2g^2h^2i^2j^2$

2.5.1 Motivation for Studying Fifth Order MCFS

The study of fifth-order Multiplicative Coupled Fibonacci Sequences (MCFS) arises from the need to explore deeper structural and multiplicative behavior in recursive number sequences. While lower-order MCFS provide foundational insights, higher-order variants such as the fifth-order scheme introduce increased complexity and exhibit richer algebraic patterns. Scheme 2.5, in particular, involves a five-fold product of previous terms, creating highly non-linear growth, which opens the door for discovering new identities with potential mathematical and applied implications.

This investigation aims to generalize the known properties of second, third, and fourth-order MCFS to a broader setting, offering a more holistic understanding of how multiplicative coupling behaves at a higher order. The chosen scheme (2.5) highlights the delicate interplay between recursive depth and initial conditions, paving the way for uncovering elegant identities and recurrence properties that could find utility in cryptography, dynamic system modeling, and combinatorial analysis.

Now we present a few results of MCFS of fifth order under Scheme no. 2.5:

2.6 Main Identities:

Theorem 2.1: For every integer $𝕞 \geq 6$:

$$¥_{𝕞-1}.Ӿ_{𝕞+5} = Ӿ_{𝕞-1}.¥_{𝕞+5}$$

Proof: We use induction hypothesis to prove this result:

If $𝕞 = 6,$ then

$¥_5.Ӿ_{11} = ¥_5.¥_{10}.¥_9.¥_8.¥_7.¥_6$ (By Scheme 2.5)

$= ¥_5.Ӿ_9.Ӿ_8.Ӿ_7.Ӿ_6.Ӿ_5.¥_9.¥_8.¥_7.¥_6$ (By Scheme 2.5)

$$=Ӿ_5.Ӿ_9.Ӿ_8.Ӿ_7.Ӿ_6.¥_9.¥_8.¥_7.¥_6.¥_5$$

$$=Ӿ_5.Ӿ_9.Ӿ_8.Ӿ_7.Ӿ_6.Ӿ_{10}$$

$$=Ӿ_5.Ӿ_{10}.Ӿ_9.Ӿ_8.Ӿ_7.Ӿ_6$$

$$=Ӿ_5.¥_{11}$$

The outcome is valid for $\mathtt{m} = 6$. Assume the identity holds for some integer $\mathtt{m}$. We now prove it holds for $\mathtt{m}+1$. Then by using Scheme no. 2.5

$$¥_{\mathtt{m}}.Ӿ_{\mathtt{m}+6} = ¥_{\mathtt{m}}.¥_{\mathtt{m}+5}.¥_{\mathtt{m}+4}.¥_{\mathtt{m}+3}.¥_{\mathtt{m}+2}.¥_{\mathtt{m}+1}$$

$$=¥_{\mathtt{m}}.Ӿ_{\mathtt{m}+4}.Ӿ_{\mathtt{m}+3}.Ӿ_{\mathtt{m}+2}.Ӿ_{\mathtt{m}+1}.Ӿ_{\mathtt{m}}.¥_{\mathtt{m}+4}.¥_{\mathtt{m}+3}.¥_{\mathtt{m}+2}.¥_{\mathtt{m}+1}$$

$$=Ӿ_{\mathtt{m}}.Ӿ_{\mathtt{m}+4}.Ӿ_{\mathtt{m}+3}.Ӿ_{\mathtt{m}+2}.Ӿ_{\mathtt{m}+1}.¥_{\mathtt{m}+4}.¥_{\mathtt{m}+3}.¥_{\mathtt{m}+2}.¥_{\mathtt{m}+1}.¥_{\mathtt{m}}$$

$$=Ӿ_{\mathtt{m}}.Ӿ_{\mathtt{m}+4}.Ӿ_{\mathtt{m}+3}.Ӿ_{\mathtt{m}+2}.Ӿ_{\mathtt{m}+1}.¥_{\mathtt{m}+5}$$

$$=Ӿ_{\mathtt{m}}.¥_{\mathtt{m}+5}.Ӿ_{\mathtt{m}+4}.Ӿ_{\mathtt{m}+3}.Ӿ_{\mathtt{m}+2}.Ӿ_{\mathtt{m}+1}$$

$$=Ӿ_{\mathtt{m}}.¥_{\mathtt{m}+6}$$

Hence the identity holds for all integers $\mathtt{m} \geq 6$

Theorem 2.2: For every integer $\mathtt{m} \geq 0$:

$$\frac{\prod_{k=0}^{\mathtt{m}+6} Ӿ_k}{\prod_{k=0}^{\mathtt{m}+6} ¥_k} = \frac{Ӿ_0Ӿ_1 \ldots\ldots Ӿ_n}{¥_0¥_1 \ldots\ldots ¥_n}$$

Proof: With the help of induction hypothesis, we prove this result:

If $\mathtt{m} = 0$ then,

$$\frac{\prod_{k=0}^{6} Ӿ_k}{\prod_{k=0}^{6} ¥_k} = \frac{Ӿ_0Ӿ_1Ӿ_2Ӿ_3Ӿ_4Ӿ_5Ӿ_6}{¥_0¥_1¥_2¥_3¥_4¥_5¥_6}$$

$$= \frac{Ӿ_0¥_6Ӿ_6}{¥_0Ӿ_6¥_6}$$

$$= \frac{Ӿ_0}{¥_0}$$

The result is hold for $\mathtt{m} = 0$. Assume the identity holds for some integer $\mathtt{m}$

We now prove it holds for $\mathtt{m}+1$. Then

$$\frac{\prod_{k=0}^{\mathtt{m}+7} Ӿ_k}{\prod_{k=0}^{\mathtt{m}+7} ¥_k} = \frac{Ӿ_{\mathtt{m}+7}Ӿ_{\mathtt{m}+6}Ӿ_{\mathtt{m}+5}Ӿ_{\mathtt{m}+4}Ӿ_{\mathtt{m}+3}Ӿ_{\mathtt{m}+2}Ӿ_{\mathtt{m}+1}Ӿ_{\mathtt{m}}Ӿ_{\mathtt{m}-1}Ӿ_{\mathtt{m}-2} \ldots .\, Ӿ_1Ӿ_0}{¥_{\mathtt{m}+7}¥_{\mathtt{m}+6}¥_{\mathtt{m}+5}¥_{\mathtt{m}+4}¥_{\mathtt{m}+3}¥_{\mathtt{m}+2}¥_{\mathtt{m}+1}¥_{\mathtt{m}}¥_{\mathtt{m}-1}¥_{\mathtt{m}-2} \ldots .\, ¥_1¥_0}$$

$$= \frac{Ӿ_{\mathtt{m}+7}¥_{\mathtt{m}+7}Ӿ_{\mathtt{m}+1}Ӿ_{\mathtt{m}} \ldots .\, Ӿ_1Ӿ_0}{¥_{\mathtt{m}+6}Ӿ_{\mathtt{m}+7}¥_{\mathtt{m}+1}¥_{\mathtt{m}} \ldots .\, ¥_1¥_0}$$

$$= \frac{Ӿ_0 Ӿ_1 \ldots\ldots Ӿ_{\mathbb{m}+1}}{¥_0 ¥_1 \ldots\ldots ¥_{\mathbb{m}+1}}$$

Hence the identity holds for all integer $\mathbb{m} \geq 0$

Theorem 2.3: For every integers $\mathbb{m} \geq 1$:

(a) $Ӿ_{6\mathbb{m}+6} = \frac{\prod_{\theta=0}^{6\mathbb{m}+5} ¥_\theta}{\prod_{\theta=0}^{6\mathbb{m}} ¥_\theta}$

(b) $¥_{6\mathbb{m}+6} = \frac{\prod_{\theta=0}^{6\mathbb{m}+5} Ӿ_\theta}{\prod_{\theta=0}^{6\mathbb{m}} Ӿ_\theta}$

(c) $Ӿ_{6\mathbb{m}+7} = \frac{\prod_{\theta=0}^{6\mathbb{m}+6} ¥_\theta}{\prod_{\theta=0}^{6\mathbb{m}+1} ¥_\theta}$

(d) $¥_{6\mathbb{m}+7} = \frac{\prod_{\theta=0}^{6\mathbb{m}+6} Ӿ_\theta}{\prod_{\theta=0}^{6\mathbb{m}+1} Ӿ_\theta}$

(e) $Ӿ_{6\mathbb{m}+8} = \frac{\prod_{\theta=0}^{6\mathbb{m}+7} ¥_\theta}{\prod_{\theta=0}^{6\mathbb{m}+2} ¥_\theta}$

(f) $¥_{6\mathbb{m}+8} = \frac{\prod_{\theta=0}^{6\mathbb{m}+7} Ӿ_\theta}{\prod_{\theta=0}^{6\mathbb{m}+2} Ӿ_\theta}$

Proof: With the help of induction hypothesis, we prove this result:

If $\mathbb{m} = 1$, then

$$\frac{\prod_{\theta=0}^{11} ¥_\theta}{\prod_{\theta=0}^{6} ¥_\theta} = \frac{¥_0 ¥_1 ¥_2 ¥_3 ¥_4 ¥_5 ¥_6 ¥_7 ¥_8 ¥_9 ¥_{10} ¥_{11}}{¥_0 ¥_1 ¥_2 ¥_3 ¥_4 ¥_5 ¥_6}$$

$$= ¥_7 ¥_8 ¥_9 ¥_{10} ¥_{11}$$

$$= Ӿ_{12}$$

The result is very for $\mathbb{m} = 1$. Assume the identity holds for some integer $\mathbb{m} + 1$. We now prove it holds for $\mathbb{m} + 2$.

$$\frac{\prod_{\theta=0}^{6\mathbb{m}+17} ¥_\theta}{\prod_{\theta=0}^{6\mathbb{m}+12} ¥_\theta} = \frac{¥_{6\mathbb{m}+6} ¥_{6\mathbb{m}+7} ¥_{6\mathbb{m}+8} \ldots\ldots. ¥_{6\mathbb{m}+17}}{¥_{6\mathbb{m}+1} ¥_{6\mathbb{m}+2} \ldots\ldots. ¥_{6\mathbb{m}+12}} \frac{\prod_{\theta=0}^{6\mathbb{m}+5} ¥_\theta}{\prod_{\theta=0}^{6\mathbb{m}} ¥_\theta}$$

$$= \frac{¥_{6\mathbb{m}+6} ¥_{6\mathbb{m}+7} ¥_{6\mathbb{m}+8} \ldots\ldots. ¥_{6\mathbb{m}+17} Ӿ_{6\mathbb{m}+6}}{¥_{6\mathbb{m}+1} ¥_{6\mathbb{m}+2} \ldots\ldots. ¥_{6\mathbb{m}+12}}$$

$$= \frac{¥_{6\mathbb{m}+13} ¥_{6\mathbb{m}+14} ¥_{6\mathbb{m}+15} ¥_{6\mathbb{m}+16} ¥_{6\mathbb{m}+17} Ӿ_{6\mathbb{m}+6}}{¥_{6\mathbb{m}+1} ¥_{6\mathbb{m}+2} ¥_{6\mathbb{m}+3} ¥_{6\mathbb{m}+4} ¥_{6\mathbb{m}+5}}$$

$$= ¥_{6\mathbb{m}+13} ¥_{6\mathbb{m}+14} ¥_{6\mathbb{m}+15} ¥_{6\mathbb{m}+16} ¥_{6\mathbb{m}+17}$$

$$= Ӿ_{6\mathbb{m}+18}$$

Hence the identity holds for all integers $\mathbb{m} \geq 1$.

Theorem 2.4: For every integers $\mathbb{m} \geq 1, \theta \geq 3, \vartheta \geq 3$:

(a) $\left(\text{Ӿ}_{\theta \mathrm{m}+\vartheta-1}\right)\left(\frac{\text{Ӿ}_{\theta \mathrm{m}+\vartheta}}{\text{¥}_{\theta \mathrm{m}+\vartheta}}\right)=\frac{\text{Ӿ}_{\theta \mathrm{m}+\vartheta}\text{Ӿ}_{\theta \mathrm{m}+\vartheta-6}}{\text{¥}_{\theta \mathrm{m}+\vartheta-1}}$

(b) $\left(\text{¥}_{\theta \mathrm{m}+\vartheta-1}\right)\left(\frac{\text{¥}_{\theta \mathrm{m}+\vartheta}}{\text{Ӿ}_{\theta \mathrm{m}+\vartheta}}\right)=\frac{\text{¥}_{\theta \mathrm{m}+\vartheta}\text{¥}_{\theta \mathrm{m}+\vartheta-6}}{\text{Ӿ}_{\theta \mathrm{m}+\vartheta-1}}$

Proof: With the help of induction hypothesis, we prove this result:

If $\mathrm{m}=1$, then

(a) $\left(\text{Ӿ}_{\theta+\vartheta-1}\right)\left(\frac{\text{Ӿ}_{\theta+\vartheta}}{\text{¥}_{\theta+\vartheta}}\right)=\frac{\text{Ӿ}_{\theta+\vartheta-1}\text{Ӿ}_{\theta+\vartheta}}{\text{Ӿ}_{\theta+\vartheta-1}\text{Ӿ}_{\theta+\vartheta-2}\text{Ӿ}_{\theta+\vartheta-3}\text{Ӿ}_{\theta+\vartheta-4}\text{Ӿ}_{\theta+\vartheta-5}}$

$$=\frac{\text{Ӿ}_{\theta+\vartheta}}{\text{Ӿ}_{\theta+\vartheta-2}\text{Ӿ}_{\theta+\vartheta-3}\text{Ӿ}_{\theta+\vartheta-4}\text{Ӿ}_{\theta+\vartheta-5}}$$

$$=\frac{\text{Ӿ}_{\theta+\vartheta}\text{Ӿ}_{\theta+\vartheta-6}}{\text{Ӿ}_{\theta+\vartheta-2}\text{Ӿ}_{\theta+\vartheta-3}\text{Ӿ}_{\theta+\vartheta-4}\text{Ӿ}_{\theta+\vartheta-5}\text{Ӿ}_{\theta+\vartheta-6}}$$

$$=\frac{\text{Ӿ}_{\theta+\vartheta}\text{Ӿ}_{\theta+\vartheta-6}}{\text{Ӿ}_{\theta+\vartheta-1}}$$

The result is very for $\mathrm{m}=1$

Assume the identity holds for some integer m. We now prove it holds for $\mathrm{m}+1$

By using Scheme no. 2.5

$$\left(\text{Ӿ}_{\theta \mathrm{m}+\theta+\vartheta-1}\right)\left(\frac{\text{Ӿ}_{\theta \mathrm{m}+\theta+\vartheta}}{\text{¥}_{\theta \mathrm{m}+\theta+\vartheta}}\right)=\frac{\text{Ӿ}_{\theta \mathrm{m}+\theta+\vartheta-1}\text{Ӿ}_{\theta \mathrm{m}+\theta+\vartheta}}{\text{Ӿ}_{\theta \mathrm{m}+\theta+\vartheta-1}\text{Ӿ}_{\theta \mathrm{m}+\theta+\vartheta-2}\text{Ӿ}_{\theta \mathrm{m}+\theta+\vartheta-3}\text{Ӿ}_{\theta \mathrm{m}+\theta+\vartheta-4}\text{Ӿ}_{\theta \mathrm{m}+\theta+\vartheta-5}}$$

$$=\frac{\text{Ӿ}_{\theta \mathrm{m}+\theta+\vartheta}}{\text{Ӿ}_{\theta \mathrm{m}+\theta+\vartheta-2}\text{Ӿ}_{\theta \mathrm{m}+\theta+\vartheta-3}\text{Ӿ}_{\theta \mathrm{m}+\theta+\vartheta-4}\text{Ӿ}_{\theta \mathrm{m}+\theta+\vartheta-5}}$$

$$=\frac{\text{Ӿ}_{\theta \mathrm{m}+\theta+\vartheta}\text{Ӿ}_{\theta \mathrm{m}+\theta+\vartheta-6}}{\text{Ӿ}_{\theta \mathrm{m}+\theta+\vartheta-2}\text{Ӿ}_{\theta \mathrm{m}+\theta+\vartheta-3}\text{Ӿ}_{\theta \mathrm{m}+\theta+\vartheta-4}\text{Ӿ}_{\theta \mathrm{m}+\theta+\vartheta-5}\text{Ӿ}_{\theta \mathrm{m}+\theta+\vartheta-6}}$$

$$=\frac{\text{Ӿ}_{\theta \mathrm{m}+\theta+\vartheta}\text{Ӿ}_{\theta \mathrm{m}+\theta+\vartheta-6}}{\text{Ӿ}_{\theta \mathrm{m}+\theta+\vartheta-1}}$$

Hence the identity holds for all integers $\mathrm{m}\geq 1$. Similar proof can be given for the remaining part (b).

2.6.1 Possible Applications of Fifth Order MCFS Identities

The identities derived in this chapter for fifth-order MCFS under Scheme 2.5 have potential relevance in various domains. The complex recurrence relations, involving multiplicative coupling over five previous terms, mirror processes in cryptographic key generation algorithms, particularly in the design of nonlinear feedback shift registers and pseudorandom number generators. The sensitivity of these sequences to initial values also makes them suitable for modeling systems with multi-layered dependencies, such as ecological population models or compound-interest-like growth phenomena. Moreover, these identities contribute to the broader mathematical landscape by offering new avenues for exploration in combinatorics and discrete mathematics. They may also serve as theoretical test cases for evaluating the behavior of multiplicative recursive algorithms in symbolic computation. By understanding these structures, researchers can better assess their potential role in secure communication protocols, error detection schemes, and the mathematical foundations of recursive sequence analysis.

2.6.2 Some Mathematical Properties of Fifth Order MCFS

The fifth-order MCFS defined under Scheme 2.5 exhibits several interesting mathematical properties arising from its recursive multiplicative structure. These properties highlight the richness and potential utility of such sequences in theoretical and applied contexts.

Monotonicity

The growth of the sequence terms is highly dependent on the initial conditions. When all initial terms are greater than or equal to 1, the terms in both $Ӿ_{\mathbb{m}}$ and $Ɏ_{\mathbb{m}}$ sequences exhibit monotonic increasing behavior. This is due to the multiplicative nature of the recurrence relation, where each new term is a product of five positive previous terms. However, this Monotonicity may not hold if any of the initial terms are less than 1 or include 0.

Boundedness

The fifth-order MCFS is generally unbounded for positive initial conditions. As the sequences grow through multiplicative recurrence, they tend to increase rapidly, leading to exponential or even super-exponential growth. However, bounded behavior may arise under specific modular constraints or when initial terms include values such as 0 or 1, which suppress growth due to multiplication by a neutral or null factor.

Symmetry and Periodicity

The structure of Scheme 2.5 does not inherently produce symmetry or periodicity in the classical sense. Unlike additive Fibonacci sequences, which can exhibit periodicity modulo m, the multiplicative fifth-order MCFS lacks evident cyclical behavior under normal conditions.

Periodicity might emerge under modular arithmetic, which could be an area of further exploration.

Sensitivity to Initial Conditions

The fifth-order MCFS is highly sensitive to initial values. A slight change in any one of the initial ten values can lead to significantly different trajectories for X_m and Y_m. This sensitivity is a result of the deep coupling and the multiplicative propagation of initial differences across iterations. Such a feature makes these sequences suitable for applications like pseudorandom number generation and cryptographic systems, where sensitivity and unpredictability are valuable traits.

2.7 Conclusion:

We presented fifth order MCFS under a particular scheme in this chapter.The exploration of the fifth-order MCFS under a specific scheme has uncovered complex mathematical structures and promising applications. This research expanded upon classical Coupled Fibonacci Sequences by incorporating multiplicative elements, adding depth and complexity to the behavior of the sequences. Detailed analysis of recurrence relations and initial values revealed distinctive patterns, highlighting how the sequence's properties are strongly influenced by the chosen scheme. This study enhances our understanding of the structural characteristics of higher-order Fibonacci sequences and underscores their sensitivity to initial conditions. We can similarly describe other fifth order schemes.

"In the fifth-order multiplicative coupled Fibonacci sequence, the interplay of multiple recurrence layers magnifies both structural complexity and generative potential, offering a mathematical landscape where unpredictability and order coexist in delicate balance."

Chapter-3

Multiplicative Triple Fibonacci Sequence of Second and Third Order

The work presented in this chapter has been published in the form of research papers entitled “Multiplicative Triple Fibonacci Sequence of Third Order” and “Multiplicative Triple Fibonacci Sequence of Second Order under Three Specific Schemes and Third Order under Nine Specific Schemes” in Scopus Indexed Journals (Q_3).

3.1. Introduction

The Triple Fibonacci Sequence, also known as the TFS, is the most recent and significant development in the field of Fibonacci sequence. A new supervision for the generalization of the Coupled Fibonacci Sequence (CFS) is represented by the TFS. Atanassov was the one who initiated the CFS for the first time [58], and he was also the one who investigated a wide variety of peculiar qualities and a fresh principle for the generalization of the Fibonacci Sequence (FS). The Fibonacci Sequence and its generalization both have a wide variety of features and applications that are seen to be provocative. The book written by Koshy [4] is an important starting point for various applications. In 1985, K.T. Attanasov [58] was the one who brought the concept of CFS to the public's attention. He brought forth a fresh design for the FTS. The FTS connects three sequences of integers, where the components of each sequence are generalizations of the components of the other two sequences.

The Multiplicative Coupled Fibonacci Sequence (MCFS) and the Additive Fibonacci Sequence (TFS) have been calculated by B. Singh and O. Sikhwal [51] with some important features. It was first presented by J. Z. Lee and J. S. Lee [48] in the Initially Additive Triple Sequence. Atanassov presents a novel concept for Additive FTS in the form of the 3-Fibonacci Sequence, which is also referred to as the 3-F Sequence.

Triple Fibonacci sequences (TFS) represent a novel approach to generalizing the Coupled Fibonacci Sequence (CFS).The TFS is a significant advancement in the field of FS and extends the CFS, offering a wide range of intriguing properties and applications. The multiplicative triple Fibonacci sequences (MTFS), an extension of the classical FS, have garnered substantial interest in recent mathematical research, particularly in the context of Second and Third-order derivations under specific Schemes. The FS, known for its ubiquity in nature and applications across diverse fields, serves as the foundation for exploring the multiplicative variations proposed in this study.

The TFS represents a fresh approach to the generalization of the CFS. It is a significant advancement in the field of FS and a generalization of the CFS, offering a wide range of fascinating properties and applications. The MTFS, an extension of the classical FS, has garnered substantial interest in recent mathematical research, particularly concerning Second and Third-order derivations under specific Schemes. The FS, known for its ubiquity in nature and applications across diverse fields, composes the foundation for exploring the multiplicative variations proposed in this study.

There has been a great deal of research on the TFS. J. Z. Lee and J. S. Lee [38] were the first to propose the TFS. Koshy's book [4] is an excellent source for these applications. In 1985, Attanasov popularized the concept of the CFS and introduced a new TFS design. The TFS connects three integer sequences, where the elements of one sequence are part of the generalization of the others, and vice versa. Singh and Sikhwal computed the MCFS and additive TFS, both have significant properties.

Under two distinct Schemes, Kiran Singh Sisodiya, Vandana Gupta, and Kiran Sisodiya [41] investigated several features of the fourth-order MCFS. Omprakash Sikhwal, Mamta Singh, and Shweta Jain examined various aspects of the fifth-order CFS. In 2014, Krishna Kumar Sharma et al. [70] formulated the additive-linked Fibonacci sequences of r^{th} order and demonstrated their diverse features. Bijendra Singh and Omprakash Sikhwal explored both the primitive aspects of Second-order TFS and several features of additive TFS. The MTFS of the Second order was examined from multiple perspectives by Mamta Singh, Shikha Bhatnagar, and Omprakash Sikhwal [52]. The properties of Second-order MTFS were extensive by Satish Kumar, Hari Kishan, and Deepak Gupta [71]. Additionally, K.S. Sisodiya, V. Gupta, and V. H. Badshah [72] illustrated different characteristics of Second-order TFS. B. Singh, Kiran Singh Sisodiya, and Kiran Sisodiya [73] further enhanced the Second-order MTFS and provided convinced fundamental characteristics. Shoukralla [74] obtained a numerical solution to the first kind of Fredholm integral equation using the matrix form of the Second-kind chebyshev polynomials.

The Second-order MTFS introduces a novel dimension to the classical sequence by incorporating three distinct initial values and employing three specific Schemes for its evolution. This extension beyond the traditional Fibonacci paradigm unveils a richer tapestry of numerical relationships and behaviors, prompting a deeper investigation into the underlying mathematical structure. Building upon this exploration, the study delves into the Third-order MTFS, expanding its complexity by introducing nine specific Schemes. This extension amplifies the intricacies of the sequence, offering a more nuanced understanding of its behavior and potential applications. The literature surrounding FS and its derivatives has witnessed a surge in interest due to their relevance in various scientific and computational domains. Previous studies have often focused on additive properties and relationships within the Fibonacci framework. However, the current research contributes significantly by extending the scope to multiplicative operations under specific Schemes, thereby paving the way for novel insights into the mathematical landscape. This literature review sets the stage for a comprehensive analysis of MTFS, emphasizing its potential impact on both theoretical Mathematics and practical applications.

Overall, the MTFS of the Second and Third order, with three and nine specific Schemes appropriately, presents a unique and intricate exploration of mathematical sequences, contributing to a broader understanding of Fibonacci-related structures and their potential applications. In the Second order, the sequence is generated by considering three initial values and using a set of rules that dictate the multiplication of the last three terms to obtain the subsequent term. Exploring different Schemes adds complexity and diversity to the sequence, uncovering unique numerical behaviors. Moving into the Third order, the investigation expands to nine distinct Schemes, each contributing to the richness and complexity of the sequence. The interplay of these Schemes yields an MTFS sequence with intricate dynamics, offering mathematicians and researchers a wealth of material for analysis and exploration.

This introduction encapsulates a pioneering study in the realm of mathematical sequences, showcasing the remarkable versatility and adaptability of the Fibonacci framework when subjected to multiplicative operations. The exploration of specific Schemes introduces a nuanced understanding of the sequence's evolution, offering a solid platform for further research and diverse applications in various mathematical and computational domains.

In this note, we present some elemental view that will be used to formulate Multiplicative Triple Fibonacci Sequence (MTFS) of Second and Third order with delightful properties.

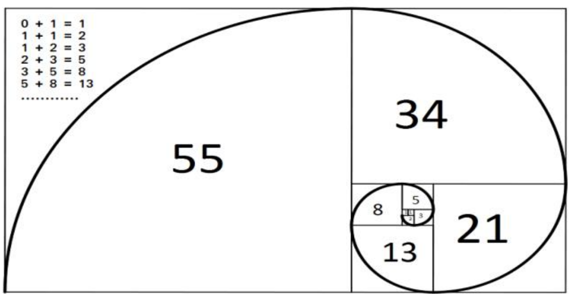


Figure 3.1: Fibonacci Numbers Spiral [77]

In Fig 3.1, The Fibonacci spiral in the figure is constructed by arranging squares whose side lengths correspond to Fibonacci numbers (1, 1, 2, 3, 5, 8, 13, 21, 34, 55, etc.). Each square's dimensions represent the sequence's increasing values. By connecting the corners of these squares with quarter-circle arcs, the figure forms a spiral. This spiral visually demonstrates the Fibonacci sequence's exponential growth pattern and its approximation of the golden ratio. Such spirals are commonly found in nature, such as in the arrangement of sunflower seeds, shells, and galaxies, highlighting the connection between Mathematics and natural phenomena.

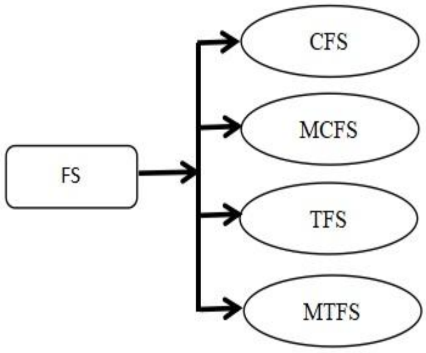


Figure 3.2: Types of Fibonacci Sequence

Fig 3.2 illustrates different variations of the Fibonacci sequence. CFS are modified versions where each term is generated based on a coupling between previous terms. MCFS variations where the relationship between the terms involves multiplication and coupling of previous terms.TFS is an extension of the Fibonacci sequence where the next term is calculated based on the previous three term instead of two.MTFS is an extension where the terms are calculated based on a multiplicative relationship among three previous terms.

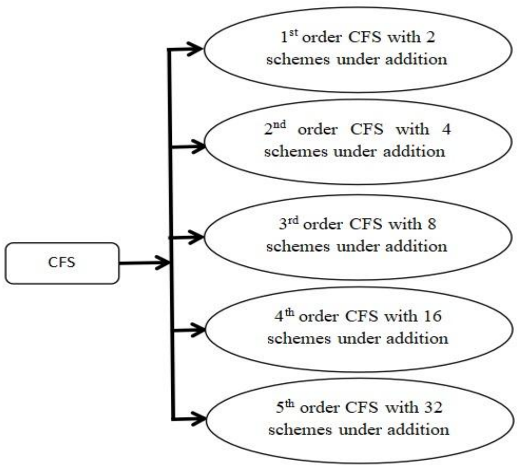


Figure 3.3: Hierarchical Structure of CFS Under Addition

Fig 3.3 illustrates hierarchical structure of the CFS under addition, with different orders and Schemes:1st order CFS represents the basic CFS with two Schemes, where the terms are derided by adding two coupled sequences.2nd order CFS are more complex sequence with four Schemes, extending the coupling process to a Second level. 3rd order CFS involves eight Schemes, further expanding the coupling and addition process. 4th order CFS are more advanced version with sixteen Schemes, counting the pattern of CFS under addition.5th order CFS are most complex, involving thirty-two Schemes, representing the highest order of coupling in this structure.

