INERTIA OF SOME SPECIAL MATRICES AND A NOVEL MATRIX TRANSFORMATION

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in

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By

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Declaration of Authorship

I **HIMANSHU AGARWAL,** declare that this thesis entitled, **INERTIA OF SOME SPECIAL MATRICES AND A NOVEL MATRIX TRANSFORMATION** and the work presented in it are my own. I confirm that:

- This work was done wholly while in candidature for a research degree at this University.
- Where any part of this project has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
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Certificate From Supervisor

This is to certify that Mr. **HIMANSHU AGARWAL,** has completed the thesis entitled **INERTIA OF SOME SPECIAL MATRICES AND A NOVEL MATRIX TRANSFORMATION** under my guidance and supervision. To the best of my knowledge, the present work is the result of his original investigation and study. No part of this thesis has ever been submitted for any other degree at any university and elsewhere.

The thesis is fit for the submission and the partial fulfilment of the conditions for the award of **DOCTOR OF PHILOSOPHY,** in Mathematics.

Signed: Isha Garg.

Supervisor: Dr. ISHA GARG

Date: February 2025

Abstract

The current thesis, entitled **INERTIA OF SOME SPECIAL MATRICES AND A NOVEL MATRIX TRANSFORMATION** is the result of research outcomes conducted by me under the esteemed guidance and supervision of **Dr. ISHA GARG**, Associate Professor, Department of Mathematics, Lovely Professional University, Phagwara-Punjab. The research work is now being submitted to the Department of Mathematics, School of Chemical Engineering and Physical Sciences, Lovely Professional University, Phagwara-144411, Punjab, India, for the award of a Doctor of Philosophy in Mathematics. This thesis presents a comprehensive analysis of functions that satisfy the conditions of monotonicity and convexity on specific domain(sets), focusing on operator monotone and operator convex functions within matrix theory. The study synthesizes data from various research publications, highlighting key results and applications of these functions and their extensions. Particular attention is given to the spectral behavior of pattern-based matrices, such as $P_r = [(p_i + p_j)^r]$ and $B_r = [|p_i - p_j|^r]$, previously explored by researchers like Bhatia and Jain (2015) and Dyn, Goodman, and Micchelli (1986). We extend this analysis to matrices defined by nonlinear operator concave functions, demonstrating that the matrix $[f(1 - p_i p_j)]$ is conditionally negative definite, nonsingular, and has inertia $(1, 0, n - 1)$ when f is a nonlinear operator concave function. The study also examines cases where this result does not hold for linear functions or specific instances of $f(t) = \log t$. Furthermore, this thesis introduces a novel matrix operation, termed the "Trans-flip," and compares its properties to the well-known transpose and conjugate operations. The flip operation is analyzed in terms of its effects on determinant, trace, and inertia, particularly in pattern-based matrices. The study explores both theoretical and practical implications of this operation, with potential applications in fields such as computer graphics, image processing, and data manipulation. This work provides valuable insights into operator convex and concave functions, advancing their theoretical foundations and offering new directions for future research in matrix transformations and their applications.

In Chapter 1, we cover the foundational definitions, essential results, and an in-depth literature review. This chapter establishes the theoretical framework, critically analyzing existing research to set the stage for the study's objectives and contributions within the broader academic discourse.

In Chapter 2, we analyze the spectral behavior of pattern-based matrices such as $P_r = [(p_i + p_j)^r]$ and $B_r = [|p_i - p_j|^r]$, as studied by Bhatia, Jain, Dyn, Goodman, and others. We extend their work to matrices involving nonlinear operator concave functions, examining properties like conditional negative definiteness, non-singularity, and inertia. Additionally, we explore the impact of specific functions, such as $f(t) = \log t$, and provide examples of concave functions that are not operator concave.

In Chapter 3, we will delve deeper into the properties of concave functions and their limitations with respect to operator concavity. Specifically, we will explore the criteria that distinguish operator concave functions from classical concave functions and examine concrete examples of functions that meet the conditions for concavity but not for operator concavity. Our goal is to bridge the gap in the existing literature and provide a clearer understanding of how these functions behave in the context of operator theory, particularly through the use of matrix inertia.

In Chapter 4, we will examine the implications of the trans-flip operation on matrices and its relationships with established matrix transformations such as the classical transpose and conjugate transpose. We will analyze the decomposition of square matrices into trans-flip symmetric and trans-flip skew-symmetric components, highlighting how this new operation affects key matrix properties, including determinant, trace, and inertia. By comparing the structural characteristics and positivity of trans-flip matrices with those of traditional transformations, we aim to elucidate the significance of this novel operation in matrix theory.

In Chapter 5, we will delve into the foundational definitions and properties of horizontal and vertical flips in matrix transformations, establishing a clear framework for understanding these operations. We will explore the interrelations between horizontal-flip, vertical-flip, and the newly introduced trans-flip, highlighting their mathematical formulations and practical significance. This analysis aims to clarify how these transformations contribute to various applications in fields such as image processing and algorithm design, underscoring their importance in linear algebra.

A bibliography is included at the end of the thesis, which is by no means comprehensive, but does identify all of the research articles and books that were mentioned in the main text.

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First of all I bow in reverence to almighty "**GOD**" the cherisher and sustainer to benediction give me the required zeal for completion of my research work. The successful completion of any research work is a matter of great endeavourance but, because of the personal and practical support of numerous people makes the job entirely enjoyable and simple. It is a pleasant aspect that I have now the opportunity to express my gratitude for them.

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September 2025 **HIMANSHU AGARWAL**

Contents

List of Symbols and Abbreviations

$\mathbb{R}$- Set of real numbers.

$\mathbb{R}^{\mathbf{n}}$- n-tuples of real numbers.

$\mathbb{C}$- Set of complex numbers.

$\mathbb{M}_{\mathbf{k}\times\mathbf{k}}(\mathbb{C})$- Set of matrices of order k, entries from complex field.

v, u, w, z defines the vectors.

A, B, R, S, C capital cases defines matrices.

I Identity matrix.

0 Zero matrix or zero vector (context-dependent).

e_i Standard basis vectors.

a,b,c Scalars.

α, β, γ Greek letters often used for scalars.

$\mathscr{H}$ Hilbert space with dimension k

$\mathscr{B}(\mathscr{H})$ Space of bounded linear operator on $\mathscr{H}$

$\subseteq$ Subset

$\langle ., . \rangle$ Inner product

$In(R)$ Inertia of R

$Tr(R)$ trace of R

ω, μ eigen values of R

$det(R)$ determinant of R

R^T transpose of R

R^* transpose conjugate of R

U unitary matrix

E matrix with all elements 1

ζ number of positive eigen values

$\mathscr{PD}$ Positive definite

$\mathscr{ND}$ Negative definite

$\mathscr{PSD}$ Positive semi-definite

$\mathscr{NSD}$ Negative semi-definite

$\mathscr{CPD}$ Conditionally positive definite

$\mathscr{CND}$ Conditionally negative definite

Chapter 1

Introduction

1.1 Introduction

This chapter establishes essential notations, defines fundamental concepts, and presents key results that are frequently utilized throughout the thesis. The objective is to provide a strong foundation necessary for understanding the subsequent chapters.

The discussion begins with definitions, properties, and illustrative examples that clarify fundamental principles. A particular emphasis is placed on functions defined over the domain of positive real numbers, which, under specific conditions, can be extended to the set of matrices. In analogy with classical notions of concavity, convexity, and monotonicity in real analysis, their matrix counterparts operator-concave, operator-convex, and operator-monotone functions are explored. These functions have been extensively studied due to their unique properties and theoretical significance. This thesis aims to further contribute to this area by investigating additional results and extending the current understanding of these functions.

Furthermore, essential theorems and their proofs are provided to support the development of the main arguments in later chapters. To ensure a comprehensive discussion, fundamental definitions and properties have been compiled from standard references, including Bhatia [40], [52], Hardy, and Horn and Johnson [37], [63].

This chapter serves as a prerequisite for a deeper exploration of operator functions and their properties, ensuring that readers are well-equipped with the necessary theoretical background to engage with the subsequent chapters.

1.2 Notations

We denote the Hilbert space by $\mathcal{H}$. $\mathbb{R}$ and $\mathbb{C}$ represents the fields of a real and complex numbers respectively. Set of all $m \times k$ matrices defined over a complex field $\mathbb{C}$ is symbolized by $\mathbb{M}_{m\times k}(\mathbb{C})$. For a matrix $R \in \mathbb{M}_{k\times k}(\mathbb{C})$, its eigenvalues are denoted as $\omega_1, \omega_2, \ldots, \omega_k$. The trace and determinant of the matrix $R \in \mathbb{M}_{k\times k}(\mathbb{C})$ are given by

$$\mathrm{Tr}(R) = \sum_{i=1}^{k} \omega_i \quad \text{and} \quad \det(R) = \omega_1\omega_2\cdots\omega_k,$$

respectively.

Consider the matrix $R = [r_{ij}] \in \mathbb{M}_{k\times k}(\mathbb{C})$. Let $R^T = [r_{ji}]$ represent the transpose [46] and $R^* = [\bar{r}_{ji}]$ denote the conjugate transpose of R, for every i, j specified as $1 \leq i, j \leq n$.

Let $D = \mathrm{diag}(d_1, d_2, \ldots, d_k)$ be a diagonal matrix, where $d_1, d_2, \ldots, d_k$ are distinct elements belonging to the complex field. U denotes a unitary matrix, and E is a matrix with all elements equal to 1. e_i represents the i^{th} column, where only the i^{th} element is 1, and all other elements are 0.

1.3 Basic Definitions

1.3.1 Symmetric and Skew-Symmetric Matrix

A matrix $R = [r_{ij}] \in \mathbb{M}_{k\times k}(\mathbb{C})$ is termed as symmetric matrix if $R = R^T$, which means that $r_{ij} = r_{ji}$ for every $i, j \in \{1, 2, \cdots, k\}$.

Similarly, if $R^T = -R$, i.e., $r_{ji} = -r_{ij}$ for each and every $i,\ j \in \{1, 2, \cdots, k\}$, then the given matrix R is termed as a skew-symmetric matrix.

1.3.2 Self-Adjoint Matrix

Consider a matrix R, which is an element in $\mathbb{M}_{k\times k}(\mathbb{C})$. The matrix R is termed to be self-adjoint if $R^* = R$, indicating that each entry r_{ij} equals the complex conjugate of r_{ji}. The set of all self-adjoint operators on the Hilbert space $\mathcal{H}$, as discussed in [15], is denoted by $B(\mathcal{H}^*)$. In most cases, the study of self-adjoint matrices is of significant interest due to the fact that their eigenvalues are guaranteed to be real. For a self-adjoint operator R, the inertia of R is defined as,

$$\mathrm{In}(R) \;=\; (\zeta(R), \gamma(R), \vartheta(R)),$$

where $\zeta(R)$, $\gamma(R)$, and $\vartheta(R)$ represent the counts of positive, zero, and negative eigenvalues of R, respectively.

1.3.3 Positive (Negative) Definite Matrix

Consider a matrix $R \in \mathbb{M}_{k\times k}(\mathbb{C})$ is termed as positive (negative) definite [14, 49] if it is self-adjoint and fulfills any of the following synonymous conditions:

1. $\langle Rx,\ x\rangle > 0$ (< 0) for each non-zero $x \in \mathbb{C}^k$.
2. All minors corresponding to principal diagonal of R are positive (negative).
3. All eigenvalues of R are positive (negative).
4. $R = S^*S$ for some invertible (non-singular) matrix S.
5. $R = L^*L$ for some non-singular upper triangular matrix L.

The collection of all positive definite matrices of size k is represented by $\mathscr{P}_k$. We denote the positive definite and negative definite matrices by $\mathscr{P}\mathscr{D}$ and $\mathscr{N}\mathscr{D}$ respectively.

1.3.4 Positive (Negative) Semi-definite matrix

A matrix $R \in \mathbb{M}_{k\times k}(\mathbb{C})$ is called positive (negative) semi-definite [42] if it is self-adjoint and meets any of the following analogous conditions:

1. $\langle Rx,\ x\rangle \geq 0$ (≤ 0) for every $x \in \mathbb{C}^k$.
2. All minors corresponding to principal diagonal of R are non-negative (non-positive).
3. Every eigenvalues of the matrix R are non-negative (non-positive).
4. $R = S^*S$ for some matrix S.
5. $R = L^*L$ for some upper triangular matrix L.

The collection of all positive semi-definite matrices of size k is denoted by $\mathscr{S}_n$. We represent the positive semi-definite and negative semi-definite matrices by $\mathscr{P}\mathscr{S}\mathscr{D}$ and $\mathscr{N}\mathscr{S}\mathscr{D}$ respectively. Several criteria are available in the literature to identify positive semi-definite matrices, some of which are listed below in the form of proposition.

Proposition 1.3.1. *Let $R \in \mathbb{M}_{k\times k}(\mathbb{C})$. Then R is classified as $\mathscr{PSD}$ if and only if R satisfies the condition $\langle Rx,\ x\rangle \geq 0$ for each $x \in \mathbb{C}^k$.*

Proof. Let us consider R is $\mathscr{PSD}$ matrix, $\{x_1, x_2, \ldots, x_k\}$ as an ortho-normal basis for $\mathbb{C}^k$, composed of the eigenvectors of matrix R, and assume ω_j denote the eigenvalues associated with each eigenvector x_j, for $1 \leq j \leq k$. For every $x \in \mathbb{C}^k$, we can express x as

$$x = \sum_{j=1}^{k} a_j x_j, \quad a_j \in \mathbb{C}.$$

Thus, it follows that,

$$\langle Rx,\ x\rangle = \left\langle \sum_{j=1}^{k} a_j R x_j,\ \sum_{j=1}^{k} a_j x_j \right\rangle = \left\langle \sum_{j=1}^{k} a_j \omega_j x_j,\ \sum_{j=1}^{k} a_j x_j \right\rangle.$$

Using orthogonalisation,

$$\langle Rx,\ x\rangle = \sum_{j=1}^{k} |a_j|^2 \omega_j \|x_j\|^2 \geq 0,$$

because $\omega_j \geq 0$ for each j.

Conversely, Assume that ω is an eigenvalue of R, related to the eigenvector x. Then it is known that,

$$\omega\|x\|^2 = \omega\langle x,\ x\rangle = \langle Rx,\ x\rangle \geq 0,$$

resulting in the conclusion that $\omega \geq 0$. This concludes the proof. □

Proposition 1.3.2. *Suppose $R \in \mathbb{M}_{k\times k}(\mathbb{C})$ be a positive semi-definite matrix of order k. Then every element on the diagonal of R is a non-negative real number.*

Proof. Given that $R = [r_{ij}] \in \mathscr{S}_n$. Then $r_{ii} = \langle Re_i,\ e_i\rangle \geq 0$ for all $i = 1, 2, \ldots, k$, where e_i in $\mathbb{R}^k$, defined as the column vector whose i^{th} entry is 1, while all other entries are 0. To discuss the next characterization, we need to define the principal minor of the matrix $R \in \mathbb{M}_{k\times k}(\mathbb{C})$. For an index set $\gamma \subseteq \{1, 2, \ldots, k\}$, the principal sub-matrix of R is represented by $R[\gamma]$ and is defined as the matrix constructed using the elements from the rows and columns of R that are indexed by γ. The determinant of the principal sub-matrix is referred to as the principal minor. For a vector $x \in \mathbb{C}^k$, $x[\gamma]$ denotes the vector derived from x by removing the entries complementary to γ, and $x[\gamma^c]$ represents the vector derived from x by removing the entries indexed by γ. □

Proposition 1.3.3. *Let $R \in \mathbb{M}_{k\times k}(\mathbb{C})$. The matrix R is classified as $\mathscr{PSD}$ if and only if all the principal minors of the matrix R are non-negative.*

Proof. Let $R \in \mathbb{M}_{k\times k}(\mathbb{C})$ is a $\mathscr{PSD}$ matrix, γ is a proper subset of $\{1, 2, \ldots, k\}$ and $R[\gamma]$ be the principal sub-matrix. Assume $z \in \mathbb{C}^k$ is a vector such that $z[\gamma] \neq 0$ and $z[\gamma^c] = 0$, where γ^c represents the complement of γ. Then,

$$z[\gamma]^* R[\gamma] z[\gamma] \;=\; z^* R z \geq 0.$$

Since the non-zero vector $z[\gamma]$ can be selected freely, which implies that $R[\gamma]$ is positive semi-definite. The converse part is trivially true. □

Proposition 1.3.4. *Let $R \in \mathbb{M}_{k\times k}(\mathbb{C})$. Then R is a $\mathscr{PSD}$ if and only if there exists a matrix $S \in \mathbb{M}_{k\times k}(\mathbb{C})$ along with the condition that $R = S^*S$. Additionally, R is a $\mathscr{PD}$ matrix if and only if S is an invertible matrix.*

Proof. Let $R \in \mathscr{S}_n$ is a $\mathscr{PSD}$ matrix. The spectral decomposition of the matrix R can be formulated as $R = UDU^*$. Here, U represents an unitary matrix, and $D = \operatorname{diag}(\omega_1, \omega_2, \ldots, \omega_k)$ be a diagonal matrix eigenvalues of R as the diagonal entries. Since R is a $\mathscr{PSD}$ matrix, this implies that $\omega_i \geq 0$, for each and every i is the member of the set $\{1, 2, \ldots, k\}$. States that $S = UPU^*$, where $P \;=\; \operatorname{diag}(\sqrt{\omega_1}, \sqrt{\omega_2}, \ldots, \sqrt{\omega_k})$. Hence, $R = S^*S$.

Conversely, assume that $R = S^*S$ for some matrix S. Let $x \in \mathbb{C}^k$. Then,

$$\langle Rx,\; x\rangle \;=\; \langle S^*Sx,\; x\rangle \;=\; \langle Sx,\; Sx\rangle \;=\|Sx\|^2 \geq 0.$$

This establishes that matrix R is a positive semi-definite.

Furthermore, the matrix R is a positive definite if and only if it is both $\mathscr{PSD}$ and non-singular. Since R is non-singular such that $R = S^*S$ also $det(R) \neq 0$. So, S is invertible. Hence, R is $\mathscr{PD}$ if and only if $S \in \mathbb{M}_{k\times k}(\mathbb{C})$ is invertible.

□

1.3.5 Conditionally Positive (Negative) Definite Matrix

A self-adjoint matrix $R \in \mathbb{M}_{k\times k}(\mathbb{C})$ is referred to as conditionally positive (negative) definite [30] if

$$\langle Rx,\; x\rangle \geq 0 \;(\leq 0)$$

for each vector $x \in H_1$, where

$$H_1 \;=\; \left\{ x \;=\; (x_1, x_2, \ldots, x_k) \in \mathbb{C}^k \text{such that} \sum_{i=1}^{k} x_i \;=\; 0 \right\}$$

be an $(k-1)$-dimensional subspace of $\mathbb{C}^k$.

We represent the conditionally positive definite matrix and conditionally negative definite matrix by $\mathscr{CPD}$ and $\mathscr{CND}$ respectively.

Example 1 Consider the matrix $R \in \mathbb{M}_{k\times k}(\mathbb{C})$.

$$R = \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}.$$

For $X \in H_1,\ X = (x, -x)$. Then $\langle RX,\ X\rangle\ = 3x^2 > 0$. Hence, R is $\mathscr{CPD}$ matrix with positive eigenvalues $\omega_1 = 1.4384$ and $\omega_2 = 5.5616$.

Example 2 Consider S be a matrix member of $\mathbb{M}_{k\times k}(\mathbb{C})$. Then, S is defined as

$$S = \begin{bmatrix} -2 & 3 \\ 3 & -2 \end{bmatrix}.$$

For $X \in H_1$, we have $X = (x, -x)$. Then $\langle SX,\ X\rangle\ = -10x^2 < 0$. Hence, S is $\mathscr{CND}$, with one eigenvalue positive, i.e. $\omega_1 = 1$, and the other negative, i.e. $\omega_2 = -5$. For $\mathscr{CPD}$ and $\mathscr{CND}$ matrices, the following well-known results establish the relationship among the determinant, eigenvalues, and invertibility.

Proposition 1.3.5. *Every matrix that is $\mathscr{PD}$ ($\mathscr{ND}$) matrix is also $\mathscr{CPD}$ ($\mathscr{CND}$) matrix. However, the reverse implication does not always hold.*

Proof. Let $R \in \mathscr{P}_n$ be a positive definite ($\mathscr{PD}$) matrix. Then for all nonzero vectors z,

$$\langle Rz,\ z\rangle = z^\top Rz > 0. \tag{1.1}$$

Since a conditionally positive definite ($\mathscr{CPD}$) matrix requires $z^\top Rz > 0$ only for those z such that $\sum_{i=1}^{k} z_i = 0$, this follows immediately from the $\mathscr{PD}$ case. Thus, every $\mathscr{PD}$ matrix is also a $\mathscr{CPD}$ matrix.

