FIBONACCI NUMBERS, POLYNOMIALS AND ITS SEQUENCES

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in

Mathematics

By

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Declaration

I, Priyanka, declare that the thesis entitled "FIBONACCI NUMBERS, POLYNOMIALS AND ITS SEQUENCES", is a document of innovative, unique, and unbiased work consummated underneath the benign supervision of Dr. Vipin Verma, Associate Professor in Mathematics Department, School of Chemical Engineering and Physical Sciences, Lovely Professional University, Phagwara, hereby, submitted in the partial fulfillment for the award of Ph.D. degree.

I, further, declare that the thesis equipped is the original study conducted by me and has not been put forward by any other university.

Priyanka 12/07/2022

Signature of Scholar

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Certified that research embodied in this thesis entitled "FIBONACCI NUMBERS, POLYNOMIALS AND ITS SEQUENCES" has been done by Ms. Priyanka at Lovely Professional University, Phagwara for the award of Ph.D. degree. The research work has been carried out under my supervision and is to- my satisfaction.

Dated: 12 July, 2022

Signature of Supervisor

Acknowledgment

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F_n	:	n^{th} Fibonacci Number
L_n	:	n^{th} Lucas Number
$\emptyset$	:	Golden Ratio
J_n	:	n^{th} Jacobsthal Number
j_n	:	n^{th} Jacobsthal-Lucas Number
Q_n	:	n^{th} Pell-Lucas Number
$H_n(x,y,z)$	:	n^{th} Trivariate Fibonacci Polynomial
$\lfloor . \rfloor$	:	Greatest Integer
$det(A)$	:	determinant of a matrix A
$\lVert . \rVert$	:	denotes the distance from the nearest integer

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Abstract

Fibonacci numbers, one of the God's best-gifted numbers, are amazing numbers researched by Leonardo of Pisa that play a crucial role in our lives. Fibonacci numbers are the result of the famous Rabbit problem demonstrated by Leonardo of Pisa, which we discuss later in the subsequent chapters of this thesis. In addition to being a part of our daily lives, these numbers also have a variety of applications in nature, music, etc. that can't be expressed in a single word.

As a whole, the thesis focuses on the concept of these beautiful divinely gifted Fibonacci numbers and the polynomials that surround these numbers. This thesis is composed of nine chapters. In the first chapter, a general introduction to the Fibonacci numbers constituting history and their applications in various fields is described. Also, we briefly recall a few definitions and well-known results of the Fibonacci numbers which fulfill the minimum prerequisite for the subsequent chapters. The section of literature review lighting on the work done by the various researchers in the field of Fibonacci numbers and its associated polynomials, is also included in this chapter. In the review, the research gap has been identified. Further, the objectives and methodology to achieve these gaps have been detailed in this chapter.

The remaining chapters are an attempt to discuss the behavior and different properties of sequences of polynomials based on Fibonacci numbers. In this thesis, we mainly work on the sequences of Generalized Fibonacci polynomials, 2-Fibonacci numbers, Generalized Lucas polynomials, Jacobsthal polynomials, Jacobsthal-Lucas polynomials, Trivariate Fibonacci polynomials, and Pell-Lucas numbers. Also, we determine the number of non-negative integral solutions of a Diophantine equation involving Pell-Lucas numbers. In order to achieve our objective, we employ various methodologies. In this thesis, we develop various summation formulae for sequences of the Generalized Fibonacci polynomials and 2-Fibonacci numbers by approaching recursive methodology. After that, an explicit formula is established by using the concept of matrices to find out the general term of "2-Fibonacci numbers and Trivariate Fibonacci polynomials". Further, reciprocal sum of "Generalized Fibonacci

polynomials, Generalized Lucas polynomials, Jacobsthal polynomials, Jacobsthal-Lucas polynomials" and their respective squares developed. Additionally, various properties involving the relationship of 2-Fibonacci sequences of numbers with the sequence of Fibonacci numbers, Cassini's identity, etc. have been demonstrated using the concept of congruences. A fractional differential equation is solved by, "Lucas polynomials and Pell-Lucas polynomials" by the concept of matrices. A matrix is developed to generate the sequence of the derivatives of Fibonacci polynomials. Also, we discuss various properties of the sequence obtained by differentiating one time to the Fibonacci polynomials. Further, we represent various sequences graphically using GeoGebra software. Also, using GeoGebra software, we discuss the sequence of "Fibonacci numbers and the sequences of 2-Fibonacci numbers" approaching the Golden ratio.

Various summation formulae has been developed for a Generalized sequence of the Fibonacci polynomials and its respective first-order derivatives with even indices, odd indices, and alternating summation formulae. In the next step, we discuss some particular cases for the Fibonacci numbers, Pell numbers, Tetrabonacci numbers by changing the initial conditions.

In the following ordinary differential equation, we discuss the convergence of its solutions:

$$y'' - y' - y = 0$$

Afterwards, an analysis of the extremum values for the Fibonacci polynomials with one variable has been done by using Descartes' Rule of the sign.

By adopting similar approach, we find the extremum values for Fibonacci polynomials of two variables or Bivariate Fibonacci polynomials. Thereafter, these polynomials have been graphically visualized using MATLAB.

Our next step is to establish an explicit formula by using the concept of matrices in order to determine the general term of sequences of the 2-Fibonacci numbers. Furthermore, by incorporating the concept of congruences, various properties of the 2-Fibonacci sequences of numbers including the generating function, relationship with

Fibonacci numbers, Cassini's identity have been explored. Afterward, we extend the concept of summation formulae for the 2-Fibonacci sequence of numbers. In the categories of closed forms of the summation formulae, the characteristics of these sequences along with their squares are developed and validated. We develop various summation formulae for 2-Fibonacci sequences of numbers having even indices, odd indices, alternating summation formulae.

After this, we continue our research work on "Generalized Fibonacci polynomials, Generalized Lucas polynomials, Jacobsthal polynomials, and Jacobsthal-Lucas polynomials". Furthermore, the integral sum of reciprocity of these polynomials and their respective squares having even indices, developed through recursive methodology. As a result, we formulate various integral sums involving the reciprocal of "Generalized Fibonacci polynomials and Generalized Lucas polynomials", having even indices. Inequalities regarding the reciprocal sum from the Jacobsthal polynomials, Jacobsthal-Lucas polynomials and their respective squares with even indices have been developed. Also, we discuss multiple results using these polynomials to accomplish our goal. We establish multiple relationships between "Generalized Fibonacci polynomials and Generalized Lucas polynomials, Jacobsthal polynomials and Jacobsthal-Lucas polynomials respectively".

After that, we work on the sequence of Trivariate Fibonacci polynomials that follows a third-order recursive relation. Using the concept of matrices, an explicit formula is developed to find out the general terms of Trivariate Fibonacci polynomials. We present a matrix whose successive determinant generates the Trivariate Fibonacci polynomials. Moreover, by using matrix methods, we derive Binet's formula and some other properties for Trivariate Fibonacci polynomials. After that, a few remarks have been made regarding n^{th} Generalized Lucas numbers. The next chapter of this thesis is devoted to solving a fractional differential equation by using Lucas and Pell-Lucas polynomials.

A fractional differential equation

$$\begin{cases} D^{\alpha}(x) + y^{(k)}(x) + y(x) = f(x) \\ y^{(r)}(0) = c_r \end{cases},$$

where $r = 0, 1, 2, \dots, m-1; k = 0, 1, 2, \dots, m; m-1 < \alpha \leq m$,

is solved by, “Lucas polynomials and Pell-Lucas polynomials” using matrices. In addition, we construct a matrix having elements as “the coefficients appearing in the expansion of the derivative of Fibonacci polynomials and the sum of the elements in the same row gives the first-order derivative of the classical Fibonacci sequence”. Furthermore, we discuss some properties of the sequence obtained by differentiating the sequence of Fibonacci polynomials.

After that, towards the ending chapter of this thesis, we work on a Diophantine equation

$$Q_n - Q_m = 2^a,$$

where Q_n and Q_m are Pell-Lucas numbers with $n > m \geq 0$. By using various linear forms for an upper bound, we completely solve the above Diophantine equation.

Chapter 1

General Introduction

1.1 Introduction

The Italian mathematician Leonardo of Pisa (1170-1250) or Fibonacci was a notable mathematician in the Middle Ages. "Fibonacci numbers", which bear his name, is the best-known work of Fibonacci. Fibonacci's mathematical background began during his many visits to North Africa, where he was introduced to the early works of algebra, arithmetic, and geometry. He also traveled to countries located in the Mediterranean and studied the mathematical systems that they were practicing.

"*Fibonacci published his first book entitled* ***Liber Abaci*** *at the age of thirty, which discusses the usefulness of the Hindu numerals and multiple mathematical problems. Liber Abaci was published for the first time in 1202 and later revised in 1228*" [1, 2]. It is said that the work of Egyptian mathematician Abu Kamil inspired Fibonacci's work in *Liber Abaci.* The beginning of his book has a statement about the Hindu-Arabic number system: "*Nine Indian figures are noted: 9, 8, 7, 6, 5, 4, 3, 2, 1. With these nine figures, and with symbol 0, any number may be written*" [3]. The problems in the book were able to illustrate for the first time the benefits of the new Hindu-Arabic numeral system. During Fibonacci's time, *Liber Abaci* was considered to be a complete source of arithmetical knowledge. The publication of this book inspired additional research in algebra and arithmetic and continued to serve as a key mathematics source for hundreds of years.