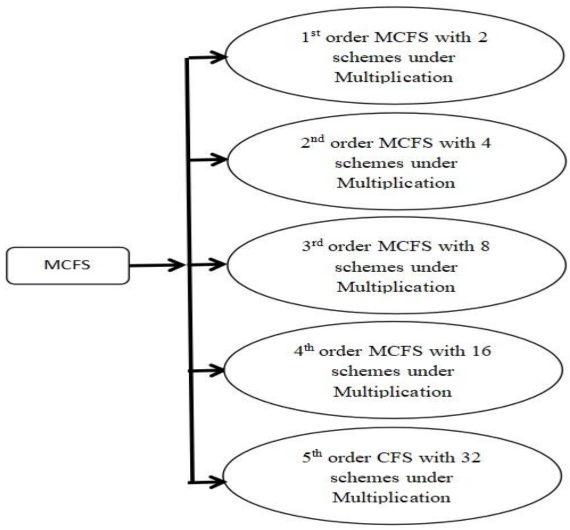


Figure 3.4: Structure of MCFS Under Multiplication

Fig. 3.4 outlines the structure of the MCFS under multiplication , showcasing different orders and Schemes. 1st order MCFS is most basic form of MCFS with two Schemes, where terms are generated using a multiplication process between coupled sequences.2nd order MCFS is more advanced version with four Schemes, extending the multiplication based coupling to a Second level.3rd order MCFS increases in complexity with eight Schemes, involving further multiplication of coupled sequence.4th order MCFS is higher level sequence with sixteen Schemes, expanding the multiplicative coupling process even further.5th order MCFS is most complex sequence, involving thirty-two Schemes, representing the highest level of multiplicative coupling in the FS structure.

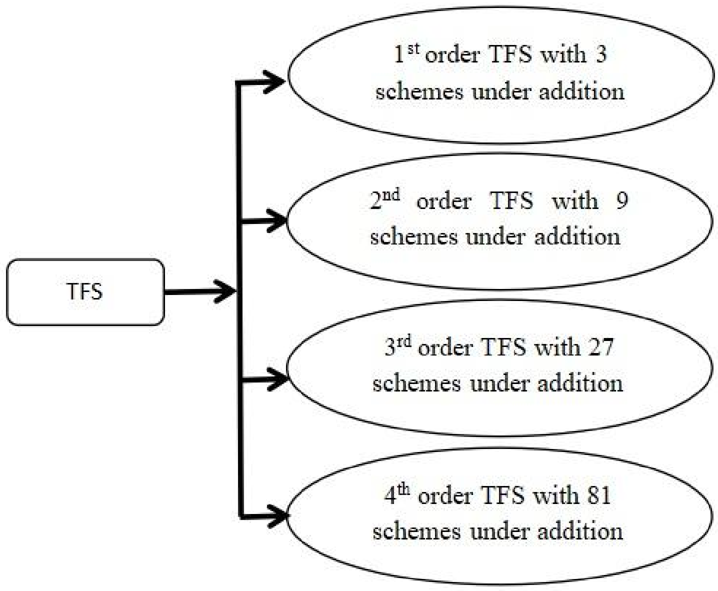


Figure 3.5: Structure of TFS

Fig. 3.5 represents the structure of the TFS under addition, featuring different orders and Schemes. 1st order TFS is the basic form of the TFS, where each term is derived from the sum of the previous three terms, with three Schemes for generating the sequence. Second order TFS is more complex extension, incorporating with nine Schemes, where the coupling and addition process are applied at the Second level. Third order TFS involves twenty-seven Schemes, expanding the addition process to further include previous terms at an even higher level. 4th order TFS is the most complex version in this series, with eighty-one Schemes, involving a highly intricate addition process across multiple levels.

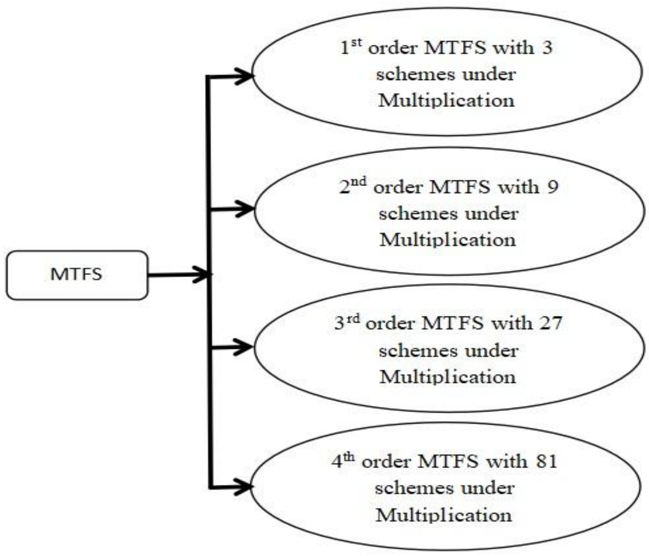


Figure 3.6: Structure of MTFS

Fig. 3.6 illustrates the structure of the MTFS, showcasing increasing complexity through various orders and Schemes. 1st order MTFS generates terms by multiplying the previous three terms, utilizing three Schemes for sequence generation. Second order MTFS is a more complex version that applies the multiplicative relationship at a Second level, incorporating nine Schemes to enhance the sequence formation. Third order MTFS further expands the multiplicative structure, using twenty-seven Schemes for generating terms through the multiplication of three previous terms in more intricate patterns. Fourth order MTFS is most advanced level in this series, with eighty-one Schemes, where the multiplicative relationships become increasingly elaborate across multiple levels.

3.1.1 Motivation for Studying Second and Third Order MTFS

Second and third-order MTFS present a distinctive mathematical framework where multiplicative recursions interact with structured generation schemes. From a theoretical perspective, their study is motivated by the richness of their nonlinear behaviour, the complexity arising from sensitivity to initial conditions, and the intricate patterns that emerge through higher-order relations. These properties invite deeper analytical exploration, encouraging the development of new mathematical tools for sequence classification, growth analysis, and stability examination. Furthermore, investigating these structures contributes to a broader understanding of how multiplicative processes can produce both predictable periodicities and unpredictable fluctuations, offering fertile ground for future theoretical advancements.

3.2 Second Order MTFS

Let $\{X_i\}_{i=0}^{\infty}$, $\{Y_i\}_{i=0}^{\infty} and \{Z_i\}_{i=0}^{\infty}$ be three infinite sequences with initial values $a, b, c, d,$ e and f which are referred to as the 3-F Sequence or TFS.

If $X_0 = a$, $Y_0 = b$, $Z_0 = c$, $X_1 = d$, $Y_1 =$ e and $Z_1 = f$

Then the there are nine different Multiplicative Triple Fibonacci Sequence Schemes, each defined by initial values a, b and c. These sequences evolve through distinct multiplicative relationships, generating unique patterns and behaviors. Additionally, we will introduce parameters d, e, and f to further enhance the complexity and richness of these sequences.

J. Z.Lee and J.S.Lee [48] defined following nine different Schemes of multiplicative triple Fibonacci sequences are as follows:

Table 3.1 Second Order MTFS Schemes

Scheme	X_{n+2}	Y_{n+2}	Z_{n+2}
1	$Y_{n+1}.\,Z_{n}$	$Z_{n+1}.\,X_{n}$	$X_{n+1}.\,Y_{n}$
2	$Z_{n+1}.\,Y_{n}$	$X_{n+1}.\,Z_{n}$	$Y_{n+1}.\,X_{n}$
3	$X_{n+1}.\,Y_{n}$	$Y_{n+1}.\,Z_{n}$	$Z_{n+1}.\,X_{n}$
4	$Y_{n+1}.\,X_{\mathbb{m}}$	$Z_{\mathbb{m}+1}.\,Y_{n}$	$X_{\mathbb{m}+1}.\,Z_{n}$
5	$X_{\mathbb{m}+1}.\,Z_{\mathbb{m}}$	$Y_{\mathbb{m}+1}.\,X_{\mathbb{m}}$	$Z_{\mathbb{m}+1}.\,Y_{\mathbb{m}}$
6	$Z_{\mathbb{m}+1}.\,X_{\mathbb{m}}$	$X_{\mathbb{m}+1}.\,Y_{\mathbb{m}}$	$Y_{\mathbb{m}+1}.\,Z_{\mathbb{m}}$
7	$X_{\mathbb{m}+1}.\,X_{\mathbb{m}}$	$Y_{\mathbb{m}+1}.\,Y_{\mathbb{m}}$	$Z_{\mathbb{m}+1}.\,Z_{\mathbb{m}}$
8	$Y_{\mathbb{m}+1}.\,Y_{\mathbb{m}}$	$Z_{\mathbb{m}+1}.\,Z_{\mathbb{m}}$	$X_{\mathbb{m}+1}.\,X_{\mathbb{m}}$
9	$Z_{\mathbb{m}+1}.\,Z_{\mathbb{m}}$	$X_{\mathbb{m}+1}.\,X_{\mathbb{m}}$	$Y_{\mathbb{m}+1}.\,Y_{\mathbb{m}}$

Properties of 7th, 8th and 9th Scheme.Below are the first few terms of the 7th Schemes:

Table 3.2: Some terms of 7th Scheme

$\mathbb{m}$	$X_{\mathbb{m}}$	$Y_{\mathbb{m}}$	$Z_{\mathbb{m}}$
0	a	b	c
1	d	e	f
2	ad	be	cf
3	ad^2	be^2	cf^2
4	a^2d^3	b^2e^3	c^2f^3
5	a^3d^5	b^3e^5	c^3f^5

The 8th Scheme's initial terms are listed below:

Table 3.3: Some terms of 8th Scheme

$\mathbb{m}$	$X_{\mathbb{m}}$	$Y_{\mathbb{m}}$	$Z_{\mathbb{m}}$
0	a	b	c
1	d	e	f
2	be	cf	ad
3	be^2	cf^2	ad^2
4	b^2e^3	c^2f^3	a^2d^3
5	b^3e^5	c^3f^5	a^3d^5

Following are the first few terms of the 9th Schemes:

Table 3.4: Some terms of 9th Scheme

$\mathbb{m}$	$X_{\mathbb{m}}$	$Y_{\mathbb{m}}$	$Z_{\mathbb{m}}$
0	a	b	c
1	d	e	f
2	cf	ad	be
3	cf^2	ad^2	be^2
4	c^2f^3	a^2d^3	b^2e^3
5	c^3f^5	a^3d^5	b^3e^5

O. P. Sikhwal, M. Singh, and S. Bhatnagar [52] examined a wide range of Second-order results.

3.3 Main Results of 2nd Order MTFS

We will present some other results on the MTFS of Second order under three specific Schemes and Third Order under nine Schemes in this chapter.

Now, under Schemes 7th, 8th and 9th, we introduce some results of the MTFS of Second Order:

Theorem 3.1: For each whole number $\mathbb{m}$:

(a) $\mathrm{X}_{\mathbb{m}+1} = \mathrm{X}_0^{\mathrm{F}_{\mathbb{m}}} \mathrm{X}_1^{\mathrm{F}_{\mathbb{m}+1}}$

(b) $\mathrm{Y}_{\mathbb{m}+1} = \mathrm{Y}_0^{\mathrm{F}_{\mathrm{n}}} \mathrm{Y}_1^{\mathrm{F}_{\mathrm{n}+1}}$

(c) $\mathrm{Z}_{\mathbb{m}+1} = \mathrm{Z}_0^{\mathrm{F}_{\mathbb{m}}} \mathrm{Z}_1^{\mathrm{F}_{\mathbb{m}+1}}$

Proof: These results are confirmed by the induction hypothesis.

(a) If $\mathbb{m} = 0$, then

$$X_1 = X_0^{F_0} X_1^{F_1}$$
$$= X_1$$

For $\mathbb{m} = 0$, the base case holds.

Assume the identity holds for some integer $\mathbb{m}$. Then for $\mathbb{m} + 1$

$$X_{\mathbb{m}+2} = X_{\mathbb{m}+1} X_{\mathbb{m}} \qquad \text{(By Scheme No. 7)}$$
$$= X_0^{F_{\mathbb{m}}} X_1^{F_{\mathbb{m}+1}} X_0^{F_{\mathbb{m}-1}} X_1^{F_{\mathbb{m}}} \qquad \text{(By given Hypothesis)}$$
$$= X_0^{F_{\mathbb{m}}+F_{\mathbb{m}-1}} X_1^{F_{\mathbb{m}+1}+F_{\mathbb{m}}}$$
$$= X_0^{F_{\mathbb{m}+1}} X_1^{F_{\mathbb{m}+2}}$$

The conclusion is valid for all integers $\mathbb{m} \geq 0$. Similar evidence is available for the remaining parts (b) and (c).

Example based on Theorem 3.1

Consider a Fibonacci sequence $F_{\mathbb{m}}$, in this sequence, each term is obtained by adding the two preceding terms, usually beginning with the initial values 0 and 1.

0, 1, 1, 2, 3, 5, 8, 13, 21.........

$F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5$ and so on..

Let $\{X_{\mathbb{m}}\}_{\mathbb{m}=0}^{\infty}$, $\{Y_{\mathbb{m}}\}_{\mathbb{m}=0}^{\infty}$ and $\{Z_{\mathbb{m}}\}_{n=0}^{\infty}$ be three sequences where each term is the product of the two preceding ones, such that

$$\{X_{\mathbb{m}}\}_{\mathbb{m}=0}^{\infty} = 1,3,3,9,27,243,\ldots\ldots\ldots\ldots$$

Where,

$X_0 = 1, X_1 = 3, X_2 = 3, X_3 = 9, X_4 = 27, X_5 = 243$ and so on...

$$\{Y_{\mathbb{m}}\}_{\mathbb{m}=0}^{\infty} = 2,3,6,18,108,1944\ldots\ldots\ldots$$

Where,

$Y_0 = 2, Y_1 = 3, Y_2 = 6, Y_3 = 18, Y_4 = 108, Y_5 = 1944$ and so on…

$$\{Z_{\mathbb{m}}\}_{\mathbb{m}=0}^{\infty} = 1,4,4,16,64,1024,\ldots\ldots\ldots$$

Where,

$Z_0 = 1, Z_1 = 4, Z_2 = 4, Z_3 = 16, Z_4 = 64, Z_5 = 1024$ and so on...

This example verifies Theorem 3.1 for the sequences defined above- The theorem provides a closed-form formula for the terms of the sequence generated under Scheme 7, expressing $X_{\mathtt{m}+1}$ in terms of the initial values X_0 and X_1 raised to powers based on the standard Fibonacci sequence. We will now check if the computed value of X_5 , Y_5 and Z_5 using the recurrence matches the value given by the theorem's formula.

Now we are going to apply the result of part (a) of theorem 3.1

$$X_{\mathtt{m}+1} = X_0^{F_{\mathtt{m}}} X_1^{F_{\mathtt{m}+1}}$$

Put $\mathtt{m} = 4$, $\quad X_{4+1} = X_0^{F_4} X_1^{F_{4+1}}$

$\Rightarrow \quad X_5 = X_0^{F_4} X_1^{F_5}$

$= (1)^3(3)^5$

$= 243$

Now we are going to apply the result of part (b) of theorem 3.1

$$Y_{\mathtt{m}+1} = Y_0^{F_{\mathtt{m}}} Y_1^{F_{\mathtt{m}+1}}$$

Put $\mathtt{m} = 4$, $\quad Y_{4+1} = Y_0^{F_4} Y_1^{F_{4+1}}$

$\Rightarrow \quad Y_5 = Y_0^{F_4} Y_1^{F_5}$

$= (2)^3(3)^5$

$= 1944$

Now we are going to apply the result of part (c) of theorem 3.1

$$Z_{\mathtt{m}+1} = Z_0^{F_{\mathtt{m}}} Z_1^{F_{\mathtt{m}+1}}$$

Put $\mathtt{m} = 4$, $\quad Z_{4+1} = Z_0^{F_4} Z_1^{F_{4+1}}$

$\Rightarrow \quad Z_5 = Z_0^{F_4} Z_1^{F_5}$

$= (1)^3(4)^5$

$= 1024$

As the calculations show, the results from the multiplicative recurrence (243, 1944, 1024) match exactly the results predicted by the formulas in Theorem 3.1. This provides a concrete numerical verification of the theorem's correctness for these specific initial conditions and demonstrates the utility of the closed-form expression in calculating terms directly without iterative multiplication.

Theorem 3.2: For each natural number $\mathtt{m}$;

$$\left(X_{\mathtt{m}} Y_{\mathtt{m}} Z_{\mathtt{m}}\right) = (X_0 Y_0 Z_0)^{F_{\mathtt{m}-1}} (X_1 Y_1 Z_1)^{F_{\mathtt{m}}}$$

Proof: We will confirm this result with the help of induction hypothesis

If $\mathtt{m} = 1$, then

$$\left(X_1 Y_1 Z_1\right) = (X_0 Y_0 Z_0)^{F_0}(X_1 Y_1 Z_1)^{F_1}$$
$$= (X_1 Y_1 Z_1)$$

For $\mathtt{m} = 1$, the base case holds.

Assume the identity holds for some integer $\mathtt{m}$. Then for $\mathtt{m}+1$

$$\left(X_{\mathtt{m}+1} Y_{\mathtt{m}+1} Z_{\mathtt{m}+1}\right) = (X_{\mathtt{m}} X_{\mathtt{m}-1})(Y_{\mathtt{m}} Y_{\mathtt{m}-1})(Z_{\mathtt{m}} Z_{\mathtt{m}-1})$$
$$= \left(X_{\mathtt{m}} Y_{\mathtt{m}} Z_{\mathtt{m}}\right)\left(X_{\mathtt{m}-1} Y_{\mathtt{m}-1} Z_{\mathtt{m}-1}\right)$$
$$= (X_0 Y_0 Z_0)^{F_{\mathtt{m}-1}}(X_1 Y_1 Z_1)^{F_{\mathtt{m}}}(X_0 Y_0 Z_0)^{F_{\mathtt{m}-2}}(X_1 Y_1 Z_1)^{F_{\mathtt{m}-1}}$$
$$=(X_0 Y_0 Z_0)^{F_{\mathtt{m}-1}+F_{\mathtt{m}-2}}(X_1 Y_1 Z_1)^{F_{\mathtt{m}}+F_{\mathtt{m}-1}}$$
$$=(X_0 Y_0 Z_0)^{F_{\mathtt{m}}}(X_1 Y_1 Z_1)^{F_{\mathtt{m}+1}}$$

The conclusion is valid for all integers $\mathtt{m} \geq 1$.

Example based on Theorem 3.2

The Fibonacci sequence $F_{\mathtt{m}}$ is defined such that each term is the sum of the two preceding ones, typically starting with 0 and 1.

0, 1, 1, 2, 3, 5, 8, 13, 21...........

$F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5$ and so on.

Let $\{X_{\mathtt{m}}\}_{\mathtt{m}=0}^{\infty}, \{Y_{\mathtt{m}}\}_{\mathtt{m}=0}^{\infty}$ and $\{Z_{\mathtt{m}}\}_{\mathtt{m}=0}^{\infty}$ be three sequences where each term is the product of the two preceding ones, such that

$$\{X_{\mathtt{m}}\}_{\mathtt{m}=0}^{\infty} = 2,4,8,32,256\ldots\ldots\ldots\ldots$$

Where,

$X_0 = 2, X_1 = 4, X_2 = 8, X_3 = 32, X_4 = 256$ and so on...

$$\{Y_{\mathtt{m}}\}_{\mathtt{m}=0}^{\infty} = 1,1,1,1,1,1\ldots\ldots.$$

Where,

$Y_0 = 1, Y_1 = 1, Y_2 = 1, Y_3 = 1, Y_4 = 1$ and so on...

$$\{Z_{\mathtt{m}}\}_{\mathtt{m}=0}^{\infty} = 2,3,6,18,108,1944,\ldots\ldots..$$

Where,

$Z_0 = 2, Z_1 = 3, Z_2 = 6, Z_3 = 18, Z_4 = 108$ and so on...

This example is designed to test the identity in Theorem 3.2- The theorem establishes a relationship for the product of the $\mathtt{m}$th terms of all three sequences$X_{\mathtt{m}}$, $Y_{\mathtt{m}}$ and $Z_{\mathtt{m}}$. It claims this product can be calculated solely from the products of the initial terms X_0, Y_0, Z_0 and X_1, Y_1,

Z_1 again using Fibonacci numbers as exponents. We will verify this for $\mathtt{m} = 4$.

Now we are going to apply the result of theorem 3.2

$$\left(X_{\mathtt{m}} Y_{\mathtt{m}} Z_{\mathtt{m}}\right) = (X_0 Y_0 Z_0)^{F_{\mathtt{m}-1}} (X_1 Y_1 Z_1)^{F_{\mathtt{m}}}$$

Put $\mathtt{m} = 4$

$$\left(X_4 Y_4 Z_4\right) = (X_0 Y_0 Z_0)^{F_{4-1}} (X_1 Y_1 Z_1)^{F_4}$$

$$\Rightarrow \qquad (256 \times 1 \times 108) = (2 \times 1 \times 2)^2 (4 \times 1 \times 3)^3$$

$$\Rightarrow \qquad 27648 = (4)^2 (12)^3$$

$$= 16 \times 1728$$

$$= 27648$$

The equality holds, confirming Theorem 3.2 for this case. It is noteworthy that even though the $Y_{\mathtt{m}}$ sequence remains constant at 1, the theorem still accurately captures the combined multiplicative growth of the three coupled sequences.

Theorem 3.3: For each whole number $\mathtt{m}$;

(a) $X_{\mathtt{m}} X_{\mathtt{m}+1} X_{\mathtt{m}+2} = {X_0}^{2F_{\mathtt{m}+1}} {X_1}^{2F_{\mathtt{m}+2}}$

(b) $Y_{\mathtt{m}} Y_{\mathtt{m}+1} Y_{\mathtt{m}+2} = {Y_0}^{2F_{\mathtt{m}+1}} {Y_1}^{2F_{\mathtt{m}+2}}$

(c) $Z_{\mathtt{m}} Z_{\mathtt{m}+1} Z_{\mathtt{m}+2} = {Z_0}^{2F_{\mathtt{m}+1}} {Z_1}^{2F_{\mathtt{m}+2}}$

Proof: These results are confirmed by the induction hypothesis.

If $\mathtt{m} = 0$, then $X_0 X_1 X_2 = {X_0}^{2F_1} {X_1}^{2F_2}$

$$= {X_0}^2 {X_1}^2$$

$$= X_0 X_0 X_1 X_1$$

$$= X_0 X_1 X_0 X_1$$

$$= X_0 X_1 X_2 \qquad \text{(By Scheme No. 7)}$$

For $\mathtt{m} = 0$, the base case holds.

Assume the identity holds for some integer $\mathtt{m}$. Then for $\mathtt{m} + 1$.

$$X_{\mathtt{m}+1} X_{\mathtt{m}+2} X_{n+3} = (X_{\mathtt{m}+1} X_{\mathtt{m}+2} X_{\mathtt{m}+1} X_{\mathtt{m}+2}) \qquad \text{(By Scheme No. 7)}$$

$$= (X_{\mathtt{m}+1} X_{\mathtt{m}+1})(X_{\mathtt{m}+2} X_{\mathtt{m}+2})$$

$$= (X_{\mathtt{m}-1} X_{\mathtt{m}} X_{\mathtt{m}+1})(X_{\mathtt{m}} X_{\mathtt{m}+1} X_{\mathtt{m}+2}) \qquad \text{(By given Hypothesis)}$$

$$= ({X_0}^{2F_{\mathtt{m}}} {X_1}^{2F_{\mathtt{m}+1}})({X_0}^{2F_{\mathtt{m}+1}} {X_1}^{2F_{\mathtt{m}+2}})$$

$$= {X_0}^{2F_{\mathtt{m}}+2F_{\mathtt{m}+1}} {X_1}^{2F_{\mathtt{m}+1}+2F_{\mathtt{m}+2}}$$

$$= {X_0}^{2F_{\mathtt{m}+2}} {X_1}^{2F_{\mathtt{m}+3}}$$

The conclusion is valid for all integers $\mathtt{m} \geq 0$.

For the remaining sections (b) and (c), comparable evidence is provided.

Example based on Theorem 3.3

The Fibonacci sequence $F_{\mathtt{m}}$ is defined such that each term is the sum of the two preceding ones, typically starting with 0 and 1.

0, 1, 1, 2, 3, 5, 8, 13, 21, ...

$F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5$ and so on...

Let $\{Ӿ_{\mathtt{m}}\}_{\mathtt{m}=0}^{\infty}$, $\{Ɏ_{\mathtt{m}}\}_{\mathtt{m}=0}^{\infty}$ and $\{Ƶ_{\mathtt{m}}\}_{\mathtt{m}=0}^{\infty}$ be three sequences where each term is the product of the two preceding ones, such that

$$\{Ӿ_{\mathtt{m}}\}_{\mathtt{m}=0}^{\infty} = 4,5,20,100,2000,200000\ldots\ldots\ldots\ldots$$

Where,

$Ӿ_0 = 4, Ӿ_1 = 5, Ӿ_2 = 20, Ӿ_3 = 100, Ӿ_4 = 2000$ and so on...

$$\{Ɏ_{\mathtt{m}}\}_{\mathtt{m}=0}^{\infty} = 1,7,7,49,343,16807\ldots\ldots\ldots$$

Where,

$Ɏ_0 = 1, Ɏ_1 = 7, Ɏ_2 = 7, Ɏ_3 = 49, Ɏ_4 = 343$ and so on...

$$\{Ƶ_{\mathtt{m}}\}_{\mathtt{m}=0}^{\infty} = 2,4,8,32,256,8192\ldots\ldots\ldots$$

Where,

$Ƶ_0 = 2, Ƶ_1 = 4, Ƶ_2 = 8, Ƶ_3 = 32, Ƶ_4 = 256$ and so on...

This example serves to validate Theorem 3.3, which provides an identity for the product of three consecutive terms of a single sequence. For instance, part (a) gives a formula for $Ӿ_{\mathtt{m}}Ӿ_{\mathtt{m}+1}Ӿ_{\mathtt{m}+2}$. We will check if this formula holds true for the given sequences at $\mathtt{m} = 2$.

Now we are going to apply the result part (a) of theorem 3.3

$$Ӿ_{\mathtt{m}}Ӿ_{\mathtt{m}+1}Ӿ_{\mathtt{m}+2} = Ӿ_0^{\,2F_{\mathtt{m}+1}} Ӿ_1^{\,2F_{\mathtt{m}+2}}$$

For $\mathtt{m} = 2$, $\quad Ӿ_2Ӿ_3Ӿ_4 = Ӿ_0^{\,2F_3} Ӿ_1^{\,2F_4}$

$\Rightarrow \quad Ӿ_2Ӿ_3Ӿ_4 = Ӿ_0^{\,2F_3} Ӿ_1^{\,2F_4}$

$\Rightarrow \quad 20 \times 100 \times 2000 = 4^4 5^6$

$\Rightarrow \quad 20000 = 20000$

Now we are going to apply the result part (b) of theorem 3.3

$$Ɏ_{\mathtt{m}}Ɏ_{\mathtt{m}+1}Ɏ_{\mathtt{m}+2} = Ɏ_0^{\,2F_{\mathtt{m}+1}} Ɏ_1^{\,2F_{\mathtt{m}+2}}$$

For $\mathtt{m} = 2$, $\quad Ɏ_2Ɏ_3Ɏ_4 = Ɏ_0^{\,2F_3} Ɏ_1^{\,2F_4}$

$\Rightarrow \quad Ɏ_2Ɏ_3Ɏ_4 = Ɏ_0^{\,2F_3} Ɏ_1^{\,2F_4}$

$$\Rightarrow \qquad 7 \times 49 \times 343 = 1^4 7^6$$

$$\Rightarrow \qquad 17649 = 117649$$

Now we are going to apply the result part (c) of theorem 3.3

$$Z_{\mathrm{m}} Z_{\mathrm{m}+1} Z_{\mathrm{m}+2} = Z_0^{\;2F_{\mathrm{m}+1}} Z_1^{\;2F_{\mathrm{m}+2}}$$

For $\mathrm{m} = 2$, $\qquad Z_2 Z_3 Z_4 = Z_0^{\;2F_3} Z_1^{\;2F_4}$

$$\Rightarrow \qquad Z_2 Z_3 Z_4 = Z_0^{\;2F_3} Z_1^{\;2F_4}$$

$$\Rightarrow \qquad 8 \times 32 \times 256 = 2^4 4^6$$

$$\Rightarrow \qquad 65536 = 65536$$

The successful verification for all three sequences strengthens the proof by induction provided for Theorem 3.3. It illustrates the theorem's application across different sequences governed by the same multiplicative scheme.

Theorem 3.4: For each whole number n and every natural no. $k \geq 2$;

(a) $X_{\mathrm{m}+k+1} Y_{\mathrm{m}+k-1} = X_{\mathrm{m}}^{F_k} X_{\mathrm{m}+1}^{F_{k+1}} Y_{\mathrm{m}}^{F_{k-2}} Y_{\mathrm{m}+1}^{F_{k-1}}$

(b) $Y_{\mathrm{m}+k+1} Z_{\mathrm{m}+k-1} = Y_{\mathrm{m}}^{F_k} Y_{\mathrm{m}+1}^{F_{k+1}} Z_{\mathrm{m}}^{F_{k-2}} Z_{\mathrm{m}+1}^{F_{k-1}}$

(c) $Z_{\mathrm{m}+k+1} X_{\mathrm{m}+k-1} = Z_{\mathrm{m}}^{F_k} Z_{\mathrm{m}+1}^{F_{k+1}} X_{\mathrm{m}}^{F_{k-2}} X_{\mathrm{m}+1}^{F_{k-1}}$

Proof: These results are confirmed by the induction hypothesis.

If $k = 2$ then $\quad X_{\mathrm{m}+3} Y_{\mathrm{m}+1} = X_{\mathrm{m}+2} X_{\mathrm{m}+1} Y_{\mathrm{m}+1}$

$$= X_{\mathrm{m}+1} X_{\mathrm{m}} X_{\mathrm{m}+1} Y_{\mathrm{m}+1}$$

$$= X_{\mathrm{m}}^1 X_{\mathrm{m}+1}^2 Y_{\mathrm{m}}^0 Y_{\mathrm{m}+1}^1$$

$$= X_{\mathrm{m}}^{F_2} X_{\mathrm{m}+1}^{F_3} Y_{\mathrm{m}}^{F_0} Y_{\mathrm{m}+1}^{F_1}$$

For $k = 2$, the base case holds.