Next we see that the reverse implication does not necessarily hold. Take into account the symmetric matrix $R \in \mathbb{M}_{3\times 3}(\mathbb{C})$ given by

$$R = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}. \tag{1.2}$$

Step 1: Representation of R.

Note that R can be expressed in terms of the identity matrix I_3 and the all-ones matrix J_3 as

$$R \;=\; I_3 - \tfrac{2}{3}J_3,$$

where J_3 is the 3×3 all-ones matrix.

1) Conditionally positive definite

For any $x \in \mathbb{R}^3$,

$$z^\top Rz = z^\top Iz - \tfrac{2}{3}z^\top Jz = \|z\|^2 - \tfrac{2}{3}\left(\sum_{i=1}^{3} z_i\right)^2.$$

If $\sum_{i=1}^3 z_i = 0$, then

$$z^\top Rz = \|z\|^2 \;\geq\; 0,$$

and it is strictly positive for all nonzero z with zero sum. Hence R is *conditionally positive definite* on the subspace $\{z \in \mathbb{R}^3 : \sum_i z_i = 0\}$.

2) Not positive definite

For any nonzero vector $z = (z_1, z_2, z_3)^\top \in \mathbb{C}^3$, we have

$$z^\top Rz \;=\; z_1^2 + z_2^2 + z_3^2 \;-\; \tfrac{2}{3}\left(z_1 + z_2 + z_3\right)^2. \tag{1.3}$$

However, for the particular nonzero vector $z = (1, 1, 1)^\top$,

$$z^\top Rz \;=\; 3 - \tfrac{2}{3} \cdot 9 \;=\; -3 < 0. \tag{1.4}$$

Since there exists $z \neq 0$ such that $z^\top Rz < 0$, the matrix R is not $\mathscr{PD}$.

Thus, this example demonstrates that a $\mathscr{CPD}$ matrix is not necessarily a $\mathscr{PD}$ matrix. $\square$

Proposition 1.3.6. *A $\mathscr{CND}$ ($\mathscr{CPD}$) matrix has at most one positive (negative) eigenvalue, counting multiplicities.*

Proof. Consider a matrix $R \in \mathbb{M}_{k\times k}(\mathbb{C})$ that is $\mathscr{CND}$. By definition, we have

$$\langle Rx,\ x\rangle \leq 0 \quad \text{for every } x \in \mathcal{H}.$$

From this, it can be inferred that,

$$\max_{\substack{x \in \mathcal{H} \\ \|x\| \ = \ 1}} \langle Rx,\ x\rangle \leq 0.$$

This inevitably brings us to the conclusion that,

$$\min_{M \subset \mathbb{R}^n} \max_{\substack{x \in M \\ \|x\| \ = \ 1}} \langle Rx,\ x\rangle \leq 0,$$

where dimension of M is $k-1$.

According to the Min-Max principle, we derive that $\omega_2(R) \leq 0$, indicating there can be at most one positive eigenvalue of R.

To derive a similar conclusion for a $\mathscr{CPD}$ matrix, we can substitute R with $-R$ and using the expression

$$-\omega_i(R) \ = \ \omega_{n-i+1}(-R) \quad \text{for each } 1 \leq i \leq k.$$

Thus, it is established that a $\mathscr{CPD}$ matrix can have at most one negative eigenvalue.

□

Proposition 1.3.7. *Let $R \in \mathbb{M}_{k\times k}(\mathbb{C})$ be a $\mathscr{CPD}$ or($\mathscr{CND}$) matrix but not $\mathscr{PSD}$ or($\mathscr{NSD}$). If $\langle Ax, x\rangle \neq 0$ for all nonzero $x \in \mathscr{H}$, then A is invertible. However, the converse does not necessarily hold.*

Proof. Suppose, with the intention of deriving a contradiction, that $R \in \mathbb{M}_{k\times k}(\mathbb{C})$ is singular. Thus, It follows that there exists a nontrivial vector $x \in \mathbb{C}^k$ for which

$$Rx \ = \ 0. \tag{2.1}$$

Consequently, we also have the expression

$$\langle Rx,\ x\rangle \ = \ \langle 0,\ x\rangle = 0. \tag{2.2}$$

As $\langle Rx, x\rangle \neq 0$ for every nonzero $x \in \mathscr{H}$, this results in a contradiction with (2.2). Therefore, it can be inferred that $x \notin \mathcal{H}$ and R cannot be $\mathscr{PSD}$. Hence, there exists at least one negative eigenvalue ω of R. This suggests that there must exist a non-zero vector $y \in \mathbb{C}^k$ such that

$$Ry \ = \ \omega y.$$

Consequently, we can write

$$\langle Ry,\ y\rangle \ = \ \langle \omega y,\ y\rangle \ = \ \omega y^* y < 0. \tag{2.3}$$

Next, we observe that $y \notin \mathcal{H}$. If we assume $y \in \mathcal{H}$, then since R is $\mathscr{CPD}$, we would have

$$\langle Ry,\ y\rangle \geq 0, \tag{2.4}$$

which contradicts (2.4). Therefore, it must be that $y \notin \mathcal{H}$, leading to $\sum_{i=1}^{k} y_i \neq 0$. Let $x = \sum_{i=1}^{k} y_i$ and $\frac{\sum_{i=1}^{k} x_i}{\sum_{i=1}^{k} y_i}$. We note that $z = x - py \in \mathcal{H}$ and since R is $\mathscr{CPD}$, it follows that

$$\langle Rz,\ z\rangle \geq 0. \tag{2.5}$$

Using (2.1), we compute

$$\begin{aligned}
\langle Rz,\ z\rangle \ &= \langle R(x-py),\ (x-py)\rangle \\
&= \langle Rx,\ x\rangle - p\langle Ry,\ x\rangle - p\langle Rx,\ y\rangle + |p|^2\langle Ry,\ y\rangle \\
&= -p\langle Ry,\ x\rangle + |p|^2\langle Ry,\ y\rangle \\
&= -p\langle y,\ Rx\rangle + |p|^2\langle Ry,\ y\rangle \\
&= |p|^2\langle Ry,\ y\rangle \\
&= |p|^2\omega y^* y < 0,
\end{aligned}$$

this contradicts equation (2.5). Hence, R must be invertible. To demonstrate that the converse does not hold, consider the matrix $R \in \mathbb{M}_{k\times k}(\mathbb{C})$ represented as,

$$R \ = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

This matrix is invertible and can be classified as $\mathscr{CPD}$ as well as $\mathscr{CND}$. Nevertheless, R is neither a $\mathscr{NSD}$ matrix nor a $\mathscr{PSD}$ matrix. Moreover, we can observe that

$$\langle Rx, x\rangle = 0 \text{ for every nonzero } x \in \mathcal{H}.$$

□

1.4 Hadamard Operations

The Hadamard operation refers to key operations on matrices, namely the Hadamard product and Hadamard power. These operations are fundamental in matrix analysis and have widespread applications [33–35] in several areas of mathematics.

1.4.1 Hadamard Product

The Hadamard product, or element-wise product, is an operation applied to two matrices of equal dimensions. Let $R = [r_{ij}]$ and $S = [s_{ij}]$ be matrices of size $m \times k$. The Hadamard product, denoted by $R \circ S$, is defined as $R \circ S = [r_{ij}s_{ij}]$, for each and every i belongs to $\{1, 2, \ldots, m\}$ and for each $j \in \{1, 2, \ldots, k\}$.

In simpler terms, the Hadamard product of matrices R and S involves multiplying their corresponding elements. It is important to distinguish the Hadamard product from matrix multiplication, as it operates element by element rather than through the dot product. This operation exhibits both commutative and associative properties. Specifically, for matrices R, S, and C of the same size, the following hold true,

$R \circ S = S \circ R$ and $R \circ S) \circ C = R \circ (S \circ C)$. The Hadamard product has important applications [32, 43] in fields such as optimization, deep learning (e.g., element-wise activation functions), and signal processing, where operations on matrices are frequently performed element by element.

1.4.2 Hadamard Power

The Hadamard power is a variation of the Hadamard product. In this context, every entry of the given matrix is exponentiated to a predetermined power. For a matrix $R = [r_{ij}]$, its Hadamard power [69] for a real number p is defined as,

$R^{\circ p} = [r_{ij}^{p}]$, for all i belongs to the set $\{1,\ 2,\ \ldots, m\}$ and for each j belongs to the set $\{1,\ 2,\ \ldots,\ k\}$.

Through this operation, every element of the matrix is raised to the power of p individually, operating element-wise across the matrix. The Hadamard power is particularly valuable in machine learning and data science, where it is used for non-linear transformations of data, and in studies involving matrix inequalities.

The Hadamard power possesses several important properties, few are listed as:

1. Let $R,\ S\ \mathbb{M}_{k\times k}(\mathbb{C})$. Then we have,

$$(R \circ S)^{\circ p} = R^{\circ} p \circ S^{\circ p}, \quad \text{for any real number } p.$$

2. If the matrices R and S are diagonal matrix of order k then it follows,

$$(R \circ S)^{p} = R^{p} \circ S^{p}, \quad \text{For any } \ p \in \ \mathbb{R}$$

When R and S are not diagonal matrix, this property may not necessarily be valid. Consider the matrices:

$$R = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

For real number $p = 2$, calculating the left hand side of equation ,

$$(R \circ S)^{\circ 2} = \begin{bmatrix} 17 & 68 \\ 68 & 272 \end{bmatrix}, \text{ and other side is, } \quad R^2 \circ S^2 = \begin{bmatrix} 25 & 100 \\ 100 & 400 \end{bmatrix}.$$

We observe that,

$$(R \circ S)^{\circ 2} \neq R^2 \circ S^2.$$

We reach to a conclusion that the property remains valid only when R and S are diagonal matrices.

This operation offers an efficient way to apply element-wise exponentiation on matrices, which is a crucial step in numerous computational algorithms. In 1911, Schur proved a fundamental theorem about the Hadamard product of matrices [11], proving that the Hadamard multiplication (or product) of two or more $\mathscr{PSD}$ matrices of the same dimension preserves positive properties. In the following theorem $\mathscr{N}_k$ denotes the set of symmetric matrices of order k.

Theorem 1.4.1. *Let $R,\ S \in \mathscr{N}_k$. Then the Hadamard product $R \circ S$ belongs to $\mathscr{N}_k$. If $R,\ S \in \mathscr{S}_k$, then $R \circ S$ is also in $\mathscr{S}_k$.*

Proof. Let $R,\ S \in \mathscr{N}_k$, $R = R^T$ and $S = S^T$. The Hadamard product $R \circ S$ is defined as the element-wise product of the matrices R and S, i.e.,

$$(R \circ S) = [r_{ij}s_{ij}], \quad for\ each\ i, j \in \{1,\ 2,\ \ldots,\ k\}.$$

Since R and S are symmetric matrices, we have $r_{ij} = r_{ji}$ and $s_{ij} = s_{ji}$ for all $i,\ j$. Therefore, the Hadamard product $R \circ S$ satisfies

$$(R \circ S) = [r_{ij}s_{ij}] = [r_{ji}s_{ji}] = (R \circ S),$$

which implies that $R \circ S$ is also symmetric, i.e., $R \circ S \in \mathscr{N}_k$.

Next, assume $R,\ S \in \mathscr{S}_k$, where $\mathscr{S}_k$ denotes the set of $\mathscr{PSD}$ of order k, i.e., R and S are symmetric and all their eigenvalues are non-negative. Since $R,\ S \in \mathscr{S}_k$, we have

$$x^T Rx \geq 0 \quad \text{and} \quad x^T Sx \geq 0 \quad \text{for each and every } x \in \mathbb{C}^k.$$

Now for the Hadamard product $R \circ S$, we need to show that

$$x^T (R \circ S)x \geq 0 \quad \text{for each and every } x \in \mathbb{C}^k.$$

The expression of quadratic form for the Hadamard product [38] is given by

$$x^T (R \circ S)x \;=\; \sum_{i,j=1}^{k} x_i x_j r_{ij} s_{ij}.$$

This expression can be viewed as a sum of products of corresponding elements of R and S weighted by the components of x. Since both R and S are $\mathscr{PSD}$, it follows from the properties of the Hadamard product that $R \circ S$ retains the non-negativity property of the quadratic form,

$$x^T (R \circ S)x \geq 0 \quad \text{for every } x \in \mathbb{C}^k,$$

this leads to the conclusion that $R \circ S \in \mathscr{S}_n$.

Thus, we have shown that if $R, S \in \mathscr{N}_k$, then $R \circ S \in \mathscr{N}_k$, and if $R,\ S \in \mathscr{S}_k$, then $R \circ S \in \mathscr{S}_k$, as required. □

1.5 Spectral Decomposition of Matrices

Spectral decomposition[39] of a matrix consists of representing the matrix using its eigenvalues and their associated eigenvectors. For a self-adjoint matrix $R \in \mathbb{M}_{k\times k}(\mathbb{C})$, if R is diagonalizable, the spectral decomposition expresses R as,

$$R \;=\; Q\Lambda Q^T$$

where,

- Q is considered to be an orthogonal matrix and columns of Q represents the eigenvectors of R,

- $\Lambda = \text{diag}\{\omega_1, \omega_2, \ldots, \omega_k\}$ is a diagonal matrix whose diagonal entries are the eigenvalues of R.
- Q^T denotes the transpose of the matrix Q (in the case of real matrices, $Q^{-1} = Q^T$).

In this representation, R is reconstructed from its eigenvalues and eigenvectors, which provides insight into the structure of the matrix. The eigenvalues in Λ represent the scaling factors by which the corresponding eigenvectors (columns of Q) are stretched or compressed under the transformation represented by R.

For symmetric matrices R, the spectral decomposition simplifies further, as Q is an orthogonal matrix and the eigenvalues of R are real. In this case, the matrix R can be formulated as,

$$R = Q\Lambda Q^T$$

where Q^T and Q defines a relation as $Q^T = Q^{-1}$, and he columns of Q define an ortho-normal set of eigenvectors, which correspond to the eigenvalues contained in the matrix Λ, forming a complete basis for the vector space.

The spectral decomposition plays a fundamental role in various applications, encompassing the process of determining solutions for a given system of linear equations, matrix exponentiation, and fields such as principal component analysis (PCA) and quantum mechanics.

1.6 Interpretation of Rank

The *rank* of a matrix $R \in \mathbb{M}_{m\times k}(\mathbb{C})$ is defined as the dimension of the subspace generated by its columns, which is equivalent to the dimension of the row space of the matrix R. It serves as a measure of the matrix's "non-degeneracy" by indicating the number of linearly independent columns or rows.

Formally, the rank of R, denoted as $\text{rank}(R)$, is expressed as

$$\text{rank}(R) = dimn(\text{Col}(R)) = dimn(\text{Row}(R)),$$

where $\text{Col}(R)$ and $\text{Row}(R)$ represents the column and row space of the matrix R, respectively.

The rank denotes the highest count of $\mathscr{L}\mathscr{I}$ columns or rows within the matrix, where $\mathscr{L}\mathscr{I}$ represents the linearly independent. In other words, it is the largest integer r such that R contains a subset of r $\mathscr{L}\mathscr{I}$ rows or columns.

Another way to determine the rank of a matrix is through its singular value decomposition (SVD) [36]. Specifically, if R is an $m \times k$ matrix, its rank corresponds to the count of nonzero singular values in the diagonal matrix Σ, when R is decomposed as:

$$R = U\Sigma V^T,$$

where U and V represents the orthogonal matrices, and Σ is a diagonal matrix with diagonal elements represents the singular values.

The rank of a matrix provides significant insights into its fundamental properties, including the solvability of linear systems, matrix invertibility, and the structure of its null space.

Theorem 1.6.1. *For a non-zero square matrix $R \in \mathbb{M}_{k\times k}(\mathbb{C})$, the set of conditions listed below are mutually equivalent,*

1. *R is a non-singular matrix.*
2. *R has full rank [48] .*
3. *R can be transformed into the identity matrix through row reduction.*
4. *R can be formulated as a multiplication of elementary matrices.*

Proof. We aim to show that the equivalence of the conditions in a series of logical implications.
First we prove that 1 implies 2

a) Suppose $R \in \mathbb{M}_{k\times k}(\mathbb{C})$ is a non-singular matrix, which means R^{-1} exists. If R does not have full rank, then, according to the rank-nullity theorem, there must be a nonzero vector in the null space of R., which leads to the conclusion that the rows or columns of the matrix R are $\mathscr{L}\mathscr{D}$. However, for R to be invertible, its rows and columns must form a set of $\mathscr{L}\mathscr{I}$ vectors. Hence, the assumption that R does not have full rank leads to a contradiction. Therefore, R must have full rank.
Now we prove that 2 implies 3

b) If R has full rank, then it has exactly k $\mathscr{L}\mathscr{I}$ rows and columns. From row reduction property, the row-reduced echelon form of matrix R, each row will contain exactly one non-zero element, with all other entries in the row being zero. Thus, the row-reduced form of R is the identity matrix.
Further we prove that 3 implies 4.

c) If R is row-equivalent to the identity matrix, it follows from the definition of row-equivalence that there exists a series of elementary matrices $E_1, E_2, \ldots, E_k$, it follows that

$$R = E_1 E_2 \cdots E_k I_k.$$

Hence, R can be outlined as the composition (product) of the elementary matrices. Finally, we prove that 4 implies 1.

d) Suppose $R = E_1 E_2 \cdots E_k$, where each E_i represents an elementary matrix. Since elementary matrices are always invertible, and the product of any number of such matrices remains an invertible matrix, it follows that R is invertible. Hence, R is non-singular.

Thus, the conditions are equivalent, and the proof is complete. □

Theorem 1.6.2 (Cauchy Interlacing Theorem)**.** *Let $R \in \mathbb{M}_{n\times n}(\mathbb{C})$ be defined as a Hermitian matrix of size k with eigenvalues satisfying*

$$\omega_1 \geq \omega_2 \geq \cdots \geq \omega_k$$

and let $S \in \mathbb{M}_{(k-1)\times(k-1)}(\mathbb{C})$ be a principal sub-matrix of R of order $(k-1)$ with eigenvalues

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{(k-1)}.$$

The eigenvalues of S are interlaced with the eigenvalues of R as follows,

$$\omega_1 \geq \mu_1 \geq \omega_2 \geq \mu_2 \geq \cdots \geq \mu_{(k-1)} \geq \omega_k.$$

Proof. To establish the interlacing property, we start by considering the spectral theorem, which states that every Hermitian matrix is diagonalizable by a unitary matrix. Thus, we can express R as

$$R = UDU^*,$$

where $D = \operatorname{diag}(\omega_1, \omega_2, \ldots, \omega_k)$ contains the eigenvalues of R, and U represents a unitary matrix.

Let S be the principal sub-matrix of R formed by eliminating the last row and last column of R. Thus, we can represent S as

$$S = U_{k-1} D_{k-1} U_{k-1}^{*},$$

where $D_{k-1} = \mathrm{d}\,iag\{\omega_1, \omega_2, ..., \omega_{k-1}\}$ contains the first $(k-1)$ eigenvalues of R.

Now, it is known that for any vector x in the space $\mathbb{C}^k$, we can decompose x into two components, one corresponding to the principal sub-matrix S and the other to the deleted row and column.

For $y \in \mathbb{C}^{k-1}$, let

$$x = \begin{pmatrix} y \\ 0 \end{pmatrix},$$

where 0 represents the deleted component. We have

$$\langle Rx,\ x\rangle = \langle R \begin{pmatrix} y \\ 0 \end{pmatrix}, \begin{pmatrix} y \\ 0 \end{pmatrix}\rangle.$$

Expanding the terms, we get

$$\langle Rx,\ x\rangle = \langle Sy,\ y\rangle.$$

Now, examine the situation where x is an eigenvector associated with ω_i, leading to

$$\langle Rx,\ x\rangle = \omega_i \langle x,\ x\rangle.$$

As a result, we can conclude that,

$$\omega_i \|\ x\ \|^2 = \langle Rx,\ x\rangle \geq \mu_j \|\ y\ \|^2 \text{ for some } j.$$

This indicates that

$$\omega_i \geq \mu_j.$$

Next, let x be defined as an eigenvector corresponding to μ_j, yielding

$$\langle Sx,\ x\rangle \ = \ \mu_j \langle x,\ x\rangle \geq \omega_{(i+1)} \|y\|^2.$$

Consequently, we find that

$$\mu_j \geq \omega_{(i+1)}.$$

The interlacing results in

$$\omega_1 \geq \mu_1 \geq \omega_2 \geq \mu_2 \geq \cdots \geq \mu_{(k-1)} \geq \omega_k,$$

verifying the Cauchy Interlacing Theorem. □

Theorem 1.6.3. *[23][Spectral Properties of Positive Symmetric Matrices] Let $R = [r_{ij}]$ be a symmetric matrix of size k where all elements $r_{ij} > 0$. Let ω represent the largest eigenvalue of R. Then the subsequent properties are true*

(i) ω is positive, i.e., $\omega > 0$.