"*We now shift the direction to Indian mathematicians and the role of they played in the Fibonacci numbers. While the Fibonacci numbers are named after Leonardo Fibonacci, who was described in detail above, it is interesting to note that knowledge about these numbers occurred long before his time. The sequence of Fibonacci numbers originates from former India. Singh claims that Indian mathematician Pingala was the first to possess knowledge of the Fibonacci numbers. It is speculated that he lived* *sometimes around 400 B.C. Acarya Virashanka, who lived between 600 and 800 A.D., is said to have been the first Indian mathematician to give a written representation of the Fibonacci numbers. Another significant figure in the role of Fibonacci numbers is Gopala, who was born sometime before 1135 A.D. Archarya Hemachandra, a great Jain writer, describes an estimation of variations in matra-vrttas in Chandonusasana. In Chandonusasana, his rule is translated and quoted from* [4] *as follows: "Sum of the last and the last but one, number of variations of the matra-vrttas coming afterward." (Matras-Vrttas are meters in which the quantity of morae remains constant and the numbers of letters are random). He continues, "From amongst the numbers 1, 2, etc. those that are last and the last apart from one is 3 which is kept afterward and is the number of variations (of meter) having 3 matras. The sum of 3 and 2 is 5, which is kept afterward and the number of variations (of the meter) having 4 matras"* [5].

Figure 1.1: Hemachandra (left) and Fibonacci (right)

The real problem demonstrated by Fibonacci (in the year 1202) was the number of pairs of rabbits that may arise under ideal circumstances given by

- Initialize with a pair of neonate bunnies
- 1 month before maturation
- A month before pregnancy
- Mimic a pair of neonates

- Intimates again and so forth
- None of the rabbits die

Now, the question arises that how many rabbit pairs after one year? This problem can be solved by constructing a table:

Table 1.1: Rabbit Problem

Month	***Youth Pairs***	***Matured Pairs***	***Total***
January	1	0	1
February	0	1	1
March	1	1	2
April	1	2	3
May	2	3	5
June	3	5	8
July	5	8	13
August	8	13	21
September	13	21	34
October	21	34	55
November	34	55	89
December	55	89	144
January	89	144	233

As a result of this, the above pairs of rabbits form a sequence of numbers referred to as Fibonacci numbers generated by,

$$F_{n+1} = F_n + F_{n-1};\ n = 1, 2, 3, \ldots, \tag{1.1}$$

with two initial seeds $F_0 = 0, F_1 = 1$.

Consequently, Fibonacci numbers are the sum of two procedural terms [3] with two initial seeds $F_0 = 0, F_1 = 1$. Now, observe that the last 2nd row of the above table is the sequence of these numbers.

Fibonacci numbers are generated [3] via Binet's formula given as:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where α and β satisfies the following equation

$$r_\gamma^2 - r_\gamma - 1 = 0$$

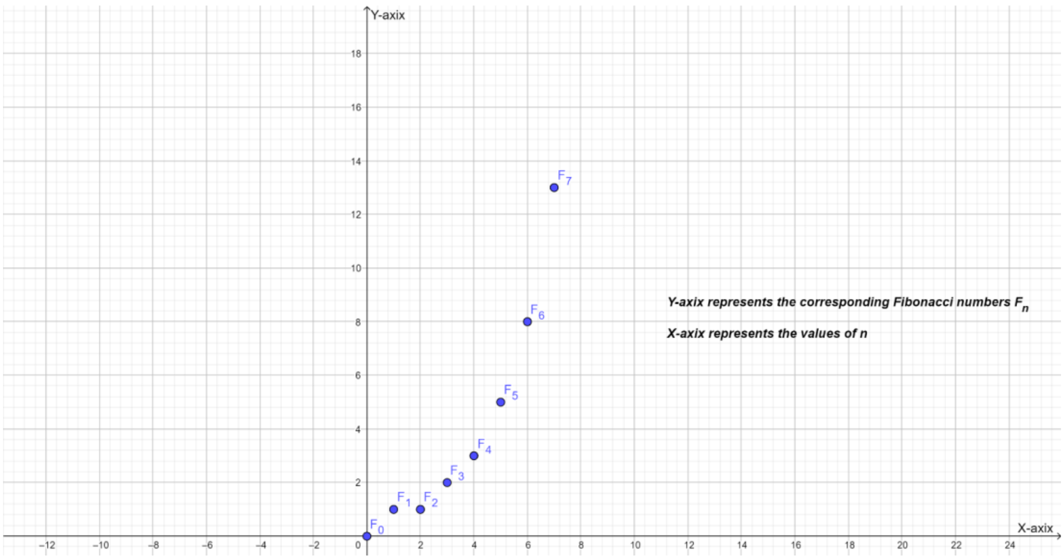


Figure 1.2: Fibonacci Numbers

Furthermore, following is generating function $g_F(u)$ for Fibonacci numbers:

$$\sum_{n=0}^{\infty} F_n u^n = g_F(u) = \frac{u}{1-u-u^2},$$

where F_n is n^{th} Fibonacci number and a generating function $g_F(u)$, first introduced in 1730, to solve the general linear recurrence problem of a sequence, say $\{c_n\}$ is a power series expansion having coefficients as the terms of a sequence $\{c_n\}$ and is given by

$$g_F(u) = \sum_{n=0}^{\infty} c_n u^n$$

1.2 Applications of Fibonacci Numbers

Besides mathematics, Fibonacci numbers are crucial to our daily life, nature and have many applications in various fields. There are many flowers of which the patella forms a sequence of the Fibonacci numbers. For instance, Lily has three petals, Buttercups have five, Delphiniums, daisies and aster have 8, 13, and 21 respectively, chicory and 34 petals in plantain, pyrethrum, etc. Additionally, some flowers exhibit spirals that are Fibonacci numbers when they are counted in a clockwise or anticlockwise.

Figure 1.3: Fibonacci Numbers in Flowers

Fibonacci numbers and music are also closely related to each other. The Fibonacci numbers are depicted by looking at the keyboard of a piano [6]. Pineapple also exhibits these numbers [1].

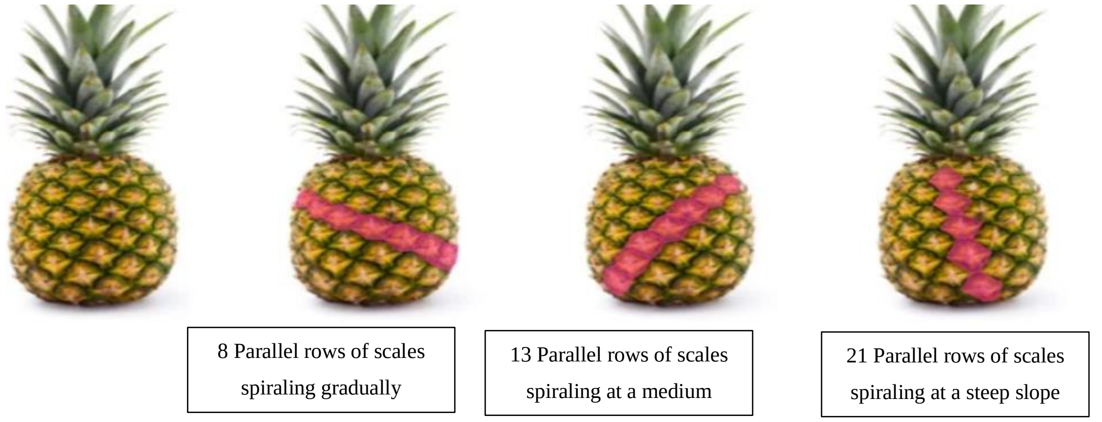


Figure 1.4: Fibonacci Numbers in Pineapples

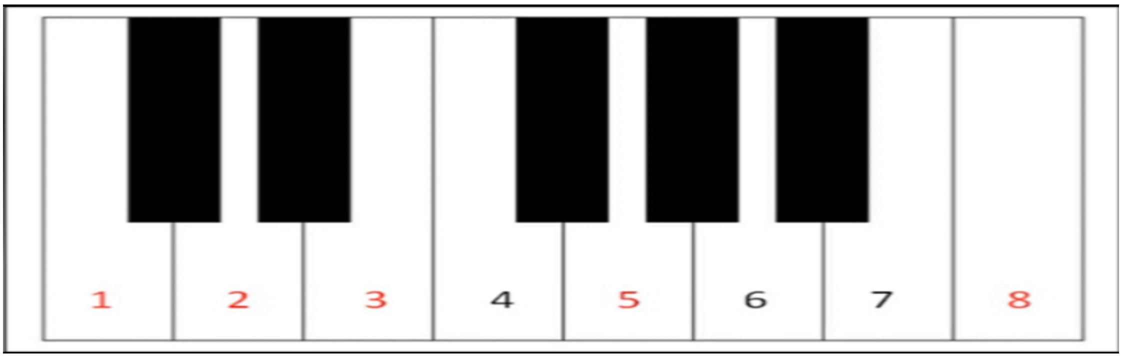


Figure 1.5: Piano and the Fibonacci Numbers

There is a lot of contribution of Fibonacci numbers in our life cycle, family trees of honey bees, growing points in trees, and in the various fields which we can't state in single words.

1.3 Basic Terminology

To accomplish our goal, some basic terminologies are used. We recall few definitions and well-known results, some of which are pinpointed as per our requirements.

1.3.1 Golden Ratio

Two numbers a and b are in the golden ratio if their proportion is equal to that of their sum to the larger among them say a, symbolized by phi ϕ [7].

Thus, a and b are forms golden ratio if

$$\frac{a+b}{a} = \frac{a}{b} = \phi$$

(1.2)

Put $\frac{a}{b} = \phi$, then (1.2) becomes

$$1 + \frac{1}{\phi} = \phi$$

Therefore, $\phi^2 - \phi - 1 = 0$.

On solving this quadratic equation, we obtain two roots given by

$$\frac{(1+\sqrt{5})}{2} = 1.6180339887\ \ldots,$$

and

$$\frac{(1-\sqrt{5})}{2} = -0.6180339887\ \ldots$$

Since ϕ is the ratio of lengths, so its value should be positive, so neglecting the negative value. Therefore,

$$\phi = \frac{(1+\sqrt{5})}{2}$$

By taking the ratio of two consecutive Fibonacci numbers larger divided by smaller, the sequence obtained approaches to ϕ.