Assume the identity holds for some integer k. Then for $k + 1$.

$$X_{\mathrm{m}+k+2} Y_{\mathrm{m}+k} = X_{\mathrm{m}+k+1} X_{\mathrm{m}+k} Y_{\mathrm{m}+k-1} Y_{\mathrm{m}+k-2}$$

$$= (X_{\mathrm{m}+k+1} Y_{\mathrm{m}+k-1})(X_{\mathrm{m}+k} Y_{\mathrm{m}+k-2})$$

$$= X_{\mathrm{m}}^{F_k} X_{\mathrm{m}+1}^{F_{k+1}} Y_{\mathrm{m}}^{F_{k-2}} Y_{\mathrm{m}+1}^{F_{k-1}} X_{\mathrm{m}}^{F_{k-1}} X_{\mathrm{m}+1}^{F_k} Y_{\mathrm{m}}^{F_{k-3}} Y_{\mathrm{m}+1}^{F_{k-2}}$$

$$= X_{\mathrm{m}}^{F_k + F_{k-1}} X_{\mathrm{m}+1}^{F_{k+1} + F_k} Y_{\mathrm{m}}^{F_{k-2} + F_{k-3}} Y_{\mathrm{m}+1}^{F_{k-1} + F_{k-2}}$$

$$= X_{\mathrm{m}}^{F_{k+1}} X_{\mathrm{m}+1}^{F_{k+2}} Y_{\mathrm{m}}^{F_{k-1}} Y_{\mathrm{m}+1}^{F_k}$$

The conclusion is valid for all integers $\mathrm{m} \geq 0, k \geq 2$.

Similar evidence is available for the remaining parts (b) and (c).

Example based on Theorem 3.4

The Fibonacci sequence $F_{\mathtt{m}}$ is defined such that each term is the sum of the two preceding ones, typically starting with 0 and 1.

0, 1, 1, 2, 3, 5, 8, 13, 21, ...

$F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5$ and so on...

Let $\{X_{\mathtt{m}}\}_{\mathtt{m}=0}^{\infty}$, $\{Y_{\mathtt{m}}\}_{\mathtt{m}=0}^{\infty}$ and $\{Z_{\mathtt{m}}\}_{\mathtt{m}=0}^{\infty}$ be three sequences whose terms is the multiplication of the two preceding ones such that

$$\{X_{\mathtt{m}}\}_{\mathtt{m}=0}^{\infty} = 1,3,3,9,27,243,\ldots\ldots\ldots\ldots$$

Where,

$X_0 = 1, X_1 = 3, X_2 = 3, X_3 = 9, X_4 = 27, X_5 = 243, X_6 = 6561$ and so on...

$$\{Y_{\mathtt{m}}\}_{\mathtt{m}=0}^{\infty} = 2,3,6,18,108,1944\ldots\ldots\ldots$$

Where,

$Y_0 = 2,\ Y_1 = 3,\ Y_2 = 6,\ Y_3 = 18,\ Y_4 = 108,\ Y_5 = 1944$ and so on...

$$\{Z_{\mathtt{m}}\}_{\mathtt{m}=0}^{\infty} = 1,4,4,16,64,1024,\ldots\ldots\ldots$$

Where,

$Z_0 = 1, Z_1 = 4, Z_2 = 4, Z_3 = 16, Z_4 = 64, Z_5 = 1024$ and so on...

This example demonstrates the more complex identity stated in Theorem 3.4- The theorem relates a term from one sequence $X_{\mathtt{m}+k+1}$ and another from a different sequence $Y_{\mathtt{m}+k-1}$ to a product of four earlier terms from both sequences, with Fibonacci number exponents. We test part (a) for specific values $\mathtt{m} = 2$ and $k = 3$.

Now we are going to apply the result of part (a) theorem 3.4

$$X_{\mathtt{m}+k+1} Y_{\mathtt{m}+k-1} = X_{\mathtt{m}}^{F_k} X_{\mathtt{m}+1}^{F_{k+1}} Y_{\mathtt{m}}^{F_{k-2}} Y_{\mathtt{m}+1}^{F_{k-1}}$$

For $\mathtt{m} = 2$ and $k = 3$

$$X_6 Y_4 = X_2^{F_3} X_3^{F_4} Y_2^{F_1} Y_3^{F_2}$$

$\Rightarrow \qquad 108 \times 6561 = 3^2 9^3 6^1 18^1$

$\Rightarrow \qquad 708588 = 708588$

The result is verified, demonstrating the intricate cross-sequence relationships that Theorem 3.4 captures. This complexity highlights the rich structure inherent in the Multiplicative Triple Fibonacci Sequences.

Similarly, we can apply the result in parts (b) and (c).

Theorem 3.5: For every integer $\mathtt{m} \geq 0, k \geq 2$;

(a) $\quad X_{\mathtt{m}+k+1} Z_{\mathtt{m}+k-1} = X_{\mathtt{m}}^{F_k} X_{\mathtt{m}+1}^{F_{k+1}} Z_{\mathtt{m}}^{F_{k-2}} Z_{\mathtt{m}+1}^{F_{k-1}}$

(b) $Y_{m+k+1}X_{m+k-1} = Y_m^{F_k} Y_{m+1}^{F_{k+1}} X_m^{F_{k-2}} X_{m+1}^{F_{k-1}}$

(c) $Z_{m+k+1}Y_{m+k-1} = Z_m^{F_k} Z_{m+1}^{F_{k+1}} Y_m^{F_{k-2}} Y_{m+1}^{F_{k-1}}$

Proof: A similar proof can be given as in theorem 3.4.

Theorem 3.6: For every integer $m \geq 0$;

(a) $X_0 X_{m+4} = X_0^{F_{m+3}-1} X_1^{F_{m+4}}$

(b) $Y_0 Y_{m+4} = Y_0^{F_{m+3}-1} Y_1^{F_{m+4}}$

(c) $Z_0 Z_{m+4} = Z_0^{F_{m+3}-1} Z_1^{F_{m+4}}$

Proof: We can prove the theorem by the method of mathematical induction.

We can also prove theorem 3.1 to theorem 3.6 with the help of Schemes 8th and 9th.

3.4 3rd ORDER MTFS

Let $\{X_i\}_{i=0}^{\infty}$, $\{Y_i\}_{i=0}^{\infty} and \{Z_i\}_{i=0}^{\infty}$ be three infinite sequences with initial values a, b, c, d, e, f, g, h and i, which are referred to as the 3-F Sequence or TFS.

If $X_0 = a, Y_0 = b, Z_0 = c, X_1 = d, Y_1 = e, Z_1 = f, X_2 = g, Y_2 = h, Z_2 = i$,

Then the following are twenty-seven different MTFS Schemes:

Table 3.5: Third Order MTFS Schemes

Scheme	X_{m+3}	Y_{m+3}	Z_{m+3}
1	$Y_{m+2}.Z_{m+1}.X_m$	$Z_{m+2}.X_{m+1}.Y_m$	$X_{m+2}.Y_{m+1}.Z_m$
2	$X_{m+2}.X_{m+1}.X_m$	$Y_{m+2}.Y_{m+1}.Y_m$	$Z_{m+2}.Z_{m+1}.Z_m$
3	$X_{m+2}.Z_{m+1}.Y_m$	$Y_{m+2}.X_{m+1}.Z_m$	$Z_{m+2}.Y_{m+1}.X_m$
4	$Z_{m+2}.Y_{m+1}.X_m$	$X_{m+2}.Z_{m+1}.Y_m$	$Y_{m+2}.X_{m+1}.Z_m$
5	$X_{m+2}.Y_{m+1}.Z_m$	$Y_{m+2}.Z_{m+1}.X_m$	$Z_{m+2}.X_{m+1}.Y_m$
6	$X_{m+2}.X_{m+1}.Y_m$	$Y_{m+2}.Y_{m+1}.Z_m$	$Z_{m+2}.Z_{m+1}.X_m$
7	$X_{m+2}.Y_{m+1}.X_m$	$Y_{m+2}.Z_{m+1}.Y_m$	$Z_{m+2}.X_{m+1}.Z_m$
8	$Y_{m+2}.X_{m+1}.X_m$	$Z_{m+2}.Y_{m+1}.Y_m$	$X_{m+2}.Z_{m+1}.Z_m$
9	$X_{m+2}.X_{m+1}.Z_m$	$Y_{m+2}.Y_{m+1}.X_m$	$Z_{m+2}.Z_{m+1}.Y_m$

10	$X_{m+2}.Z_{m+1}.X_m$	$Y_{m+2}.X_{m+1}.Y_m$	$Z_{m+2}.Y_{m+1}.Z_m$
11	$Z_{m+2}.X_{m+1}.X_m$	$X_{m+2}.Y_{m+1}.Y_m$	$Y_{m+2}.Z_{m+1}.Z_m$
12	$Y_{m+2}.Y_{m+1}.Z_m$	$Z_{m+2}.Z_{m+1}.X_m$	$X_{m+2}.X_{m+1}.Y_m$
13	$Y_{m+2}.Z_{m+1}.Y_m$	$Z_{m+2}.X_{m+1}.Z_m$	$X_{m+2}.Y_{m+1}.X_m$
14	$Z_{m+2}.Y_{m+1}.Y_m$	$X_{m+2}.Z_{m+1}.Z_m$	$Y_{m+2}.X_{m+1}.X_m$
15	$Y_{m+2}.Z_{m+1}.Z_m$	$Z_{m+2}.X_{m+1}.X_m$	$X_{m+2}.Y_{m+1}.Y_m$
16	$Z_{m+2}.Y_{m+1}.Z_m$	$X_{m+2}.Z_{m+1}.X_m$	$Y_{m+2}.X_{m+1}.Y_m$
17	$Z_{m+2}.Z_{m+1}.Y_m$	$X_{m+2}.X_{m+1}.Z_m$	$Y_{m+2}.Y_{m+1}.X_m$
18	$Z_{m+2}.X_{m+1}.Y_m$	$X_{m+2}.Y_{m+1}.Z_m$	$Y_{m+2}.Z_{m+1}.X_m$
19	$Y_{m+2}.X_{m+1}.Y_m$	$Z_{m+2}.Y_{m+1}.Z_m$	$X_{m+2}.Z_{m+1}.X_m$
20	$X_{m+2}.Y_{m+1}.Y_m$	$Y_{m+2}.Z_{m+1}.Z_m$	$Z_{m+2}.X_{m+1}.X_m$
21	$Y_{m+2}.Y_{m+1}.X_m$	$Z_{m+2}.Z_{m+1}.Y_m$	$X_{m+2}.X_{m+1}.Z_m$
22	$X_{m+2}.Z_{m+1}.Z_m$	$Y_{m+2}.X_{m+1}.X_m$	$Z_{m+2}.Y_{m+1}.Y_m$
23	$Z_{m+2}.X_{m+1}.Z_m$	$X_{m+2}.Y_{m+1}.X_m$	$Y_{m+2}.Z_{m+1}.Y_m$
24	$Z_{m+2}.Z_{m+1}.X_m$	$X_{m+2}.X_{m+1}.Y_m$	$Y_{m+2}.Y_{m+1}.Z_m$
25	$Y_{m+2}.X_{m+1}.Z_m$	$Z_{m+2}.Y_{m+1}.X_m$	$X_{m+2}.Z_{m+1}.Y_m$
26	$Y_{m+2}.Y_{m+1}.Y_m$	$Z_{m+2}.Z_{m+1}.Z_m$	$X_{m+2}.X_{m+1}.X_m$
27	$Z_{m+2}.Z_{m+1}.Z_m$	$X_{m+2}.X_{m+1}.X_m$	$Y_{m+2}.Y_{m+1}.Y_m$

Table 3.6: Some terms of 1st Scheme of Third order

m	X_m	Y_m	Z_m
0	a	b	c
1	d	e	f
2	g	h	i
3	hfa	idb	gec
4	i^2d^2b	g^2e^2c	h^2f^2a
5	$g^4e^3c^2$	$h^4f^3a^2$	$i^4d^3b^2$

Table 3.7: Some terms of 2nd Scheme of Third order

m	X_m	Y_m	Z_m

0	a	b	c
1	d	e	f
2	g	h	i
3	adg	beh	cfi
4	ad^2g^2	be$^2h^2$	cf^2i^2
5	a^2d^3g^4	b^2e$^3h^4$	$c^2f^3i^4$

Table 3.8: Some terms of 3rd Scheme of Third order

$\mathfrak{m}$	$X_{\mathfrak{m}}$	$Y_{\mathfrak{m}}$	$Z_{\mathfrak{m}}$
0	a	b	c
1	d	e	f
2	g	h	i
3	gfb	hdc	iea
4	g$fbic$	$hdcgf$	$ieahd$

Table 3.9: Some terms of 4th Scheme of Third order

$\mathfrak{m}$	$X_{\mathfrak{m}}$	$Y_{\mathfrak{m}}$	$Z_{\mathfrak{m}}$
0	a	b	c
1	d	e	f
2	g	h	i
3	$a$ei	bgf	cdh
4	cd^2h^2	ae$^2i^2$	bf^2g^2

Table 3.10: Some terms of 5th Scheme of Third order

$\mathfrak{m}$	$X_{\mathfrak{m}}$	$Y_{\mathfrak{m}}$	$Z_{\mathfrak{m}}$
0	a	b	c
1	d	e	f
2	g	h	i
3	ge$c$	hfa	idb
4	h^2f^2a	i^2d^2b	g^{2}e^2c

Table 3.11: Some terms of 18th Scheme of Third order

m	X_m	Y_m	Z_m
0	a	b	c
1	d	e	f
2	g	h	i
3	idb	gec	hfa
4	g^2e^2c	h^2f^2a	i^2d^2b

Table 3.12: Some terms of 25th Scheme of Third order

m	X_m	Y_m	Z_m
0	a	b	c
1	d	e	f
2	g	h	i
3	hdc	iea	gfb

Table 3.13: Some terms of 26th Scheme of Third order

m	X_m	Y_m	Z_m
0	a	b	c
1	d	e	f
2	g	h	i
3	beh	cfi	adg
4	be^2h^2	cf^2i^2	ad^2g^2
5	$b^2e^3h^4$	$c^2f^3i^4$	$a^2d^3g^4$

Table 3.14: Some terms of 27th Scheme of Third order

m	X_m	Y_m	Z_m
0	a	b	c
1	d	e	f
2	g	h	i
3	cfi	adg	beh
4	cf^2i^2	ad^2g^2	be^2h^2
5	$c^2f^3i^4$	$a^2d^3g^4$	$b^2e^3h^4$

3.5 Main Results of 3rd Order MTFS

Now we present some results of MTFS of Third order under 1st, 2nd, 3rd, 4th, 5th, 18th, 25th, 26th and 27th:

Theorem 3.7: For each natural no. $\mathtt{m} \geq 2$:

$$\frac{\prod_{k=0}^{\mathtt{m}} Ӿ_{k+6}¥_{k+6}Ƶ_{k+6}}{\prod_{k=0}^{\mathtt{m}} Ӿ_{k+4}¥_{k+4}Ƶ_{k+4}} = \frac{(Ӿ_{\mathtt{m}+5}¥_{\mathtt{m}+5}Ƶ_{\mathtt{m}+5})(Ӿ_{\mathtt{m}+6}¥_{\mathtt{m}+6}Ƶ_{\mathtt{m}+6})}{(Ӿ_4¥_4Ƶ_4)(Ӿ_5¥_5Ƶ_5)}$$

Proof: We demonstrate these findings through induction hypothesis:

If $\mathtt{m} = 2$, then

$$\frac{\prod_{k=0}^{2} Ӿ_{k+6}¥_{k+6}Ƶ_{k+6}}{\prod_{k=0}^{2} Ӿ_{k+4}¥_{k+4}Ƶ_{k+4}} = \frac{(Ӿ_6¥_6Ƶ_6)(Ӿ_7¥_7Ƶ_7)(Ӿ_8¥_8Ƶ_8)}{(Ӿ_4¥_4Ƶ_4)(Ӿ_5¥_5Ƶ_5)(Ӿ_6¥_6Ƶ_6)}$$

$$= \frac{(Ӿ_7¥_7Ƶ_7)(Ӿ_8¥_8Ƶ_8)}{(Ӿ_4¥_4Ƶ_4)(Ӿ_5¥_5Ƶ_5)}$$

For $\mathtt{m} = 2$, the conclusion is correct.

We'll proceed by assuming that the outcome is accurate for some integer $\mathtt{m}$. Then for $\mathtt{m} + 1$

$$\frac{\prod_{k=0}^{\mathtt{m}+1} Ӿ_{k+6}¥_{k+6}Ƶ_{k+6}}{\prod_{k=0}^{\mathtt{m}+1} Ӿ_{k+4}¥_{k+4}Ƶ_{k+4}} = \frac{(Ӿ_{\mathtt{m}+7}¥_{\mathtt{m}+7}Ƶ_{\mathtt{m}+7})\prod_{k=0}^{\mathtt{m}} Ӿ_{k+6}¥_{k+6}Ƶ_{k+6}}{(Ӿ_{\mathtt{m}+5}¥_{\mathtt{m}+5}Ƶ_{\mathtt{m}+5})\prod_{k=0}^{\mathtt{m}} Ӿ_{k+4}¥_{k+4}Ƶ_{k+4}}$$

$$= \frac{(Ӿ_{\mathtt{m}+7}¥_{\mathtt{m}+7}Ƶ_{\mathtt{m}+7})(Ӿ_{\mathtt{m}+5}¥_{\mathtt{m}+5}Ƶ_{\mathtt{m}+5})(Ӿ_{\mathtt{m}+6}¥_{\mathtt{m}+6}Ƶ_{\mathtt{m}+6})}{(Ӿ_{\mathtt{m}+5}¥_{\mathtt{m}+5}Ƶ_{\mathtt{m}+5})(Ӿ_4¥_4Ƶ_4)(Ӿ_5¥_5Ƶ_5)}$$

$$= \frac{(Ӿ_{\mathtt{m}+6}¥_{\mathtt{m}+6}Ƶ_{\mathtt{m}+6})(Ӿ_{\mathtt{m}+7}¥_{\mathtt{m}+7}Ƶ_{\mathtt{m}+7})}{(Ӿ_4¥_4Ƶ_4)(Ӿ_5¥_5Ƶ_5)}$$

The conclusion is valid for all integers $\mathtt{m} \geq 0$.

Theorem 3.8: For each whole no. $\mathtt{m}$:

$$\frac{(Ӿ_{\mathtt{m}}¥_{\mathtt{m}}Ƶ_{\mathtt{m}})(Ӿ_{\mathtt{m}+1}¥_{\mathtt{m}+1}Ƶ_{\mathtt{m}+1})}{(Ӿ_{\mathtt{m}+3}¥_{\mathtt{m}+3}Ƶ_{\mathtt{m}+3})} = \frac{1}{(Ӿ_{\mathtt{m}+2}¥_{\mathtt{m}+2}Ƶ_{\mathtt{m}+2})}$$

Proof: By induction hypothesis, we have

If $\mathtt{m} = 0$, then

$$\frac{(Ӿ_0¥_0Ƶ_0)(Ӿ_1¥_1Ƶ_1)}{(Ӿ_3¥_3Ƶ_3)} = \frac{(Ӿ_0¥_0Ƶ_0)(Ӿ_1¥_1Ƶ_1)}{(¥_2Ƶ_1Ӿ_0)(Ƶ_2Ӿ_1¥_0)(Ӿ_2¥_1Ƶ_0)}$$

$$= \frac{(Ӿ_0¥_0Ƶ_0)(Ӿ_1¥_1Ƶ_1)}{(Ӿ_0¥_0Ƶ_0)(Ӿ_1¥_1Ƶ_1)(Ӿ_2¥_2Ƶ_2)}$$

$$= \frac{1}{(Ӿ_2 Ɏ_2 Ƶ_2)}$$

For $\mathbb{m} = 0$, the base case holds.

Assume the identity holds for some integer $\mathbb{m}$. Then for $\mathbb{m} + 1$.

$$\frac{(Ӿ_{\mathbb{m}+1} Ɏ_{\mathbb{m}+1} Ƶ_{\mathbb{m}+1})(Ӿ_{\mathbb{m}+2} Ɏ_{\mathbb{m}+2} Ƶ_{\mathbb{m}+2})}{(Ӿ_{\mathbb{m}+4} Ɏ_{\mathbb{m}+4} Ƶ_{\mathbb{m}+4})} = \frac{(Ӿ_{\mathbb{m}+1} Ɏ_{\mathbb{m}+1} Ƶ_{\mathbb{m}+1})(Ӿ_{\mathbb{m}+2} Ɏ_{\mathbb{m}+2} Ƶ_{\mathbb{m}+2})}{(Ӿ_{\mathbb{m}+3} Ӿ_{\mathbb{m}+2} Ӿ_{\mathbb{m}+1})(Ɏ_{\mathbb{m}+3} Ɏ_{\mathbb{m}+2} Ɏ_{\mathbb{m}+1})(Ƶ_{\mathbb{m}+3} Ƶ_{\mathbb{m}+2} Ƶ_{\mathbb{m}+1})}$$

$$= \frac{(Ӿ_{\mathbb{m}+1} Ɏ_{\mathbb{m}+1} Ƶ_{\mathbb{m}+1})(Ӿ_{\mathbb{m}+2} Ɏ_{\mathbb{m}+2} Ƶ_{\mathbb{m}+2})}{(Ӿ_{\mathbb{m}+1} Ɏ_{\mathbb{m}+1} Ƶ_{\mathbb{m}+1})(Ӿ_{\mathbb{m}+2} Ɏ_{\mathbb{m}+2} Ƶ_{\mathbb{m}+2})(Ӿ_{\mathbb{m}+3} Ɏ_{\mathbb{m}+3} Ƶ_{\mathbb{m}+3})}$$

$$= \frac{1}{(Ӿ_{\mathbb{m}+3} Ɏ_{\mathbb{m}+3} Ƶ_{\mathbb{m}+3})}$$

The conclusion is valid for all integers $\mathbb{m} \geq 0$.

Theorem 3.9: For every integer $\mathbb{m} \geq 0$:

$$\frac{(Ӿ_{\mathbb{m}} Ɏ_{\mathbb{m}} Ƶ_{\mathbb{m}})(Ӿ_{\mathbb{m}+1} Ɏ_{\mathbb{m}+1} Ƶ_{\mathbb{m}+1})}{(Ӿ_{\mathbb{m}+3} Ɏ_{\mathbb{m}+3} Ƶ_{\mathbb{m}+3})} = \frac{1}{(Ӿ_{\mathbb{m}+2} Ɏ_{\mathbb{m}+2} Ƶ_{\mathbb{m}+2})}$$

Proof: These results are confirmed by the induction hypothesis.

If $\mathbb{m} = 0$, then

$$\frac{(Ӿ_0 Ɏ_0 Ƶ_0)(Ӿ_1 Ɏ_1 Ƶ_1)}{(Ӿ_3 Ɏ_3 Ƶ_3)} = \frac{(Ӿ_0 Ɏ_0 Ƶ_0)(Ӿ_1 Ɏ_1 Ƶ_1)}{(Ɏ_2 Ƶ_1 Ӿ_0)(Ƶ_2 Ӿ_1 Ɏ_0)(Ӿ_2 Ɏ_1 Ƶ_0)}$$

$$= \frac{(Ӿ_0 Ɏ_0 Ƶ_0)(Ӿ_1 Ɏ_1 Ƶ_1)}{(Ӿ_0 Ɏ_0 Ƶ_0)(Ӿ_1 Ɏ_1 Ƶ_1)(Ӿ_2 Ɏ_2 Ƶ_2)}$$

$$= \frac{1}{(Ӿ_2 Ɏ_2 Ƶ_2)}$$

For $\mathbb{m} = 0$, the base case holds.

Assume the identity holds for some integer $\mathbb{m}$. Then for $\mathbb{m} + 1$.

$$\frac{(Ӿ_{\mathbb{m}+1} Ɏ_{\mathbb{m}+1} Ƶ_{\mathbb{m}+1})(Ӿ_{\mathbb{m}+2} Ɏ_{\mathbb{m}+2} Ƶ_{\mathbb{m}+2})}{(Ӿ_{\mathbb{m}+4} Ɏ_{\mathbb{m}+4} Ƶ_{\mathbb{m}+4})} = \frac{(Ӿ_{\mathbb{m}+1} Ɏ_{\mathbb{m}+1} Ƶ_{\mathbb{m}+1})(Ӿ_{\mathbb{m}+2} Ɏ_{\mathbb{m}+2} Ƶ_{\mathbb{m}+2})}{(Ӿ_{\mathbb{m}+3} Ӿ_{\mathbb{m}+2} Ӿ_{\mathbb{m}+1})(Ɏ_{\mathbb{m}+3} Ɏ_{\mathbb{m}+2} Ɏ_{\mathbb{m}+1})(Ƶ_{\mathbb{m}+3} Ƶ_{\mathbb{m}+2} Ƶ_{\mathbb{m}+1})}$$

$$= \frac{(Ӿ_{\mathbb{m}+1} Ɏ_{\mathbb{m}+1} Ƶ_{\mathbb{m}+1})(Ӿ_{\mathbb{m}+2} Ɏ_{\mathbb{m}+2} Ƶ_{\mathbb{m}+2})}{(Ӿ_{\mathbb{m}+1} Ɏ_{\mathbb{m}+1} Ƶ_{\mathbb{m}+1})(Ӿ_{\mathbb{m}+2} Ɏ_{\mathbb{m}+2} Ƶ_{\mathbb{m}+2})(Ӿ_{\mathbb{m}+3} Ɏ_{\mathbb{m}+3} Ƶ_{\mathbb{m}+3})}$$

$$= \frac{1}{(Ӿ_{\mathbb{m}+3} Ɏ_{\mathbb{m}+3} Ƶ_{\mathbb{m}+3})}$$

The conclusion is valid for all integers $\mathbb{m} \geq 0$.

now we will present the identities of 3[rd] order MTFS under Scheme no. 1.

Theorem 3.10: For each integer $\mathbb{m} \geq 0$

(a) $X_{m+9} = X_{m+6}^4 . Y_{m+5}^3 . Z_{m+4}^2$

(b) $Y_{m+9} = Y_{m+6}^4 . Z_{m+5}^3 . X_{m+4}^2$

(c) $Z_{m+9} = Z_{m+6}^4 . X_{m+5}^3 . Y_{m+4}^2$

Proof: These results are confirmed by the induction hypothesis.

If $m = 0$ then $X_9 = Y_8 . Z_7 . X_6$ (By Scheme No. 1)

$= Z_7 . X_6 . Y_5 . Z_7 . X_6$ (By Scheme No. 1)

$= X_6^2 . Y_5 . Z_7^2$

$= X_6^2 . Y_5 . X_6 . Y_5 . Z_4 . X_6 . Y_5 . Z_4$ (By Scheme No. 1)

$= X_6^4 . Y_5^3 . Z_4^2$

For $m = 0$, the base case holds.

Assume the identity holds for some integer m. Then for $m + 1$.

$X_{m+10} = Y_{m+9} . Z_{m+8} . X_{m+7}$ (By Scheme No. 1)

$= Z_{m+8} . X_{m+7} . Y_{m+6} . Z_{m+8} . X_{m+7}$ (By Scheme No. 1)

$= X_{m+7}^2 . Y_{m+6} . Z_{m+8}^2$

$= X_{m+7}^2 . Y_{m+6} . (X_{m+7} . Y_{m+6} . Z_{m+5})^2$ (By Scheme No. 1)

$= X_{m+7}^4 . Y_{m+6}^3 . Z_{m+5}^2$

The conclusion is valid for all integers $m \geq 0$.

Similar evidence is available for the remaining parts (b) and (c).

Theorem 3.11: For every integer $m \geq 0$

(a) $\prod_{k=0}^{m} X_{2k+10} = \prod_{k=0}^{m} Y_{2k+9} . Z_{2k+8} . X_{2k+7}$

(b) $\prod_{k=0}^{m} Y_{2k+10} = \prod_{k=0}^{m} Z_{2k+9} . X_{2k+8} . Y_{2k+7}$

(c) $\prod_{k=0}^{m} Z_{2k+10} = \prod_{k=0}^{m} X_{2k+9} . Y_{2k+8} . Z_{2k+7}$

Proof. These results are confirmed by the induction hypothesis.

For $m = 0$ then $X_{10} = Y_9 . Z_8 . X_7$

This is true by first Scheme.

We'll proceed by assuming that the outcome is accurate for some integer $m = 1$.

Hence $\prod_{k=0}^{l} X_{2k+10} = \prod_{k=0}^{l} Y_{2k+9} . Z_{2k+8} . X_{2k+7}$

Now for $m = l + 1$. Then

$$\prod_{k=0}^{l+1} Ɏ_{2k+9} \,.\, Ƶ_{2k+8} .\, Ӿ_{2k+7} = Ɏ_{2(l+1)+9} .\, Ƶ_{2(l+1)+8} .\, Ӿ_{2(l+1)+7} .\prod_{k=0}^{l} Ɏ_{2k+9} \,.\, Ƶ_{2k+8} .\, Ӿ_{2k+7}$$

$$= Ӿ_{2(l+1)+10} .\, \prod_{k=0}^{l} Ӿ_{2k=10} \quad \text{(By induction hypothesis)}$$

$$= \prod_{k=0}^{l+1} Ӿ_{2k=10}$$

Thus, the result is true for $\mathtt{m} = l + 1$. Hence by induction method the result is true for any positive integer $\mathtt{m}$.

Similar proof can be given for remaining parts (b) and (c).

Theorem 3.12: For each integer $\mathtt{m} \geq 0$

(a) $\prod_{k=0}^{\mathtt{m}} Ӿ_{3k+10} = \prod_{k=0}^{\mathtt{m}} Ɏ_{3k+9} \,.\, Ƶ_{3k+8} .\, Ӿ_{3k+7}$

(b) $\prod_{k=0}^{\mathtt{m}} Ɏ_{3k+10} = \prod_{k=0}^{\mathtt{m}} Ƶ_{3k+9} \,.\, Ӿ_{3k+8} .\, Ɏ_{3k+7}$

(c) $\prod_{k=0}^{\mathtt{m}} Ƶ_{3k+10} = \prod_{k=0}^{\mathtt{m}} Ӿ_{3k+9} \,.\, Ɏ_{3k+8} .\, Ƶ_{3k+7}$

Induction can also be used to support this.