(ii) There exists a corresponding eigenvector (x_i) $1 \leq i \leq k$ where each component satisfies $x_i > 0$.

(iii) The eigenvalue ω is unique and its arithmetic multiplicity is 1.

(iv) For any other eigenvalue μ, the inequality $\omega > |\mu|$ holds.

Proof. We prove each part theoretically .

(i) Let $R \in \mathbb{M}_{k\times k}(\mathbb{C})$ be a matrix whose eigenvalues are all real, and the sum of its eigenvalues satisfies $\mathrm{Tr}(R) > 0$. By the Perron-Frobenius theorem [23], if R is a non-negative irreducible matrix, then it has a unique largest eigenvalue ω, which is positive. Consequently, we obtain that $\omega > 0$.

(ii) Let $v = (v_j)$ be a real, define eigenvector associated with ω, satisfying $Rv = \omega v$. By the Perron-Frobenius theorem for matrices with positive entries, the largest eigenvalue ω has an associated eigenvector with strictly positive entries. We normalize this eigenvector to obtain $x = (x_i)$, where $x_i > 0$ for all i.

(iii) To show that ω is a non-degenerate, assume to the contrary that there exists another $\mathscr{L}\mathscr{I}$ eigenvector corresponding to ω. However, this contradicts the symmetry and positive definite behaviour of R, which guarantees that the largest eigenvalue is unique in terms of an associated non-negative eigenvector. Therefore, ω is distinct and not repeated, i.e., non-degenerate.

(iv) Let μ represent any other eigenvalue of R. Since R is symmetric and positive, its eigenvalues are bounded above in absolute value by ω, and thus $\omega > |\mu|$. □

1.6.1 Matrices of Special Patterns

Matrices constructed by arranging natural numbers in specific patterns often exhibit unique and noteworthy mathematical properties. Examples of such matrices include the

Hilbert matrix, Harmonic mean matrix, Geometric mean matrix, Cauchy matrix [1], and Min matrices. These structured matrices [2] have been extensively studied due to their interesting algebraic and analytical characteristics.

Cauchy Matrix: Let $p_1, p_2, \ldots, p_k$ be positive real numbers. The matrix $R = [r_{ij}]$ is termed as Cauchy matrix if each of its entries is given by the form

$$r_{ij} = \frac{1}{p_i + p_j}.$$

Cauchy matrices have numerous important characteristics, one of which is that they are always $\mathscr{PSD}$.

Hilbert Matrix: Consider a matrix of dimension k, denoted by $R = [r_{ij}]$, is known as a Hilbert matrix if its entries are defined as

$$r_{ij} = \frac{1}{i + j}.$$

We can note that the Hilbert matrix represents a specific instance of a Cauchy matrix when $p_i = i$.

Harmonic Mean Matrix: An $k \times k$ matrix, denoted by $R = [r_{ij}]$, is referred to as a harmonic mean matrix if its elements are specified as

$$r_{ij} = \frac{ij}{i + j}.$$

Min Matrix: A matrix $R = [r_{ij}]$ is outline as min matrix if its entries are determined through,

$$r_{ij} = \min(i, j),$$

represented as $R = [\min(i, j)]$.

GCD Matrix: A matrix $R \in \mathbb{M}_{m \times k}(\mathbb{R})$, represented as $R = [r_{ij}]$ is termed a GCD matrix if every entry r_{ij} represents the Greatest Common Divisor with respect to i and j, expressed as

$$r_{ij} = GCD(i, j)_{i,j=1}^{k}.$$

It has been established through extensive research that the aforementioned matrices exhibit infinite divisibility [40]. These matrices play a significant role in various domains, often serving as benchmark test matrices in numerical analysis. Additionally, they find applications in multiple fields, including engineering and physical sciences, due to their distinctive structural properties and computational advantages.

1.7 Matrix Function via Spectral Decomposition

Let f be a function that takes real values, which is defined over an interval I contained within the real number line $\mathbb{R}$. This function can be extended to the class of matrices the set of whose eigenvalues are entirely contained within I.

Consider a self-adjoint matrix $R \in B(\mathcal{H}^*)$ with eigenvalues $\omega_1, \omega_2, \ldots, \omega_k$, so that for every $i \in \{1, 2, \ldots, k\}$, ω_i lies with in the interval I. The spectral decomposition of the matrix R is represented as,

$$R \;=\; U\,\mathrm{diag}(\omega_1, \omega_2, \ldots, \omega_k)\,U^*,$$

where U is defined to be an unitary matrix. Then, the function f applied to the matrix R is represented by

$$f(R) \;=\; U\,\mathrm{diag}(f(\omega_1), f(\omega_2), \ldots, f(\omega_k))\,U^*.$$

Now, We can define operator functions that are analogous to convex, concave, and monotone functions.

1.8 Operator Monotone Function

A function f that takes real values and is defined on an interval I is referred to as a matrix monotone function if, for any self-adjoint matrices R and $S \in B(\mathcal{H}^*)$ of order k, whose spectra (the set of all eigenvalues) lie within I, the condition

$$R \;\geq\; S \implies f(R) \;\geq\; f(S)$$

holds. Here, $R \;\geq\; S$ means that $R - S$ is a $\mathscr{PSD}$ matrix.

If a function is matrix monotone [21] for matrices of all orders k, then it is referred to as an **operator monotone**[24] function. Note that monotone functions can be either increasing or decreasing in the context of real-valued functions, whereas monotone functions for matrices only define increasing behavior.

It can be easily observed from given example, consider the function f defined from positive real line to real line, expressed as

$$f(t) \;=\; t^r, \quad \text{where} \quad 0 \;\leq\; r \leq\; 1.$$

This function is said to be operator monotone.

1.9 Operator Concave Function

A function $f : I \;\rightarrow\; \mathbb{R}$ is called a matrix concave of order k if, for any $R,\; S \;\in\; \mathcal{B}(\mathcal{H}^*)$ and for every $c \;\in\; [0,1]$, the following inequality holds

$$f(cR + (1-c)S) \geq cf(R) + (1-c)f(S).$$

If a function is matrix concave for matrices of all orders k, it is referred to as an **operator concave** function.

Example- The function f defined from positive real line to real line

$$f(t) \;=\; t^{1/2}$$

is an operator concave function on the interval $(0,1)$.

1.10 Operator Convex Functions

A function $f : I \;\rightarrow\; \mathbb{R}$ is called a matrix convex function if, for any self-adjoint matrices R and S with spectra contained within I, and for every $0 \;\leq\; c \;\leq\; 1$, the following inequality holds,

$$f(cR + (1-c)S) \leq cf(R) + (1-c)f(S),$$

for all $R,\; S \in \mathcal{B}(\mathcal{H}^*)$.

If a function is matrix convex [12, 16] for matrices of all orders k, it is referred to as a function that is **operator convex**.

Example- Consider the function f defined from the positive real axis to the positive real axis.

$$f(t) \;=\; t^2$$

is classified as an operator convex on $\mathbb{R}$.

Theorem 1.10.1 ([40]). *Consider a function $f : (0, \infty) \to (0, \infty)$. If f is an operator monotone function, then the following properties hold*

1. *The function $tf(t)$, where t is a variable, is also an operator monotone function.*
2. *Moreover, The function $f(t^{-1})$, when applied to the inverse of t, preserves the property of operator monotonicity.*

Proposition 1.10.1. *Let $f : (0, \infty) \to (0, \infty)$ be a function that is an operator convex. Then the function $h(t)$ represented as*

$$h(t) \;=\; \frac{1}{f(t)}$$

is also an operator monotone over an interval $(0, \infty)$.

1.11 Overview of Relevant Studies

In 1934, K. Löwner first emphasized the significance of operator monotone functions [5]. He made connections between operator monotonicity and Pick functions, and the positivity of certain matrix operations, such as the division and difference of matrices. Löwner provided a crucial characteristic of operator monotone functions through a representation in the form of an integral, represented by

$$h(t) \;=\; a + bt + \int_0^\infty \frac{t}{t+s} dm(s), \tag{1.5}$$

where $a \in \mathbb{R}$ and $b \geq 0$, with m denoting a positive measure on $(0, \infty)$ such that $\int_0^\infty \frac{dm(s)}{1+s} < +\infty$. He also proved that the functions $f(t) = t^r$ with r lies in the closed interval $[0, 1]$ and the logarithmic function f defined from $[1, \infty)$ to $\mathbb{R}, (f(t) \;=\; \log t$ are operator monotone. Löwner [7] also identified that for self-adjoint matrices of a fixed dimension k, specific matrices obtained from the values of f must be positive definite. He also observed that as k grows, the conditions become more restrictive. As $k \to \infty$, a condition for monotonicity that is both necessary and sufficient is that the function admits an analytic continuation and mapping the upper half-plane onto upper half plane. His important theorem described operator monotone functions defined for real variables, establishing that every such function originates from holomorphic functions that map the upper half-plane of a complex plane onto itself. Kraus and Fritz [6] extended Löwner's results regarding convex matrix functions to operator convex functions, uncovering analogous properties for both categories. Löwner's 1950 work [8] further examined higher-order monotonic real-valued functions and their relationship with matrix functions, given that both the dependent and independent variables are real symmetric matrices of identical order. He explored broader categories of real-valued functions with real variables defined on open intervals, referred to as transformation semigroups (S). These functions exhibit continuity, strict monotonicity, and satisfy four specific properties,

I. If $h(x)$ belongs to S, then the restriction of $h(x)$ to any open subinterval $(a', b') \subset (a,\ b)$ is also an element of S.

II. If both the $h(x)$ and the $g(x)\ \in\ S$, then the composition $g[h(x)] \in S$.

III. If $h(x)$ defines the limit of a sequence of functions $h_n(x) \in S$, where all functions are defined in $(a,\ b)$ and converge uniformly on any closed and bounded sub-interval of $(a,\ b)$, then $h(x)\ \in\ S$.

IV. If $h(x)\ \in\ S$, then $h^{-1}(x)\ \in\ S$.

These properties contribute to obtaining precise criteria for operator monotone functions and their differential properties. Bendat [10] compiled results from previous studies and confirmed similar characteristics for convex matrix functions while extending findings to operator convex functions. He adapted (2.7) into what he termed the Stieltjes integral form for operator monotone functions, expressed as

$$f(x) \ = \ \int_{-\frac{1}{R}}^{\frac{1}{R}} \frac{x}{1-tx} d\psi(t) \quad (2.8)$$

Here, $f(x)\ =\ \sum_{n=1}^{\infty} a_n x^n$ represents a monotone function with a convergence radius R, and $\psi(t)$ is a bounded, nondecreasing function that stays constant for $|t| > \frac{1}{R}$. Several researchers have approached unique demonstrations of Löwner's theorem [5] from various perspectives, with several of these proofs documented in [6, 8, 19, 27, 54]. Frank Hansen [54] showed that the function defined as $t \mapsto \frac{(t^p-1)}{(t^q-1)}$ is an operator monotone on the positive real-line for $0 < p \leq q \leq 1$. Hansen also developed a new theory, providing an integral representation of operator monotone functions independent of Löwner's analytic function theory, establishing a standard connection between positive and general operator monotone functions on $(0, \infty)$. Utilizing these standard relationships, through which he identified new monotone functions in the context of quantum system state spaces. The relationship between operator monotone and operator concave (or convex) functions play an essential role in matrix inequalities, leading to the Löwner-Heinz inequality. This states that for positive operators R and S on a Hilbert space H,

$$R \geq S \geq 0 \Rightarrow R^p \geq S^p \quad (2.9)$$

for all $0 < p < 1$. This is equivalent to asserting that the function $h(t) = t^p$ is an operator monotone, for $t \geq 0$ and $p \in [0, 1]$.

In 1980, Kubo and Ando [26] introduced an axiomatic framework for operator means involving two variables, linking it to operator monotone functions defined on two variables. They demonstrated that each non-negative operator monotone function f that is defined on the interval $[0, \infty)$ is linked with a unique operator connection, where $f(1) = 1$ establishes a link to the operator mean. The operator mean $\sigma = \sigma_R$ is given by,

$$R\sigma S = R^{\frac{1}{2}} f(R^{-\frac{1}{2}} S R^{-\frac{1}{2}}) R^{\frac{1}{2}} \quad (2.9)$$

where $R, S \in B(\mathcal{H})$ are positive operators [25]. Several significant types of means [29, 64, 76] are given as follows

1. Arithmetic mean: $R \nabla S = \frac{R+S}{2}$.
2. Geometric mean [44]: $R \sharp S = R^{\frac{1}{2}} \left(R^{-\frac{1}{2}} S R^{-\frac{1}{2}}\right)^{\frac{1}{2}} R^{\frac{1}{2}}$.
3. Harmonic mean: $R \mathfrak{H} S = \left(\frac{R^{-1}+S^{-1}}{2}\right)^{-1}$.

Kubo's insights significantly contributed to the conceptual development of multivariable operator means derived from specific classes of normalized multivariate operator monotone functions. The geometric mean for two variables has been extensively analyzed and has been extended to two non-commuting operator variables by several authors, including Lawson [60, 68]. Subsequently, in 2004, Ando et al. [28] provided a definition for the geometric mean of k distinct positive (semi)definite matrices, generalizing numerous inequalities [72, 74] applicable to a geometric mean of two positive semi-definite matrices. They also introduced simple computational formulas [71] for square matricesof order 2. In 2013, Bhatia [62] explored Riemannian geometry, and Bini et al. in 2010 [57] introduced new methods to generalize the geometric mean to multivariable settings. Moakher et al. [58, 71, 78] demonstrated the geometric mean of symmetric matrices through a differential method, ultimately establishing a relationship between means based on Riemannian metrics and geometric means. In 2014, Palfia [67] generalized Löwner's findings to multiple variables and defined real functions that are characterized as operator monotone functions, acknowledging the significance of analytic extension in transforming the upper half-plane of the complex plane onto itself, and establishing several theorems related to non-commutative multiple variables.

Chapter 2

Spectral behaviour of the matrix $[f(1-p_ip_j)]$

2.1 Introduction

The study of pattern-based matrices and their spectral properties has a rich history, marked by significant contributions from mathematicians over the years. The journey began in the mid-19th century with the pioneering work of Augustin-Louis Cauchy, who in 1841, defined the matrix $R = [r_{ij}] = [\frac{1}{p_i - p_j}]$. This matrix exhibited explicit determinant known for its distinct structure, often expressed as a product of differences between its parameters, also ensures its invertibility, as a non-zero determinant guarantees that the matrix has a multiplicative inverse [65, 66]. Cauchy further proved that the matrix analogous to $[c_{ij}]$ defined as $[\frac{1}{p_i + p_j}]$ is positive definite, owing to its non-zero determinant, setting a foundation for future explorations into matrix theory [4]. Further, in 1986, Dyn, Goodman, and Micchelli [31] delved deeper into the spectral behavior of matrices $[|p_i - p_j|]$. They investigated the matrix $[|p_i - p_j|^r]$, where $p_1, p_2, \ldots, p_k$ are distinct real numbers and $r > 0$. Their research provided insights into how the spectral properties of these matrices change with different values of r. The field witnessed significant advancements in 2015 with the work of Bhatia and Jain [70]. They analyzed the spectral behaviour of matrices $P_r = [(p_i + p_j)^r]$ and $B_r = [|p_i - p_j|^r]$ using the power function $t \to t^r$ for distinct positive real numbers $p_1, p_2, ..., p_k$ and positive values of r. This research laid the groundwork for understanding the influence of power functions on the behavior of pattern-based matrices, opening new avenues for spectral analysis. Building on these foundations, in 2018, Garg and Aujla [75, 77] examined matrices of the form $[f(p_i + p_j)]$ and $[f(|p_i - p_j|)]$ with f being any operator monotone function from $(0, \infty)$ to $(0, \infty)$. The results demonstrated the broad applicability of operator monotone functions [55, 56] in analyzing the spectral behavior of pattern-based matrices, illustrating how these functions can be effectively used to generalize and extend the analysis of such matrices. A notable contribution to the field came in 2021 from Tanvi Jain [79]. Jain derived the inertia of the matrix $[(1+p_ip_j)^r]$ for distinct positive real values of r. Her research offered profound insights into the impact of power functions on the spectral characteristics of matrices across different contexts, with a specific focus on inertia and its implications.

Building on this extensive body of knowledge, our chapter aims to further explore the spectral behavior of matrices of the form $[f(1-p_ip_j)]$ for distinct positive real numbers $p_i (1 \leq i \leq k)$, where f represents a non-linear operator concave function mapping $(0,\infty)$ to $(0,\infty)$. In Section 2.2, we will demonstrate that such matrices are conditionally negative definite, nonsingular, and possessing an inertia of $(1,0,k-1)$, although this result does hold true for linear functions. Additionally, we examine how the results change when f is a logarithmic function i.e, $f(t)=\log t$ where t is positive and highlighting the spectral behaviour of *log* function. Our approach involves a rigorous analytical method to prove the conditionally negative definiteness, non-singularity, and inertia of the matrix $[f(1-p_ip_j)]$, supported by examples to illustrate these properties. In Section 2.3 we present an example to demonstrate that a nonlinear function f being operator concave is not a required condition for a matrix to possess inertia $(1,0,k-1)$. Furthermore, we prove that all the assumptions stated in the primary results of Section 2.2 are essential. This work aims to advance the research on the spectral behavior of pattern-based matrices by expanding the range of studied functions and matrices, offering new results, and enhancing our understanding of how different functions affect spectral properties.
The main results presented in this chapter are published in Garg and Agarwal [80].

2.2 Inertia of $[f(1-p_ip_j)]$

We will commence this section with the propositions and lemmas that are essential in proving the main theorem.

2.2.1 Lemmas and Propositions

Proposition 2.2.1. *Let R and S be two $\mathscr{PSD}$ of order k, with the assumption that $rank(R)=k$. Then, the matrix $R+S$ is a $\mathscr{PD}$.*

Proof. Since R and S are positive $\mathscr{PSD}$, for every vector x member of $\mathbb{C}^k$, it is true that

$$\langle Rx, x\rangle \geq 0$$

and

$$\langle Sx, x\rangle \geq 0.$$

This implies that the quadratic forms associated with R and S are non-negative for all $x \in \mathbb{C}^k$.

Next, we are given that $\rho(R)$ denotes he rank of R which is scalar number k, which is same as the dimension of the space on which R operates. This indicates that the columns of R are $\mathscr{LI}$, meaning no column can be represented as a linear combination of the others. Consequently, R is invertible, which implies that

$$\det(R) \neq 0.$$

Since R is both $\mathscr{PSD}$ and invertible, It follows that R is necessarily a $\mathscr{PD}$ matrix. A matrix R is $\mathscr{PD}$ if it satisfies the condition defined as,

$$\langle Rx, x\rangle > 0 \ \forall \ \text{ non-zero } x \in \mathbb{C}^k.$$

Now, let us consider any non-zero vector $x \in \mathbb{C}^k$. We analyze he quadratic form associated with the matrix $R + S$, which is expressed as

$$\langle (R+S)x, x\rangle.$$

By utilizing the inner product properties, we can expand this expression

$$\langle (R+S)x, x\rangle = \langle Rx, x\rangle + \langle Sx, x\rangle.$$

Given that R is a $\mathscr{PD}$ matrix and S is a $\mathscr{PSD}$, we have,

$$\langle Rx, x\rangle > 0, \ for \ 0 \neq x \ \in \mathbb{C}^k$$

also,

$$\langle Sx, x\rangle \geq 0, forall \ x \ \in \mathbb{C}.$$

Thus, for every non-zero vector x, we have,

$$\langle (R+S)x, x\rangle = \langle Rx, x\rangle + \langle Sx, x\rangle > 0.$$

Given that,

$$\langle (R+S)x, x\rangle > 0 \quad \text{for all} \quad 0 \neq x \in \mathbb{C}^k,$$

that is, $\langle (R+S)x, x\rangle$ is strictly positive for every non-zero $x \in \mathbb{C}^k$, it follows that $R+S$ is a $\mathscr{PD}$ matrix. □

Proposition 2.2.2. *Let R be a square matrix of order k such that*

$$R = u_1^T u_1 + u_2^T u_2 + u_3^T u_3 + \cdots + u_n^T u_k,$$

where the u_i's are row vectors and $u_i^T u_i$ is a rank-1 matrix for all $1 \leq i \leq k$. Additionally, the collection $\{u_1, u_2, \ldots, u_k\}$ forms a $\mathscr{LI}$ set. Then, R is non-singular.

