EFFECTIVE METHODS FOR SOLVING FUZZY FRACTIONAL PROGRAMMING PROBLEM USING VARIOUS FUZZY NUMBERS

Thesis Submitted for the Award of the Degree of

DOCTOR OF PHILOSOPHY

in
Mathematics

By
Ravinder Kaur

Registration Number: 41800074

Supervised By

Dr. Rakesh Kumar (UID-19437)
Department of Mathematics (Associate Professor)
Lovely Professional University, Phagwara, Punjab, India.

LOVELY PROFESSIONAL UNIVERSITY, PUNJAB
2025

DECLARATION

I, hereby declared that the presented work in the thesis entitled " Effective Methods for Solving Fuzzy Fractional Programming Problems using Various Fuzzy Numbers" in fulfilment of degree of **Doctor of Philosophy (Ph. D.)** is outcome of research work carried out by me under the supervision of Dr. Rakesh Kumar, working as Associate Professor in the Department of Mathematics School of Chemical and Physical Sciences of Lovely Professional University, Punjab, India. In keeping with general practice of reporting scientific observations, due acknowledgements have been made whenever work described here has been based on findings of other investigator. This work has not been submitted in part or full to any other University or Institute for the award of any degree.

(Signature of Scholar)

Name of the scholar: Ravinder Kaur
Registration No.: 41800074
Department/school: Department of Mathematics, School of Chemical and Physical Sciences
Lovely Professional University,
Punjab, India.

CERTIFICATE

This is to certify that the work reported in the Ph. D. thesis entitled Effective Methods for Solving Fuzzy Fractional Programming Problems using Various Fuzzy Numbers" submitted in fulfillment of the requirement for the award of degree of **Doctor of Philosophy (Ph.D.)** in the Department of Mathematics School of Chemical and Physical Sciences, is a research work carried out by Ravinder Kaur, 41800074, is bonafide record of his/her original work carried out under my supervision and that no part of thesis has been submitted for any other degree, diploma or equivalent course.

(Signature of Supervisor)
Name of supervisor: Dr. Rakesh Kumar
Designation: Associate Professor
Department/school: Department of Mathematics, School of Chemical and Physical Sciences
Lovely Professional University,
Punjab, India.

Abstract

Fractional programming (FP) is a method of optimization applied in the process of decision making. One particularly unique kind of fractional programming that optimizes the ratio of two linear functions under specific restrictions is linear FP. The structure of the restrictions is a linear inequalities form. Many elements in the field of organizational sciences give rise to fractional programming problems. Though they are rarely exact, clear-cut conditions typically need to be expressed using fuzzy coefficients. It has become more and more important recently. From now on, numerous researchers in several disciplines apply fractional programming problems. A number of realistic optimization problems in which the objective functions are divided into two functions used to get the maximum ratio of actual expenses to standard expenses, workers/salary, results/students, patients/doctors, vaccinated/medication inventory to sales minimization, and so on, where the ratio reflects the highest efficiency of a system, can help one find the main motivation for fractional programming problems. Practitioners and researchers alike need FFP more than ever since real-world decision-making situations are getting more complicated and less precise. This thesis suggests and puts into action a variety of successful ways to solve FLFPP utilizing different fuzzy numbers. It focuses on creating hybrid methodologies and situation-based ambiguous representations that better reflect the uncertainty that comes up in real-world situations.

The paper starts by looking at ways to defuzzify triangular and trapezoidal fuzzy numbers. It then shows a component-wise tri-objective design utilizing the $(\mathscr{s}, \ell, \mathscr{r})$ representation. Combining a weighted sum technique with the modified Dinkelbach's algorithm makes this even better. The suggested hybrid technique is shown to work well in a manufacturing planning model, where comparisons and ranking studies show how strong and useful it is in real life.

To deal with the problems of modeling imprecision even more, an α,β cuts based bi-objective an interval model is created to find interval-valued fuzzy solutions. Using regression-based substitute model in Excel to see and check how fuzziness affects decision factors and outcomes backs up this method. The study shows that using real-world tools can help people comprehend and use fuzzy solutions better.

A Extended Intuitionistic Fuzzy Approach (EIFA) is built on this base. It combines membership functions (MF) and non-membership functions (NMF) in linear, parabolic, and exponential forms. Using benchmark issues and an agricultural case investigation from the literature, we carefully examine a lot of different combinations over nine situations. The results show that IF structures are better at dealing with uncertainty and hesitation than typical fuzzy sets, which are less flexible.

Also, a new situation-based S-shaped fuzzy number is suggested and used in the context of goal programming. We check our method using secondary data sources from the garment sector and compare its results to those of Zimmerman's linear membership function. The comparison indicates that the S-shaped fuzziness is better at reflecting trade-offs and degrees of stakeholder satisfaction in the actual world.

The results of the research show that the suggested hybrid approaches, expanded uncertain models, and situation-based fuzzy values are good, flexible, and useful ways to solve fuzzy fractional programming challenges. The created strategies have been shown to work in real life in areas including industrial organizing, agriculture, and the clothing industry.

This work adds to what we already know by filling in gaps in methodology, proving new fuzzy frameworks through multiple applications, and pointing out areas for further research, such as combining transformed metaheuristic algorithms to machine learning-based surrogate models, and applying them to a wider range of industrial settings. Researchers and practitioners are expected to be able to use the knowledge gathered from this study to help them deal with decision-making difficulties that are becoming more complicated and ambiguous.

Acknowledgements

Finally, the day has come when I can pause, look back, and express my deepest gratitude to those who have walked beside me — in spirit, in love, and in encouragement — throughout this journey.

First and foremost, I bow my head with deep reverence to the Almighty, whose blessings, grace, and divine guidance have given me the strength, patience, and willpower to complete this work. His silent presence has been my anchor in moments of struggle and uncertainty.

My heartfelt thanks go to my research supervisor, Dr. Rakesh Kumar, whose constant support, expert advice, and thoughtful feedback have been crucial in shaping this research. His guidance has not only improved the quality of this work but has also enriched my academic growth.

A special tribute is reserved for my late father, whose memory continues to be the foundation of my strength. Though he is no longer with me, his dreams, his belief in education, and his quiet courage are deeply woven into every step of this journey. I carry his love like a whisper in my heart, and his blessings have lit my way even in the darkest of hours.

To my mother, the heartbeat behind all my efforts — your sacrifices, silent prayers, and unshakable belief in me have been the greatest gift I could ever ask for. Your love is my shelter, and your strength is what I’ve leaned on through every challenge. No words can ever fully capture what you mean to me.

I also wish to thank my sister and brother-in-law, and my two little rays of sunshine — Harnoor and Noor. Your innocent laughter, warm hugs, and cheerful presence brought comfort and joy during stressful days. You may not know it yet, but your love gave me more motivation than you can imagine.

I am deeply grateful to my respected Heads of Department, Dr. Deepak Kumar and Dr. Gurpreet Singh Bhatia**,** for their guidance, encouragement, and willingness to help whenever needed. Their support contributed significantly to the smooth progress of this work.

A heartfelt thanks is also due to our family friend, Sarvesh Kaushal, whose thoughtful support and steady motivation have been truly reassuring.

To all my friends who stood by me — including my colleagues and dear friends Deepti and Dr. Prince — thank you for your patience, understanding, and constant encouragement. Your presence, conversations, and support helped me stay grounded and hopeful, especially during challenging phases. A special thanks to Dr. Vishal Thakur, whose valuable insights, motivation, and genuine guidance were a constant source of inspiration throughout this research. He was the rare kind of mentor who sometimes cared about my deadlines even more than I did — pushing, reminding, and believing in me when I needed it the most.

Lastly, I express my heartfelt appreciation to everyone — family, friends, mentors, and well-wishers — who, in ways big or small, helped me along the way. Your kindness and support have made this journey not only possible but truly meaningful.

Notations

L	Linear
NL	Non-Linear
P	Parabolic
E	Exponential
OF	Objective Function
FS	Fussy Set
IF	Intuitionistic Fuzzy
FN	Fuzzy Number
TFN	Triangular Fuzzy Number
TrFN	Trapezoidal Fuzzy Number
LP	Linear Programming
FP	Fractional Programming
GP	Goal Programming
MF	Membership Function
NMF	Non-Membership Function
LPP	Linear Programming Problem
LFP	Linear Fractional Programming
FLFP	Fuzzy Linear Fractional Programming
IFA	Intuitionistic Fuzzy Approach
FPP	Fractional Programming Problem
LFPP	Linear Fractional Programming Problem
EIFA	Extended Intuitionistic Fuzzy Approach
SOLFPP	Single Objective Linear Fractional Programming Problem
MOOP	Multi Objective Optimization Problem
MOLPP	Multi Objective Linear Programming Problem
MOLFPP	Multi Objective Linear Fractional Programming Problem
FLFPP	Fuzzy Linear Fractional Programming Problem
FFLFPP	Fully Fuzzy Linear Fractional Programming Problem
MOFLFPP	Multi Objective Fuzzy Linear Fractional Programming Problem
DM	Decision Maker

Table of Contents

List of Figures

List of Tables

Chapter 1

Introduction

In mathematics, optimal decision-making comprises the selection of the most efficient alternative from a collection of alternatives by taking into consideration the limitations of the system for making decisions and the decision-maker's own choices. Optimization is crucial to many processes in research and technology. Mathematical optimization helps create ideas and models that convert real-world decision-making challenges into mathematics problems. These models provide long-term planning and instruments to predict future trends. Modeling real mathematical optimization problems is tough since the data isn't always correct and distinct. Uncertainties in economic, industrial, and social systems can take many different forms, such as event uncertainty, insufficient system data, and language ambiguity. These can arise from several sources, such as measurement errors, inadequate information, poor expression, and human subjectivity. Consequently, optimization has the challenging task of meaningfully recognizing uncertain input and converting it into suitable mathematical values due to the lack of explicit knowledge of the parameters. Mathematical programming maximizes results with defined goals under given limitations. The more general definition of mathematical programming is the study and practice of determining the optimal use of finite resources. Industrial planning, scheduling, allocating resources, decision-making, and many more areas rely heavily on mathematical programming.

Fuzzy optimization problems may be formulated to solve many real-world challenges. Researchers have taken a keen interest in developing mathematical models for fuzzy optimization in the setting of decision-making. Despite existing mathematical models for fuzzy optimization, more effective models must be created to address uncertainties. Fractional programming is a subset of non-linear optimization problems whose objective is to determine the ideal ratio of two economical or physical variables [35]. It is usual practice to express objective tasks in fractional units so that it can fairly represent many diverse practical situations.

During the process of decision-making, it is common for a person or group to experience difficulty in maintaining the appropriate ratios among a number of essential components. LFPP in Charnes and Cooper [2] is an important tool for real-world tasks like planning production, choosing media, admitting students, and the entity in question, such as a hospital or health care organization, an air force logistics unit, a bank branch, and so on. The following ratios should be optimized due to specific technical constraints: real cost/standard cost, department/equity, profit/cost, inventory/sales, output/employee, student/teacher, cost/student, and tenured/non-tenured facility. Investigation of LFPP is crucial. The decision-makers (DMs) must identify the best ways to accomplish a number of goals, just like in many real-world issues. Because of changing circumstances, the DM must deal with a number of inaccurate numerical quantities when making a decision.

1.1 Fuzzy Sets Theory and Preliminaries

Several types of ambiguities may arise in modern life. However, traditional set theory is unable to resolve this kind of uncertainty. As a result, knowing fuzzy logic and fuzzy sets is crucial. Fuzzy logic handles inaccurate notions in fuzzy set theory. A hypothesis to explain how multi-stage decision-making occurs in fuzzy environments was proposed by Bellman & Zadeh [6]. The first person to introduce fuzzy sets was Zadeh [3]. His idea has given a new dimension in understanding uncertainty. The idea of uncertainty has been studied and applied a lot in many fields, such as artificial intelligence, psychological science, physics, chemistry, engineering, operations research, computer science, robotics, and medical science. Classical set theory allows just two pairs of set components either the item is a member or not. A fuzzy set utilizes a $[0, 1]$ assessment level to denote membership.

1.1.1 Fuzzy Set

A fuzzy set $\tilde{\mathrm{F}}$ [10] defined on set X is a set of ordered pairs expressed below

$$\tilde{\mathrm{F}} = \{(\varkappa,\ \mu_{\tilde{\mathrm{F}}}(\varkappa)) : \varkappa \in \mathrm{X}\}, \tag{1.1}$$

where $\mu_{\tilde{\mathrm{F}}}$ is called the MF or degree of membership for $\varkappa \in \tilde{\mathrm{F}}$.

1.1.2 Membership Functions

The current research work uses both Linear MF as well as Non-Linear MF for various fuzzy approaches Zimmerman [11]. Consider the minimum tolerance value of $\varkappa = \varkappa^{min}$ and the maximum tolerance value of $\varkappa = \varkappa^{max}$.

(i)*Linear:* It is a linear function that describes a straight line and its membership function for the maximization problem is described as:

$$\mu_{\tilde{F}}(\varkappa) = \begin{cases} 1\,, & if \quad \varkappa \geq \varkappa^{max}, \\ \dfrac{\varkappa - \varkappa^{min}}{\varkappa^{max} - \varkappa^{min}}\,, & if \ \varkappa^{min} \leq \varkappa \leq \varkappa^{max}, \\ 0\,, & if \quad \varkappa \leq \varkappa^{min}. \end{cases} \tag{1.2}$$

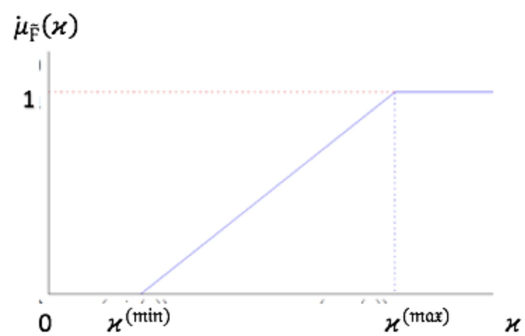


Figure 1.1: Linear membership function

(ii) Parabolic: The parabolic membership function for the maximization problem is described as:

$$\mu_{\tilde{F}}(\varkappa) = \begin{cases} 1\,, & if \quad \varkappa \geq \varkappa^{max}, \\ \left(\dfrac{\varkappa - \varkappa^{min}}{\varkappa^{max} - \varkappa^{min}}\right)^2, & if \ \varkappa^{min} \leq \varkappa \leq \varkappa^{max}, \\ 0\,, & if \quad \varkappa \leq \varkappa^{min}. \end{cases} \tag{1.3}$$

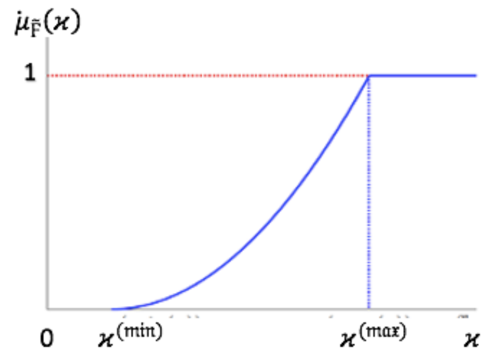


Figure 1.2: Parabolic membership function

(iii) Exponential: The exponential membership function for the maximization problem is describes as:

$$\dot{\mu}_{\tilde{F}}(\varkappa) = \begin{cases} 1, & if \ \varkappa \geq \varkappa^{max}, \\ \eta\left[1 - exp\left(-\rho \dfrac{\varkappa - \varkappa^{min}}{\varkappa^{max} - \varkappa^{min}}\right)\right], & if \ \varkappa^{min} < \varkappa < \varkappa^{max}, \\ 0, & if \ \varkappa \leq \varkappa^{min}. \end{cases} \tag{1.4}$$

where η is scaling constant and ρ shape parameter.

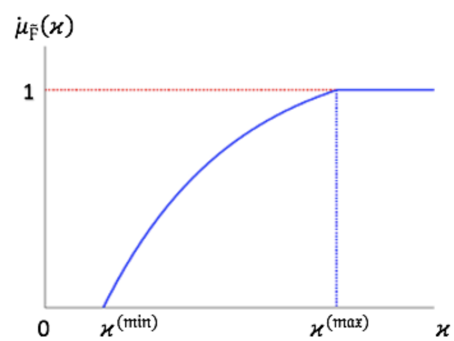


Figure 1.3: Exponential membership function

(iv) Sigmoidal: The sigmoidal membership function for the maximization problem where , A is a scale constant controlling the curve width, and α determines the steepness of the S-shaped MF is described as:

$$\acute{\mu}_{\tilde{F}}(\varkappa) = \begin{cases} 1, & if \ \varkappa \geq \varkappa^{max}, \\ 1 - \left(\dfrac{1}{1 + Ae^{\alpha\left(\frac{\varkappa - \varkappa^{min}}{\varkappa^{max} - \varkappa^{min}}\right)}} \right), & if \ \varkappa^{min} < \varkappa < \varkappa^{max}, \\ 0, & if \ \varkappa \leq \varkappa^{min}. \end{cases} \tag{1.5}$$

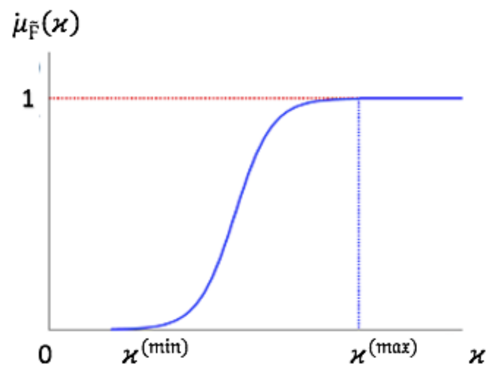


Figure 1.4: Sigmoidal membership function

1.1.3 Support

The collection of all points $\varkappa$ in X that satisfy the condition $\acute{\mu}_{\tilde{F}}(\varkappa) > 0$ is termed as support of FS $\tilde{F}$.

1.1.4 Normal

If there exists at least one $\varkappa$ in X such that $\acute{\mu}_{\tilde{F}}(\varkappa) = 1$ then the FS $\tilde{F}$ is called as normal

1.1.5 ά-Cut

For the FS $\tilde{F}$, the $\acute{\alpha}$-cut is defined as collection $\{\varkappa\colon \acute{\mu}_{\tilde{F}}(\varkappa) \geq \alpha\}$, and it is denoted as $\tilde{F}_{\acute{\alpha}}$.

1.1.6 Convex Fuzzy Set

A FS $\tilde{F}$ defined on X is said to be convex iff the following condition holds:

$$\acute{\mu}_{\tilde{F}}(\lambda\varkappa_1 + (1-\lambda)\varkappa_2) \geq min(\acute{\mu}_{\tilde{F}}(\varkappa_1), \acute{\mu}_{\tilde{F}}(\varkappa_2))$$

for all $\varkappa_1, \varkappa_2 \in$ X, with parameter $\lambda \in [0, 1]$.

1.1.7 Fuzzy Number (FN)

A fuzzy number is defined as a normal and convex fuzzy subset $\tilde{F}$ with MF $\dot{\mu}_{\tilde{F}}: \mathbb{R} \rightarrow [0, 1]$ and bounded support.

1.1.8 Triangular Fuzzy Numbers (TFN)

A triangular fuzzy number in Zimmermann [11] denoted by $\tilde{a} = (a_1, a_2, a_3)$ such that $a_1 \leq a_2 \leq a_3$ has membership function $\dot{\mu}_{\tilde{a}}(\varkappa)$ is defined by:

$$\dot{\mu}_{\tilde{a}}(\varkappa) = \begin{cases} \dfrac{\varkappa - a_1}{a_2 - a_1}, & if\ a_1 \leq \varkappa \leq a_2\,, \\ \dfrac{a_3 - \varkappa}{a_3 - a_2}, & if\ a_2 \leq \varkappa \leq a_3, \\ 0, & otherwise. \end{cases} \tag{1.6}$$

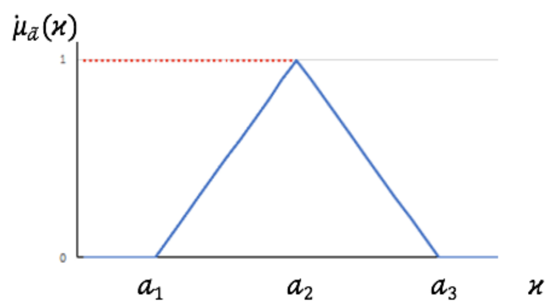


Figure 1.5: Triangular fuzzy number

1.1.9 Trapezoidal Fuzzy Number (TrFN)

A trapezoidal fuzzy number in Zimmermann [11], denoted by $\tilde{a} = (a_1, a_2\,, a_3, a_4)$ such that $a_1 \leq a_2 \leq a_3 \leq a_4$ with MF $\dot{\mu}_{\tilde{a}}(\varkappa)$ is defined by:

$$\dot{\mu}_{\tilde{a}}(\varkappa) = \begin{cases} \dfrac{(\varkappa - a_1)}{(a_2 - a_1)}, & if\ a_1 \leq \varkappa \leq a_2, \\ 1\,, & if\ a_2 \leq \varkappa \leq a_3, \\ \dfrac{(a_4 - \varkappa)}{(a_4 - a_3)}, & if\ a_3 \leq \varkappa \leq a_4, \\ 0, & otherwise. \end{cases} \tag{1.7}$$

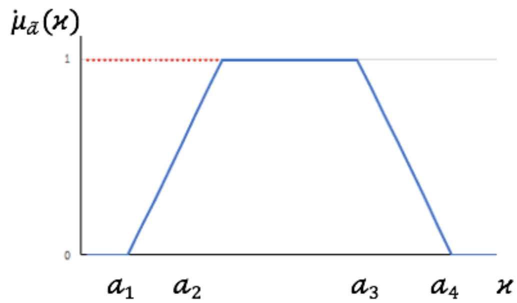


Figure 1.6: Trapezoidal fuzzy number

1.1.10 Arithmetic Operations (A.O.) on FN

Chen [14] developed the Function Principle to reduce the A.O. on FN while maintaining the original kind of MF.

The function principle was employed by Chen [14] to execute operations on imprecise numbers with a step form membership function.

1.1.10.1 Arithmetic Operations on Triangular Fuzzy Numbers

Consider two triangular fuzzy numbers $\tilde{a} = (a_1, a_2, a_3)$ and $\tilde{b} = (b_1, b_2, b_3)$ with $a_1, a_2, a_3, b_1, b_2, b_3$ as a real numbers . Then

(i) Addition: $\tilde{a} + \tilde{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$

(ii) Subtraction: $\tilde{a} - \tilde{b} = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$

(iii) Multiplication: $\tilde{a} \times \tilde{b} = (a_1 b_1,\ a_2 b_2,\ a_3 b_3)$

(iv) Division: $\frac{\tilde{a}}{\tilde{b}} = \left(\frac{a_1}{b_3}, \frac{a_1}{b_2}, \frac{a_3}{b_1}\right)$,

where b_1, b_2, b_3 non-zero positive real numbers.

(v) For any real number k:

$k\tilde{a} = (ka_1, ka_2, ka_3)$, if $k \geq 0$,

$k\tilde{a} = (ka_3, ka_2, ka_1)$, if $k < 0$.

1.1.11 Defuzzification: Ranking Function (RF)

To deal with the FLFPP the method of converting the problem into a concrete value LFPP. For this, one can apply fuzzy number ranking methods. For a fuzzy set, defuzzification produces a single, clear value. Many mathematical models depend on the fundamental step of ranking fuzzy numbers, and several ranking techniques have been proposed to do so.

RF $\mathfrak{R}$ is a mapping that maps each FN to the real line and defined as:

$$\mathfrak{R}: \mathcal{F}(\mathbb{R}) \rightarrow \mathbb{R},$$

where $\mathcal{F}(\mathbb{R})$ is the set of fuzzy numbers defined on the set of real numbers $\mathbb{R}$.

1.1.11.1 Arithmetic mean / Score function

The function $\mathfrak{R}: \mathcal{F}(\mathbb{R}) \rightarrow \mathbb{R}$, defines the RF as arithmetic mean in Zimmerman [11] for TFN of the type $\tilde{a} = (a_1, a_2, a_3)$ as

$$\mathfrak{R}(\tilde{a}) = \frac{a_1 + 2a_2 + a_3}{4} \tag{1.8}$$

1.1.11.2 Ranking using $(\mathscr{s}, \ell, \mathscr{r})$

For two TFNs $\tilde{p} = (\mathscr{s}, \ell, \mathscr{r})$ and $\tilde{q} = (\mathscr{s}', \ell', \mathscr{r}')$the relations $\approx$, $\prec$ and $\precsim$ defined as

(i) $\tilde{p} \approx \tilde{q}$ iff $\mathscr{s} = \mathscr{s}'$; $\mathscr{s} - \ell = \mathscr{s}' - \ell'$; $\mathscr{s} + \mathscr{r} = \mathscr{s}' + \mathscr{r}'$

(ii) $\tilde{p} \prec \tilde{q}$ iff $\mathscr{s} \prec \mathscr{s}'$; $\mathscr{s} - \ell \prec \mathscr{s}' - \ell'$; $\mathscr{s} + \mathscr{r} \prec \mathscr{s}' + \mathscr{r}'$

(iii) $\tilde{p} \succ \tilde{q}$ iff $\mathscr{s} \succ \mathscr{s}'$; $\mathscr{s} - \ell \succ \mathscr{s}' - \ell'$; $\mathscr{s} + \mathscr{r} \succ \mathscr{s}' + \mathscr{r}'$

Partial order $\precsim$ represented by $\tilde{\mathcal{A}} \precsim \tilde{\mathcal{B}}$ iff Max $(\tilde{p}, \tilde{q}) \approx \tilde{q}$

Also, operation addition and multiplication defines as

(iv) $(\mathscr{s}, \ell, \mathscr{r}) + (\mathscr{s}', \ell', \mathscr{r}') = (\mathscr{s} + \mathscr{s}', \ell + \ell', \mathscr{r} + \mathscr{r}')$

(v) $(\mathscr{s}, \ell, \mathscr{r})z = (\mathscr{s}z, \ell z, \mathscr{r}z)$.

1.1.11.3 $\boldsymbol{\acute{\alpha} - cut}$

(i) $\acute{\alpha} - cut$ of TFN $\tilde{a} = (a_1, a_2, a_3)$ defined as

$$a_{\acute{\alpha}} = [(a_2 - a_1)\acute{\alpha} + a_1,\ a_3 - (a_3 - a_2)\acute{\alpha}]=[a_{\acute{\alpha}}{}^{\mathcal{L}},\ a_{\acute{\alpha}}{}^{\mathcal{U}}]. \tag{1.9}$$

(ii) Arithmetic operations defines on the $\acute{\alpha} - cut$s of the FN as follows:

$Let\ \tilde{a} = [\, a_{\acute{\alpha}}{}^{\mathcal{L}},\ a_{\acute{\alpha}}{}^{\mathcal{U}}]$ and $\tilde{b} = [\, b_{\acute{\alpha}}{}^{\mathcal{L}},\ b_{\acute{\alpha}}{}^{\mathcal{U}}]$ then

(i) Addition: $(\tilde{a} + \tilde{b})_{\acute{\alpha}} = [a_{\acute{\alpha}}{}^{\mathcal{L}} + b_{\acute{\alpha}}{}^{\mathcal{L}},\ a_{\acute{\alpha}}{}^{\mathcal{U}} + b_{\acute{\alpha}}{}^{\mathcal{U}}]$.

(ii) Scalar multiplication : $(k\,\tilde{a})_{\acute{\alpha}} = [k a_{\acute{\alpha}}{}^{\mathcal{L}}, k a_{\acute{\alpha}}{}^{\mathcal{U}}]$.

(iii) Multiplication: $(\tilde{a}.\tilde{b})_{\acute{\alpha}} =$

$\left[min(a_{\acute{\alpha}}{}^{\mathcal{L}}\, b_{\acute{\alpha}}{}^{\mathcal{L}}, a_{\acute{\alpha}}{}^{\mathcal{L}}\, b_{\acute{\alpha}}{}^{\mathcal{U}},\ a_{\acute{\alpha}}{}^{\mathcal{U}}\, b_{\acute{\alpha}}{}^{\mathcal{L}},\ a_{\alpha}{}^{\mathcal{U}}\, b_{\acute{\alpha}}{}^{\mathcal{U}}), \max(a_{\alpha}{}^{\mathcal{L}}\, b_{\acute{\alpha}}{}^{\mathcal{L}}, a_{\acute{\alpha}}{}^{\mathcal{L}}\, b_{\acute{\alpha}}{}^{\mathcal{U}},\ a_{\acute{\alpha}}{}^{\mathcal{U}}\, b_{\acute{\alpha}}{}^{\mathcal{L}},\ a_{\acute{\alpha}}{}^{\mathcal{U}}\, b_{\acute{\alpha}}{}^{\mathcal{U}})\right]$.

(iv) Division : $\left(\frac{\tilde{a}}{\tilde{b}}\right)_{\acute{\alpha}} = \left[min\left(\frac{a_{\acute{\alpha}}{}^{\mathcal{L}}}{b_{\acute{\alpha}}{}^{\mathcal{L}}}, \frac{a_{\acute{\alpha}}{}^{\mathcal{L}}}{b_{\acute{\alpha}}{}^{\mathcal{U}}}, \frac{a_{\acute{\alpha}}{}^{\mathcal{U}}}{b_{\acute{\alpha}}{}^{\mathcal{L}}}, \frac{a_{\acute{\alpha}}{}^{\mathcal{U}}}{b_{\acute{\alpha}}{}^{\mathcal{U}}}\right),\ max\left(\frac{a_{\acute{\alpha}}{}^{\mathcal{L}}}{b_{\acute{\alpha}}{}^{\mathcal{L}}}, \frac{a_{\acute{\alpha}}{}^{\mathcal{L}}}{b_{\acute{\alpha}}{}^{\mathcal{U}}}, \frac{a_{\acute{\alpha}}{}^{\mathcal{U}}}{b_{\acute{\alpha}}{}^{\mathcal{L}}}, \frac{a_{\acute{\alpha}}{}^{\mathcal{U}}}{b_{\acute{\alpha}}{}^{\mathcal{U}}}\right)\right]$.

1.1.12 Intuitionistic Fuzzy Set

Atanassov [43] first presented the idea of IFSs in 1983, and they are an extension of the concept of regular fuzzy sets.

$$\tilde{F}_I = \left\{\left(\varkappa,\ \dot{\mu}_{\tilde{F}_I}(\varkappa),\ \dot{\nu}_{\tilde{F}_I}(\varkappa)\right);\ 0 \le\ \dot{\mu}_{\tilde{F}_I}(\varkappa) + \dot{\nu}_{\tilde{F}_I}(\varkappa) \le 1, \forall\, \varkappa \in X\right\} \tag{1.10}$$

where $\dot{\mu}_{\tilde{F}_I}$ is a MF defined from X to [0,1] and $\dot{\nu}_{\tilde{F}_I}$ is a NMF defined from X to [0,1].

When equation $\dot{\mu}_{\tilde{F}_I}(\varkappa) + \dot{\nu}_{\tilde{F}_I}(\varkappa) = 1$ is satisfied, the quantities in these sets are different contributions and failing to associate functions. This generates fuzzy accumulations from intuitionistic fuzzy sets.

1.2 Linear Programming Problem

Linear programming (LP) is a powerful way to solve mathematical problems that involve optimization of a linear objective function (OF) within a set of linear constraints. Under simple presumptions, this approach has proved helpful in addressing complex optimization problems and provides effective answers for practical challenges. Fundamentally, linear programming follows linear equations and linear inequalities. This feature enables linear programming to reflect complex linkages discovered in practical situations, therefore guiding the best solutions. The LPP approach may be used for a great range of FLPP seeks to find the decision variables that meet the linear limits on the choice variables for a wide range of problems in companies, government sectors, health care institutions, library systems, and other configurations. Linear programming is fundamentally based on its ability to represent complicated systems linear processes. Following linearity in both restrictions and goals helps the approach open straightforward and understandable solution paths. This simplicity of approach does not lessen its efficacy rather, it simplifies the road to identify ideal places inside the provided limitations. The LPP model generally has the following form:

$$\max/\min \mathcal{F}(x) = \sum_{j=1}^{n} c_j x_j \tag{1.11}$$

s.t.

$$\sum_{j=1}^{n} a_{ij} x_j (\leq, =, \geq)\ \mathrm{b}_i, \tag{1.12}$$

where i ranges from 1 to m and j ranges from 1 to n with $x_j \geq 0,\ x \in \mathbb{R}^n$.