Theorem 3.13: For each integer $\mathtt{m} \geq 0,\ ɍ \geq 1$.

(a) $\prod_{k=0}^{\mathtt{m}} Ӿ_{ɍk+10} = \prod_{k=0}^{\mathtt{m}} Ɏ_{ɍk+9} \,.\, Ƶ_{ɍk+8} .\, Ӿ_{ɍk+7}$

(b) $\prod_{k=0}^{\mathtt{m}} Ɏ_{ɍk+10} = \prod_{k=0}^{\mathtt{m}} Ƶ_{ɍk+9} \,.\, Ӿ_{ɍk+8} .\, Ɏ_{ɍk+7}$

(c) $\prod_{k=0}^{\mathtt{m}} Ƶ_{ɍk+10} = \prod_{k=0}^{\mathtt{m}} Ӿ_{ɍk+9} \,.\, Ɏ_{ɍk+8} .\, Ƶ_{ɍk+7}$

Induction can also be used to support this.

Theorem 3.14: For every integer $\mathtt{m} \geq 0, \mathtt{s} \geq 0$

(a) $\prod_{k=0}^{\mathtt{m}} Ӿ_{2k+3+\mathtt{s}} = \prod_{k=0}^{\mathtt{m}} Ɏ_{2k+2+\mathtt{s}} \,.\, Ƶ_{2k+1+\mathtt{s}} .\, Ӿ_{2k+\mathtt{s}}$

(b) $\prod_{k=0}^{\mathtt{m}} Ɏ_{2k+3+\mathtt{s}} = \prod_{k=0}^{\mathtt{m}} Ƶ_{2k+2+\mathtt{s}} \,.\, Ӿ_{2k+1+\mathtt{s}} .\, Ɏ_{2k+\mathtt{s}}$

(c) $\prod_{k=0}^{\mathtt{m}} Ƶ_{2k+3+\mathtt{s}} = \prod_{k=0}^{\mathtt{m}} Ӿ_{2k+2+\mathtt{s}} \,.\, Ɏ_{2k+1+\mathtt{s}} .\, Ƶ_{2k+\mathtt{s}}$

These results are confirmed by the induction hypothesis.

For $\mathtt{m} = 0$, we have $Ӿ_{\mathtt{s}+3} = Ɏ_{\mathtt{s}+2} .\, Ƶ_{\mathtt{s}+1} .\, Ӿ_{\mathtt{s}}$ which is true by the first Scheme.

We'll proceed by assuming that the outcome is accurate for some integer $\mathtt{m} \geq l$.

Hence $\prod_{k=0}^{l} Ӿ_{2k+10} = \prod_{k=0}^{l} Ɏ_{2k+9} \,.\, Ƶ_{2k+8} .\, Ӿ_{2k+7}$

Now for $\mathtt{m} = l + 1$. Then

$$\prod_{k=0}^{l+1} Ɏ_{2k+s+2} \,.\, Ƶ_{2k+s+1} .\, Ӿ_{2k+s} = Ɏ_{2(l+1)+s+2} .\, Ƶ_{2(l+1)+s+1} .\, Ӿ_{2(l+1)+s} .\prod_{k=0}^{l} Ɏ_{2k+s+2} \,.\, Ƶ_{2k+s+1} .\, Ӿ_{2k+s}$$

$$= Ӿ_{2(l+1)+s+3} .\, \prod_{k=0}^{l} Ӿ_{2k+s+3} \qquad \text{(By induction hypothesis)}$$

$$= \prod_{k=0}^{l+1} Ӿ_{2k+s+3} = \text{L.H.S.}$$

The conclusion is valid for all integers $\mathbb{m} = l + 1$.

Therefore, according to the induction process, the conclusion is valid for all positive integers $\mathbb{m}$.

Similar evidence is available for the remaining parts (b) and (c).

Theorem 3.15: For every integer $\mathbb{m} \geq 0, ɍ \geq 0, s \geq 0$

(a) $\prod_{k=0}^{\mathbb{m}} Ӿ_{ɍk+s+3} = \prod_{k=0}^{\mathbb{m}} Ɏ_{ɍk+s+2} \,.\, Ƶ_{ɍk+s+1} .\, Ӿ_{ɍk+s}$

(b) $\prod_{k=0}^{\mathbb{m}} Ɏ_{ɍk+s+3} = \prod_{k=0}^{\mathbb{m}} Ƶ_{ɍk+s+2} \,.\, Ӿ_{ɍk+s+1} .\, Ɏ_{ɍk+s}$

(c) $\prod_{k=0}^{\mathbb{m}} Ƶ_{ɍk+s+3} = \prod_{k=0}^{\mathbb{m}} Ӿ_{ɍk+s+2} \,.\, Ɏ_{ɍk+s+1} .\, Ƶ_{ɍk+s}$

Proof. These results are confirmed by the induction hypothesis:

For $\mathbb{m} = 0$, we have $Ӿ_{s+3} = Ɏ_{s+2} .\, Ƶ_{s+1} .\, Ӿ_{s}$ this is true by first Scheme.

We'll proceed by assuming that the outcome is accurate for some integer $\mathbb{m} = l$.

Hence $\prod_{k=0}^{l} Ӿ_{ɍk+10} = \prod_{k=0}^{l} Ɏ_{ɍk+9} \,.\, Ƶ_{ɍk+8} .\, Ӿ_{ɍk+7}$

Now for $\mathbb{m} = l + 1$. Then

$$\prod_{k=0}^{l+1} Ɏ_{ɍk+s+2} \,.\, Ƶ_{ɍk+s+1} .\, Ӿ_{ɍk+s} = Ɏ_{ɍ(l+1)+s+2} .\, Ƶ_{ɍ(l+1)+s+1} .\, Ӿ_{ɍ(l+1)+s} .\prod_{k=0}^{l} Ɏ_{ɍk+s+2} \,.\, Ƶ_{ɍk+s+1} .\, Ӿ_{ɍk+s}$$

$$= Ӿ_{ɍ(l+1)+s+3} .\, \prod_{k=0}^{l} Ӿ_{ɍk+s+3} \qquad \text{(By induction hypothesis)}$$

$$= \prod_{k=0}^{l+1} Ӿ_{ɍk+s+3}$$

$$= \text{L.H.S.}$$

The conclusion is valid for all integers $\mathbb{m} = l + 1$.

Consequently, by induction, the result is valid for any positive integer $\mathbb{m}$.

Similar evidence is available for the remaining parts (b) and (c).

Theorem 3.16: For every integer $\mathbb{m} \geq 2$,

$$(Ӿ_0 Ɏ_0 Ƶ_0)^{\mathbb{m}} (Ӿ_1 Ɏ_1 Ƶ_1)^{\mathbb{m}+1} (Ӿ_2 Ɏ_2 Ƶ_2)^{\mathbb{m}+2} = \left(Ӿ_3 Ɏ_3 Ƶ_3\right)^{\mathbb{m}-2} (Ӿ_5 Ɏ_5 Ƶ_5)$$

Proof: These results are confirmed by the induction hypothesis.

For $\mathbb{m} = 2$, we have

$$(Ӿ_0¥_0Ƶ_0)^2(Ӿ_1¥_1Ƶ_1)^3(Ӿ_2¥_2Ƶ_2)^4 = (Ӿ_1¥_1Ƶ_1)(Ӿ_2¥_2Ƶ_2)^2(Ӿ_3¥_3Ƶ_3)^2 \qquad \text{(By Scheme No. 1)}$$

$$= (Ӿ_2¥_2Ƶ_2)(Ӿ_3¥_3Ƶ_3)(Ӿ_4¥_4Ƶ_4) \qquad \text{(By Scheme No. 1)}$$

$$= (Ӿ_5¥_5Ƶ_5)$$

For $\mathbb{m} = 2$, the base case holds.

Assume the identity holds for some integer $\mathbb{m} = \mathrm{l}$. Then for$\mathbb{m} = \mathrm{l} + 1$.

Hence $(Ӿ_0¥_0Ƶ_0)^{\mathrm{l}}(Ӿ_1¥_1Ƶ_1)^{\mathrm{l}+1}(Ӿ_2¥_2Ƶ_2)^{\mathrm{l}+2} = \left(Ӿ_3¥_3Ƶ_3\right)^{\mathrm{l}-2}(Ӿ_5¥_5Ƶ_5)$

Now for $\mathbb{m} = \mathrm{l} + 1$. Then

$$\left(Ӿ_0¥_0Ƶ_0\right)^{\mathrm{l}+1}\left(Ӿ_1¥_1Ƶ_1\right)^{\mathrm{l}+2}\left(Ӿ_2¥_2Ƶ_2\right)^{\mathrm{l}+3}$$

$$= \left(Ӿ_0¥_0Ƶ_0\right)^{\mathrm{l}}(Ӿ_0¥_0Ƶ_0)(Ӿ_1¥_1Ƶ_1)^{\mathrm{l}+1}(Ӿ_1¥_1Ƶ_1)(Ӿ_2¥_2Ƶ_2)^{\mathrm{l}+2}(Ӿ_2¥_2Ƶ_2)$$

$$= \left(Ӿ_0¥_0Ƶ_0\right)^{\mathrm{l}}(Ӿ_1¥_1Ƶ_1)^{\mathrm{l}+1}(Ӿ_2¥_2Ƶ_2)^{\mathrm{l}+2}(Ӿ_0¥_0Ƶ_0)(Ӿ_1¥_1Ƶ_1)(Ӿ_2¥_2Ƶ_2)$$

$$= \left(Ӿ_3¥_3Ƶ_3\right)^{\mathrm{l}-2}(Ӿ_5¥_5Ƶ_5)(Ӿ_0¥_0Ƶ_0)(Ӿ_1¥_1Ƶ_1)(Ӿ_2¥_2Ƶ_2) \qquad \text{(By hypothesis)}$$

$$= \left(Ӿ_3¥_3Ƶ_3\right)^{\mathrm{l}-2}(Ӿ_5¥_5Ƶ_5)(Ӿ_3¥_3Ƶ_3) \qquad \text{(By Scheme No. 1)}$$

$$= \left(Ӿ_3¥_3Ƶ_3\right)^{\mathrm{l}-1}(Ӿ_5¥_5Ƶ_5)$$

The conclusion is valid for all integers $\mathbb{m} = \mathrm{l} + 1$.

So, the answer is true for any positive number n using the induction method.

Theorem 3.17: For each integer $\mathbb{m} \geq 0$:

(a) $Ӿ_{6\mathbb{m}+4} = \dfrac{\prod_{ɍ=0}^{6\mathbb{m}+3} Ӿ_ɍ}{\prod_{ɍ=0}^{6\mathbb{m}} Ӿ_ɍ}$

(b) $¥_{6\mathbb{m}+4} = \dfrac{\prod_{ɍ=0}^{6\mathbb{m}+3} ¥_ɍ}{\prod_{ɍ=0}^{6\mathbb{m}} ¥_ɍ}$

(c) $Ƶ_{6\mathbb{m}+4} = \dfrac{\prod_{ɍ=0}^{6\mathbb{m}+3} Ƶ_ɍ}{\prod_{ɍ=0}^{6\mathbb{m}} Ƶ_ɍ}$

Proof: We can prove the result with the help of mathematical induction.

Theorem 3.18: For every integer $\mathbb{m} \geq 0$:

(a) $Ӿ_{\mathrm{l}\mathbb{m}+\mathrm{m}} = \dfrac{\prod_{ɍ=0}^{6\mathbb{m}+\mathrm{m}-1} Ӿ_ɍ}{\prod_{ɍ=0}^{6\mathbb{m}+\mathrm{m}-4} Ӿ_ɍ}$

(b) $¥_{\mathrm{l}\mathbb{m}+\mathrm{m}} = \dfrac{\prod_{ɍ=0}^{6\mathbb{m}+\mathrm{m}-1} ¥_ɍ}{\prod_{ɍ=0}^{6\mathbb{m}+\mathrm{m}-4} ¥_ɍ}$

(c) $Ƶ_{\mathrm{l}\mathbb{m}+\mathrm{m}} = \dfrac{\prod_{ɍ=0}^{6\mathbb{m}+\mathrm{m}-1} Ƶ_ɍ}{\prod_{ɍ=0}^{6\mathbb{m}+\mathrm{m}-4} Ƶ_ɍ}$

Proof: With the aid of mathematical induction, we can demonstrate the conclusion.

Also, we can prove theorem 3.7 to theorem 3.17 with the help of Scheme no. 2^{nd}, 3^{rd}, 4^{th}, 5^{th}, 18^{th}, 25^{th}, 26^{th} and 27^{th}.

3.5.1 Possible Applications of Second and Third Order MTFS

While the theoretical aspects of MTFS are compelling, their structural characteristics also support a range of practical uses. In cryptography, the unpredictable evolution of such sequences can underpin secure key generation, resilient hashing algorithms, and robust pseudorandom number systems. Communication technologies may exploit their recursive complexity for data protection methods, including encrypted transmission protocols and watermark embedding in digital content. In scientific and engineering contexts, MTFS models can be adapted to represent systems with compound growth, cyclic feedback, or iterative construction examples include certain biological population models, network traffic simulations, and recursive algorithm design.

3.5.2 Significance of the Derived Identities

The identities established in this chapter for second- and third-order Multiplicative Triple Fibonacci Sequences (MTFS) are central to understanding the algebraic structure and functional behavior of these extended recursive systems. By expressing explicit relationships among the terms of the coupled sequences, these identities help clarify how the multiplicative nature of the recurrence interacts with the initial conditions and the specific scheme chosen. Such formulations provide insight into the inherent patterns that may not be immediately apparent from the recursive definitions alone.

From a theoretical standpoint, these identities reinforce the internal consistency of the MTFS framework and offer a formal basis for analyzing its general behavior. They also enable a more systematic investigation into the properties of the sequences, such as growth trends, symmetry, and sensitivity to initial inputs, all of which are crucial in the study of coupled nonlinear systems.

Beyond their theoretical relevance, the identities contribute to practical aspects of recursive modeling. They facilitate computational efficiency by reducing reliance on stepwise calculations and can be used to verify algorithmic implementations of MTFS-based processes.

Furthermore, these results hold potential for adaptation in areas such as cryptographic constructions, coding theory, and recursive data generation. Overall, the proven identities not only enrich the mathematical landscape of MTFS but also support its extension to higher-order systems and applied domains.

3.6 Conclusion:

The study of the MTFS of the Second order under three specific Schemes and the Third order under nine specific Schemes has illuminated a fascinating realm of mathematical intricacies and potential applications. This investigation into these extended FS has deepened our understanding of their structural properties. The explorations of the MTFS of the Second order under three specific Schemes and the Third order under nine specific Schemes have revealed a rich tapestry of mathematical intricacies and potential applications. The study not only extended the classical TFS but also introduced multiplicative factors that add a layer of complexity and depth to the sequences' behavior. Through a systematic analysis of the recurrence relations and initial conditions, we observed the emergence of distinct patterns under each specific Scheme. The Second-order MTFS exhibited unique properties influenced by carefully designed Schemes, demonstrating the sensitivity of the sequence to the choice of initial conditions. Expanding our exploration to the Third-order case, introducing nine specific Schemes further diversified the mathematical landscape.

"Within the second- and third-order multiplicative triple Fibonacci sequences lies a progression from simplicity to layered complexity, where each increase in order reveals new algebraic identities and deeper structural patterns."

Chapter-4

Multiplicative Triple Fibonacci Sequence of Fourth Order

The work presented in this chapter has been published in the form of a research paper entitled “Multiplicative Triple Fibonacci Sequence of Fourth Order” in the Scopus Indexed.

4.1. Introduction

The Fibonacci Triple Sequence is a recent guideline for the universality of the Coupled Fibonacci Sequence. The Fibonacci numbers and their underlying abstract principle can be used in almost every area of science. Koshy's book [4] is the best reason why this is important. K.T. Atanassov was the first to set up the Coupled Fibonacci Sequence. He also looked into many interesting properties and a current protocol for generalizing the Fibonacci Sequence.First Additive Triple Sequence was approved by J. Z. Lee and J.S. Lee [48]. Atanassov came up with a new idea for the Additive Triple Fibonacci Sequence, which he called the 3-Fibonacci Sequence or the 3-F Sequence.

4.1.1 Motivation for Studying Fourth-Order MTFS

The fourth-order Multiplicative Triple Fibonacci Sequence (MTFS) extends the conceptual scope of its lower-order forms, introducing a richer framework of interactions among the three component sequences. Unlike the second- and third-order cases, where multiplicative coupling follows comparatively simpler pathways, the fourth order reveals a more layered and intricate interplay, giving rise to previously unseen identities and complex recurrence relations. Investigating this higher order not only deepens the theoretical understanding of multiplicative systems but also clarifies how increasing structural depth influences growth dynamics and algebraic behavior. Such insights may, in turn, inform advanced cryptographic methods, refined mathematical models, and recursive algorithmic designs where complexity and unpredictability are essential features.

4.2. MTFS of Fourth order:

Let $\{Ӿ_i\}_{i=0}^{\infty}\{Ұ_i\}_{i=0}^{\infty}$ and $\{Ƶ_i\}_{i=0}^{\infty}$ be three infinite sequences and called 3-F Sequence or Triple Fibonacci Sequence with initial value $a, b, c, d, e, f, g, h, i, j, k$ and l be given. $Ӿ_0 = a, Ұ_0 = b, Ƶ_0 = c, Ӿ_1 = d, Ұ_1 = e, Ƶ_1 = f, Ӿ_2 = g, Ұ_2 = h, Ƶ_2 = i, Ӿ_3 = j, Ұ_3 = k, Ƶ_3 = l$. Then there are 81 Schemes of MTFS of fourth order. Here, we are presenting some identities of fourth order under nine specific Schemes and these nine Schemes are as follows:

Table 4.1: Some Schemes of 4th order MTFS we worked on

Scheme	$Ӿ_{m}$	$Ұ_{m}$	$Ƶ_{m}$
1	$Ӿ_{m+3}.Ӿ_{m+2}.Ӿ_{m+1}.Ӿ_{m}$	$Ұ_{m+3}.Ұ_{m+2}.Ұ_{m+1}.Ұ_{m}$	$Ƶ_{m+3}.Ƶ_{m+2}.Ƶ_{m+1}.Ƶ_{m}$

2	$Ұ_{\mathbb{m}+3}.Ұ_{\mathbb{m}+2}.Ұ_{\mathbb{m}+1}.Ұ_{\mathbb{m}}$	$Ƶ_{\mathbb{m}+3}.Ƶ_{\mathbb{m}+2}.Ƶ_{\mathbb{m}+1}.Ƶ_{\mathbb{m}}$	$Ӿ_{\mathbb{m}+3}.Ӿ_{\mathbb{m}+2}.Ӿ_{\mathbb{m}+1}.Ӿ_{\mathbb{m}}$
3	$Ƶ_{\mathbb{m}+3}.Ƶ_{\mathbb{m}+2}.Ƶ_{\mathbb{m}+1}.Ƶ_{\mathbb{m}}$	$Ӿ_{\mathbb{m}+3}.Ӿ_{\mathbb{m}+2}.Ӿ_{\mathbb{m}+1}.Ӿ_{\mathbb{m}}$	$Ұ_{\mathbb{m}+3}.Ұ_{\mathbb{m}+2}.Ұ_{\mathbb{m}+1}.Ұ_{\mathbb{m}}$
4	$Ӿ_{\mathbb{m}+3}.Ұ_{\mathbb{m}+2}.Ƶ_{\mathbb{m}+1}.Ӿ_{\mathbb{m}}$	$Ұ_{\mathbb{m}+3}.Ƶ_{\mathbb{m}+2}.Ӿ_{\mathbb{m}+1}.Ұ_{\mathbb{m}}$	$Ƶ_{\mathbb{m}+3}.Ӿ_{\mathbb{m}+2}.Ұ_{\mathbb{m}+1}.Ƶ_{\mathbb{m}}$
5	$Ƶ_{\mathbb{m}+3}.Ӿ_{\mathbb{m}+2}.Ұ_{\mathbb{m}+1}.Ƶ_{\mathbb{m}}$	$Ӿ_{\mathbb{m}+3}.Ұ_{\mathbb{m}+2}.Ƶ_{\mathbb{m}+1}.Ӿ_{\mathbb{m}}$	$Ұ_{\mathbb{m}+3}.Ƶ_{\mathbb{m}+2}.Ӿ_{\mathbb{m}+1}.Ұ_{\mathbb{m}}$
6	$Ұ_{\mathbb{m}+3}.Ƶ_{\mathbb{m}+2}.Ӿ_{\mathbb{m}+1}.Ұ_{\mathbb{m}}$	$Ƶ_{\mathbb{m}+3}.Ӿ_{\mathbb{m}+2}.Ұ_{\mathbb{m}+1}.Ƶ_{\mathbb{m}}$	$Ӿ_{\mathbb{m}+3}.Ұ_{\mathbb{m}+2}.Ƶ_{\mathbb{m}+1}.Ӿ_{\mathbb{m}}$
7	$Ӿ_{\mathbb{m}+3}.Ƶ_{\mathbb{m}+2}.Ұ_{\mathbb{m}+1}.Ӿ_{\mathbb{m}}$	$Ұ_{\mathbb{m}+3}.Ӿ_{\mathbb{m}+2}.Ƶ_{\mathbb{m}+1}.Ұ_{\mathbb{m}}$	$Ƶ_{\mathbb{m}+3}.Ұ_{\mathbb{m}+2}.Ӿ_{\mathbb{m}+1}.Ƶ_{\mathbb{m}}$
8	$Ұ_{\mathbb{m}+3}.Ӿ_{\mathbb{m}+2}.Ƶ_{\mathbb{m}+1}.Ұ_{\mathbb{m}}$	$Ƶ_{\mathbb{m}+3}.Ұ_{\mathbb{m}+2}.Ӿ_{\mathbb{m}+1}.Ƶ_{\mathbb{m}}$	$Ӿ_{\mathbb{m}+3}.Ƶ_{\mathbb{m}+2}.Ұ_{\mathbb{m}+1}.Ӿ_{\mathbb{m}}$
9	$Ƶ_{\mathbb{m}+3}.Ұ_{\mathbb{m}+2}.Ӿ_{\mathbb{m}+1}.Ƶ_{\mathbb{m}}$	$Ӿ_{\mathbb{m}+3}.Ƶ_{\mathbb{m}+2}.Ұ_{\mathbb{m}+1}.Ӿ_{\mathbb{m}}$	$Ұ_{\mathbb{m}+3}.Ӿ_{\mathbb{m}+2}.Ƶ_{\mathbb{m}+1}.Ұ_{\mathbb{m}}$

Table 4.2: Some terms of MTFS of 4th order of 1st Scheme

$\mathbb{m}$	$Ӿ_{\mathbb{m}}$	$Ұ_{\mathbb{m}}$	$Ƶ_{\mathbb{m}}$
0	a	b	c
1	d	e	f
2	g	h	i
3	j	k	l
4	$adgj$	$behk$	$cfil$
5	$ad^2g^2j^2$	$be^2h^2k^2$	$cf^2i^2l^2$

Table 4.2: Some terms of MTFS of 4th order of 2nd Scheme

$\mathbb{m}$	$Ӿ_{\mathbb{m}}$	$Ұ_{\mathbb{m}}$	$Ƶ_{\mathbb{m}}$
0	a	b	c
1	d	e	f
2	g	h	i
3	j	k	l
4	$behk$	$cfil$	$adgj$
5	$be^2h^2k^2$	$cf^2i^2l^2$	$ad^2g^2j^2$

Table 4.3: Some terms of MTFS of 4th order of 3rd Scheme

$\mathbb{m}$	$Ӿ_{\mathbb{m}}$	$Ұ_{\mathbb{m}}$	$Ƶ_{\mathbb{m}}$

0	a	b	c
1	d	e	f
2	g	h	i
3	j	k	l
4	$cfil$	$adgj$	$behk$
5	$cf^2i^2l^2$	$ad^2g^2j^2$	$be^2h^2k^2$

Now we present the identities of 4th order MTFS under Scheme no. 1st, 2nd, 3rd, 4th, 5th, 6th, 7th, 8th and 9th.

4.3 Main Results of 3rd Order MTFS

We will prove all the results by using Scheme no. 1

Theorem 4.1: For every natural number $\mathbb{m} \geq 2$,

$$(\mathbb{X}_0\mathbb{Y}_0\mathbb{Z}_0)^{\mathbb{m}}(\mathbb{X}_1\mathbb{Y}_1\mathbb{Z}_1)^{\mathbb{m}+1}(\mathbb{X}_2\mathbb{Y}_2\mathbb{Z}_2)^{\mathbb{m}+2}(\mathbb{X}_3\mathbb{Y}_3\mathbb{Z}_3)^{\mathbb{m}+3} = (\mathbb{X}_3\mathbb{Y}_3\mathbb{Z}_3)\left(\mathbb{X}_4\mathbb{Y}_4\mathbb{Z}_4\right)^{\mathbb{m}-2}(\mathbb{X}_6\mathbb{Y}_6\mathbb{Z}_6)$$

Proof: The induction method allows us to demonstrate the aforementioned:

For $\mathbb{m} = 2$, we have

$$\left(\mathbb{X}_0\mathbb{Y}_0\mathbb{Z}_0\right)^2\left(\mathbb{X}_1\mathbb{Y}_1\mathbb{Z}_1\right)^3\left(\mathbb{X}_2\mathbb{Y}_2\mathbb{Z}_2\right)^4\left(\mathbb{X}_3\mathbb{Y}_3\mathbb{Z}_3\right)^5$$

$$= (\mathbb{X}_1\mathbb{Y}_1\mathbb{Z}_1)(\mathbb{X}_2\mathbb{Y}_2\mathbb{Z}_2)^2(\mathbb{X}_3\mathbb{Y}_3\mathbb{Z}_3)^3(\mathbb{X}_4\mathbb{Y}_4\mathbb{Z}_4)^2$$

$$= (\mathbb{X}_2\mathbb{Y}_2\mathbb{Z}_2)(\mathbb{X}_3\mathbb{Y}_3\mathbb{Z}_3)^2(\mathbb{X}_4\mathbb{Y}_4\mathbb{Z}_4)(\mathbb{X}_5\mathbb{Y}_5\mathbb{Z}_5)$$

$$= (\mathbb{X}_3\mathbb{Y}_3\mathbb{Z}_3)(\mathbb{X}_6\mathbb{Y}_6\mathbb{Z}_6)$$

For $\mathbb{m} = 2$, the conclusion is correct.

We'll proceed by assuming that the outcome is accurate for some integer $\mathbb{m}$.

Then for $\mathbb{m} + 1$

$$\left(\mathbb{X}_0\mathbb{Y}_0\mathbb{Z}_0\right)^{\mathbb{m}+1}\left(\mathbb{X}_1\mathbb{Y}_1\mathbb{Z}_1\right)^{\mathbb{m}+2}\left(\mathbb{X}_2\mathbb{Y}_2\mathbb{Z}_2\right)^{\mathbb{m}+3}\left(\mathbb{X}_3\mathbb{Y}_3\mathbb{Z}_3\right)^{\mathbb{m}+4}$$

$$= \left(\mathbb{X}_0\mathbb{Y}_0\mathbb{Z}_0\right)(\mathbb{X}_1\mathbb{Y}_1\mathbb{Z}_1)(\mathbb{X}_2\mathbb{Y}_2\mathbb{Z}_2)(\mathbb{X}_3\mathbb{Y}_3\mathbb{Z}_3)\left(\mathbb{X}_0\mathbb{Y}_0\mathbb{Z}_0\right)^{\mathbb{m}}\left(\mathbb{X}_1\mathbb{Y}_1\mathbb{Z}_1\right)^{\mathbb{m}+1}$$

$$(\mathbb{X}_2\mathbb{Y}_2\mathbb{Z}_2)^{\mathbb{m}+2}(\mathbb{X}_3\mathbb{Y}_3\mathbb{Z}_3)^{\mathbb{m}+3}$$

$$= (\mathbb{X}_4\mathbb{Y}_4\mathbb{Z}_4)(\mathbb{X}_3\mathbb{Y}_3\mathbb{Z}_3)\left(\mathbb{X}_4\mathbb{Y}_4\mathbb{Z}_4\right)^{\mathbb{m}-2}(\mathbb{X}_6\mathbb{Y}_6\mathbb{Z}_6) \qquad \text{(By hypothesis)}$$

$$= (\mathbb{X}_3\mathbb{Y}_3\mathbb{Z}_3)\left(\mathbb{X}_4\mathbb{Y}_4\mathbb{Z}_4\right)^{\mathbb{m}-1}(\mathbb{X}_6\mathbb{Y}_6\mathbb{Z}_6)$$

The conclusion is valid for all integers $\mathbb{m} + 1$. Therefore, according to the induction process, the conclusion is valid for every positive integer $\mathbb{m} \geq 2$.

Theorem 4.2: For every even integer $\mathbb{m} \geq 2$,

$$\left(\mathbb{X}_{\mathbb{m}}\mathbb{Y}_{\mathbb{m}}\mathbb{Z}_{\mathbb{m}}\right)^{\left[\frac{\mathbb{m}}{2}\right]}\left(\mathbb{X}_{\mathbb{m}+1}\mathbb{Y}_{\mathbb{m}+1}\mathbb{Z}_{\mathbb{m}+1}\right)^{\left[\frac{\mathbb{m}}{2}\right]+1}\left(\mathbb{X}_{\mathbb{m}+2}\mathbb{Y}_{\mathbb{m}+2}\mathbb{Z}_{\mathbb{m}+2}\right)^{\left[\frac{\mathbb{m}}{2}\right]+2}(\mathbb{X}_{\mathbb{m}+3}\mathbb{Y}_{\mathbb{m}+3}\mathbb{Z}_{\mathbb{m}+3})^{[\frac{\mathbb{m}}{2}]+3}$$
$$= (\mathbb{X}_{\mathbb{m}+2}\mathbb{Y}_{\mathbb{m}+2}\mathbb{Z}_{\mathbb{m}+2})\left(\mathbb{X}_{\mathbb{m}+3}\mathbb{Y}_{\mathbb{m}+3}\mathbb{Z}_{\mathbb{m}+3}\right)^2(\mathbb{X}_{\mathbb{m}+4}\mathbb{Y}_{\mathbb{m}+4}\mathbb{Z}_{\mathbb{m}+4})$$

Proof:

These results are confirmed by the induction hypothesis.For $\mathbb{m} = 2$, we have

$$\Rightarrow \left(\mathbb{X}_2\mathbb{Y}_2\mathbb{Z}_2\right)\left(\mathbb{X}_3\mathbb{Y}_3\mathbb{Z}_3\right)^2\left(\mathbb{X}_4\mathbb{Y}_4\mathbb{Z}_4\right)^3\left(\mathbb{X}_5\mathbb{Y}_5\mathbb{Z}_5\right)^4 = (\mathbb{X}_3\mathbb{Y}_3\mathbb{Z}_3)(\mathbb{X}_4\mathbb{Y}_4\mathbb{Z}_4)^2\left(\mathbb{X}_5\mathbb{Y}_5\mathbb{Z}_5\right)^3(\mathbb{X}_6\mathbb{Y}_6\mathbb{Z}_6)$$
$$(\mathbb{X}_4\mathbb{Y}_4\mathbb{Z}_4)\left(\mathbb{X}_5\mathbb{Y}_5\mathbb{Z}_5\right)^2(\mathbb{X}_7\mathbb{Y}_7\mathbb{Z}_7)$$

For $\mathbb{m} = 2$, the conclusion is correct.