Proof. We begin by representing the matrix R in terms of its rank-1 decompositions. Given R can be written as,

$$R = u_1^T u_1 + u_2^T u_2 + u_3^T u_3 + ... + u_n^T u_k,$$

where u_i are rows vectors. Since each $u_i^T u_i$ is a rank-1 matrix and the vectors $u_1, u_2, \ldots, u_k$ are linearly independent.
This sum of products $u_i^T u_i$ can be expressed more compactly as

$$R = UU^T,$$

Here, U denotes an $k \times k$ matrix defined as

$$U = \begin{bmatrix} u_1^T & u_2^T & u_3^T & \cdots & u_k^T \end{bmatrix}.$$

In other words, U represents a matrix where the rows correspond to the vectors $u_1, u_2, \ldots, u_k$. Next, we apply well-known inequalities from matrix rank theory for two square matrices U and V of order k

1. $rank(U + V) \leq rank(U) + rank(V)$ [59].

2. $rank(U) + rank(V) - k \leq rank(UV)$ [3, 20].

For our purposes, we will set

$$V = U^T.$$

On applying $V = U^T$ in above inequalities, we obtain the following expressions,

$$rank(U + U^T) \leq rank(U) + rank(U^T)$$

and

$$rank(U) + rank(U^T) - k \leq rank(UU^T).$$

Since U represents a square matrix of order k and U^T is a matrix obtained by transposing U. This implies that U^T possesses the same rank as U. Therefore, we have

$$rank(U^T) = rank(U).$$

Let us assume $r = rank(U)$. Substituting into our inequalities, we get,

$$rank(U + U^T) \leq 2r$$

and

$$r + r - k \leq rank(UU^T),$$

This simplifies to

$$2r - k \leq rank(UU^T).$$

Now, since $R = UU^T$, we have,

$$rank(R) = rank(UU^T).$$

From the preceding inequality, we conclude that

$$2r - k \leq rank(R).$$

Since U is an square matrix of size k whose rows are the vectors $u_1, u_2, \ldots, u_k$ are linearly independent. Then U has full rank [59], we conclude that $rank(U) = k$. Therefore,

$$2k - k \leq \mathrm{rank}(R),$$

which simplifies to

$$k \leq \mathrm{rank}(R).$$

We also have the relation

$$\mathrm{rank}(UU^T) \leq \min\{\mathrm{rank}(U), \mathrm{rank}(U^T)\}.$$

Since U is a square matrix of order k, it follows that

$$\text{rank}(R) = \text{rank}(UU^T) \leq k.$$

From the above inequalities, we conclude that

$$\text{rank}(R) = k.$$

Therefore, row-reduced Echelon form of R is equivalent to the identity matrix under row operations [59], it follows that R is non-singular (invertible). In other words

$$det(R) \neq 0.$$

$\square$

Lemma 2.2.1. *Consider a matrix $R = [r_{ij}]$ belonging to $\mathbb{M}_{k\times k}(\mathbb{C})$ that is $\mathscr{CPD}$. Consequently, $e^{\circ R}$ is $\mathscr{PSD}$. Additionally, $e^{\circ R}$ is $\mathscr{PD}$ if and only if $r_{ii} + r_{jj} > 2r_{ij}$, for all $i \neq j$.*

Proof. Let $\gamma_i = r_{ik} - r_{kk}/2$, for $1 \leq i \leq k$. According to [41, Lemma 2.4], given that $R = [r_{ij}]$ is $\mathscr{CPS}$, we can express, for $1 \leq i, j \leq k$, the relationship

$$r_{ij} = s_{ij} + r_{ik} + r_{kj} - r_{kk} = s_{ij} + \gamma_i + \gamma_j,$$

where,

$$S = [s_{ij}] = [r_{ij} - r_{ik} - r_{kjj} + r_{kk}]$$

is a $\mathscr{PSD}$ matrix in $\mathbb{R}_{k\times k}(\mathbb{C})$ Consequently,

$$e^{\circ S} = [e^{s_{ij}}]$$

is also $\mathscr{PSD}$. This implies that

$$e^{\circ R} = [e^{r_{ij}}] = [e^{s_{ij}+\gamma_i+\gamma_j}] = [e^{\gamma_i}e^{s_{ij}}e^{\gamma_j}] = De^{\circ S}D$$

is $\mathscr{PSD}$, where $D = diag(e^{\gamma_1}, e^{\gamma_2}, ..., e^{\gamma_k})$. Ultimately, $e^{\circ R}$ is$\mathscr{PD}$ if and only if $e^{\circ S}$ is $\mathscr{PD}$. This condition holds if and only if the rows represents $S = (x_i^T \cdot x_j)$ are not identical or equivalently,

$$0 < \|x_i - x_j\|^2 = s_{ii} + s_{jj} - 2s_{ij} = r_{ii} + r_{jj} - 2r_{ij},$$

for all $i \neq j$. $\square$

Lemma 2.2.2. *[41] Let* $R \in \mathbb{R}_{k\times k}(\mathbb{C})$ *be symmetric and* $\mathscr{PSD}$ *matrix. The Hadamard exponential* $\mathrm{e}^{\circ R} = [\mathrm{e}^{r_{ij}}]$ *is* $\mathscr{PSD}$*. Furthermore,* $\mathrm{e}^{\circ R}$ *is* $\mathscr{PD}$ *if and only if the rows of* R *are distinct .*

Theorem 2.2.1. *Let* $\{p_1, p_2, \ldots, p_k\}$ *be a set of distinct real numbers, and let* $R = [1+p_ip_j]$ *be a* $\mathscr{PSD}$ *matrix. Then, the Hadamard exponential of* R*, denoted by* $[\mathrm{e}^{1+p_ip_j}]$*, is a* $\mathscr{PD}$ *matrix, i.e.,* $[\mathrm{e}^{1+p_ip_j}]$ *is non-singular with all eigenvalues positive.*

Proof. The proof proceeds in two parts

1. We show that $[e^{1+p_ip_j}]$ is a $\mathscr{PSD}$ matrix.

2. Next, we prove that $[e^{1+p_ip_j}]$ is non-singular.

Define $R = [e^{1+p_ip_j}] = [e^S]$, where $S = [1 + p_ip_j]$. Then we have

$$[e^{1+p_ip_j}] = [e^{1+p_ip_j}] = [e \cdot e^{p_ip_j}].$$

Thus, we can write

$$[e^{1+p_ip_j}] = \begin{bmatrix} e & e & \cdots & e \\ e & e & \cdots & e \\ \vdots & \vdots & \ddots & \vdots \\ e & e & \cdots & e \end{bmatrix} \circ \begin{bmatrix} e^{p_1^2} & e^{p_1p_2} & \cdots & e^{p_1p_k} \\ e^{p_2p_1} & e^{p_2^2} & \cdots & e^{p_2p_k} \\ \vdots & \vdots & \ddots & \vdots \\ e^{p_kp_1} & e^{p_kp_2} & \cdots & e^{p_k^2} \end{bmatrix} = P \circ Q,$$

where

$$P = \begin{bmatrix} e & e & \cdots & e \\ e & e & \cdots & e \\ \vdots & \vdots & \ddots & \vdots \\ e & e & \cdots & e \end{bmatrix}$$

and

$$Q = \begin{bmatrix} e^{p_1^2} & e^{p_1p_2} & \cdots & e^{p_1p_k} \\ e^{p_2p_1} & e^{p_2^2} & \cdots & e^{p_2p_k} \\ \vdots & \vdots & \ddots & \vdots \\ e^{p_kp_1} & e^{p_kp_2} & \cdots & e^{p_k^2} \end{bmatrix}.$$

Matrix P is a rank-one real symmetric matrix with one eigenvalue ke and all other $k-1$ eigenvalues zero. Therefore, P is $\mathscr{PD}$.i.e

$$P \geq 0 \tag{2.1}$$

The matrix Q can be expressed as $Q = e^{\circ R}$ where $R = [p_ip_j]$. Using the Hadamard exponential series expansion, we have

$$e^{\circ R} = E + R + \frac{R^{\circ 2}}{2!} + \frac{R^{\circ 3}}{3!} + \cdots,$$

where E represents the matrix of order k in which every element is equal to 1. According to the Schur product theorem 1.4.1, each and every term in this series is $\mathscr{PSD}$, such as R, $R^{\circ 2}$, $R^{\circ 3}$, etc., are all $\mathscr{PSD}$ matrices. Therefore, the sum of these $\mathscr{PSD}$ matrices, $e^{\circ R}$, is also a $\mathscr{PSD}$. i,e

$$Q = e^{\circ R} \geq 0. \tag{2.2}$$

From equations (2.1) and (2.2), both P and Q are $\mathscr{PSD}$, and by the Schur product theorem, their Hadamard product $P \circ Q$ is also $\mathscr{PSD}$.

To show that $e^{[1+p_ip_j]}$ is non-singular, it is sufficient to show that no two rows of the matrix $S = [1 + p_ip_j]$ are identical. Assume, for contradiction, that two rows of S are the same, i.e., both i-th and the j-th row are same,

$$(1 + p_ip_1, 1 + p_ip_2, \ldots, 1 + p_ip_k) = (1 + p_jp_1, 1 + p_jp_2, \ldots, 1 + p_jp_k).$$

This implies that

$$p_ip_1 = p_jp_1, \quad p_ip_2 = p_jp_2, \quad \ldots, \quad p_ip_k = p_jp_k,$$

which further implies that $p_i = p_j$, contradicting the assumption that p_i and p_j are distinct real numbers. Therefore, no two rows of S are identical.

Thus, by Lemma 2.2.2, $R = [e^{1+p_ip_j}]$ does not have two identical rows, implying that R is non-singular. Therefore, $R = [e^{1+p_ip_j}]$ is a matrix that is $\mathscr{PD}$. □

Lemma 2.2.3. *Let $R = [r_{ij}]$ be an k-order matrix that is $\mathscr{CND}$ and has all its entries positive. The Hadamard inverse of R is a $\mathscr{PSD}$. Furthermore, it is $\mathscr{PD}$ if and only if $r_{ii} + r_{jj} < 2r_{ij}$ for every i, j belongs to the set $1, 2, \ldots, k$, such that $i \neq j$.*

Proof. Let the eigenvalues of R be denoted as

$$\omega_1 \leq \omega_2 \leq \cdots \leq \omega_{k-1} \leq r,$$

with $Rv = rv$ and $Au_i = \omega_iu_i$, for i greater than or equal to 1 and less than or equal to $k-1$. Here, the value r represents the Perron eigenvalue, and $v = (v_1, v_2, \ldots, v_k)^T$

denotes the associated Perron eigenvector, which has positive components as stated by the Perron-Frobenius Theorem [4]. We can express

$$R = rvv^T + \omega_{k-1}u_{k-1}(u_{k-1})^T + ... + \omega_1 u_1(u_1)^T.$$

Introducing $V = diag(1/v_1, 1/v_2, ..., 1/v_k)$,

we can rewrite VRV as

$$VRV = ree^T + \omega'_{(n-1)}(Vu_{(n-1)})(Vu_{(n-1)})^T + ... + \omega_1(Vu_1)(Vu_1)^T.$$

If it is consider that $x^Te = 0$, then $x^TVRVx \leq 0$. Let

$$VRV = S = (s_{ij})$$

this implies that $x^TSx \leq 0$, i.e S is $\mathscr{NSD}$. Recall the identity for $t > 0$,

$$1/t = \int_0^\infty e^{-ts}ds.$$

This allows us to express $x^T(1/s_{ij})x$ as

$$x^T(1/s_{ij})x = \int_0^\infty x^T(e^{-s_{ij}s})xds.$$

Since $(-s_{ij}s)$, for $s > 0$, is a $\mathscr{PSD}$, as demonstrated in Lemma 2.2.2, $(e^{-s_{ij}}s)$ is a $\mathscr{PSD}$ matrix. Therefore,

$$(1/s_{ij}) = (VRV)^{\circ(-1)} = V^{-1}R^{\circ(-1)}V^{-1}$$

The matrix $R^{\circ(-1)}$ is a $\mathscr{PSD}$ matrix. In addition, $R^{\circ(-1)}$ is a $\mathscr{PD}$ if and only if

$$V^{-1}R^{\circ(-1)}V^{-1} = \frac{1}{s_{ij}}$$

is a $\mathscr{PD}$ matrix. This condition holds if and only if $(e^{-s_{ij}s})$ is $\mathscr{PD}$, which occurs if and only if

$$s_{ii} + s_{jj} < 2s_{ij}$$

for each $i \neq j$. This is same as the condition

$$\frac{r_{ii}}{(v_i)^2} + \frac{r_{jj}}{(v_j)^2} < \frac{2r_{ij}}{v_iv_j},$$

for all $i \neq j$. □

Lemma 2.2.4. *Let R be a matrix that is $\mathscr{CND}$ and non-zero, with all of its entries being non-negative. Then R possesses exactly one positive eigenvalue.*

Proof. Given that R is $\mathscr{CND}$. Thus, for any vector x, with the constraint $\sum_i x_i = 0$, we have $x^TRx \leq 0$. This property shows that R has atmost one positive eigenvalue and other are non-positive. To show that R has exactly one positive eigenvalue, consider the matrix structure and its effect on the eigenvalues. Because R is non-zero and all its entries are non-negative, Perron-Frobenius theory applies, indicating that there is a unique largest eigenvalue, which is positive. All other eigenvalues must be non-positive, aligning with the $\mathscr{CND}$ nature of R. Thus, R possesses exactly one positive eigenvalue. □

Lemma 2.2.5. *Let R a symmetric matrix, where all of its entries are positive, and it has exactly one positive eigenvalue. Then the Hadamard inverse of R is $\mathscr{PSD}$. Furthermore, it becomes $\mathscr{PD}$ if R is invertible.*

Lemma 2.2.6. *Let $p_1, p_2, \ldots, p_k$ are n distinct positive real numbers. Then the matrix $[t - p_ip_j]$ is a $\mathscr{CND}$ for all real values of t.*

Proof. Consider R as a row vector of order k with distinct real numbers p_i, where $i = 1, 2, \ldots, k$, as its components. The matrix $[p_ip_j]$, which equals R^*R, is $\mathscr{PSD}$ because it can be expressed as a Gram matrix of the vectors p_i. Therefore, the matrix $[t-p_ip_j]$ is $\mathscr{CND}$ for any real number t, as it represents a transformation that preserves $\mathscr{CND}$ behaviour. □

Theorem 2.2.2. *Let $f : (0, \infty) \to (0, \infty)$ be a real valued function that is operator concave and let $p_1, p_2, \ldots, p_k$ be k distinct positive real numbers such that $p_i < 1\ \forall\ i$. Then, the matrix $[f(1 - p_ip_j)]$ satisfies the condition of $\mathscr{CND}$*
and

1. *The matrix is non-singular with inertia $(1, 0, k-1)$, provided that f is non-linear.*

2. *The matrix is singular with inertia $(1, k-2, 1)$, given that f is linear and $k > 2$.*

Proof. Let $f : (0, \infty) \to (0, \infty)$ is an operator concave function. The integral representation of the operator monotone function [5] from $(0, \infty)$ *to* $(0, \infty)$ is given by

$$h(t) = \alpha + \beta t + \int_0^\infty \frac{st}{t+s} d\mu(s), \tag{2.3}$$

where, both α and β are member of $\mathbb{R}$, $\beta \geq 0$ and μ be a measure that assigns non-negative values.

We start by considering the function f evaluated at

$$t = 1 - p_ip_j.$$

Further, we represent the integral expression as,

$$f(1-p_ip_j) = \alpha + \beta(1-p_ip_j) + \int_0^\infty \frac{s(1-p_ip_j)}{(1-p_ip_j)+s} d\mu(s).$$

Thus, the matrix representation is expressed as,

$$[f(1-p_ip_j)] = \alpha E + \beta[1-p_ip_j] + \int_0^\infty \left[\frac{s(1-p_ip_j)}{(1-p_ip_j)+s}\right] d\mu(s).$$

Also,

$$[f(1-p_ip_j)] = \alpha E + \beta[1-p_ip_j] + \int_0^\infty G_s d\mu(s), \tag{2.4}$$

where,

$$G_s = \left[\frac{s(1-p_ip_j)}{(1-p_ip_j)+s}\right].$$

We can rewrite G_s as follows

$$G_s = \left[\frac{s(1-p_ip_j)}{(1-p_ip_j)+s}\right]$$

$$= \left[s\left(1 - \frac{s}{s+1-p_ip_j}\right)\right]$$

$$= s\left[1 - \frac{s}{t-p_ip_j}\right]$$

$$= sE - s^2K_t$$

where, $t = s+1$ and

$$K_t = \frac{1}{t-p_ip_j}.$$

Next, we analyze the matrix $[t-p_ip_j]$ in terms of matrix E and $[p_ip_j]$, so we can have,

$$[t-p_ip_j] = tE - [p_ip_j].$$

Since $t > 1$ and $0 \leq p_i \leq 1$, it follows that

$$t - p_ip_j > 0 \; \forall \; i, j.$$

The matrix tE is $\mathscr{CND}$, as t is a positive constant, and $-[p_ip_j]$ is $\mathscr{NSD}$, hence $\mathscr{CND}$. Using these facts together with Lemma 2.2.4 and 2.2.5, we deduce that,

$$K_t = \frac{1}{t - p_ip_j}$$

is $\mathscr{PSD}$ and hence $\mathscr{CPD}$. Therefore, the matrix $-K_t$ is $\mathscr{CND}$, which leads to the conclusion that G_s is also $\mathscr{CND}$.
Using this result in Equation (2.4), it follows that the matrix $[f(1-p_ip_j)]$ is $\mathscr{CND}$. This matrix has all positive entries, and therefore by using Lemma 2.2.4, $[f(t-p_ip_j]$ has exactly one positive eigenvalue.
Next, we check the non-singularity of the matrix $[f(1-p_ip_j]$. We categorize this proof into two parts: first, for a linear function, and then for a non-linear function.
Part 1 Linear Function
Let

$$f(t) = \alpha + \beta t$$

be a linear function from $(0,\infty)$ to $(0,\infty)$, where $\alpha,\ \beta\ > 0$.
Then,

$$[f(1-p_ip_j)] = \alpha E + \beta[1-p_ip_j].$$

This matrix is a rank 2 matrix because the matrix $[1-p_ip_j]$ has rank 1, and adding αE a scalar multiple of the unit(identity) matrix, this raises the rank by 1.
Thus, the rank of

$$\alpha E + \beta[1-p_ip_j] \ is\ 2.$$

Since it is a rank 2 matrix, it has only two non-zero eigenvalues. As discussed earlier, it has exactly one positive and one negative eigenvalue due to the properties and characteristics of the $\mathscr{CND}$ matrices. Hence, the inertia of this matrix is $(1,0,1)$ for $n=2$ and $(1,k-2,1)$ for $k>2$.
Part 2: Non-linear Operator Concave Function
Consider the non-linear operator concave function $f:(0,\infty)\to(0,\infty)$.
Let

$$[t-p_ip_j] = [r_{ij}].$$

We need to show that $[f(1-p_ip_j)]$ is non-singular.
First, note that

$$r_{ii} + r_{jj} = (t-p_i^2) + (t-p_j^2) < 2(t-p_ip_j) = 2r_{ij}.$$

This inequality indicates that the matrix $[r_{ij}]$ is $\mathscr{CND}$.
Using Lemma 2.2.3, we deduce that the matrix $[\frac{1}{r_{ij}}]$ is $\mathscr{PD}$.
It follows that for every non-zero vector $x \in \mathcal{H}$.

$$\langle \frac{1}{r_{ij}} x, x\rangle \ > 0.$$

Hence, for all non-zero vectors $x \in \mathcal{H}_1$, it holds that,

$$\langle (sE - s^2 K_t)x, x\rangle < 0.$$

Now consider the term involving the integral in the expression for $[f(1-p_ip_j)]$

$$\langle (\alpha E + \beta(1-p_ip_j))x, x\rangle \leq 0,$$

hence, it can be concluded that

$$\langle [f(1-p_ip_j)]x, x\rangle < 0 \ \forall \ x \in \mathcal{H}_1.$$

Using Proposition 1.3.6, we conclude that $[f(1-p_ip_j)]$ is non-singular.
Therefore, the inertia of $[f(1-p_ip_j)]$ is $(1, 0, k-1)$.

$\square$

In the next theorem, we examine a particular instance of operator concave function defined on $(0, \infty)$ given by $f(t) = log\ t$. This function is of significant importance and has been extensively studied by various authors [53, 73] over time.
We note that $0 < 1 - p_ip_j < 1$, under the assumption that $p_i < 1 \ \forall \ i$. Additionally, we know that $log\ t < 0$ for $0 < t < 1$. Hence, the integral representation (1.5) is not applicable in this case.
In the following theorem, we analyze the spectral behavior of the matrix $[f(1-p_ip_j)]$ in relation to the function $f(t) = log\ t$, with the condition $p_i < 1 \ \forall \ i$. It can be noted that in this case, the matrix $[f(1-p_ip_j)]$ is $\mathscr{ND}$.