Thus,

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi$$

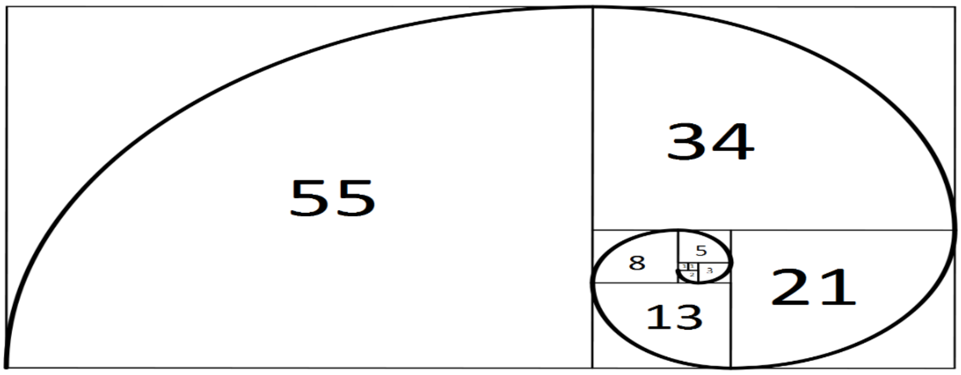


Figure 1.6: Fibonacci Spiral

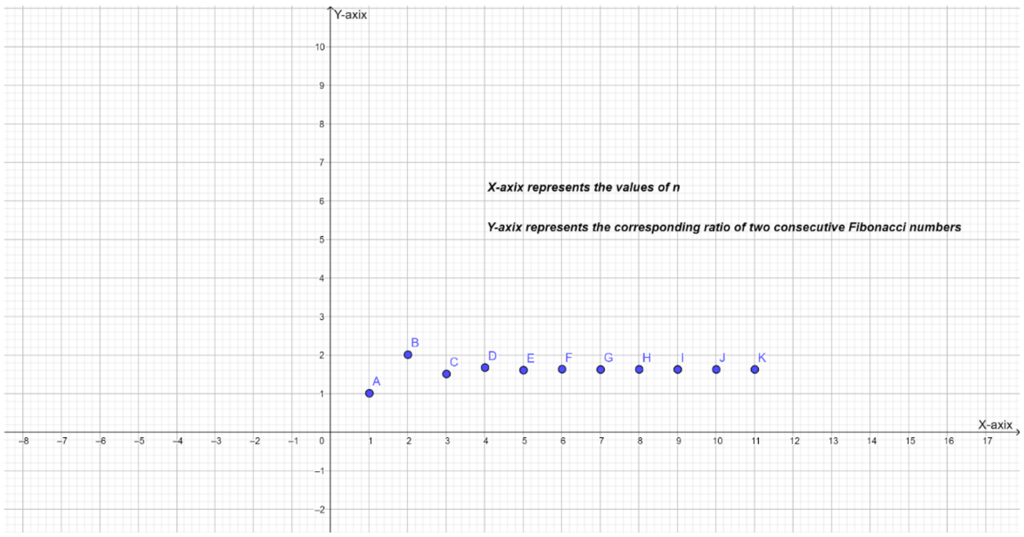


Figure 1.7: Fibonacci Numbers Approaching to Golden Ratio

1.3.2 Extension of Fibonacci Numbers to Negative Integers

Abramovich [8] extended the concept of Fibonacci sequence to the set of integers by considering a problem based on Fibonacci-like sequence defined by

$$f_{n+1} = f_n + f_{n-1}, \tag{1.3}$$

with $f_1 = x$ and $f_2 = y$ and n is an integer ≥ 2.

Now the problem was:

Problem [8]: ***"Given integer N as the 5th term of a Fibonacci-Like sequence, find all possible non-negative initial values of such a sequence."***

To solve the above problem, author again consider a problem with $N = 20$. Now, the solution of the above problem lies in solving the following Diophantine equation for integers $x, y \geq 0$:

$$2x + 3y = 20$$

	A	B	C	D	E	F	G	H	I	J
1										
2										
3		n	F_{n-2}	F_{n-1}	N	Total				
4		5	2	3	20	4				
5	x	y								
6	0									
7	1	6	7	13	20	33	53	86	139	225
8	2									
9	3									
10	4	4	8	12	20	32	52	84	136	220
11	5									
12	6									
13	7	2	9	11	20	31	51	82	133	215
14	8									
15	9									
16	10	0	10	10	20	30	50	80	130	210

Figure 1.8: Solutions of $2x + 3y = 20$

On the basis of the solutions of above Diophantine equation, following Fibonacci-like sequences with 20 as the 5th terms can be possible,

$$10, 0, 10, 10, 20, 30, \ldots$$

$$7, 2, 9, 11, 20, 31, \ldots$$

$$4, 4, 8, 12, 20, 32, \ldots$$

$$1, 6, 7, 13, 20, 33, \ldots$$

Abramovich arranged the above sequences as below in table 1.2:

Table 1.2: Quadruples forming Arithmetic Sequences

10	0	10	10	20	30	50	80	130	210
7	2	9	11	20	31	51	82	133	215
4	4	8	12	20	32	52	84	136	220
1	6	7	13	20	33	53	86	139	225
-3	**2**	**-1**	**1**	**0**	**1**	**1**	**2**	**3**	**5**

and he observes that, "these quadruples form arithmetic sequence, the differences of which are displayed in the bottom row of table 1.2. One can observe that to the right of the zero difference (after all the four Fibonacci-like sequences met at the number 20), consecutive Fibonacci numbers appear. Furthermore, to the left of the zero difference, the absolute values of the differences are consecutive Fibonacci numbers as well. This observation, motivated the author to an idea of extending Fibonacci numbers to negative subscripts through the following definition:

$$F_0 = 0, F_{-n} = (-1)^{n+1} F_n$$

One can note that to the right of zero all Fibonacci numbers are positive and to the left of zero Fibonacci numbers alternate signs. In order to generate Fibonacci numbers in both directions, one has to have the triple of numbers$(1, 0, 1)$. Then, to the right and to the left of zero, the recursions

$$F_{n+1} = F_n + F_{n-1}, \text{ and } F_{n-1} = F_{n+1} - F_n,$$

should be used, respectively". Thus, by combining above these, Abramovich extends Fibonacci sequence to the set of integers through the following definition

$$F_{\mathfrak{p}+1} = F_{\mathfrak{p}} + F_{\mathfrak{p}-1},$$

with $F_0 = 0, F_1 = F_{-1} = 1$, and $\mathfrak{p}$ is any integer.

1.3.3 Fibonacci Polynomials

Belgian mathematician Eugene Charles Catalan developed the concept of Fibonacci numbers in 1883 by studying Fibonacci polynomials $\{F_w(x)\}_{w\geq 0}$ [9] defined by

$$F_w(x) = xF_{w-1}(x) + F_{w-2}(x), \tag{1.4}$$

with $F_0(x) = 0, F_1(x) = 1$, for integral values of $w \geq 2$.

When $x = 1$, then the above sequence of polynomials becomes that of the Fibonacci numbers. Also, generating function $g_F(u)$ is

$$\sum_{w=0}^{\infty} F_w(x)u^w = g_F(u) = \frac{u}{1 - xu - u^2}$$

If $\alpha = \frac{x+\sqrt{x^2+4}}{2}$, and $\beta = \frac{x-\sqrt{x^2+4}}{2}$, then for $w \geq 0$,

$$F_w(x) = \frac{\alpha^w - \beta^w}{\alpha - \beta}$$

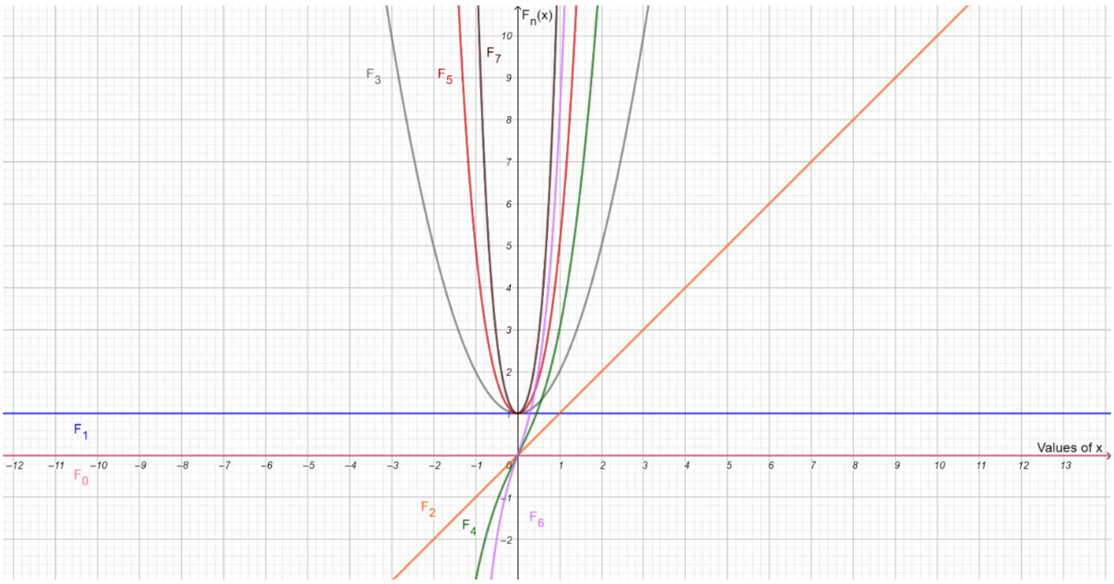


Figure 1.9: Fibonacci Polynomials

1.3.4 $h(x)$ −Fibonacci Polynomials

Nalli and Haukkanen [10] generalized Fibonacci polynomials (1.4) by introducing $h(x)$ −Fibonacci polynomials for a real coefficient polynomial $h(x)$ described as

$$F_{h,\mathfrak{e}+2}(x) = h(x)F_{h,\mathfrak{e}+1}(x) + F_{h,\mathfrak{e}}(x), \tag{1.5}$$

where $\mathfrak{e} \geq 0$, and $F_{h,0}(x) = 0, F_{h,1}(x) = 1$.