We'll proceed by assuming that the outcome is accurate for some integer $\mathbb{m}$.

Then for $\mathbb{m} + 1$

$$(\mathbb{X}_{\mathbb{m}+2}\mathbb{Y}_{\mathbb{m}+2}\mathbb{Z}_{\mathbb{m}+2})^{[\frac{\mathbb{m}+2}{2}]}(\mathbb{X}_{\mathbb{m}+3}\mathbb{Y}_{\mathbb{m}+3}\mathbb{Z}_{\mathbb{m}+3})^{[\frac{\mathbb{m}+2}{2}]+1}(\mathbb{X}_{\mathbb{m}+4}\mathbb{Y}_{\mathbb{m}+4}\mathbb{Z}_{\mathbb{m}+4})^{[\frac{\mathbb{m}+2}{2}]+2}(\mathbb{X}_{\mathbb{m}+5}\mathbb{Y}_{\mathbb{m}+5}\mathbb{Z}_{\mathbb{m}+5})^{[\frac{\mathbb{m}+2}{2}]+3}$$
$$=(\mathbb{X}_{\mathbb{m}+2}\mathbb{Y}_{\mathbb{m}+2}\mathbb{Z}_{\mathbb{m}+2})^{\left[\frac{\mathbb{m}}{2}\right]+1}(\mathbb{X}_{\mathbb{m}+3}\mathbb{Y}_{\mathbb{m}+3}\mathbb{Z}_{\mathbb{m}+3})^{[\frac{\mathbb{m}}{2}]+2}(\mathbb{X}_{\mathbb{m}+4}\mathbb{Y}_{\mathbb{m}+4}\mathbb{Z}_{\mathbb{m}+4})^{[\frac{\mathbb{m}}{2}]+3}(\mathbb{X}_{\mathbb{m}+5}\mathbb{Y}_{\mathbb{m}+5}\mathbb{Z}_{\mathbb{m}+5})^{[\frac{\mathbb{m}}{2}]+4}$$

………..(4.1)

Now we will bring each part of this equation by solving and putting its value here.

$$(\mathbb{X}_{\mathbb{m}+2}\mathbb{Y}_{\mathbb{m}+2}\mathbb{Z}_{\mathbb{m}+2})^{\left[\frac{\mathbb{m}}{2}\right]+1}$$
$$= (\mathbb{X}_{\mathbb{m}+1}\mathbb{Y}_{\mathbb{m}+1}\mathbb{Z}_{\mathbb{m}+1})^{[\frac{\mathbb{m}}{2}]+1}(\mathbb{X}_{\mathbb{m}}\mathbb{Y}_{\mathbb{m}}\mathbb{Z}_{\mathbb{m}})^{[\frac{\mathbb{m}}{2}]+1}(\mathbb{X}_{\mathbb{m}-1}\mathbb{Y}_{\mathbb{m}-1}\mathbb{Z}_{\mathbb{m}-1})^{[\frac{\mathbb{m}}{2}]+1}(\mathbb{X}_{\mathbb{m}-2}\mathbb{Y}_{\mathbb{m}-2}\mathbb{Z}_{\mathbb{m}-2})^{[\frac{\mathbb{m}}{2}]+1}$$

………(4.2)

$$(\mathbb{X}_{\mathbb{m}+3}\mathbb{Y}_{\mathbb{m}+3}\mathbb{Z}_{\mathbb{m}+3})^{[\frac{\mathbb{m}}{2}]+2}$$
$$= (\mathbb{X}_{\mathbb{m}+2}\mathbb{Y}_{\mathbb{m}+2}\mathbb{Z}_{\mathbb{m}+2})^{[\frac{\mathbb{m}}{2}]+2}(\mathbb{X}_{\mathbb{m}+1}\mathbb{Y}_{\mathbb{m}+1}\mathbb{Z}_{\mathbb{m}+1})^{[\frac{\mathbb{m}}{2}]+2}(\mathbb{X}_{\mathbb{m}}\mathbb{Y}_{\mathbb{m}}\mathbb{Z}_{\mathbb{m}})^{[\frac{\mathbb{m}}{2}]+2}(\mathbb{X}_{\mathbb{m}-1}\mathbb{Y}_{\mathbb{m}-1}\mathbb{Z}_{\mathbb{m}-1})^{[\frac{\mathbb{m}}{2}]+2}$$

………(4.3)

$$(\mathbb{X}_{\mathbb{m}+4}\mathbb{Y}_{\mathbb{m}+4}\mathbb{Z}_{\mathbb{m}+4})^{[\frac{\mathbb{m}}{2}]+3}$$
$$= (\mathbb{X}_{\mathbb{m}+3}\mathbb{Y}_{\mathbb{m}+3}\mathbb{Z}_{\mathbb{m}+3})^{[\frac{\mathbb{m}}{2}]+3}(\mathbb{X}_{\mathbb{m}+2}\mathbb{Y}_{\mathbb{m}+2}\mathbb{Z}_{\mathbb{m}+2})^{[\frac{\mathbb{m}}{2}]+3}(\mathbb{X}_{\mathbb{m}+1}\mathbb{Y}_{\mathbb{m}+1}\mathbb{Z}_{\mathbb{m}+1})^{[\frac{\mathbb{m}}{2}]+3}(\mathbb{X}_{\mathbb{m}}\mathbb{Y}_{\mathbb{m}}\mathbb{Z}_{\mathbb{m}})^{[\frac{\mathbb{m}}{2}]+3}$$

………(4.4)

$$(\mathbb{X}_{\mathbb{m}+5}\mathbb{Y}_{\mathbb{m}+5}\mathbb{Z}_{\mathbb{m}+5})^{[\frac{\mathbb{m}}{2}]+4}$$
$$= (\mathbb{X}_{\mathbb{m}+4}\mathbb{Y}_{\mathbb{m}+4}\mathbb{Z}_{\mathbb{m}+4})^{[\frac{\mathbb{m}}{2}]+4}(\mathbb{X}_{\mathbb{m}+3}\mathbb{Y}_{\mathbb{m}+3}\mathbb{Z}_{\mathbb{m}+3})^{[\frac{\mathbb{m}}{2}]+4}(\mathbb{X}_{\mathbb{m}+2}\mathbb{Y}_{\mathbb{m}+2}\mathbb{Z}_{\mathbb{m}+2})^{[\frac{\mathbb{m}}{2}]+4}(\mathbb{X}_{\mathbb{m}+1}\mathbb{Y}_{\mathbb{m}+1}\mathbb{Z}_{\mathbb{m}+1})^{[\frac{\mathbb{m}}{2}]+4}$$

………(4.5)

now putting the value of equation (4.2), (4.3), (4.4) and (4.5) in equation (4.1), we get

$$(\mathbb{X}_{\mathbb{m}+2}\mathbb{Y}_{\mathbb{m}+2}\mathbb{Z}_{\mathbb{m}+2})^{\left[\frac{\mathbb{m}}{2}\right]+1}(\mathbb{X}_{\mathbb{m}+3}\mathbb{Y}_{\mathbb{m}+3}\mathbb{Z}_{\mathbb{m}+3})^{[\frac{\mathbb{m}}{2}]+2}(\mathbb{X}_{\mathbb{m}+4}\mathbb{Y}_{\mathbb{m}+4}\mathbb{Z}_{\mathbb{m}+4})^{[\frac{\mathbb{m}}{2}]+3}(\mathbb{X}_{\mathbb{m}+5}\mathbb{Y}_{\mathbb{m}+5}\mathbb{Z}_{\mathbb{m}+5})^{[\frac{\mathbb{m}}{2}]+4}$$
$$= (\mathbb{X}_{\mathbb{m}+1}\mathbb{Y}_{\mathbb{m}+1}\mathbb{Z}_{\mathbb{m}+1})^{[\frac{\mathbb{m}}{2}]+1}(\mathbb{X}_{\mathbb{m}}\mathbb{Y}_{\mathbb{m}}\mathbb{Z}_{\mathbb{m}})^{\left[\frac{\mathbb{m}}{2}\right]+1}(\mathbb{X}_{\mathbb{m}-1}\mathbb{Y}_{\mathbb{m}-1}\mathbb{Z}_{\mathbb{m}-1})^{\left[\frac{\mathbb{m}}{2}\right]+1}(\mathbb{X}_{\mathbb{m}-2}\mathbb{Y}_{\mathbb{m}-2}\mathbb{Z}_{\mathbb{m}-2})^{\left[\frac{\mathbb{m}}{2}\right]+1}$$
$$(\mathbb{X}_{\mathbb{m}+2}\mathbb{Y}_{\mathbb{m}+2}\mathbb{Z}_{\mathbb{m}+2})^{[\frac{\mathbb{m}}{2}]+2}(\mathbb{X}_{\mathbb{m}+1}\mathbb{Y}_{\mathbb{m}+1}\mathbb{Z}_{\mathbb{m}+1})^{[\frac{\mathbb{m}}{2}]+2}(\mathbb{X}_{\mathbb{m}}\mathbb{Y}_{\mathbb{m}}\mathbb{Z}_{\mathbb{m}})^{[\frac{\mathbb{m}}{2}]+2}(\mathbb{X}_{\mathbb{m}-1}\mathbb{Y}_{\mathbb{m}-1}\mathbb{Z}_{\mathbb{m}-1})^{[\frac{\mathbb{m}}{2}]+2}$$

$$(Ӿ_{\mathbb{m}+3}Ұ_{\mathbb{m}+3}Ƶ_{\mathbb{m}+3})^{\left[\frac{\mathbb{m}}{2}\right]+3}(Ӿ_{\mathbb{m}+2}Ұ_{\mathbb{m}+2}Ƶ_{\mathbb{m}+2})^{\left[\frac{\mathbb{m}}{2}\right]+3}(Ӿ_{\mathbb{m}+1}Ұ_{\mathbb{m}+1}Ƶ_{\mathbb{m}+1})^{\left[\frac{\mathbb{m}}{2}\right]+3}(Ӿ_{\mathbb{m}}Ұ_{\mathbb{m}}Ƶ_{\mathbb{m}})^{\left[\frac{\mathbb{m}}{2}\right]+3}$$

$$\left(Ӿ_{\mathbb{m}+4}Ұ_{\mathbb{m}+4}Ƶ_{\mathbb{m}+4}\right)^{\left[\frac{\mathbb{m}}{2}\right]+4}\left(Ӿ_{\mathbb{m}+3}Ұ_{\mathbb{m}+3}Ƶ_{\mathbb{m}+3}\right)^{\left[\frac{\mathbb{m}}{2}\right]+4}\left(Ӿ_{\mathbb{m}+2}Ұ_{\mathbb{m}+2}Ƶ_{\mathbb{m}+2}\right)^{\left[\frac{\mathbb{m}}{2}\right]+4}\left(Ӿ_{\mathbb{m}+1}Ұ_{\mathbb{m}+1}Ƶ_{\mathbb{m}+1}\right)^{\left[\frac{\mathbb{m}}{2}\right]+4}$$

$$= (Ӿ_{\mathbb{m}+1}Ұ_{\mathbb{m}+1}Ƶ_{\mathbb{m}+1})^{\left[\frac{\mathbb{m}}{2}\right]+1}(Ӿ_{\mathbb{m}+2}Ұ_{\mathbb{m}+2}Ƶ_{\mathbb{m}+2})^{\left[\frac{\mathbb{m}}{2}\right]+2}(Ӿ_{\mathbb{m}+3}Ұ_{\mathbb{m}+3}Ƶ_{\mathbb{m}+3})^{\left[\frac{\mathbb{m}}{2}\right]+3}\left(Ӿ_{\mathbb{m}+4}Ұ_{\mathbb{m}+4}Ƶ_{\mathbb{m}+4}\right)^{\left[\frac{\mathbb{m}}{2}\right]+4}$$

$$(Ӿ_{\mathbb{m}}Ұ_{\mathbb{m}}Ƶ_{\mathbb{m}})^{\left[\frac{\mathbb{m}}{2}\right]+1}(Ӿ_{\mathbb{m}+1}Ұ_{\mathbb{m}+1}Ƶ_{\mathbb{m}+1})^{\left[\frac{\mathbb{m}}{2}\right]+2}(Ӿ_{\mathbb{m}+2}Ұ_{\mathbb{m}+2}Ƶ_{\mathbb{m}+2})^{\left[\frac{\mathbb{m}}{2}\right]+3}\left(Ӿ_{\mathbb{m}+3}Ұ_{\mathbb{m}+3}Ƶ_{\mathbb{m}+3}\right)^{\left[\frac{\mathbb{m}}{2}\right]+4}$$

$$(Ӿ_{\mathbb{m}-1}Ұ_{\mathbb{m}-1}Ƶ_{\mathbb{m}-1})^{\left[\frac{\mathbb{m}}{2}\right]+1}(Ӿ_{\mathbb{m}}Ұ_{\mathbb{m}}Ƶ_{\mathbb{m}})^{\left[\frac{\mathbb{m}}{2}\right]+2}(Ӿ_{\mathbb{m}+1}Ұ_{\mathbb{m}+1}Ƶ_{\mathbb{m}+1})^{\left[\frac{\mathbb{m}}{2}\right]+3}\left(Ӿ_{\mathbb{m}+2}Ұ_{\mathbb{m}+2}Ƶ_{\mathbb{m}+2}\right)^{\left[\frac{\mathbb{m}}{2}\right]+4}$$

$$(Ӿ_{\mathbb{m}-2}Ұ_{\mathbb{m}-2}Ƶ_{\mathbb{m}-2})^{\left[\frac{\mathbb{m}}{2}\right]+1}(Ӿ_{\mathbb{m}-1}Ұ_{\mathbb{m}-1}Ƶ_{\mathbb{m}-1})^{\left[\frac{\mathbb{m}}{2}\right]+2}(Ӿ_{\mathbb{m}}Ұ_{\mathbb{m}}Ƶ_{\mathbb{m}})^{\left[\frac{\mathbb{m}}{2}\right]+3}\left(Ӿ_{\mathbb{m}+1}Ұ_{\mathbb{m}+1}Ƶ_{\mathbb{m}+1}\right)^{\left[\frac{\mathbb{m}}{2}\right]+4}$$

now we use given hypothesis for every line,

$$= (Ӿ_{\mathbb{m}+3}Ұ_{\mathbb{m}+3}Ƶ_{\mathbb{m}+3})\left(Ӿ_{\mathbb{m}+4}Ұ_{\mathbb{m}+4}Ƶ_{\mathbb{m}+4}\right)^{\mathbb{m}-2}(Ӿ_{\mathbb{m}+5}Ұ_{\mathbb{m}+5}Ƶ_{\mathbb{m}+5})$$

$$(Ӿ_{\mathbb{m}+2}Ұ_{\mathbb{m}+2}Ƶ_{\mathbb{m}+2})\left(Ӿ_{\mathbb{m}+3}Ұ_{\mathbb{m}+3}Ƶ_{\mathbb{m}+3}\right)^{\mathbb{m}-2}(Ӿ_{\mathbb{m}+4}Ұ_{\mathbb{m}+4}Ƶ_{\mathbb{m}+4})$$

$$\left(Ӿ_{\mathbb{m}+1}Ұ_{\mathbb{m}+1}Ƶ_{\mathbb{m}+1}\right)\left(Ӿ_{\mathbb{m}+2}Ұ_{\mathbb{m}+2}Ƶ_{\mathbb{m}+2}\right)^{\mathbb{m}-2}\left(Ӿ_{\mathbb{m}+3}Ұ_{\mathbb{m}+3}Ƶ_{\mathbb{m}+3}\right)$$

$$\left(Ӿ_{\mathbb{m}}Ұ_{\mathbb{m}}Ƶ_{\mathbb{m}}\right)\left(Ӿ_{\mathbb{m}+1}Ұ_{\mathbb{m}+1}Ƶ_{\mathbb{m}+1}\right)^{\mathbb{m}-2}\left(Ӿ_{\mathbb{m}+2}Ұ_{\mathbb{m}+2}Ƶ_{\mathbb{m}+2}\right)$$

$$= (Ӿ_{\mathbb{m}+4}Ұ_{\mathbb{m}+4}Ƶ_{\mathbb{m}+4})\left(Ӿ_{\mathbb{m}+5}Ұ_{\mathbb{m}+5}Ƶ_{\mathbb{m}+5}\right)^{\mathbb{m}-2}(Ӿ_{\mathbb{m}+6}Ұ_{\mathbb{m}+6}Ƶ_{\mathbb{m}+6})$$

The conclusion is valid for all integers $\mathbb{m}+2$. As a result, using the induction method, the conclusion holds for any positive even integer$\mathbb{m} \geq 1$.

Theorem 4.3: For every odd integer $\mathbb{m} \geq 1$,

$$(Ӿ_{\mathbb{m}}Ұ_{\mathbb{m}}Ƶ_{\mathbb{m}})^{\left[\frac{\mathbb{m}}{2}\right]}(Ӿ_{\mathbb{m}+1}Ұ_{\mathbb{m}+1}Ƶ_{\mathbb{m}+1})^{\left[\frac{\mathbb{m}}{2}\right]+1}(Ӿ_{\mathbb{m}+2}Ұ_{\mathbb{m}+2}Ƶ_{\mathbb{m}+2})^{\left[\frac{\mathbb{m}}{2}\right]+2}(Ӿ_{\mathbb{m}+3}Ұ_{\mathbb{m}+3}Ƶ_{\mathbb{m}+3})^{\left[\frac{\mathbb{m}}{2}\right]+3}$$
$$= (Ӿ_{\mathbb{m}+1}Ұ_{\mathbb{m}+1}Ƶ_{\mathbb{m}+1})\left(Ӿ_{\mathbb{m}+2}Ұ_{\mathbb{m}+2}Ƶ_{\mathbb{m}+2}\right)^{2}(Ӿ_{\mathbb{m}+3}Ұ_{\mathbb{m}+3}Ƶ_{\mathbb{m}+3})$$

Proof: These results are confirmed by the induction hypothesis.

For $\mathbb{m} = 1$,we have

$$\left(Ӿ_1Ұ_1Ƶ_1\right)^{0}\left(Ӿ_2Ұ_2Ƶ_2\right)\left(Ӿ_3Ұ_3Ƶ_3\right)^{2}\left(Ӿ_4Ұ_4Ƶ_4\right)^{3} = (Ӿ_2Ұ_2Ƶ_2)(Ӿ_3Ұ_3Ƶ_3)^2(Ӿ_4Ұ_4Ƶ_4)$$

For each odd number $\mathbb{m} = 1$, the conclusion is correct.

We'll proceed by assuming that the outcome is accurate for some odd integer $\mathbb{m}$.

Then for $\mathbb{m}+1$

$$(Ӿ_{\mathbb{m}+2}Ұ_{\mathbb{m}+2}Ƶ_{\mathbb{m}+2})^{\left[\frac{\mathbb{m}+2}{2}\right]}(Ӿ_{\mathbb{m}+3}Ұ_{\mathbb{m}+3}Ƶ_{\mathbb{m}+3})^{\left[\frac{\mathbb{m}+2}{2}\right]+1}(Ӿ_{\mathbb{m}+4}Ұ_{\mathbb{m}+4}Ƶ_{\mathbb{m}+4})^{\left[\frac{\mathbb{m}+2}{2}\right]+2}(Ӿ_{\mathbb{m}+5}Ұ_{\mathbb{m}+5}Ƶ_{\mathbb{m}+5})^{\left[\frac{\mathbb{m}+2}{2}\right]+3}$$

$$=(Ӿ_{\mathbb{m}+2}Ұ_{\mathbb{m}+2}Ƶ_{\mathbb{m}+2})^{\left[\frac{\mathbb{m}}{2}\right]+1}(Ӿ_{\mathbb{m}+3}Ұ_{\mathbb{m}+3}Ƶ_{\mathbb{m}+3})^{\left[\frac{\mathbb{m}}{2}\right]+2}(Ӿ_{\mathbb{m}+4}Ұ_{\mathbb{m}+4}Ƶ_{\mathbb{m}+4})^{\left[\frac{\mathbb{m}}{2}\right]+3}(Ӿ_{\mathbb{m}+5}Ұ_{\mathbb{m}+5}Ƶ_{\mathbb{m}+5})^{\left[\frac{\mathbb{m}}{2}\right]+4}$$

………..(4.6)

Now we will bring each part of this equation by solving and putting its value here.

$$(\text{Ӿ}_{\mathbb{m}+2}\text{¥}_{\mathbb{m}+2}\text{Ƶ}_{\mathbb{m}+2})^{\left[\frac{\mathbb{m}}{2}\right]+1}$$
$$= (\text{Ӿ}_{\mathbb{m}+1}\text{¥}_{\mathbb{m}+1}\text{Ƶ}_{\mathbb{m}+1})^{\left[\frac{\mathbb{m}}{2}\right]+1}(\text{Ӿ}_{\mathbb{m}}\text{¥}_{\mathbb{m}}\text{Ƶ}_{\mathbb{m}})^{\left[\frac{\mathbb{m}}{2}\right]+1}(\text{Ӿ}_{\mathbb{m}-1}\text{¥}_{\mathbb{m}-1}\text{Ƶ}_{\mathbb{m}-1})^{\left[\frac{\mathbb{m}}{2}\right]+1}(\text{Ӿ}_{\mathbb{m}-2}\text{¥}_{\mathbb{m}-2}\text{Ƶ}_{\mathbb{m}-2})^{\left[\frac{\mathbb{m}}{2}\right]+1}$$
………(4.7)

$$(\text{Ӿ}_{\mathbb{m}+3}\text{¥}_{\mathbb{m}+3}\text{Ƶ}_{\mathbb{m}+3})^{\left[\frac{\mathbb{m}}{2}\right]+2}$$
$$= (\text{Ӿ}_{\mathbb{m}+2}\text{¥}_{\mathbb{m}+2}\text{Ƶ}_{\mathbb{m}+2})^{\left[\frac{\mathbb{m}}{2}\right]+2}(\text{Ӿ}_{\mathbb{m}+1}\text{¥}_{\mathbb{m}+1}\text{Ƶ}_{\mathbb{m}+1})^{\left[\frac{\mathbb{m}}{2}\right]+2}(\text{Ӿ}_{\mathbb{m}}\text{¥}_{\mathbb{m}}\text{Ƶ}_{\mathbb{m}})^{\left[\frac{\mathbb{m}}{2}\right]+2}(\text{Ӿ}_{\mathbb{m}-1}\text{¥}_{\mathbb{m}-1}\text{Ƶ}_{\mathbb{m}-1})^{\left[\frac{\mathbb{m}}{2}\right]+2}$$
………(4.8)

$$(\text{Ӿ}_{\mathbb{m}+4}\text{¥}_{\mathbb{m}+4}\text{Ƶ}_{\mathbb{m}+4})^{\left[\frac{\mathbb{m}}{2}\right]+3}$$
$$= (\text{Ӿ}_{\mathbb{m}+3}\text{¥}_{\mathbb{m}+3}\text{Ƶ}_{\mathbb{m}+3})^{\left[\frac{\mathbb{m}}{2}\right]+3}(\text{Ӿ}_{\mathbb{m}+2}\text{¥}_{\mathbb{m}+2}\text{Ƶ}_{\mathbb{m}+2})^{\left[\frac{\mathbb{m}}{2}\right]+3}(\text{Ӿ}_{\mathbb{m}+1}\text{¥}_{\mathbb{m}+1}\text{Ƶ}_{\mathbb{m}+1})^{\left[\frac{\mathbb{m}}{2}\right]+3}(\text{Ӿ}_{\mathbb{m}}\text{¥}_{\mathbb{m}}\text{Ƶ}_{\mathbb{m}})^{\left[\frac{\mathbb{m}}{2}\right]+3}$$
………(4.9)

$$(\text{Ӿ}_{\mathbb{m}+5}\text{¥}_{\mathbb{m}+5}\text{Ƶ}_{\mathbb{m}+5})^{\left[\frac{\mathbb{m}}{2}\right]+4} =$$
$$(\text{Ӿ}_{\mathbb{m}+4}\text{¥}_{\mathbb{m}+4}\text{Ƶ}_{\mathbb{m}+4})^{\left[\frac{\mathbb{m}}{2}\right]+4}(\text{Ӿ}_{\mathbb{m}+3}\text{¥}_{\mathbb{m}+3}\text{Ƶ}_{\mathbb{m}+3})^{\left[\frac{\mathbb{m}}{2}\right]+4}(\text{Ӿ}_{\mathbb{m}+2}\text{¥}_{\mathbb{m}+2}\text{Ƶ}_{\mathbb{m}+2})^{\left[\frac{\mathbb{m}}{2}\right]+4}(\text{Ӿ}_{\mathbb{m}+1}\text{¥}_{\mathbb{m}+1}\text{Ƶ}_{\mathbb{m}+1})^{\left[\frac{\mathbb{m}}{2}\right]+4}$$
………(4.10)

now putting the value of equation (4.7), (4.8), (4.9) and (4.10) in equation (4.6), we get

$$(\text{Ӿ}_{\mathbb{m}+2}\text{¥}_{\mathbb{m}+2}\text{Ƶ}_{\mathbb{m}+2})^{\left[\frac{\mathbb{m}}{2}\right]+1}(\text{Ӿ}_{\mathbb{m}+3}\text{¥}_{\mathbb{m}+3}\text{Ƶ}_{\mathbb{m}+3})^{\left[\frac{\mathbb{m}}{2}\right]+2}(\text{Ӿ}_{\mathbb{m}+4}\text{¥}_{\mathbb{m}+4}\text{Ƶ}_{\mathbb{m}+4})^{\left[\frac{\mathbb{m}}{2}\right]+3}(\text{Ӿ}_{\mathbb{m}+5}\text{¥}_{\mathbb{m}+5}\text{Ƶ}_{\mathbb{m}+5})^{\left[\frac{\mathbb{m}}{2}\right]+4}$$
$$= (\text{Ӿ}_{\mathbb{m}+1}\text{¥}_{\mathbb{m}+1}\text{Ƶ}_{\mathbb{m}+1})^{\left[\frac{\mathbb{m}}{2}\right]+1}(\text{Ӿ}_{\mathbb{m}}\text{¥}_{\mathbb{m}}\text{Ƶ}_{\mathbb{m}})^{\left[\frac{\mathbb{m}}{2}\right]+1}(\text{Ӿ}_{\mathbb{m}-1}\text{¥}_{\mathbb{m}-1}\text{Ƶ}_{\mathbb{m}-1})^{\left[\frac{\mathbb{m}}{2}\right]+1}(\text{Ӿ}_{\mathbb{m}-2}\text{¥}_{\mathbb{m}-2}\text{Ƶ}_{\mathbb{m}-2})^{\left[\frac{\mathbb{m}}{2}\right]+1}$$
$$(\text{Ӿ}_{\mathbb{m}+2}\text{¥}_{\mathbb{m}+2}\text{Ƶ}_{\mathbb{m}+2})^{\left[\frac{\mathbb{m}}{2}\right]+2}(\text{Ӿ}_{\mathbb{m}+1}\text{¥}_{\mathbb{m}+1}\text{Ƶ}_{\mathbb{m}+1})^{\left[\frac{\mathbb{m}}{2}\right]+2}(\text{Ӿ}_{\mathbb{m}}\text{¥}_{\mathbb{m}}\text{Ƶ}_{\mathbb{m}})^{\left[\frac{\mathbb{m}}{2}\right]+2}(\text{Ӿ}_{\mathbb{m}-1}\text{¥}_{\mathbb{m}-1}\text{Ƶ}_{\mathbb{m}-1})^{\left[\frac{\mathbb{m}}{2}\right]+2}$$
$$(\text{Ӿ}_{\mathbb{m}+3}\text{¥}_{\mathbb{m}+3}\text{Ƶ}_{\mathbb{m}+3})^{\left[\frac{\mathbb{m}}{2}\right]+3}(\text{Ӿ}_{\mathbb{m}+2}\text{¥}_{\mathbb{m}+2}\text{Ƶ}_{\mathbb{m}+2})^{\left[\frac{\mathbb{m}}{2}\right]+3}(\text{Ӿ}_{\mathbb{m}+1}\text{¥}_{\mathbb{m}+1}\text{Ƶ}_{\mathbb{m}+1})^{\left[\frac{\mathbb{m}}{2}\right]+3}(\text{Ӿ}_{\mathbb{m}}\text{¥}_{\mathbb{m}}\text{Ƶ}_{\mathbb{m}})^{\left[\frac{\mathbb{m}}{2}\right]+3}$$
$$(\text{Ӿ}_{\mathbb{m}+4}\text{¥}_{\mathbb{m}+4}\text{Ƶ}_{\mathbb{m}+4})^{\left[\frac{\mathbb{m}}{2}\right]+4}(\text{Ӿ}_{\mathbb{m}+3}\text{¥}_{\mathbb{m}+3}\text{Ƶ}_{\mathbb{m}+3})^{\left[\frac{\mathbb{m}}{2}\right]+4}(\text{Ӿ}_{\mathbb{m}+2}\text{¥}_{\mathbb{m}+2}\text{Ƶ}_{\mathbb{m}+2})^{\left[\frac{\mathbb{m}}{2}\right]+4}(\text{Ӿ}_{\mathbb{m}+1}\text{¥}_{\mathbb{m}+1}\text{Ƶ}_{\mathbb{m}+1})^{\left[\frac{\mathbb{m}}{2}\right]+4}$$
$$= (\text{Ӿ}_{\mathbb{m}+1}\text{¥}_{\mathbb{m}+1}\text{Ƶ}_{\mathbb{m}+1})^{\left[\frac{\mathbb{m}}{2}\right]+1}(\text{Ӿ}_{\mathbb{m}+2}\text{¥}_{\mathbb{m}+2}\text{Ƶ}_{\mathbb{m}+2})^{\left[\frac{\mathbb{m}}{2}\right]+2}(\text{Ӿ}_{\mathbb{m}+3}\text{¥}_{\mathbb{m}+3}\text{Ƶ}_{\mathbb{m}+3})^{\left[\frac{\mathbb{m}}{2}\right]+3}(\text{Ӿ}_{\mathbb{m}+4}\text{¥}_{\mathbb{m}+4}\text{Ƶ}_{\mathbb{m}+4})^{\left[\frac{\mathbb{m}}{2}\right]+4}$$
$$(\text{Ӿ}_{\mathbb{m}}\text{¥}_{\mathbb{m}}\text{Ƶ}_{\mathbb{m}})^{\left[\frac{\mathbb{m}}{2}\right]+1}(\text{Ӿ}_{\mathbb{m}+1}\text{¥}_{\mathbb{m}+1}\text{Ƶ}_{\mathbb{m}+1})^{\left[\frac{\mathbb{m}}{2}\right]+2}(\text{Ӿ}_{\mathbb{m}+2}\text{¥}_{\mathbb{m}+2}\text{Ƶ}_{\mathbb{m}+2})^{\left[\frac{\mathbb{m}}{2}\right]+3}(\text{Ӿ}_{\mathbb{m}+3}\text{¥}_{\mathbb{m}+3}\text{Ƶ}_{\mathbb{m}+3})^{\left[\frac{\mathbb{m}}{2}\right]+4}$$
$$(\text{Ӿ}_{\mathbb{m}-1}\text{¥}_{\mathbb{m}-1}\text{Ƶ}_{\mathbb{m}-1})^{\left[\frac{\mathbb{m}}{2}\right]+1}(\text{Ӿ}_{\mathbb{m}}\text{¥}_{\mathbb{m}}\text{Ƶ}_{\mathbb{m}})^{\left[\frac{\mathbb{m}}{2}\right]+2}(\text{Ӿ}_{\mathbb{m}+1}\text{¥}_{\mathbb{m}+1}\text{Ƶ}_{\mathbb{m}+1})^{\left[\frac{\mathbb{m}}{2}\right]+3}(\text{Ӿ}_{\mathbb{m}+2}\text{¥}_{\mathbb{m}+2}\text{Ƶ}_{\mathbb{m}+2})^{\left[\frac{\mathbb{m}}{2}\right]+4}$$
$$(\text{Ӿ}_{\mathbb{m}-2}\text{¥}_{\mathbb{m}-2}\text{Ƶ}_{\mathbb{m}-2})^{\left[\frac{\mathbb{m}}{2}\right]+1}(\text{Ӿ}_{\mathbb{m}-1}\text{¥}_{\mathbb{m}-1}\text{Ƶ}_{\mathbb{m}-1})^{\left[\frac{\mathbb{m}}{2}\right]+2}(\text{Ӿ}_{\mathbb{m}}\text{¥}_{\mathbb{m}}\text{Ƶ}_{\mathbb{m}})^{\left[\frac{\mathbb{m}}{2}\right]+3}(\text{Ӿ}_{\mathbb{m}+1}\text{¥}_{\mathbb{m}+1}\text{Ƶ}_{\mathbb{m}+1})^{\left[\frac{\mathbb{m}}{2}\right]+4}$$