1.3 Fractional Programming Problem

LFP has several appealing and useful features, including linear programming (LP). This is relevant for practical applications. We can convert an LP problem to an LFPP by substituting two ratio-form linear functions for an LPP. Moreover, from an applications point, such a perfect solution is often more attractive and desirable than the one produced from the LPP due to its higher efficiency. LFP issues arise when one wants to optimize efficiency among the conflicting objectives. These days, the use of such exact criteria is becoming more essential because of a shortage of natural resources. Applying LFP to maximize efficiency in practical LP problems increasingly common.

A specific area of mathematical programming known as fractional programming has grown and expanded significantly in the last several years. This growth has shown up in both theoretical study and the practical applications that many experts have created. This monograph explores fractional programming, which optimizes performance metrics like cost/time, cost/quantity, and cost/profit to assess an organization's effectiveness. Take industrial systems productivity, is the ratio of the resources used during a certain time to the services the system provides. To evaluate the efficiency of such systems, this ratio is considered by many to be among the most trustworthy metrics. These types of problems naturally lead to fractional programming models, where the objective function is stated as a ratio of functions. Fractional programming is useful for modeling decision-making processes in several domains, including management science, operations research, and economics. It offers a strong framework for optimizing systems with a ratio of two linear functions as the target.

The LFPP's general form can be expressed as:

1.3.1 Single-objective (SO)-LFPP

$$\max/\min \mathcal{Z}(\varkappa) = \frac{\mathcal{P}(\varkappa)}{\mathcal{Q}(\varkappa)} = \frac{p^{\mathrm{T}}\varkappa + d}{q^{\mathrm{T}}\varkappa + e} \tag{1.13}$$

s.t.

$$\varkappa \in \mathcal{S} = \{\varkappa \in \mathbb{R}^n : \mathcal{A}\varkappa (\leq, =, \geq) \mathcal{B} \text{ or } \sum_{j=1}^{n} a_{ij}\varkappa_j (\leq, =, \geq) \mathrm{b}_i, \varkappa \geq 0\}, \tag{1.14}$$

with

$$\mathcal{A} = (a_{ij}) \in \mathbb{R}^{m\times n}, \mathcal{B} = (\mathrm{b}_i) \in \mathbb{R}^m, p, q \in \mathbb{R}^n \text{ and } d, e \in \mathbb{R}. \tag{1.15}$$

To keep the values of $\varkappa$ from becoming zero, the structure of $\mathcal{A}\varkappa \leq \mathcal{B}$ with $\varkappa \geq 0$ must have either $\mathcal{Q}(\varkappa) > 0$ or $\mathcal{Q}(\varkappa) < 0$. For the purpose of ease, we make the assumption that LFP meets the requirements that preserve $\mathcal{Q}(\varkappa) > 0$.

Note: If $\mathcal{P}(\varkappa)$ is concave on $\mathcal{S}$ and $\mathcal{Q}(\varkappa)$ convex and positive on $\mathcal{S}$, then (1.13) is a classic concave-convex programming problem.

1.3.2 Multi-objective (MO)-LFPP

$$\max/\min \mathcal{Z}_k(\varkappa) = \frac{p^{\mathrm{T}}{}_k\varkappa + d_k}{q^{\mathrm{T}}{}_k\varkappa + e_k}, k = 1,2,\ldots,\mathcal{K} \tag{1.16}$$

s.t.

$$\varkappa \in \mathcal{S} = \{\varkappa \in \mathbb{R}^n : \mathcal{A}\varkappa (\leq, =, \geq) \mathcal{B}, \varkappa \geq 0\}, \tag{1.17}$$

$$with\ \mathcal{A} \in \mathbb{R}^{m\times n}, \mathcal{B} \in \mathbb{R}^m, p_k, q_k \in \mathbb{R}^n \text{ and } d_k, e_k \in \mathbb{R}. \tag{1.18}$$

1.3.3 Charnes and Cooper Method

Using the Charnes and Cooper's [2] approach, we can find the standard form of a conventional LFPP as follows:

$$\max/\min \mathcal{Z}(\varkappa) = \frac{p^T\varkappa + d}{q^T\varkappa + e} \tag{1.19}$$

where p *and* $q \in \mathbb{R}^n$ and $d, e \in \mathbb{R}$,

s.t.

$$\mathcal{A}\varkappa \le \mathcal{B}, \varkappa \ge 0, \varkappa \in \mathbb{R}^n, \text{ with } \mathcal{A} \in \mathbb{R}^{m\times n}, \mathcal{B} \in \mathbb{R}^m. \tag{1.20}$$

Let $S = \{\mathcal{A}\varkappa \le \mathcal{B}, \varkappa \ge 0\}$ and $q^T\varkappa + e > 0$ for each $\varkappa \in \mathbb{R}^n$.

Let $\frac{1}{q^T\varkappa+e} = \tau$ and $y = \tau\varkappa$.

Then above LFPP reduces to LPP. The reduced LPP can be solved using classical methods , to retrieve optimal solution and by using the relation $\varkappa = \frac{y}{\tau}$ the optimal solution of LFPP can be reduced as:

$$\max/\min \mathcal{Z}(y) = p^T y + d\tau \tag{1.21}$$

s.t.

$$\mathcal{A}y - \tau\mathcal{B} \le 0, \tag{1.22}$$

$$q^T y + e\tau = 1, \tag{1.23}$$

$$y \ge 0, \tau \ge 0. \tag{1.24}$$

1.4 Fuzzy Goal Programming (FGP)

The aim of FGP is to provide the decision maker (DM) with multiple choices to select an optimal or compromise solution when dealing with SO/MO-FLFPP. In classical GP, by Mohamad [15] the objectives or aspiration levels are regarded as exact and predictable. However, realistically numerous circumstances happen where it becomes difficult for the DM to accurately develop the values of the goals. The imprecision or fuzziness of the targets depends on the kind of objective functions used in the decision-making process. Many times, the components of an issue are regarded as fuzzy or stochastic as they are unknown.

Consider the MOLFPP setup discussed above in section 1.3.2.

Set g_k as desired output for the kth OF $\mathcal{Z}_k(\varkappa)$ and let ℓ_k be the lowest level of acceptance.

Then the associated MF defines as follows:

$$\mu(\mathcal{Z}_k(x)) = \begin{cases} 1, & if\ \mathcal{Z}_k(x) \geq g_k, \\ \dfrac{\mathcal{Z}_k(x) - \ell_k}{g_k - \ell_k}, & if\ \ell_k < \mathcal{Z}_k(k) < g_k, \\ 0, & if\ \mathcal{Z}_k(x) \leq \ell_k. \end{cases} \quad (1.25)$$

Fuzzy programming methods allow MFs up to 1.

According to Mohamed [15], the MF mentioned above might be considered as flexible membership goals with a preferred level 1.

$$\frac{Z_k(X) - (l_k)}{g_k - l_k} + d_k^- - d_k^+ = 1 \quad (1.26)$$

where d_k^- under deviated value and d_k^+ over deviated value, with their product equals to 0.

1.5 Fuzzy Linear Fractional Programming Problem (FLFPP)

Any physical problem modeled and solved with exact knowledge of the decision parameters. In the actual world, most decisions are made in situations when the objectives, limitations, and execution of potential actions are all unknown. Since the observed value in issue is uncertain, an observed value cannot be a precise (certain) numerical value. Uncertain data is information that is dubious, erroneous, or doubtful. The fundamental LFPP becomes a FLFPP when any of decision parameters of the issue are not known by portraying it as fuzzy integers. Interpreting the cost of the goal functions, the technical coefficients, and the resources themselves as fuzzy values would be more natural given the occurrence of uncertainty in actual world events. The notion of fuzzy LFPP is not unique as the uncertainty may show in a LFPP in different ways, let's say out of parameters p , q, d , e , $\mathcal{A}, \mathcal{B}$ in (1.13) and (1.14) some or all may be of fuzzy integers. The MOFLFPP arises when the task has more than one objective. Many practical issues are addressed with MOFLFPP.

1.6 Literature Review

Due to the significance of the fractional programming problem, a number of scholars have already focused their attention on the issue in order to provide support for this field. Additionally, due to the absence of a software program that is specifically designed to address the LFPP, numerous researchers have explored this field. Charnes and Cooper [2] pioneered a variable transformation that transforms linear fractional programming into linear programming allowing for easy evaluation using a consistent simplex technique. There are more variables and constraints after this transformation, which changes the framework of the original constraints. The fundamental issue with this adjustment is that it increases the amount of variables and constraints, therefore altering the original constraint structure. This approach is not appropriate for every situation, that is to say, for an assignment or a transportation difficulty. The fractional programming challenges surface in many decision-making contexts; for instance, they are applied in game theory, network flows, traffic planning, and field of production planning.

Parametric programming, as proposed by Dinkelbach [5], is among the most popular and versatile approaches to fractional programming, second only in prevalence to the Charnes and Cooper [2]. This approach reduces LFP to LPPs, finding the optimal solution through iterations using the goal function or a real parameter in the numerator or denominator. When compared to the parametric model, the Charnes and Cooper approach appears to be more computationally friendly.

Bitran and Novaes [9] developed a breakthrough approach that outperforms the previous models due to its non-modification of constraints, no variable transformation, and no extra constraints or variables. This method has a fundamental flaw whereby the possible answer is continually improved. Following that, Swarup [4] devised a method to solve LF function without translating them into LPP. Different improvements of ambiguous Programming have arisen with time; for instance, fuzzy extend programming, ambiguous stochastic programming, FGP, and so on and have been utilized regressively to organize MOLPP as recommended by the problem.

First the fuzzy set was added by Zadeh [3], [6] in 1965 fuzzy set as an excellent mathematical tool for portraying real-word inaccuracies inconsistencies, and imprecision.Bellman & Zadeh [6] first proposed the concept of making decisions in a fuzzy environment. Tanaka [8] first covered linear programming under uncertainty. He suggested the initial idea of broad fuzzy mathematical programming. First presenting fuzzy linear programming as conventional linear programming, Zimmermann [11] thought about LPP considering fuzzy goals and fuzzy constraints. A comparable LPP to fuzzy LPP using linear membership functions and min operator as an aggregator for these functions. Luhandjula [12] turned the original issue into a multiple goal linear program via approximate values or change of variables. Based on the linear programming approach**,** Sakawa & Yano [25] proposed a fresh interactive decision making tool to help the decision maker find the answer. Adding an extra non-membership degree and hesitation degree, Atanassov [43] presented the concept of intuitionistic fuzzy set. Chakraborty [19] described the fuzzy-based MF FMOFPP approach employing variable transformation.

Pal & Moitra [21] GP approach for achievement for highest membership value for each goal. Surapati [42] provides an overview of the formulation and applications of the FGP model by Mohamed [15] came up with a flexible goal programming (FGP) method to deal with MOLFPP. Srinivasan [67] used the FGP approach, which is reliant on the Taylor series, to evaluate MOLFPP. For piecewise LFPP, Sahni & Pandey [86] put up a basic simplex method. To solve an LFPP Tantawy [27] proposed a duality technique and a workable direction. Baky [29] by reducing their deviational factors, FGP by describing MF at all levels achieves the maximum level of each of the a membership aims suggested computer techniques and approximative method for fractional stochastic programming issue using FGP.

Using Charnes-Cooper approach, Stanojevic & Stancu-Minasian [35] converted the linear fractional programming into an LPP to solve problem deterministicly. Mehlawat & Kumar [34] presented vertex-following technique with linear ranking. Guzel et al. [33] converted a fractional transportation issue with interval coefficients into a classical transportation problem by use of transformations. Developed technique to solve MOLFPP based on the theorem that deals with nonlinear fractional programming with single objective Effati & Pakdaman [32] presented an LFPP with interval values based on parameters. Stanojevic [38] remark on Taylor approximation provide a better approach ensuring the effectiveness of the solution it generates. Jain & Arya [37] suggested a reverse approach to LFPP so that produced effective solution becomes the most suitable one. Based on a theory addressing nonlinear fractional programming with single objective function, Guzel [36] offered a fresh approach to the multi-objective LFPP. Mishra et al.[41] applied GP in FLFPP in agriculture use of land for planting problem with varied aspiration level using every parameter and variables as triangular fuzzy integers, Safaei [40] developed a new approach to solve the totally fuzzy LFPP. Mathematical operations and a partial ordering relations. Veeramani et al.[39],[44],[45] used single objective

FLFPP with TFN to transform into deterministic MOLFPP and worked on bi-objective create a technique for obtaining the (α, r) suitable optimum value for an LFPP with uncertain parameters and ambiguous decision variables as well as their application. Singh & Yadav [46], [62] used non linear MF and linear ranking function to turn the intuitionistic fuzzy model into a crisp model to deal with MOFPP under uncertainty, and ambiguous decision variables as well as their application. Chang [51] examines user utility functions as well as binary behavior in multi-criteria environments. Das et al.[61] provided basic ranking technique between two FLFPP triangular fuzzy numbers. Arya & Singh [60], [65] contributed on branch and bound methods and also presented an iterative fuzzy approach to get efficient solution for MOLFPP.Some of the limitations and each of the stated fuzzy goals for every objective have fuzzy numbers as MOFLFPP uses a methodology based on superior and inferior measurements discussed by Yang et al. [66]. Srinivasan [67] defined the clear link among the target objective and choosing the settings to solve the resulting programming problem, thereby getting a fair and optimal answer to the problem at hand. Stanojević [77] worked on real-world solutions to a certain type of FFLFPP and used trapezoidal fuzzy numbers to characterize the parameters and find the real-world form of the MF of the desired function's ideal values of the problem. Moges et al.[87] discussed weighted IF GP, IF non-dominant, and Pareto-optimal solutions for agricultural land allocation. Mahajan & Gupta [73] suggested FIFMOP with non-linear functions, optimistic, pessimistic and hybrid approaches. Valipour & Yaghoobi [76] examined various strategies currently in use for dealing with fuzzy objectives on a set of clear constraints, by transforming the initial MOLFP issue into an LPP or MOLPP, these methods generate a single efficient or moderately efficient solution. Sahoo et al. [83] proposed a substitute method to address intuitionistic fuzzy MOFPP transformed into crisp MOLFP and studied on MOLFPP involving pentagonal IFN. Fathy et al. [88] recommended solution that converts each level of the multi-level MOLFPP into five crisp LPP with a constraint on extra limited variables and treats the top problems optimization variables as parameters. Zhang [52], [89] developed fuzzy credibility-based MOFPP and investigated crop area planning in relation to agricultural water-food-environment nexus. Malik and Gupta [103] employed IF numbers to solve MOLFPP in the scenarios of pessimistic, optimistic and mixed fuzzy programming. Karthick & Saraswathi [102] discussed about methodology solving intuitionistic MOLFPP without converting into crisp model. Akhtar & Islam [100] worked on defuzzification of interval based trapezoidal bipolar numbers that are fuzzier using ranking function and Chauhan et al. [104] proposed modified operator method for MOLFPP.

1.7 Research Motivation

In real life, it is often hard to see what our objectives and constraints are since information changes and we don't always know everything. FLFPP is a flexible way to solve these issues that uses the strengths of both fuzzy set theory and fractional programming. To fully overcome these problems, though, we need new techniques to deal with uncertainty from diverse angles, especially for sectors that are complex and subject to change.

The purpose of this study is to come up with robust hybrid methods that can handle different kinds of uncertain values, MF and NMF that can vary, and different kinds of objective structures. This work tries to construct situation-based not certain numbers and combine them with advanced defuzzification and optimization algorithms to come up with more realistic and adaptable answers to real-world problems like planning production, allocating resources, and setting goals

1.8 Research Gap and Novelty

The present research reveals that there are various approaches to fix fuzzy programming problems. However, the majority of studies only look at particular types of fuzzy numbers, including triangular or trapezoidal numbers, and typical defuzzification methods. There aren't many studies that look at all the many sorts of fuzzy numbers, sophisticated non-membership and membership function constructs, and hybrid solution strategies all at once. Also, not much effort has been done to systematically compare these hybrid approaches using various instances from real life to see how well they operate in different situations. This gap demonstrates that we need innovative ways to combine multiple fuzzification and defuzzification approaches, flexible multiple goals formulations, and practical implementation with real data set.

This study fills up these gaps by offering new hybrid approaches that combine component-wise investigation, weighted sum methods, and revised Dinkelbach techniques to better solve fuzzy fractional programming challenges. It includes scenario-based fuzzy numbers, such S-shaped fuzzy numbers, and customized intuitionistic non-membership and membership functions to better show uncertainties that are unique to a certain situation. The study also looks at surrogate interpreting as a tool to make sense of unclear data by combining bi-objective interval demonstration and regression analysis with α-cut and β-cut methods. The study indicates that the recommended frameworks are adaptable, practical, and can be utilized in more circumstances by using them in many different kinds of real-world and comparisons instances. This is a major step ahead for FPP.

1.9 Objectives of the Proposed Work

1. To propose hybrid techniques for solving fuzzy fractional programming problems with fuzzy numbers
2. To create situation-based fuzzy numbers and develop a new methodology for fuzzy fractional programming.
3. Analysis of real-world data sets to determine the optimal solution by employing the proposed techniques.

1.10 Thesis Outline

The thesis is divided into six parts that explain in a logical way how to create and use efficient approaches for solving FLFPP using various fuzzy numbers.

Chapter 1 gives an introduction, some fundamental terminology, and a review of the literature. It also talks about the study's research gap, research goal, and what makes it new. Chapter 2 talks about several ways to defuzzify triangular and a trapezoidal fuzzy numbers. First one with TFN shows how to turn them into the component-wise tri-objective design, and how to use a hybrid weighted sum based on the modified Dinkelbach technique in a production planning model .

Chapter 3 discussed about an α-cut, β-cut based bi-objective interval model and shows how to use Excel to do surrogate regression modeling.

Chapter 4 shows an extended intuitionistic fuzzy approach (EIFA) by creating and studying many both membership and non- membership functions (linear, parabolic, exponential) in diverse situations, as well as benchmark and real-life agricultural case studies.

Chapter 5 suggests a situation-based S-shaped fuzzy number algorithm in a goal development framework and compares its performance to that of traditional linear methods using data from the garment industry.

Chapter 6 wraps up the thesis by going over the main points, showing how they fit with the research goals, pointing out the study's shortcomings, and recommending areas for further study.

Chapter 2

Hybrid Weighted-sum with Modified Dinkelbach Approach to Solve Linear Fractional Programming Problems

2.1 Introduction and Motivation

In this chapter, a hybrid method that incorporates weighted Dinkelbach approach is developed as part of this research to overcome identified shortcomings. This hybrid method provides a solution framework that is more adaptive and efficient for fractional optimization issues that occur in the real world. Dinkelbach method discussed by Guzel [36] and the weighted sum Nayak et al. [47], [90] is being utilized. The purpose of this research is to formulate the efficient methodologies to obtain optimal solution for FLFPP with triangular fuzzy numbers. In this algorithm to deal with defuzzification we implemented Kumar et al. [53] $(\mathscr{s}, \ell, \mathscr{r})$ technique where $\mathscr{s}$ is the static value with ℓ and $\mathscr{r}$ specifying it the spread to the left and right, respectively. The method has two phases. In the first phase the FLFPP is decomposed into three LFPPs with crisp coefficient using $(\mathscr{s}, \ell, \mathscr{r})$ defuzzification method and the Charnes & Cooper [2] transformation is implemented to obtain ideal solution for each component. In the second phase each fractional component is the linearized using the Dinkelbach [5] method given by Guzel [36] and weighted mean approach is used to convert them into single objective assigning appropriate weight to each component objective function. The incorporation of weight elements into Dinkelbach's method improves the equilibrium in optimization, which in turn leads to an increase in both the computing efficiency and the quality of the solution.

The chapter is organized as follows: In section 2.2 is about $(\mathscr{s}, \ell, \mathscr{r})$ defuzzification technique for FLFPP. Section 2.3 provides a description of the suggested technique. The effectiveness of the suggested method is proved by the utilization of a numerical example that is based on real-world data. This example is described in further detail in section 2.4, and the analysis of the results is offered in section 2.5. Lastly, the article is wrapped up in section 2.6 with some thoughts and potential avenues for further study.

2.2 Defuzzification of FLFPP

2.2.1 *Defuzzification of (ꜱ, ℓ, r) form of TFN*

Three real numbers (s, ℓ, r) representing TFN. The meanings of these numbers are shown in Figure 2.1[s- static value , ℓ- left spread , r- right spread].

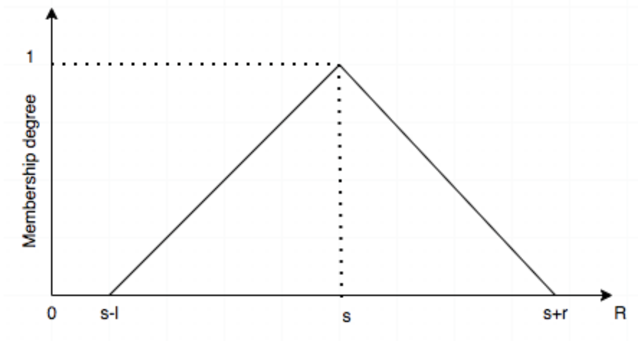


Figure 2.1: Triangular (s, ℓ, r) fuzzy number

From Kumar of (s, ℓ, r) [53]where s, ℓ, r type represented the following form of LPP

$$\max f(x) \tag{2.1}$$

s.t.

$$\sum_{j=1}^{n} (s_{ij}, \ell_{ij}, r_{ij})\, x_j \leq (t_i, u_i, v_i), \tag{2.2}$$

for all $x_j \geq 0$.

The model (2.1) reduced into

"As established by Kumar [53], a triangular fuzzy coefficient $(s_{ij}, \ell_{ij}, r_{ij})$ is interpreted through its left, central, and right bounds. The fuzzy inequality in (2.2) is satisfied only when each bound is satisfied individually. Hence, it is equivalently decomposed into the following crisp constraints."

$$\max f(x) \tag{2.3}$$

s.t.

$$\sum_{j=1}^{n} s_{ij} x_j \leq t_i, \tag{2.4}$$

$$\sum_{j=1}^{n}(s_{ij} - \ell_{ij})x_j \leq t_i - u_i\,, \tag{2.5}$$

$$\sum_{j=1}^{n}(s_{ij} + r_{ij})x_j \leq t_i + v_i\,, \tag{2.6}$$

$$\text{for all } x_j \geq 0.$$

2.2.2 General form of FLFPP

The general form of FLFPP is shown below:

$$max\ \tilde{Z}(x) = \frac{\tilde{P}(x)}{\tilde{Q}(x)} = \frac{\tilde{p}^{T}x + \tilde{d}}{\tilde{q}^{T}x + \tilde{e}} = \frac{\sum_{j=1}^{n}\widetilde{p_j}\,x_j + \widetilde{d_j}}{\sum_{j=1}^{n}\widetilde{q_j}x_j + \widetilde{e_j}}. \tag{2.7}$$

s.t.

$$\sum_{j=1}^{n}\widetilde{a_{ij}}x_j \leq \tilde{b}_i. \tag{2.8}$$

conditions on region discussed in (1.15).

2.2.3 Modified Dinkelbach's theorem

$$z_k^* = \frac{\mathcal{P}_k(x^*)}{\mathcal{Q}_k(x^*)} = \max\left\{\frac{\mathcal{P}_k(x)}{\mathcal{Q}_k(x)};\ x \in \mathcal{S}\right\} \tag{2.9}$$

$$\text{iff } f(z_k^*) = f(z_k^*, x^*) = \max\{\mathcal{P}_k(x) - z_k^*\mathcal{Q}_k(x);\ x \in \mathcal{S}\} = 0. \tag{2.10}$$

Above approach of Nuren [36] on modified Dinkelbach incorporated along with weights is proposed in hybrid method to provide efficient solution $\bar{x}$ for multi objective LFPP as follows:

$$max\left\{\sum_{k=1}^{m} w_k(\mathcal{P}_k(x) - z_k^*\mathcal{Q}_k(x);\ x \in \mathcal{S}\right\} \tag{2.11}$$

$$\text{where } z_k^* = \frac{\mathcal{P}_k(x^*)}{\mathcal{Q}_k(x^*)} = \max\left\{\frac{\mathcal{P}_k(x)}{\mathcal{Q}_k(x)}; x \in S\right\}, \text{ for all } k = 1,2,\ldots.\ m. \tag{2.12}$$

2.2.4 Reduction of FLFPP into MOLFPP

The defuzzification process of (s, ℓ, r) discussed above in section 2.2.1 can be employed to reduce single objective FLFPP (2.7) into MOLFPP as follows:

Let $\tilde{p}_j = (p_j^s, p_j^\ell, p_j^r)$; $\tilde{q}_j = (q_j^s, q_j^\ell, q_j^r)$; $\tilde{b}_i = (b_i^s, b_i^\ell, b_i^r)$; $\tilde{a}_{ij} = (a_{ij}^s, a_{ij}^\ell, a_{ij}^r)$; $\tilde{d} = (d^s, d^\ell, d^r)$; $\tilde{e} = (e^s, e^\ell, e^r)$

Then (10-11) becomes:

$$max\ \tilde{Z}(x) = \frac{\sum(p_j^s, p_j^\ell, p_j^r)\ x_j + (d^s, d^\ell, d^r)}{\sum(q_j^s, q_j^\ell, q_j^r)\ x_j + (e^s, e^\ell, e^r)} \tag{2.13}$$

s.t.

$$\sum(a_{ij}^s, a_{ij}^\ell, a_{ij}^r)x_j \le (b_i^s, b_i^\ell, b_i^r), \tag{2.14}$$

$$x_j \ge 0. \tag{2.15}$$

where i ranges from 1 to m and j ranges from 1 to n.

After applying defuzzification discussed in section 2.2.1, the problem (2.13)-(2.15) may be restated as a triple LFPP follows:

$$\max \tilde{Z}(x) = \left\{\frac{\sum(p_j^s - p_j^\ell)x_j + (d^s - d^\ell)}{\sum(q_j^r + q_j^s)x_j + (e^r + e^s)}, \frac{\sum p_j^s x_j + d^s}{\sum q_j^s x_j + e^s}, \frac{\sum(p_j^r + p_j^s)x_j + (d^r + d^s)}{\sum(q_j^s - q_j^\ell)x_j + (e^s - e^\ell)}\right\} \tag{2.16}$$

s.t.

$$\sum a_{ij}^s\, x_j \le b_i^s, \tag{2.17}$$

$$\sum(a_{ij}^s - a_{ij}^\ell)x_j \le (b_i^s - b_i^\ell), \tag{2.18}$$

$$\sum(a_{ij}^r + a_{ij}^s)\, x_j \le (b_i^r + b_i^s), \tag{2.19}$$

$$x_j \ge 0. \tag{2.20}$$

where i ranges 1 to m and j ranges 1 to n.

FLPP can be further modelled into three crisp LFPP to provide lower bounds Z_l, Z_m *and* Z_u of objective function:

$$\text{Max } \mathcal{Z}_l = \frac{\sum(p_j^s - p_j^\ell)\varkappa_j + (d^s - d^\ell)}{\sum(q_j^r + q_j^s)\varkappa_j + (e^r + e^s)} = \frac{\mathcal{P}_l(\varkappa)}{\mathcal{Q}_l(\varkappa)} \tag{2.21}$$

s.t.

same as (2.17)-(2.20),

where i varies from 1 to m and j varies from 1 to n.

$$\text{Max} \mathcal{Z}_m = \frac{\sum p_j^s \varkappa_j + d^s}{\sum q_j^s \varkappa_j + e^s} = \frac{\mathcal{P}_m(\varkappa)}{\mathcal{Q}_m(\varkappa)} \tag{2.22}$$

s.t.

same as (2.17)-(2.20).

And

$$\text{Max } \mathcal{Z}_u = \frac{\sum(p_j^r + p_j^s)\varkappa_j + (d^r + d^s)}{\sum(q_j^s - q_j^\ell)\varkappa_j + (e^s - e^\ell)} = \frac{\mathcal{P}_u(\varkappa)}{\mathcal{Q}_u(\varkappa)} \tag{2.23}$$

s.t.

same as (2.17)-(2.20).

2.3 Proposed hybrid Method:

The proposed hybrid method is described in two stages.

Consider the fuzzy fractional optimization problem of maximize form:

$$\max \tilde{\mathcal{Z}}(\varkappa) = \frac{\tilde{\mathcal{P}}(\varkappa)}{\tilde{\mathcal{Q}}(\varkappa)} = \frac{\sum_{j=1}^{n} \widetilde{p_j}\, \varkappa_j + \widetilde{d_j}}{\sum_{j=1}^{n} \widetilde{q_j} \varkappa_j + \widetilde{e_j}}. \tag{2.24}$$

s.t.

$$\sum_{j=1}^{n} \widetilde{a_{ij}} \varkappa_j \leq \tilde{b}_i, \varkappa \geq 0. \tag{2.25}$$

Stage I : In stage one of hybrid method convert fuzzy modelled equation (2.24) to classical crisp tri-objective optimization problem using (s, ℓ, r) as follows:

$$\max \mathcal{Z}(\varkappa) = \max\left\{\mathcal{Z}_l = \frac{\mathcal{P}_l(\varkappa)}{\mathcal{Q}_l(\varkappa)}, \mathcal{Z}_m = \frac{\mathcal{P}_m(\varkappa)}{\mathcal{Q}_m(\varkappa)}, \mathcal{Z}_u = \frac{\mathcal{P}_u(\varkappa)}{\mathcal{Q}_u(\varkappa)}\right\} \tag{2.26}$$

s.t. (2.17)-(2.20).

Tri-objective LFPP (2.26) is solved using Cooper method to find three individual ideal solutions as

$$\mathcal{Z}_l = z_l^*;\ \ \mathcal{Z}_m = z_m^*;\ \ \mathcal{Z}_u = z_u^*. \tag{2.27}$$

Stage II: In stage two of hybrid approach weighted Dinkelbach method is used to construct the equivalent single objective LP by assigning weight w_k (lower, middle and upper) with $\sum_{k=1}^{3} w_k = 1$. These weights could be fixed (or they are chosen in a subjective manner, taking into consideration the decision-maker's preferences and the domain knowledge they possess. There is no need for explicit mathematical explanation of this subjectivity in weighted Dinkelbach formulations.)

$$max\{\sum_{k=1}^{3} w_k(\mathcal{P}_k(\varkappa) - z_k^* \mathcal{Q}_k(\varkappa);\ \varkappa \in S\},\ k = 1,2,3 \tag{2.28}$$

or

$$\text{Max } \mathcal{Z} = \max\ [w_l\big(\mathcal{P}_l(\varkappa) - z_l^* \mathcal{Q}_l(\varkappa)\big) + w_m(\mathcal{P}_m(\varkappa) - z_m^* \mathcal{Q}_m(\varkappa)) + w_u(\mathcal{P}_u(\varkappa) - z_u^* \mathcal{Q}_u(\varkappa))] \tag{2.29}$$

s.t. (2.17)-(2.20).

An effective solution for MOLFP that also takes into account its counterpart FLFPP is given by the solution of LPP (2.29).

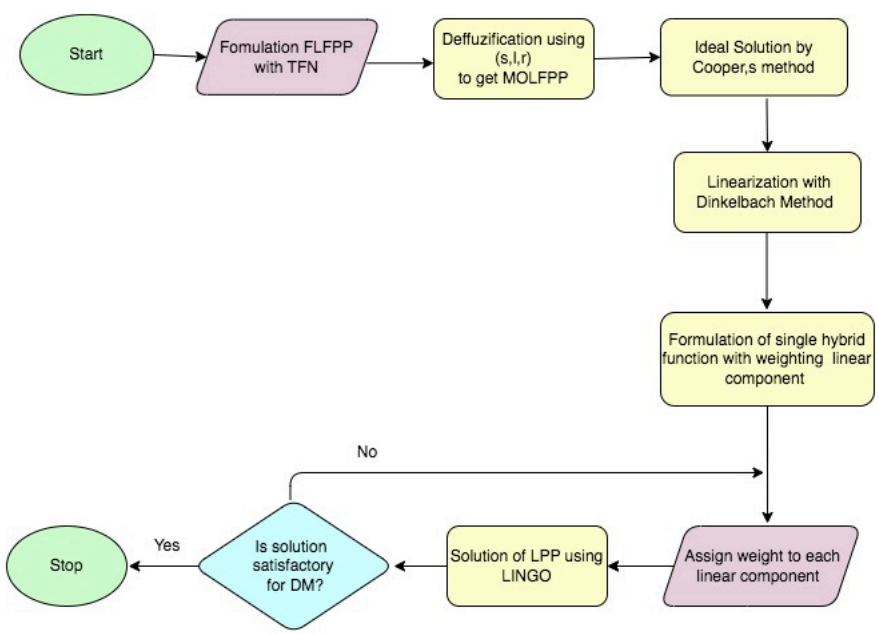


Figure 2.2: Flowchart to solve FLFPP

2.4 Case Study: Production Planning

To show how the suggested FLFPP technique might be used in real life, we use data from [61] to show a production planning scenario with two items, A and B. In this case, triangular fuzzy numbers are used to show the earnings, expenses, raw material needs, and man-hours. This is done to show how imprecise market circumstances and resource availability may be in real life.