On taking $h(x) = 1$, sequence given by (1.5) becomes that of Fibonacci numbers.

If $h(x) = x$, then the above sequence become that of Fibonacci polynomials respectively. And, the generating function $g_F(\mathfrak{u})$ is

$$\sum_{\mathfrak{e}=0}^{\infty} F_{h,\mathfrak{e}}(x)\mathfrak{u}^{\mathfrak{e}} = g_F(\mathfrak{u}) = \frac{\mathfrak{u}}{1 - h(x)\mathfrak{u} - \mathfrak{u}^2},$$

and the $\mathfrak{e}^{th}$ term of this sequence is:

$$F_{h,\mathfrak{e}}(x) = \frac{\alpha_1^{\mathfrak{e}} - \alpha_2^{\mathfrak{e}}}{\alpha_1 - \alpha_2},$$

where $\alpha_1 = \frac{h(x)+\sqrt{h^2(x)+4}}{2}$, and $\alpha_2 = \frac{h(x)-\sqrt{h^2(x)+4}}{2}$.

1.3.5 Bivariate Fibonacci polynomials

The concept of Fibonacci polynomials to two variables was extended by introducing Bivariate Fibonacci polynomials by Catalan [11] that are defined by

$$F_n(\mathfrak{z}_1, \mathfrak{t}_1) = \mathfrak{z}_1 F_{n-1}(\mathfrak{z}_1, \mathfrak{t}_1) + \mathfrak{t}_1 F_{n-2}(\mathfrak{z}_1, \mathfrak{t}_1), \tag{1.6}$$

for integral values of $n \geq 2$, with $F_0(\mathfrak{z}_1, \mathfrak{t}_1) = 0, F_1(\mathfrak{z}_1, \mathfrak{t}_1) = 1$. By using $\mathfrak{z}_1 = \mathfrak{t}_1 = 1$, it becomes the sequence of the Fibonacci numbers. Bivariate Fibonacci polynomials are generated by the following Binet's formula:

$$F_n(\mathfrak{z}_1, \mathfrak{t}_1) = \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2},$$

where $\alpha_1 = \frac{\mathfrak{z}_1+\sqrt{(\mathfrak{z}_1)^2+4t_1}}{2}$, and $\alpha_2 = \frac{\mathfrak{z}_1-\sqrt{(\mathfrak{z}_1)^2+4t_1}}{2}$.

1.3.6 Generalized Fibonacci Polynomials

Researchers introduced following Generalized Fibonacci polynomials [12] by extending $h(x)$ −Fibonacci polynomials:

$$F_{k,q,\mathfrak{n}+1}(g) = k(g)F_{k,q,\mathfrak{n}}(g) + q(g)F_{k,q,\mathfrak{n}-1}(g), \text{ for } \mathfrak{n} = 2, 3, \ldots, \quad (1.7)$$

with $F_{k,q,0}(g) = 0$, $F_{k,q,1}(g) = 1$, for real coefficients polynomials $k(g)$ and $q(g)$. For $k(g) = q(g) = 1$, sequence given by (1.7) becomes that of Fibonacci numbers. Further,

$$\sum_{\mathfrak{n}=0}^{\infty} F_{k,q,\mathfrak{n}}(g)\mathfrak{u}^{\mathfrak{n}} = g_F(\mathfrak{u}) = \frac{\mathfrak{u}}{1 - k(g)\mathfrak{u} - q(g)\mathfrak{u}^2}$$

1.3.7 Lucas Sequence

Now, the n^{th} term of Fibonacci numbers which is named as Binet's Formula [3] can be written as

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where $\alpha = \frac{1+\sqrt{5}}{2}$, and $\beta = \frac{1-\sqrt{5}}{2}$.

Now,

$$\alpha^2 = \frac{3 + \sqrt{5}}{2},$$

$$\alpha^3 = \frac{4 + 2\sqrt{5}}{2},$$

$$\alpha^4 = \frac{7 + 3\sqrt{5}}{2},$$

and so on.

Thus, one can see that coefficient of $\sqrt{5}$ in $2\alpha^n$ forms a sequence of the Fibonacci numbers while the remaining terms arises a new sequence given by

$$1, 3, 4, 7, \ldots$$

These sequences motivated the idea of Lucas numbers by supplying the initial term as 2. Thus, Lucas numbers [13], derived name from the mathematician François Édouard Anatole Lucas, follows a recursive relation similar to that of Fibonacci numbers, but differ only in seeds. Thus, Lucas numbers follow relation:

$$L_{n+2} = L_{n+1} + L_n, \tag{1.8}$$

for $n = 0, 1, 2, \ldots$, with $L_0 = 2$, and $L_1 = 1$.Thus, the Lucas sequence is

$$2, 1, 3, 4, 7, \ldots$$

Lucas sequences can be produced [13] using Binet's formula:

$$L_n = \alpha^n + \beta^n \,; n \geq 0$$

where α and β satisfies:

$$\nu^2 - \nu - 1 = 0$$

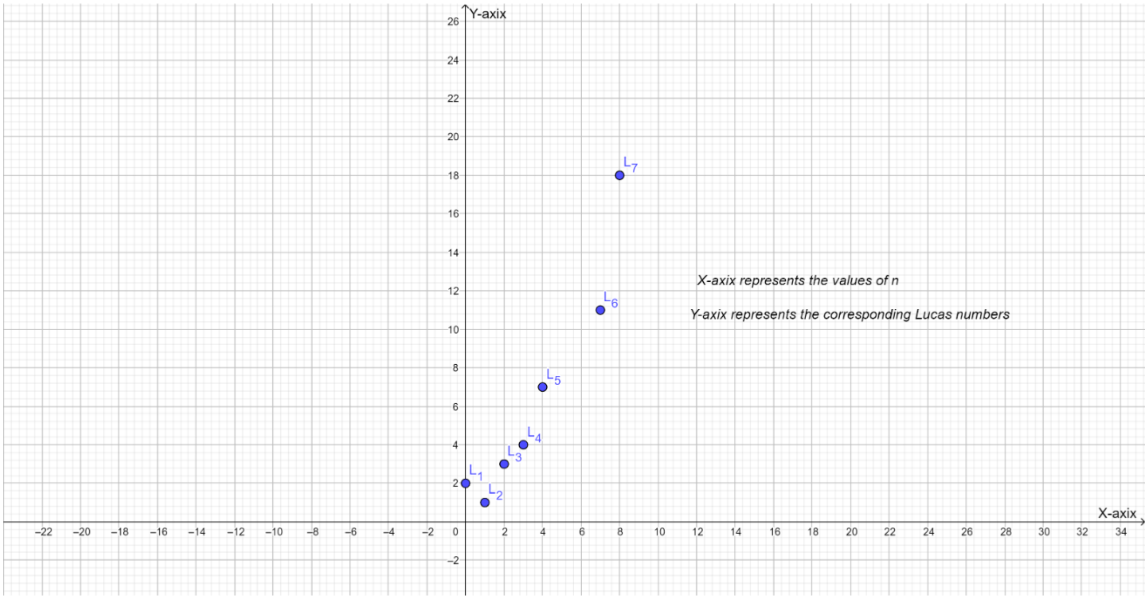


Figure 1.10: Lucas Numbers

1.3.8 Lucas Pseudoprimes

For any positive integers $\mathcal{R}_1$ and $\mathcal{S}_1$ with $\mathcal{R}_1 > 0$, and $D = \mathcal{R}_1^2 - 4\mathcal{S}_1$, the corresponding Lucas sequences [14] defined by

$$U_n(\mathcal{R}_1, \mathcal{S}_1) = \mathcal{R}_1 U_{n-1}(\mathcal{R}_1, \mathcal{S}_1) - \mathcal{S}_1 U_{n-2}(\mathcal{R}_1, \mathcal{S}_1),$$

and

$$V_n(\mathcal{R}_1, \mathcal{S}_1) = \mathcal{R}_1 V_{n-1}(\mathcal{R}_1, \mathcal{S}_1) - \mathcal{S}_1 V_{n-2}(\mathcal{R}_1, \mathcal{S}_1),$$

with $U_0(\mathcal{R}_1, \mathcal{S}_1) = 0, U_1(\mathcal{R}_1, \mathcal{S}_1) = 1, V_0(\mathcal{R}_1, \mathcal{S}_1) = 2$, and $V_1(\mathcal{R}_1, \mathcal{S}_1) = \mathcal{R}_1$, and $n = 2, 3, 4, \ldots$

Let $\left(\frac{D}{n}\right)$ be the Jacobi symbol defined by:

$$\left(\frac{D}{n}\right) = \left(\frac{D}{q_1}\right)^{\alpha_1} \left(\frac{D}{q_2}\right)^{\alpha_2} \ldots \left(\frac{D}{q_k}\right)^{\alpha_k},$$

where $n = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_k^{\alpha_k}$ is the prime factorization of n. For odd prime q, the Legendre symbol $\left(\frac{D}{q}\right)$ is defined by

$$\left(\frac{D}{q}\right) = \begin{cases} 0 \text{ if } D \equiv 0(mod\ q) \\ 1 \text{ if } D \not\equiv 0(mod\ q), \text{and } D \equiv x^2(mod\ q), \text{for some integer } x \\ -1 \text{ if } D \not\equiv 0(mod\ q), \text{there is no such } x: D \equiv x^2(mod\ q) \end{cases}$$

A Lucas pseudoprime for a composite integer n satisfy:

$$U_{\delta(n)} \equiv 0(mod\ n),$$

where $\delta(n) = n - \left(\frac{D}{n}\right)$.

For instance, 119 is a Lucas pseudoprime if, $\mathcal{R}_1 = 3, \mathcal{S}_1 = -1$, and $D = 13$.