Now we use given hypothesis for every line,

$$= (\text{Ӿ}_{\mathbb{m}+2}\text{¥}_{\mathbb{m}+2}\text{Ƶ}_{\mathbb{m}+2})(\text{Ӿ}_{\mathbb{m}+3}\text{¥}_{\mathbb{m}+3}\text{Ƶ}_{\mathbb{m}+3})^{\mathbb{m}-2}(\text{Ӿ}_{\mathbb{m}+4}\text{¥}_{\mathbb{m}+4}\text{Ƶ}_{\mathbb{m}+4})$$
$$(\text{Ӿ}_{\mathbb{m}+1}\text{¥}_{\mathbb{m}+1}\text{Ƶ}_{\mathbb{m}+1})(\text{Ӿ}_{\mathbb{m}+2}\text{¥}_{\mathbb{m}+2}\text{Ƶ}_{\mathbb{m}+2})^{\mathbb{m}-2}(\text{Ӿ}_{\mathbb{m}+3}\text{¥}_{\mathbb{m}+3}\text{Ƶ}_{\mathbb{m}+3})$$
$$(\text{Ӿ}_{\mathbb{m}}\text{¥}_{\mathbb{m}}\text{Ƶ}_{\mathbb{m}})(\text{Ӿ}_{\mathbb{m}+1}\text{¥}_{\mathbb{m}+1}\text{Ƶ}_{\mathbb{m}+1})^{\mathbb{m}-2}(\text{Ӿ}_{\mathbb{m}+2}\text{¥}_{\mathbb{m}+2}\text{Ƶ}_{\mathbb{m}+2})$$
$$(\text{Ӿ}_{\mathbb{m}-1}\text{¥}_{\mathbb{m}-1}\text{Ƶ}_{\mathbb{m}-1})(\text{Ӿ}_{\mathbb{m}}\text{¥}_{\mathbb{m}}\text{Ƶ}_{\mathbb{m}})^{\mathbb{m}-2}(\text{Ӿ}_{\mathbb{m}+1}\text{¥}_{\mathbb{m}+1}\text{Ƶ}_{\mathbb{m}+1})$$
$$= (\text{Ӿ}_{\mathbb{m}+3}\text{¥}_{\mathbb{m}+3}\text{Ƶ}_{\mathbb{m}+3})(\text{Ӿ}_{\mathbb{m}+4}\text{¥}_{\mathbb{m}+4}\text{Ƶ}_{\mathbb{m}+4})^{\mathbb{m}-2}(\text{Ӿ}_{\mathbb{m}+5}\text{¥}_{\mathbb{m}+5}\text{Ƶ}_{\mathbb{m}+5})$$

The conclusion is valid for all integers $\mathbb{m}+2$. As a result, using the induction method, the conclusion holds for any positive odd integer $\mathbb{m} \geq 1$.

Theorem 4.4: For every integer $\mathbb{m} \geq 0$,

(a) $\sqrt{\prod_{k=0}^{10\mathbb{m}+4} X_k} = X_4 X_9 X_{14} \ldots\ldots X_{10\mathbb{m}+4}$

(b) $\sqrt{\prod_{k=0}^{10\mathbb{m}+4} Y_k} = Y_4 Y_9 Y_{14} \ldots\ldots Y_{10\mathbb{m}+4}$

(c) $\sqrt{\prod_{k=0}^{10\mathbb{m}+4} Z_k} = Z_4 Z_9 Z_{14} \ldots\ldots Z_{10\mathbb{m}+4}$

Proof: These results are confirmed by the induction hypothesis.

For $\mathbb{m} = 1$ then

$$\sqrt{\prod_{k=0}^{14} X_k} = \sqrt{X_0 X_1 X_2 X_3 X_4 X_5 X_6 X_7 X_8 X_9 X_{10} X_{11} X_{12} X_{13} X_{14}}$$

$$= \sqrt{X_4^2 X_9^2 X_{14}^2}$$

$$= X_4 X_9 X_{14}$$

For each odd number $\mathbb{m} = 1$, the conclusion is correct.

We'll proceed by assuming that the outcome is accurate for some odd integer $\mathbb{m}$.

Then for $\mathbb{m}+1$

$$\sqrt{\prod_{k=0}^{10\mathbb{m}+14} X_k}$$

$$= \sqrt{X_{10n+5} X_{10n+6} X_{10n+7} X_{10n+8} X_{10n+9} X_{10n+10} X_{10n+11} X_{10n+12} X_{10n+13} X_{10n+14} \prod_{k=0}^{10n+4} X_k}$$

$$= \sqrt{X_{10\mathbb{m}+9}^2 X_{10\mathbb{m}+14}^2} X_4 X_9 X_{14} \ldots\ldots X_{10\mathbb{m}+4}$$

$$= X_4 X_9 X_{14} \ldots\ldots X_{10\mathbb{m}+14}$$

The conclusion is valid for all integers $\mathbb{m}+1$. As a result, using the induction method, the conclusion holds for any positive odd integer $\mathbb{m} \geq 1$.

Theorem 4.5: For every integer $\mathbb{m} \geq 1$,

(a) $\sqrt{\prod_{k=0}^{10\mathbb{m}-1} X_k} = X_9 X_{19} \ldots\ldots X_{10\mathbb{m}-1}$

(b) $\sqrt{\prod_{k=0}^{10m-1} ¥_k} = ¥_9 ¥_{19} \ldots\ldots ¥_{10m-1}$

(c) $\sqrt{\prod_{k=0}^{10m-1} Ƶ_k} = Ƶ_9 Ƶ_{19} \ldots\ldots Ƶ_{10m-1}$

Proof: Similar to the theorem above, this one can be proved through mathematical induction.

Also, we can use Scheme no. 2nd, 3rd, 4th, 5th, 6th, 7th, 8th and 9th to prove theorem no. 4.1 to 4.5

4.3.1 Significance of the Derived Identities

The identities established in this chapter for fourth-order MTFS hold significant theoretical and practical value. At the core, these identities reveal how complex multiplicative relationships evolve across three interlinked sequences under higher-order recurrence. Their derivation through mathematical induction not only validates the internal structure of the sequence but also ensures logical consistency across various recurrence schemes.

These results contribute meaningfully to the general theory of coupled recursive sequences. By identifying fixed patterns, multiplicative symmetries, and functional dependencies, the identities provide deeper insight into how initial conditions and scheme selection influence long-term behavior. This understanding becomes essential when considering the sequences' use in algorithmic modeling or theoretical studies of growth dynamics.

From a practical standpoint, these identities can be applied in computational contexts where recursive, nonlinear processes are used such as in cryptographic key design, pseudorandom number generators, or simulations involving multiple interacting systems. Furthermore, they help in classifying different fourth-order schemes based on algebraic behavior, opening the door to future generalizations or modular extensions in higher-order MTFS research.

4.3.2 Possible Applications of Fourth-Order MTFS

The fourth-order MTFS exhibits notable behaviors and identity patterns across various recurrence schemes, indicating several areas where these sequences may find meaningful applications. Their rapid growth and high sensitivity to initial conditions make them particularly suitable for roles in secure data transmission, cryptographic key generation, and pseudorandom number generation. Furthermore, the structured yet adaptable nature of these recursions allows them to be employed in modeling complex systems—such as multiphase population dynamics, recursive financial systems, or computational algorithms that rely on layered feedback mechanisms. The flexibility offered by the multiple scheme variations also enables customization of sequence behavior to meet specific mathematical or practical requirements in algorithmic and security contexts.

4.4 Conclusion:

The study of the fourth-order Multiplicative Triple Fibonacci Sequence (MTFS) across nine distinct Schemes has brought to light intricate mathematical behaviors and valuable application prospects. By embedding multiplicative factors into classical Triple Fibonacci Sequences, the research introduced enhanced complexity and dynamic variations. Careful analysis of recurrence formulas and initial terms revealed unique sequence patterns, showing the strong influence of both initial conditions and the specific Scheme applied. This work significantly contributes to a deeper understanding of the structural properties of advanced Fibonacci sequences and emphasizes their diverse mathematical potential.

"Just as relationships in life grow more complex with each new connection, the fourth-order multiplicative triple Fibonacci sequence shows how adding layers of interaction transforms simple beginnings into intricate and unpredictable outcomes."

Chapter-5

Coupled Lucas Sequence of Second order and Fibonacci Lucas Sequence's Determinantal Identities

The work presented in this chapter has been partially published in the form of a research paper entitled "Application of Coupled Lucas Sequence of Second Order" in a Scopus Indexed Journal (Q_3), and partially presented orally in an International Conference related to "Fibonacci–Lucas Sequence's Determinantal Identities."

5.1 Introduction

The Fibonacci numbers and polynomials are important concepts that are utilized in a variety of mathematical disciplines, such as algebra, combinatorics, approximation theory, geometry, graph theory, and number theory itself. The rabbit-reed problem is possibly the most well-known application of the Fibonacci numbers. It was initially published by Leonardo de Pisa in his book "Liber Abaci" in the year 1202, and it is believed to be named after him. A large number of authors have written on their varied and lovely properties as well as their many and varied uses. As seen in the graphic on page [4] of the book by Koshy, the mathematical sequences associated with the Fibonacci numbers and the Lucas numbers are among the most intriguing ever discovered. Numerous identifiers have been catalogued in the form of a comprehensive list that may be found in Vajda's work [75].

In order to study Fibonacci numbers, a long form of unity matrices and determinants is used. Cahill and Narayan [56] looked into the Fibonacci and Lucas numbers' historical background as determinants of several tridiagonal matrices. Atanassov and Suman, Amitava, and K. Sisodiya[76], respectively, present the interconnected Jacobsthal Sequence y and the correlated Second order recurrence relation by creating two sequences, $\{\text{Ӿ}_i\}_{i=0}^{\infty}$ and $\{\text{¥}_i\}_{i=0}^{\infty}$, which they refer to as 2F Sequences. The way they accomplish this is by creating two interconnected sequences.

T. Koshy is the author of a book that consists of two chapters and focuses on the application of matrices and determinants in relation to the Fibonacci numbers. The creation of classes of identities for Generalized Fibonacci numbers was accomplished by Bicknell-Johnson and Spears [62] by the application of fundamental matrix operations and determinants. One can find a variety of helpful and amazing techniques for determining the future in the excellent survey articles. A significant amount of focus has been placed on the interpretation of matrices, in particular when their entries are presented in a recursive fashion. The sequence of numbers known as the Fibonacci numbers is made up of integers 0 and 1, with each succeeding term in the series being calculated as the sum of the two terms before it.

i.e. $F_{\mathbb{m}} = F_{\mathbb{m}-1} + F_{\mathbb{m}-2},\ \mathbb{m} \geq 2$ and $F_0 = 0, F_1 = 1$

The Lucas sequence is also thought to have a similar perception. The recurrence relation confirms that the Lucas sequence [61] is genuine.

$L_{\mathbb{m}} = L_{\mathbb{m}-1} + L_{\mathbb{m}-2},\ \mathbb{m} \geq 2$ and $L_0 = 2, L_1 = 1$

We use recurrence to illustrate the Generalized Fibonacci sequence $\{Ƀ_{\mathbb{m}}\}_{\mathbb{m}=0}^{\infty}$ in this area:

$Ƀ_{\mathbb{m}} = Ƀ_{\mathbb{m}-1} + Ƀ_{\mathbb{m}-2},\ \mathbb{m} \geq 2$ and $Ƀ_0 = 2b, Ƀ_1 = s$ Where *b* and *s* must both be non-negative integers.

Table 5.1: Some terms of CLS

$\mathbb{m}$	$Ƀ_{\mathbb{m}}$
0	$2b$
1	s
2	$2b + s$
3	$2b + 2s$
4	$4b + 3s$
5	$6b + 5s$
6	$10b + 8s$
7	$16b + 13s$
8	$26b + 21s$
9	$42b + 34s$

$t^2 - t - 1 = 0$ is the defining equation of the recurrence relation. which actually has two roots.

$$Ӿ = \frac{1+\sqrt{5}}{2}, ¥ = \frac{1-\sqrt{5}}{2}$$

Now, $Ӿ¥ = -1, Ӿ + ¥ = 1, Ӿ - ¥ = \sqrt{5}, Ӿ^2 + ¥^2 = 3$.

According to the Scheme

$Ӿ_{\mathbb{m}+2} = ¥_{\mathbb{m}+1} + ¥_{\mathbb{m}},\ \ \mathbb{m} \geq 0$

$¥_{\mathbb{m}+2} = Ӿ_{\mathbb{m}+1} + Ӿ_{\mathbb{m}},\ \ \mathbb{m} \geq 0$

Taking $Ӿ_0 = \mathrm{a}, ¥_0 = \mathrm{b}, Ӿ_1 = \mathrm{c}, ¥_1 = \mathrm{d}$ where $\mathrm{a}, \mathrm{b}, \mathrm{c}$ and d are integers.

Hirschhorn provides clear answers to the long-standing issues with Atanassov's Second and Third order recurrence relations. Recently, coupled recurrence relations of order five were found by Singh, Sikhwal, and Jain. Additionally, Carlitz et al. [65] provided a representation for a unique sequence. The "Coupled Lucas Sequence of Second Order" emerges as a captivating exploration within the domain of number theory, building upon the foundations laid by the classical LS. This innovative extension introduces a dynamic interplay between two distinct Second-order LS, weaving a tapestry of numerical relationships that transcend the conventional boundaries of sequence theory. As a testament to the continuous evolution of mathematical inquiry, this study delves into the intricacies of the coupled sequences, unraveling a myriad of patterns, properties and applications. At its core, the Second-order LS, defined as an integer sequence generated by a recurrence relation, forms the basis for the coupled exploration. By introducing coupling mechanisms between two such sequences, a new and intriguing mathematical entity emerges. This coupled relationship manifests as a simultaneous evolution of two interconnected sequences, influencing each other's progression in a harmonious dance of numerical dynamics. Cahill and Narayan analyzed the origins of the Fibonacci and Lucas numbers as determinants of certain tridiagonal matrices. By creating the sequences $\{Ӿ_i\}_{i=0}^{\infty}$ and $\{¥_i\}_{i=0}^{\infty}$, Atanassov and Suman, Amitava, K. Sisodiya [76] introduce the interrelated Second order recurrence relation and interlinked Jacobsthal Sequence, respectively, referring to them as 2F Sequences.

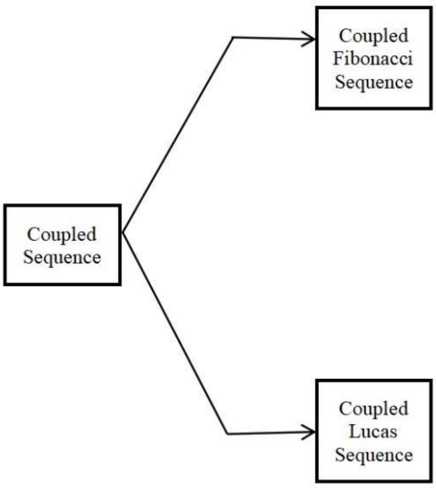


Figure 5.1: Structure of Coupled Sequence

Fig. 5.1 illustrates the hierarchical relationship between sequences. At the top level is the "Coupled Sequence", which branches into two distinct types: Coupled Fibonacci Sequence. One branch leads to the "Coupled Fibonacci Sequence" suggesting it is a variant or extension of the traditional Fibonacci sequence, possibly modified by a coupling rule or relationship. Coupled Lucas Sequence has the branch leads to the "Coupled Lucas Sequence" indicating a similar variant or extension of the Lucas sequence, also with some form of coupling rule. This structure shows that the "Coupled Sequence" serves as a foundational concept that can lead to either a coupled version of the Fibonacci or Lucas sequences, depending on the branching path.

5.1.1 Motivation for Studying CLS and Their Identities

The Second-Order CLS builds upon the classical Lucas sequence by introducing a system of two interdependent sequences that evolve simultaneously. This coupling results in more intricate algebraic relationships and reveals properties that are absent in the standard Lucas or Fibonacci sequences. Through the study of such coupled structures, one can uncover meaningful identities and determinant-based expressions that shed light on the underlying interconnections within recursive systems. Additionally, due to its responsiveness to initial conditions, CLS has potential applications in areas such as mathematical modeling, cryptographic design, and algorithmic computation. This chapter aims to establish a collection of fundamental identities that reflect both the theoretical richness and the applied relevance of these sequences.

5.2 Coupled Lucas Sequence Of Second Order

The sequences $\{L_i\}_{i=0}^{\infty}$ and $\{M_i\}_{i=0}^{\infty}$ will coincide and the sequence $\{L_i\}_{i=0}^{\infty}$ will turn into a generalized Lucas sequence if

we set $a = b$ and $c = d$.

Where,

$L_0(a,c) = a$, $L_1(a,c) = c$

$L_{m+2}(a,c) = M_{m+1}(a,c) + 2M_m(a,c)$,

$L_m = a, b, d + 2c, b + 2a + 2d$

$M_m = c, d, b + 2a, d + 2c + 2b$

Following are the first few terms.

Table 5.2 First few terms of Second order coupled Lucas sequence

m	L_m	M_m
0	a	c
1	b	d
2	$d + 2c$	$b + 2a$
3	$b + 2a + 2d$	$d + 2c + 2b$

4	$d + 2c + 4b + 4a$	$b + 2a + 4d + 4c$
5	$6d + 8c + 5b + 2a$	$5d + 2c + 6b + 8a$

Taking Lucas sequence

$\mathcal{L}_{m+2} = \mathcal{L}_{m+1} + 2\mathcal{L}_{m},\ m \geq 0$

$\mathcal{M}_{m+2} = \mathcal{M}_{m+1} + 2\mathcal{M}_{m},\ m \geq 0$

We defined 2-L Sequences as coupled order recurrence relations for Lucas numbers and Lucas sequences.

$\mathcal{L}_{m+2} = \mathcal{M}_{m+1} + 2\mathcal{M}_{m},\ m \geq 0$

$\mathcal{M}_{m+2} = \mathcal{L}_{m+1} + 2\mathcal{L}_{m},\ m \geq 0$

$\mathcal{L}_0 = a, \mathcal{L}_1 = b, \mathcal{M}_0 = c, \mathcal{M}_1 = d$

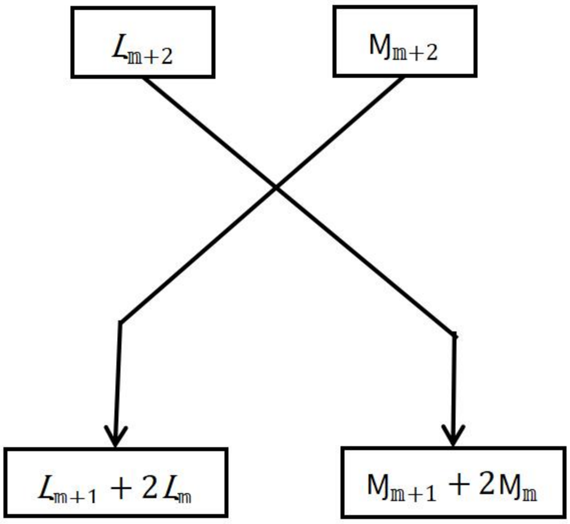


Figure 5.2: Structure of Scheme of CLS

Fig 5.2 illustrates the hierarchical structure of the Scheme of CLS under addition. 2^{nd} order CLS represents the basic CLS with one Scheme, where the terms are derided by adding last term and twice the Second to last term of the sequence.

5.2.1 Significance of the Derived Identities of CLS

The identities established in this chapter provide a rigorous framework for understanding the algebraic dynamics inherent in the Second-Order CLS. By articulating precise relationships between the terms of two mutually dependent sequences, these results illuminate the structural behavior that arises from their coupling. Such identities not only support analytical predictions and simplifications but also serve to distinguish the CLS from other classes of recursive systems.

On a theoretical level, the identities deepen insight into coupled, nonlinear recurrence relations and clarify how dual-sequence interactions can give rise to more complex behavior than that seen in single-sequence models. From an applied perspective, these identities offer potential utility in areas where controlled complexity and deterministic evolution are essential—such as in algorithm design, secure data transmission, and recursive computations. Notably, the determinantal identities bridge the study of recursive sequences with matrix theory, offering a multidimensional viewpoint on the algebraic structure of such systems. Collectively, these contributions reinforce the mathematical significance of CLS and highlight its potential for further exploration in both pure and applied contexts.

5.3 Main Identities

We can derive the following properties from the above terms:

Theorem 5.1: For every odd number $\mathbb{m} \geq 3$.

$$\frac{\mathcal{L}_{\mathbb{m}} - \mathcal{L}_1}{2} = (\mathcal{L}_0 + \mathrm{M}_1 + \mathcal{L}_2 + \mathrm{M}_3 + \ldots\ldots\ldots\ldots \mathcal{L}_{\mathbb{m}-3} + \mathrm{M}_{\mathbb{m}-2})$$

Proof: We will use a mathematical induction method to demonstrate this conclusion.

For $\mathbb{m} = 3$,

$$\frac{\mathcal{L}_3 - \mathcal{L}_1}{2} = \frac{\mathrm{M}_2 + 2\mathrm{M}_1 - \mathcal{L}_1}{2}$$

$$= \frac{\mathcal{L}_1 + 2\mathcal{L}_0 + 2\mathrm{M}_1 - \mathcal{L}_1}{2}$$

$$= \frac{2\mathcal{L}_0 + 2\mathrm{M}_1}{2}$$

$$= \mathcal{L}_0 + \mathrm{M}_1$$

The result is accurate for $\mathbb{m} = 2$, therefore we suppose the same for $\mathbb{m}$. We will now demonstrate that for $\mathbb{m} + 2$.

$$\frac{\mathcal{L}_{\mathbb{m}+2} - \mathcal{L}_1}{2} = \frac{\mathrm{M}_{\mathbb{m}+1} + 2\mathrm{M}_{\mathbb{m}} - \mathcal{L}_1}{2}$$

$$= \frac{\mathcal{L}_{\mathbb{m}} + 2\mathcal{L}_{\mathbb{m}-1} + 2\mathrm{M}_{\mathbb{m}} - \mathcal{L}_1}{2}$$

$$= \frac{\mathcal{L}_{\mathbb{m}} - \mathcal{L}_1}{2} + \frac{2\mathcal{L}_{\mathbb{m}-1} + 2\mathrm{M}_{\mathbb{m}}}{2}$$

$= (L_0 + M_1 + L_2 + M_3 + \dots\dots\dots\dots L_{m-3} + M_{m-2}) + (L_{m-1} + M_m)$

$$= (L_0 + M_1 + L_2 + M_3 + \dots\dots\dots\dots L_{m-1} + M_m)$$

Hence the result is true for $m + 2$.

Example based on Theorem 5.1

Let $\{L_m\}_{m=0}^{\infty}$and $\{M_m\}_{m=0}^{\infty}$ be two infinite sequences.

$L_{m+2} = M_{m+1} + 2M_m,\ \ m \geq 0$

$M_{m+2} = L_{m+1} + 2L_m,\ \ m \geq 0$. Such that

Table 5.3: Initial terms of the coupled Lucas sequence of Second order

m	M_m	L_m
0	2	1
1	4	3
2	5	8
3	14	13
4	29	24
5	50	57
6	105	108
7	222	205
8	421	432
9	842	865
10	1729	1684

Now we will apply the theorem on this example

$$\frac{L_m - L_1}{2} = (L_0 + M_1 + L_2 + M_3 + \dots\dots\dots\dots L_{m-3} + M_{m-2})$$

For $m = 3$ in L.H.S.

$$\Rightarrow \qquad \frac{L_3 - L_1}{2} = \frac{13 - 3}{2}$$

$$\Rightarrow \qquad = 5$$

Now $m = 3$in R.H.S

$L_0 + M_1 + L_2 + M_3 + \dots\dots\dots\dots L_{m-3} + M_{m-2} = L_0 + M_1$

$$= 1 + 4$$

$$= 5\text{=L.H.S}$$

For $\mathbb{m} = 5$ in L.H.S

$$\Rightarrow \qquad \frac{L_5 - L_1}{2} = \frac{57 - 3}{2}$$

$$\Rightarrow \qquad = 27$$

Now $\mathbb{m} = 5$in R.H.S.

$$L_0 + M_1 + L_2 + M_3 + \ldots\ldots\ldots\ldots L_{\mathbb{m}-3} + M_{\mathbb{m}-2} = L_0 + M_1 + L_2 + M_3$$

$$= 1 + 4 + 8 + 14$$

$$= 27 = \text{L.H.S}$$

Hence the conclusion is valid.

For $\mathbb{m} = 7$ in L.H.S

$$\Rightarrow \qquad \frac{L_7 - L_1}{2} = \frac{205 - 3}{2}$$

$$\Rightarrow \qquad = 101$$

Now $\mathbb{m} = 7$in R.H.S.

$$L_0 + M_1 + L_2 + M_3 + \ldots\ldots\ldots\ldots L_{\mathbb{m}-3} + M_{\mathbb{m}-2} = L_0 + M_1 + L_2 + M_3 + L_4 + M_5$$

$$= 1 + 4 + 8 + 14 + 24 + 50$$

$$= 101 = \text{L.H.S}$$

Hence, the conclusion is valid for every odd number $\mathbb{m} \geq 3$

Theorem 5.2: For every even number $\mathbb{m} \geq 2$.

$$\frac{L_{\mathbb{m}} - M_1}{2} = (M_0 + L_1 + M_2 + L_3 + \ldots\ldots\ldots\ldots L_{\mathbb{m}-3} + M_{\mathbb{m}-2})$$

Proof: We will use a mathematical induction method to demonstrate this conclusion.

For $\mathbb{m} = 2$,

$$\frac{L_2 - M_1}{2} = \frac{M_1 + 2M_0 - M_1}{2}$$

$$= M_0$$

The result is accurate for $\mathbb{m} = 2$, therefore we suppose the same for n .We will now demonstrate that for $\mathbb{m} + 2$,

$$\frac{L_{\mathbb{m}+2} - M_1}{2} = \frac{M_{\mathbb{m}+1} + 2M_{\mathbb{m}} - M_1}{2}$$

$$= \frac{L_{\mathbb{m}} + 2L_{\mathbb{m}-1} + 2M_{\mathbb{m}} - M_1}{2}$$

$$= \frac{L_{\mathbb{m}} - M_1}{2} + \frac{2L_{\mathbb{m}-1} + 2M_{\mathbb{m}}}{2}$$

$$= (M_0 + L_1 + M_2 + L_3 + \ldots\ldots\ldots\ldots + L_{m-3} + M_{m-2}) + (L_{m-1} + M_m)$$

$$= (M_0 + L_1 + M_2 + L_3 + \ldots\ldots\ldots\ldots L_{m-1} + M_m)$$

Thus, the outcome is accurate for $m + 2$.