Let $\varkappa_1$ represents the quantity of product A unit and $\varkappa_2$ as the quantity of product B units that resulted in this situation. In this way, the problem that was indicated before may be stated:

Now, take into account the following FLFPP

$$Max\mathcal{Z} = \frac{(\tilde{5})\varkappa_1 + (\tilde{3})\varkappa_2}{(\tilde{5})\varkappa_1 + (\tilde{2})\varkappa_2 + (\tilde{1})} \tag{2.30}$$

$s.t.$

$$\left.\begin{array}{c} (\tilde{3})\varkappa_1 + (\tilde{5})\varkappa_2 \le (\widetilde{15}) \\ (\tilde{5})\varkappa_1 + (\tilde{2})\varkappa_2 \le (\widetilde{10}) \\ \varkappa_1, \varkappa_2 \ge 0 \end{array}\right\} \tag{2.31}$$

Writing the above fuzzy coefficients model in (s, ℓ, r) form with ℓ- left spread , r- right spread, the model will be expressed as follows:

$$Max\mathcal{Z} = Max\frac{(5,2,2)\varkappa_1 + (3,1,1)\varkappa_2}{(5,1,1)\varkappa_1 + (2,1,1)\varkappa_2 + (1,1,1)} \tag{2.32}$$

s.t.

$$(3,1,1)\varkappa_1 + (5,2,2)\varkappa_2 \le (15,4,4), \tag{2.33}$$

$$(5,1,1)\varkappa_1 + (2,1,1)\varkappa_2 \le (10,2,2); \varkappa_1, \varkappa_2 \ge 0. \tag{2.34}$$

FLFPP (2.32) reduced to the equivalent tri-objective LFPP as below:

$$Max\mathcal{Z} = \left\{\mathcal{Z}_l = \frac{3\varkappa_1 + 2\varkappa_2}{6\varkappa_1 + 3\varkappa_2 + 2};\ \mathcal{Z}_m = \frac{5\varkappa_1 + 3\varkappa_2}{5\varkappa_1 + 2\varkappa_2 + 1};\ \mathcal{Z}_u = \frac{7\varkappa_1 + 4\varkappa_2}{4\varkappa_1 + \varkappa_2}\right\} \tag{2.35}$$

s.t.

$$2\varkappa_1 + 3\varkappa_2 \le 11; 3\varkappa_1 + 5\varkappa_2 \le 15 ; 4\varkappa_1 + 7\varkappa_2 \le 19. \tag{2.36}$$

$$4x_1 + x_2 \le 8\,;\, 5x_1 + 2x_2 \le 10\,;6x_1 + 3x_2 \le 12\,\,;x_1, x_2 \ge 0. \tag{2.37}$$

Apply Charnes Cooper's Method on (2.35) by using transformation $y_1 = \tau x_1$ and $y_2 = \tau x_2$ the equivalent model will be expressed as

$$MaxZ = \{Z_l = 3y_1 + 2y_2\,, Z_m = 5y_1 + 3y_2, Z_u = 7y_1 + 4y_2\} \tag{2.38}$$

s.t.

$$2y_1 + 3y_2 - 11\tau \le 0;\, 3y_1 + 5y_2 - 15\tau \le 0\,;4y_1 + 7y_2 - 19\tau \le 0, \tag{2.39}$$

$$4y_1 + y_2 - 8\tau \le 0;\; 5y_1 + 2y_2 - 10\tau \le 0\,;6y_1 + 3y_2 - 12\tau \le 0, \tag{2.40}$$

$$6y_1 + 3y_2 + 2\tau \le 1\,;5y_1 + 2y_2 + \tau \le 1;\, 4y_1 + y_2 \le 1, \tag{2.41}$$

$$y_1, y_2, \tau \ge 0. \tag{2.42}$$

Above model solved by coopers technique to get the following ideal results:

$$z_l^* = 0.5352\,,\;\; z_m^* = 1.266,\;\;\; z_u^* = 3.9$$

The corresponding Linear Programming problem as mentioned in (2.24) of the algorithm which is equivalent to the (2.28) is formulated as follows

$$\begin{aligned} MaxZ &= w_l[3x_1 + 2x_2 - 0.5352(6x_1 + 3x_2 + 2)] \\ &+ w_m\,[5x_1 + 3x_2 - 1.266(5x_1 + 2x_2 + 1)] + w_u[7x_1 + 4x_2 \\ &- 3.9(4x_1 + x_2) \end{aligned} \tag{2.43}$$

s.t. (2.36)-(2.37).

The table below shows how to get the best solution by calculating the value of the objective function Z for different choices of the weight parameter w.

2.5 Result Analysis and Comparison

Table 2.1: 10 cases of weights for calculation of Z and distance from ideal solution.

Cases	w_l	w_m	w_u	calculated Z	Distance from ideal solution $Z^* = 4$
1	1	0	0	3.82	0.18
2	0	1	0	0.93	3.07
3	0	0	1	3.77	0.23
4	0.25	0.25	0.5	3.07	0.91
5	0.5	0.25	0.25	3.09	0.91
6	0.25	0.5	0.25	2.36	1.64
7	0.3	0.4	0.3	2.81	1.19
8	0.5	0.5	0	2.38	1.62
9	0	0.5	0.5	2.35	1.65
10	0.5	0	0.5	2.38	1.62

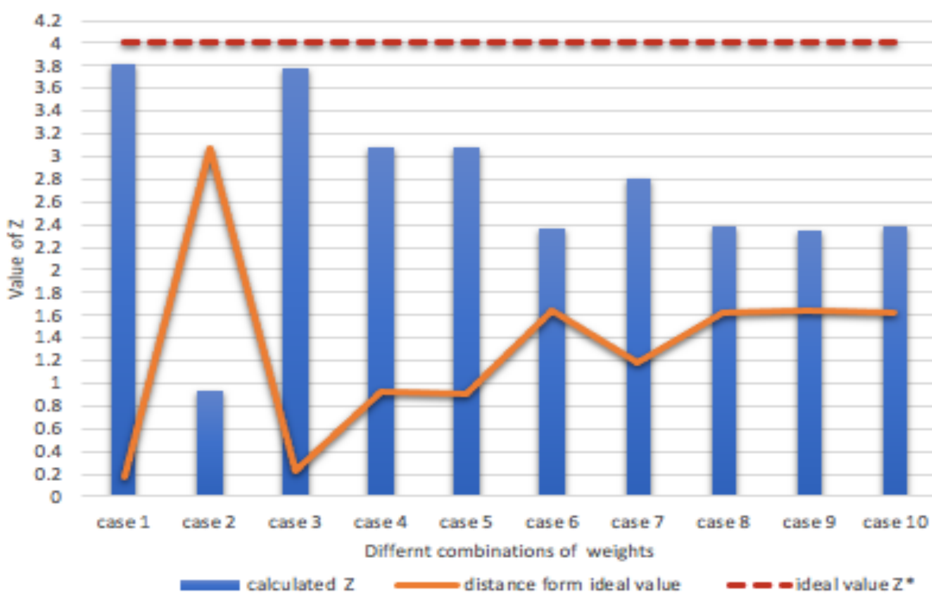


Figure 2.3: Comparison between different combinations of weights

Observations from the Table 2.1 and from Figure 2.3:

(i) Since the ideal solution is $\mathcal{Z}^* = 4$. The proximity of $\mathcal{Z}$ to 4 correlates positively with performance in the distance column indicates the deviation of each weighting from this aim.

(ii) Only the scenarios that are totally optimistic (lower or upper bound) are considered to be the best:

- $\mathcal{Z}$=3.82 and distance=0.18 pertain to Case 1 (w =(1,0,0)).
- In Case 3 (w = (0,0,1)), $\mathcal{Z}$=3.77 and distance=0.23.

Therefore, near-ideal output is achieved when the manufacturing scenario exhibits behavior close to the upper or lower fuzzy boundaries.

(iii) The entirely pessimistic scenario (w =(0,1,0)) is the most unfavorable:

Case2: $\mathcal{Z}$=0.93 and distance=3.07. If one just prepares for the worst-case scenario, performance is significantly suboptimal.

(iv) Risk is reduced but $\mathcal{Z}$ is lowered through balanced trade-offs:

The result is more conservative the more we spread weights over lower, middle, and upper $\mathcal{Z}$ values range from 2.3 to 3.1 for cases 4–10, with distances ranging from around 0.9 to 1.6. This demonstrates how a decision-maker may modify their attitude toward risk and manage ambiguity.

2.5.1 Comparative Perspective with Recent Hybrid Fuzzy Approaches

In the best-case scenario, the proposed hybrid approach provides the decision maker with more efficient and diverse options than the one compromised answer proposed by S.K. Das et al. [61] for the identical problem (0.50, 1.0, 2.4). The recommended method is different from earlier ones since it uses (s, ℓ, r) based decomposition together with Charnes-Cooper and weighted Dinkelbach transformations. This chapter, however, does not have the space to fully compare all of the modern hybrid fuzzy approaches. Most comparable methods do not have this built-in means of dealing with fractional fuzzy targets. The quick comparison to other methods helps to put the contribution in context.

2.5.2 Computational Complexity and Runtime Discussion

The proposed method is computationally efficient because both phases use polynomial-time linear and fractional programming techniques. The Charnes–Cooper transformation and weighted Dinkelbach iterations converge quickly due to the structure of the LFPP. Even with ten weighting scenarios, the execution time remained low for the test problem, showing that the method works well for small- to medium-sized cases. While larger industrial FLFPPs may need further testing, this case study suggests that runtime is not a major concern for practical use.

2.6 Conclusion

The chapter presented a hybrid optimization technique that efficiently solves Fuzzy Linear Fractional Programming Problems (FLFPP) by combining Weighted Dinkelbach's Theorem with tri-objective decomposition. There are two steps to the process. During the first stage, the fuzzy LFPP that has been provided is converted into three Linear Fractional Programming Problems (LFPP) by employing the Upper-Middle-Lower (UML) methodology for triangular fuzzy numbers. These problems are then addressed by employing the Charnes-Cooper method. In the second stage, the optimal solutions that were achieved in the first stage are utilized to form a single Linear Programming Problem (LPP) by utilizing Weighted Dinkelbach's Theorem. This ensures that the computing efficiency is increased while also improving the precision of the solution. This hybrid strategy is shown to be effective through the use of a numerical example, which demonstrates that it is capable of providing answers that are both stable and accurate. A further refinement and expansion of this methodology can be accomplished through the exploration of sophisticated fuzzy models and the inclusion of machine learning in future research.

Chapter 3

An Approach to Linear Fractional Programming Using a Hybrid Fuzzy Interval-Regression Model

3.1 Introduction and Motivation

LFPP is a mathematical method that may be used in a number of scenarios, including mathematical modeling, to allocate resources to various tasks in the best possible way based on preset optimality criteria. Its applications are utilized in numerous industries, including production, finance, business, healthcare, etc. The fuzzy set theory, which was first put forward by Zadeh [3] , has gotten a lot of attention because it can deal with uncertainty and later contribution given by Zimmermann [10],[11]. Numerous authors discussed the FLFPP, Atanassov [17] described the multi objective problem in fuzzy scenario, further Chen[14], Pop & Stancu-Minasian [26], Nachammai &Thangaraj[30], Dubey et al. [31], Mehlawat & Kumar [34], Mitlif [48], Das & Mandal [54],Ambika et al.[55] and Agnihotri et al.[80] provided their extensions. Pal et al.[21] developed a two-dimensional, geometric, interval-based approach to solving FLP problems with fuzzy constraints and fuzzy goals, where the parameters are represented as TFN to generalize Chen[14].

The fractional programming issue, or FPP, is a difficulty that comes up when you want to make the most of a limited ratio. It takes careful thought and planning. LFPP is a way to use arithmetic to best distribute resources to different activities based on set standards of optimality by Elaibi & Nasseri [96], Jerbi & Muanmer [97] and Akhtar et al.[100]. It may be used in a number of situations, such as mathematical modeling. They also employed fuzzy numbers to solve FFLFP problems. Fuzzy sets in Veeramani et al. [39],[49],[69] discovered the optimum solution when the formulation of bi-objective model is done for FFLFP. Various fuzzification strategies have been investigated by researchers over the years with the aim of making optimization models more realistic when faced with uncertainty. α-cut methods are commonly used to convert FN into interval representations, which may then be solved using traditional optimization methods. Several notable works have contributed different techniques to modeling fuzzy objectives, constraints, and solution ranking methods. These includes work by Safaei [40],Das et al.[61], [79], Stanojevic [77], Chauhan et al.[94] , among others.

Researchers have suggested a number of uncertain non-linear and linear programming frameworks during the past few decades. These research have given us useful information, but most of the methods Khalifa et al.[78] that are already out there either employ one or a few α-cuts or just fuzzify the goal or constraints but not both at the same time in a lot of different ways which only give pointwise solutions that don't show the complete variety of possible possibilities.. Veeramani et.al [44] use line graphs to show the objective value's lower, higher, and membership functions for only two r-levels. They don't systematically discussed different α-β combinations or give full tabulated satisfaction ratings for each scenario, but it does show the fuzzy solution space. The present research takes this a step further by using a versatile $\alpha - \beta$ grid method and presenting the findings in detailed surface plots to aid in practical decision-making. But in real life, making decisions typically needs a stronger method that takes into account how fuzziness affects more than one part of the model. This means that we need a bigger framework that can look at how varying degrees of confidence affect both the acceptable range and the best solution in a systematic way. Also, decision-makers may benefit from both interval solutions and knowing how satisfied people are with varied situations, which many current approaches don't adequately measure. Some recent research has started to look into hybrid optimization methods with data-driven surrogates, but there isn't much work being done on how to combine machine learning models with fuzzy fractional programming.

This study resolves these constraints by creating an expanded FILFP framework that employs a parametric α-cut and β-cut approach to systematically investigate the combined uncertainty in OF and restrictions. It does this by employing systematic α-cuts to deal with TFN in the goal function and β-cuts to provide restrictions. This method does a two-dimensional a parametric sweep, which makes a full surface containing interval solutions for every (α, β) pair. We get both the lower bound of $\mathcal{Z}^{l}_{\alpha\beta}$ and the upper bound $\mathcal{Z}^{u}_{\alpha\beta}$ for each pair. This gives us a strong bi-objective interval representation instead of just one clear point. The method also uses Zimmerman’s fuzzy choice-making approach to creates strong interval solutions that show how uncertainty affects choice variables and the amount of satisfaction (λ). The framework lets decision-makers examine how different levels of fuzziness affect the trade-off between the best objective values and the most flexible solution by showing every aspect of the surface in a 3D & 2D display. To make it even more useful in real life, an intuitive supervened regression framework is built on the parameterized sweep results. This model acts as a machine learning-based stand-in to quickly guess λ for every new (α, β) situation. An actual λ vs. predicted λ study shows that this surrogate is reliable. It gives decision makers a useful and easy-to-understand way to do scenario analyses without having to solve all the sharp subproblems over and over again. A production planning scenario based on existing literature shows how the framework is new, easy to use, and has the possibility to be expanded in the future.

The major goal of this study is to create and show how this extended fuzzy intervals parametric method may be used on any real-world FPP with uncertain coefficients. The practical use of the technique in the production planning scenario is an example that shows how the suggested model gives those who make decisions a flexible and realistic way to deal with uncertainty. The suggested method makes it possible to connect complex fuzzy fractional modeling with simple excel surrogates in a useful way, which helps people make strong judgments when they do not know what can be done.

The rest of the Chapter is arranged as follows: The suggested technique, which includes the satisfaction measure and a framework that combines α-cut and β-cut, is detailed in section 3.2 Within the section 3.3, a case study that illustrates the technique is presented. Using in-depth tables and three-dimensional graphs the regression surrogate model. Section 3.5 discuss result analysis and discussion examines the outcomes and evaluates the satisfaction levels and solution ranges. In the end, section 3.6 wraps up the study and gives an outline of probable future research subjects.

3.2 Proposed Method

In this method all the coefficients are FLFPP considered as triangular fuzzy numbers, we have considered symmetric case wherein the defuzzification is done using α-cut and β-cut. To create an fuzzy interval model for objective function using this parametric sweep of α, β .The Zimmerman's max-min approach is incorporated in this methodology with linear MF to investigate the level of satisfaction in different cases obtained with the α-cut and β-cut combos.

We also employed a multiple linear regression in Excel with a 95% confidence level. To make it more stable, a regression model was trained on an interval grid without the endpoints. The results provides the goodness-of-fit statistics and coefficients.

3.2.1 Algorithm for the Proposed Methodology

Step 1: Problem description and mathematical formulation:

Describe the general FLFP model and explain the fuzzy numbers that are shaped like triangles in the objective function and the constraints. Convert real life problem of fuzzy fractional optimization problem into the maximize form as mentioned below:

$$max\mathcal{Z} = \frac{\tilde{p}^{\mathrm{T}}\varkappa + \tilde{d}}{\tilde{q}^{\mathrm{T}}\varkappa + \tilde{e}} = \frac{\sum_{j=1}^{n} \widetilde{p_j}\, \varkappa_j + \widetilde{d_j}}{\sum_{j=1}^{n} \widetilde{q_j}\varkappa_j + \widetilde{e_j}}. \tag{3.1}$$

s.t.

$$\tilde{\mathcal{A}}x \leq \widetilde{\mathcal{B}}, x \geq 0. \tag{3.2}$$

Step 2: Fuzzification, using $\boldsymbol{\alpha}$-cut and $\boldsymbol{\beta}$-cut representation:

Describe how to apply the α-cut method to the goal coefficients and the β-cut method to the restrictions. Conversion of these into interval as expressed below :

Using $\alpha - cut$ of triangular fuzzy number $\widetilde{p} = (l, m, n)$ defined as $p_\alpha = [(m-l)\alpha + l\,, n-(n-m)\alpha]$ to transform (3.1) into interval form of crisp bi-objective optimization problem as follows:

$$\text{Max } \mathcal{Z}_{\alpha\beta} = \frac{[p^l, p^u]\varkappa + [d^l, d^u]}{[q^l, q^u]\varkappa + [e^l, e^u]} \tag{3.3}$$

s.t.

$$[\mathcal{A}_\beta^l, \mathcal{A}_\beta^u]\varkappa \leq [\mathcal{B}_\beta^l, \mathcal{B}_\beta^u] \text{ and } \varkappa \geq 0. \tag{3.4}$$

The α-cut regulates how fuzzy the objective coefficients are, while the β-cut controls how fuzzy the constraints are. Together, they create an interval combination $(\mathcal{Z}_\alpha^l, \mathcal{Z}_\alpha^u)$ by making both the numerator and denominator more or less fuzzy at the same time.

Step 3: Bi-objective deterministic model :

Using ranking and interval algebra, the aforementioned model may be stated as follows.:

$$\text{Max } \mathcal{Z}_{\alpha\beta}(\varkappa) = \max\,[\mathcal{Z}_{\alpha}^{l}(\varkappa),\ \mathcal{Z}_{\alpha}^{u}(\varkappa)] \quad (3.5)$$

$$\mathcal{Z}_{\alpha}^{l}(\varkappa) = \frac{(p^{l}\varkappa + d^{l})}{(q^{u}\varkappa + e^{u})} \; and \; \mathcal{Z}_{\alpha}^{u}(\varkappa) = \frac{(p^{u}\varkappa + d^{u})}{(q^{l}\varkappa + e^{l})} \quad (3.6)$$

s.t.

$$\mathcal{A}_{\beta}^{l}\varkappa \leq \mathcal{B}_{\beta}^{l};\ \mathcal{A}_{\beta}^{u}\varkappa \leq \mathcal{B}_{\beta}^{u} \text{ and } \varkappa \geq 0. \quad (3.7)$$

Step 4: Formulation of equivalent linear model :

. The Charnes-Cooper variable transformation discussed in (1.21)- (1.24) converts fractional programming into a linear bi-objective models that can be evaluated using classical approaches.

Step 5: Perform a systematic parametric sweep with multiple α-β combinations

Change α and β from 0 to 1 in increments of 0.1 (flexible as needed), get the interval limits $\mathcal{Z}_{\alpha}^{l}$ and $\mathcal{Z}_{\alpha}^{u}$, and then find the level of satisfaction levels for every pair.

Step 6: Degree of satisfaction:

Zimmerman's max-min Approach calculates each interval's degree of satisfaction λ, letting decision-makers weigh fuzziness and performance.

Step7:Excel regression model on sweep data.

The parametric grid has exceptionally α and β levels that are 0 and 1, but the end result model of regression may be fitted to the feasible intervals [0.1, 0.9] to make it more broad and eliminate problems that could come up with theoretical edge cases.

Step 8:Result analysis and discussion:

(i)Examine interval solutions and satisfaction ratings λ for various α and β combinations. Discuss how fuzziness levels impact the feasible region, lower-upper bound trade-offs, and robust, risk-informed decision-making. Plots and tables can be used to enhance planners' practical interpretations and show the fuzzy viable zone.

(ii) To make a strong supervened regression substitute, a hybrid approach was trained with and without extreme α and β values ranging from [0,1]. The endpoints were left out of the final prediction model since they correlate to very low levels of satisfaction ($\lambda = 0$). This made the model better at generalizing and making predictions for more realistic degrees of fuzziness. This gave a R^2 of 0.95, which shows that it suited quite well.

3.3 Illustration

Case Study: Production Planning

To show how the suggested FILFP technique might be used in real life, we use data from [54] to show a production planning scenario with two items, A and B. Here, TFN are used to showcase the earnings, expenses, raw material needs, and man-hours. This is done to show how imprecise market circumstances and resource availability may be in real life.

We may write $\varkappa_1$ as the quantity of product A produced and $\varkappa_2$ as the quantity of product B produced in this situation. The problem can be stated as follows:

$$Max\mathcal{Z} = \frac{\tilde{5}\varkappa_1 + \tilde{3}\varkappa_2}{\tilde{5}\varkappa_1 + \tilde{2}\varkappa_2 + \tilde{1}} \qquad (3.8)$$

$s.t.$

$$\tilde{3}\varkappa_1 + \tilde{5}\varkappa_2 \leq \widetilde{15}, \qquad (3.9)$$

$$\tilde{5}\varkappa_1 + \tilde{2}\varkappa_2 \leq \widetilde{10}, \qquad (3.10)$$

$$\varkappa_1, \varkappa_2 \geq 0. \qquad (3.11)$$

Where $\tilde{5} = (3,5,7)$,$\tilde{3} = (2,3,4)$,$\tilde{5} = (4,5,6)$,$\tilde{2} = (1,2,3)$, $\tilde{1} = 0,1,2)$, $\widetilde{15} = (11,15,19)$, $\widetilde{10} = (8,10,12)$.

Then (3.8)-(3.11) expressed as follows:

$$Max\ \mathcal{Z} = \frac{(3,5,7)\varkappa_1 + (2,3,4)\varkappa_2}{(4,5,6)\varkappa_1 + (1,2,3)\varkappa_2 + (0,1,2)} \qquad (3.12)$$

$$(2,3,4)\varkappa_1 + (3,5,7)\varkappa_2 \leq (11,15,19), \quad (3.13)$$

$$(4,5,6)\varkappa_1 + (1,2,3)\varkappa_2 \leq (8,10,12), \quad (3.14)$$

$$\varkappa_1, \varkappa_2 \geq 0. \quad (3.15)$$

After apply $\alpha - cut$ and algebraic operations to (3.12) we obtained

$$max\mathcal{Z}_{\alpha\beta} = \left[\frac{(3+2\,\alpha\,,7-2\,\alpha)\varkappa_1+(2+\,\alpha\,,4-\,\alpha)\varkappa_2}{(4+\,\alpha\,,6-\,\alpha)\varkappa_1+(1+\,\alpha\,,3-\,\alpha)\varkappa_2+(\,\alpha\,,2-\,\alpha)}\right] \quad (3.16)$$

s.t.

$$(2+\beta\,,4-\,\beta)\varkappa_1 + (3+2\beta\,,7-2\,\beta)\varkappa_2 \;\leq (11+4\beta\,,19-4\,\beta), \quad (3.17)$$

$$(4+\beta\,,6-\,\beta)\varkappa_1 + (1+\beta\,,3-\,\beta)\varkappa_2 \;\leq (8+2\beta\,,12-2\,\beta), \quad (3.18)$$

$$\varkappa_1, \varkappa_2 \geq 0. \quad (3.19)$$

(3.16) can be expresses FILFPP as fractional interval of lower and upper programming problems as follows:

$$max\mathcal{Z}_{\alpha\beta} = \left[\frac{(3+2\,\alpha\,)\varkappa_1+(2+\,\alpha\,)\varkappa_2}{(6-\,\alpha)\varkappa_1+(3-\,\alpha)\varkappa_2+(\,2-\,\alpha)}\,,\frac{(7-2\,\alpha)\varkappa_1+(\,4-\,\alpha)\varkappa_2}{(4+\,\alpha\,)\varkappa_1+(1+\,\alpha\,)\varkappa_2+\alpha}\right] = [\mathcal{Z}_\alpha^l\,,\mathcal{Z}_\alpha^u\,] \quad (3.20)$$

s.t.

$$(2+\beta\,)\varkappa_1 + (3+2\beta\;\;)\varkappa_2 \;\leq (11+4\beta\,), \quad (3.21)$$

$$(\,4-\,\beta)\varkappa_1 + (\,7-2\,\beta)\varkappa_2 \;\leq (19-4\,\beta), \quad (3.22)$$

$$(4+\beta\,)\varkappa_1 + (1+\beta\,)\varkappa_2 \;\leq (8+2\beta\,), \quad (3.23)$$

$$(6-\,\beta)\varkappa_1 + (3-\,\beta)\varkappa_2 \;\leq (\,12-2\,\beta), \quad (3.24)$$

$$\varkappa_1, \varkappa_2 \geq 0. \quad (3.25)$$

(3.20) can be expressed as two objectives in FILFPP lower and upper bound as follows:

$$max\mathcal{Z}_\alpha^l = \frac{(3+2\,\alpha\,)\varkappa_1+(2+\,\alpha\,)\varkappa_2}{(6-\,\alpha)\varkappa_1+(3-\,\alpha)\varkappa_2+(\,2-\,\alpha)}\,;\, max\mathcal{Z}_\alpha^u = \frac{(7-2\,\alpha)\varkappa_1+(\,4-\,\alpha)\varkappa_2}{(4+\,\alpha\,)\varkappa_1+(1+\,\alpha\,)\varkappa_2+\alpha} \quad (3.26)$$

s.t.

$$(2+\beta)\varkappa_1 + (3+2\beta\;\;)\varkappa_2 \;\leq (11+4\beta), \quad (3.27)$$

$$(4 - \beta)\varkappa_1 + (7 - 2 \beta)\varkappa_2 \leq (19 - 4 \beta), \quad (3.28)$$

$$(4 + \beta)\varkappa_1 + (1 + \beta)\varkappa_2 \leq (8 + 2\alpha), \quad (3.29)$$

$$(6 - \beta)\varkappa_1 + (3 - \beta)\varkappa_2 \leq (12 - 2 \beta), \quad (3.30)$$

$$\varkappa_1, \varkappa_2 \geq 0. \quad (3.31)$$

Now the above equation (3.26) as a crisp LFPP in $\alpha, \beta - cut$ parameter converted to crisp LPP in $\alpha, \beta - cut$ parameter using Charnes and Coopers method putting $y_1 = \tau\varkappa_1, y_2 = \tau\varkappa_2$ that becomes

$$max\mathcal{Z}_\alpha^l = (3 + 2 \alpha)y_1 + (2 + \alpha)y_2 \,; max\mathcal{Z}_\alpha^u = (7 - 2 \alpha)y_1 + (4 - \alpha)y_2 \quad (3.32)$$

s.t.

$$(2 + \beta)y_1 +.(3 + 2\beta)y_2 - (11 + 4\beta)\tau \leq 0 \,, \quad (3.33)$$

$$(4 - \beta)y_1 + (7 - 2\beta)y_2 - (19 - 4\beta)\tau \leq 0, \quad (3.34)$$

$$(4 + \beta)y_1 + (1 + \beta)y_2 - (8 + 2\beta)\tau \leq 0, \quad (3.35)$$

$$(6 - \beta)y_1 + (3 - \beta)y_2 - (12 - 2 \beta)\tau \leq 0, \quad (3.36)$$

$$(6 - \beta)y_1 + (3 - \beta)y_2 + (2 - \beta)\tau \leq 1, \quad (3.37)$$

$$(4 + \beta)y_1 + (1 + \beta)y_2 + \beta\tau \leq 1, \quad (3.38)$$

$$y_1, y_2, \tau \geq 0. \quad (3.39)$$

Now, given different values of α and β in [0,1] LINGO may be used to find the best solution for both lower and upper functions in (3.32). For varied values of α, β =0,0.1,0.2,0.3,0.4,0.6, 0.7,0.8,0.9,1, we obtained the various optimal solutions as lower value $\mathcal{Z}_\alpha^l$ and upper value $\mathcal{Z}_\alpha^u$ at multiple (α, β) combinations provided in following tables and figures i=under result and discussion section.

3.3.1 Calculations and Interpretation

In this part, we provide the comprehensive computational findings derived from the FILFP approach, such as how the Z-bounds change for different α–β values, a degree of satisfaction λ, and the corresponds to regression surrogate. Using the surface charts and statistical analysis to help with decision-making, the results are explained. Interval results $\mathcal{Z}_{\alpha\beta}^l$ and $\mathcal{Z}_{\alpha\beta}^u$ and interpretations:

Table 3.1: Optimal value of lower bound $\mathcal{Z}^{l}_{\alpha\beta}$

$\beta\backslash\alpha$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0.564	0.614	0.669	0.727	0.790	0.859	0.935	1.017	1.108	1.208	1.320
0.1	0.562	0.612	0.666	0.724	0.787	0.856	0.931	1.013	1.104	1.204	1.315
0.2	0.559	0.609	0.663	0.721	0.784	0.853	0.927	1.010	1.100	1.200	1.311
0.3	0.557	0.607	0.660	0.718	0.781	0.850	0.924	1.006	1.096	1.196	1.307
0.4	0.555	0.605	0.658	0.716	0.779	0.847	0.922	1.003	1.093	1.193	1.304
0.5	0.553	0.603	0.656	0.714	0.776	0.844	0.918	1.000	1.090	1.189	1.300
0.6	0.551	0.601	0.654	0.711	0.774	0.842	0.916	0.997	1.087	1.186	1.297
0.7	0.550	0.599	0.652	0.709	0.772	0.839	0.913	0.995	1.084	1.183	1.294
0.8	0.548	0.597	0.650	0.708	0.770	0.837	0.911	0.992	1.082	1.181	1.291
0.9	0.547	0.596	0.649	0.706	0.768	0.835	0.909	0.990	1.079	1.178	1.288
1	0.439	0.583	0.635	0.691	0.752	0.819	0.892	0.972	1.060	1.158	1.267

Table 3.2: Optimal value of upper bound $\mathcal{Z}^{u}_{\alpha\beta}$

$\beta\backslash\alpha$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	4	3.497	3.029	2.678	3.386	2.139	1.928	1.745	1.586	1.445	1.320
0.1	4	3.457	3.025	2.673	2.381	2.134	1.923	1.740	1.581	1.440	1.315
0.2	4	3.455	3.022	2.669	2.376	2.129	1.918	1.735	1.576	1.436	1.311
0.3	4	3.453	3.018	2.665	2.371	2.124	1.913	1.731	1.572	1.431	1.307
0.4	4	3.451	3.016	2.661	2.368	2.121	1.910	1.727	1.568	1.428	1.304
0.5	4	3.449	3.012	2.657	2.364	2.116	1.905	1.723	1.564	1.424	1.300
0.6	4	3.447	3.009	2.654	2.360	2.113	1.902	1.719	1.560	1.421	1.297
0.7	4	3.446	3.007	2.651	2.357	2.109	1.898	1.716	1.557	1.417	1.294
0.8	4	3.444	3.004	2.648	2.354	2.106	1.895	1.713	1.554	1.415	1.291
0.9	4	3.443	3.002	2.645	2.351	2.103	1.892	1.710	1.551	1.412	1.288
1	4	3.311	2.803	2.413	2.104	1.853	1.645	1.470	1.321	1.192	1.080

Tables 3.1 and 3.2 show the best values for $\mathcal{Z}^{l}_{\alpha}$and $\mathcal{Z}^{u}_{\alpha}$ at the lower as well as upper boundaries for various arrangements of α-cut and β-cut. These findings show how changing the amount of fuzziness in the desired function and restrictions affects the range of possible solutions. The feasible interval gets smaller when the α-cut or β-cut gets bigger (which means more confidence and less fuzziness). This is because the boundaries get tighter and the uncertainty

goes down. This behavior shows the trade-off between being flexible and being strong in fuzzy fractional decision-making.