1.3.9 n^{th} Generalized Lucas Numbers

n^{th} Generalized Lucas numbers [15] follows relation given as:

$$j_n(x) = j_{n-1}(x) + xj_{n-2}(x), \tag{1.9}$$

with $n = 2, 3, 4, \ldots$, and x is any positive integer with $j_0(x) = 2$, and $j_1(x) = 1$.

1.3.10 Lucas Polynomials

Lucas polynomials and Fibonacci polynomials are closely related to each other as both have same recursive relation, differ only in initial conditions. In 1970, Bicknell introduced the sequence of Lucas polynomials $\{L_n(x)\}_{n\geq 0}$ [9] with $L_0(x) = 2, L_1(x) = x$, and follows by recursive relation

$$L_n(x) = xL_{n-1}(x) + L_{n-2}(x), \tag{1.10}$$

for $n = 2, 3, 4, \ldots$

On taking $x = 1,2$, the above sequence of polynomials becomes the sequence of Lucas numbers and Pell-Lucas numbers respectively. And, the generating function $g_F(\mathfrak{u})$ for this sequence of polynomials is

$$\sum_{n=0}^{\infty} L_n(x)\mathfrak{u}^n = g_F(\mathfrak{u}) = \frac{2 - \mathfrak{u}}{1 - x\mathfrak{u} - \mathfrak{u}^2}$$

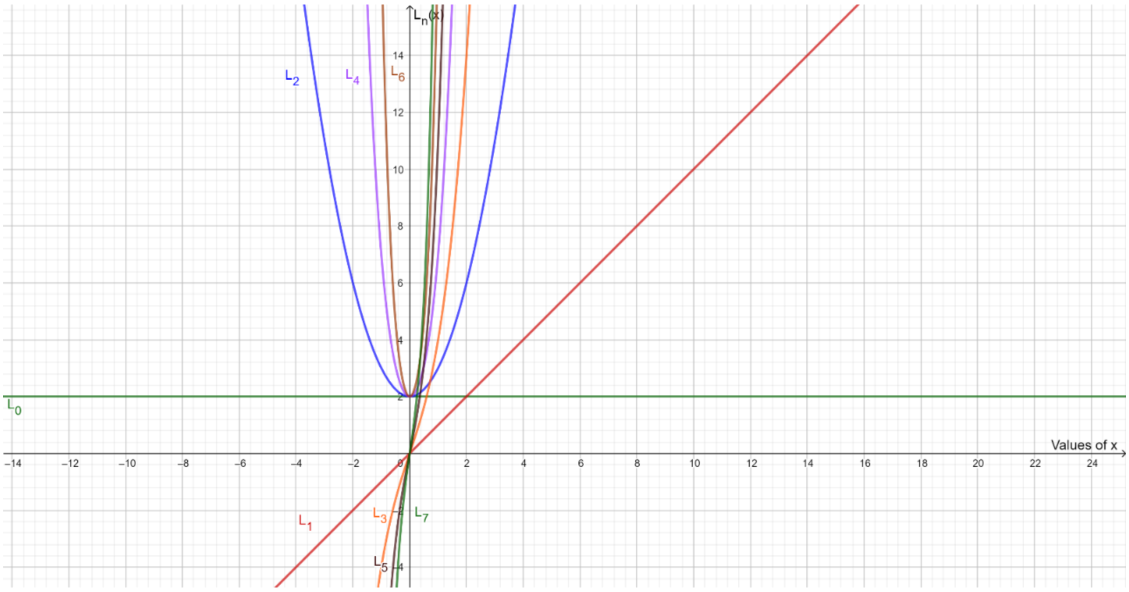


Figure 1.11: Lucas Polynomials

1.3.11 $h(x)$ −Lucas Polynomials

Nalli and Haukkanen [10] generalized Lucas polynomials by introducing $h(x)$ −Lucas polynomials for a real coefficient polynomial $h(x)$ described by

$$L_{h,\mathfrak{w}+2}(x) = h(x)L_{h,\mathfrak{w}+1}(x) + L_{h,\mathfrak{w}}(x), \tag{1.11}$$

where $\mathfrak{w} = 0, 1, 2, \ldots$

with $L_{h,0}(x) = 2$, and $L_{h,1}(x) = h(x)$.

On taking $h(x) = 1$, sequences given by (1.11) becomes that of Lucas numbers respectively. If $h(x) = x$, then the above sequences become that of Lucas polynomials. And, the generating function $g_F(\mathfrak{u})$ for this sequence of polynomials is

$$\sum_{\mathfrak{w}=0}^{\infty} L_{h,\mathfrak{w}}(x)\mathfrak{u}^{\mathfrak{w}} = g_F(\mathfrak{u}) = \frac{2 - h(x)\mathfrak{u}}{1 - h(x)\mathfrak{u} - \mathfrak{u}^2}$$

1.3.12 Generalized Lucas Polynomials

Generalized Lucas polynomials [12] are the generalization $h(x) -$ Lucas polynomials and defined by

$$L_{k,q,n+1}(g) = k(g)L_{k,q,n}(g) + q(g)L_{k,q,n-1}(g), \text{ for } n = 1, 2, 3, \ldots, \tag{1.12}$$

with $L_{k,q,0}(g) = 2$, $L_{k,q,1}(g) = k(g)$.

Here, $k(g)$ and $q(g)$ are the polynomials having real coefficients.

For $k(g) = q(g) = 1$, sequence given by (1.12) becomes that of Lucas numbers. Also,

$$\sum_{n=0}^{\infty} L_{k,q,n}(g)\mathrm{u}^n = g_F(\mathrm{u}) = \frac{2 - k(g)\mathrm{u}}{1 - k(g)\mathrm{u} - q(g)\mathrm{u}^2}$$

1.3.13 Jacobsthal Numbers

In [16], Jacobsthal numbers $\{J_n\}_{n\geq 0}$ named after Ernst Jacobsthal, are the numbers having the same initial conditions as that of Fibonacci numbers i.e., $J_0 = 0$ and $J_1 = 1$ but follows a different recursive relation given by

$$J_{n+2} = J_{n+1} + 2J_n, \tag{1.13}$$

where n is non-negative integer.

Jacobsthal numbers satisfies:

$$\mathfrak{r}_\delta^2 - \mathfrak{r}_\delta - 2 = 0,$$

with roots $\alpha = 2, \beta = -1$.

And these numbers can be generated by following formula

$$J_n = \frac{\alpha^n - \beta^n}{3}$$

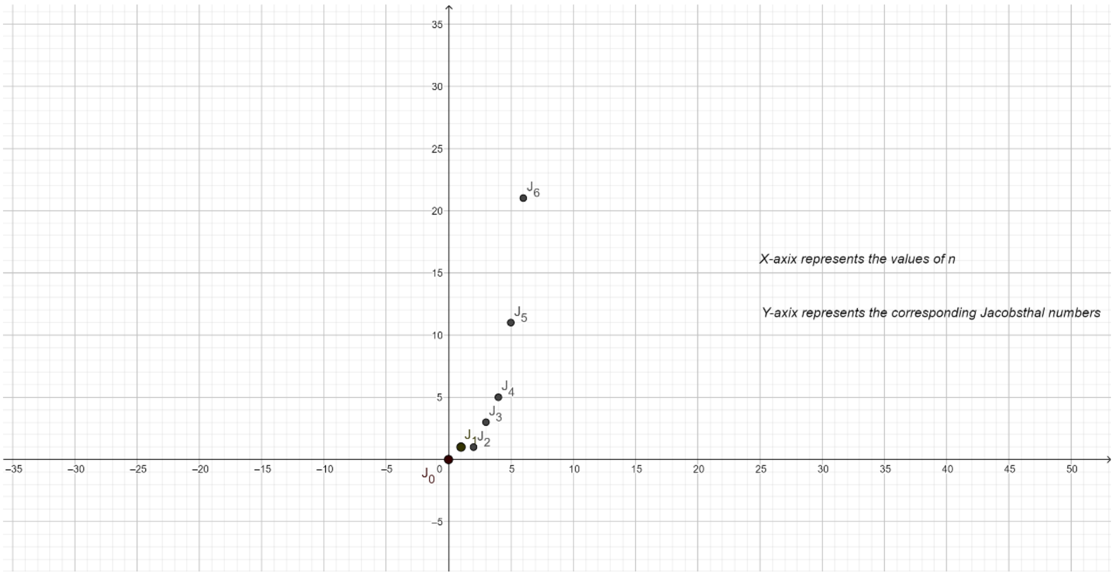


Figure 1.12: Jacobsthal Numbers

1.3.14 Jacobsthal Polynomial

Jacobsthal polynomials [10] studied by E. Jacobsthal, is an extension of Jacobsthal numbers and forms a recursive relation as given below

$$J_n(x) = \begin{cases} 0, \text{if } n = 0 \\ 1, \text{if } n = 1 \\ J_{n-1}(x) + xJ_{n-2}(x), \text{if } n \geq 2 \end{cases} \quad (1.14)$$

The sequence of Fibonacci polynomials differs from that of Jacobsthal polynomials only in the position of x. The n^{th} term of Jacobsthal polynomial is obtained by the formula

$$J_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where $\alpha = \frac{1+\sqrt{1+4x}}{2}$, and $\beta = \frac{1-\sqrt{1+4x}}{2}$.