Example based on Theorem 5.2

Let $\{L_m\}_{m=0}^{\infty}$ and $\{M_m\}_{m=0}^{\infty}$ be two infinite sequences.

$L_{m+2} = M_{m+1} + 2M_m, \ m \geq 0$

$M_{m+2} = L_{m+1} + 2L_m, \ m \geq 0$. Such that

Table 5.4 Second-order Coupled Lucas sequence's initial few terms

m	M_m	L_m
0	1	2
1	3	4
2	8	5
3	13	14
4	24	29
5	57	50
6	108	105
7	205	222
8	432	421
9	865	842
10	1684	1729

Now we will apply the theorem on this example

$$\frac{L_m - M_1}{2} = (M_0 + L_1 + M_2 + L_3 + \ldots\ldots\ldots\ldots L_{m-3} + M_{m-2})$$

For $n = 4$ in L.H.S.

$$\Rightarrow \qquad \frac{L_4 - M_1}{2} = \frac{29 - 3}{2}$$

$$\Rightarrow \qquad = 13$$

Now $m = 4$in R.H.S

$M_0 + L_1 + M_2 + L_3 + \ldots\ldots\ldots\ldots L_{m-3} + M_{m-2} = M_0 + L_1 + M_2$

$$= 1 + 4 + 8$$

$$= 13$$

=L.H.S

For $\mathbb{m} = 6$ in L.H.S.

$\Rightarrow$ $$\frac{L_6 - M_1}{2} = \frac{105 - 3}{2}$$

$\Rightarrow$ $$= 51$$

Now $\mathbb{m} = 6$ in R.H.S

$$M_0 + L_1 + M_2 + L_3 + \ldots\ldots\ldots\ldots L_{\mathbb{m}-3} + M_{\mathbb{m}-2} = M_0 + L_1 + M_2 + L_3 + M_4$$

$$= 1 + 4 + 8 + 14 + 24$$

$$= 51$$

=L.H.S

For $\mathbb{m} = 8$ in L.H.S

$\Rightarrow$ $$\frac{L_8 - M_1}{2} = \frac{421 - 3}{2}$$

$\Rightarrow$ $$= 209$$

Now $\mathbb{m} = 8$ in R.H.S

$$M_0 + L_1 + M_2 + L_3 + \ldots\ldots\ldots\ldots L_{\mathbb{m}-3} + M_{\mathbb{m}-2} = M_0 + L_1 + M_2 + L_3 + M_4 + L_5 + M_6$$

$$= 1 + 4 + 8 + 14 + 24 + 50 + 108$$

$$= 209$$

=L.H.S

For $\mathbb{m} = 10$ in L.H.S

$\Rightarrow$ $$\frac{L_{10} - \ell_1}{2} = \frac{1729 - 3}{2}$$

$\Rightarrow$ $$= 863$$

Now $\mathbb{m} = 10$in R.H.S

$$M_0 + L_1 + M_2 + L_3 + \ldots\ldots\ldots\ldots L_{\mathbb{m}-3} + M_{\mathbb{m}-2} = M_0 + L_1 + M_2 + L_3 + M_4 + L_5 + M_6 + L_7 + M_8$$

$$= 1 + 4 + 8 + 14 + 24 + 50 + 108 + 222 + 432$$

$$= 863$$

=L.H.S

Hence the conclusion is valid.

Theorem 5.3: For every odd number $\mathbb{m} \geq 3$.

$$\frac{M_{\mathbb{m}} - M_1}{2} = (M_0 + L_1 + M_2 + L_3 + \ldots\ldots\ldots\ldots M_{\mathbb{m}-3} + L_{\mathbb{m}-2})$$

OR

$$\mathbb{M}_{\mathbb{m}} - \mathbb{M}_1 = 2(\mathbb{M}_0 + \mathcal{L}_1 + \mathbb{M}_2 + \mathcal{L}_3 + \ldots\ldots\ldots\mathbb{M}_{\mathbb{m}-3} + \mathcal{L}_{\mathbb{m}-2})$$

Example based on Theorem 5.3

Let $\{\mathcal{L}_{\mathbb{m}}\}_{\mathbb{m}=0}^{\infty}$and $\{\mathbb{M}_{\mathbb{m}}\}_{\mathbb{m}=0}^{\infty}$ be two infinite sequences.

$\mathcal{L}_{\mathbb{m}+2} = \mathbb{M}_{\mathbb{m}+1} + 2\mathbb{M}_{\mathbb{m}},\ \ \mathbb{m} \geq 0$

$\mathbb{M}_{\mathbb{m}+2} = \mathcal{L}_{\mathbb{m}+1} + 2\mathcal{L}_{\mathbb{m}},\ \ \mathbb{m} \geq 0.$

Such that

Table 5.6 Initial terms of the Second-order Coupled Lucas sequence

$\mathbb{m}$	$\mathbb{M}_{\mathbb{m}}$	$\mathcal{L}_{\mathbb{m}}$
0	1	2
1	2	3
2	7	4
3	10	11
4	19	24
5	46	39
6	87	84
7	162	179
8	347	336
9	694	671
10	1343	1388

Now we will apply the theorem on this example

$$\frac{\mathbb{M}_{\mathbb{m}} - \mathbb{M}_1}{2} = (\mathbb{M}_0 + \mathcal{L}_1 + \mathbb{M}_2 + \mathcal{L}_3 + \ldots\ldots\ldots\mathbb{M}_{\mathbb{m}-3} + \mathcal{L}_{\mathbb{m}-2})$$

For $\mathbb{m} = 5$ in R.H.S

$\Rightarrow \qquad \frac{\mathbb{M}_5 - \mathbb{M}_1}{2} = \frac{46 - 2}{2}$

$\Rightarrow \qquad 22 =$L.H.S

Now $\mathbb{m} = 5$in L.H.S

$\mathcal{M}_0 + \mathcal{L}_1 + \mathcal{M}_2 + \mathcal{L}_3 + \ldots\ldots\ldots\ldots \mathcal{M}_{\mathbb{m}-3} + \mathcal{L}_{\mathbb{m}-2} = \mathcal{M}_0 + \mathcal{L}_1 + \mathcal{M}_2 + \mathcal{L}_3$

$= 1 + 3 + 7 + 11 = 22$=R.H.S

For $\mathbb{m} = 7$ in R.H.S

$$\Rightarrow \quad \frac{\mathcal{M}_7 - \mathcal{M}_1}{2} = \frac{162 - 2}{2}$$

$$\Rightarrow \quad = 80$$

=L.H.S

Now $\mathbb{m} = 7$in L.H.S

$\mathcal{M}_0 + \mathcal{L}_1 + \mathcal{M}_2 + \mathcal{L}_3 + \ldots\ldots\ldots\ldots \mathcal{M}_{\mathbb{m}-3} + \mathcal{L}_{\mathbb{m}-2} = \mathcal{M}_0 + \mathcal{L}_1 + \mathcal{M}_2 + \mathcal{L}_3 + \mathcal{M}_4 + \mathcal{L}_5$

$= 1 + 3 + 7 + 11 + 19 + 39$

$= 80$

=R.H.S

For $\mathbb{m} = 9$ in R.H.S

$$\Rightarrow \quad \frac{\mathcal{M}_9 - \mathcal{M}_1}{2} = \frac{694 - 2}{2}$$

$$\Rightarrow \quad = 346$$

=L.H.S

Now $\mathbb{m} = 9$in L.H.S

$\mathcal{M}_0 + \mathcal{L}_1 + \mathcal{M}_2 + \mathcal{L}_3 + \ldots\ldots\ldots\ldots \mathcal{M}_{\mathbb{m}-3} + \mathcal{L}_{\mathbb{m}-2} = \mathcal{M}_0 + \mathcal{L}_1 + \mathcal{M}_2 + \mathcal{L}_3 + \mathcal{M}_4 + \mathcal{L}_5 + \mathcal{M}_6 + \mathcal{L}_7$

$= 1 + 3 + 7 + 11 + 19 + 39 + 87 + 179$

$= 346$

=R.H.S

Hence the conclusion is valid.

Theorem 5.4: For every even number $\mathbb{m} \geq 2$.

$$\frac{\mathcal{M}_{\mathbb{m}} - \mathcal{L}_1}{2} = (\mathcal{L}_0 + \mathcal{M}_1 + \mathcal{L}_2 + \mathcal{M}_3 + \ldots\ldots\ldots\ldots \mathcal{M}_{\mathbb{m}-3} + \mathcal{L}_{\mathbb{m}-2})$$

Example based on Theorem 5.4

Let $\{\mathcal{L}_{\mathbb{m}}\}_{\mathbb{m}=0}^{\infty}$and $\{\mathcal{M}_{\mathbb{m}}\}_{\mathbb{m}=0}^{\infty}$ be two infinite sequences.

$\mathcal{L}_{\mathbb{m}+2} = \mathcal{M}_{\mathbb{m}+1} + 2\mathcal{M}_{\mathbb{m}},\ \ \mathbb{m} \geq 0$

$\mathcal{M}_{\mathbb{m}+2} = \mathcal{L}_{\mathbb{m}+1} + 2\mathcal{L}_{\mathbb{m}},\ \ \mathbb{m} \geq 0$

Such that

Table 5.7 First few terms of Second order coupled Lucas sequence

$\mathbb{m}$	$\mathcal{M}_{\mathbb{m}}$	$\mathcal{L}_{\mathbb{m}}$
0	1	2
1	1	2
2	6	3
3	7	8
4	14	19
5	35	28
6	66	63
7	119	136
8	262	251
9	523	500
10	1002	1047

Now we will apply the theorem on this example

$$\frac{\mathcal{M}_{\mathbb{m}} - \mathcal{L}_1}{2} = (\mathcal{L}_0 + \mathcal{M}_1 + \mathcal{L}_2 + \mathcal{M}_3 + \ldots\ldots\ldots\ldots \mathcal{M}_{\mathbb{m}-3} + \mathcal{L}_{\mathbb{m}-2})$$

For $\mathbb{m} = 6$ in L.H.S

$$\Rightarrow \quad \frac{\mathcal{M}_8 - \mathcal{L}_1}{2} = \frac{66 - 2}{2}$$

$$\Rightarrow \quad = 32$$

$$=\text{R.H.S}$$

Now $\mathbb{m} = 6$ in R.H.S

$$\mathcal{L}_0 + \mathcal{M}_1 + \mathcal{L}_2 + \mathcal{M}_3 + \ldots\ldots\ldots\ldots \mathcal{M}_{\mathbb{m}-3} + \mathcal{L}_{\mathbb{m}-2} = \mathcal{L}_0 + \mathcal{M}_1 + \mathcal{L}_2 + \mathcal{M}_3 + \mathcal{L}_4$$

$$= 2 + 1 + 3 + 7 + 19$$

$$=32$$

$$=\text{L.H.S}$$

Hence the conclusion is valid.

Theorem 5.5: For every positive integer $\mathbb{m}$.

$$\frac{\mathcal{L}_{\mathbb{m}+2}\mathcal{L}_{\mathbb{m}+1} - \mathcal{M}_{\mathbb{m}+2}\mathcal{M}_{\mathbb{m}+1}}{\mathcal{M}_{\mathbb{m}+2}\mathcal{M}_{\mathbb{m}} - \mathcal{L}_{\mathbb{m}+2}\mathcal{L}_{\mathbb{m}}} = 2$$

Proof: We will prove this result by method of mathematical induction

For $\mathbb{m} = 1$,

$$\frac{L_3 L_2 - M_3 M_2}{M_3 M_1 - L_3 L_1} = \frac{(M_2 + 2M_1)L_2 - (L_2 + 2L_1)M_2}{(L_2 + 2L_1)M_1 - (M_2 + 2M_1)L_1}$$

$$= \frac{M_2 L_2 + 2M_1 L_2 - L_2 M_2 - 2L_1 M_2}{L_2 M_1 + 2L_1 M_1 - M_2 L_1 - 2M_1 L_1}$$

$$= \frac{2M_1 L_2 - 2L_1 M_2}{L_2 M_1 - M_2 L_1}$$

$$= 2\left[\frac{M_1 L_2 - L_1 M_2}{L_2 M_1 - M_2 L_1}\right]$$

$$= 2$$

The result is accurate for $\mathbb{m} = 1$.

Therefore we suppose the same for $\mathbb{m}$.

We will now demonstrate that for $\mathbb{m} + 1$,

$$\frac{L_{\mathbb{m}+3} L_{\mathbb{m}+2} - M_{\mathbb{m}+3} M_{\mathbb{m}+2}}{M_{\mathbb{m}+3} M_{\mathbb{m}+1} - L_{\mathbb{m}+3} L_{\mathbb{m}+1}} = \frac{(M_{\mathbb{m}+2} + 2M_{\mathbb{m}+1})L_{\mathbb{m}+2} - (L_{\mathbb{m}+2} + 2L_{\mathbb{m}+1})M_{\mathbb{m}+2}}{(L_{\mathbb{m}+2} + 2L_{\mathbb{m}+1})M_{\mathbb{m}+1} - (M_{\mathbb{m}+2} + 2M_{\mathbb{m}+1})L_{\mathbb{m}+1}}$$

$$= \frac{M_{\mathbb{m}+2} L_{\mathbb{m}+2} + 2M_{\mathbb{m}+1} L_{\mathbb{m}+2} - L_{\mathbb{m}+2} M_{\mathbb{m}+2} - 2L_{\mathbb{m}+1} M_{\mathbb{m}+2}}{L_{\mathbb{m}+2} M_{\mathbb{m}+1} + 2L_{\mathbb{m}+1} M_{\mathbb{m}+1} - M_{\mathbb{m}+2} L_{\mathbb{m}+1} - 2M_{\mathbb{m}+1} L_{\mathbb{m}+1}}$$

$$= \frac{2M_{\mathbb{m}+1} L_{\mathbb{m}+2} - 2L_{\mathbb{m}+1} M_{\mathbb{m}+2}}{L_{\mathbb{m}+2} M_{\mathbb{m}+1} - M_{\mathbb{m}+2} L_{\mathbb{m}+1}}$$

$$= 2\left[\frac{M_{\mathbb{m}+1} L_{\mathbb{m}+2} - L_{\mathbb{m}+1} M_{\mathbb{m}+2}}{L_{\mathbb{m}+2} M_{\mathbb{m}+1} - M_{\mathbb{m}+2} L_{\mathbb{m}+1}}\right]$$

$$= 2$$

Hence the result is true for $\mathbb{m} + 1$.

Example based on Theorem 5.5

Let $\{L_{\mathbb{m}}\}_{\mathbb{m}=0}^{\infty}$ and $\{M_{\mathbb{m}}\}_{\mathbb{m}=0}^{\infty}$ be two infinite sequences.

$L_{\mathbb{m}+2} = M_{\mathbb{m}+1} + 2M_{\mathbb{m}},\ \ \mathbb{m} \geq 0$

$M_{\mathbb{m}+2} = L_{\mathbb{m}+1} + 2L_{\mathbb{m}},\ \ \mathbb{m} \geq 0$

Such that

Table 5.8 introductory terms of the Second-order Coupled Lucas sequence

$\mathbb{m}$	$M_{\mathbb{m}}$	$L_{\mathbb{m}}$
0	2	1

1	2	1
2	3	6
3	8	7
4	19	14
5	28	35
6	63	66
7	136	119
8	251	262
9	500	523
10	1047	1002

Now we will apply the theorem on this example

$$\frac{\mathcal{L}_{\mathbb{m}+2}\mathcal{L}_{\mathbb{m}+1} - \text{Ɱ}_{\mathbb{m}+2}\text{Ɱ}_{\mathbb{m}+1}}{\text{Ɱ}_{\mathbb{m}+2}\text{Ɱ}_{\mathbb{m}} - \mathcal{L}_{\mathbb{m}+2}\mathcal{L}_{\mathbb{m}}} = 2$$

Put $\mathbb{m} = 1$,

$$\frac{\mathcal{L}_3\mathcal{L}_2 - \text{Ɱ}_3\text{Ɱ}_2}{\text{Ɱ}_3\text{Ɱ}_1 - \mathcal{L}_3\mathcal{L}_1} = \frac{(11 \times 4) - (10 \times 7)}{(10 \times 2) - (11 \times 3)}$$

$$= \frac{44 - 70}{20 - 33}$$

$$= 2$$

Put $\mathbb{m} = 2$,

$$\frac{\mathcal{L}_4\mathcal{L}_3 - \text{Ɱ}_4\text{Ɱ}_3}{\text{Ɱ}_4\text{Ɱ}_2 - \mathcal{L}_4\mathcal{L}_2} = \frac{(24 \times 11) - (19 \times 10)}{(19 \times 7) - (24 \times 4)}$$

$$= \frac{(264) - (190)}{(133) - (96)}$$

$$= 2$$

Put $\mathbb{m} = 3$,

$$\frac{\mathcal{L}_5\mathcal{L}_4 - \text{Ɱ}_5\text{Ɱ}_4}{\text{Ɱ}_5\text{Ɱ}_3 - \mathcal{L}_5\mathcal{L}_3} = \frac{(39 \times 24) - (46 \times 19)}{(46 \times 10) - (39 \times 11)}$$

$$= \frac{(936) - (874)}{(460) - (429)}$$

$$= 2$$

Hence the conclusion is valid for $\mathbb{m} = 1, 2, 3, \ldots\ldots\ldots\ldots$

Theorem 5.6: For every integer $\mathbb{m} \geq 0$

$$\begin{vmatrix} Ƀ_{\mathbb{m}+1} & Ƀ_{\mathbb{m}+1}^2 & Ƀ_{\mathbb{m}+1}^3 \\ Ƀ_{\mathbb{m}+2} & Ƀ_{\mathbb{m}+2}^2 & Ƀ_{\mathbb{m}+2}^3 \\ Ƀ_{\mathbb{m}+3} & Ƀ_{\mathbb{m}+3}^2 & Ƀ_{\mathbb{m}+3}^3 \end{vmatrix} = Ƀ_{\mathbb{m}} Ƀ_{\mathbb{m}+1}^2 Ƀ_{\mathbb{m}+2}^2 Ƀ_{\mathbb{m}+3}$$

Proof:

$$\text{Let } A = \begin{vmatrix} Ƀ_{\mathbb{m}+1} & Ƀ_{\mathbb{m}+1}^2 & Ƀ_{\mathbb{m}+1}^3 \\ Ƀ_{\mathbb{m}+2} & Ƀ_{\mathbb{m}+2}^2 & Ƀ_{\mathbb{m}+2}^3 \\ Ƀ_{\mathbb{m}+3} & Ƀ_{\mathbb{m}+3}^2 & Ƀ_{\mathbb{m}+3}^3 \end{vmatrix}$$

Taking common out $B_{\mathbb{m}+1},\ B_{\mathbb{m}+2},$ and $B_{\mathbb{m}+3}$ from 1st, 2nd and 3rd row respectively,

$$A = Ƀ_{\mathbb{m}+1} Ƀ_{\mathbb{m}+2} Ƀ_{\mathbb{m}+3} \begin{vmatrix} 1 & Ƀ_{\mathbb{m}+1} & Ƀ_{\mathbb{m}+1}^2 \\ 1 & Ƀ_{\mathbb{m}+2} & Ƀ_{\mathbb{m}+2}^2 \\ 1 & Ƀ_{\mathbb{m}+3} & Ƀ_{\mathbb{m}+3}^2 \end{vmatrix}$$

Applying $R_2 = R_2 - R_1$ and $R_3 = R_3 - R_1$

$$A = Ƀ_{\mathbb{m}+1} Ƀ_{\mathbb{m}+2} Ƀ_{\mathbb{m}+3} \begin{vmatrix} 1 & Ƀ_{\mathbb{m}+1} & Ƀ_{\mathbb{m}+1}^2 \\ 0 & Ƀ_{\mathbb{m}} & Ƀ_{\mathbb{m}} Ƀ_{\mathbb{m}+3} \\ 0 & Ƀ_{\mathbb{m}+2} & Ƀ_{\mathbb{m}+2}(Ƀ_{\mathbb{m}+3} + Ƀ_{\mathbb{m}+1}) \end{vmatrix}$$

Taking common out $Ƀ_{\mathbb{m}}$ and $Ƀ_{\mathbb{m}+2}$ from 2nd and 3rd row respectively,

$$A = Ƀ_{\mathbb{m}} Ƀ_{\mathbb{m}+1} Ƀ_{\mathbb{m}+2}^2 Ƀ_{\mathbb{m}+3} \begin{vmatrix} 1 & Ƀ_{\mathbb{m}+1} & Ƀ_{\mathbb{m}+1}^2 \\ 0 & 1 & Ƀ_{\mathbb{m}+3} \\ 0 & 1 & (Ƀ_{\mathbb{m}+3} + Ƀ_{\mathbb{m}+1}) \end{vmatrix}$$

Applying $R_3 = R_3 - R_2$

$$A = Ƀ_{\mathbb{m}} Ƀ_{\mathbb{m}+1} Ƀ_{\mathbb{m}+2}^2 Ƀ_{\mathbb{m}+3} \begin{vmatrix} 1 & Ƀ_{\mathbb{m}+1} & Ƀ_{\mathbb{m}+1}^2 \\ 0 & 1 & Ƀ_{\mathbb{m}+3} \\ 0 & 0 & Ƀ_{\mathbb{m}+1} \end{vmatrix}$$

$$= Ƀ_{\mathbb{m}} Ƀ_{\mathbb{m}+1} Ƀ_{\mathbb{m}+2}^2 Ƀ_{\mathbb{m}+3}$$

Theorem 5.7: For every integer $\mathbb{m} \geq 0$

$$\begin{vmatrix} Ƀ_{\mathbb{m}} + F_{\mathbb{m}} & Ƀ_{\mathbb{m}+1} + F_{\mathbb{m}+1} & Ƀ_{\mathbb{m}+2} + F_{\mathbb{m}+2} \\ Ƀ_{\mathbb{m}+3} + F_{\mathbb{m}+3} & Ƀ_{\mathbb{m}+4} + F_{\mathbb{m}+4} & Ƀ_{\mathbb{m}+5} + F_{\mathbb{m}+5} \\ Ƀ_{\mathbb{m}+6} + F_{\mathbb{m}+6} & Ƀ_{\mathbb{m}+7} + F_{\mathbb{m}+7} & Ƀ_{\mathbb{m}+8} + F_{\mathbb{m}+8} \end{vmatrix} = 0$$

Proof:

$$\text{Let } A = \begin{vmatrix} Ƀ_{\mathbb{m}} + F_{\mathbb{m}} & Ƀ_{\mathbb{m}+1} + F_{\mathbb{m}+1} & Ƀ_{\mathbb{m}+2} + F_{\mathbb{m}+2} \\ Ƀ_{\mathbb{m}+3} + F_{\mathbb{m}+3} & Ƀ_{\mathbb{m}+4} + F_{\mathbb{m}+4} & Ƀ_{\mathbb{m}+5} + F_{\mathbb{m}+5} \\ Ƀ_{\mathbb{m}+6} + F_{\mathbb{m}+6} & Ƀ_{\mathbb{m}+7} + F_{\mathbb{m}+7} & Ƀ_{\mathbb{m}+8} + F_{\mathbb{m}+8} \end{vmatrix}$$

Applying $C_1 = C_1 + C_2$

$$A = \begin{vmatrix} Ƀ_{\mathbb{m}} + F_{\mathbb{m}} + Ƀ_{\mathbb{m}+1} + F_{\mathbb{m}+1} & Ƀ_{\mathbb{m}+1} + F_{\mathbb{m}+1} & Ƀ_{\mathbb{m}+2} + F_{\mathbb{m}+2} \\ Ƀ_{\mathbb{m}+3} + F_{\mathbb{m}+3} + Ƀ_{\mathbb{m}+4} + F_{\mathbb{m}+4} & Ƀ_{\mathbb{m}+4} + F_{\mathbb{m}+4} & Ƀ_{\mathbb{m}+5} + F_{\mathbb{m}+5} \\ Ƀ_{\mathbb{m}+6} + F_{\mathbb{m}+6} + Ƀ_{\mathbb{m}+7} + F_{\mathbb{m}+7} & Ƀ_{\mathbb{m}+7} + F_{\mathbb{m}+7} & Ƀ_{\mathbb{m}+8} + F_{\mathbb{m}+8} \end{vmatrix}$$

$$A = \begin{vmatrix} Ƀ_{\mathbb{m}} + Ƀ_{\mathbb{m}+1} + F_{\mathbb{m}} + F_{\mathbb{m}+1} & Ƀ_{\mathbb{m}+1} + F_{\mathbb{m}+1} & Ƀ_{\mathbb{m}+2} + F_{\mathbb{m}+2} \\ Ƀ_{\mathbb{m}+3} + Ƀ_{\mathbb{m}+4} + F_{\mathbb{m}+3} + F_{\mathbb{m}+4} & Ƀ_{\mathbb{m}+4} + F_{\mathbb{m}+4} & Ƀ_{\mathbb{m}+5} + F_{\mathbb{m}+5} \\ Ƀ_{\mathbb{m}+6} + Ƀ_{\mathbb{m}+7} + F_{\mathbb{m}+6} + F_{\mathbb{m}+7} & Ƀ_{\mathbb{m}+7} + F_{\mathbb{m}+7} & Ƀ_{\mathbb{m}+8} + F_{\mathbb{m}+8} \end{vmatrix}$$

$$A = \begin{vmatrix} Ƀ_{\mathbb{m}+2} + F_{\mathbb{m}+2} & Ƀ_{\mathbb{m}+1} + F_{\mathbb{m}+1} & Ƀ_{\mathbb{m}+2} + F_{\mathbb{m}+2} \\ Ƀ_{\mathbb{m}+5} + F_{\mathbb{m}+5} & Ƀ_{\mathbb{m}+4} + F_{\mathbb{m}+4} & Ƀ_{\mathbb{m}+5} + F_{\mathbb{m}+5} \\ Ƀ_{\mathbb{m}+8} + F_{\mathbb{m}+8} & Ƀ_{\mathbb{m}+7} + F_{\mathbb{m}+7} & Ƀ_{\mathbb{m}+8} + F_{\mathbb{m}+8} \end{vmatrix}$$

Since 1st and 3rd columns are identical, thus we obtained the required result.