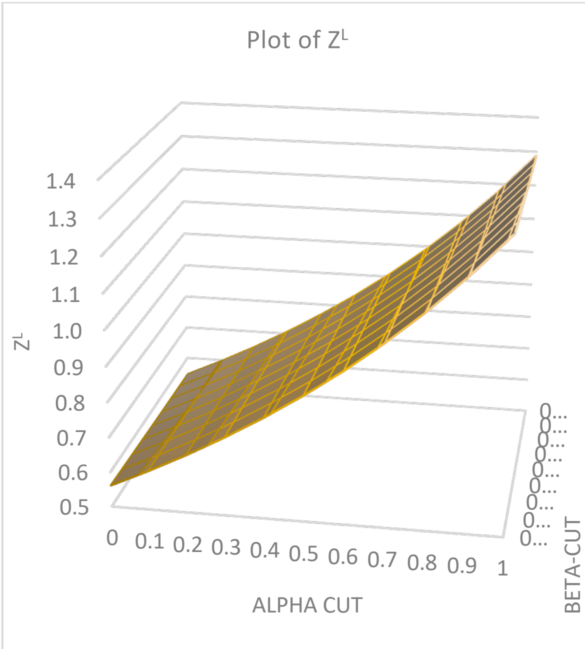


Figure 3.1: Surface plot optimal value of lower bound $\mathcal{Z}^{l}_{\alpha\beta}$ based on table 3.1

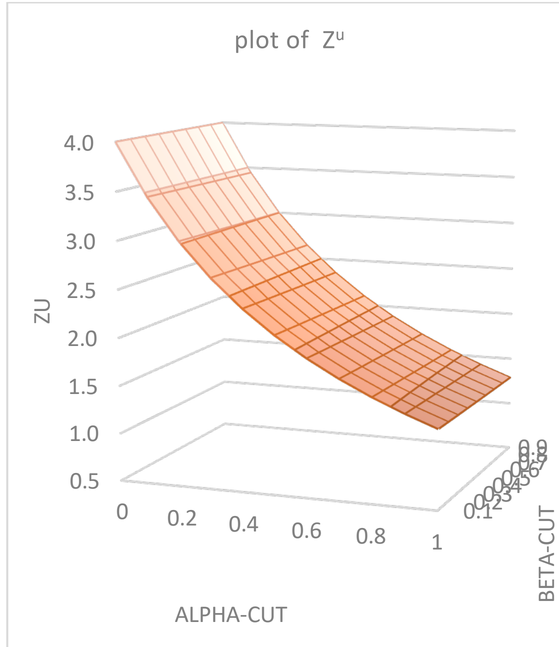


Figure 3.2: Surface plot optimal value of upper bound $\mathcal{Z}^{u}_{\alpha\beta}$ based on table 3.2

Figures 3.1 and 3.2 demonstrate the manner in which the lower and upper limits vary when α and β change, which helps to show the patterns in Tables 3.1 and 3.2. Surface graphs reveal that increasing lower and lower bounds (α and β) reduces the feasible interval. This gives those who make decisions a clear idea of how levels of confidence impact the conservative viewpoint and risk of a solution.

3.3.2 Analysis for Level of Satisfaction λ

Table 3.3: Level of satisfaction λ for $\mathcal{Z}^{l}_{\alpha\beta}$ and $\mathcal{Z}^{u}_{\alpha\beta}$ at different combination of α, β

$\beta \backslash \alpha$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	0.1848	0.2515	0.3237	0.3836	0.2919	0.2142	0.1472	0.0889	0.0379
0.2	0.1861	0.2528	0.3251	0.3842	0.2927	0.2148	0.1479	0.0896	0.0384
0.3	0.1873	0.2541	0.3264	0.3849	0.2934	0.2155	0.1485	0.0903	0.0391
0.4	0.1886	0.2555	0.3279	0.3856	0.2941	0.2163	0.1493	0.0998	0.0398

0.5	0.1899	0.2569	0.3294	0.3864	0.2949	0.217	0.15	0.0917	0.0405
0.6	0.1913	0.2585	0.3312	0.3872	0.2957	0.2179	0.1508	0.0925	0.0413
0.7	0.1928	0.2601	0.3328	0.388	0.2966	0.2187	0.1517	0.0933	0.042
0.8	0.1944	0.2618	0.3346	0.3889	0.2975	0.2196	0.1525	0.0942	0.0428
0.9	0.1961	0.2636	0.3365	0.3899	0.2984	0.2205	0.1535	0.0951	0.047

Table 3.4: Comparison of level of satisfaction λ attained for $(\mathcal{Z}_\alpha^l, \mathcal{Z}_\alpha^U)$ at different α $(\forall\beta \in [0,1])$

α	$\varkappa_1$	$\varkappa_2$	$\mathcal{Z}_\alpha^l$	$\mathcal{Z}_\alpha^U$	λ
0	0	3.67	0.55	4.00	0.00
0.1	0	3.56	0.60	3.44	0.19
0.2	0	3.47	0.65	3.00	0.26
0.3	0	3.39	0.71	2.65	0.33
0.4	**0**	**3.33**	**0.77**	**2.35**	**0.39**
0.5	0	3.25	0.84	2.10	0.29
0.6	0	3.19	0.91	1.89	0.22
0.7	0	3.14	0.99	1.71	0.15
0.8	0	3.09	1.08	1.55	0.09
0.9	0	3.04	1.18	1.41	0.04
1	0	3	1.29	1.29	0.00

The Tables 3.3 and 3.4 demonstrate the computed degree of satisfaction λ for different combinations of (α, β) and compare the level of satisfaction across the boundaries. The results show that increased uncertainty (lower α and β) usually means less satisfaction, whereas more cautious cuts mean higher λ values. This shows that changing the fuzziness settings has a direct effect on the balance between solution accuracy and adaptability.

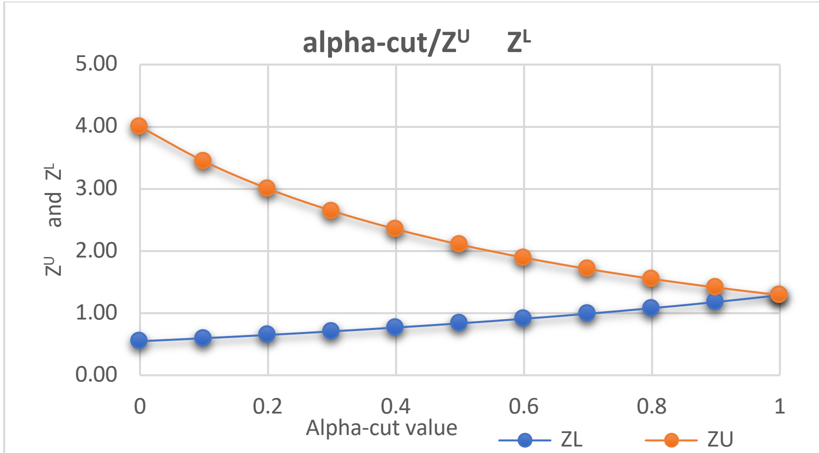


Figure 3.3: Graph for comparison between Z_α^l *and* Z_α^u at α ($\forall\beta \in [0,1]$)

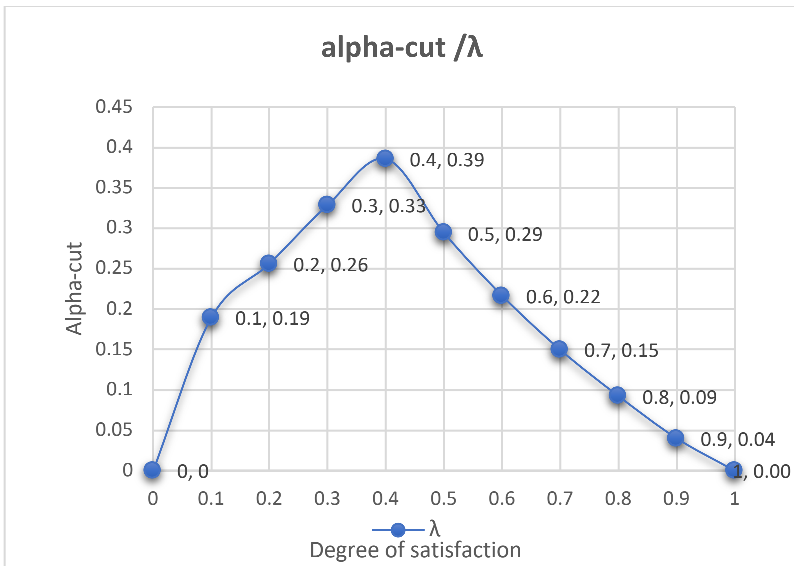


Figure 3.4: Degree of satisfaction associated with Z_α^l *and* Z_α^u *at different values of* α ($\forall\beta \in [0,1]$)

Figures 3.3 and 3.4 show how the calculated Z-bounds relate to the amount of satisfaction λ that results. These graphs show that when the possible range gets smaller, the level of satisfaction goes greater. This supports the numerical patterns that have been shown. This kind of information helps planners choose α, β levels that are in line with acceptable risk levels.

3.4 Regression predictive Model for λ

Table 3.5: Summary regression output for input parameters(α, β, Z_{α}^{l}, Z_{α}^{u}) and λ

(i)

Regression Statistics	
Multiple R	0.974
R Square	0.949
Adjusted R Square	0.946
Standard Error	0.025
Observations	81.000

(ii)

ANOVA

	df	SS	MS	F	Significance F
Regression	4	0.8611	0.2153	354.0865	0.00000
Residual	76	0.0462	0.0006		
Total	80	0.9074			

(iii)

	Coefficient	Standard Error	t Stat	P-value	Lower 95%	Upper 95%	Lower 95.0%	Upper 95.0%
Intercept	2.5983	0.0818	31.7818	0.00000	2.4355	2.7612	2.4355	2.7612
alpha	-5.4813	0.5912	-9.2722	0.0000	-6.6587	-4.3039	-6.6587	-4.3039
beta	0.0577	0.0153	3.7774	0.0003	0.0273	0.0881	0.0273	0.0881
Z(L)	3.3250	0.5314	6.2575	0.0000	2.2667	4.3832	2.2667	4.3832
Z(U)	-1.1344	0.0851	-13.3265	0.0000	-1.3039	-0.9649	-1.3039	-0.9649

A simple linear regression was used because the association between λ and the four predictors (α, β, $Z\alpha^{L}$, $Z\alpha^{u}$) was shown to be monotonic and roughly linear over the parametric grid, which made linear regression a good and understandable option.

The final substitute model that fits is:

$\lambda = 2.60 - 5.48\alpha + 0.0577\beta + 3.32\mathcal{Z}_{\alpha}^{l} - 1.13\mathcal{Z}_{\alpha}^{u}$; ($R^2 = 0.949$).

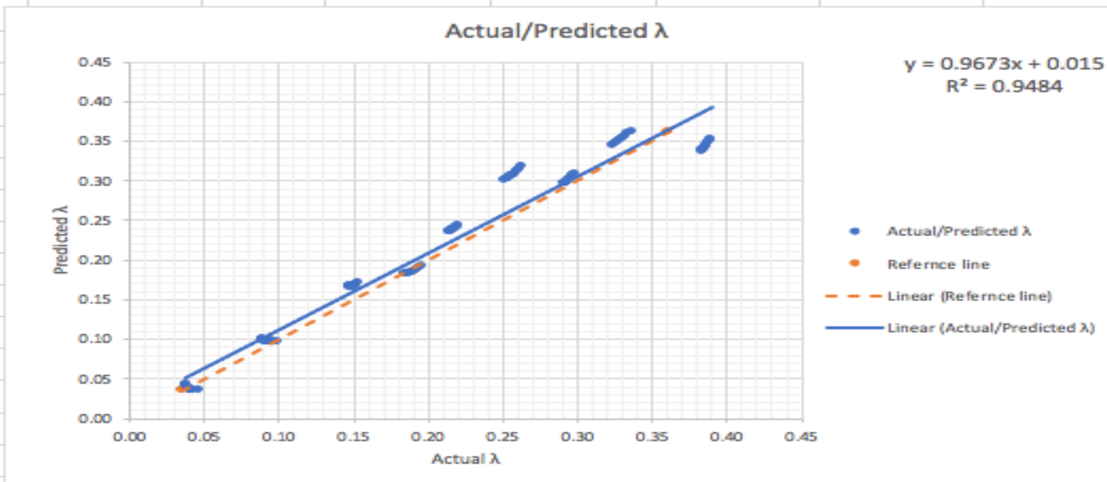


Figure 3.5: Actual/Predicted value of λ

Table 3.5 provides an overview of the regression model that uses α, β, $\mathcal{Z}_{\alpha}^{l}$ and $\mathcal{Z}_{\alpha}^{u}$ to predict λ. The coefficients show that higher degrees of fuzziness make people less satisfied, whereas firmer lower limits make them more satisfied. Figure 3.5 displays the Actual λ vs. Predicted λ. The points are quite near to the reference line, which means that the predictions are very accurate ($R^2 = 0.949$). This shows that the surrogate model is effective in real life for quickly analyzing scenarios without having to do parametric sweeps over and over.

3.5 Result Analysis and Discussion

The interval solutions for several (α, β) pairings show how the lower and upper limits of an objective function shift in a predictable way when the fuzziness levels change. As α and β go up, the solution intervals get smaller and smaller, coming closer to the clear optimal value at $\alpha = \beta = 1$. This is shown in the boundary plots (Figures 3.2 and 3.3). The satisfaction levels calculated using Zimmerman's method reveal that larger degrees of fuzziness lower the overall level of satisfaction. This shows the trade-off among resilience and risk tolerance. The findings in the table show that the α -cut for the aim and the β -cut for the restrictions work together to create a complete fuzzy viable region. We trained a supervened regression surrogate to estimate levels of satisfaction directly from the uncertainty factors (α, β) and the range bounds $(\mathcal{Z}^l, \mathcal{Z}^u)$ to help with real-world decision-making.

A standard error of 0.0247 was recorded across 81 observations, and the hybrid regression substitute that was fitted to the (α, β, $\mathcal{Z}_\alpha^l$, $\mathcal{Z}_\alpha^u$) dataset produced a R^2 of 0.949 (adjusted R^2 = 0.946). The significance of the model was validated by the total ANOVA ($F = 354.08$, $p < 0.0001$). of the predictors were of significance at $p < 0.001$, which means that both the uncertainty levels and the interval limits are very good at explaining the degree of satisfaction. The final substitute surrogate model that fits is :

$\lambda = 2.60 - 5.48\alpha + 0.0577\beta + 3.32\mathcal{Z}_\alpha^l - 1.13\mathcal{Z}_\alpha^u$ ($R^2 = 0.949$).

Because of this, decision-makers are able to immediately evaluate the level of satisfaction for every new combination of α and β without having to restart the entire fractional program. Excel multiple linear regression was performed at 95% confidence; Results provide goodness-of-fit statistics and coefficients

The final model, which was fitted without any trivial end cases when λ goes to zero, had a R^2 of 0.95, which shows that it was quite well at predicting. This hybrid approach lets decision-makers quickly evaluate "what-if" possibilities and get a rough idea of how satisfied people are without having to run the entire fractional programming every time. Overall, the findings show that the suggested fuzzy interval-regression hybrid is a versatile and dependable method for robust optimization when parameters are not known.

A short sensitivity assessment was done by changing the regression coefficients between the 95% confidence ranges. The λ values that came out of the regression varied just a little (≤ 2–3%), which means that regression uncertainty does not have a big effect on the overall FILFPP results.

The results show that the suggested FILFP framework accurately models imprecise coefficients via systematic α-cut, β-cuts, and the regression surrogate model makes it easy to quickly figure out how satisfied people are in any situation. This mixed method gives decision-makers both clarity and speed, which helps with strong analysis of tradeoffs in real-world planning situations.

3.6 Conclusion

This research has created a more advanced FILFP framework to help solve real-world choice issues when both goal function and constraint coefficients are not precise. The discussed process creates strong bi-objective a interval solutions that accurately show how fuzziness affects the amount of satisfaction (λ) by methodically incorporating a parametric α-cut for the goal and a β-cuts for the constraints. The Cooper's transformation and Zimmerman's method work well to deal with the fractional structure and measure degrees of satisfaction in different situations. To make it even more useful in real life, an effortless supervened regression model was built on the parameterized sweep data. This model can quickly estimate λ using machine learning. The Actual vs. Predicted λ study shows that the surrogate is accurate and easy to understand. This gives decision makers a useful tool for scenario analysis that doesn't need them to solve the same sharp subproblems over and over again.

The suggested hybrid fuzzy interval–regression methodology connects conventional fuzzy fractional programming with current data driven prediction. It offers an adaptable, risk-based in order, and computationally efficient way to help people make strong judgments when they don't know what to do. A production planning example shows how useful the framework is. In the future, the technique may be used for more complicated real-world problems and advanced machine learning surrogates.

Chapter 4

Multi-Objective Linear Fractional Programming Problem Solving with Hybrid Extended Intuitionistic Fuzzy Approach

4.1 Introduction and Motivation

Many actual world problems and challenges are dependent upon the fraction of economic or physical worth, like price over time, price over volume, profit over price, or any other or any other figure that evaluates a system's effectiveness, can be given in the form of linear functions. This is where LFPP comes into play. Moreover, when addressing some real issues, it is more wise to maximize the ratios of several criteria rather than improving the functions of each criteria individually. One assumes, conventionally, that the decision variables and parameters of an optimization model have precise values. Practically, the DM often fails to give input information with explicit deterministic constraints due to uncontrollable factors like market conditions., sudden climate changes, mistakes of judgment, data recording problems. Later on the idea used by many researcher to create various methods to deal with FLFP can be seen in Buckley [18], Kumar et al.[82] .In recent past numerous growth has been shown by researchers in the direction of fuzzy optimization Arora & Gupta [20], Lee et al.[24].

In very complicated real-life decision-making procedures, the DM has some uncertainty in choosing the last one within a realistic time constraint. Atanassov[17] thus extended the FS theory framework to IF sets, where each element is specified by a resistance level in addition to its acceptance and rejection degrees; this includes hesitancy in addition to data imprecision. Angelov [16] worked on IF conversion of constraints and OF in crisp model, Dubey et al. [31] provided interval based IF methodology , Singh & Yadav [62] worked on ranking techniques in IFA. In recent year many worked on IF MOFPP Mishra et al.[41], Malik & Gupta [103] and Chauhan et al.[104].Researchers interested in addressing the several genuine issues with information that takes the form of IF values rather of the determinate values have drawn attention to this generalization Kumar et al.[82], Gulia et al.[99].Fuzzy approach to handle MOLFPP was initiated by Chakraborty & Gupta [19]. Later on many modifications and extensions were done and that can be seen in Yang [66] provided a way to solve fuzzy MOLFP that uses measurements of superiority and inferiority in agriculture.

Most of the work done in the area of IFA used IF coefficients only and case of MF only linear models are being used in Debnath & Gupta[64], Mukherjee et al.[68], Dombi et al.[91], [72], Dinkelbach[5], Pal et al.[21], Dey et al.[42], Jain & Arya [37] these research fuzzify parameters or goals to account for data uncertainty and the fact that decision-makers are unsure. These works did a good job of showing how IFS may be used to describe input data ambiguity, but they don't take the intuitionistic structure all the way to the solution stage Ehsani et al.[56], Huang[57]. The research, Agarwal et al.[92], Gulia et al.[98],[99] on the other hand, demonstrates that the application of IFS in FP models is still restricted and frequently just looks at basic coefficient fuzzification without looking at how MF and NMF might be combined in different ways. We studied from reviewing literature that no one has tried an organized intuitionistic fuzzy technique with more than one non-linear and linear MF/NMF form on crisp LFPP from past few years. This gap is what led to this work, which suggests an EIFA that has a clear fractional programming designs but adds an amalgamated intuitionistic layer with numerous MF/NMF combinations. This gives a better way to quantify satisfaction, discontent, and net certainty for making strong decisions.

Given this gap, the goal of this work is to create and evaluate an EIFA over MOLFPP. The EIFA systematically blends linear and non-linear (parabolic and exponential) MF/NMF to create nine hybrid scenarios that may be utilized to gauge satisfaction, discontent, and net decision confidence when there is uncertainty. Most IFS implementations just look at fuzzified coefficients, while the suggested model keeps a clear underlying structure and adds an IF phase at the solution stage Medina et al. [58], Amer [59]. We tested the method contrary to the famous Chakraborty & Gupta [19] fuzzy min–max problem solving and used it to plan a real-life agricultural project Yang [66]. This introduces a versatile and easy-to-understand technique that may be used with many types of MF or combined with fuzzified coefficients for even more practical applications.

The subsequent sections constitute the remainder of this chapter. Section 4.2 consisting of MF and NMF required for proposed EIFA model. Section 4.3 described the steps of the methodology under EIFA. Section 4.4 and 4.5 covered benchmark illustration with calculation result analysis and comparison. Whereas the real life application case study on agriculture planning based on secondary data is covered under section 4.6 and 4.7 with calculation, result analysis and comparison. Finally section 4.8 described the conclusion and future work.

4.1.1 Single objective (SO)-LFPP model

$$\max \mathcal{Z} = \frac{p^{\mathrm{T}}\varkappa + d}{q^{\mathrm{T}}\varkappa + e} \tag{4.1}$$

$s.t.$

$$\varkappa \in \mathcal{S} = \{\varkappa \in \mathbb{R}^n : .\mathcal{A}\varkappa(\leq, =, \geq)\,\mathcal{B}, \varkappa \geq 0\} \tag{4.2}$$

$\mathcal{A}$ as $m \times n\ matrix$ and $\mathcal{B}\ as\ m$x1 matrix ; $p\,, q\ \in\ \mathbb{R}^n$ and $d, e \in\ \mathbb{R}$.

4.1.2 Multi-objective (MO)-LFPP

$$\max \mathcal{Z}_k(\varkappa)\ = \frac{p^{\mathrm{T}}{}_k\ \varkappa + d_k}{q^{\mathrm{T}}{}_k\varkappa + e_k}\ \ k = 1,2,\ldots,\mathcal{K} \tag{4.3}$$

$s.t.$

$$\mathcal{A}\varkappa(\leq, =, \geq)\mathcal{B}, \varkappa \geq 0. \tag{4.4}$$

(4.3) will become single objective if $\mathcal{K} = 1$

4.2 MF and NMF:

In this chapter, we constructed the MF and NMF for intuitionistic method in conjunction using both linear and non-linear association functions. We are going to incorporate linear(L) , parabolic(P) and exponential(E) function for the construction of hybrid combination of these fuzzy numbers. In chapter 1 we have discussed about the membership function $\mu_{\tilde{\mathrm{F}}}(\mathcal{Z})$ for all of the above mentioned functions in equations (1.8)-(1.10). Consider the minimum tolerance value of $\mathcal{Z} = \mathcal{Z}^{min}$ and the maximum tolerance value of $\mathcal{Z} = \mathcal{Z}^{max}$. NMF is denoted as $\nu_{\tilde{\mathrm{F}}}(\mathcal{Z})$ and defined for the given set of associated functions as follows:

4.2.1 Linear NMF

$$\dot{\nu}_{\tilde{F}}(\mathcal{Z}) = \begin{cases} 0, & if \quad \mathcal{Z} \geq \mathcal{Z}^{max}, \\ 1 - \dfrac{\mathcal{Z} - \mathcal{Z}^{min}}{\mathcal{Z}^{max} - \mathcal{Z}^{min}}, & if \ \mathcal{Z}^{min} \leq \mathcal{Z} \leq \mathcal{Z}^{max}, \\ 1, & if \ \ \mathcal{Z} \leq \mathcal{Z}^{min}. \end{cases} \tag{4.5}$$

4.2.2 Parabolic NMF

$$\dot{\nu}_{\tilde{F}}(\mathcal{Z}) = \begin{cases} 0, & if \ \ \mathcal{Z} \geq \mathcal{Z}^{max}, \\ 1 - \left(\dfrac{\mathcal{Z} - \mathcal{Z}^{min}}{\mathcal{Z}^{max} - \mathcal{Z}^{min}}\right)^2, & if \ \mathcal{Z}^{min} \leq \mathcal{Z} \leq \mathcal{Z}^{max}, \\ 1, & if \ \mathcal{Z} \leq \mathcal{Z}^{min}. \end{cases} \tag{4.6}$$

4.2.3 Exponential NMF

$$\dot{\nu}_{\tilde{F}}(\mathcal{Z}) = \begin{cases} 0, & if \quad \mathcal{Z} \geq \mathcal{Z}^{max}, \\ 1 - \eta\left[1 - exp\left(-\rho\dfrac{\mathcal{Z} - \mathcal{Z}^{min}}{\mathcal{Z}^{max} - \mathcal{Z}^{min}}\right)\right], & if \ \mathcal{Z}^{min} < \mathcal{Z} < \mathcal{Z}^{max}, \\ 1, & if \ \ \mathcal{Z} \leq \mathcal{Z}^{min}. \end{cases} \tag{4.7}$$

Non-membership functions are usually specified by opponents. Dissatisfaction decreases as satisfaction increases from 0 to 1. NMF is defined as one minus the MF.

4.3 Mathematical Model development for EIFA

Step1: ***Formulation of S/MO-LFPP***

Deciding the decision variables , objective functions with associated ratio to be optimize and the constrains as per the restrictions over the resources as expressed in form of (4.1) or (4.3). as P1.

Step2: Defuzzification and Reduction to Equivalent S/MO-LPP

(i)

For fuzzy coefficients based fractional programming problem using appropriate ranking function first convert it into the crisp model of maximization type.

(ii)

Use of Charnes-Cooper method with linear transformation $y = \tau x$, the MOLFPP reduces to equivalent deterministic MO-LPP as follows.

$$P2: \max \mathcal{Z}_k = \max \tau \mathcal{P}_k(y/\tau) \quad (4.8)$$

$$\mathcal{A}\left(\frac{y}{\tau}\right) - \mathcal{B} \leq 0 \quad (4.9)$$

$$\tau Q_k(y/\tau) \leq 1 \quad (4.10)$$

$$y \geqq 0 \text{ and } \tau > 0 \quad (4.11)$$

Step 3: Pay-off Matrix

Solve (4.8) for k =1,2,…, $\mathcal{K}$ to find solutions for each $\mathcal{Z}_k$. Using Pay-off matrix each kth objective find target point as $(\mathcal{Z}_k)^{max.}$, pareto- lower bound as $(\mathcal{Z}_k)^{min.}$, and global minimum (worst) point as $(\mathcal{Z}_k)^{w}$. Considering $(\mathcal{Z}_k)^{max.}$ and $(\mathcal{Z}_k)^{min.}$ as upper and lower respect. for MF whereas $(\mathcal{Z}_k)^{max.}$ and $\mathcal{Z}_k)^{w}$ as upper and lower value respect. for NMF constructed for EIFA setup.

Step 4: Formation of MF and NMF

Define for each $\mathcal{Z}_k$ a MF as $\dot{\mu}_k(\mathcal{Z}_k)$ and NMF as $\dot{\nu}_k(\mathcal{Z}_k)$ as follows:

(i)L-MF

$$\dot{\mu}_k(\mathcal{Z}_k) = \begin{cases} 1, & if \ \ \mathcal{Z}_k \geq (\mathcal{Z}_k)^{max.}, \\ \dfrac{\mathcal{Z}_k - \mathcal{Z}_k^{min.}}{\mathcal{Z}_k^{max.} - \mathcal{Z}_k^{min.}}, & if \ \mathcal{Z}_k^{min.} < \mathcal{Z}_k < (\mathcal{Z}_k)^{max.}, \\ 0, & if \ \mathcal{Z}_k \leq \mathcal{Z}_k^{min.}. \end{cases} \quad (4.12)$$

(ii) L-NMF

$$\dot{\nu}_k(\mathcal{Z}_k) = \begin{cases} 0, & if \ \ \mathcal{Z}_k \geq (\mathcal{Z}_k)^{max.}, \\ \dfrac{\mathcal{Z}_k^{max.} - \mathcal{Z}_k}{\mathcal{Z}_k^{max.} - \mathcal{Z}_k^{w.}}, & if \ \mathcal{Z}_k^{w.} < \mathcal{Z}_k < (\mathcal{Z}_k)^{max.}, \\ 1, & if \ \mathcal{Z}_k \leq \mathcal{Z}_k^{w.}. \end{cases} \quad (4.13)$$

(iii) P-MF

$$\dot{\mu}_{k}(\mathcal{Z}_{k}) = \begin{cases} 1, & if \quad \mathcal{Z}_{k} \geq (\mathcal{Z}_{k})^{max.}, \\ \left(\dfrac{\mathcal{Z}_{k} - \mathcal{Z}_{k}^{min.}}{\mathcal{Z}_{k}^{max.} - \mathcal{Z}_{k}^{min.}}\right)^{2}, & if \ \mathcal{Z}_{k}^{min.} < \mathcal{Z}_{k} < (\mathcal{Z}_{k})^{max.}, \\ 0, & if \quad \mathcal{Z}_{k} \leq \mathcal{Z}_{k}^{min.}. \end{cases} \tag{4.14}$$

(iv)P-NMF

$$\dot{\nu}_{k}(\mathcal{Z}_{k}) = \begin{cases} 0, & if \quad \mathcal{Z}_{k} \geq (\mathcal{Z}_{k})^{max.}, \\ \left(\dfrac{\mathcal{Z}_{k}^{max.} - \mathcal{Z}_{k}}{\mathcal{Z}_{k}^{max.} - \mathcal{Z}_{k}^{w.}}\right)^{2}, & if \ \mathcal{Z}_{k}^{w} < \mathcal{Z}_{k} < (\mathcal{Z}_{k})^{max.}, \\ 1, & if \ \ \mathcal{Z}_{k} \leq \mathcal{Z}_{k}^{w.}. \end{cases} \tag{4.15}$$

(v) E-MF

$$\dot{\mu}_{k}(\mathcal{Z}_{k} = \begin{cases} 1, & if \quad \mathcal{Z}_{k} \geq (\mathcal{Z}_{k})^{max.}, \\ \eta_{k}\left[1 - exp\left(\dfrac{-\rho_{k}(\mathcal{Z}_{k} - \mathcal{Z}_{k}^{min.})}{(\mathcal{Z}_{k}^{max.} - \mathcal{Z}_{k}^{min.})}\right)\right], & if \ \mathcal{Z}_{k}^{min.} < \mathcal{Z}_{k} < (\mathcal{Z}_{k})^{max.}, \\ 0, & if \quad \mathcal{Z}_{k} \leq \mathcal{Z}_{k}^{min.}. \end{cases} \tag{4.16}$$

(vi) E-NMF

$$\dot{\nu}_{k}(\mathcal{Z}_{k}) = \begin{cases} 0, & if \quad \mathcal{Z}_{k} \geq (\mathcal{Z}_{k})^{max.}, \\ 1 - \eta_{k}\left[1 - exp\left(\dfrac{-\rho_{k}\, \mathcal{Z}_{k}^{max.} - \mathcal{Z}_{k})}{\mathcal{Z}_{k}^{max.} - \mathcal{Z}_{k}^{w.})}\right)\right], & if \ \mathcal{Z}_{k}^{w} < \mathcal{Z}_{k} < (\mathcal{Z}_{k})^{max.}, \\ 1 & if \ \mathcal{Z}_{k} \leq \mathcal{Z}_{k}^{w.}. \end{cases} \tag{4.17}$$

Step 5: Intuitionistic Fuzzy Approach (IFA)

The reference function method developed by Angelov (1997) can be used to produce an effective solution for MO-LPP. For that we need to define functions:

$\lambda = \min_{k}\{(\dot{\mu}_{k}(\mathcal{Z}_{k})$ and $\delta = \max_{k}\{\dot{\nu}_{k}(\mathcal{Z}_{k})\}$

Now, Angelov's theory says that the objective is to lower the highest degree of rejection and raise the lowest level of acceptance. This makes the problem in IFA

IFA-1: $\max\lambda,\ min\delta$

s.t.

$$\dot{\mu}_{k}(\mathcal{Z}_{k}) \geq \lambda, \; k = 1 \text{ to } \mathcal{K}$$
$$\dot{\nu}_{k}(\mathcal{Z}_{k}) \leq \delta, \; k = 1 \text{to } \mathcal{K}$$
$$0 \leq \delta \leq \lambda \leq \lambda + \delta \leq 1, \tag{4.18}$$
constraint (4.9)-(4.11).