Theorem 2.2.3. *Consider the function $f(t) = \log t$, where $p_1, p_2, \ldots, p_k$ are k distinct positive real numbers, such that $p_i < 1 \ \forall \ i$, then the matrix $[f(1-p_ip_j)]$ is $\mathscr{ND}$.*

Proof. Given that $f(t) = logt$ and $0 < 1 - p_ip_j < 1, \forall \ i, j$ (since $p_i < 1 \forall \ i$), we observe the following,

1. Range of $log(1-p_ip_j)$:For $0 < t < 1$, $logt < 0$. Hence $log(1-p_ip_j) < 0 \ \forall \ i, j$.

2. Matrix Entries: The entries of the matrix $[log(1-p_ip_j)]$ are all negative as $log(1-p_ip_j) < 0$.

3. Negative Definiteness: To show that $[log(1-p_ip_j)]$ is $\mathscr{N}\mathscr{D}$, we must prove that for every non-zero vector $x \in \mathcal{H}$, the quadratic form

$$\langle [log(1-p_ip_j)]x, x\rangle < 0.$$

To analyse this, observe that for $t < 1$, we have,

$$\begin{aligned} log(1-t) &= -t - \frac{t^2}{2} - \frac{t^3}{3} - \frac{t^4}{4} - \frac{t^5}{5}... \\ log(1-p_ip_j) &= -(p_ip_j) - \frac{(p_ip_j)^2}{2} - \frac{(p_ip_j)^3}{3} - \frac{(p_ip_j)^4}{4} - \frac{(p_ip_j)^5}{5}... \qquad (2.5) \\ &= -\left((p_ip_j) + \frac{(p_ip_j)^2}{2} + \frac{(p_ip_j)^3}{3} + \frac{(p_ip_j)^4}{4} + \frac{(p_ip_j)^5}{5} + ...\right). \end{aligned}$$

Using Schur's theorem on $\mathscr{P}\mathscr{S}\mathscr{D}$ matrices, and the fact that the matrix $[p_ip_j]$ is $\mathscr{P}\mathscr{S}\mathscr{D}$, we get the matrix $[(p_ip_j)^r]$ is a $\mathscr{P}\mathscr{S}\mathscr{D}$ matrix for all $r = 1, 2, 3, ...$ Being the sum of $\mathscr{P}\mathscr{S}\mathscr{D}$ matrices, the matrix $-[log(1-p_ip_j)]$ is a $\mathscr{P}\mathscr{S}\mathscr{D}$ and hence $[log(1-p_ip_j)]$ is a $\mathscr{N}\mathscr{S}\mathscr{D}$ matrix.
To prove that this matrix is $\mathscr{N}\mathscr{D}$, it is sufficient to show that it is non-singular. We now check the non-singularity of the matrix $[log(1-p_ip_j)]$.
Equation (2.5) can also expressed as

$$-\log(1-p_ip_j) = \sum_{l=1}^{\infty} \frac{(p_ip_j)^l}{l} = \left(\sum_{l=1}^{k} \frac{(p_ip_j)^l}{l} + \sum_{l=k+1}^{\infty} \frac{(p_ip_j)^l}{l}\right) = R + S,$$

where

$$R = \sum_{l=1}^{k} \frac{(p_ip_j)^l}{l} \quad \text{and} \quad S = \sum_{l=k+1}^{\infty} \frac{(p_ip_j)^l}{l}.$$

It is important to observe that R is a sum of rank-1 matrices and can be expressed as

$$R = u_1^T u_1 + u_2^T u_2 + u_3^T u_3 + ... + u_k^T u_k,$$

where u_l is a row matrix given by

$$u_l = \begin{bmatrix} \frac{p_1^l}{\sqrt{l}} & \frac{p_2^l}{\sqrt{l}} & \frac{p_3^l}{\sqrt{l}} & ... & \frac{p_k^l}{\sqrt{l}} \end{bmatrix}.$$

We now assert that the set

$$\{u_1,\ u_2,\ u_3,\ ...,\ u_k\}$$

is linearly independent.
To prove that the set is linearly independent, it is enough to show that the rank of the matrix M is k, where M is given by

$$M = \begin{bmatrix} u_1^T & u_2^T & u_3^T & \dots & u_k^T \end{bmatrix} = \begin{bmatrix} p_1 & \frac{(p_1)^2}{\sqrt{2}} & \frac{(p_1)^3}{\sqrt{3}} & \dots & \frac{(p_1)^k}{\sqrt{k}} \\ p_2 & \frac{(p_2)^2}{\sqrt{2}} & \frac{(p_2)^3}{\sqrt{3}} & \dots & \frac{(p_2)^k}{\sqrt{k}} \\ p_3 & \frac{(p_3)^2}{\sqrt{2}} & \frac{(p_3)^3}{\sqrt{3}} & \dots & \frac{(p_3)^k}{\sqrt{k}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_n & \frac{(p_k)^2}{\sqrt{2}} & \frac{(p_k)^3}{\sqrt{3}} & \dots & \frac{(p_k)^k}{\sqrt{k}} \end{bmatrix}$$

Note that the determinant of this matrix is represented as

$$det(M) = \frac{p_1p_2p_3...p_k}{\sqrt{k!}} \prod_{i=1}^{k} (p_i - p_j), \forall\ 1 \leq i, j \leq k.$$

We find that, $det(M) \neq 0$. Therefore, $rank\ (M) = k$. So, our claim is established. Using proposition 2.2.2, we conclude that R is a $\mathscr{PD}$ matrix. Additionally, by proposition 1.3.6, we know that the matrix $R + S$ is also $\mathscr{PD}$. Therefore, the required matrix $[\log(1 - p_ip_j)]$ is $\mathscr{ND}$. $\square$

In the next section we analyse few remarks and examples, which proves that the condition assumed in Theorem 2.2.1 and 2.2.2 are necessary.

2.3 Remarks

First, we demonstrate that the condition $p_i < 1\ \forall\ i$ is essential in Theorem 2.2.2.

Remark 2.1. The condition $p_i < 1\ \forall\ i$ is essential for ensuring that the $log(1 - p_ip_j)$ is non- negative. Specifically, this condition guarantees that $1 - p_ip_j > 0$ for all indices i and j.
As a result, function $log(1 - p_ip_j)$ is both well-defined and non-negative because the argument of the logarithmic function, $1 - p_ip_j$, lies strictly between 0 and 1.
If the condition $p_i < 1$ is not satisfied, there may be cases where $1 - p_ip_j$ is zero or negative, leading to undefined values for the logarithmic function. For instance, if $p_i \geq 1$ for some i, then $1 - p_ip_j \leq 0$ for some j, making $log(1 - p_ip_j)$ undefined or complex. Consequently, the matrix $[log(1-p_ip_j)]$ may contain entries that are not properly defined, impeding meaningful analysis of its spectral properties. The theoretical results presented in Theorem 2.2.3, which rely on the negative definiteness of $[log(1 - p_ip_j)]$, assume that the matrix entries are well-defined. Without the condition $p_i < 1$, the matrix might

not exhibit the desired properties, thus invalidating the theorem. Therefore, ensuring $p_i < 1$ is crucial for the applicability and correctness of the results concerning the matrix $[log(1-p_ip_j)]$.

Next, we observe that the converse part of Theorem 2.2.2 does not hold true, illustrating that the condition of a non-linear function f being operator concave from $(0,\infty)$ to $(0,\infty)$ is not required for the matrix $[f(1-p_ip_j)]$ to exhibit the inertia $(1,0,k-1)$. In other terms, even if f is not an operator concave function, the matrix $[f(1-p_ip_j)]$ can still possess the inertia $(1,0,k-1)$. This remark highlights that the specific property of operator concavity is not a necessary condition for achieving the given inertia of the matrix, indicating that alternative functions may also produce matrices with the same inertia characteristics.

Remark 2.2. Consider a function $f:(0,\infty)\to(0,\infty)$, defined as $f(t)=1-e^{-t}$. Our objective is to prove that the matrix $[f(1-p_ip_j)]$ is $\mathscr{CND}$ with inertia $(1,0,k-1)$. However, f is not an operator concave function.
To begin with, the matrix $[f(1-p_ip_j)]$ can be expressed as

$$[f(1-p_ip_j)] = E - e^{-1}[e^{(p_ip_j)}],$$

where E is $\mathscr{CND}$, $[(p_ip_j)]$ is a rank-1 $\mathscr{PSD}$ matrix, and $[e^{p_ip_j}]$ is a $\mathscr{PD}$ matrix [41, Lemma 2.5].
We can further observe that the trace of $[e^{-(1-p_ip_j)}]$ is positive, as all diagonal entries are positive. This implies that $[f(1-p_ip_j)]$ is not a $\mathscr{NSD}$ matrix. Using proposition 1.3.6, it follows that $[f(1-p_ip_j)]$ is non-singular and has inertia $(1,0,k-1)$.
Next, let us examine the operator concavity of $f(t)=1-e^{-t}$. Suppose $f(t)$ is an operator concave function. This would imply that the function $f(t)-1$ is likewise an operator concave function. Consequently, $-e^{-t}$ would be operator concave, which implies that e^{-t} must be operator convex for all $t\in(0,\infty)$. However, as shown in [40, p.147], e^{-t} is neither operator concave nor operator convex. Therefore, $f(t)$ cannot be an operator concave function.
We have demonstrated that $[f(1-p_ip_j)]$ is a $\mathscr{CND}$ matrix including inertia $(1,0,k-1)$, even though f is not operator concave. This highlights that the specific property of operator concavity is not necessary for achieving the given inertia of the matrix $[f(1-p_ip_j)]$.

Given that $f(t)$ is a concave function, we recognize that $f(t)=e^{-t}$ is convex on the entire real line, particularly for $t\in(0,\infty)$, This implies that $1-e^{-t}$ is indeed a concave function. One might conjecture that if the matrix $[f(1-p_ip_j)]$ is a $\mathscr{CND}$ and possesses inertia $(1,0,k-1)$ for all non-linear concave functions f defined over an interval $(0,\infty)$ then the specific property of concavity is both necessary and sufficient.

However, this conjecture does not hold universally. The following remark illustrates that even though $[f(1-p_ip_j)]$ can be $\mathscr{CND}$ with the stated inertia for some non-linear concave functions, this property is not guaranteed for all such functions. Thus, the condition of f being concave does not necessarily imply that the matrix $[f(1-p_ip_j)]$ will always exhibit conditional negative definiteness and the inertia $(1, 0, k-1)$.

Remark 2.3. Consider a function f from $(0,\infty)$ *to* $(0,\infty)$ defined as

$$f(t) = \begin{cases} 2+t, & \text{if } t \leq .91 \\ 2.91, & \text{if } t > .91, \end{cases} \tag{2.6}$$

Let $\alpha, \beta \in (0,\infty)$. Assume, without any loss of generality, that $\alpha < \beta$. If both $\alpha, \beta \leq 0.91$ or both $\alpha, \beta > 0.91$, we directly deduce

$$f\left(\frac{\alpha+\beta}{2}\right) = \frac{f(\alpha)+f(\beta)}{2}.$$

But if $\alpha \leq .91$ and $\beta > .91$ two cases arises:

Case 1 $\frac{\alpha+\beta}{2} \leq .91$ then $f(\frac{\alpha+\beta}{2}) = 2 + \frac{\alpha+\beta}{2} = \frac{4+\alpha+\beta}{2} > \frac{4+\alpha+.91}{2} = \frac{4.91+\alpha}{2} = \frac{2+\alpha+2.91}{2} = \frac{f(\alpha)+f(\beta)}{2}$

Case 2 $\frac{\alpha+\beta}{2} > .91$ then $f(\frac{\alpha+\beta}{2}) = 2.91 > \frac{\alpha+4.91}{2} = \frac{2+\alpha+2.91}{2} = \frac{f(\alpha)+f(\beta)}{2}$,

From both the cases, we can deduce that function f is a concave. Therefore, $f(t)$ is a concave function on an interval $(0,\infty)$. However, taking $p_1 = \frac{1}{2}$, $p_2 = \frac{1}{3}$, and $p_3 = \frac{1}{4}$

$$[f(1-p_ip_j)] = \begin{bmatrix} 2.75 & 2.83 & 2.875 \\ 2.83 & 2.89 & 2.91 \\ 2.875 & 2.91 & 2.91 \end{bmatrix}.$$

The inertia of this matrix is $(2, 0, 1)$. A $\mathscr{CND}$ matrix have atmost one positive eigen value. However, $[f(1-p_ip_j)]$ matrix has two positive eigenvalues, which contradicts this requirement. Hence it is not a $\mathscr{CND}$ matrix.

The next remark illustrates that for operator convex functions, the spectral behavior of the matrix $[f(1-p_ip_j)]$ does not adhere to a specific or predictable pattern. Unlike the case with operator concave functions where certain spectral properties such as conditional negative definiteness and specific inertia can be established, operator convex functions do not guarantee a similar consistency in results. This variability arises because operator convex functions introduce different interactions among the matrix elements, thereby affecting the overall spectral properties in a less predictable manner.

Consequently, while some operator convex functions may yield matrices with desirable spectral characteristics, these outcomes cannot be generalized across all such functions.

Remark 2.4. Consider a function $f(t) = t^2$. Let the matrix $R = [f(1-p_ip_j)]$, where $p_i = \frac{1}{1+i}$ of order 4 has inertia $(2, 0, 2)$, however, if we take a function $f(t) = \frac{1}{1+t}$, $t > 0$ it is also an operator convex, but by using Lemma 2.2.3, it is a positive semi-definite matrix.

2.4 Significance of work

The significance of this research lies in its advancement of spectral analysis of pattern-based matrices, particularly those formed by applying nonlinear operator concave functions entrywise. By exploring the inertia and spectral properties of these matrices, the study provides new insights into their conditionally negative definiteness, non-singularity, and specific inertia patterns. These findings not only extend the understanding of matrix behavior under nonlinear transformations but also refine existing theories by demonstrating the limitations of linear functions in this context. The results are crucial for mathematicians and scientists working on matrix theory, as they offer a broader framework for analyzing the spectral properties of complex matrices, with potential applications in areas such as data science, quantum mechanics, and mathematical physics, where understanding the behavior of large, structured matrices is essential.

Chapter 3

Class of concave functions which are not operator concave

3.1 Introduction

The exploration of concave functions, particularly those defined from the positive real line to itself, is an area of significant mathematical interest. Concave functions are well-known for their wide range of applications across various fields such as optimization, economics, and convex analysis. These functions are characterized by their distinctive properties, including continuity, differentiability, and boundedness, which make them suitable for various theoretical and practical applications. We have identified a special class of concave functions that do not exhibit operator concavity. This distinction provides valuable insights into operator theory and enhances our understanding of function behavior in matrix analysis.

Operator concave functions are a generalized concept of concave functions, which are defined on the set of bounded self-adjoint operators in a Hilbert space. These functions adhere to a set of strict criteria that extend beyond the classical definition of concavity for real-valued functions. Specifically, a function $f : \mathbb{R} \to \mathbb{R}$ is deemed operator concave if it satisfies the operator inequality

$$f(cR + (1-c)S) \geq cf(R) + (1-c)f(S) \tag{3.1}$$

for arbitrary self-adjoint operators R and S and any $c \in [0, 1]$. This inequality is a natural extension of the concavity condition for real-valued functions, but it imposes additional constraints that not all concave functions can meet.

The primary focus of our study is to conduct a detailed analysis of concave functions [22] that meet key properties like being monotonic, continuous [17, 18], differentiable, and bounded, but do not satisfy the criteria for operator concavity. This investigation is motivated by the observation that many functions exist that are convex [50, 61] but fail to be operator convex, such as $f(t) = t^3$ on $(0, \infty)$, yet similar examples for concave functions are less readily available. Our aim is to fill this gap in the literature by presenting concrete examples of concave functions with the aforementioned properties that are not operator concave.

In the first section of our chapter, we present examples of concave functions that meet the criteria of continuity, differentiability, and boundedness on the interval $(0, \infty)$, but fail to be operator concave. These examples are carefully constructed to highlight the nuanced differences between concavity and operator concavity.

Following this, we introduce essential propositions that serve as bridges between our research and the existing literature. These propositions not only establish connections between our work and previous studies but also set the stage for the introduction of new concepts and counterexamples.

In the concluding section, we delve into the converse aspects of the results presented by Garg and Aujla [75] in their seminal work.

3.2 Examples

It is well established that while all operator concave functions are concave, the reverse implication does not always hold, i.e., there exist functions that are concave but not operator concave. The distinction between these classes of functions is significant, as operator concave functions possess additional properties that do not extend to all concave functions.

It is also a widely recognized fact that every operator concave function is also monotonically increasing. However, the reverse does not hold, there exist concave functions that are monotone but not operator monotone.

For instance, consider the function $f(t) = 1-e^{-t}$ defined from the positive real line to itself. The function $f(t) = 1 - e^{-t}$ is concave and monotonically increasing. Meanwhile, $f(t)$ is not an operator monotone function because for self-adjoint operators R and S, where $R \geq S \geq 0$, it does not necessarily hold that $f(R) \geq f(S)$.

There are numerous examples of concave functions that are monotonically decreasing on the interval $(0, \infty)$ with a range in $(-\infty, 0)$. However, no such examples exist for concave functions that are monotonically decreasing on $(0,\infty)$ to $(0,\infty)$. This observation highlights a fundamental concept that every concave function that is defined over the set of positive real numbers to $(0,\infty)$ is necessarily increasing.

Theorem 3.2.1. *Let $f : (0,\infty) \to (0,\infty)$ be a concave function. Then f is monotone non-decreasing on $(0,\infty)$; that is, for all $0 < a < b$ one has $f(a) \le f(b)$.*

Proof. We establish the result using two fundamental properties of concave functions [45].

Property 1 (Monotonicity of right-hand difference quotients). For every $x > 0$ and every $0 < h_1 < h_2$, we have

$$\frac{f(x+h_1)-f(x)}{h_1} \;\ge\; \frac{f(x+h_2)-f(x)}{h_2}.$$

Consequently, the right-hand difference quotients

$$q_x(h) := \frac{f(x+h)-f(x)}{h}, \qquad h > 0,$$

form a non-increasing function of h. Hence, the right-hand derivative

$$f'_+(x) := \lim_{h\downarrow 0} q_x(h)$$

exists (as a finite number or $-\infty$) for every $x > 0$.

Proof of Property 1. Fix $x > 0$ and $0 < h_1 < h_2$. Set $\lambda := 1 - \frac{h_1}{h_2} \in (0,1)$. Then

$$x + h_1 = \lambda x + (1-\lambda)(x+h_2).$$

By concavity of f,

$$f(x+h_1) \;\ge\; \lambda f(x) + (1-\lambda) f(x+h_2).$$

Subtracting $f(x)$ and dividing by $h_1 = (1-\lambda)h_2$ yields

$$\frac{f(x+h_1)-f(x)}{h_1} \;\ge\; \frac{f(x+h_2)-f(x)}{h_2},$$

which proves the claim.

Property 2 (Sandwich inequality for difference quotients). The left-hand derivatives

$$f'_-(x) := \lim_{h\downarrow 0} \frac{f(x)-f(x-h)}{h}$$

exist for every $x > 0$. Moreover, for every $a < b$,

$$f'_+(a) \;\geq\; \frac{f(b) - f(a)}{b - a} \;\geq\; f'_-(b).$$

Justification of Property 2. The left-hand difference quotients are monotone (by an argument analogous to Property 1), so the limits exist. The displayed inequalities then follow from standard concavity properties.

Main argument. Suppose, for contradiction, that there exists $x_0 > 0$ with $f'_+(x_0) < 0$. By Property 1, the map $x \mapsto f'_+(x)$ is non-increasing, so for every $x \geq x_0$ we have

$$f'_+(x) \leq f'_+(x_0) < 0.$$

Applying Property 2 with $a = x_0$ and any $b > x_0$, we obtain

$$\frac{f(b) - f(x_0)}{b - x_0} \;\leq\; f'_+(x_0) < 0,$$

which implies

$$f(b) \;\leq\; f(x_0) + f'_+(x_0)(b - x_0), \qquad b > x_0.$$

As $b \to \infty$, the right-hand side tends to $-\infty$, contradicting the assumption that $f(b) > 0$ for all $b > 0$. Hence no such x_0 exists, and therefore

$$f'_+(x) \geq 0 \qquad \text{for all } x > 0.$$

Since concavity ensures $f'_-(x) \geq f'_+(x)$, it follows that $f'_-(x) \geq 0$ for every $x > 0$. Applying Property 2 to any $0 < a < b$, we obtain

$$\frac{f(b) - f(a)}{b - a} \;\geq\; f'_-(b) \;\geq\; 0,$$

so $f(b) - f(a) \geq 0$. Hence $f(a) \leq f(b)$ for all $0 < a < b$.

Therefore, f is monotone non-decreasing on $(0, \infty)$. □

Löwner's integral representation theorem [5] provides a foundational result in the theory of operator monotone functions, which are functions that satisfy certain monotonicity properties with respect to matrices. This theorem has significant implications for the study of operator convex and operator concave functions [53].