The graphical representation of Jacobsthal polynomial is:

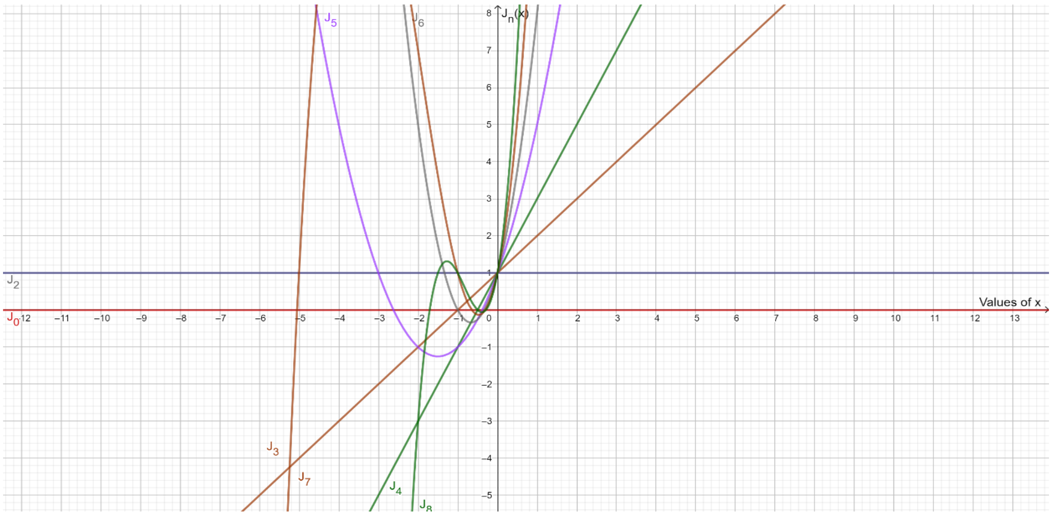


Figure 1.13: Jacobsthal Polynomials

1.3.15 Jacobsthal-Lucas Numbers

Jacobsthal-Lucas [16] numbers $\{j_n\}_{n\geq 0}$ has the same recursive relationship as that of Jacobsthal numbers $\{J_n\}_{n\geq 0}$ but with different initial seeds. Thus, Jacobsthal-Lucas numbers $\{j_n\}_{n\geq 0}$ are defined by

$$j_{n+2} = j_{n+1} + 2j_n \,,$$

with $j_0 = 2$, and $j_1 = 1$, and n is a non-negative integer.

Thus, Jacobsthal-Lucas numbers satisfies the same characteristic equation as that of Jacobsthal-Numbers

$$r_\gamma^2 - r_\gamma - 2 = 0,$$

with roots $\alpha = 2, \beta = -1$.

And these numbers can be generated by following formula:

$$J_n = \alpha^n + \beta^n$$

Graphical representation of Jacobsthal-Lucas number is:

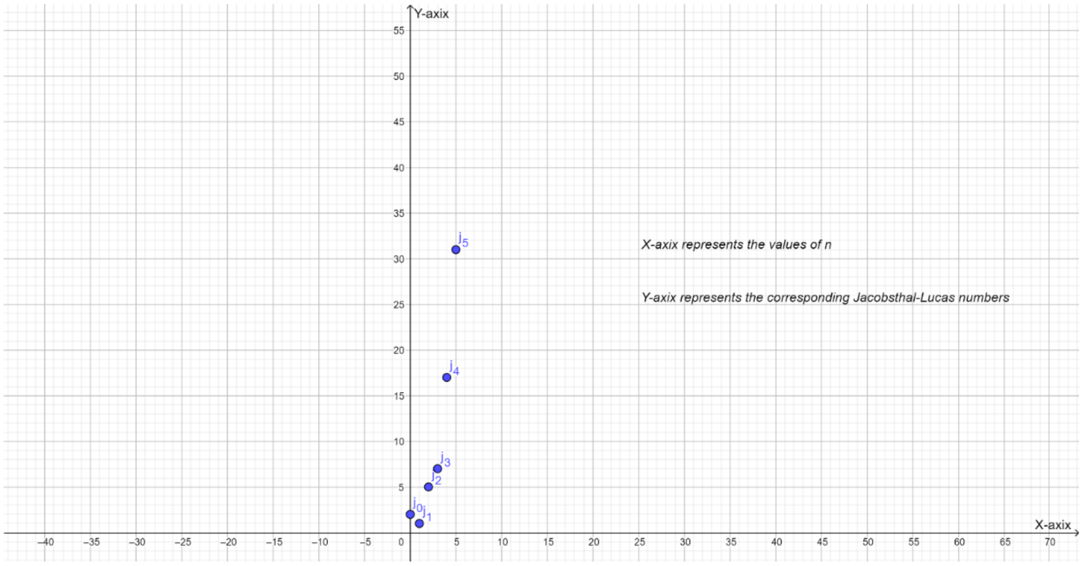


Figure 1.14: Graphical Representation of the Jacobsthal-Lucas Numbers

1.3.16 Jacobsthal-Lucas Polynomials

Jacobsthal-Lucas polynomials [17] is an extension of Jacobsthal-Lucas numbers and follows the recursive relation

$$j_n(x) = \begin{cases} 2, \text{if } n = 0 \\ 1, \text{if } n = 1 \\ j_{n-1}(x) + xj_{n-2}(x), \text{if } n \geq 2 \end{cases} \quad (1.15)$$

On taking $x = 1$ and $x = 2$, it generates the sequences of the Lucas and Jacobsthal-Lucas numbers respectively. The sequence of Lucas polynomials differs from that of Jacobsthal-Lucas polynomials only in the position of x. The n^{th} term of Jacobsthal-Lucas polynomial is obtained by the formula

$$j_n(x) = \alpha^n + \beta^n,$$

where $\alpha = \frac{1+\sqrt{1+4x}}{2}$, and $\beta = \frac{1-\sqrt{1+4x}}{2}$.

The graphical representation of Jacobsthal-Lucas polynomial is represented by Figure 1.15:

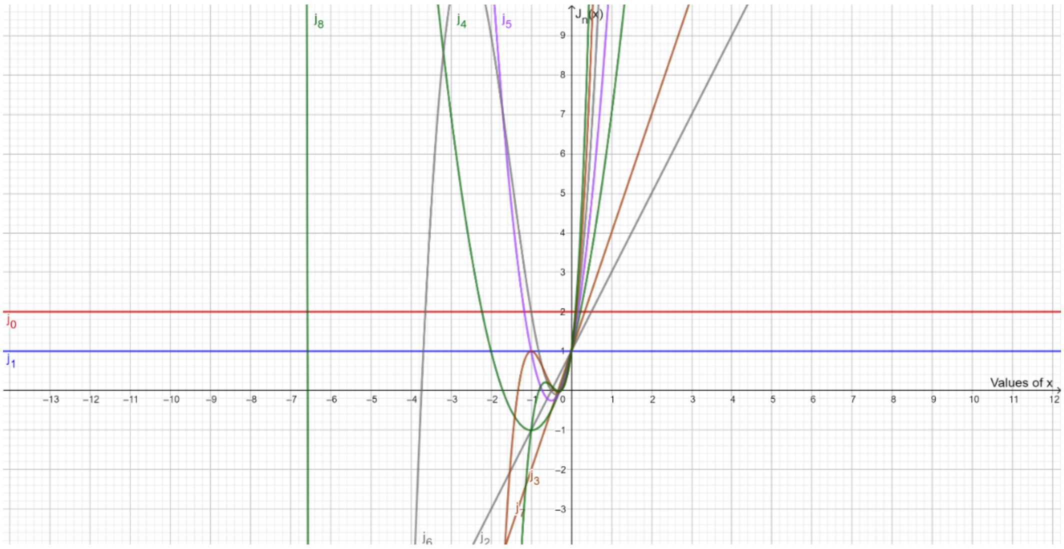


Figure 1.15: Graphical Representation of the Jacobsthal-Lucas Polynomials

1.3.17 Pell-Numbers

Pell-numbers [18] derived by John Pell are given by the recursive relation

$$P_n = 2P_{n-1} + P_{n-2};\ n = 2, 3, 4, \ldots, \tag{1.16}$$

with $P_0 = 0, P_1 = 1$.

Thus, Pell- Numbers and Jacobsthal-Numbers differs only in the position of 2. Jacobsthal- numbers are the sum of its previous terms and twice of the term before that. On the other hand, Pell-Numbers are the sum of twice of its previous term and the term before it.

Pell-numbers can be generated by the following formula

$$P_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}},$$

where α and β satisfies the equation

$$r_\epsilon^2 - 2r_\epsilon - 1 = 0$$

Graphical representation of Pell-numbers is:

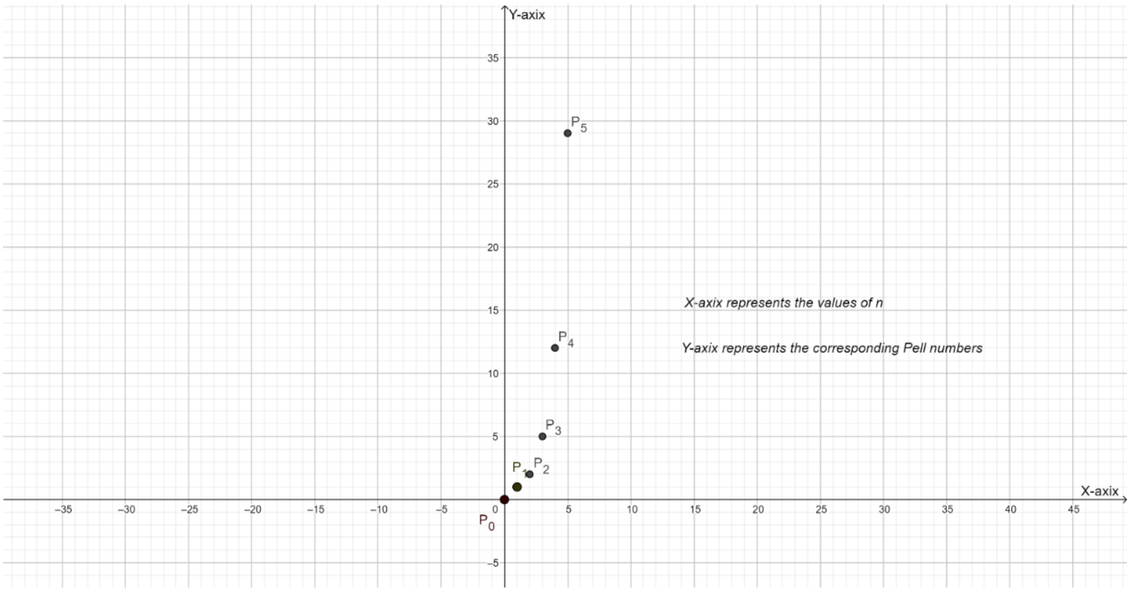


Figure 1.16: Graphical Representation of the Pell Numbers

1.3.18 Pell-Lucas Numbers

The Pell-Lucas [18] numbers $\{Q_n\}_{n\geq 0}$ has the same recursive relation as that of Pell-Numbers but with different initial conditions. Thus, Pell-Lucas numbers satisfies the following recursive relationship

$$Q_{n+1} = 2Q_n + Q_{n-1},\ \forall\, n = 1, 2, 3, \dots, \tag{1.17}$$

with $Q_0 = Q_1 = 2$.