This above model could potentially further reformulated as a deterministic problem with only one objective as a IFA

IFA-2:

$$\max(\lambda - \delta)$$

s.t.

$$\dot{\mu}_{k}(\mathcal{Z}_{k}) \geq \lambda, \; k = 1,2,\ldots, \mathcal{K}$$
$$\dot{\nu}_{k}(\mathcal{Z}_{k}) \leq \delta, \; k = 1,2,\ldots, \mathcal{K}$$
$$0 \leq \delta \leq \lambda \leq \lambda + \delta \leq 1, \tag{4.19}$$
constraint (4.9)-(4.11).

Step 6: Situation based Extended Intuitionistic Fuzzy Approach (EIFA):

Here in this model we are considering three associate functions linear, parabolic and exponential so one can select MF in three different ways and NMF also as three different ways. So we can build 9 situations by taking their different hybrid combinations in the EIFA as mentioned below:

Situation 1:L-MF/L-NMF

$$\max(\lambda - \delta)$$
$$\frac{\mathcal{Z}_{k} - \mathcal{Z}_{k}^{(\min)}}{\mathcal{Z}_{k}^{(\max)} - \mathcal{Z}_{k}^{(\min)}} \geq \lambda, \; k = 1 \text{to } \mathcal{K}$$
$$\frac{\mathcal{Z}_{k}^{(\max)} - \mathcal{Z}_{k}}{\mathcal{Z}_{k}^{(\max)} - \mathcal{Z}_{k}^{(w)}} \leq \delta, \; k = 1 \text{to } \mathcal{K} \tag{4.20}$$
$$0 \leq \delta \leq \lambda \leq \lambda + \delta \leq 1,$$

constraint (4.9)-(4.11).

Situation 2: L-MF/P-NMF

$$\max(\lambda - \delta)$$

$$\frac{z_k - z_k^{(min)}}{z_k^{(max)} - z_k^{(min)}} \geq \lambda, \ k = 1 \text{to} \ \mathcal{K}$$

$$\left(\frac{z_k^{(max)} - z_k}{z_k^{(max)} - z_k^{(w)}}\right)^2 \leq \delta, \ k = 1 \text{to} \ \mathcal{K} \tag{4.21}$$

$$0 \leq \delta \leq \lambda \ \leq \lambda + \delta \leq 1,$$

constraint (4.9)-(4.11).

Situation 3: L-MF/E-NMF

$$\max(\lambda - \delta)$$

$$\frac{z_k - z_k^{(min)}}{z_k^{(max)} - z_k^{(min)}} \geq \lambda, \ k = 1 \text{to} \ \mathcal{K}$$

$$1 - \eta_k \left[1 - exp\left(-\rho_k \frac{(z_k - z_k^{(w)})}{(z_k^{(max)} - z_k^{(w)})}\right)\right] \leq \delta, \ k = 1 \text{to} \ \mathcal{K} \tag{4.22}$$

$$0 \leq \delta \leq \lambda \ \leq \lambda + \delta \leq 1,$$

constraint (4.9)-(4.11).

Situation 4: P-MF/L-NMF

$$\max(\lambda - \delta)$$

$$\left(\frac{z_k - z_k^{(min)}}{z_k^{(max)} - z_k^{(min)}}\right)^2 \geq \lambda, \ k = 1 \text{to} \ \mathcal{K}$$

$$\frac{z_k^{(max)} - z_k}{z_k^{(max)} - z_k^{(w)}} \leq \delta, \ k = 1 \text{to} \ \mathcal{K} \tag{4.23}$$

$$0 \leq \delta \leq \lambda \ \leq \lambda + \delta \leq 1,$$

constraint (4.9)-(4.11).

Situation 5: P-MF/P-NMF

$$\max(\lambda - \delta)$$

$$\left(\frac{\mathcal{Z}_k-\mathcal{Z}_k^{(\min)}}{\mathcal{Z}_k^{(\max)}-\mathcal{Z}_k^{(\min)}}\right)^2 \geq \lambda,\ k = 1 \text{ to } \mathcal{K}$$

$$\left(\frac{\mathcal{Z}_k^{(\max)}-\mathcal{Z}_k}{\mathcal{Z}_k^{(\max)}-\mathcal{Z}_k^{(w)}}\right)^2 \leq \delta,\ k = 1 \text{ to } \mathcal{K} \tag{4.24}$$

$$0 \leq \delta \leq \lambda \leq \lambda + \delta \leq 1,$$

constraint (4.9)-(4.11).

Situation 6:P-MF/E-NMF

$$\max(\lambda - \delta)$$

$$\left(\frac{\mathcal{Z}_k-\mathcal{Z}_k^{(\min)}}{\mathcal{Z}_k^{(\max)}-\mathcal{Z}_k^{(\min)}}\right)^2 \geq \lambda,\ k = 1 \text{ to } \mathcal{K}$$

$$1 - \eta_k\left[1 - exp\left(-\rho_k \frac{(\mathcal{Z}_k-\mathcal{Z}_k^{(w)})}{(\mathcal{Z}_k^{(\max)}-\mathcal{Z}_k^{(w)})}\right)\right] \leq \delta,\ k = 1 \text{ to } \mathcal{K} \tag{4.25}$$

$$0 \leq \delta \leq \lambda \leq \lambda + \delta \leq 1,$$

constraint (4.9)-(4.11).

Situation 7:E-MF/L-NMF

$$\max(\lambda - \delta)$$

$$\eta_k\left[1 - exp\left(-\rho_k \frac{(\mathcal{Z}_k-\varkappa^{(\min)})}{(\mathcal{Z}_k^{(\max)}-\mathcal{Z}_k^{(\min)})}\right)\right] \geq \lambda,\ k = 1 \text{ to } \mathcal{K}$$

$$\frac{\mathcal{Z}_k^{(\max)}-\mathcal{Z}_k}{\mathcal{Z}_k^{(\max)}-\mathcal{Z}_k^{(w)}} \leq \delta,\ k = 1 \text{ to } \mathcal{K} \tag{4.26}$$

$$0 \leq \delta \leq \lambda \leq \lambda + \delta \leq 1,$$

constraint (4.9)-(4.11).

Situation 8:E-MF/P-NMF

$$\max(\lambda - \delta)$$

$$\eta_k\left[1 - exp\left(-\rho_k \frac{(\mathcal{Z}_k-\varkappa^{(\min)})}{(\mathcal{Z}_k^{(\max)}-\mathcal{Z}_k^{(\min)})}\right)\right] \geq \lambda,\ k = 1,2,\ldots,\mathcal{K}$$

$$\left(\frac{\mathcal{Z}_k^{(\max)}-\mathcal{Z}_k}{\mathcal{Z}_k^{(\max)}-\mathcal{Z}_k^{(w)}}\right)^2 \leq \delta,\ 1 \text{ to } \mathcal{K} \tag{4.27}$$

$$0 \leq \delta \leq \lambda \leq \lambda + \delta \leq 1,$$

constraint (4.9)-(4.11).

Situation 9:E-MF/E-NMF

$$\max(\lambda - \delta)$$

$$\eta_k \left[1 - exp\left(-\rho_k \frac{(Z_k - \varkappa^{(min)})}{(Z_k^{(max)} - Z_k^{(min)})}\right)\right] \geq \lambda,\ k = 1,2,\ldots,\mathcal{K}$$

$$1 - \eta_k \left[1 - exp\left(-\rho_k \frac{(Z_k - Z_k^{(w)})}{(Z_k^{(max)} - Z_k^{(w)})}\right)\right] \leq \delta,\ k = 1,2,\ldots,\mathcal{K} \tag{4.28}$$

$$0 \leq \delta \leq \lambda \leq \lambda + \delta \leq 1,$$

constraint (4.9)-(4.11).

Step 7: Comparitive Study and Result Analysis among nine-situations and with existing approaches.

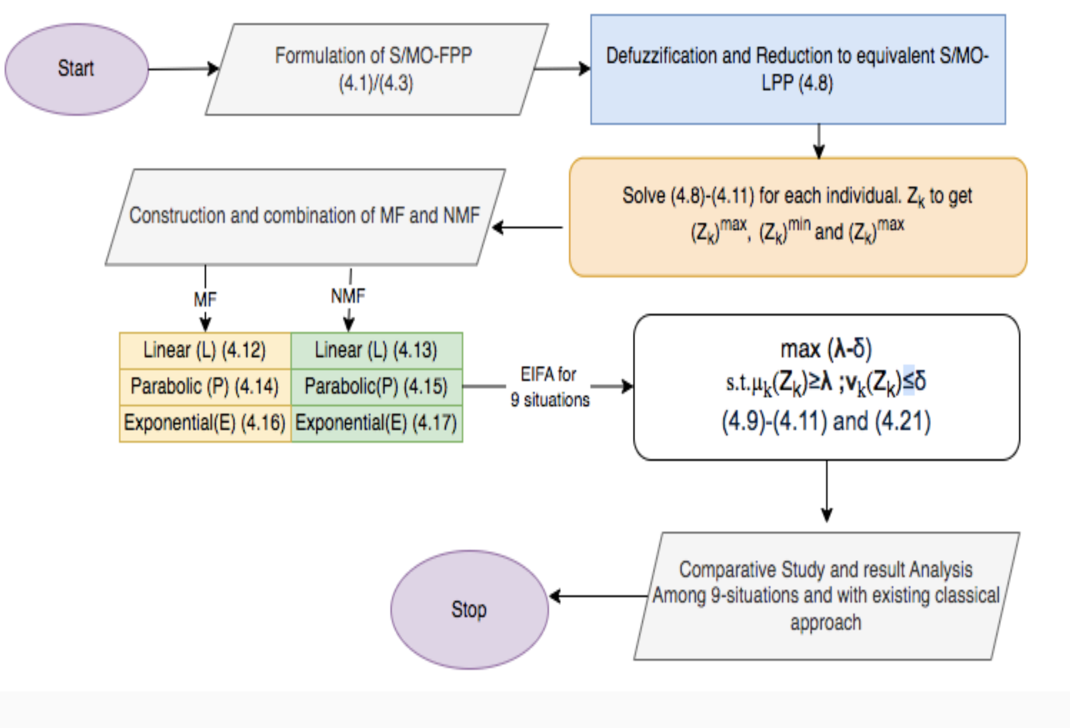


Figure 4.1: Flowchart for EIFA

4.4 Numerical Example:

To demonstrate the effectiveness of the suggested strategy, consider the following example taken from the literature M. Chakraborty et al. [19] Consider the MOLFPP consisting of two objectives as follows:

$$P1: \max \mathcal{Z} = \{ \mathcal{Z}_1(\varkappa) = \frac{-3\,\varkappa_1 + 2\varkappa_2}{\varkappa_1 + \varkappa_2 + 3}, \mathcal{Z}_1(\varkappa) = \frac{7\,\varkappa_1 + \varkappa_2}{5\varkappa_1 + 2\varkappa_2 + 1} \} \tag{4.29}$$

$s.t.$

$$\varkappa_1 - \varkappa_2 \geq 1, \tag{4.30}$$

$$2\varkappa_1 + 3\varkappa_2 \leq 15, \tag{4.31}$$

$$\varkappa_1 \geq 3, \tag{4.32}$$

$$\varkappa_1, \varkappa_2 \geq 0. \tag{4.33}$$

P-1 in eqn. (4.29) reduces to equivalent deterministic MO-LPP using Charnes-Cooper method with linear transformation $y = \tau\varkappa$, as follows

$$P2: \ \max \mathcal{Z} = \mathcal{Z}_1 = y_1 + y_2 + 3\tau; \mathcal{Z}_2 = 7y_1 + y_2 \tag{4.34}$$

$$y_1 - y_2 - \tau \geq 0, \tag{4.35}$$

$$2y_1 + 3y_2 - 15\tau \leq 0, \tag{4.36}$$

$$y_1 - 3\tau \geq 0, \tag{4.37}$$

$$3y_1 - y_2 \leq 0, \tag{4.38}$$

$$5y_1 + 2y_2 + \tau \leq 1, \tag{4.39}$$

$$y_1, y_2, \tau \geq 0. \tag{4.40}$$

Once both objectives have been resolved independently with constraints using LINGO we obtained following results:

Max $\mathcal{Z}_1 = \mathcal{Z}_1^{(\max)} = 0.3801$ at point (0.14876,0.107438,0.041322)

Max $\mathcal{Z}_2 = \mathcal{Z}_2^{(\max)}$ =1.4 at point (0.2,0,0)

Minimal acceptable point $\mathcal{Z}_1 = \mathcal{Z}_1^{(\min)} = 0.2$

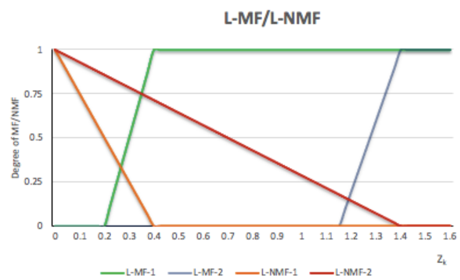
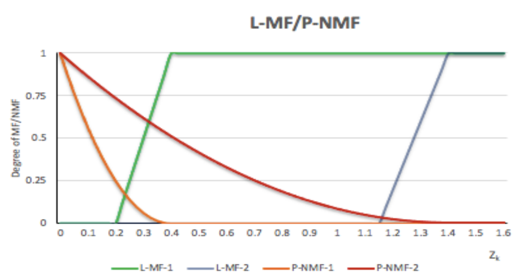
Minimal acceptable point $\mathcal{Z}_1 = \mathcal{Z}_1^{(\min)}$=1.15

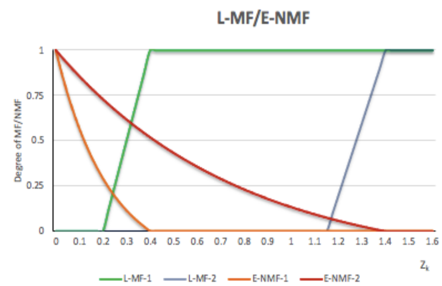
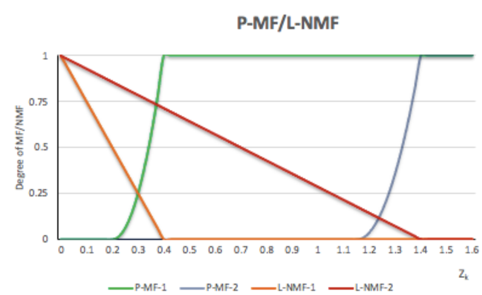
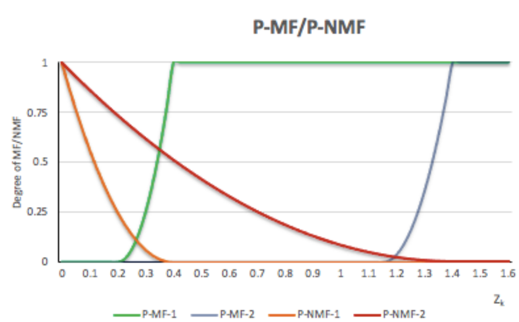
Worst point at $\mathcal{Z}_1 = \mathcal{Z}_1^{(w)}$ =0

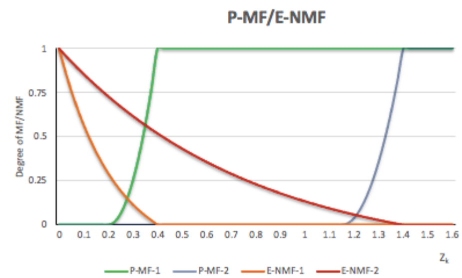
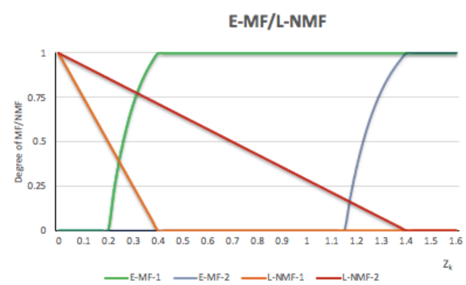
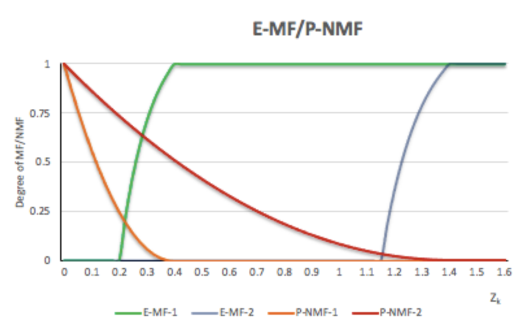
Worst point at $\mathcal{Z}_2 = \mathcal{Z}_2^{(w)}$ =0.

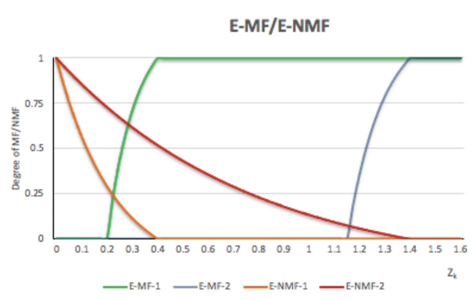
Following the assignment of the MF and NMF to both functions, the MOLPP is transformed into a SO problem in accordance with the suggested situation-based hybrid EIFA, as demonstrated in the following with associated figures.

Table 4.1: Mathematical model and graphs of EIFA 9 situations.

Hybrid (MF/NMF) combinations	Figures of MF/NMF for EIFA of Example
Situation 1:L-MF/L-NMF $$\max(\lambda-\delta)$$ $\frac{\mathcal{Z}_1-0.2}{0.4-0.2}\geq\lambda;\ \frac{\mathcal{Z}_2-1.15}{1.4-1.15}\geq\lambda,$ $\frac{0.4-\mathcal{Z}_1}{0.4-0}\leq\delta;\ \frac{1.4-\mathcal{Z}_2}{1.4-0}\leq\delta,$ $0\leq\delta\leq\lambda\ \leq\lambda+\delta\leq 1;$ Constraints (4.35)-(4.40).	 Figure 4.2: L-MF/L-NMF for EIFA
Situation 2:L-MF/P-NMF $$\max(\lambda-\delta)$$ $\frac{\mathcal{Z}_1-0.2}{0.4-0.2}\geq\lambda;\ \frac{\mathcal{Z}_2-1.15}{1.4-1.15}\geq\lambda,$ $\left(\frac{0.4-\mathcal{Z}_1}{0.4-0}\right)^2\leq\delta\ ;\ \left(\frac{1.4-\mathcal{Z}_2}{1.4-0}\right)^2\leq\delta,$ $0\leq\delta\leq\lambda\ \leq\lambda+\delta\leq 1;$ Constraints (4.35)-(4.40).	 Figure 4.3: L-MF/P-NMF for EIFA
Situation 3:L-MF/E-NMF $$\max(\lambda-\delta)$$ $\frac{\mathcal{Z}_1-0.2}{0.4-0.2}\geq\lambda;\ \frac{\mathcal{Z}_2-1.15}{1.4-1.15}\geq\lambda,$	

$1-\ 1.2\left[1-exp\left(-1.8\frac{(\mathcal{Z}_1-0)}{(0.4-0)}\right)\right]\leq\delta,$ $1-\ 1.2\left[1-exp\left(-1.8\frac{(\mathcal{Z}_2-0)}{1.4-0)}\right)\right]\leq\delta,$ $0\leq\delta\leq\lambda\ \leq\lambda+\delta\leq1;$ Constraints (4.35)-(4.40).	 Figure 4.4: L-MF/E-NMF for EIFA
Situation 4:P-MF/L-NMF $$max(\lambda-\delta)$$ $\left(\frac{\mathcal{Z}_1-0.2}{0.4-0.2}\right)^2\geq\lambda;\ \left(\frac{\mathcal{Z}_2-1.15}{1.4-1.15}\right)^2\geq\lambda,$ $\frac{0.4-\mathcal{Z}_1}{0.4-0}\leq\delta;\ \frac{1.4-\mathcal{Z}_2}{1.4-0}\leq\delta,$ $0\leq\delta\leq\lambda\ \leq\lambda+\delta\leq1;$ Constraints (4.35)-(4.40).	 Figure 4.5: P-MF/L-NMF for EIFA
Situation 5:P-MF/P-NMF $$max(\lambda-\delta)$$ $\left(\frac{\mathcal{Z}_1-0.2}{0.4-0.2}\right)^2\geq\lambda;\ \left(\frac{\mathcal{Z}_2-1.15}{1.4-1.15}\right)^2\geq\lambda,$ $\left(\frac{0.4-\mathcal{Z}_1}{0.4-0}\right)^2\leq\delta\ \ ;\ \left(\frac{1.4-\mathcal{Z}_2}{1.4-0}\right)^2\leq\delta,$ $0\leq\delta\leq\lambda\ \leq\lambda+\delta\leq1;$ Constraints (4.35)-(4.40).	 Figure 4.6: P-MF/P-NMF for EIFA
Situation 6:P-MF/E-NMF	

$max(\lambda-\delta)$ $\left(\frac{\mathcal{Z}_1-0.2}{0.4-0.2}\right)^2 \ge \lambda;\ \left(\frac{\mathcal{Z}_2-1.15}{1.4-1.15}\right)^2 \ge \lambda,$ $1-\ 1.2\left[1-exp\left(-1.8\frac{(\mathcal{Z}_1-0)}{(1.4-0)}\right)\right] \le \delta,$ $1-\ 1.2\left[1-exp\left(-1.8\frac{(\mathcal{Z}_2-0)}{(1.4-0)}\right)\right] \le \delta,$ $0 \le \delta \le \lambda\ \le \lambda+\delta \le 1;$ Constraints (4.35)-(4.40).	 Figure 4.7: P-MF/E-NMF for EIFA
Situation 7:E-MF/L-NMF $max(\lambda-\delta)$ $1.2\left[1-exp\left(-1.8\frac{\mathcal{Z}_1-0.2}{0.4-0.2}\right)\right] \ge \lambda,$ $1.2\left[1-exp\left(-1.8\frac{\mathcal{Z}_2-1.15}{1.4-1.15}\right)\right] \ge \lambda,$ $\frac{0.4-\mathcal{Z}_1}{0.4-0} \le \delta;\ \frac{1.4-\mathcal{Z}_2}{1.4-0} \le \delta,$ $0 \le \delta \le \lambda\ \le \lambda+\delta \le 1;$ Constraints (4.35)-(4.40).	 Figure 4.8: E-MF/L-NMF for EIFA
Situation 8:E-MF/P-NMF $max(\lambda-\delta)$ $1.2\left[1-exp\left(-1.8\frac{\mathcal{Z}_1-0.2}{0.4-0.2}\right)\right] \ge \lambda,$ $1.2\left[1-exp\left(-1.8\frac{\mathcal{Z}_2-1.15}{1.4-1.15}\right)\right] \ge \lambda,$ $\left(\frac{0.4-\mathcal{Z}_1}{0.4-0}\right)^2 \le \delta\ ;\ \left(\frac{1.4-\mathcal{Z}_2}{1.4-0}\right)^2 \le \delta,$	 Figure 4.9: E-MF/P-NMF for EIFA

$0 \leq \delta \leq \lambda \ \leq \lambda + \delta \leq 1;$ Constraints (4.35)-(4.40).	
Situation 9:E-MF/E-NMF $\max(\lambda - \delta)$ $1.2\left[1 - exp\left(-1.8\frac{\mathcal{Z}_1 - 0.2}{0.4 - 0.2}\right)\right] \geq \lambda,$ $1.2\left[1 - exp\left(-1.8\frac{\mathcal{Z}_2 - 1.15}{1.4 - 1.15}\right)\right] \geq \lambda,$ $1 - \ 1.2\left[1 - exp\left(-1.79\frac{(\mathcal{Z}_1 - 0)}{(0.4 - 0)}\right)\right] \leq \delta,$ $1 - \ 1.2\left[1 - exp\left(-1.8\frac{(\mathcal{Z}_2 - 0)}{(1.4 - 0)}\right)\right] \leq \delta,$ $0 \leq \delta \leq \lambda \ \leq \lambda + \delta \leq 1;$ Constraints (4.35)-(4.40).	 Figure 4.10: E-MF/E-NMF for EIFA

4.5 Result Analysis and Comparative Study :

We used LINGO to solve the 9 models and obtained results of all the parameters that includes level of satisfaction/dissatisfaction along with difference. The comparative analysis of these three cases presented in the tabular form discussed below.

i) level of satisfaction: least of both membership degrees

$$\lambda = \min\{\ \dot{\mu}(\mathcal{Z}_1),\ \dot{\mu}(\mathcal{Z}_2)\} \tag{4.41}$$

ii) Level of dissatisfaction: the highest of two non-membership degrees

$$\delta = \max\{\ \dot{\nu}(\mathcal{Z}_1),\ \dot{\nu}(\mathcal{Z}_2)\} \tag{4.42}$$

iii) Their difference:

$$\lambda - \delta \tag{4.43}$$

Table 4.2 Degree of satisfaction(λ) of 9-situations

λ	Linear (MF)	Parabolic (MF)	Exponential (MF)
L-(MF)	0.5524	0.55244	0.5524
P-(MF)	0.3052	0.3052	0.3052
E-(MF)	0.8677	0.9825	0.9993

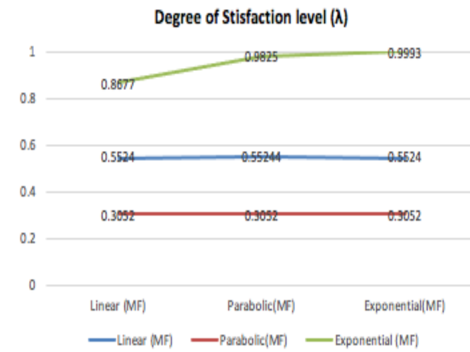


Figure 4.11: Level of satisfaction (λ)

Table 4.3: Degree of dissatisfaction(δ) of 9-situations

δ	Linear (NMF)	Parabolic (NMF)	Exponential (NMF)
L-(NMF)	0.2237	0.5007	0.0010
P-(NMF)	0.2238	0.0501	0.0010
E-(NMF)	0.1322	0.0174	0.0010

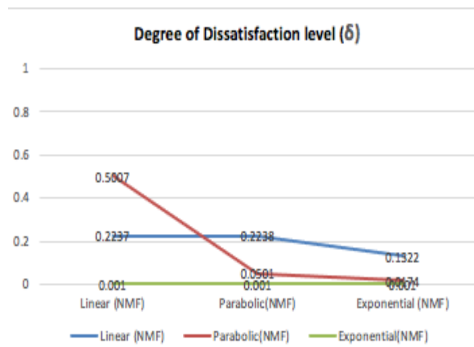


Figure 4.12: Level of dissatisfaction (δ)

Table 4.4: Difference between satisfaction and dissatisfaction level ($\lambda - \delta$)

$\lambda - \delta$	Linear (NMF)	Parabolic (NMF)	Exponential (NMF)
L-(MF)	0.3286	0.5024	0.5514
P-(MF)	0.0814	0.2551	0.3042
E-(MF)	0.7355	0.9650	0.9980

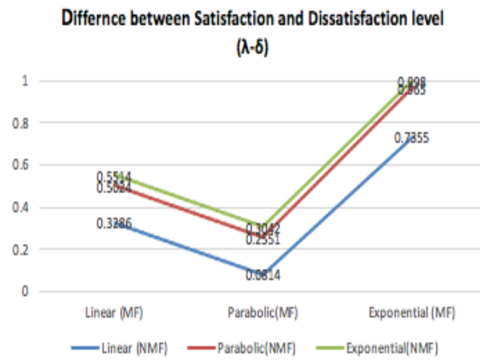


Figure 4.13: Difference in level ($\lambda - \delta$)

The comparative results for satisfaction (λ), discontent (δ), and net difference ($\lambda - \delta$) are presented in Table 4.2-4.4 with graphs presented in Fig 4.11-4.13. The results represented nine different combinations of MF and NMF. It is clear that the most satisfying outcome (0.9993) and least dissatisfying (0.0010), leading to the greatest net difference (0.9980), is consistently produced by combining E-MF/E-NMF. This implies that the E-MF/E-NMF arrangement offers the highest level of solution certainty with the least amount of hesitancy. Combinations of P-MF/L-NMF produce reduced satisfaction (0.3052) and comparatively larger discontent (0.2238), resulting in the smallest net difference among hybrid situations (0.0814). Compared to linear or parabolic versions, exponential functions for MF and NMF greatly improve decision-maker satisfaction and minimize discontent.

Employing Chakraborty's [19]conventional fuzzy methodology, the attained maximum level of satisfaction (λ) is 0.8. This signifies the decision-maker's assurance inside a singular fuzzy model.

- Comparison with EIFA's λ:

"Within the EIFA arrangement, the highest satisfaction levels (s) vary from 0.3052 to 0.9993 for the nine MF/NMF combos. Significantly, the E–MF/E–NMF combination attains a s of 0.9993, markedly surpassing the benchmark λ. Even the minimal satisfaction (P–MF) combination yields a satisfaction level comparable to the benchmark, suggesting that this hybrid intuitionistic structure may maintain or enhance satisfaction levels".

- Comparison with EIFA's δ:

"In contrast to the conventional method, EIFA additionally quantifies dissatisfaction (δ), offering a more precise assessment of uncertainty." Exponential non-membership forms provide the minimal discontent (as low as 0.0010), but some linear or parabolic non-membership forms exhibit elevated dissatisfaction levels. This knowledge is absent in the conventional model yet is essential for sound decision-making.

- Comparison with EIFA's (λ- δ):

The intuitionistic differences (λ- δ) in EIFA attains a maximum of 0.9980 for the E–MF/E–NMF combo, exceeding the benchmark's λ value. This indicates that EIFA can improve net decision assurance beyond the capabilities of a singular min-max strategy. Linear-linear combinations closely align with the benchmark, demonstrating consistency with conventional outcomes while emphasizing the advantages of the hybridization structure.

The comparison research demonstrates that the EIFA not just corresponds with Chakraborty's model but also enhances its understanding by offering a multi-faceted perspective on pleasure and reluctance. This confirms EIFA's capability to provide more resilient and accessible solutions to uncertain multi-objective challenges.

4.6 Case Study: Agriculture Land Allocation

For agricultural planting structure optimization, considering the secondary data from Yang et al. [66]. Every inland river basin has water as a key constraint to agricultural and economic growth. To address irrigation water supply unpredictability, water-saving agriculture was implemented. Reduced water use by crops is an inevitable byproduct of water-conscious farming practices in the area. A "water-saving crops planning" strategy is an agricultural planting pattern that aims to maximize the environmental, social, and economic advantages of water-constrained crop production. Although there are many well-defined and perhaps accurate components to optimize in agricultural planting structures, there are still many unknowns included but not limited to time, money, capacity, and demands. The goals are very important for yield efficiency measures because they show how earnings are related to the resources that are used, like water, to make crops economically valuable. Models often include linear fractional objectives that consider the highest efficiency per unit.

In a case study on irrigation the suggested method is used to improve a planting arrangement for agricultural purposes. Three main crops are grown in this area: cotton, summer corn, and winter grain . Winter grain and summer corn are grown using a double-cropping strategy, so their agricultural land area is shared. As a result, summer corn is believed to have the same farmed area as winter grain.