Theorem 5.8: For every integer $n \geq 0$

$$\begin{vmatrix} 1 + Ƀ_{\mathbb{m}} & 1 + B_{\mathbb{m}+1} & 1 + Ƀ_{\mathbb{m}+2} \\ 1 + Ƀ_{\mathbb{m}+3} & 1 + Ƀ_{\mathbb{m}+4} & 1 + Ƀ_{\mathbb{m}+5} \\ 1 + Ƀ_{\mathbb{m}+6} & 1 + Ƀ_{\mathbb{m}+7} & 1 + Ƀ_{\mathbb{m}+8} \end{vmatrix} = 8(Ƀ_{\mathbb{m}}^2 - Ƀ_{\mathbb{m}+1}^2 + Ƀ_{\mathbb{m}}Ƀ_{n+1})$$

Proof:

Let $A = \begin{vmatrix} 1 + Ƀ_{\mathbb{m}} & 1 + B_{\mathbb{m}+1} & 1 + Ƀ_{\mathbb{m}+2} \\ 1 + Ƀ_{\mathbb{m}+3} & 1 + Ƀ_{\mathbb{m}+4} & 1 + Ƀ_{\mathbb{m}+5} \\ 1 + Ƀ_{\mathbb{m}+6} & 1 + Ƀ_{\mathbb{m}+7} & 1 + Ƀ_{\mathbb{m}+8} \end{vmatrix}$

Applying $C_1 = C_1 + C_2$

$$A = \begin{vmatrix} 2 + Ƀ_{\mathbb{m}} + Ƀ_{\mathbb{m}+1} & 1 + Ƀ_{\mathbb{m}+1} & 1 + Ƀ_{\mathbb{m}+2} \\ 2 + Ƀ_{\mathbb{m}+3} + Ƀ_{\mathbb{m}+4} & 1 + Ƀ_{\mathbb{m}+4} & 1 + Ƀ_{\mathbb{m}+5} \\ 2 + Ƀ_{\mathbb{m}+6} + Ƀ_{\mathbb{m}+7} & 1 + Ƀ_{\mathbb{m}+7} & 1 + Ƀ_{\mathbb{m}+8} \end{vmatrix}$$

$$A = \begin{vmatrix} 2 + Ƀ_{\mathbb{m}+2} & 1 + Ƀ_{\mathbb{m}+1} & 1 + Ƀ_{\mathbb{m}+2} \\ 2 + Ƀ_{\mathbb{m}+5} & 1 + Ƀ_{\mathbb{m}+4} & 1 + Ƀ_{\mathbb{m}+5} \\ 2 + Ƀ_{\mathbb{m}+8} & 1 + Ƀ_{\mathbb{m}+7} & 1 + Ƀ_{\mathbb{m}+8} \end{vmatrix}$$

Applying $C_1 = C_1 - C_3$

$$A = \begin{vmatrix} 1 & 1 + Ƀ_{\mathbb{m}+1} & 1 + Ƀ_{\mathbb{m}+2} \\ 1 & 1 + Ƀ_{\mathbb{m}+4} & 1 + Ƀ_{\mathbb{m}+5} \\ 1 & 1 + Ƀ_{\mathbb{m}+7} & 1 + Ƀ_{\mathbb{m}+8} \end{vmatrix}$$

Applying $C_2 = C_2 - C_1$ and $C_3 = C_3 - C_2$

$$A = \begin{vmatrix} 1 & Ƀ_{\mathbb{m}+1} & Ƀ_{\mathbb{m}} \\ 1 & Ƀ_{\mathbb{m}+4} & Ƀ_{\mathbb{m}+3} \\ 1 & Ƀ_{\mathbb{m}+7} & Ƀ_{\mathbb{m}+6} \end{vmatrix}$$

Applying $R_2 = R_2 - R_1$ and $R_3 = R_3 - R_2$

$$A = \begin{vmatrix} 1 & Ƀ_{\mathbb{m}+1} & Ƀ_{\mathbb{m}} \\ 0 & 2Ƀ_{\mathbb{m}+2} & 2Ƀ_{\mathbb{m}+1} \\ 0 & 2Ƀ_{\mathbb{m}+5} & 2Ƀ_{\mathbb{m}+4} \end{vmatrix}$$

Taking common out 2 from 2nd and 3rd row,

$$A = 4\begin{vmatrix} 1 & Ƀ_{\mathbb{m}+1} & Ƀ_{\mathbb{m}} \\ 0 & Ƀ_{\mathbb{m}+2} & Ƀ_{\mathbb{m}+1} \\ 0 & Ƀ_{\mathbb{m}+5} & Ƀ_{\mathbb{m}+4} \end{vmatrix}$$

Applying $R_3 = R_3 - R_2$

$$A = 4\begin{vmatrix} 1 & Ƀ_{\mathbb{m}+1} & Ƀ_{\mathbb{m}} \\ 0 & Ƀ_{\mathbb{m}+2} & Ƀ_{\mathbb{m}+1} \\ 0 & 2Ƀ_{\mathbb{m}+3} & 2Ƀ_{\mathbb{m}+2} \end{vmatrix}$$

Taking common out 2 from 3rd row,

$$A = 8\begin{vmatrix} 1 & Ƀ_{\mathbb{m}+1} & Ƀ_{\mathbb{m}} \\ 0 & Ƀ_{\mathbb{m}+2} & Ƀ_{\mathbb{m}+1} \\ 0 & Ƀ_{\mathbb{m}+3} & Ƀ_{\mathbb{m}+2} \end{vmatrix}$$

Again applying $R_3 = R_3 - R_2$

$$A = 8\begin{vmatrix} 1 & Ƀ_{\mathbb{m}+1} & Ƀ_{\mathbb{m}} \\ 0 & Ƀ_{\mathbb{m}+2} & Ƀ_{\mathbb{m}+1} \\ 0 & Ƀ_{\mathbb{m}+1} & Ƀ_{\mathbb{m}} \end{vmatrix}$$

Again applying $R_1 = R_1 - R_3$

$$A = 8\begin{vmatrix} 1 & 0 & 0 \\ 0 & Ƀ_{\mathbb{m}+2} & Ƀ_{\mathbb{m}+1} \\ 0 & Ƀ_{\mathbb{m}+1} & Ƀ_{\mathbb{m}} \end{vmatrix}$$

Again applying $R_2 = R_2 - R_3$

Again applying $R_1 = R_1 - R_3$

$$A = 8\begin{vmatrix} 1 & 0 & 0 \\ 0 & Ƀ_{\mathbb{m}} & Ƀ_{\mathbb{m}+1} - Ƀ_{\mathbb{m}} \\ 0 & Ƀ_{\mathbb{m}+1} & Ƀ_{\mathbb{m}} \end{vmatrix}$$

$$= 8(Ƀ_{\mathbb{m}}^2 - Ƀ_{\mathbb{m}+1}^2 + Ƀ_{\mathbb{m}}Ƀ_{n+1})$$

Theorem 5.9: For every integer $\mathbb{m} \geq 0$

$$\begin{vmatrix} Ƀ_{\mathbb{m}}F_{\mathbb{m}} & Ƀ_{\mathbb{m}}F_{\mathbb{m}+1} & Ƀ_{\mathbb{m}}F_{\mathbb{m}+2} \\ Ƀ_{\mathbb{m}+1}F_{\mathbb{m}} & Ƀ_{\mathbb{m}+1}F_{\mathbb{m}+1} & Ƀ_{\mathbb{m}+1}F_{\mathbb{m}+2} \\ Ƀ_{\mathbb{m}+2}F_{\mathbb{m}} & Ƀ_{\mathbb{m}+2}F_{\mathbb{m}+1} & Ƀ_{\mathbb{m}+2}F_{\mathbb{m}+2} \end{vmatrix} = 0$$

Proof:

$$\text{Let } A = \begin{vmatrix} Ƀ_{m}F_{m} & Ƀ_{m}F_{m+1} & Ƀ_{m}F_{m+2} \\ Ƀ_{m+1}F_{m} & Ƀ_{m+1}F_{m+1} & Ƀ_{m+1}F_{m+2} \\ Ƀ_{m+2}F_{m} & Ƀ_{m+2}F_{m+1} & Ƀ_{m+2}F_{m+2} \end{vmatrix}$$

Taking common out $Ƀ_{m}$, $Ƀ_{m+1}$ and $Ƀ_{m+2}$ from 1st, 2nd and 3rd row and F_{m}, F_{m+1} and F_{m+2} from 1st, 2nd and 3rd column respectively,

$$A = Ƀ_{m}Ƀ_{m+1}Ƀ_{m+2}F_{m}F_{m+1}F_{m+2} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

Since all the rows and columns are identical, thus we obtained the required result.

5.4 Conclusion:

The exploration of the Coupled Lucas Sequence of Second Order offers a profound insight into the broader field of sequence theory, especially in relation to other well-known sequences such as the Fibonacci sequence and the Generalized Fibonacci-Lucas sequence. Throughout the history of mathematical research, these sequences have been recognized for their fascinating properties and significant applications in various fields such as number theory, computer science, cryptography, and even nature. The current study of the Coupled Lucas Sequence of Second Order continues in this tradition, pushing the boundaries of what we know about recursive sequences and their applications.

The Lucas sequences, much like the Fibonacci sequences, are defined by a set of recurrence relations. However, the unique feature of the Coupled Lucas Sequence of Second Order lies in its coupling mechanism, which intertwines two independent sequences into a single structure. This coupling adds a layer of complexity and elegance, as each term in one sequence depends not only on the preceding terms of its own sequence but also on the corresponding terms of the other sequence. This disconnectedness introduces intricate patterns and dependencies, leading to behaviors that are much more complex than those observed in simple sequences like Fibonacci or Lucas on their own.

Through theoretical analysis, it has been demonstrated that these coupled sequences possess unique identities and properties that distinguish them from other known sequences. By utilizing inductive reasoning and computational methods, it is possible to uncover new identities and relationships within the Coupled Lucas Sequence. Inductive reasoning, in particular, plays a crucial role in predicting novel outcomes, as it allows researchers to extrapolate from known patterns to discover previously unrecognized properties of the sequence.

The initial values of the two sequences in the Coupled Lucas Sequence of Second Order play a significant role in determining their behavior. These initial values act as seeds that define the growth and evolution of the sequences over time. Small changes in these initial conditions can lead to vastly different outcomes, revealing the sensitivity and complexity of the system. The recurrence relations, which govern the progression of the sequences, ensure that each term is calculated based on a fixed formula, but the interaction between the two sequences adds an additional layer of unpredictability and complexity to the system.

One of the most intriguing aspects of this research is the way in which the simultaneous evolution of the two sequences creates a harmonious relationship between them. Each term in the sequence is intricately linked not only to the preceding terms of its own sequence but also to the corresponding terms in the coupled sequence. This disconnectedness suggests that the sequences are working together in tandem, each influencing the other's progression in a delicate balance. This relationship introduces a deeper level of structure to the sequences, which could have far-reaching implications for other areas of Mathematics, especially in the study of dynamical systems and complexity theory.

The investigation has also revealed practical applications of the Coupled Lucas Sequence of Second Order. Beyond its theoretical significance, the sequence can be applied in fields such as cryptography, where the complex relationships between terms in the sequence could be used to generate secure encryption keys. Additionally, the sequence's intricate patterns and behaviors could have applications in computer science, particularly in algorithms related to recursive functions and optimization problems.

In conclusion, the study of the Coupled Lucas Sequence of Second Order has unveiled a rich mathematical structure that blends theoretical elegance with practical applications. The combination of recurrence relations and coupling mechanisms introduces new complexities that challenge our understanding of traditional sequences, offering new avenues for research and discovery. By continuing to explore the properties of these sequences, mathematicians can gain deeper insights into the nature of recursion, interdependence, and complexity, enriching the broader field of sequence theory. The potential for uncovering new identities and applications within this framework remains vast, promising exciting developments in both theoretical and applied Mathematics.Moreover, the practical applications of the coupled sequence extend into various domains. The coupling mechanism, while inherently mathematical, holds promise for applications in cryptography, optimization, and other areas where the dynamic interplay of numerical relationships can be harnessed for practical purposes. This underscores the relevance of pure mathematical exploration, demonstrating that seemingly abstract concepts can find meaningful applications in the real world.

Moreover, the practical applications of the coupled sequence extend into various domains. The coupling mechanism, while inherently mathematical, holds promise for applications in cryptography, optimization, and other areas where the dynamic interplay of numerical relationships can be harnessed for practical purposes. This underscores the relevance of pure mathematical exploration, demonstrating that seemingly abstract concepts can find meaningful applications in the real world.

"Just as the rhythm of the seasons follows an unseen order, the coupled Lucas sequence and Fibonacci–Lucas determinantal identities reveal how separate elements can move in harmony, creating patterns that are both predictable and profound."

Chapter-6

Summary and Conclusions

6.1 Summary of the Research Work

This thesis presents a comprehensive study of various coupled extensions of Fibonacci and Lucas sequences, focusing on their multiplicative forms, higher-order constructions, and the identities they satisfy. The investigation spans multiple orders and structural schemes, offering a systematic development of algebraic results in this area.

In Chapter 2, we explored the fifth-order Multiplicative Coupled Fibonacci Sequence (MCFS) under a specific recurrence scheme. Several novel identities were derived and proved through mathematical induction. The results demonstrated the influence of the selected scheme on the behavior of the sequences and provided a foundation for deeper analysis.

Chapter 3 addressed the second- and third-order Multiplicative Triple Fibonacci Sequences (MTFS). Here, we introduced recurrence relations involving three coupled sequences and established identities based on these interactions. The derivations highlighted the role of initial conditions and recurrence structure in generating sequence patterns and symmetries.

In Chapter 4, the study was extended to fourth-order MTFS. By analyzing a variety of recurrence schemes, we obtained a broader class of identities. The work also examined structural behavior across schemes, revealing distinctions in algebraic complexity and sensitivity to initial values.

Chapter 5 focused on the Coupled Lucas Sequences (CLS) of the second order and their generalizations. Identities involving both additive and multiplicative properties were established, including determinantal identities that linked the sequence behavior with matrix algebra. These results connected recursive sequence theory with linear representations and provided further scope for mathematical modeling.

Each chapter applied methods such as mathematical induction, determinant expansions, and combinatorial logic to develop and validate the proposed identities.

6.2 Major Contributions of the Study

This research contributes several key results to the field of recurrence sequences and their generalizations:

- **Development of new identities** for coupled Fibonacci and Lucas sequences, especially in their multiplicative and higher-order forms.
- **Introduction of structurally varied schemes** that highlight the dependence of identity formation on recurrence rules.
- **Demonstration of sensitivity and complexity** in coupled sequences due to interactions across multiple sequences.
- **Application of determinant methods** to formulate compact and generalized identities, linking sequence behavior with linear algebraic structures.
- **Classification and comparison of schemes**, illustrating how different formulations yield distinct algebraic properties.

These findings not only enhance the theoretical understanding of coupled recursive systems but also pave the way for future applications in mathematical modeling and computational algorithms.

6.3 Concluding Remarks

The work undertaken in this thesis has led to a rich collection of identities and theoretical insights into coupled and multiplicative Fibonacci and Lucas sequences. Through rigorous derivation and scheme-wise comparison, the study establishes a strong foundation for future research in generalized recursive structures. The identities and formulations presented here not only expand the mathematical framework of such sequences but also highlight their versatility in modeling complex systems and supporting computational methods. The results serve as a bridge between classical sequence theory and modern applications, reinforcing the relevance and adaptability of recurrence relations in contemporary mathematical discourse.

Scope for Future Research

Although the present study covers a wide range of coupled sequence types and recurrence schemes, several directions remain open for further exploration:

- Extension to sixth-order and hybrid coupled sequences, such as those combining Fibonacci and Lucas characteristics.
- Investigation of modular behavior, periodicity, and convergence in different arithmetic settings.
- Analysis of the computational complexity and algorithmic implementation of these sequences in real-world applications.
- Exploration of their potential in cryptographic systems, pseudorandom number generation, and error correction codes.
- Study of matrix representations, spectral properties, and connections with linear transformations in higher dimensions.
- We can prove the result of CFS of order 2^{nd}, 3^{rd}, 4^{th} and 5^{th}, also the results on some special Scheme with the help of Mathematical induction.
- Coupled Fibonacci sequence of 2^{nd} and 3^{rd} order can be obtained as MTFS.
- Coupled Lucas sequence of 3^{rd}, 4^{th} and 5^{th} can be derived as 2^{nd} order.
- Determinantal identities can be obtained with the help of MCFS and MTFS.

These paths offer promising avenues to deepen both theoretical and applied aspects of recurrence sequence research.

List of Publications

- Vikas Ranga and A.K. Awasthi, “Application of second order Coupled Lucas Sequence” in IAENG International Journal of Applied Mathematics. (2025). (Scopus)Q3
- A.K. Awasthi and Vikas Ranga, “Multiplicative Triple Fibonacci sequence of second order under three Specific Schemes and third order under nine Specific Schemes” in IAENG International Journal of Applied Mathematics. (2024). (Scopus)Q3
- Vikas Ranga and Vipin Verma, “Multiplicative Coupled Fibonacci sequence of fifth order” in Journal of Physics: Conference Series (2022). (Scopus)Q4
- Vikas Ranga and Vipin Verma, “Multiplicative Triple Fibonacci sequence of fourth order under nine Specific Schemes” in Journal of Algebraic Statistics (2022). (SCI & Scopus)
- Vipin Verma and Vikas Ranga, “Multiplicative Triple Fibonacci sequence of third order” in Turkish Journal of Computer and Mathematics Education (2021). (Scopus)

Published Matter

IAENG International Journal of Applied Mathematics

Application of Second Order Coupled Lucas Sequence

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Abstract - **Fibonacci numbers and polynomials have been widely studied due to their importance in mathematics, physics, and business. The Coupled Fibonacci Sequence (CFS) and Multiplicative Coupled Fibonacci Sequence (MCFS) contain useful identities but depend on previous terms for computation. The Lucas Sequence (LS) also displays notable properties in number theory. This study investigates the second-order Coupled Lucas Sequence (CLS), in which two interdependent sequences evolve in tandem. Through mathematical analysis and simulations, we uncover patterns, periodicities, and structural relationships within the sequence. Additionally, the research explores its promising applications in cryptography, optimization, and algorithm design. A deeper understanding of CLS enhances number theory and offers insights into broader mathematical systems. This study contributes to mathematical research by revealing intricate connections between sequences and emphasizing the elegance and utility of coupled sequences across disciplines.**

Index Terms- **LS, FS, CFS, MCFS, CLS.**

I. INTRODUCTION

Numerous fields, including algebra, combinatorics, approximation theory, geometry, graph theory, and number theory itself, have benefited from it., the Fibonacci numbers and polynomials play a crucial role. Perhaps the most well-known application of the Fibonacci numbers is in the rabbit breeding puzzle, which Leonardo de Pisa first presented in his book "Liber-Abaci" in 1202. Numerous authors have explored their various characteristics and broadened usefulness. The Fibonacci and Lucas numbers are undoubtedly two of the most fascinating mathematical sequences, as illustrated in Koshy's book [1]. A long list of identities can be found in Vajda's book [2] and includes numerous identities. There is a long form of unity matrices and determinants to study Fibonacci numbers. A.K. Awasthi, Vikas Ranga, and Kamal Dutt [14] discuss the extension of Fibonacci sequences using specific multiplicative schemes.

Manuscript received December 1, 2024; revised June 14, 2025.

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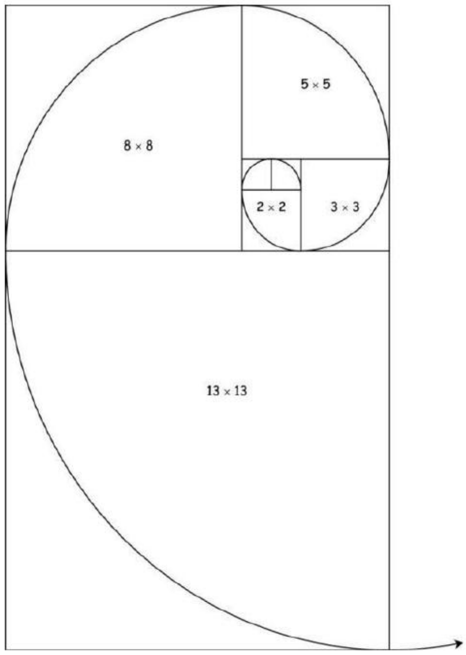

Fig 1. Fibonacci Spiral with golden ratio

Fig 1, represents the Fibonacci Spiral, formed using squares with side lengths following the Fibonacci sequence (2×2, 3×3, 5×5, 8×8,…). A quarter-circle arc inside each square creates a spiral-like curve, approximating the Golden Spiral, seen in nature, art, and architecture. It visually demonstrates the connection between the Fibonacci sequence and the Golden Ratio, showcasing proportional and symmetrical growth patterns.

The "Coupled Lucas Sequence of Second Order" emerges as a captivating exploration within the domain of number theory, building upon the foundations laid by the classical LS. This innovative extension introduces a dynamic interplay between two distinct second-order LS, weaving a tapestry of numerical relationships that transcend the conventional boundaries of sequence theory. As a testament to the continuous evolution of mathematical inquiry, this study delves into the intricacies of the coupled sequences, unraveling a myriad of patterns, properties, and applications. By introducing coupling mechanisms between two such sequences, a new and intriguing mathematical entity emerges. This coupled relationship manifests as a simultaneous evolution of two interconnected sequences, influencing each other's progression in a harmonious dance of numerical dynamics.

Volume 55, Issue 9, September 2025, Pages 2668-2677

Multiplicative Triple Fibonacci Sequence of Second Order under Three Specific Schemes and Third Order under Nine Specific Schemes

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***Abstract*—The Fibonacci sequence (FS) can be found in various aspects of nature. This sequence has applications in multiple fields of mathematics and real-world scenarios. The FS is used to build various algebraic structures, including the Fibonacci group, Fibonacci graph, Fibonacci lattice, Fibonacci quaternion and Fibonacci octonion. This theory has gained significant attention recently and is now considered a major area of number theory. In recent years, there has been considerable interest in the growth of knowledge in the general area of Fibonacci numbers and related mathematical problems. Triple Fibonacci sequences (TFS) have gained popularity recently, although multiplicative triple equations of recurrence relations are less well-known. In 1202, Leonardo of Pisa, also known as Fibonacci (which means "son of Bonacci"), introduced the results of his investigation into expanding a rabbit population. The FS is recognized as a sequence with astonishing properties. In 1985, K.T. Attanasov introduced the Coupled Fibonacci Sequence (CFS), and further developments were made in 1987. However, compared to the additive form of TFS, the multiplicative form of TFS is less well-known. The multiplicative triple Fibonacci sequences (MTFS) of the second and third order represent a novel extension of the classical FS, introducing three specific schemes for the second order and nine specific schemes for the third order. This mathematical study explores the intricate relationships between numbers in a multiplicative context, revealing fascinating patterns and properties.**

***Index Terms*- FS, CFS, TFS, MTFS.**

I. INTRODUCTION

One well-known integer sequence is the Fibonacci sequence (FS). Mathematicians have long been fascinated by this series. The FS has applications in numerous fields, including architecture, engineering, computer science, physics, nature, art, and more. By altering the recurrence relation, the initial condition, or both, the FS can be generalized. This broader form is known as the generalized Fibonacci sequence. Several authors have explored second-order generalized Fibonacci sequences in the literature. The Fibonacci numbers appear in many remarkable scenarios andare abundant in nature, often represented in images of fruits, vegetables, and flowers. Mathematical scholars have been deeply interested in the study of Fibonacci numbers and related mathematics for centuries.

Triple Fibonacci sequences (TFS) represent a novel approach to generalizing the Coupled Fibonacci Sequence (CFS). The TFS is a significant advancement in the field of FS and extends the CFS, offering a wide range of intriguing properties and applications. The multiplicative triple Fibonacci sequences (MTFS), an extension of the classical FS, have garnered substantial interest in recent mathematical research, particularly in the context of second and third-order derivations under specific schemes. The FS, known for its ubiquity in nature and applications across diverse fields, serves as the foundation for exploring the multiplicative variations proposed in this study.

The TFS represents a fresh approach to the generalization of the CFS. It is a significant advancement in the field of FS and a generalization of the CFS, offering a wide range of fascinating properties and applications. The MTFS, an extension of the classical FS, has garnered substantial interest in recent mathematical research, particularly concerning second and third-order derivations under specific schemes. The FS, known for its ubiquity in nature and applications across diverse fields, composes the foundation for exploring the multiplicative variations proposed in this study.

There has been a great deal of research on the TFS. J. Z. Lee and J. S. Lee [1] were the first to propose the TFS. Koshy's book [2] is an excellent source for these applications. In 1985, Attanasov [3, 4] popularized the concept of the CFS and introduced a new TFS design. The TFS connects three integer sequences, where the elements of one sequence are part of the generalization of the others, and vice versa. Singh and Sikhwal [4, 7] computed the MCFS and additive TFS, both have significant properties.

Under two distinct schemes, Kiran Singh Sisodiya, Vandana Gupta, and Kiran Sisodiya [8] investigated several features of the fourth-order MCFS. Omprakash Sikhwal, Mamta Singh, and Shweta Jain [6] examined various aspects of the fifth-order CFS. In 2014, Krishna Kumar Sharma et al. [13] formulated the additive-linked Fibonacci sequences of r-th order and demonstrated their diverse features. Bijendra Singh and Omprakash Sikhwal [9] explored both the primitive aspects of second-order TFS and several features of additive TFS. The MTFS of the second order was examined from multiple perspectives by Mamta Singh, Shikha Bhatnagar, and Omprakash Sikhwal [10]. The properties of

Manuscript received June 8, 2024; revised September 17, 2024.
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Multiplicative Coupled Fibonacci Sequence of Fifth Order

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Abstract: From the last few years, there has consequential diversion in progress of Coupled Fibonacci Sequence, additive Coupled Fibonacci Sequence are modished but Multiplicative Coupled Fibonacci Sequence of Recurrence Relation are less are not much improved. K. T. Atanassov transplanted the notion of Coupled Fibonacci Sequence in 1985. He contemplates Multiplicative Coupled Fibonacci Sequence of second order in 1995.

In the Fibonacci sequence, this paper defined and a craving to outstretch the results of Multiplicative Coupled Fibonacci sequence.

A lot of work has been terminated on Multiplicative Coupled Fibonacci Sequence of second, third and fourth order. We offer some identities of Multiplicative Coupled Fibonacci Sequence of fifth order under one specific scheme.

Keywords- Fibonacci, Sequence, Coupled, Multiplicative

1.Introduction

In the recent, much work has been done Fibonacci Sequence its Multiplicative Fibonacci Sequence is less known. Firstly, K.T. Atanassov [5] worked in the properties and generalization of Fibonacci Sequence [1][2] and [4]. K.T. Atanassov [9] notify four different ways of Multiplicative Coupled Fibonacci Sequence. P. Glaister [6] and P. Hope [7] also studied on Multiplicative Fibonacci Sequence.

Let $\{J_i\}_{i=0}^{\infty}$ and $\{K_i\}_{i=0}^{\infty}$be two infinite sequences and called 2-F Sequence or Coupled Fibonacci Sequence with basic value a, b, c and d then all the distinct schemes of Multiplicative Coupled Fibonacci Sequence are as follows:

$$J_0 = a, K_0 = b, J_1 = c, K_1 = d$$

First Scheme:

$$\begin{aligned} &J_0 = a, K_0 = b, J_1 = c, K_1 = d \\ &J_{n+2} = K_{n+1}.K_n, \quad n \geq 0 \qquad (1.1) \\ &K_{n+2} = J_{n+1}.J_n, \quad n \geq 0 \end{aligned}$$

Second Scheme:

$$\begin{aligned} &J_0 = a, K_0 = b, J_1 = c, K_1 = d \\ &J_{n+2} = J_{n+1}.K_n, \quad n \geq 0 \qquad (1.2) \\ &K_{n+2} = K_{n+1}.J_n, \quad n \geq 0 \end{aligned}$$

Multiplicative Triple Fibonacci Sequence of Third Order

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Article History: Received: 11 January 2021; Revised: 12 February 2021; Accepted: 27 March 2021; Published online: 10 May 2021

Abstract:
The Coupled Fibonacci Sequence are Firstly established by K.T. Atanassov in 1985. The abstractions of Fibonacci Triple Sequence are considered in 1987. Fibonacci Sequence stands as a kind of super sequence with Fabulous properties. This is the explosive development in the region of Fibonacci Sequence. Fibonacci was advanced by Leonardo de Pisa (whose nickname was Fibonacci, which means son of Bonacci) in 1202 as a result of his inspection on the growth of a population of rabbits. The consecutive Fibonacci numbers are produced by adding together the two previous numbers in the sequence, after specifying suitable initial conditions. In the last years Triple Fibonacci Sequence are hype, but Multiplicative Triple Sequence of Recurrence Relations are less known. Much work has been done to study on Fibonacci Triple Sequence in Additive form. In 1995, Multiplicative Coupled Fibonacci Sequence are contemplated. Our purpose of this paper to present some results of Multiplicative Triple Fibonacci Sequence of third order under one specific scheme.
This paper expanded out of a curiosity in the Fibonacci sequence and a craving to spread the results of Multiplicative Coupled Fibonacci sequence. Ever since Fibonacci (Leonardo of Pisa) wrote his Liber Abbaci in 1202, his fascinating sequence has transfixed men through the centuries, not only for its inborn mathematical riches, but also for its applications in art and nature. Indeed, it is almost true to say that the research produced by its nearly amounts to the quantity of off- spring generated by the mythical pair of rabbits who started Fibonacci off on the problem.

Keywords: Fibonacci Sequence, Multiplicative Triple Fibonacci sequence.

1. Introduction: The Fibonacci Triple Sequence is a new direction in generalization of Coupled Fibonacci sequence. Fibonacci sequence and their generalization have many attracting applications and properties to every field of science. Koshy's book [9] is a good origin for these applications. The Coupled Fibonacci Sequence was first inaugurated by K. T. Atnassov [4] and also examined many curious properties and a new guideline of generalization of Fibonacci Sequence [2, 5, 6].
J. Z. Lee and J. S. Lee established Firstly Additive Triple Sequence [3]. K. T. Atnassov delineate new notion for Additive Triple Fibonacci Sequence [7, 8] and called 3-Fibonacci Sequence or 3-F Sequence.
Let $\{\alpha_i\}_{i=0}^{\infty}$ $\{\beta_i\}_{i=0}^{\infty}$ and $\{\gamma_i\}_{i=0}^{\infty}$ be three infinite sequences and called 3-F Sequence or Triple Fibonacci Sequence with initial value a, b, c, d, e and f.
If $\alpha_0 = a, \beta_0 = b, \gamma_0 = c, \alpha_1 = d, \beta_1 = e, \gamma_1 = f$, then nine different schemes of Multiplicative Triple Fibonacci Sequence are as follows:

First Scheme:

$$\alpha_{n+2} = \beta_{n+1}\cdot\gamma_n$$
$$\beta_{n+2} = \gamma_{n+1}\cdot\alpha_n$$
$$\gamma_{n+2} = \alpha_{n+1}\cdot\beta_n$$

Second Scheme:

$$\alpha_{n+2} = \gamma_{n+1}\cdot\beta_n$$
$$\beta_{n+2} = \alpha_{n+1}\cdot\gamma_n$$
$$\gamma_{n+2} = \beta_{n+1}\cdot\alpha_n$$

Third Scheme:

$$\alpha_{n+2} = \alpha_{n+1}\cdot\beta_n$$
$$\beta_{n+2} = \beta_{n+1}\cdot\gamma_n$$
$$\gamma_{n+2} = \gamma_{n+1}\cdot\alpha_n$$

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 3830-3839
https://publishoa.com
ISSN: 1309-3452

Multiplicative Triple Fibonacci Sequence of Fourth Order Under Nine Specific Schemes

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Received 2022 April 02**; Revised** 2022 May 20**; Accepted** 2022 June 18**.**

Abstract
K.T. Atanassov are Firstly established the Coupled Fibonacci Sequence in 1985. In 1987, The essence of Fibonacci Triple Sequences are examined. Fibonacci Sequence stand out as a kind of super sequence with amazing properties. This is the meteoric expansion in the province of Fibonacci Sequence. Leonardo de Pisa foremost Fibonacci's observation on the growth of the rabbit population as a result in 1202.
Triple Fibonacci Sequence are hype in the last years, but Multiplicative Triple Sequence of Recurrence Relations are less known. Extravagant work has been done to course on Fibonacci Triple Sequence in Additive form. In 1995, Multiplicative Coupled Fibonacci Sequence are treated. Our wish of this paper to offer some results of Multiplicative Triple Fibonacci Sequence of fourth order under nine specific schemes.
Keywords- Fibonacci Sequence, Multiplicative Triple Fibonacci sequence

1. **Introduction**

The Fibonacci Triple Sequence is a current guidance in universality of Coupled Fibonacci sequence. Fibonacci sequence and their abstract principle have umpteen tempting utilization and properties to every field of science. The best motive for this relevance is Koshy's book [9]. The Coupled Fibonacci Sequence was first installed by K.T. Atanassov [4] and also investigated many inquisitive properties and a modern protocol of generalization of Fibonacci Sequence [2,5,6].
J.Z. Lee and J.S. Lee ratified Firstly Additive Triple Sequence [3]. K.T. Atanassov lay out new notion for Additive Triple Fibonacci Sequence [7,8] and called 3-Fibonacci Sequence or 3-F Sequence.

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in the International Conference on "Recent Advances in Fundamental and Applied Sciences" (RAFAS 2021) held on June 25-26, 2021, organized by School of Chemical Engineering and Physical Sciences, Lovely Faculty of Technology and Sciences, Lovely Professional University, Punjab.

Date of Issue : 15-07-2021
Place of Issue: Phagwara (India)

Prepared by
(Administrative Officer-Records)

Organizing Secretary
(RAFAS 2021)

Convener
(RAFAS 2021

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