Therefore, all the Pell-Lucas numbers are even. These numbers can be generated by the following formula

$$Q_n = \alpha^n + \beta^n,$$

where α and β satisfies the equation

$$r_\eta^2 - 2r_\eta - 1 = 0$$

Graphical representation of Pell-Lucas numbers is:

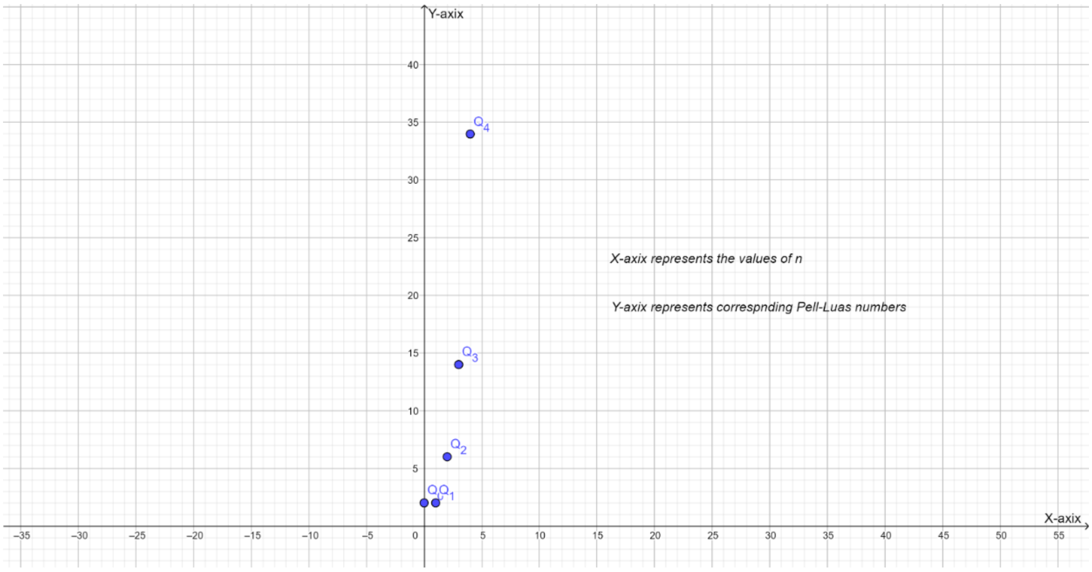


Figure 1.17: Graphical Representation of the Pell-Lucas Numbers

1.3.19 2-Fibonacci sequences

K.T. Atanassov et al., introduced four methods to construct two sequences $\{\alpha_m\}_{m=0}^{\infty}$ and $\{\beta_m\}_{m=0}^{\infty}$ of numbers named as 2-Fibonacci sequences [19] described as

$$\alpha_{m+2} = \beta_{m+1} + \beta_m, \beta_{m+2} = \alpha_{m+1} + \alpha_m, \quad (1.18)$$

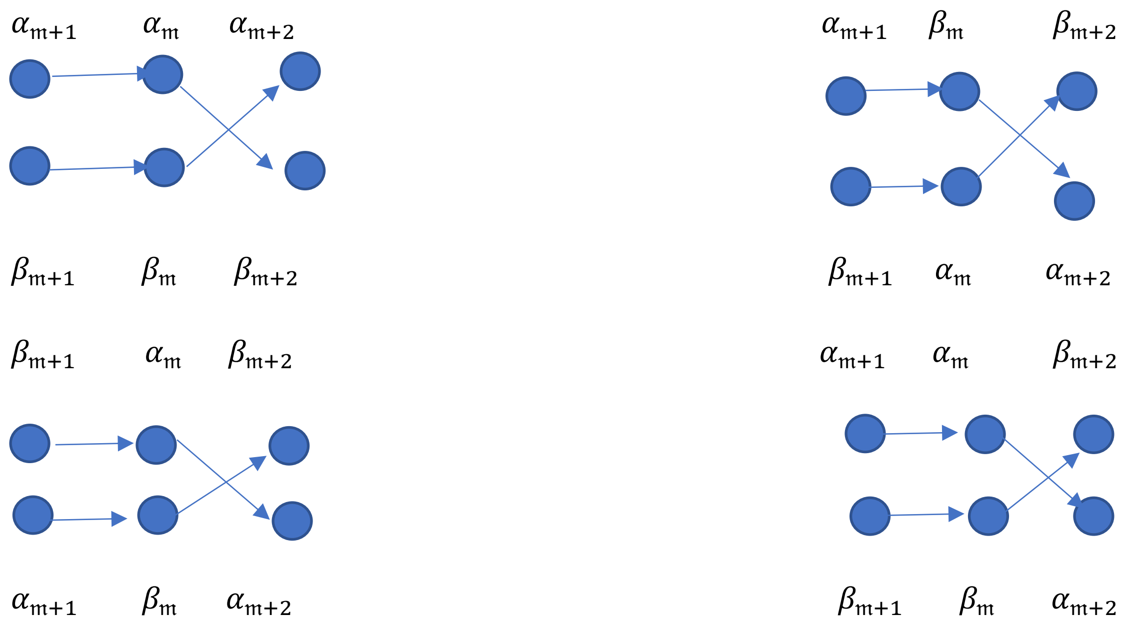
$$\alpha_{m+2} = \alpha_{m+1} + \beta_m, \ \beta_{m+2} = \beta_{m+1} + \alpha_m, \quad (1.19)$$

$$\alpha_{m+2} = \beta_{m+1} + \alpha_m, \ \beta_{m+2} = \alpha_{m+1} + \beta_m, \quad (1.20)$$

$$\alpha_{m+2} = \alpha_{m+1} + \alpha_m, \ \beta_{m+2} = \beta_{m+1} + \beta_m, \quad (1.21)$$

with real numbers $\alpha_0 = a, \beta_0 = b, \alpha_1 = c, \ \beta_1 = d,$ and $m = 0, 1, 2, \ldots$

Recurrence relation (1.21) generates Fibonacci and Lucas numbers respectively on taking $\alpha_0 = 0, \ \beta_0 = 1, \alpha_1 = 2, \ \beta_1 = 1.$



1.3.20 Generalized Tribonacci Numbers

The Generalized Tribonacci [20] numbers $\{D_n\}_{n\geq 0}$ is a third-order recurrence relation that generalizes the concept of Fibonacci numbers and satisfy

$$D_{\mathfrak{n}} = r_\alpha D_{\mathfrak{n}-1} + s_\beta D_{\mathfrak{n}-2} + t_\gamma D_{\mathfrak{n}-3}\,; D_0 = a_1, D_1 = b_1, D_2 = c_1, \mathfrak{n} = 3, 4, 5, \ldots,$$

for complex numbers a_1, b_1 and c_1 and for real numbers r_α, s_β and t_γ.

If we take $a_1 = 0, b_1 = 1,\ r_\alpha = s_\beta = 1, t_\gamma = 0$, then the above sequence becomes that of the Fibonacci numbers. On taking $a_1 = 2, b_1 = 1, r_\alpha = 1, s_\beta = 1, t_\gamma = 0$, it arises the sequence of the Lucas numbers. Similarly, by changing the initial conditions and values of r_α, s_β and t_γ, we get various types of sequences.

1.3.21 Generalized Tetrabonacci Sequence

The Generalized Tetrabonacci sequence [21] is a fourth order recurrence relation that generalizes the concept of Generalized Fibonacci numbers and forms a recursive relation defined by

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4}\,; W_0 = a, W_1 = b, W_2 = c, W_3 = d, n \geq 4,$$

for complex numbers a, b, c and d and real numbers r, s, t and u. If $u = 0$, then it becomes the sequence of the "Generalized Tribonacci numbers. Fibonacci numbers, Lucas numbers, Pell numbers, Pell-Lucas numbers, Jacobsthal numbers and Jacobsthal-Lucas numbers are particular cases of Generalized Tetrabonacci sequence".

1.3.22 Gaussian Generalized Fibonacci Numbers

Gaussian Generalized Fibonacci (Horadam) numbers [22] are defined by

$$GW_n = rGW_{n-1} + sGW_{n-2},$$

with the conditions

$$GW_0 = W_0 + \left(-\frac{r}{s}GW_0 + \frac{1}{s}GW_1\right)i, GW_1 = W_1 + W_0 i,$$

where the sequence $\{W_n\}_{n\geq 0}$ is:

$$W_n = rW_{n-1} + sW_{n-2},$$

where W_n and W_n are arbitrary complex numbers with $W_0 = a, W_1 = b$, for $n = 2, 3, 4, \ldots.$

1.3.23 Trivariate Fibonacci polynomials

For any variable quantities e', m', f' and for integer $n \geq 3$, Trivariate Fibonacci [23] polynomials $H_n(e', m', f')$ is an extension of Generalized Fibonacci polynomials (1.7) and follows a third-order recursive relation given by

$$H_n(e', m', f') = e'H_{n-1}(e', m', f') + m'H_{n-2}(e', m', f') + f'H_{n-3}(e', m', f'), \tag{1.22}$$

with $H_0(e', m', f') = 0, H_1(e', m', f') = 1$, and $H_2(e', m', f') = e'$.