4.6.1 Numerical Data Available

As per the given data in Yang [66] with a total cost of almost 770000000 (INR), the grain crops have to be planted on more than 65% of the 59056 hectares that are fit for farming. Rest of the required information is given in the table4.5 below:

Table 4.5: Data for different crops

Crops	Average price/kg(INR)	Production / Hectare(Kg)	Irrigation/ Hectare (m^3)	Expenditure cost (INR)
Winter grain	2.4	6600	2850	7500
Summer corn	2.2	8100	600	5250
cotton	14.7	1125	525	22500

4.6.2 Formulation of MOLFPP

(a)Decision variables

There are two decision variables required $\varkappa_1$ and $\varkappa_2$. Let $\varkappa_1$ be the land in hectare allocated for double cropping for winter grain and summer corn and $\varkappa_2$ area in hectare for cotton.

(b) Objective functions (maximization)

There are two objectives to the problem. First and foremost, the objective is to increase the economic value of each cubic meter of irrigation water. It can help improve the efficiency of water resources. The problem's objective can be articulated as follows.

$$\mathcal{Z}_1(\varkappa) = \frac{2.4(6600)\varkappa_1 + 2.2(8100)\varkappa_1 + 14.7(1125)\varkappa_2}{2850\varkappa_1 + 600\varkappa_1 + 525\varkappa_2}$$

$$\mathcal{Z}_1(\varkappa) = \frac{33600\varkappa_1 + 16537.5\varkappa_2}{3450\varkappa_1 + 525\varkappa_2} \tag{4.44}$$

The second objective is to increase the production of food per acre in order to meet the demands of the population.

$$\mathcal{Z}_2(\varkappa) = \frac{33600\varkappa_1 + 16537.5\varkappa_2}{\varkappa_1 + \varkappa_2} \tag{4.45}$$

(c) Constraints of problem

(i) Land availability

$$\varkappa_1 + \varkappa_2 \leq 59056$$

(ii) Food security

$$\frac{\varkappa_1}{\varkappa_1 + \varkappa_2} \geq 65\%$$
$$0.35\varkappa_1 + 0.65\varkappa_2 \geq 0. \tag{4.46}$$

(iii) Investment cost

$$7500\varkappa_1 + 5250\varkappa_1 + 22500\varkappa_2 \leq 770000000 \tag{4.47}$$

(iv) Non-negative

$$\varkappa_1, \varkappa_2 \geq 0. \tag{4.48}$$

There are two main objectives to seek: One is to optimize financial return per cubic meters of water used for irrigation to encourage better water resource use. Second is to maximize food output per unit of land area to meet population's rising needs. Built with these objectives in mind, the optimization model looks for the most effective crop allocation approach under the particular circumstances.

Therefore the MOLFPP can be formulated as:

$$\max \mathcal{Z} = \{\mathcal{Z}_1(\varkappa), \mathcal{Z}_2(\varkappa)\} \tag{4.49}$$

$$\mathcal{Z}_1(\varkappa) = \frac{33600\varkappa_1 + 16537.5\varkappa_2}{3450\varkappa_1 + 525\varkappa_2} \tag{4.50}$$

$$\mathcal{Z}_2(\varkappa) = \frac{33600\varkappa_1 + 16537.5\varkappa_2}{\varkappa_1 + \varkappa_2} \tag{4.51}$$

s.t.

$$\varkappa_1 + \varkappa_2 \leq 59056, \tag{4.52}$$

$$0.35\varkappa_1 + 0.65\varkappa_2 \geq 0, \tag{4.53}$$

$$7500\varkappa_1 + 5250\varkappa_1 + 22500\varkappa_2 \leq 770000000, \tag{4.54}$$

$$\varkappa_1, \varkappa_2 \geq 0. \tag{4.55}$$

Solving the above model using Charnes and Cooper method to get the following solutions

Ideal solution for $\mathcal{Z}_1 = \mathcal{Z}_1^{(\max)} = 11.4$ at (31151.51,18031)

Ideal solution for $\mathcal{Z}_2 = \mathcal{Z}_2^{(\max)} = 33660$ at (59056,0)

Minimal acceptable point $\mathcal{Z}_1 = \mathcal{Z}_1^{(\min)} = 9.76$ at (59056,0)

Minimal acceptable point $\mathcal{Z}_2 = \mathcal{Z}_2^{(\min)} = 27667.12$ at

Worst point at $\mathcal{Z}_1 = \mathcal{Z}_1{}^{(w)}$ =0 at (0,0)

Worst point at $\mathcal{Z}_2 = \mathcal{Z}_2{}^{(w)}$ =0 at (0,0).

Following the assignment of the MF and NMF to both functions, the MOLPP is transformed into a SO problem in accordance with the suggested situation-based hybrid EIFA as follows:

Situation 1: L-MF/L-NMF

$$\left.\begin{array}{c} \max(\lambda-\delta) \\ \dfrac{\mathcal{Z}_1-9.76}{11.4-9.76} \geq \lambda;\ \dfrac{\mathcal{Z}_2-27667.12}{33660-27667.12} \geq \lambda, \\ \dfrac{11.4-\ \mathcal{Z}_1}{11.4-0} \leq \delta; \dfrac{33660-\ \mathcal{Z}_2}{33660-0} \leq \delta, \\ 0 \leq \delta \leq \lambda\ \leq \lambda+\delta \leq 1, \\ constraints\ (4.52)-(4.55). \end{array}\right\} \tag{4.56}$$

Situation 2: L-MF/P-NMF

$$\left.\begin{array}{c} \max(\lambda-\delta) \\ \dfrac{\mathcal{Z}_1-9.76}{11.4-9.76} \geq \lambda;\ \dfrac{\mathcal{Z}_2-27667.12}{33660-27667.12} \geq \lambda, \\ \left(\dfrac{11.4-\ \mathcal{Z}_1}{11.4-0}\right)^2 \leq \delta; \left(\dfrac{33660-\ \mathcal{Z}_2}{33660-0}\right)^2 \leq \delta, \\ 0 \leq \delta \leq \lambda\ \leq \lambda+\delta \leq 1, \\ constraints\ (4.52)-(4.55). \end{array}\right\} \tag{4.57}$$

Situation 3: L-MF/E-NMF

$$\left.\begin{array}{c}\mathfrak{max}(\lambda-\delta)\\ \dfrac{\mathcal{Z}_1-9.76}{11.4-9.76}\geq\lambda;\;\; \dfrac{\mathcal{Z}_2-27667.12}{33660-27667.12}\geq\lambda,\\ 1-\;1.2\left[1-exp\left(-1.8\dfrac{(\mathcal{Z}_1-0)}{(11.4-0)}\right)\right]\leq\delta,\\ 1-\;1.2\left[1-exp\left(-1.8\dfrac{(\mathcal{Z}_2-0)}{33660-0)}\right)\right]\leq\delta,\\ 0\leq\delta\leq\lambda\;\leq\lambda+\delta\leq 1,\\ constraints\;(4.52)-(4.55).\end{array}\right\}\tag{4.58}$$

Situation 4: P-MF/L-NMF

$$\left.\begin{array}{c}\mathfrak{max}(\lambda-\delta)\\ \left(\dfrac{\mathcal{Z}_1-9.76}{11.4-9.76}\right)^2\geq\lambda;\;\; \left(\dfrac{\mathcal{Z}_2-27667.12}{33660-27667.12}\right)^2\geq\lambda,\\ \dfrac{11.4-\;\mathcal{Z}_1}{11.4-0}\leq\delta;\dfrac{33660-\;\mathcal{Z}_2}{33660-0}\leq\delta,\\ 0\leq\delta\leq\lambda\;\leq\lambda+\delta\leq 1,\\ constraints\;(4.52)-(4.55).\end{array}\right\}\tag{4.59}$$

Situation 5: P-MF/P-NMF

$$\left.\begin{array}{c}\mathfrak{max}(\lambda-\delta)\\ \left(\dfrac{\mathcal{Z}_1-9.76}{11.4-9.76}\right)^2\geq\lambda;\;\; \left(\dfrac{\mathcal{Z}_2-27667.12}{33660-27667.12}\right)^2\geq\lambda,\\ \left(\dfrac{11.4-\;\mathcal{Z}_1}{11.4-0}\right)^2\leq\delta;\left(\dfrac{33660-\;\mathcal{Z}_2}{33660-0}\right)^2\leq\delta,\\ 0\leq\delta\leq\lambda\;\leq\lambda+\delta\leq 1,\\ constraints\;(4.52)-(4.55).\end{array}\right\}\tag{4.60}$$

Situation 6: P-MF/E-NMF

$$
\left.\begin{array}{c}
\max(\lambda-\delta) \\
\left(\frac{\mathcal{Z}_1-9.76}{11.4-9.76}\right)^2 \geq \lambda;\ \left(\frac{\mathcal{Z}_2-27667.12}{33660-27667.12}\right)^2 \geq \lambda, \\
1-\ 1.2\left[1-exp\left(-1.8\frac{(\mathcal{Z}_1-0)}{(33660-0)}\right)\right] \leq \delta, \\
1-\ 1.2\left[1-exp\left(-1.8\frac{(\mathcal{Z}_2-0)}{(33660-0)}\right)\right] \leq \delta, \\
0 \leq \delta \leq \lambda\ \leq \lambda+\delta \leq 1, \\
constraints\ (4.52)-(4.55).
\end{array}\right\} \quad (4.61)
$$

Situation 7: E-MF/L-NMF

$$
\left.\begin{array}{c}
\max(\lambda-\delta) \\
1.2\left[1-exp\left(-1.8\frac{\mathcal{Z}_1-9.76}{11.4-9.76}\right)\right] \geq \lambda, \\
1.2\left[1-exp\left(-1.8\frac{\mathcal{Z}_2-27667.12}{33660-27667.12}\right)\right] \geq \lambda, \\
\frac{11.4-\ \mathcal{Z}_1}{11.4-0} \leq \delta; \frac{33660-\ \mathcal{Z}_2}{33660-0} \leq \delta, \\
0 \leq \delta \leq \lambda\ \leq \lambda+\delta \leq 1, \\
constraints\ (4.52)-(4.55).
\end{array}\right\} \quad (4.62)
$$

Situation 8: E-MF/P-NMF

$$
\left.\begin{array}{c}
\max(\lambda-\delta) \\
1.2\left[1-exp\left(-1.8\frac{\mathcal{Z}_1-9.76}{11.4-9.76}\right)\right] \geq \lambda, \\
1.2\left[1-exp\left(-1.8\frac{\mathcal{Z}_2-27667.12}{33660-27667.12}\right)\right] \geq \lambda, \\
\left(\frac{11.4-\ \mathcal{Z}_1}{11.4-0}\right)^2 \leq \delta; \left(\frac{33660-\ \mathcal{Z}_2}{33660-0}\right)^2 \leq \delta, \\
0 \leq \delta \leq \lambda\ \leq \lambda+\delta \leq 1, \\
constraints\ (4.52)-(4.55).
\end{array}\right\} \quad (4.63)
$$

Situation 9: E-MF/E-NMF

$$\left.\begin{array}{c}\mathfrak{max}(\lambda-\delta)\\ 1.2\left[1-exp\left(-1.8\dfrac{\mathcal{Z}_1-9.76}{11.4-9.76}\right)\right]\geq\lambda,\\ 1.2\left[1-exp\left(-1.8\dfrac{\mathcal{Z}_2-27667.12}{33660-27667.12}\right)\right]\geq\lambda,\\ 1-\ 1.2\left[1-exp\left(-1.8\dfrac{(\mathcal{Z}_1-0)}{(11.4-0)}\right)\right]\leq\delta,\\ 1-\ 1.2\left[1-exp\left(-1.8\dfrac{(\mathcal{Z}_2-0)}{(33660-0)}\right)\right]\leq\delta,\\ 0\leq\delta\leq\lambda\ \leq\lambda+\delta\leq1,\\ constraints\ (4.52)-(4.55).\end{array}\right\}\quad(4.64)$$

Using Lingo software we solved the above 9 situation based problems under proposed method of EIFA to get the results of level of satisfaction/dissatisfaction and their difference. The obtained result are expressed in following table

Table 4.6: Results obtained from EIFA for all 9 situations

Situations	λ	δ	$\lambda-\delta$	Value of $\mathcal{Z}_1$	Value of $\mathcal{Z}_2$
L-MF/L-NMF	0.45598	0.09686	0.35912	10.50758	30400.65
L-MF/P-NMF	0.45598	0..00938	0.4466	10.50781	30399.81
L-MF/E-NMF	0.45598	0.00000	0.45598	10.50781	30399.81
P-MF/L-NMF	0.50111	0.49889	0.00220	0.00000	0.00000
P-MF/P-NMF	0.20792	0.00938	0.19854	10.50782	30399.78
P-MF/E-NMF	1.00000	0.00000	1.00000	0.00000	0.00000
E-MF/L-NMF	0.91351	0.08649	0.82700	10.41392	30748.50

E-MF/P-NMF	0.99252	0.00748	0.985036	10.41392	30748.48
E-MF/E-NMF	1.00000	0.00000	1.00000	11.40325	27667.13

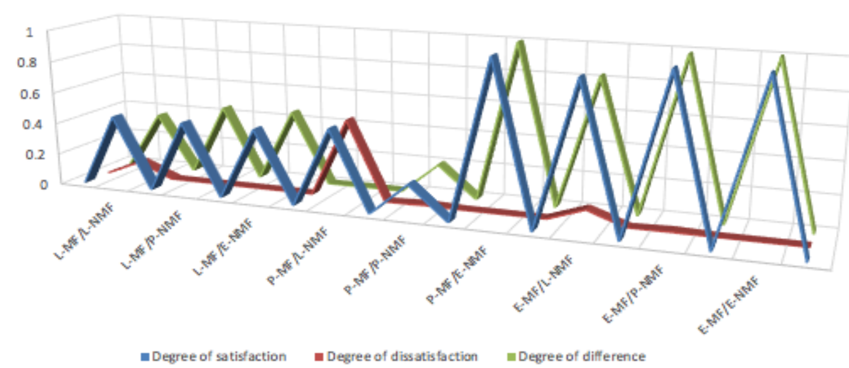


Figure 4.14: Graphical representation level of satisfaction/dissatisfaction/their difference

4.7 Comparative Study and Result Analysis

The comparative analysis of these three cases of 9-situations is presented in the tabular form and graphically discussed below.

i) Level of satisfaction:

$$\lambda = \min\{\ \dot{\mu}(\mathcal{Z}_1),\ \dot{\mu}(\mathcal{Z}_2)\} \tag{4.65}$$

Table 4.7: Satisfaction values for 9 different situations

λ	Linear (NMF)	Parabolic (NMF)	Exponential (NMF)
Linear (MF)	0.45598	0.45598	0.45598
Parabolic(MF)	0.50111	0.20792	1

Exponential (MF)	0.91351	0.99252	1

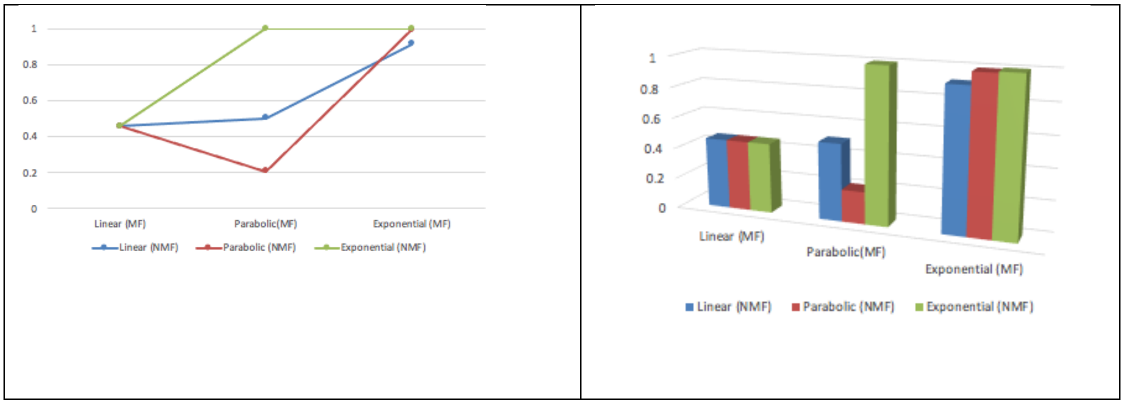


Figure 4.15: Degree of satisfaction level for 9 situations

ii) Level of dissatisfaction :

$$\delta = \max\{\ \dot{\nu}\,(\mathcal{Z}_1),\ \dot{\nu}(\mathcal{Z}_2)\} \tag{4.66}$$

Table 4.8: Dissatisfaction values for 9 different situations

δ	Linear (NMF)	Parabolic (NMF)	Exponential (NMF)
Linear (MF)	0.09686	0..00938	0.0
Parabolic(MF)	0.49889	0.00938	0.0
Exponential (MF)	0.08649	0.00748	0.0

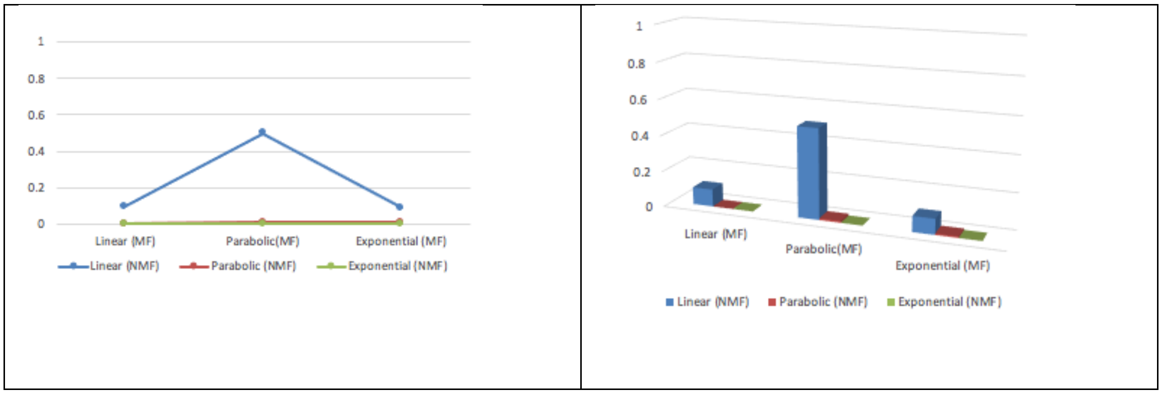


Figure 4.16: Degree of dissatisfaction for 9 situations

iii) Difference between satisfaction and dissatisfaction level:

$$\lambda - \delta \tag{4.67}$$

Table 4.9: Difference between λ and δ values for 9 different scenario

$\lambda - \delta$	Linear (NMF)	Parabolic (NMF)	Exponential (NMF)
Linear (MF)	0.35912	0.4466	0.45598
Parabolic(MF)	0.00220	0.19854	1
Exponential(MF)	0.82700	0.985036	1

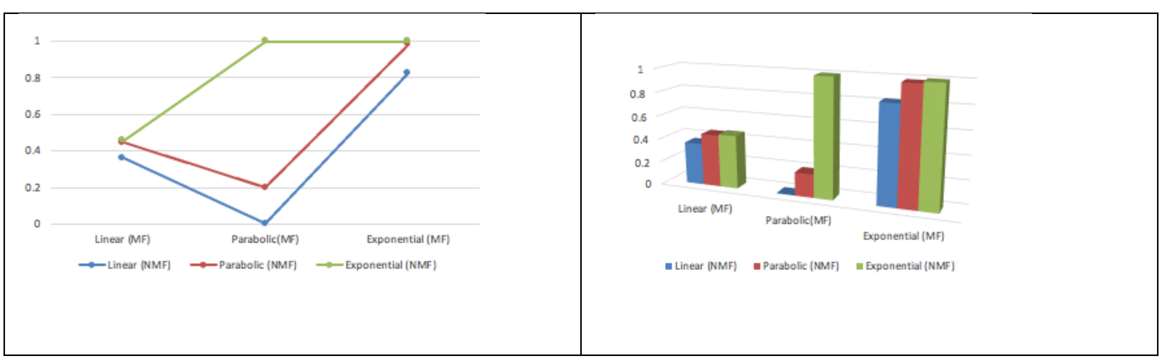


Figure 4.17: Degree of difference between satisfaction and dissatisfaction values for 9 different situations

Within the context of agricultural planning, the findings from the tables (4.5-4.8) and figures (4.14-4.17) demonstrate how various combinations of MF and NMF functions have an effect on the level of satisfaction as well as the objective outcomes. We looked at nine different permutations including exponential, parabolic, and linear forms. The combination of E-MF and E-MF produces the highest degree of satisfaction (λ = 1.0) and zero discontent, which results in the greatest difference between λ and δ, which is 1.0. In comparison to other high-satisfaction combinations, this likewise yields the greatest value of $\mathcal{Z}_1$ (11.40325), although $\mathcal{Z}_2$somewhat drops to 27,667.13. Also doing well are the E-MF/L-NMF and E-MF/P-NMF combos, with $\lambda - \delta$ values of 0.82700 and 0.98504 correspondingly. They provide a satisfactory equilibrium, preserving high values of the objective function ($\mathcal{Z}_2$ is around 30,748.5) while also guaranteeing minimal degrees of discontent. The mixtures of L-MF and E-NMF and L-MF and P-NMF show average levels of satisfaction, with $\lambda - \delta$ differences of 0.45598 and 0.44660, respectively. However, they reach stable objective values, which means they have a strong solution for L-MF with nonlinear NMF. The P-MF/E-NMF combination is noteworthy since it satisfies all requirements ($\lambda - \delta$=1.0), yet it fails to accomplish either goal, suggesting that it is not viable to work within the limits of the issue. A general conclusion may be drawn from the findings that exponential forms, particularly when applied to NMF, contribute to an increase in the levels of satisfaction under intuitionistic scenario. But the trade-offs between getting the most joy and keeping realistic, achievable goals must be carefully thought through. When it comes to agricultural planning, a combination like E-MF/P-NMF and E-MF/L-NMF show the optimum equilibrium among high satisfaction and good objective values.

The real-life case study of agricultural planning shows how the EIFA builds on current FFP approaches. For example, Yang employed a SIMM to find the best cropping structure by solving an FMOLFPP. The SIMM method dealt with uncertainty by setting fuzzy objectives and fuzzy restrictions got target values for function $\mathcal{Z}_1$=11.4 and $\mathcal{Z}_2$=27,667.1.

Yang's strategy is based on maximizing membership function only. The EIFA model suggested in this research, on the other hand, uses a hybrid MF/NMF arrangement to the crisp a fractional model to measure both satisfaction and discontent and find the difference ($\lambda - \delta$). This gives decision-makers more confidence in their choices by helping them think about not just how well they are meeting their goals, but also how much danger or hesitancy they could feel when planning for unknown situations. The EIFA has similar goal values and higher net satisfaction, which shows that it might be a useful and more reliable option for planning agriculture when there is uncertainty.

Table 4.10: Summary of MM/NMF combinations and outcomes

MF-NMF Combination	$\lambda - \delta$	Behaviour of Objectives	Interpretation
E-MF/E-NMF	1	Highest $\mathcal{Z}_1 = 11.40325, \mathcal{Z}_2 = 27{,}667.13$	Best overall performance, zero discontent.
E-MF/L-NMF	0.82700	$\mathcal{Z}_2 \approx 30{,}784.5$	High satisfaction with good objective balance
E-MF/P-NMF	0.985036	$\mathcal{Z}_2 \approx 30{,}784.5$	Very high satisfaction and strong stability
L-MF/E-NMF	0.45598	Stable objective values	Moderate satisfaction, reliable solution
L-MF/P-NMF	0.4466	Stable objective values	Moderate satisfaction, reliable solution
P-MF/E-NMF	1	Does not achieve either $\mathcal{Z}_1$ or $\mathcal{Z}_2$	Meets satisfaction criteria but impractical, violates problem limits.

4.8 Conclusion

This chapter presents an EIFA for resolving MOLFPP under uncertainty. The most important thing that EIFA does is combine linear and nonlinear MF and NMF in a structured way, using IF frameworks. These can be linear, parabolic, or exponential forms. The hybrid structure that has been presented allows for greater flexibility in representing real-world imprecision, in contrast to conventional models, which often rely on a single kind of fuzzification. The EIFA was initially validated using a classical Zimmerman's min-max fuzzy fractional programming problem, as mentioned by Chakraborty, illustrating that the proposed model aligns with established results while offering further insights via intuitionistic evaluations of satisfaction, dissatisfaction, and their differences. The model was then employed to address a practical agricultural planning issue utilizing secondary data initially presented by Yang with SIMM approach. The findings from nine MF/NMF combinations indicate that hybrid models, particularly those employing exponential NMF, generally enhance net decision-makers satisfaction while reconciling opposing objectives for productivity, irrigation, and land utilization. The EIFA addresses imprecision and contradictory criteria in MOLPP in a unique, flexible, and resilient manner. This study utilized mainly linear, a parabolic and exponential functions in order to create MF/NMF pairings, however, the EIFA may easily be adapted to other functional forms, including hyperbolic form, sigmoidal, or the Gaussian shapes. This creates opportunities for customizing the technique for more intricate decision-making contexts, hence increasing its applicability in many practical scenarios in addition to agriculture, such as the design of supply chains, the distribution of resources, or the planning of sustainable development.

Chapter 5

Multi-objective Fractional Programming Problem Analysis in the Manufacturing Sector Using Situation-Based S-Shaped Nonlinear Fuzzy Numbers

5.1 Introduction and Motivation

In this chapter we discussed a method for solving linear fractional programming problems by incorporating a non-linear S-shaped fuzzy membership function into a fuzzy goal programming approach. The comparison of the proposed methodology based on S-shaped membership function is also done with existing technique which is based on linear membership function. The main motivation behind the enhancement of the membership function for fuzzy goal programming from linear to non-linear is that if the result obtained by using the linear membership function is not satisfactory for decision-makers, there should be a scope to switch to a non-linear version of it.

To address various optimization problems with multiple objectives, an extensive number of strategies have been proposed in the literature Gulia et al.[85]. Mohamad [15] introduced the relationship between fuzzy programming and goal programming and their similarities and also discussed how one can leads to the other .The extensions and modification is being discussed by researchers discussed in Maiti [71] and Das et al.[61] proposed fuzzy logic approaches to solve linear fractional programing problem with numerical point of view. Multi-fractional programming refers to the resolution of multi-objective problems that involve ratios Shan et al.[74]. Chang [22] applied absolute function on fractional programming with goal programming approach and Mishra et al. [41] proposed multi objective goal programming in optimization of land use in agriculture.

A fundamental principle of goal programming (GP) is to identify a solution that is both feasible and satisfies the given constraints. GP approaches quantify the degree of variation between a solution and each objective. The goal is to maximize the ratio of profit to expenditure in the manufacture of different items with restrictions or constraints. Bal and Pal [13] discussed on dynamic programming. In a fuzzy decision-making context, achieving the necessary levels of objective goals is primarily determined by attaining the highest possible degree of their related membership values and also the result of goal programming is being use by Deb [50]. Membership functions and numerous fuzzy techniques have also been implemented. Companies and organizations from a variety of industries, such as technology and finance, need to make sure that logistics-related operations have a positive ratio Shakirullah et al.[70]. To create the desired goods in the industrial sector, a manufacturing system integrates a number of resources, some of which may have competing needs. This research demonstrates how to resolve a multi-objective optimization problem using the concept of fractional fuzzy programming. Industrial production systems use diverse resources with conflicting needs that must be integrated to produce products. A nonlinear functions used in multi-objective linear function in Shivani et al.[95] that has also grabbed the interest of researchers from throughout the world. So far most of research work used Zimmermann's approach [5] of linear membership function in fuzzy goal programming.

In this paper we enhanced Zimmermann's approach by applying non- linear S-shaped membership function instead of linear function in goal programming. The multi objective linear fractional programing is discussed in fuzzy environment using non-linear S-shaped Zhao et al. [74] membership function and the solutions and results compared with the usual Zimmerman approach on linear membership function .The proposed methodology is applied to a case study of a knit garment manufacturing unit, utilizing secondary data Shakirullah et al. [70] and LINGO software is used to solve fractional programming problems directly without reducing them into Linear form. In fractional programming, to address uncertainty, the majority of studies focus on linear fuzzy numbers, such as triangular and trapezoidal fuzzy numbers. In numerous intricate real world scenarios, non-linear fuzzy numbers may yield more advantageous solutions compared to linear fuzzy numbers in Sadeghieh [63] . This study addresses the issue by employing a non-linear fuzzy number specifically, an S-shaped fuzzy number Kumar et al. [82], Gulia et al. [99],[98]. We modified our methodology to encompass the S-shaped fuzzy number, and our proposed approach, which employs S-shaped fuzzy numbers, produces better outcomes than linear fuzzy numbers.

The layout of chapter is as follows: section 5.2 dealt with methodology where we discussed both the Zimmerman as well as non-linear function based proposed methodology [6]. In section 5.3 one numerical example is discussed with both methods and in section 5.4 these methodologies applied on a case study based on garment industry. In section 5.5 result analysis is done in tabular and graphical form followed by conclusion in section 5.6.

5.2 Methodology

The basic problem of linear fractional programming with assumption is presented as follows.

$$max\ Z_k(x) = \frac{p_k \varkappa + d_k}{q_k \varkappa + e_k}\quad k = 1, 2, \ldots, \mathcal{K} \tag{5.1}$$

$$\text{s.t. } \varkappa \in \mathcal{S} = \{\varkappa \in \mathbb{R}^n : \mathcal{A}\varkappa \leq \mathcal{B}, \varkappa \geq 0\}.$$

$with\ \mathcal{A} \in \mathbb{R}^{m \times n}$, $\mathcal{B} \in \mathbb{R}^m$, $p_k, q_k \in \mathbb{R}^n$ and $d_k, e_k \in \mathbb{R}$.

Let us assume the u_k assigned as the aspiration level to the objective function $Z_k(\varkappa)$ and let ℓ_k be the lower bound limit to the fuzzy goal then (5.1) can be stated as follow:

$$Z_k(x) = \frac{p_k \varkappa + d_k}{q_k \varkappa + e_k} \geq u_k \tag{5.2}$$

Now we find the solution for (5.2) 1st by the existing approach and then we will apply the proposed nonlinear membership and will compare the results obtained from these two methodologies.

I. By Zimmermann's Approach, the fuzzy objective (5.2) describes the linear membership function as follows:

$$\mu(Z_k(\varkappa)) = \begin{cases} 1, & if\ Z_k(\varkappa) \geq u_k, \\ \dfrac{Z_k(\varkappa) - l_k}{g_k - l_k}, & if\ \ell_k \leq Z_k(k) \leq u_k, \\ 0, & if\ Z_k(\varkappa) \leq \ell_k. \end{cases} \tag{5.3}$$

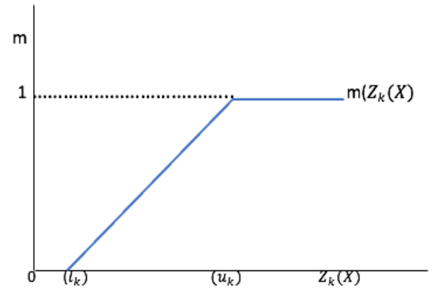


Figure 5.1: Linear membership function

Figure 5.1 describes the Zimmerman's linear membership function. Since the largest value is 1 for the membership function mentioned in (5.3) can be expressed as following model by converting into single objective function problem

$$\text{Min}\psi$$

s.t.

$$\frac{Z_k(X) - (l_k)}{u_k - l_k} + D_k^- - D_k^+ = 1 \tag{5.4}$$

$$\mathcal{A}x \leq \mathcal{B},$$

where $D_k^- \geq 0, D_k^+ \geq 0$ and $D_k^-.D_k^+ \geq 0.$

$$0 \leq \psi \leq 1,$$

$$\psi \geq D_k^-, D_k^+ \geq 0. \tag{5.5}$$

II. By proposed methodology, the fuzzy objective function (2) using non-linear (S-shaped) membership function with is defined as follows:

$$\mu(Z_k(x)) = \begin{cases} 1, & \text{if } Z_k(x) \geq u_k, \\ 1 - \left(\dfrac{1}{1 + Ae^{\alpha\left(\frac{Z_k(x) - l_k}{g_k - l_k}\right)}} \right), & \text{if } \ell_k < Z_k(x) < u_k, \\ 0, & \text{if } Z_k(x) \leq \ell_k. \end{cases} \tag{5.6}$$

Here , A is a scale constant controlling the curve width, and α determines the steepness of the S-shaped MF.