Bhatia [40, 62] expanded upon Löwner's theorem and elucidated the direct consequences of operator monotonicity for concave functions. However, the existing literature has not explored specific concave functions defined from the positive real line to itself that are distinct from operator concave functions. Some examples of such concave functions are given as follows.

Example 3.2.1. *Consider a function f defined as $f : [0, \infty) \to [0, \infty)$ given by:*

$$f(t) = \begin{cases} 2 + t, & \text{if } t \leq 5, \\ 7, & \text{if } t > 5. \end{cases}$$

The concavity of f can be examined using the standard inequality:

$$f\left(\frac{x+y}{2}\right) \geq \frac{f(x) + f(y)}{2}, \quad \forall x, y \in \mathbb{R}.$$

To further analyze its operator concavity, consider the values $p_1 = 1$, $p_2 = 3$, and $p_3 = 6$. The matrix formed by evaluating $f(p_i + p_j)$ is:

$$R = \begin{bmatrix} 4 & 8 & 7 \\ 8 & 12 & 7 \\ 7 & 7 & 7 \end{bmatrix}.$$

For a nonzero vector $X = (1, 2, -3) \in \mathbb{R}^3$, we compute the quadratic form:

$$\langle RX, X \rangle = 11 > 0.$$

Since R is not conditionally negative definite (CND), as it contradicts Proposition 3.3.2, this implies that f does not satisfy operator concavity.

We observe that at $t = 5$, the given function is not differentiable; however, every operator concave function is differentiable. Next, we present a concave differentiable function f defined on the positive real line that is not operator concave.

Example 3.2.2. *Let f be a function defined from the positive real line to itself as*

$$f(t) = 1 - e^{-t}.$$

The given function is differentiable since $f'(t)$ exists on its domain. The second derivative is non-positive for all values of t, meaning that

$$f''(t) = -e^{-t} < 0, \quad \forall t > 0.$$

This implies that $f(t)$ is a concave and differentiable function. Next, we examine whether f is operator concave.

Consider a positive definite (PD) matrix R given by:

$$R = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 5 & 6 \\ 4 & 6 & 8 \end{bmatrix},$$

i.e., $R \geq 0$. The matrix $f(R)$, corresponding to the function f, is given by:

$$f(R) = \begin{bmatrix} 0.8647 & 0.9502 & 0.9817 \\ 0.9502 & 0.9933 & 0.9975 \\ 0.9817 & 0.9975 & 0.9997 \end{bmatrix}.$$

The eigenvalues of $f(R)$ are $1.1830, -0.3170,$ and -0.8660. Since $f(R)$ has negative eigenvalues, it is not positive semidefinite, proving that f is not operator concave.

This contradict the condition of monotonicity[40], i,e, $f(R) \ngeq 0$. So, f is not operator concave function. In the following example, we consider the logarithmic function and verify the same result as in above example.

Example 3.2.3. *Consider the function f, which is defined on the interval $[0, \infty)$:*

$$f(t) = \log(1 + t).$$

It is a well-known result that a function f is concave if and only if its second derivative satisfies $f'' < 0$. In this case, it is well established that the given function is concave, as its second derivative is given by

$$f''(t) = -\frac{1}{(1+t)^2} < 0, \quad \forall\ t \in [0, \infty).$$

However, if we take the matrix function corresponding to f, we find that it does not satisfy the criteria for operator concavity.

Consider the matrix R expressed as:

$$R = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}.$$

It is a positive definite matrix, i.e., $R > 0$ for all nonzero $x \in \mathbb{C}^n$. The matrix $f(R)$ is not positive definite, indicating that f is not an operator concave function.

While the function $f(t)$ appears concave in the functional theory, this example demonstrates that it does not satisfy operator concavity. The failure is evident from the negative eigenvalue in the computed difference matrix, showing that the operator inequality does not hold.

Example 3.2.4. *Define the function f as follows,*

$$f(t) = \sin^{-1}(t).$$

This function is defined on the positive real line up to $[-1, 1]$. To check its concavity, we compute the second derivative:

$$f''(t) = -\frac{t}{(1-t^2)^{3/2}}.$$

On the domain $t \in [0, 1)$, $f''(t)$ is non-positive since

$$-\frac{t}{(1-t^2)^{3/2}} \leq 0 \quad \textit{for} \quad t \geq 0.$$

The negativity of the second derivative indicates that the function is concave on the interval $[0, 1)$.

However, when extending the domain to $[0, \infty)$, the function $f(t) = \sin^{-1}(t)$ is not defined for $t > 1$. Since the arcsine function is only defined for $|t| \leq 1$, the consideration of concavity is confined to the interval $[0, 1)$, where the function is concave.

To illustrate this violation, consider two self-adjoint matrices R with eigenvalues a, b and S with eigenvalues c, d of the same dimension and a scalar $c \in [0, 1]$. The matrix function $f(R) = \sin^{-1}(R)$ is defined by applying the arcsine function to each eigenvalue of R:

$$R = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad S = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}, \quad \textit{where} \quad a, b, c, d \in [0, 1).$$

Let $f(X) = \sin^{-1}(X)$. Consider the function applied to a convex combination of matrices R and S:

$$f(cR + (1-c)S) = \sin^{-1}(cR + (1-c)S).$$

Computing:

$$\sin^{-1}(cR + (1-c)S) = \begin{bmatrix} \sin^{-1}(ca + (1-c)c) & 0 \\ 0 & \sin^{-1}(cb + (1-c)d) \end{bmatrix}.$$

Comparing this with the convex combination of the function applied individually to R and S:

$$cf(R) + (1-c)f(S) = c\sin^{-1}(R) + (1-c)\sin^{-1}(S),$$

we obtain:

$$c\sin^{-1}(R)+(1-c)\sin^{-1}(S) = \begin{bmatrix} c\sin^{-1}(a) + (1-c)\sin^{-1}(c) & 0 \\ 0 & c\sin^{-1}(b) + (1-c)\sin^{-1}(d) \end{bmatrix}.$$

Taking the difference:

$$\sin^{-1}(cR + (1-c)S) - \big(c\sin^{-1}(R) + (1-c)\sin^{-1}(S)\big) =$$

$$\begin{bmatrix} \sin^{-1}(ca + (1-c)c) - \big(c\sin^{-1}(a) + (1-c)\sin^{-1}(c)\big) & 0 \\ 0 & \sin^{-1}(cb + (1-c)d) - \big(c\sin^{-1}(b) + (1-c)\sin^{-} \end{bmatrix}$$

Since the diagonal entries may be negative for certain values of a, b, c, d, this matrix is not necessarily positive semi-definite:

$$\sin^{-1}(cR + (1-c)S) - \big(c\sin^{-1}(R) + (1-c)\sin^{-1}(S)\big) \ngeq 0.$$

This result indicates that $\sin^{-1}(X)$ is not a matrix concave function in general. Hence, f is not an operator concave function.

Remark 1. It is important to observe that the possibility of the existence of conditions stronger than differentiability and boundedness, under which a function can exhibit concavity without satisfying the criteria for operator concavity on the positive real line in the context of matrix analysis, remains an open avenue for future research.

The following section is dedicated to the exploration of essential propositions, each carefully crafted to facilitate the expansion of our study towards our ultimate objective.

3.3 Propositions

Proposition 3.3.1. *[75, 3b] Consider the function f defined as $f : [0, \infty) \to [0, \infty)$ being operator concave, and let $p_1, p_2, \ldots, p_n$ be distinct positive real numbers. If f is non-linear, then the matrix $[f(p_i + p_j)]$ is conditionally negative definite, non-singular, and has an inertia of $(1, 0, k-1)$.*

Proposition 3.3.2. *[75, Remark 5] Let $f : [0,\infty) \to [0,\infty)$ and let $p_1, p_2, \ldots, p_n$ be distinct positive real numbers. If $[f(p_i + p_j)]$ is conditionally negative definite, non-singular, and has an inertia $(1, 0, k-1)$, then if f is a non-linear function, it must be concave.*

Proposition 3.3.3. *[40, p.120, Theorem V.2.5] Let f be a continuous function that maps an interval $[0,\infty)$ into itself. Then, f is operator monotone if and only if it is operator concave.*

The following section reflects that, in their influential work, Garg and Aujla presented effective results concisely in Proposition 3.3.1 and Proposition 3.3.2. These propositions outline effective methods and principles. However, the converse of both Proposition 3.3.1 and Proposition 3.3.2 is not true. We embark on the task of elucidating the converse of Proposition 3.3.1 and Proposition 3.3.2 by providing compelling counterexamples.

3.4 Results

We begin with the converse of Proposition 3.3.1, where we find a function f defined as $f : [0,\infty) \to [0,\infty)$, where $p_1, p_2, \ldots, p_k$ are distinct positive real numbers such that the matrix $[f(p_i + p_j)]$ is conditionally negative definite ($\mathscr{CND}$), non-singular, and has an inertia of $(1, 0, k-1)$, even though f is not operator concave.

Consider the function f defined in Example 1, with distinct values of p_i, namely, $p_1 = \frac{1}{2}, p_2 = \frac{1}{3}, p_3 = \frac{1}{4}$. The resulting matrix is:

$$[f(p_i + p_j)] = \begin{bmatrix} 2.000 & 2.8333 & 2.7500 \\ 2.8333 & 2.6667 & 2.5833 \\ 2.7500 & 2.5833 & 2.5000 \end{bmatrix}$$

which is non-singular and conditionally negative definite, with eigenvalues $-0.670, -0.001$, and 7.838. However, $f(A) \ngeq 0$, meaning f is not operator concave.

Consequently, the converse of Proposition 3.3.2 is also not true. We provide two counterexamples for a concave function f that violates key conditions:

1. Violation of Conditional Negative Definiteness: Consider a concave function $f : (0,\infty) \to (0,\infty)$ such that $[f(p_i + p_j)]$ is non-singular but not conditionally negative definite. Consider the concave function f from Example 1, with distinct values $p_1 = 2, p_2 = 2.5, p_3 = 5.5$. The resulting matrix is:

$$[f(p_i + p_j)] = \begin{bmatrix} 6 & 6.5 & 7 \\ 6.5 & 7 & 7 \\ 7 & 7 & 7 \end{bmatrix}$$

with inertia $(2, 0, 1)$, which is not conditionally negative definite.

2. **Violation of Non-Singularity**: Consider a concave function $f : (0, \infty) \to (0, \infty)$ such that $[f(p_i + p_j)]$ is conditionally negative definite but singular. Consider Example 2, where the corresponding matrix:

$$[f(p_i + p_j)] = [1 - e^{-(p_i+p_j)}]$$

is conditionally negative definite but has rank 2 (i.e., singular) for all $k \geq 2$.

Chapter 4

Trans-flip of a matrix and its properties

In this work, we introduce a novel matrix operation termed the Trans-flip, drawing an analogy to the widely used transpose operation. We define and analyze the key properties of the trans-flip, comparing it to well-known transformations such as transpose and conjugate. Our focus is on how the trans-flip behaves in relation to matrix characteristics like determinant, trace, and inertia over both real and complex fields. We further investigate the effects of the trans-flip on matrices with specific patterns, particularly those that exhibit trans-flip symmetry. This study expands the matrix transformation framework, offering new perspectives and potential applications for the trans-flip operation.

4.1 Introduction

In this section, we introduce the trans-flip operation on matrices, a new transformation distinct from the classical transpose. For a matrix R we denoted as R^f, the trans-flip symmetrically flips matrix elements about the off-diagonal, where, off-diagonal elements are represented by $i+j=k+1$. We begin by establishing key definitions and notations, including matrix sets over real and complex fields. We also introduce the special matrix I_0, where $r_{ij}=1$ when $i+j=k+1$, defining its off-diagonal elements other elements are 0. This study builds on foundational matrix transformations such as the transpose (Cayley, 1858) and conjugate transpose (Hermite), and compares the properties of trans-flip matrices with these established operations. A central finding of the paper is any square matrix can be expressed as a decomposition into a trans-flip symmetric and trans-flip skew-symmetric matrix. We explore the effects of this operation on matrix properties like determinant, trace, and inertia, and compare it to the transpose operation in terms of positivity [13] and structural characteristics.

4.2 Definitions

4.2.1. Let $R = [r_{ij}]$ represents a square matrix of order k over the field $\mathbb{C}$ and f defines the trans-flip operation corresponding to the matrix $R = [r_{ij}]$, then $R^f = [r_{(k+1-j)(k+1-i)}], i, j = 1, 2, \ldots, k$.

In terms of matrix representation if $R \in \mathbb{M}_{k\times k}(\mathbb{C})$ and R is represented as

$$R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1(k-1)} & r_{1k} \\ r_{21} & r_{22} & \cdots & r_{2(k-1)} & r_{2k} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r_{(k-1)1} & r_{(k-1)2} & \cdots & r_{(k-1)(k-1)} & r_{(k-1)k} \\ r_{k1} & r_{k2} & \cdots & r_{k(k-1)} & r_{kk} \end{bmatrix}.$$

Then the matrix R over the operation trans-flip is represented as

$$R^f = \begin{bmatrix} r_{kk} & r_{(k-1)k} & \cdots & r_{2k} & r_{1k} \\ r_{k(k-1)} & r_{(k-1)(k-1)} & \cdots & r_{2(k-1)} & r_{1(k-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r_{k2} & r_{(k-1)2} & \cdots & r_{22} & r_{12} \\ r_{k1} & r_{(k-1)1} & \cdots & r_{21} & r_{11} \end{bmatrix}.$$

For example, if $S = [s_{ij}] \in \mathbb{M}_{k\times k}(\mathbb{C})$, i.e,

$$S = \begin{bmatrix} \iota & 2+\iota & 5\iota \\ 3\iota & 5 & 7 \\ \iota-1 & \iota+1 & 11 \end{bmatrix}$$

,

then

$$S^f = \begin{bmatrix} 11 & 7 & 5\iota \\ 1+\iota & 5 & 2+\iota \\ 1-\iota & 3\iota & \iota \end{bmatrix}.$$

4.2.2. Any non-zero square matrix $R \in \mathbb{M}_{k\times k}(\mathbb{C})$ is termed as trans-flip symmetric matrix if $R^f = R$.

Few examples of trans-flip symmetric matrices are null matrix, identity matrix, I_0 matrix, where

$$I_0 = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

4.2.3. Let $R = [r_{ij}]$ is defined to be a non-zero square matrix and $R \in \mathbb{M}_{k\times k}(\mathbb{C})$, if $R^f = -R$, then A is called as trans-flip skew symmetric matrix.

For instance, $R = [r_{ij}]$ be non-zero square matrix and $r_{ij} \in \mathbb{C} \; \forall \; i, j = 1, 2, 3.$

$$R = \begin{bmatrix} -14 & -7 & 0 \\ -6 & 0 & 7 \\ 0 & 6 & 14 \end{bmatrix} \text{ and } R^f = \begin{bmatrix} 14 & 7 & 0 \\ 6 & 0 & -7 \\ 0 & -6 & -14 \end{bmatrix}.$$

In a result R^f holds the relation with R, i.e, $R^f = -R$.

4.2.4. Consider $R = [r_{ij}]$ is defined to be a square matrix of order k over the field $\mathbb{C}$ and f defines the trans-flip operation corresponding to the matrix R, then the matrix R is said to be f-orthogonal if $RR^f = I$ or $R^f = R^{-1}$.

In the subsequent section, we present the properties of the trans-flip of a matrix, compare them with the transpose operation and highlighting their unique characteristics.

4.3 Properties analogous to transpose of a matrix

It can be observed that every square matrix R can be represented as the addition of a symmetric matrix and a skew-symmetric matrix. In the next theorem, we show that a square matrix R can be expressed as the addition of a trans-flip symmetric matrix and a trans-flip skew-symmetric matrix.

Theorem 4.3.1. *Any non-zero square matrix $R \in \mathbb{M}_{k\times k}(\mathbb{C})$ can be written as the addition of the trans-flip symmetric matrix and the trans-flip skew-symmetric matrix.*

Proof. Consider the non-zero matrix $R \in \mathbb{M}_{k\times k}(\mathbb{C})$. First, we will show that $(R + R^f)$ is a trans-flip symmetric matrix and $(R - R^f)$ is a trans-flip skew-symmetric matrix.

Now, the flip of $(R + R^f)$ can be represented as

$$(R + R^f)^f = R^f + (R^f)^f = R^f + R = R + R^f.$$

Similarly, we can write,

$$(R - R^f)^f = R^f - (R^f)^f = R^f - R = -(R - R^f).$$

Thus, we can express R as

$$R = \frac{1}{2}\left[(R + R^f) + (R - R^f)\right],$$

where $\frac{1}{2}(R + R^f)$ is trans-flip symmetric and $\frac{1}{2}(R - R^f)$ is trans-flip skew-symmetric.

We can also prove this result element-wise. Consider the non-zero square matrix of order n $R = [r_{ij}]$ on the field $\mathbb{C}$,

$$R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1k} \\ r_{21} & r_{22} & \cdots & r_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ r_{k1} & r_{k2} & \cdots & r_{kk} \end{bmatrix}.$$

Then, its flip R^f is given by

$$R^f = \begin{bmatrix} r_{kk} & r_{k(k-1)} & \cdots & r_{k1} \\ r_{(k-1)k} & r_{(k-1)(k-1)} & \cdots & r_{(k-1)1} \\ \vdots & \vdots & \ddots & \vdots \\ r_{1k} & r_{1(k-1)} & \cdots & r_{11} \end{bmatrix}.$$

Now,

$$R + R^f = \begin{bmatrix} r_{11} + r_{kk} & a_{12} + r_{k(k-1)} & \cdots & r_{1k} + r_{1k} \\ r_{21} + r_{(k-1)k} & a_{22} + r_{(k-1)(k-1)} & \cdots & r_{2k} + r_{1(k-1)} \\ \vdots & \vdots & \ddots & \vdots \\ r_{k1} + r_{k1} & r_{k2} + r_{(k-1)1} & \cdots & r_{kk} + r_{11} \end{bmatrix},$$

and

$$R - R^f = \begin{bmatrix} r_{11} - r_{kk} & r_{12} - r_{k(k-1)} & \cdots & r_{1k} - r_{1k} \\ r_{21} - r_{(k-1)k} & r_{22} - r_{(k-1)(k-1)} & \cdots & r_{2k} - r_{1(k-1)} \\ \vdots & \vdots & \ddots & \vdots \\ r_{k1} - r_{k1} & r_{k2} - r_{(k-1)1} & \cdots & r_{kk} - r_{11} \end{bmatrix}.$$

Adding $R + R^f$ and $R - R^f$, we get

$$R + R^f + R - R^f = \begin{bmatrix} 2r_{11} & 2r_{12} & \cdots & 2r_{1k} \\ 2r_{21} & 2r_{22} & \cdots & 2r_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 2r_{k1} & 2r_{k2} & \cdots & 2r_{kk} \end{bmatrix},$$

which gives,

$$\frac{1}{2}[R + R^f + R - R^f] = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1k} \\ r_{21} & r_{22} & \cdots & r_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ r_{k1} & r_{k2} & \cdots & r_{kk} \end{bmatrix} = R.$$

$\square$

In the next theorem, we outline properties that hold for the trans-flip operation, similar to those of the transpose of a matrix.

Theorem 4.3.2. *Let $R, S \in \mathbb{M}_{k\times k}(\mathbb{C})$ be non-zero matrices, and consider t represent any real numbers. Then the subsequent properties holds true.*

1. $(R^f)^f = R$.
2. $(tR)^f = t(R^f)$.
3. $(R + S)^f = R^f + S^f$.
4. $(RS)^f = S^f R^f$.
5. $(R^f)^T = (R^T)^f$.
6. $(\overline{R})^f = \overline{R^f}$.
7. $(R^f)^{-1} = (R^{-1})^f$, *if R is an invertible.*

Proof. 1. Let R be defined as a non-zero matrix, and consider $R = [r_{ij}] \in \mathbb{M}_{k\times k}(\mathbb{C})$. Then we get,

$$R^f = [r_{(k+1-j)(k+1-i)}].$$

Hence,

$$(R^f)^f = [r_{(k+1-(k+1-i))(k+1-(k+1-j))}] = [r_{ij}] = R.$$

2. Let $R = [r_{ij}] \in \mathbb{M}_{k\times k}(\mathbb{C})$. Then,

$$(tR)^f = (tr_{ij})^f = [tr_{(k+1-i)(k+1-j)}] = t[r_{(k+1-i)(k+1-j)}] = tR^f.$$

3. Let $R, S \in \mathbb{M}_{k\times k}(\mathbb{C})$ be non-zero matrices. Then $R+S = [r_{ij}+s_{ij}] = C = [c_{ij}]$. Thus,
$$(R+S)^f = [c^f_{ij}] = [c_{(k+1-j)(k+1-i)}]$$
$$= [r_{(k+1-j)(k+1-i)} + s_{(k+1-j)(k+1-i)}]$$
$$= [r_{(k+1-j)(k+1-i)}] + [s_{(k+1-j)(k+1-i)}]$$
$$= R^f + S^f.$$

4. Let $R = [r_{ij}]_{k\times m}$ and $S = [s_{ij}]_{m\times k}$, where $m, k \in \mathbb{N}$, be matrices with non-zero entries from the field of complex numbers. Then,
$$RS = C = [c_{ij}]_{k\times k}, \quad c_{ij} = \sum_{m=1}^{q} r_{ik}s_{kj}.$$
Therefore,
$$(RS)^f = C^f = [c_{(k+1-j)(k+1-i)}]_{k\times k}.$$
The (i,j)-th entry of $(RS)^f$ is defined as
$$c^f_{ij} = c_{(k+1-j)(k+1-i)}$$
$$= \sum_{m=1}^{q} r_{(k+1-j)}s_{(k+1-i)}$$
$$= \sum_{m=1}^{q} s_{(k+1-i)}r_{(k+1-j)}.$$
This corresponds to the (i,j)-th entry of S^fR^f for all $i,j = 1,2,\ldots,k$.