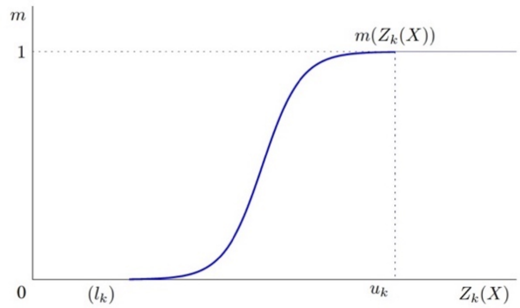


Figure 5.2: Non-linear membership function

Above fig 5.2 is of non-linear membership function. Since the largest value is 1 for the membership functions mentioned in (5.6) then the single-objective problem can be formulated as follows:

$$\text{Min}\psi$$

s.t.

$$\frac{1-\left(\frac{1}{1+Ae^{\alpha\left(\frac{Z_{k}(\varkappa)-(l_{k})}{g_{k}-l_{k}}\right)}}\right)-(l_{k})}{g_{k}-l_{k}}+D_{k}^{-}-D_{k}^{+}=1 \tag{5.7}$$

$$\mathcal{A}\varkappa \leqq \mathcal{B}$$

where $D_{k}^{-} \geq 0$, $D_{k}^{+} \geq 0$.

$$0 \leq \psi \leq 1 \text{ , } \psi \geq D_{k}^{-}, D_{k}^{+}$$
$$D_{k}^{-}.D_{k}^{+} \geq 0 \text{ ,} \varkappa \geq 0. \tag{5.8}$$

Following is the flow chart explaining the procedure to follow using propose methodology based on non-linear S-shaped membership function.

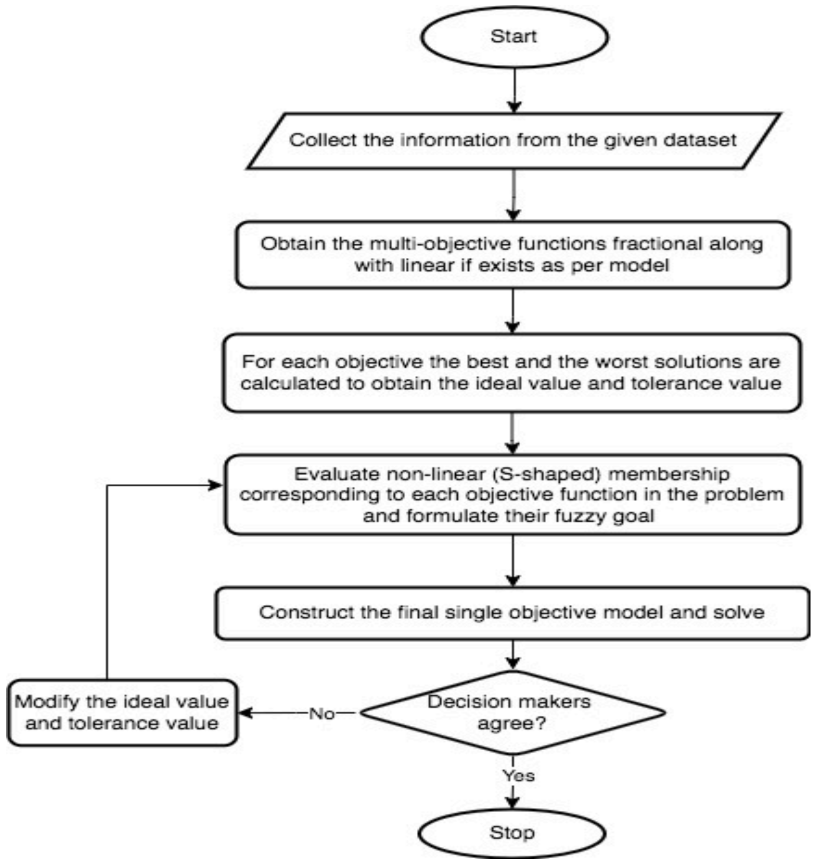


Figure 5.3 Flow chart to explain proposed methodology using S-shaped membership function.

5.3 Numerical Example

Let us investigate a MOLFPP discussed in [103] having the following objectives:

$$Max\ Z(\varkappa) = \left(Z_1(\varkappa) = \frac{-3\varkappa_1 + 2\varkappa_2}{\varkappa_1 + \varkappa_2 + 3}\ ;\ Z_2(\varkappa) = \frac{7\varkappa_1 + \varkappa_2}{5\varkappa_1 + 2\varkappa_2 + 1} \right) \tag{5.9}$$

s.t.

$$\varkappa_1 - \varkappa_2 \geq 1, \tag{5.10}$$

$$2\varkappa_1 + 3\varkappa_2 \leq 15, \tag{5.11}$$

$$\varkappa_1 \geq 3, \tag{5.12}$$

$$\varkappa_1, \varkappa_2 \geq 0. \tag{5.13}$$

For setting aspiration levels for $Z_1(\varkappa)$ and $Z_2(\varkappa)$ selected as $maxZ_1(\varkappa)$ =1/2 and $maxZ_2(\varkappa)$ =7/5 by setting lower value for $Z_1(\varkappa)$ and $Z_2(\varkappa)$ as 0.

By Zimmerman's approach

Linear Membership function formulation:

$$\mu(Z_1(\varkappa)) = \begin{cases} 1, & \text{if } Z_1(\varkappa) \geq 1/2, \\ \dfrac{\dfrac{-3\varkappa_1 + 2\varkappa_2}{\varkappa_1 + \varkappa_2 + 3} - 0}{1/2}, & \text{if } 0 < Z_1(\varkappa) < 1/2, \\ 0, & \text{if } Z_1(\varkappa) \leq 0. \end{cases} \tag{5.14}$$

$$\mu(Z_2(\varkappa)) = \begin{cases} 1, & \text{if } Z_2(\varkappa) \geq 7/5, \\ \dfrac{\dfrac{7\varkappa_1 + \varkappa_2}{5\varkappa_1 + 2\varkappa_2 + 1} - 0}{7/5}, & \text{if } 0 < Z_2(\varkappa) < 7/5, \\ 0, & \text{if } Z_2(\varkappa) \leq 0. \end{cases} \tag{5.15}$$

Converting into single objective function problem:

$$\text{Min } \Psi \tag{5.16}$$

$$\frac{-6\varkappa_1+\varkappa_2}{\varkappa_1+\varkappa_2+3} + D_1^- - D_1^+ = 1, \tag{5.17}$$

$$\frac{35\varkappa_1+5\varkappa_2}{35\varkappa_1+14\varkappa_2+7} + D_2^- - D_2^+ = 1, \tag{5.18}$$

$$0 \leq \Psi \leq 1, \tag{5.19}$$

$$\Psi \geq D_1^-, D_1^+, D_2^-, D_2^+ \geq 0, \tag{5.20}$$

$$D_1^- . D_1^+, D_2^- . D_2^+ = 0, \tag{5.21}$$

$$\varkappa_1 - \varkappa_2 \geq 1,$$

$$2\varkappa_1 + 3\varkappa_2 \leq 15,$$

$$\varkappa_1 \geq 3,$$

$$\varkappa_1, \varkappa_2 \geq 0. \tag{5.22}$$

By solving through LINGO software, we got $\varkappa_1 = 3$, $\varkappa_2 = 0.4268153$, $D_1^- = 0.4268153$, $D_2^- = 0.42$ and $Z_1(\varkappa) = -0.80076$and $Z_2(\varkappa) = 1.27134$.

By proposed methodology

Non-linear (S-shaped) membership function:

$$\dot{\mu}(Z_1(\varkappa)) = \begin{cases} 1, & \text{if } Z_1(\varkappa) \geq 1/2, \\ 1 - \left(\dfrac{1}{1 + 0.001001e^{13.813\left(\frac{\frac{-3\varkappa_1+2\varkappa_2}{\varkappa_1+\varkappa_2+3}-0}{0.5}\right)}} \right), & \text{if } 0 < Z_1(\varkappa) < \dfrac{1}{2}, \\ 0, & \text{if } Z_1(\varkappa) \leq 0. \end{cases} \tag{5.23}$$

$$\dot{\mu}\big(Z_2(\varkappa)\big) = \begin{cases} 1, & \text{if } Z_2(\varkappa) \geq \dfrac{7}{5}, \\ \left(1 - \left(\dfrac{1}{1 + 0.001001e^{13.813\left(\frac{\frac{7\varkappa_1+\varkappa_2}{5\varkappa_1+2\varkappa_2+1}-0}{\frac{7}{5}}\right)}} \right) \right), & \text{if } 0 < Z_2(\varkappa) < \dfrac{7}{5}, \\ 0, & \text{if } Z_2(\varkappa) \leq 0. \end{cases} \tag{5.24}$$

Converting into single objective function problem:

$$\text{Min } \Psi \tag{2.25}$$

$$1 - \left(\frac{1}{1+0.001001e^{13.813\left(\frac{\frac{-3\varkappa_1+2\varkappa_2}{\varkappa_1+\varkappa_2+3}-0}{0.5}\right)}} \right) + D_1^- - D_1^+ = 1, \tag{5.26}$$

$$1 - \left(\frac{1}{1 + 0.001001e^{13.813\left(\frac{\frac{7\varkappa_1+\varkappa_2}{5\varkappa_1+2\varkappa_2+1}-0}{\frac{7}{5}}\right)}} \right) + D_2^- - D_2^+ = 1, \tag{5.27}$$

$$0 \leq \Psi \leq 1, \tag{5.28}$$

$$\Psi \geq D_1^-, D_1^+, D_2^-, D_2^+ \geq 0, \tag{5.29}$$

$$D_1^- . D_1^+, D_2^- . D_2^+ = 0, \tag{5.30}$$

$$\begin{gathered} \varkappa_1 - \varkappa_2 \geq 1, \\ 2\varkappa_1 + 3\varkappa_2 \leq 15, \\ \varkappa_1 \geq 3, \\ \varkappa_1, \varkappa_2 \geq 0. \end{gathered} \tag{5.31}$$

By solving through LINGO software, we got $\varkappa_1 = 3$, $\varkappa_2 = 2$, $D_1^- = 1$, $D_2^- = 0.11$ and $Z_1(\varkappa) = -0.62$ and $Z_2(\varkappa) = 1.15$

5.4 Case Study

We worked with secondary data used in (Desktop/ ProfitOptimizationofanApparelIndustryin-BangladeshPublished-AJAM.pdf). In the present study, a knit garment manufacturing unit from Bangladesh has been taken into consideration. This plant is located in the Gazipur district of Bangladesh[70]. From the case industry, information has been gathered that includes the monthly resource usage amount, product volume, and profit per unit for a variety of product categories. The case industry manufactures a wide variety of knitted garments. The collated data served as the input for the linear programming model that was proposed. There are currently eight different kinds of clothing that are being manufactured by the company that we are examining. The purpose of this study is to determine the current level of resource consumption, production cost, time utilized, and monthly profit, and then compare these values to the ideal answer that was reached by solving the FLPP model that we built. We used LINGO solver to solve the model.

The following table 5.1, table 5.2, and table 5.3 of the paper provide a summary of the pertinent information that was acquired from the case company. This information includes the amount of time required to make various items, the monthly production, the profit, the cost, and the material utilization per unit.

According to the information:

Table 5.1: Details of industry production, profit and time utilization required for different products

Product	Industry Production	Profit (Per piece)	Profit by industry	Time			
				Cutting	Sewing	Trimming	Finishing
GT	8000	42	336000	0.4	6.4	0.4	0.4
KBLJ	14000	36	504000	0.3	5.3	0.3	0.3
BCH	8000	40	320000	0.5	5.5	0.5	0.5
BUW	7000	30	210000	0.2	5.2	0.2	0.2
GC	12000	35	420000	0.6	7.6	0.3	0.6
GL	10000	40	400000	0.5	6.5	0.3	0.5
GCS	12000	30	360000	0.4	5.4	0.4	0.4
GTL/S	11000	25	275000	0.5	7.5	0.4	0.5
Total			2825000				

Restrictions:

Table 5.2: Material and cost requirement for fabric thread and labour

Over	Material	Over	Cost
Fabric	12156000	Labour	2853000
Tread	11295000	Material	13194000

Decision variable:

Table 5.3: Symbols and notations for different garment products

Symbol (number of product)	Product
$\varkappa_1$	T-shirt for girls(GT)
$\varkappa_2$	Long-johns made by Keiki Boy(KBLJ)

$\varkappa_3$	Hoodie for Boys in College(BCH)
$\varkappa_4$	Underwear for Boys(BUW)
$\varkappa_5$	Cardigan for girls (GC)
$\varkappa_6$	Leggings for girls(GL)
$\varkappa_7$	College shirt for girls(GCS)
$\varkappa_8$	T-shirt for girls, L/S (GTL/S)

Objective function: Profit/Investment

$$\text{Max P}(\varkappa) = \frac{42\varkappa_1+36\varkappa_2+40\varkappa_3+30\varkappa_4+35\varkappa_5+40\varkappa_6+30\varkappa_7+25\varkappa_8}{159\varkappa_1+169\varkappa_2+285\varkappa_3+142\varkappa_4+300\varkappa_5+150\varkappa_6+185\varkappa_7+165\varkappa_8} \quad (5.32)$$

Time utilization:

Cutting:

$$\text{Max } C(\varkappa) = 0.4\varkappa_1 + 0.3\varkappa_2 + 0.5\varkappa_3 + 0.2\varkappa_4 + 0.6\varkappa_5 + 0.5\varkappa_6 + 0.4\varkappa_7 + 0.5\varkappa_8 \quad (5.33)$$

Sewing:

$$\text{Max} S(\varkappa) = 6.4\varkappa_1 + 5.3\varkappa_2 + 5.5\varkappa_3 + 5.2\varkappa_4 + 7.6\varkappa_5 + 6.5\varkappa_6 + 5.4\varkappa_7 + 7.5\varkappa_8 \quad (5.34)$$

Trimming:

$$\text{Max} T(\varkappa) = 0.4\varkappa_1 + 0.3\varkappa_2 + 0.5\varkappa_3 + 0.2\varkappa_4 + 0.3\varkappa_5 + 0.3\varkappa_6 + 0.4\varkappa_7 + 0.4\varkappa_8 \quad (5.35)$$

Finishing:

$$\text{Max} F(\varkappa) = 0.4\varkappa_1 + 0.3\varkappa_2 + 0.5\varkappa_3 + 0.2\varkappa_4 + 0.6\varkappa_5 + 0.5\varkappa_6 + 0.4\varkappa_7 + 0.5\varkappa_8 \quad (5.36)$$

Restrictions:

Budget (labour cost):

$$28\varkappa_1 + 25\varkappa_2 + 65\varkappa_3 + 22\varkappa_4 + 60\varkappa_5 + 25\varkappa_6 + 30\varkappa_7 + 25\varkappa_8 \leq 2853000 \quad (5.37)$$

Budget (material cost):

$$131\varkappa_1 + 144\varkappa_2 + 220\varkappa_3 + 120\varkappa_4 + 240\varkappa_5 + 125\varkappa_6 + 155\varkappa_7 + 140\varkappa_8 \leq 13194000 \quad (5.38)$$

Fabric:

$$128\varkappa_1 + 121\varkappa_2 + 246\varkappa_3 + 100\varkappa_4 + 180\varkappa_5 + 131\varkappa_6 + 155\varkappa_7 + 120\varkappa_8 \leq 12156000 \quad (5.39)$$

Thread:

$$120\varkappa_1 + 110\varkappa_2 + 220\varkappa_3 + 70\varkappa_4 + 220\varkappa_5 + 120\varkappa_6 + 120\varkappa_7 + 115\varkappa_8 \leq 11295000 \quad (5.40)$$

Cutting Time:

$$0.4\varkappa_1 + 0.3\varkappa_2 + 0.5\varkappa_3 + 0.2\varkappa_4 + 0.6\varkappa_5 + 0.5\varkappa_6 + 0.4\varkappa_7 + 0.5\varkappa_8 \leq 35300 \quad (5.41)$$

Sewing time:

$$6.4\varkappa_1 + 5.3\varkappa_2 + 5.5\varkappa_3 + 5.2\varkappa_4 + 7.6\varkappa_5 + 6.5\varkappa_6 + 5.4\varkappa_7 + 7.5\varkappa_8 \leq 509300 \quad (5.42)$$

Trimming time:

$$0.4\varkappa_1 + 0.3\varkappa_2 + 0.5\varkappa_3 + 0.2\varkappa_4 + 0.3\varkappa_5 + 0.3\varkappa_6 + 0.4\varkappa_7 + 0.4\varkappa_8 \leq 28600 \quad (5.43)$$

Finishing and packing time:

$$0.4\varkappa_1 + 0.3\varkappa_2 + 0.5\varkappa_3 + 0.2\varkappa_4 + 0.6\varkappa_5 + 0.5\varkappa_6 + 0.4\varkappa_7 + 0.5\varkappa_8 \leq 35300 \quad (5.44)$$

Non-negativity restrictions:

$$\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4, \varkappa_5, \varkappa_6, \varkappa_7, \varkappa_8 \geq 0. \quad (5.45)$$

Max P($\varkappa$)=$\bar{P}$=0.2666667 at point (0,0,0,0,0,70600,0,0)

Max C($\varkappa$)= $\bar{C}$=35300 at point (0,0,25281,0,0,45319,0,0)

Max S($\varkappa$)=$\bar{S}$= 509299.5 at point (0,0,1,0,0,1,0,67905)

Max T($\varkappa$)=$\bar{T}$=28600 at point (0,0,31570,1,0,0,0,32037)

Worst point of P^w =0.11667

Worst point of C^w=0

Worst point of S^w =0

Worst point of T^w =0.

By Zimmerman's approach

Linear Membership function formulation:

$$\mu(P(\varkappa)) = \begin{cases} 1, & \text{if } \mathrm{P}(\varkappa) \geq 0.2666667, \\ \dfrac{\mathrm{P}(\varkappa) - 0.11687}{0.149997}, & 0.11687 < \mathrm{P}(\varkappa) < 0.2666667, \\ 0, & \text{if } \mathrm{P}(\varkappa) \leq 0.11687. \end{cases}$$

(5.46)

$$\mu\big(C(\varkappa)\big) = \begin{cases} 1, & \text{if } C(\varkappa) \geq 35300, \\ \dfrac{C(\varkappa)}{35300}, & 0 < C(\varkappa) < 35300, \\ 0, & 0. \end{cases}$$

(5.47)

$$\mu(S(\varkappa)) = \begin{cases} 1, & \text{if } S(\varkappa) \geq 509299.5, \\ \dfrac{S(\varkappa)}{509299.5}, & \text{if } 0 < \mathrm{S}(\varkappa) < 509299.5, \\ 0, & \text{if } \mathrm{S}(\varkappa) \leq 0. \end{cases}$$

(5.48)

$$\mu(\mathrm{T}(\varkappa)) = \begin{cases} 1, & \text{if } T(\varkappa) \geq 28600, \\ \dfrac{\mathrm{T}(\varkappa)}{28600}, & \text{if } 21180 < T(\varkappa) < 28600, \\ 0, & \text{if } T(\varkappa) \leq 21180. \end{cases} \quad (5.49)$$

Converting into single objective function problem:

$$\text{Min } \Psi \quad (5.50)$$

$$\text{s.t.} \quad \frac{\frac{42\varkappa_1+36\varkappa_2+40\varkappa_3+30\varkappa_4+35\varkappa_5+40\varkappa_6+30\varkappa_7+25\varkappa_8}{159\varkappa_1+169\varkappa_2+285\varkappa_3+142\varkappa_4+300\varkappa_5+150\varkappa_6+185\varkappa_7+165\varkappa_8} - 0.11687}{0.149997} + \mathrm{D}_1^- - \mathrm{D}_1^+ = 1, \quad (5.51)$$

$$\frac{0.4\varkappa_1 + 0.3\varkappa_2 + 0.5\varkappa_3 + 0.2\varkappa_4 + 0.6\varkappa_5 + 0.5\varkappa_6 + 0.4\varkappa_7 + 0.5\varkappa_8}{35300} + \mathrm{D}_2^- - \mathrm{D}_2^+ = 1, \quad (5.52)$$

$$\frac{6.4\varkappa_1 + 5.3\varkappa_2 + 5.5\varkappa_3 + 5.2\varkappa_4 + 7.6\varkappa_5 + 6.5\varkappa_6 + 5.4\varkappa_7 + 7.5\varkappa_8}{509299.5} + \mathrm{D}_3^- - \mathrm{D}_3^+ = 1, \quad (5.53)$$

$$\frac{0.4\varkappa_1+0.3\varkappa_2+0.5\varkappa_3+0.2\varkappa_4+0.6\varkappa_5+0.5\varkappa_6+0.4\varkappa_7+0.5\varkappa_8}{28600}+D_4^- - D_4^+ = 1, \tag{5.54}$$

$$0 \le \Psi \le 1, \tag{5.55}$$

$$\Psi \ge D_1^-, D_1^+, D_2^-, D_2^+, D_3^-, D_3^+, D_4^-, D_4^+, \tag{5.56}$$

$$D_1^-, D_1^+, D_2^-, D_2^+, D_3^-, D_3^+, D_4^-, D_4^+ \ge 0, \tag{5.57}$$

$$D_1^-. D_1^+, D_2^-. D_2^+, D_3^-. D_3^+, D_4^-. D_4^+ = 0, \tag{5.58}$$

$$128\varkappa_1 + 121\varkappa_2 + 246\varkappa_3 + 100\varkappa_4 + 180\varkappa_5 + 131\varkappa_6 + 155\varkappa_7 + 120\varkappa_8 \le 12156000,$$

$$120\varkappa_1 + 110\varkappa_2 + 220\varkappa_3 + 70\varkappa_4 + 220\varkappa_5 + 120\varkappa_6 + 120\varkappa_7 + 115\varkappa_8 \le 11295000,$$

$$28\varkappa_1 + 25\varkappa_2 + 65\varkappa_3 + 22\varkappa_4 + 60\varkappa_5 + 25\varkappa_6 + 30\varkappa_7 + 25\varkappa_8 \le 2853000,$$

$$131\varkappa_1 + 144\varkappa_2 + 220\varkappa_3 + 120\varkappa_4 + 240\varkappa_5 + 125\varkappa_6 + 155\varkappa_7 + 140\varkappa_8 \le 13194000,$$

$$0.4\varkappa_1 + 0.3\varkappa_2 + 0.5\varkappa_3 + 0.2\varkappa_4 + 0.6\varkappa_5 + 0.5\varkappa_6 + 0.4\varkappa_7 + 0.5\varkappa_8 \le 35300,$$

$$6.4\varkappa_1 + 5.3\varkappa_2 + 5.5\varkappa_3 + 5.2\varkappa_4 + 7.6\varkappa_5 + 6.5\varkappa_6 + 5.4\varkappa_7 + 7.5\varkappa_8 \le 509300,$$

$$0.4\varkappa_1 + 0.3\varkappa_2 + 0.5\varkappa_3 + 0.2\varkappa_4 + 0.3\varkappa_5 + 0.3\varkappa_6 + 0.4\varkappa_7 + 0.4\varkappa_8 \le 28600,$$

$$\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4, \varkappa_5, \varkappa_6, \varkappa_7, \varkappa_8 \ge 0. \tag{5.59}$$

By solving through LINGO software P= 0.2651431, C= 34941.3, S= 509299.3, T= 28309.4 at point (45892,0,9,0,0,33159,0,1). Value of deviation parameter is 0.01016084. According to which profit is 3254209 and investment is 12273408.

By proposed methodology

Non-linear (S-shaped) membership function:

$$\mu(P(\varkappa)) = \begin{cases} 1, & \text{if } P(\varkappa) \ge 0.2666667, \\ 1-\left(\dfrac{1}{1+0.001001e^{13.813\left(\frac{P(\varkappa)-0.11687}{0.149997}\right)}}\right), & \text{if } 0.11687 < P(\varkappa) < 0.2666667, \\ 0, & \text{if } P(\varkappa) \le 0.11687. \end{cases} \tag{5.60}$$

$$\mu(C(\varkappa)) = \begin{cases} 1, & \text{if } C(\varkappa) \geq 35300, \\ \left(1 - \left(\dfrac{1}{1 + 0.001001e^{13.813\left(\frac{C(\varkappa)}{35300}\right)}}\right)\right), & \text{if } 0 < C(\varkappa) < 35300, \\ 0, & \text{if } C(\varkappa) \leq 0. \end{cases} \tag{5.61}$$

$$\mu(S(\varkappa)) = \begin{cases} 1, & \text{if } S(\varkappa) \geq 509299.5, \\ 1 - \left(\dfrac{1}{1 + 0.001001e^{13.813\left(\frac{S(\varkappa)}{509299.5}\right)}}\right), & \text{if } 0 < S(\varkappa) < 509299.5, \\ 0, & \text{if } S(\varkappa) \leq 0. \end{cases} \tag{5.62}$$

$$\mu(\mathrm{T}(\varkappa)) = \begin{cases} 1, & \text{if } \mathrm{T}(\varkappa) \geq 28600, \\ \left(1 - \left(\dfrac{1}{1 + 0.001001e^{13.813\left(\frac{\mathrm{T}(\varkappa)}{28600}\right)}}\right)\right), & \text{if } 21180 < \mathrm{T}(\varkappa) < 28600, \\ 0, & \text{if } \mathrm{T}(\varkappa) \leq 21180. \end{cases} \tag{5.63}$$

Converting into single objective function problem:

$$\text{Min } \Psi \tag{5.64}$$

s.t.

$$1 - \left(\frac{1}{1 + 0.001001e^{13.813\left(\frac{\frac{42\varkappa_1+36\varkappa_2+40\varkappa_3+30\varkappa_4+35\varkappa_5+40\varkappa_6+30\varkappa_7+25\varkappa_8}{159\varkappa_1+169\varkappa_2+285\varkappa_3+142\varkappa_4+300\varkappa_5+150\varkappa_6+185\varkappa_7+165\varkappa_8}-0.11687}{0.149997}\right)}}\right) + \mathrm{D}_1^- - \mathrm{D}_1^+ = 1, \tag{5.65}$$

$$1 - \left(\frac{1}{1+0.001001e^{13.813\left(\frac{0.4\varkappa_1+0.3\varkappa_2+0.5\varkappa_3+0.2\varkappa_4+0.6\varkappa_5+0.5\varkappa_6+0.4\varkappa_7+0.5\varkappa_8}{35300}\right)}}\right) + \mathrm{D}_2^- - \mathrm{D}_2^+ = 1, \tag{5.66}$$

$$1 - \left(\frac{1}{1+0.001001e^{13.813\left(\frac{6.4\varkappa_1+5.3\varkappa_2+5.5\varkappa_3+5.2\varkappa_4+7.6\varkappa_5+6.5\varkappa_6+5.4\varkappa_7+7.5\varkappa_8}{509299.5}\right)}}\right) + \mathrm{D}_3^- - \mathrm{D}_3^+ = 1, \tag{5.67}$$

$$1 - \left(\frac{1}{1+0.001001e^{13.813\left(\frac{0.4\varkappa_1+0.3\varkappa_2+0.5\varkappa_3+0.2\varkappa_4+0.6\varkappa_5+0.5\varkappa_6+0.4\varkappa_7+0.5\varkappa_8}{28600}\right)}}\right) + \mathrm{D}_4^- - \mathrm{D}_4^+ = 1, \tag{5.68}$$

$$0 \leq \Psi \leq 1, \tag{5.69}$$

$$\Psi \geq \mathrm{D}_1^-, \mathrm{D}_1^+, \mathrm{D}_2^-, \mathrm{D}_2^+, \mathrm{D}_3^-, \mathrm{D}_3^+, \mathrm{D}_4^-, \mathrm{D}_4^+, \tag{5.70}$$

$$D_1^-, D_1^+, D_2^-, D_2^+, D_3^-, D_3^+, D_4^-, D_4^+ \geq 0, \tag{5.71}$$

$$D_1^- . D_1^+, D_2^- . D_2^+, D_3^- . D_3^+, D_4^- . D_4^+ = 0, \tag{5.72}$$

$$128\varkappa_1 + 121\varkappa_2 + 246\varkappa_3 + 100\varkappa_4 + 180\varkappa_5 + 131\varkappa_6 + 155\varkappa_7 + 120\varkappa_8 \leq 12156000,$$

$$120\varkappa_1 + 110\varkappa_2 + 220\varkappa_3 + 70\varkappa_4 + 220\varkappa_5 + 120\varkappa_6 + 120\varkappa_7 + 115\varkappa_8 \leq 11295000,$$

$$28\varkappa_1 + 25\varkappa_2 + 65\varkappa_3 + 22\varkappa_4 + 60\varkappa_5 + 25\varkappa_6 + 30\varkappa_7 + 25\varkappa_8 \leq 2853000,$$

$$131\varkappa_1 + 144\varkappa_2 + 220\varkappa_3 + 120\varkappa_4 + 240\varkappa_5 + 125\varkappa_6 + 155\varkappa_7 + 140\varkappa_8 \leq 13194000,$$

$$0.4\varkappa_1 + 0.3\varkappa_2 + 0.5\varkappa_3 + 0.2\varkappa_4 + 0.6\varkappa_5 + 0.5\varkappa_6 + 0.4\varkappa_7 + 0.5\varkappa_8 \leq 35300,$$

$$6.4\varkappa_1 + 5.3\varkappa_2 + 5.5\varkappa_3 + 5.2\varkappa_4 + 7.6\varkappa_5 + 6.5\varkappa_6 + 5.4\varkappa_7 + 7.5\varkappa_8 \leq 509300,$$

$$0.4\varkappa_1 + 0.3\varkappa_2 + 0.5\varkappa_3 + 0.2\varkappa_4 + 0.3\varkappa_5 + 0.3\varkappa_6 + 0.4\varkappa_7 + 0.4\varkappa_8 \leq 28600,$$

$$\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4, \varkappa_5, \varkappa_6, \varkappa_7, \varkappa_8 \geq 0. \tag{5.73}$$

By solving through LINGO software C= 34939.5, S= 509278.5, T= 28307.9, P = 0.2651461 at point (45886, 0,1,0,0,33158,14,0). Value of deviation parameter is 0.001151189. According to which profit is 3253992 and investment is 12272449.

5.5 Result Analysis:

Table 5.4 :Deviation of objective functions from ideal points

Objective Function	Deviation From	Ideal Points
	Linear membership	Non-linear membership
Profit/Investment(P)	0.571349%	0.15206%
Cutting time(C)	1.0161473%	1.021246%
Sewing time (S)×10^4	0.000039269%	0.000041233105%
Trimming time (T)	1.0160839%	1.0213286%
Total	2.603619469%	2.194675833%

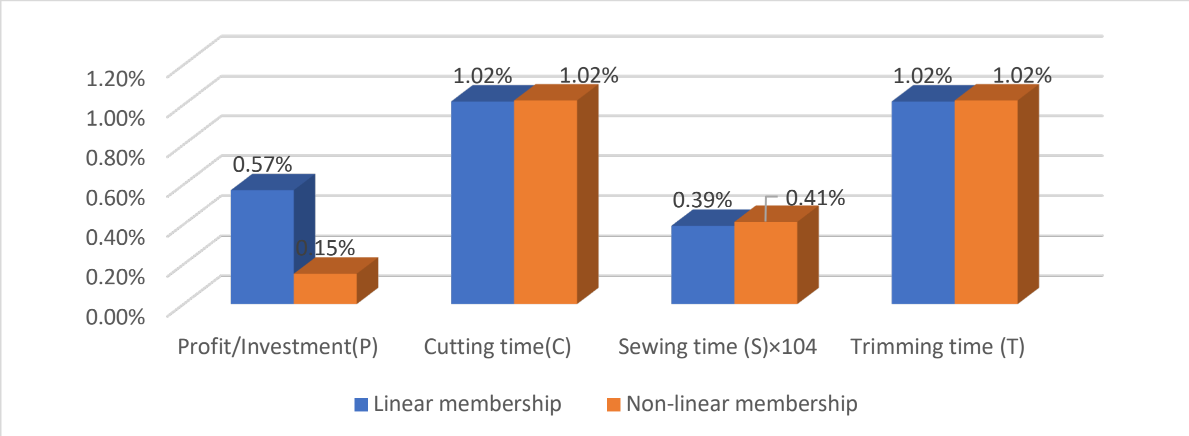


Figure 5.4: Comparison between linear and nonlinear membership functions over deviation from ideal solution

In table 5.4 we analyzed that deviation of objective function from its ideal point is reduced for non-linear membership function as compared to linear membership function. We can see that the deviation of objective function profit over investment using linear membership function for fuzzy goal programming is 0.571349% from it ideal point but it reduces to 0.15206% when we considered the same problem for non-linear membership function in fuzzy goal programming. The comparison between the linear membership and non-linear membership function over the deviation from its ideal solution can be seen in figure 5.4, where we can see that there is huge gap between two for the objective function profit over investment. Lesser the deviation value from it ideal point better is the result.