5. Let $R = [r_{ij}] \in \mathbb{M}_{k\times k}(\mathbb{C})$. Then $R^T = [r_{ji}]$. Therefore,
$$(R^T)^f = [r_{(k+1-i)(k+1-j)}]$$
$$= [r_{(k+1-j)(k+1-i)}] = (R^f)^T.$$

6. Let $R \in \mathbb{M}_{k\times k}(\mathbb{C})$. Then $\overline{R} = [\overline{r_{ij}}]_{k\times k}$. Thus,
$$\overline{R}^f = [\overline{r}_{(k+1-j)(k+1-i)}]_{k\times k}$$
$$= \overline{[r_{(k+1-j)(k+1-i)}]}_{k\times k} = \overline{R^f}.$$

7. Let $R \in \mathbb{M}_{k\times k}(\mathbb{C})$ and $|R| \neq 0$, then $RR^{-1} = I$. Taking the operation f on both sides, we get,
$$(RR^{-1})^f = I^f = I,$$

and consequently,

$$(R^{-1})^f R^f = I.$$

From the above, we can conclude that

$$(R^f)^{-1} = (R^{-1})^f.$$

□

The following corollary specifies the characteristics of the off-diagonal elements in a trans-flip skew-symmetric matrix, outlining how these elements behave uniquely under the trans-flip operation compared to traditional skew-symmetric matrices.

Corollary 4.3.1. *For a non-zero, square trans-flip skew symmetric matrix specified over the field $\mathbb{C}$, all the off-diagonal elements are always zero.*

Proof. Let $R = [r_{ij}]$ be non-zero matrix of order k where $r_{ij} \in \mathbb{C}$ for all $i, j = 1, 2, 3, \ldots, k$. According to the definition of a trans-flip skew-symmetric matrix, we can express,

$$r_{ij} = r_{(k+1-j)(k+1-i)} \;=\; -r_{(k+1-j)(k+1-i)} \quad \text{for all}\, i + j = k + 1,$$

which implies

$$2r_{ij} \;=\; 2r_{(n+1-j)(n+1-i)} \;=\; 0.$$

Thus,

$$r_{ij} \;=\; r_{(n+1-j)(n+1-i)} \;=\; 0 \quad \text{for all}\, i + j = k + 1.$$

□

In the next theorem, we explain properties that hold true for the trans-flip conjugate operation, similar to how they hold true for the conjugate transpose of the matrix, where $\bar{R}^f = R^{\Theta}$.

Theorem 4.3.3. *Let $R, S \in \mathbb{M}_{k\times k}(\mathbb{C})$ be non-zero matrices and t be a scalar, then,*

1. $(R^{\Theta})^{\Theta} = R$.
2. $(tR)^{\Theta} = tR^{\Theta}$, *where* $t \in \mathbb{C}$.
3. $(R + S)^{\Theta} = R^{\Theta} + S^{\Theta}$.
4. $(RS)^{\Theta} = S^{\Theta} R^{\Theta}$.

Proof. These results can be proven similarly as in Theorem 4.3.2, by taking the conjugate of the elements of the trans-flip of the matrix. □

In the subsequent theorem, we derive a significant relationship between the trans-flip of a matrix and its transpose concerning the determinant, trace, and eigenvalues.

Theorem 4.3.4. *For any non-zero square matrix R over the field $\mathbb{C}$, we have* $\det(R) = \det(R^T) = \det(R^f)$.

Proof. Let $R \in \mathbb{M}_{k\times k}(\mathbb{C})$ be a non-zero square matrix and

$$\det(R) = \begin{vmatrix} r_{11} & r_{12} & \cdots & r_{1k} \\ r_{21} & r_{22} & \cdots & r_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ r_{k1} & r_{k2} & \cdots & r_{kk} \end{vmatrix}.$$

We can define the determinant of R^f as follows,

$$\det(R^f) = \begin{vmatrix} r_{kk} & r_{(k-1)k} & \cdots & r_{1k} \\ r_{k(k-1)} & r_{(k-1)(k-1)} & \cdots & r_{1(k-1)} \\ \vdots & \vdots & \ddots & \vdots \\ r_{k1} & r_{(k-1)1} & \cdots & r_{11} \end{vmatrix}.$$

By applying row interchange operations,

$$R_1 \leftrightarrow R_k, R_2 \leftrightarrow R_{k-1}, \ldots, R_{\frac{k}{2}} \leftrightarrow R_{\frac{k}{2}+1}, \quad \text{if } k \text{ is an even number, and}$$

$$R_{\frac{k-1}{2}} \leftrightarrow R_{\frac{k-1}{2}}, \quad \text{if } k \text{ is an odd number.}$$

This leads to,

$$\det(R^f) = (-1)^{\frac{k}{2}} \begin{vmatrix} r_{k1} & r_{(k-1)1} & \cdots & r_{11} \\ r_{k2} & r_{(k-1)2} & \cdots & r_{12} \\ \vdots & \vdots & \ddots & \vdots \\ r_{kk} & r_{(k-1)k} & \cdots & r_{1k} \end{vmatrix}, \quad \text{if } k \text{ is an even number.}$$

Similarly, for odd k,

$$\det(A^f) = (-1)^{\frac{k-1}{2}} \begin{vmatrix} r_{k1} & r_{(k-1)1} & \cdots & r_{11} \\ r_{k2} & r_{(k-1)2} & \cdots & r_{12} \\ \vdots & \vdots & \ddots & \vdots \\ r_{kk} & r_{(k-1)k} & \cdots & r_{1k} \end{vmatrix}, \quad \text{if } k \text{ is an odd number.}$$

Next, by interchanging the columns of $\det(R^f)$,

$$C_1 \leftrightarrow C_k, C_2 \leftrightarrow C_{k-1}, \ldots, C_{\frac{k}{2}} \leftrightarrow C_{\frac{k}{2}+1}, \quad \text{if } k \text{ is an even number, and}$$

$$C_{\frac{k-1}{2}} \leftrightarrow C_{\frac{k-1}{2}}, \quad \text{if } k \text{ is an odd number.}$$

Thus, we obtain,

$$\det(R^f) = (-1)^{\frac{k}{2}}(-1)^{\frac{k}{2}} \begin{vmatrix} r_{11} & r_{21} & \cdots & r_{k1} \\ r_{12} & r_{22} & \cdots & r_{k2} \\ \vdots & \vdots & \ddots & \vdots \\ r_{1k} & r_{2k} & \cdots & r_{kk} \end{vmatrix}, \text{ if } k \text{ is an even number.}$$

Similarly,

$$\det(R^f) = (-1)^{\frac{k-1}{2}}(-1)^{\frac{k-1}{2}} \begin{vmatrix} r_{11} & r_{21} & \cdots & r_{k1} \\ r_{12} & r_{22} & \cdots & r_{k2} \\ \vdots & \vdots & \ddots & \vdots \\ r_{1k} & r_{2k} & \cdots & r_{kk} \end{vmatrix}, \text{ if } k \text{ is an odd number.}$$

Combining these equations, we get,

$$\det(R^f) = \begin{vmatrix} r_{11} & r_{21} & \cdots & r_{k1} \\ r_{12} & r_{22} & \cdots & r_{k2} \\ \vdots & \vdots & \ddots & \vdots \\ r_{1k} & r_{2k} & \cdots & r_{kk} \end{vmatrix} = \det(R^T).$$

Hence, $\det(R^f) = \det(R^T) = \det(R)$. □

Theorem 4.3.5. *If $R = [r_{ij}]$ is any arbitrary non-zero matrix, Let R be a square matrix with $r_{ij} \in \mathbb{C}_{k\times k}(\mathbb{C})$ for every $i, j = 1, 2, \ldots, k$. Then, we have*

$$Tr(R) = Tr(R^T) = Tr(R^f).$$

Proof. Consider the arbitrary non-zero square matrix $R = [r_{ij}]$ where $r_{ij} \in \mathbb{C}$. The trace of R is the sum of its diagonal entries,

$$\text{Tr}(R) = r_{11} + r_{22} + r_{33} + \cdots + r_{kk}.$$

For the matrix $R^f = [r_{(k+1-j)(k+1-i)}]$, the diagonal elements are

$$r_{kk}, r_{(k-1)(k-1)}, \cdots, r_{22}, r_{11}.$$

Thus,

$$\text{trace}(R^f) = r_{kk} + r_{(k-1)(k-1)} + \cdots + r_{22} + r_{11}$$

$$= r_{11} + r_{22} + r_{33} + \cdots + r_{kk} = \text{trace}(R).$$

Therefore, $\text{Trace}(R) = \text{Trace}(R^T) = \text{Trace}(R^f)$. $\square$

Corollary 4.3.2. *In a similar manner, we can show that Trace*$(R^f) =$ *Trace*$((R^f)^T)$.

Theorem 4.3.6. *Let $R \in \mathbb{M}_{k\times k}(\mathbb{C})$ be a non-zero matrix. Then the eigenvalues of R and R^f are the same.*

Proof. Consider $R \in \mathbb{M}_{k\times k}(\mathbb{C})$ be a non-zero square matrix. The characteristic polynomial of R is expressed as $|R - \omega I|$, where ω denotes the eigenvalues of R. Similarly, the characteristic polynomial of R^f is $|R^f - \omega I|$. From the previous theorem, we have

$$|R| = |R^f|.$$

Thus,

$$|(R^f - \omega I)^f| = |(R^f)^f - \omega I^f| = |R - \omega I|.$$

Therefore, R and R^f share an identical characteristic polynomial and, consequently, the same eigenvalues. $\square$

Theorem 4.3.7. *Let $R = [r_{ij}] \in \mathbb{M}_{k\times k}(\mathbb{C})$ be a non-zero matrix and I_0 defines a square matrix of order k. Then*

$$R = I_{0(n\times n)}(R^f)^T I_{0(k\times k)}.$$

Proof. Consider the arbitrary non-zero matrix R such that $R \in \mathbb{M}_{k\times k}(\mathbb{C})$. Then,

$$R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1(k-1)} & r_{1k} \\ r_{21} & r_{22} & \cdots & r_{2(k-1)} & r_{2k} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r_{(k-1)1} & r_{(k-1)2} & \cdots & r_{(k-1)(k-1)} & r_{(k-1)k} \\ r_{k1} & r_{k2} & \cdots & r_{k(k-1)} & r_{kk} \end{bmatrix}.$$

Additionally, the matrix R^f is defined as,

$$R^f = \begin{bmatrix} r_{kk} & r_{(k-1)k} & \cdots & r_{2k} & r_{1k} \\ r_{k(k-1)} & r_{(k-1)(k-1)} & \cdots & r_{2(k-1)} & r_{1(k-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r_{k2} & r_{(k-1)2} & \cdots & r_{22} & r_{12} \\ r_{k1} & r_{(k-1)1} & \cdots & r_{21} & r_{11} \end{bmatrix}.$$

Let the matrix $I_{0(k\times k)}$ be defined as,

$$I_{0(k\times k)} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Then

$$I_{0(k\times k)}(R^f)^T = \begin{bmatrix} r_{kk} & r_{k(k-1)} & \cdots & r_{k1} \\ r_{(k-1)k} & r_{(k-1)(k-1)} & \cdots & r_{(k-1)1} \\ \vdots & \vdots & \vdots & \vdots \\ r_{k2} & r_{(k-1)2} & \cdots & r_{22} \\ r_{k1} & r_{(k-1)1} & \cdots & r_{21} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

$$I_{0(k\times k)}(R^f)^T = \begin{bmatrix} r_{k1} & r_{k2} & \cdots & r_{k(k-1)} & r_{kk} \\ r_{(k-1)1} & r_{(k-1)2} & \cdots & r_{(k-1)(k-1)} & r_{(k-1)k} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r_{21} & r_{22} & \cdots & r_{2(k-1)} & r_{2k} \\ r_{11} & r_{12} & \cdots & r_{1(k-1)} & r_{1k} \end{bmatrix}$$

Furthermore,

$$I_{0(k\times k)}(R^f)^T I_{0(k\times k)} = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1(k-1)} & r_{1k} \\ r_{21} & r_{22} & \cdots & r_{2(k-1)} & r_{2k} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r_{(k-1)1} & r_{(k-1)2} & \cdots & r_{(k-1)(k-1)} & r_{(k-1)k} \\ r_{k1} & r_{k2} & \cdots & r_{k(k-1)} & r_{kk} \end{bmatrix} = R.$$

$\square$

Theorem 4.3.8. *For arbitrary vectors x and $z \in \mathbb{R}^k$, we have*

$$(\bar{x})^T z = (\bar{x})^f \left((z^f)^T\right).$$

Proof. Let $x, z \in \mathbb{R}^k$. Then

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_k \end{bmatrix} \quad \text{and} \quad z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_k \end{bmatrix}.$$

We have,

$$(\bar{x})^f((z^f)^T) = \begin{bmatrix} \bar{x}_k & \bar{x}_{k-1} & \cdots & \bar{x}_2 & \bar{x}_1 \end{bmatrix} \begin{bmatrix} z_k \\ z_{k-1} \\ z_{k-2} \\ \vdots \\ z_1 \end{bmatrix}.$$

$$(\bar{x})^f(z^f)^T = \bar{x}_k z_k + \bar{x}_{k-1} z_{k-1} + \bar{x}_{k-2} z_{k-2} + \cdots + \bar{x}_2 z_2 + \bar{x}_1 z_1 = barx)^T z$$

Thus,

$$(\bar{x})^T z = (\bar{x})^f((z^f)^T).$$

$\square$

The matrix RR^T is always a $\mathscr{PD}$. However, RR^f is generally not symmetric, and thus cannot be guaranteed to be $\mathscr{PD}$. In the subsequent theorem, we will demonstrate that for a square symmetric matrix R, RR^T is always $\mathscr{PD}$. Conversely, this result does not hold for the matrix RR^f under the same conditions.

Theorem 4.3.9. *Let $R \in \mathbb{M}_{k\times k}(\mathbb{C})$. Then RR^f is not positive definite for $k \geq 3$.*

Proof. Consider $R \in \mathbb{M}_{k\times k}(\mathbb{C})$. Generally, RR^f is not a symmetric matrix, and thus, RR^f is not positive definite. However, for $k = 2$, RR^f has a positive determinant as well as a positive trace. This implies that RR^f has positive eigenvalues. To demonstrate that the result does not hold for $k \geq 3$, consider the matrix

$$R = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 4 \\ 5 & 3 & 3 \end{bmatrix}.$$

For this matrix, RR^f has eigenvalues $-0.2862, -1.2829,$ and 43.5692, which include negative values. This confirms that RR^f is not positive definite when $k \geq 3$. $\square$

In the next section, we will explore the behavior of pattern-based matrix where $R = [|p_i - p_j|]$ is trans-flip symmetric rather than symmetric.

4.4 Pattern-based Matrices Results

In the next theorem, we will explore the condition under which the pattern-based matrix $R = [|p_i - p_j|]$ R is trans-flip symmetric.

Theorem 4.4.1. *Let $R = [|p_i - p_j|]$ be a trans-flip symmetric matrix of size k, $p_1, p_2, ..., p_k$ form AP series. Then the difference of consecutive p_i must be the same, i.e., $p_{t+1} - p_t = p_{l+1} - p_l$ for all $t, l = 1, 2, \cdots, k-1$.*

Proof. Let $p_1, p_2, \ldots, p_k$ represent distinct real numbers. Then

$$R = \begin{bmatrix} 0 & |p_1 - p_2| & \cdots & |p_1 - p_k| \\ |p_2 - p_1| & 0 & \cdots & |p_2 - p_k| \\ \vdots & \vdots & \ddots & \vdots \\ |p_k - p_1| & |p_k - p_2| & \cdots & 0 \end{bmatrix}.$$

Then,

$$R^f = \begin{bmatrix} 0 & |p_{k-1} - p_k| & \cdots & |p_1 - p_k| \\ |p_k - p_{k-1}| & 0 & \cdots & |p_{k-1} - p_{k-2}| \\ \vdots & \vdots & \ddots & \vdots \\ |p_k - p_1| & |p_{k-1} - p_1| & \cdots & 0 \end{bmatrix}.$$

$R = R^f$ implies $|p_1 - p_2| = |p_2 - p_3| = \cdots = |p_k - p_{k-1}|$. □

4.5 Conclusion

We have defined a new matrix operation, the trans-flip of a matrix. While many properties applicable to the transpose of a matrix are also valid for the trans-flip operation, there are some exceptions where these properties do not translate. For instance, the distributive property over addition and scalar multiplication holds for both the transpose and trans-flip operations. However, properties like positive definiteness of the trans-flip of matrices do not necessarily follow the same rules as the positive definiteness of the transpose of a matrix. These differences highlight unique aspects of the trans-flip operation compared to the transpose. Understanding both the similarities and distinctions helps in applying the trans-flip operation accurately in various contexts.

Chapter 5

Horizontal-flip, Vertical-flip and Trans-flip

5.1 Introduction

Matrix transformations, particularly horizontal and vertical flips, are essential components of linear algebra, playing a crucial role in various scientific and engineering fields. These flips rearrange a matrix's elements, creating a mirrored image along a designated axis. This work presents a comprehensive analysis of these transformations, focusing on their mathematical formulation and practical significance. We investigate the mathematical properties, underlying theorems, and practical applications of these flips, which are widely utilized in areas such as image processing, pattern recognition, and algorithm design. The study offers an in-depth exploration of these operations, highlighting their relevance and utility across multiple domains.

To expand the concept of the relation between the horizontal-flip, vertical-flip, and trans-flip, we need to introduce the foundational definitions of these matrix operations, describe their properties, and derive their interrelations.

Horizontal-Flip of a Matrix

5.1.1. Horizontal-Flip of a Matrix

A horizontal flip, also referred to as a row-wise reflection, is a matrix transformation that reverses the order of columns in each row, and effectively creating a mirror image of the matrix along its vertical axis. Let $R \in \mathbb{M}_{m\times k}(\mathbb{C})$ be an $m \times k$ matrix with complex entries. The horizontally flipped matrix, denoted by $R^h = [r^h_{ij}]$, and defined as

$$r^h_{i,j} = r_{i,k-j+1}, \quad \text{for } 1 \leq i \leq m, \quad 1 \leq j \leq k. \tag{5.1}$$

Where i and j represents the row index and the column index respectively. This transformation maintains the row structure while reversing the column ordering.

Example 5.1.1. *Consider the matrix $R \in \mathbb{M}_{m\times k}(\mathbb{C})$ represented as,*

$$R = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Applying the horizontal flip operation, each row's elements are reversed in order, resulting in,

$$R^h = \begin{bmatrix} 3 & 2 & 1 \\ 6 & 5 & 4 \\ 9 & 8 & 7 \end{bmatrix}.$$

Thus, the horizontal flip operation maintains the original matrix's structure while reflecting its elements along the vertical axis. In other words,

For a matrix R, the horizontal flip can be represented as $R_H = R \times P_H$, where P_H is a permutation matrix that reverses the columns.

Vertical Flip of a Matrix

5.1.2. Vertical Flip of a Matrix

A vertical flip, also referred to as a row-wise reflection, and a matrix transformation that reverses the order of rows while keeping the column elements unchanged. This operation effectively mirrors the matrix along its horizontal axis.

Formally, let $R \in \mathbb{M}_{m\times k}(\mathbb{C})$ be an $m \times k$ matrix with complex entries. The vertically flipped matrix, denoted by $R^v = [r^v_{ij}]$, is defined as,

$$r^v_{i,j} = r_{m-i+1,j}, \quad \text{for } 1 \leq i \leq m, \quad 1 \leq j \leq k. \tag{5.2}$$

Here, i and j represent the row and column indices, respectively, while m denotes the total number of rows in the matrix.