Table 5.5: Percentage change in single objective and multi-objective LPP

	In Single objective LPP	In Multi-objective LPP	Percentage of Change
Profit	3420618	3253992	-4.871%
Cost	15542248	12272449	21.038%
Fabric	11386410	10219522	10.248%
Thread	10381550	9487180	8.6149%
Cutting time utilization	30002.2	34939.5	14.1309%

Sewing time utilization	509299.2	509278.5	-0.004644%
Trimming time utilization	28600	28307.9	-1.0213%
Total			49.1459%

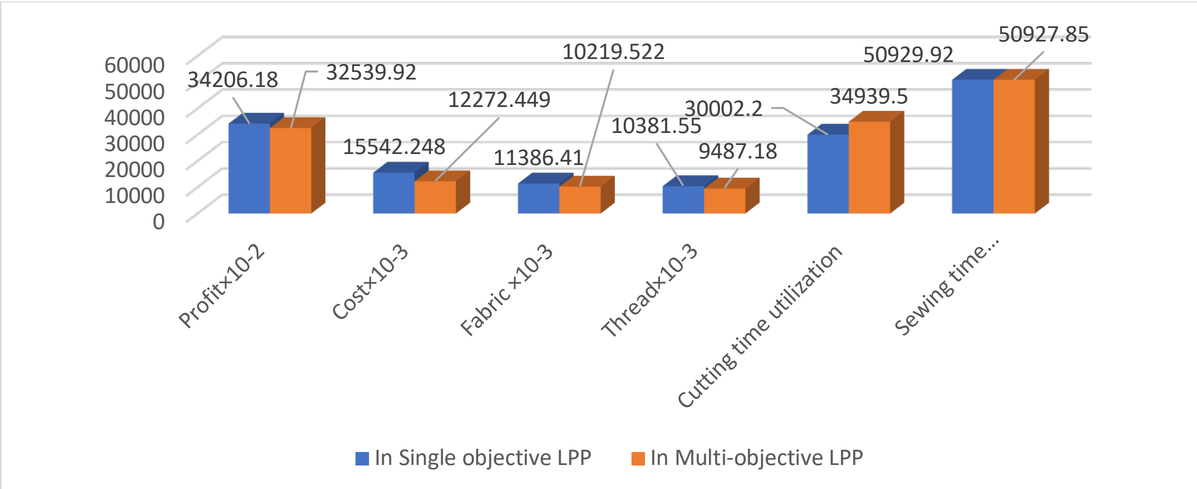


Figure 5.5: Graphical representation of percentage change between single objective to multi-objective.

In table 5.5 we compared the results obtained in single objective LPP with multi-objective LPP and analyzed that overall, the percentage change between them is 49.11459% that describes that optimal utility of all the resources is better in case of multi-objective programming problem as compare to the single objective problem. Even though the profit in single objective is little higher as compared to multi-objective but as we can see from table that cost and other important parameters reduced with higher amount in case of multi-objective LPP than the single objective LPP which is always required for optimal solution. The results are also presented graphically in figure 5.5.

Table 5.6: Percentage change in profit and cost components via linear and non-linear functions

	Linear	Non-linear	Percentage of change
Profit	3254209	3253992	-0.000066683%

Cost	12273408	12272449	0.000078136%
Total			0.000011453%

In table 5.6 described the profit and cost values obtained by linear function and by non-linear function and the net percentage change of these factors by two methodologies. We observed that overall percentage of change is positive.

Table 5.7: Percentage change in Fabric and Thread via linear and non-linear functions

	Linear	Non-linear	Percentage of change
Fabric	10220339	10219522	0.000079938%
Thread	9488215	9487180	0.010908%
Total			0.0109879%

In table 5.7 we have discussed about the results obtained in two important component of garment industry fabric and thread and we observed that the overall as well as individual percentage changes between linear and non-linear is positive that describes that non-linear methodology has enhanced the results as compare to the linear function-based methodology.

5.6 Conclusion

In this chapter we used the non-linear S-shaped membership function to find an optimal solution to the fuzzy multiple-purpose linear fractional programming Problems. We compared the results and solutions with the Zimmerman approach using the linear membership function. We found that when an S-shaped membership function solves the objective function profit over the investment, the percentage deviation from the ideal point is lower than the deviation achieved by the linear membership function technique. In the future, we will extend our proposed approach to more advanced fuzzy sets that provide the ideal solution to many real-world issues.

To provide a quick overview of the constraints, the chapter does not investigate a comprehensive range of S-shaped parameter choices, nor does it carry out an extended sensitivity analysis. Furthermore, the case study is dependent on secondary data. Future research can go into more detail about these points.

Chapter 6

Conclusion And Future Scope

This thesis designed, implemented, and validated different hybrid ways to solve FLFPP using fuzzy numbers, meeting the stated objectives.

A comprehensive introduction and literature study were conducted in Unit one in order to establish the fundamental features. This was accomplished by determining the research gaps that existed and establishing the reason for the suggested techniques.

The second unit took a close look at a variety of defuzzification techniques for trapezoidal and triangular fuzzy integers. An organized method for dealing with fuzziness in the objective function was offered by the component-wise tri-objective approach that was created utilizing the (s, ℓ, r) representation. This was improved even further by combining the Guzel-modified Dinkelbach's method with a weighted sum technique, which was effectively used on a production planning model, results are demonstrated in table 2.1 under section 2.5 of Result analysis and comparison. For trapezoidal a complementary model was based on linearization methodology with single crisp value based ranking approach. Comparative data and method ratings showed that this hybrid approach was reliable, which opens the door for more research into other defuzzification methods. In general, the hybrid strategy made solutions more stable and less likely to change across the three circumstances shown in Table 2.1.

In unit three, a bi-objective interval model that is based on α, β cuts was introduced in order to provide fuzzy solutions to be represented as intervals. A further demonstration of how practical tools may enhance the understanding of uncertainty in model results was provided by the implementation of regression surrogate modeling that was based on Excel.The α–β sweep made the interval width consistently smaller, which showed that the robustness improved at all grid levels.

In the fourth unit, MF and NMF were constructed in both linear and non-linear (parabolic and exponential) forms in order to establish an extended intuitionistic fuzzy approach (EIFA).We used benchmark problem and real databased case study from the existing literature to conduct an analysis of hybrid combinations under nine distinct situations. The comparison study showed that hybridized intuitionistic frameworks may solve uncertain issues realistically and adaptably. The summary Tables 4.6–4.10 indicate the differences in performance between MF/NMF pairings in numbers, with exponential forms showing a noticeable improvement.

Using secondary data from the garment industry, unit five added an S-shaped random number that changes based on the situation to a goal programming structure. Comparing Zimmerman's linear membership function to the S-shaped function showed that the suggested technique is more flexible and better represents goal-based decision-making's real-world trade-offs. In Section 5.5, we can see from the numerical comparison that the S-shaped strategy outperformed the linear case in terms of satisfaction ratio.

The thesis methods work well in case studies, but they have limitations. The computing cost is increased due to the need for numerous α-β assessments in interval models. The S-shaped approach relies on parameter selections that may differ across applications, and certain case studies utilize secondary data, thereby limiting their generalizability. These features may be enhanced in future expansions.

Together, the five units show how defuzzification, interval mathematical modeling, intuitionistic expansions, and nonlinear membership structures can all work together to deal with different kinds of uncertainty in FLFPP. The thesis therefore presents an integrated array of tools that enhance modeling depth while ensuring practical relevance across diverse decision-making contexts.

The following courses of action are suggested for future research:

Among the potential extensions, the primary focus is to investigate more expressive fuzzy representations, subsequently followed by the development of computational strategies that enhance scalability for large-scale FLFPP models.

To investigate other defuzzification techniques and create more flexible membership functions, such as interval-valued fuzzy numbers or type-2 fuzzy numbers. Integrating methods that are based on artificial intelligence or metaheuristics in order to solve complicated fuzzy fractional programming challenges. The methodologies that have been developed should be applied to growing industries such as sustainable supply chains, healthcare optimization, and renewable energy. Creation of iterative stakeholder feedback-based interactive fuzzy decision-making systems in FLFPP models. To handle issues with more dynamic uncertainty and greater degrees of imprecision, broaden the intuitionistic and S-shaped fuzzy frameworks.

In conclusion, this study provides a big addition to the burgeoning area of FFPP by suggesting and testing viable hybrid approaches, creating new FN that are based on real-life situations, and showing how they may be used in real life through case studies. The knowledge gathered and the approaches created here can be a useful starting point for future scholars and professionals who want to solve decision-making situations that are getting more complicated and unpredictable.

Bibliography

[1] Sbell, J. R., & Marlow, W. H. (1956). Attrition games. *Naval Research Logistics Quarterly*, 71.

[2] Charnes,A.,&Cooper,W. W. (1962). Programming with linear fractional functionals. *Naval Research logistics quarterly*, *9*(3-4), 181-186

[3] Zadeh, L. A. (1965). Fuzzy sets. *Information and control*, *8*(3), 338-353.

[4] Swarup,K.(1965).Linear fractional functionals programming. *Operations Research*, *13*(6), 1029-1036.

[5] Dinkelbach,W. (1967). On nonlinear fractional programming. *Management science*, *13*(7), 492-498.

[6] RE, B. (1970). Decision making in a fuzzy environment. *Manage Sci*, *17*(4), 209-215.

[7] Bracken, J., & McGill, J. T. (1973). Mathematical programs with optimization problems in the constraints. *Operations research*, *21*(1), 37-44.

[8] Tanaka, H., Okuda, T., & Asai, K. (1973). Fuzzy mathematical programming. *Transactions of the society of instrument and control engineers*, *9*(5), 607-613.

[9] Bitran, G. R., & Novaes, A. G. (1973). Linear programming with a fractional objective function. *Operations research*, *21*(1), 22-29.

[10] Zimmermann, H. J. (1975). Description and optimization of fuzzy systems. *International journal of general System*, *2*(1), 209-215.

[11] Zimmermann, H. J. (1978). Fuzzy programming and linear programming with several objective functions. *Fuzzy sets and systems*, *1*(1), 45-55.

[12] Luhandjula,M. K.(1984).Fuzzy approaches for multiple objective linear fractional optimization . *Fuzzy sets and systems*, *13*(1), 11-23.

[13] Pal,B.B.,&Basu,I.(1995). A goal programming method for solving fractional programming problems via dynamic programming. *Optimization*, *35*(2), 145-157.

[14] Li, D., & Chen, S. (1996). A fuzzy programming approach to fuzzy linear fractional programming with fuzzy coefficients. *Journal of fuzzy mathematics*, *4*, 829-834.

[15] Mohamed, R. H. (1997). The relationship between goal programming and fuzzy programming. *Fuzzy sets and systems*, *89*(2), 215-222.

[16] Angelov, P. P. (1997). Optimization in an intuitionistic fuzzy environment. *Fuzzy sets and Systems*, *86*(3), 299-306.

[17] Atanassov, K. T. (1999). Intuitionistic fuzzy sets. In *Intuitionistic fuzzy sets: theory and applications* (pp. 1-137). Heidelberg: Physica-Verlag HD.

[18] Buckley, J. J., & Feuring, T. (2000). Evolutionary algorithm solution to fuzzy problems: fuzzy linear programming. *Fuzzy sets and systems*, *109*(1), 35-53.

[19] Chakraborty, M., & Gupta, S. (2002). Fuzzy mathematical programming for multi objective linear fractional programming problem. *Fuzzy sets and systems*, *125*(3), 335-342.

[20] Arora, S. R., & Gupta, R. (2003). A hybrid fuzzy-goal programming approach to multiple objective linear fractional programming problems. *Opsearch*, *40*(3), 202-218.

[21] Pal, B. B., Moitra, B. N., & Maulik, U. (2003). A goal programming procedure for fuzzy multi-objective linear fractional programming problem. *Fuzzy sets and systems*, *139*(2), 395-405.

[22] Chang, C. T. (2005). Fractional programming with absolute-value functions: a fuzzy goal programming approach. *Applied mathematics and computation*, *167*(1), 508-515.

[23] Güzel, N., & Sivri, M. (2005). Taylor Series Solution of Multi Objective Linear Fractional Programming Problem. *Trakya University Journal of Science*, *6*, 91-98.

[24] Abd El-Wahed, W. F., & Lee, S. M. (2006). Interactive fuzzy goal programming for multi-objective transportation problems. *Omega*, *34*(2), 158-166.

[25] Sakawa, M., Yano, H., & Yumine, T. (2007). An interactive fuzzy satisficing method for multi-objective linear-programming problems and its application. *IEEE Transactions on Systems, Man, and Cybernetics*, *17*(4), 654-661.

[26] Pop, B., & Stancu-Minasian, I. (2008). A method of solving fully fuzzified linear fractional programming problems. *Journal of applied mathematics and computing*, *27*(1), 227-242.

[27] Tantawy, S. F. (2008). A new procedure for solving linear fractional programming problems. *Mathematical and computer modelling*, *48*(5-6), 969-973.

[28] Chang, C. T. (2009). A goal programming approach for fuzzy multiobjective fractional programming problems. *International journal of systems science*, *40*(8), 867-874

[29] Baky, I. A. (2010). Solving multi-level multi-objective linear programming problems through fuzzy goal programming approach. *Applied Mathematical Modelling*, *34*(9), 2377-2387.

[30] Nachammai, A., & Thangaraj, P. (2012). Solving fuzzy linear fractional programming problem using metric distance ranking. *Applied Mathematical Sciences*, *6*(26), 1275-1285

[31] Dubey, D., Chandra, S., & Mehra, A. (2012). Fuzzy linear programming under interval uncertainty based on IFS representation. *Fuzzy sets and systems*, *188*(1), 68-87.

[32] Effati, S., & Pakdaman, M. (2012). Solving the interval-valued linear fractional programming problem. *American journal of computational mathematics*, *2*(1), 51-55.

[33] Güzel, N., Emiroglu, Y., Tapci, F., Guler, C., & Syvry, M. (2012). A solution proposal to the interval fractional transportation problem. *Applied mathematics & information sciences*, *6*(3).

[34] Mehlawat, M. K., & Kumar, S. (2012, April). A solution procedure for a linear fractional programming problem with fuzzy numbers. In *Proceedings of the International Conference on Soft Computing for Problem Solving (SocProS 2011) December 20-22, 2011: Volume 1* (pp. 1037-1049). India: Springer India.

[35] Stanojević, B., & Stancu-Minasian, I. (2012). Evaluating fuzzy inequalities and solving fully fuzzified linear fractional programs. *YUJOR*, *22*(1), 41-50.

[36] Güzel, N. (2013). A proposal to the solution of multi-objective linear fractional programming problem. In *Abstract and applied analysis* (Vol. 2013, No. 1, p. 435030). Hindawi Publishing Corporation.

[37] Jain,S.,&Arya,N.(2013).Inverse optimization for linear fractional programming. *International Journal of Physical and Mathematical Sciences*, *4*(1).

[38] Stanojević, B. (2013). A note on 'Taylor series approach to fuzzy multiple objective linear fractional programming'. *Information Sciences*, *243*, 95-99.

[39] Veeramani, C., & Sumathi, M. (2014). Fuzzy Mathematical Programming approach for Solving Fuzzy Linear Fractional Programming Problem. *RAIRO-Operations research*, *48*(1), 109-122.

[40] Safaei, N.(2014). A new method for solving fully fuzzy linear fractional programming with a triangular fuzzy numbers. *Applied mathematics and computational intelligence (AMCI)*, *3*(1), 273-281

[41] Mishra, B., Nishad, A. K., & Singh, S.R. (2014). Fuzzy multi-fractional programming for land use planning in agricultural production system. *Fuzzy Information and Engineering*, *6*(2), 245-262.

[42] Dey, P. P., Pragmatic, S., & Giri, B. C. (2014). TOPSIS approach to linear fractional bi-level based on fuzzy goal programming. *Journal of industrial engineering international*, *10*(4), 173-184.

[43] Atanassov, K. (2016). Intuitionistic fuzzy sets. *International journal bioautomation*, *20*, 1.

[44] Chinnadurai, V., & Muthukumar, S. (2016). Solving the linear fractional programming problem in a fuzzy environment: Numerical approach. *Applied Mathematical Modelling*, *40*(11-12), 6148-6164.

[45] Veeramani, C., & Sumathi, M. (2016). A new method for solving fuzzy linear fractional programming problems. *Journal of intelligent & fuzzy systems*, *31*(3), 1831-1843.

[46] Singh, S. K., & Yadav, S. P. (2016). Fuzzy programming approach for solving intuitionistic fuzzy linear fractional programming problem. *International Journal of Fuzzy Systems*, *18*(2),263-269.

[47] Nayak, S., & Ojha, A. K. (2016, March). An approach to solve multi-objective linear fractional programming problem. In *Proceedings of Fifth International Conference on Soft Computing for Problem Solving: SocProS 2015, Volume 1* (pp. 711-721).

[48] Mitlif, R. J. (2016). Solving fuzzy fractional linear programming problems by ranking function methods, Journal of college of education, 1, 93-108.

[49] Chinnadurai, V., & Muthukumar, S. (2016). Solving the linear fractional programming problem in a fuzzy environment: Numerical approach. *Applied Mathematical Modelling*, *40*(11-12), 6148-6164.

[50] De, P. K., & Deb, M. (2016, December). Using goal programming approach to solve fuzzy multi-objective linear fractional programming problems. In *2016 IEEE international conference on computational intelligence and computing research (ICCIC)* (pp. 1-5).

[51] Chang, C. T. (2017). Fuzzy linearization strategy for multiple objective linear fractional programming with binary utility functions. *Computers & Industrial Engineering*, *112*, 437-446.

[52] Zhang, C., & Guo, P. (2017). A generalized fuzzy credibility-constrained linear fractional programming approach for optimal irrigation water allocation under uncertainty. *Journal of Hydrology*, *553*, 735-749.

[53] Chandrawat, R. K., Kumar, R., Garg, B. P., Dhiman, G., & Kumar, S. (2017, February). An analysis of modeling and optimization production cost through fuzzy linear programming problem with symmetric and right angle triangular fuzzy number. In *Proceedings of Sixth International Conference on Soft Computing for Problem Solving: SocProS 2016, Volume 1* (pp. 197-211).

[54] Das, S. K., & Mandal, T. (2017). A new model for solving fuzzy linear fractional programming problem with ranking function. *Journal of applied research on industrial engineering*, *4*(2), 89-96.

[55] Muruganandam, S., & Ambika, P. (2017). A new approach for solving fully fuzzy linear fractional programming problem. *Middle-East Journal of Scientific Research*, *25*(3), 647-652.

[56] Ehsani, E., Kazemi, N., Olugu, E. U., Grosse, E. H., & Schwindl, K. (2017). Applying fuzzy multi-objective linear programming to a project management decision with nonlinear fuzzy membership functions. *Neural computing and applications, 28*(8), 2193-2206.

[57] He, Y., He, Z., & Huang, H. (2017). Decision making with the generalized intuitionistic fuzzy power interaction averaging operators. *Soft Computing*, *21*(5), 1129-1144.

[58] Medina-González, S. A., Rojas-Torres, M. G., Ponce-Ortega, J. M., Espuña, A., & Guillén-Gosálbez, G. (2018). Use of nonlinear membership functions and the water stress index for the environmentally conscious management of urban water systems: Application to the city of Morelia. *ACS Sustainable Chemistry & Engineering*, *6*(6), 7752-7760

[59] Amer, A. H. (2018). An interactive intuitionistic fuzzy non-linear fractional programming problem. *Int. J. Appl. Eng. Res*, *13*(10), 8116-8125.

[60] Arya, R., Singh, P., & Bhati, D. (2018). A fuzzy based branch and bound approach for multi- objective linear fractional (MOLF) optimization problems. *Journal of computational science*, 24,54-64.

[61] Das, S. K., Edalatpanah, S. A., & Mandal, T. (2018). A proposed model for solving fuzzy linear fractional programming problem: Numerical Point of View. *Journal of computational science*, *25*, 367-375.

[62] Singh, S. K., & Yadav, S. P. (2018). Intuitionistic fuzzy multi-objective linear programming problem with various membership functions. *Annals of operations research*, *269*(1), 693-707.

[63] Goli, A., Zare, H. K., Tavakkoli-Moghaddam, R., & Sadeghieh, A. (2019). Hybrid artificial intelligence and robust optimization for a multi-objective product portfolio

problem Case study: The dairy products industry. *Computers & industrial engineering*, *137*, 106090.

[64] Debnath, I. P., & Gupta, S. K. (2019). Exponential membership function and duality gaps for I-fuzzy linear programming problems. *Iranian Journal of Fuzzy Systems*, *16*(2), 147-163.

[65] Arya, R., & Singh, P. (2019). Fuzzy efficient iterative method for multi-objective linear fractional programming problems. *Mathematics and Computers in Simulation*, *160*, 39-54.

[66] Yang, G., Li, X., Huo, L., & Liu, Q. (2020). A solving approach for fuzzy multi-objective linear fractional programming and application to an agricultural planting structure optimization problem. *Chaos, solitons & fractals*, *141*, 110352.

[67] Srinivasan, R. (2020). On solving fuzzy linear fractional programming in material aspects. *Materials Today: Proceedings*, *21*, 155-157.

[68] Bhattacharyya, R., & Mukherjee, S. (2020). Fuzzy membership function evaluation by non- linear regression: An algorithmic approach. *Fuzzy information and engineering*, *12*(4), 412-434.

[69] Veeramani, C., Sharanya, S., & Ebrahimnejad, A. (2020). Optimization for multi-objective sum of linear and linear fractional programming problem: fuzzy nonlinear programming approach. *Mathematical Sciences*, *14*(3), 219-233.

[70] Shakirullah, F. M., Ahammad, M. U., & Uddin, M. F. (2020). Profit optimization of an apparel industry in Bangladesh by linear programming model. *American Journal of Applied Mathematics*, *8*(4), 182-189.

[71] Maiti, I., Mandal, T., Pramanik, S., & Das, S. K. (2021). Solving multi-objective linear fractional programming problem based on Stanojevic's normalisation technique under fuzzy environment. *International journal of operational research*, *42*(4), 543-564.

[72] Ji, Y., & Lv, L. (2021). Two identification methods for a nonlinear membership function. *Complexity*, *2021*(1), 5515888.

[73] Mahajan, S., & Gupta, S. K. (2021). On optimistic, pessimistic and mixed approaches under different membership functions for fully intuitionistic fuzzy multiobjective nonlinear programming problems. *Expert Systems with Applications*, *168*, 114309.

[74] Shan, Z., Wu, W., Lei, Y., & Zhao, B. (2022). A new fuzzy rule based multi-objective optimization method for cross-scale injection molding of protein electrophoresis microfluidic chips. *Scientific Reports*, *12*(1), 13159.

[75] Oh, H., & Cho, J. (2022). A fractional programming model for improving multiplicative consistency of intuitionistic fuzzy preference relations. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, 30*(05), 879-896.

[76] Valipour, E., & Yaghoobi, M. A. (2022). On fuzzy linearization approaches for solving multi-objective linear fractional programming problems. *Fuzzy Sets and Systems*, *434*, 73-87.

[77] Stanojević, B. (2022). Extension principle-based solution approach to full fuzzy multi-objective linear fractional programming. *Soft Computing, 26*(11), 5275-5282.

[78] Abd El-Wahed Khalifa, H., Kumar, P., & Alodhaibi, S. S. (2022). Application of fuzzy random-based multi-objective linear fractional programming to inventory management problem. *Systems science & Control engineering*, *10*(1), 90-103.

[79] Das, S. K. (2022). An approach to optimize the cost of transportation problem based on triangular fuzzy programming problem. *Complex & intelligent systems*, *8*(1), 687-699.

[80] Agnihotri, S., & Dhodiya, J. M. (2022). VARIANTS OF GENETIC ALGORITHM TO SOLVE MULTI-OBJECTIVE INTERVAL SOLID TRANSPORTATION PROBLEM. *Palestine Journal of Mathematics*, *11*.

[81] Malik, M., & Gupta, S. K. (2022). An application of fully intuitionistic fuzzy multi-objective linear fractional programming problem in e-education system. *International Journal of Fuzzy Systems*, *24*(8), 3544-3563.

[82] Kumar, R., Khepar, J., Yadav, K., Kareri, E., Alotaibi, S. D., Viriyasitavat, W., & Dhiman, G. (2022). A systematic review on generalized fuzzy numbers and its applications: past, present and future. *Archives of Computational Methods in Engineering*, *29*(7), 5213-5236.

[83] Sahoo, D., Tripathy, A. K., & Pati, J. K. (2022). Study on multi-objective linear fractional programming problem involving pentagonal intuitionistic fuzzy number. *Results in Control and Optimization*, *6*, 100091

[84] Akarçay, Ö., & Pehlivan, N. Y. (2023). Fuzzy Multi-Objective Nonlinear Programming Problems Under Various Membership Functions: A Comparative Analysis. *Mühendislik Bilimleri ve Tasarım Dergisi*, *11*(3), 857-872.

[85] Gulia, P., Kumar, R., Viriyasitavat, W., Aledaily, A. N., Yadav, K., Kaur, A., & Dhiman, G. (2023). A systematic review on fuzzy-based multi-objective linear programming methodologies: concepts, challenges and applications. *Archives of Computational Methods in Engineering*, *30*(8), 4983-5022

[86] Sahni, A. K., Pandey, J. T., & Mishra, R. K. (2023). On fuzzy projective modules-its modified classes and estimation of projective dimension. *Palestine Journal of Mathematics*, *12*, 89.

[87] Moges, D. M., Mushi, A. R., & Wordofa, B. G. (2023). A new method for intuitionistic fuzzy multi-objective linear fractional optimization problem and its application in agricultural land allocation problem. *Information Sciences*, *625*, 457-475.

[88] Fathy, E., Ammar, E., & Helmy, M. A. (2023). Fully intuitionistic fuzzy multi-level linear fractional programming problem. *Alexandria Engineering Journal*, *77*, 684-694.

[89] Zhang, C., Yang, G., Wang, C., & Huo, Z. (2023). Linking agricultural water-food-environment nexus with crop area planning: a fuzzy credibility-based multi-objective linear fractional programming approach. *Agricultural Water Management*, *277*, 108135.

[90] Nayak, S., & Maharana, S. (2023). An efficient fuzzy mathematical approach to solve multi- objective fractional programming problem under fuzzy environment. *Journal of Applied Mathematics and Computing*, *69*(3), 2873-2899.

[91] Dombi, J., & Rigó, P. R. (2023). The construction of multidimensional membership functions and its application to feasibility problems. *Fuzzy Sets and Systems*, *469*, 108634.

[92] Agarwal, D., Singh, P., & El Sayed, M. A. (2023). The Karush–Kuhn–Tucker (KKT) optimality conditions for fuzzy-valued fractional optimization problems. *Mathematics and Computers in Simulation*, *205*, 861-877.

[93] Ebrahimnejad, A., Ghiyasi, M., Nikhbakht, M., & Najjarian, H. (2023). FUZZY GOAL PROGRAMMING APPROACH FOR IDENTIFYING TARGET UNIT IN COMBINED-ORIENTED DEA MODELS: APPLICATION IN BANK INDUSTRY. *Economic Computation & Economic Cybernetics Studies & Research*, *57*(1).

[94] Chauhan, A., Mahajan, S., Ahmad, I., & Al-Homidan, S. (2023). On fuzzy linear fractional programming problems via α-cut-based method with application in transportation sector. *Symmetry*, *15*(2), 419.

[95] Shivani, Rani, D., Ebrahimnejad, A., & Gupta, G. (2024). Multi-objective non-linear programming problem with rough interval parameters: an application in municipal solid waste management. *Complex & Intelligent Systems*, *10*(2), 2983-3002.

[96] Elaibi, W. M., & Nasseri, H. S. (2024). A new approach to multi objective fuzzy fractional linear programming. *Journal of applied research on industrial engineering*, *11*(4), 635-651.

[97] Al Jerbi, T., & Muamer, M. (2024, October). Solving linear fractional programming problems with triangular LR fuzzy numbers coefficients. In *Sebha University Conference Proceedings* (Vol. 3, No. 2, pp. 227-232).

[98] Gulia, P., & Kumar, R. (2024, February). Distance-based methods for multi-objective optimization problems: Fuzzy and non-fuzzy approaches. In *AIP Conference Proceedings* (Vol. 2986, No. 1, p. 030139).

[99] Gulia, P., Sharma, V., Rakhra, M., Jairath, K., Suri, S., & Kumar, R. (2024, September). Transportation Planning Decision-Making Using Intuitionistic Fuzzy Programming. In *2024 7th International Conference on Contemporary Computing and Informatics (IC3I)* (Vol. 7, pp. 1-6).

[100] Akhtar, N., & Islam, S. (2024). Linear fractional transportation problem in bipolar fuzzy environment. *Results in Control and Optimization*, *17*, 100482.

[101] Bas, S. A., & Ozkok, B. A. (2024). An iterative approach for the solution of fully fuzzy linear fractional programming problems via fuzzy multi-objective optimization. *AIMS Math*, *9*(6), 15361-15384.

[102] Karthick, S., & Saraswathi, A. (2025). A METHOD FOR SOLVING INTUITIONISTIC FUZZY LINEAR FRACTIONAL PROGRAMMING PROBLEM. Dynamics of Continuous, Discrete and Impulsive Systems Series B: Applications and Algorithms.32,59-69

[103] Malik, M., & Gupta, S. K. (2025). On optimistic, pessimistic and mixed fuzzy-programming based approaches to solve multi-objective fully intuitionistic fuzzy linear fractional programming problems. *Annals of Operations Research*, *346*(2), 1399-1443.

[104] Kaur, S., Mahajan, S., & Chauhan, A. (2025). Modified γ-operator technique for intuitionistic fuzzy multi-objective nonlinear programming problems with application in agriculture. *Expert Systems with Applications*, *273*, 126874

[105] Kumar, A., & Mishra, B. (2025). A new method to solve multi-objective linear fractional programming problem in fuzzy stochastic environment. *Mathematical Modelling and Analysis*, *30*(3), 480-503.

Publications And Conferences:

1. R. Kaur, R. Kumar, and P. Gulia, “Multi-objective Fractional Programming Problem Analysis in the Manufacturing Sector Using Situation-Based S-Shaped Nonlinear Fuzzy Numbers” *Palestine Journal of Mathematics* 14 (2025).**(Published)**
2. R. Kaur, R. Kumar, ”Multi-Objective Linear Fractional Programming Problem Solving with Hybrid Extended Intuitionistic Fuzzy Approach” *Mathematics and Computational Sciences (July 2025 communicated ,Under review)*
3. R. Kaur, R. Kumar, V. Sharma, V. Thakur,” Hybrid Weighted-sum with Modified Dinkelbach Approach to Solve Linear Fractional Programming Problems” *Palestine Journal of Mathematics* .(Communicated)
4. R. Kaur, R. Kumar, V. Thakur, “An Approach to Linear Fractional Programming Using a Hybrid Fuzzy Interval-Regression Predictive Model” Edelweiss applied science and technology.(communicated)
5. R. Kaur, R. Kumar, and H. Günerhan, “Study of the Ranking-function-based Fuzzy Linear Fractional Programming Problem: Numerical Approaches,” in *Advance Numerical Techniques to Solve Linear and Nonlinear Differential Equations*, River Publishers, 2024.(Published-Book Chapter)
6. Participated and presented paper entitled “A Study of the Fuzzy Linear Fractional Programming Problem Utilizing Rank Based Fuzzy Number” in International Conference on communication, security and artificial intelligence, ICCSAI-2022 By IEEE held on 23rd and 24th Dec.2022.
7. Participated and presented paper entitled “A Parametric Approach with $\alpha -$ cuts to Solve Fuzzy Linear Fractional Programming Problem with Fuzzy Coefficients” in 4th International Conference on Recent Advances In Fundamental And Applied Sciences (RAFAS 2023).