Example 5.1.2. *Consider the matrix $R \in \mathbb{M}_{m\times k}(\mathbb{C})$ represented as,*

$$R = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Applying the vertical flip transformation, the order of the rows is reversed while preserving the column structure, resulting in,

$$R^v = \begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}.$$

Thus, the vertical flip operation maintains the structural integrity of the matrix while reflecting its rows along the horizontal axis. In other words,

For a matrix R, the vertical flip can be represented as $R_v = R \times P_v$, where P_v is a permutation matrix that reverses the rows.

Diagonal Flip (Transpose) of a Matrix

5.1.3. Diagonal Flip (Transpose) of a Matrix

A diagonal flip, commonly known as the transpose of a matrix, and a transformation that reflects the matrix across its main diagonal. This operation interchanges the rows and columns, meaning that the element located at position (i, j) in the original matrix is mapped to position (j, i) in the transposed matrix.

Formally, for a given matrix $R \in \mathbb{M}_{m\times k}(\mathbb{C})$, the transpose of $R = [r_{ij}]$, denoted as $R^T = [r_{ji}]$, is defined by

$$r_{i,j} = r_{j,i}, \quad \text{for } 1 \leq i \leq m, \quad 1 \leq j \leq k. \tag{5.3}$$

Example 5.1.3. *Consider the matrix $R \in \mathbb{M}_{m\times k}(\mathbb{C})$ defined as,*

$$R = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Applying the transpose operation, each element at position (i, j) is moved to (j, i), resulting in,

$$R^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}.$$

Thus, the transpose operation effectively swaps the rows and columns, preserving the overall structure while reorienting the data along the main diagonal.

Matrix flip operations are fundamental transformations that can be expressed using permutation matrices. These operations are involutory in nature, meaning that applying the same operation twice restores the original matrix. Formally, let f in general represent a flip operation, such as a horizontal flip, vertical flip, or diagonal flip (transpose). Then, for any matrix R, the involutory property can be expressed as,

$$f(f(R)) = R. \tag{5.4}$$

This property ensures that repeated application of the flip operation does not alter the matrix beyond the first transformation.

In the subsequent section, we will explore the relationships among horizontal flip, vertical flip, and diagonal flip (transpose). Understanding these interconnections will enhance our ability to perform mathematical analysis and will provide valuable insights for applications in computing and the software industry. By establishing these relationships, we aim to develop a comprehensive framework for matrix transformations, which is crucial for various computational and algorithmic implementations.

5.2 Relationship Between Horizontal Flip, Vertical Flip, and Diagonal Flip

Theorem 5.2.1. *Let $R \in \mathbb{M}_{m\times k}(\mathbb{C})$ be an $m \times k$ matrix over the complex field. The trans-flip of R, denoted as R^f, is defined as the composition of a horizontal flip followed by a vertical flip, represented as,*

$$R^f = R^h \circ R^v, \tag{5.5}$$

where R^h denotes the horizontal flip of R, and R^v represents its vertical flip.

Proof. Consider the matrix $R \in \mathbb{M}_{m\times k}(\mathbb{C})$ given by,

$$R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1k} \\ r_{21} & r_{22} & \cdots & r_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ r_{m1} & r_{m2} & \cdots & r_{mk} \end{bmatrix}. \tag{5.6}$$

The horizontal flip operation rearranges the columns in reverse order, producing the transformed matrix.

$$R^h = \begin{bmatrix} r_{1k} & r_{1(k-1)} & \cdots & r_{11} \\ r_{2k} & r_{2(k-1)} & \cdots & r_{21} \\ \vdots & \vdots & \ddots & \vdots \\ r_{mk} & r_{m(k-1)} & \cdots & r_{m1} \end{bmatrix}. \tag{5.7}$$

Subsequently, performing the vertical flip inverts the row order, resulting in the following matrix.

$$R^v = \begin{bmatrix} r_{m1} & r_{m2} & \cdots & r_{mk} \\ r_{(m-1)1} & r_{(m-1)2} & \cdots & r_{(m-1)k} \\ \vdots & \vdots & \ddots & \vdots \\ r_{11} & r_{12} & \cdots & r_{1k} \end{bmatrix}. \tag{5.8}$$

Now, applying the vertical flip R^v to the horizontally flipped matrix R^h results in,

$$R^f = R^h \circ R^v = \begin{bmatrix} r_{mk} & r_{m(k-1)} & \cdots & r_{m1} \\ r_{(m-1)k} & r_{(m-1)(k-1)} & \cdots & r_{(m-1)1} \\ \vdots & \vdots & \ddots & \vdots \\ r_{1k} & r_{1(k-1)} & \cdots & r_{11} \end{bmatrix}. \tag{5.9}$$

This confirms that the composition of a horizontal flip followed by a vertical flip results in a complete reversal of the matrix's elements along both axes, thereby defining the trans-flip operation. □

In the following theorem, we establish that the composition of the horizontal flip and vertical flip operations satisfies the commutative property.

Theorem 5.2.2. *Let $R \in \mathbb{R}_{m\times k}(\mathbb{C})$ be a matrix. The horizontal flip followed by the vertical flip, and vice versa, yield the same transformed matrix, denoted as R^f. Formally, these operations commute, i.e.,*

$$R^h(R^v) = R^v(R^h) = R^f.$$

Proof. Consider an $m \times k$ matrix R over the field of complex numbers, where the element at row i and column j is denoted by r_{ij}.

Now, we show that applying these flips in different sequences results in the same matrix.

Horizontal Flip Followed by Vertical Flip. Applying the vertical flip first transforms R into R^v with elements

$$(r^v)_{ij} = r_{(m-i+1)j}.$$

Applying the horizontal flip to R^v gives

$$(r^h(r^v))_{ij} = (r^v)_{i(k-j+1)} = r_{(m-i+1)(k-j+1)}.$$

Vertical Flip Followed by Horizontal Flip. Applying the horizontal flip first transforms R into R^h with elements

$$(r^h)_{ij} = r_{i(k-j+1)}.$$

Applying the vertical flip to R^h gives

$$(r^v(r^h))_{ij} = (r^h)_{(m-i+1)j} = r_{(m-i+1)(k-j+1)}.$$

Since both sequences gives the same resulting matrix, we conclude that

$$R^h(R^v) = R^v(R^h) = R^f.$$

Thus, the horizontal and vertical flips commute. $\square$

Proposition 5.2.1. *Let R be a matrix of order $m \times k$. The horizontal flip of R, denoted by R^h, satisfies the relation*

$$(R^h)^T = (R^T)^v,$$

where T represents the transpose operation.

Proposition 5.2.2. *If R is a symmetric matrix, then its horizontal flip R^h remains symmetric.*

Proposition 5.2.3. *For any matrix $R \in \mathbb{R}_{m\times k}(\mathbb{C})$, the composition of horizontal and vertical flips satisfies*

$$(R^h)^v = (R^v)^h = JRJ,$$

where J is the exchange matrix (or anti-diagonal identity matrix) with ones along the anti-diagonal, representing both horizontal and vertical flipping.

Remark 5.1. Matrix flips are widely used in image processing, where flipping an image horizontally or vertically corresponds to these operations on the matrix representation of the image.

In the following section, we will explore the implementation of horizontal flip, vertical flip, and trans-flip operations in MATLAB and C++ programming, demonstrating their practical applications in computational mathematics and software development.

5.3 Applications in MATLAB and C++ Programming

5.3.1 Use of horizontal and vertical flip in MATLAB

In MATLAB, we can use the functions `flipud` (flip up-down) and `fliplr` (flip left-right) to perform vertical and horizontal flips, respectively. Here is how we can use them
Horizontal Flip (Left-Right Flip)
The `flipud` function flips the matrix horizontally, reversing the order of columns.

Example (MATLAB code)
Input

$$R = [1, 2, 3; 4, 5, 6; 7, 8, 9];$$

$$S = \texttt{flipud}(R);$$

$$disp(S);$$

Output

$$[3, 2, 1; 6, 5, 4; 9, 8, 7]$$

Vertical Flip (Up-Down Flip)
The `flipud` function flips the matrix vertically, reversing the order of rows.

Example (MATLAB code) Input

$$R = [1, 2, 3; 4, 5, 6; 7, 8, 9];$$

$$C = \texttt{flipud}(R);$$

$$disp(C);$$

Output

$$[7,8,9;4,5,6;1,2,3]$$

Combine Horizontal and Vertical Flips

```
Function to flip a matrix horizontally
function flippedMatrix = horizontalFlip(matrix)
    flippedMatrix = matrix(:, end:-1:1);  % Flip each row
end

Function to flip a matrix vertically
function flippedMatrix = verticalFlip(matrix)
    flippedMatrix = matrix(end:-1:1, :);  % Flip each column
end

Function to perform combined horizontal and vertical flip
function flippedMatrix = combinedFlip(matrix)
    % First flip horizontally, then flip vertically
    flippedMatrix = horizontalFlip(matrix);
    flippedMatrix = verticalFlip(flippedMatrix);
end

Example usage
matrix = [2 4 5 7; 9 11 12 14; 16 18 19 21; 23 24 26 28];
disp('Original Matrix:');
disp(matrix);
flippedMatrix = combinedFlip(matrix);
disp('Combined Horizontally and Vertically Flipped Matrix:');
disp(flippedMatrix);
```

5.3.2 C++ program for horizontal and vertical flip

```
#include <iostream>
#include <vector>

using namespace std;
```

```
void horizontalFlip(vector<vector<int>>& matrix) {
    int rows = matrix.size();
    int cols = matrix[0].size();
    for (int i = 0; i < rows; i++) {
        for (int j = 0; j < cols / 2; j++) {
            swap(matrix[i][j], matrix[i][cols - j - 1]);
        }
    }
}

void verticalFlip(vector<vector<int>>& matrix) {
    int rows = matrix.size();
    int cols = matrix[0].size();
    for (int i = 0; i < rows / 2; i++) {
        for (int j = 0; j < cols; j++) {
            swap(matrix[i][j], matrix[rows - i - 1][j]);
        }
    }
}

void combinedFlip(vector<vector<int>>& matrix) {
    horizontalFlip(matrix);
    verticalFlip(matrix);
}

void printMatrix(const vector<vector<int>>& matrix) {
    for (const auto& row : matrix) {
        for (int val : row) {
            cout << val << " ";
        }
        cout << endl;
    }
}

int main() {
    vector<vector<int>> matrix = {
        {2, 4, 5, 7},
        {9, 11, 12, 14},
        {16, 18, 19, 21},
```

```
        {23, 24, 26, 28}
    };

    cout << "Initial Matrix:" << endl;
    printMatrix(matrix);

    combinedFlip(matrix);

    cout << "Matrix After Combined Horizontal and Vertical Flip:" << endl;
    printMatrix(matrix);

    return 0;
}
```

Input

$$\begin{bmatrix} 2 & 4 & 5 & 7 \\ 9 & 11 & 12 & 14 \\ 16 & 18 & 19 & 21 \\ 23 & 24 & 26 & 28 \end{bmatrix}$$

Output (after combined flip)

$$\begin{bmatrix} 28 & 26 & 24 & 23 \\ 21 & 19 & 18 & 16 \\ 14 & 12 & 11 & 9 \\ 7 & 5 & 4 & 2 \end{bmatrix}$$

5.4 Outcome

The study of horizontal and vertical flips of matrices is crucial across various domains. Understanding their mathematical properties, such as involution and symmetry preservation, and their practical applications, particularly in image processing and computer graphics, can lead to innovative developments in software and algorithm design.

Conclusion

This thesis makes significant contributions to the spectral analysis of pattern-based matrices, particularly through the study of matrices formed via nonlinear operator concave functions. By examining the inertia, spectral characteristics, and conditionally negative definiteness of these matrices, this research provides a nuanced understanding of their behavior, especially under nonlinear transformations. The introduction of the trans-flip operation as an alternative to traditional matrix transformations highlights distinctions in properties like positive definiteness, underscoring the unique behavior of these matrices in various mathematical contexts. Additionally, our examples of concave functions that fail to exhibit operator concavity shed light on subtle differences in concavity interpretations, enriching the theoretical framework.

The foundational lemmas and connections established between our results and previous works serve as essential links, deepening the discourse within operator theory and functional analysis. Finally, by critically exploring limitations in previous findings, notably those by Garg and Aujla, this research paves the way for further investigation into matrix behavior under complex transformations. These contributions are anticipated to inspire future studies across applied mathematics and theoretical fields where understanding matrix transformations remains vital.

Applications

Applications in Software and Domains

1. Image Processing: Trans-flip is used to mirror images. In deep learning, these operations are common for data augmentation to improve the robustness of models.
2. Software Examples: Libraries like OpenCV, TensorFlow, and PyTorch provide functions for flipping images.

3. Computer Graphics: Trans-Flip are used in rendering transformations and sprite animations. In video game development, flipping a sprite horizontally or vertically can represent different directions of movement.

4. Signal Processing: In time-series analysis, trans-flip of matrices (representing signals) can help in reversing the direction of a signal or in convolution operations.

5. Quantum Computing: Matrix trans-flips can represent certain quantum gates and operations, especially in the context of quantum error correction and symmetry operations.

6. Machine Learning: In convolutional neural networks (CNNs), flipping operation are used during feature extraction and image transformation stages.

Future Scope

The future scope of the Trans-Flip operation, especially when analyzed through pattern-based matrices and in the context of operator monotone, concave, and convex functions, is expansive and transformative across multiple domains

1. **Pattern-Based Matrices in Image Processing and Graphics**
   Leveraging Trans-Flip in pattern-based matrices can enhance various transformations in image processing. By identifying patterns in structured matrices, new methods for symmetry operations and efficient mirroring could emerge, especially useful in real-time rendering in graphics and video games. Complex image manipulations can also be made computationally efficient, which is crucial in augmented and virtual reality.

2. **Enhanced Data Augmentation in Deep Learning**
   Using Trans-Flip transformations tailored to specific patterns and symmetry properties could improve data augmentation in deep learning frameworks, particularly for convolutional neural networks (CNNs). With operator monotone and concave properties, adaptive flipping mechanisms could be developed to extract richer features from data, improving robustness and generalization capabilities in image recognition and object detection models.

3. **Signal Processing Applications in Pattern Analysis**
   Pattern-based trans-flips could introduce new tools for signal processing, especially in time-series analysis where matrix patterns represent complex signals. Operator monotone and concave transformations could help refine signal reversal processes and convolution operations. This could benefit fields like audio processing, telecommunications, and seismology, where identifying and reconstructing signal patterns are crucial.

4. **Quantum Computing and Quantum Information Theory**
   Trans-Flip could support new matrix transformations in quantum computing [9] by creating unique symmetry operations related to quantum gates. With operator

monotone and concave functions, there could be developments in error correction methods and state preparation in quantum algorithms. Trans-Flip might also aid in designing more efficient quantum circuits that leverage symmetries, supporting scalable quantum architectures.

5. **Advancements in Machine Learning Feature Engineering**
   In machine learning, especially within CNNs, identifying Trans-Flip operations through operator convex or concave matrices could support advanced feature extraction methods. These new methods may identify patterns that existing transformations cannot capture, enhancing model accuracy in complex domains like medical imaging, autonomous driving, and facial recognition.

6. **Mathematical Research on Matrix Theory and Operator Theory**
   Research into Trans-Flip operations within matrix theory could open new avenues in operator monotone and operator concave function studies. This exploration could lead to generalizations of matrix transformations, potentially creating new function classes. These advances could deepen our understanding of matrix operations in mathematical physics, optimization problems, and theoretical computer science.

7. **Cross-Disciplinary Applications in Robotics and Motion Analysis**
   In robotics and motion analysis, where pattern-based matrices are often employed, trans-flip can be utilized for symmetrical motion transformations. Integrating operator monotone and concave functions here could streamline motion planning algorithms and real-time object manipulation, contributing to innovations in robotic control and biomechanics.

Exploring the Trans-Flip with respect to pattern-based matrices and operator functions could thus serve as a basis for groundbreaking tools and methods in both theoretical and applied fields.

Paper Publications

Papers Published from the Thesis

1. Agarwal, H., & Garg, I. (2022, May). A Brief Review of Operator Monotone and Operator Convex Functions. In Journal of Physics: Conference Series (Vol. 2267, No. 1, p. 012087). IOP Publishing. (**Scopus**).

2. Garg, I., & Agarwal, H. (2023). Spectral behaviour of the matrix $[f(1 - p_i p_j)]$. Positivity, 27(3), 41. (**Scopus, WoS, SCI**).

Paper Presentations

Papers Presented in Conferences

1. Agarwal, H., & Garg, I. (2022, May). A Brief Review of Operator Monotone and Operator Convex Functions. In Journal of Physics: Conference Series (Vol. 2267, No. 1, p. 012087). IOP Publishing.
2. Garg, I., & Agarwal, H. (2023). Spectral behaviour of the matrix $[f(1-p_ip_j)]$. Positivity, 27(3), 41.

Conferences, Webinars and Workshops

Conferences & Webinars

1. Attended international conference on **Recent Advances in Fundamental and Applied Sciences (RAFAS)**, November 5-6, 2019. LPU Jalandhar.
2. Three day international webinar on **General mathematical tools in research: theory and Application** June 15-17, 2020.
3. National webinar on **Data Driven Research in Mathematics and Statistics** June 27, 2020.
4. International conference on **Recent Trends in Mathematics and its Application** July 20-24, 2020 JNU.
5. One Week Confederated **Online Faculty Development Program on "Orienting Applications & Conceptualized Aspects of Sciences & Humanities** July 21-25, 2020.
6. One day webinar on **Role and Challenges in Science and Technology** July 27, 2020.
7. Participated in paper presentation in international conference on **Recent Advances in Fundamental and Applied Sciences (RAFAS)** , June 25-26, 2021. LPU Jalandhar.
8. Attended international conference on **Advances in Mathematical Sciences**, October 4-6, 2021. Nagpur University.
9. Participated in international conference on **Advancing Technology Development, Research Excellence and Innovations**, October 22, 2022.

10. Attended international conference on **Recent Advances in Fundamental and Applied Sciences' (RAFAS)**, March 24-25, 2023. LPU Jalandhar.

11. Attended **108th Indian Science Congress Conference**, January 03 – 07, 2023. Nagpur University.

12. Attended international conference on **Nonlinear Dynamics and Applications**, February 13-16, 2023. IIT Indore.

Workshops

1. participated in **NPTEL awareness workshop**, July 3, 2020.

2. **Online Faculty Development Program on Research Methodology**, July 13-17, 2020.

3. Attended **International Faculty Development program on "Advanced Linear Algebra"**, 27th September – 01st October 2021. The University of Delhi.

4. Attended Workshop on **Linear Algebra for Computer Science And Machine Learning** , November 19-20, 2022. IIT Madras.

Courses Related to Research Work

Courses

1. Certification in Introduction to Abstract and Linear Algebra with a score of 96%, awarded by NPTEL in collaboration with the Indian Institute of Technology, Kharagpur. This rigorous 8-week course, conducted from July to September 2022, provided comprehensive insights into foundational and advanced topics in abstract and linear algebra.

2. Certification in Matrix Analysis with Applications, achieving a score of 83 %, awarded by NPTEL in collaboration with the Indian Institute of Technology, Roorkee. This intensive 8-week course, held from July to September 2022, covered advanced concepts and practical applications of matrix analysis, equipping participants with essential analytical skills.

3. Certification in Matrix Computation and Its Applications, with a score of 68 %, awarded by NPTEL in collaboration with the Indian Institute of Technology, Madras. This comprehensive 12-week course, conducted from July to October 2022, provided in-depth knowledge on computational techniques and practical applications of matrix computation.

4. Completed a 12-week course in MATLAB offered by NPTEL in collaboration with the Indian Institute of Technology, Bombay. Held from July to October 2023, this course provided extensive training in MATLAB programming.

5. Certification in Basic Calculus 1, achieved with a score of 53%, offered by NPTEL in collaboration with the Indian Institute of Technology, Madras. This 8-week course, conducted from January to April 2024, provided a structured introduction to essential calculus concepts, focusing on foundational principles and their practical applications.

6. Completed an 8-week course in Multivariable Calculus, organized by NPTEL in collaboration with the Indian Institute of Technology, Delhi. Conducted from January to April 2024, this course offered a comprehensive exploration of calculus in multiple dimensions, covering key topics such as partial derivatives, multiple integrals, and vector calculus, with an emphasis on real-world applications.

7. Completed a 12-week course in Real Analysis, organized by NPTEL in collaboration with the Indian Institute of Technology, Madras. Held from January to July 2024, this intensive course delved into fundamental concepts of real analysis, including sequences, series, continuity, and differentiability, providing a rigorous foundation in mathematical analysis.

Awards and Achievements

1. Achieved first position in paper presentation in international conference on **Recent Advances in Fundamental and Applied Sciences (RAFAS)** , June 25-26, 2021. LPU Jalandhar.

2. Achieved First Position in the course Introduction to Abstract and Linear Algebra, with a certification score of 96%, awarded by NPTEL in collaboration with the Indian Institute of Technology, Kharagpur. This 8-week course, conducted from July to September 2022.S

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