Table 1.3: Sequences by taking different values of e', m', f'

Sr. No	Values of e', m', f'	Sequences
1	$e' = 1, m' = 1, f' = 0$	Fibonacci Numbers
2	$e' = 2, m' = 1, f' = 0$	Pell-Numbers
3	$e' = 1, m' = 2, f' = 0$	Jacobsthal Numbers
4	$m' = 1, f' = 0$	Fibonacci Polynomials
5	$e' = 1, f' = 0$	Jacobsthal Polynomials

1.3.24 Fractional Differential Equation

Fractional differential equation [24] with constant coefficients of order (n, q_1) defined by

$$[D_t^{(n)\alpha} + a_1 D_t^{(n-1)\alpha} + \cdots + a_{n-1} D_t^{\alpha} + a_n] y(t) = f(t), t \geq 0,$$

where $\alpha = \frac{1}{q_1}$ is a rational number with and $D_t^{(n)\alpha} = D_t^{\alpha} D_t^{\alpha} \ldots D_t^{\alpha}$, and $a_n, a_{n-1}, \ldots, a_1$ are real numbers.

If $q_1 = 1$, we get simply an ordinary differential equation of order n.

Here,

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t - \rho)^{-\alpha} (f(\rho) - f(0)) d\rho, 0 < \alpha < 1$$

and $D_t^{\alpha} f(t) = \frac{d^n}{dt^n} \left(D^{\alpha - n} f(t) \right), n < \alpha < n + 1$ and $n \geq 1$.

1.3.25 Extremum Values of a Function

The extremum value of a function corresponds to that point (called extrema) where the function has either greatest value or lowest value. Further, at the critical point, first derivative of a differentiable function f vanishes. If function f goes increasing to decreasing, then it arises maxima while if it changes from decreasing to increasing, then it arises maxima.

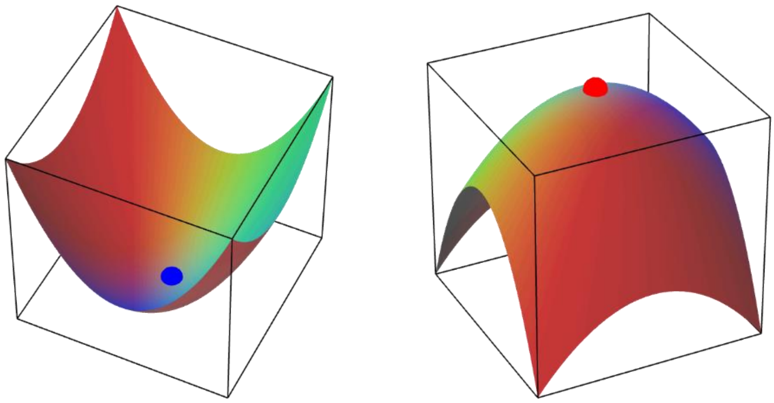


Figure 1.18: Local Minima (left) and Local Maxima (right)

1.3.26 Diophantine Equation

A Diophantine equation [3] initiated by mathematician Diophantus is a polynomial equation, usually involving more than one unknown, so that only integral solutions are of interest. For instance,

$$P_\ell + P_m + P_n = 2^a,$$

with integers $\ell, m, n, \mathfrak{a} \geq 0$ such that $\ell \geq m \geq n$ and P_ℓ, P_m, P_n are Pell-Numbers that follows the recursive relation defined by (1.16).

1.3.27 Continued Fraction

A finite continued fraction [3] notated by $[b_0;\ b_1, b_2, \ldots, b_n]$ and introduced by Fibonacci is a fraction given as

$$\mathscr{b}_0 + \cfrac{1}{\mathscr{b}_1 + \cfrac{1}{\mathscr{b}_2 + \cfrac{1}{\ddots + \cfrac{1}{\mathscr{b}_n}}}}$$

for positive real numbers $\mathscr{b}_1, \mathscr{b}_2, \ldots., \mathscr{b}_n$, except possibly for real number $\mathscr{b}_0$ and these are called as partial denominators of this fraction. For instance,

$$\frac{170}{53} = 3 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{2}}}}$$

1.3.28 Convergent of Continued Fraction

The continued fraction [3] derived from $[b_0;\ b_1, b_2, \ldots., b_n]$ by taking after the k^{th} partial denominator b_k is known as k^{th} convergent of the continued fraction $[b_0;\ b_1, b_2, \ldots., b_n]$ and is denoted by C_k i.e.,

$$C_k = [b_0;\ b_1, \ldots, b_k],$$

where $1 \leq k \leq n$. Clearly, the zeroth convergent is denoted by C_0. For instance, the first three convergents of $\frac{17}{15}$ denoted by C_0, C_1 and C_2 are given by

$$C_0 = 1$$

$$C_1 = [1; 7] = 1 + \frac{1}{7} = \frac{8}{7}$$

and

$$C_2 = [1; 7, 2] = 1 + \cfrac{1}{7 + \cfrac{1}{2}} = \frac{17}{15}$$

1.3.29 Tridiagonal Matrix

Matrix $A(n) = [\rho_{lm}]_{n\times n}$is called a tridiagonal matrix when all the entries below the sub diagonal and above the super diagonal are all zero. Henceforth,

$$\rho_{lm} = 0 \text{ for } m > l+1, \text{ and for } l > m+1.$$

For instance, a 4×4 tridiagonal matrix is given by

$$A(4) = \begin{pmatrix} \rho_{11} & \rho_{12} & 0 & 0 \\ \rho_{21} & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & \rho_{34} \\ 0 & 0 & \rho_{43} & \rho_{44} \end{pmatrix}$$

1.4 Literature Review

Many researchers had worked in the field of the Fibonacci numbers. Various summation formulae of the, "Generalized Fibonacci and Gaussian Fibonacci numbers, Pell and Pell-Lucas numbers" are developed in [22], [25, 26] respectively. Similar properties of, "Fibonacci, Tribonacci, Tetranacci, Pentanacci, Hexanacci numbers", [9, 27, 28], [29, 30], [31, 32, 33], [34, 35] and [36] respectively has been detailed.

In [37], the authors presented the various summation formulae for Generalized Fibonacci numbers defined as

$$W_n = rW_{n-1} + sW_{n-2}\,; W_0 = a, W_1 = b, \text{ for } n = 2, 3, 4, \ldots$$

Similar work has been done for different sequences [38].

In [39], the author studied various properties of 2-Fibonacci sequences defined by

$$\alpha_{n+2} = \alpha_{n+1} + \beta_n, \text{ and } \beta_{n+2} = \beta_{n+1} + \alpha_n$$

with $\alpha_0 = a,\ \beta_0 = b, \alpha_1 = c,\ \beta_1 = d$, and $n = 0, 1, 2, \ldots$

In [40, 41], the author introduced new schemes of 2-Fibonacci sequences defined as

$$\alpha_{n_1+2} = \frac{\alpha_{n_1+1}+\beta_{n_1+1}}{2} + \beta_{n_1}, \beta_{n_1+2} = \frac{\beta_{n_1+1}+\alpha_{n_1+1}}{2} + \alpha_{n_1},$$

and

$$\alpha_{n_1+2} = \frac{\alpha_{n_1}+\beta_{n_1}}{2} + \beta_{n_1+1}, \beta_{n_1+2} = \alpha_{n_1} + \frac{\beta_{n_1}+\alpha_{n_1}}{2},$$

with $n_1 = 0,1,2,\ldots$, and $\alpha_0 = 2a,\ \beta_0 = 2b,\ \alpha_1 = 2c,\ \beta_1 = 2d$, for any real numbers $a,\ b, c$ and d and he established various relationship of these sequences with Generalized Fibonacci sequence defined by

$$F_{\mathfrak{n}+2}(d_\delta, c_\gamma) = F_{\mathfrak{n}+1}(d_\delta, c_\gamma) + F_{\mathfrak{n}}(d_\delta, c_\gamma),$$

with $F_0(d_\delta, c_\gamma) = d_\delta,\ F_1(d_\delta, c_\gamma) = c_\gamma$, and $\mathfrak{n} = 0, 1, 2, \ldots$

Also, listed various properties of these sequences with the help of an integer function σ described by

$$\sigma(j + 2) + \sigma(j) = 0; j = 0, 1, 2, \ldots,$$

with $\sigma(0) = 0$, and $\sigma(1) = 1$.

Similar work has been done by the authors [42, 43] for different schemes of sequences.

In [44], the author derived the following formulae

$$\mathcal{U}_{\mathfrak{y}} = \frac{\alpha_q^{\mathfrak{y}+1}}{(\alpha_q - \beta_q)(\alpha_q - \gamma_q)} + \frac{\beta_q^{\mathfrak{y}+1}}{(\beta_q - \alpha_q)(\beta_q - \gamma_q)} + \frac{\gamma_q^{\mathfrak{y}+1}}{(\gamma_q - \beta_q)(\gamma_q - \alpha_q)},$$

and

$$\mathcal{V}_{\mathfrak{y}} = \alpha_q^{\mathfrak{y}+1} + \beta_q^{\mathfrak{y}+1} + \gamma_q^{\mathfrak{y}+1},$$

where

$$\alpha_q = \frac{1 + \left(\sqrt[3]{19 + 3\sqrt{33}}\right) + \left(\sqrt[3]{19 - 3\sqrt{33}}\right)}{3},$$

$$\beta_q = \frac{1 + \omega\left(\sqrt[3]{19 + 3\sqrt{33}}\right) + \omega^2\left(\sqrt[3]{19 - 3\sqrt{33}}\right)}{3},$$

$$\gamma_q = \frac{1 + \omega^2\left(\sqrt[3]{19 + 3\sqrt{33}}\right) + \omega\left(\sqrt[3]{19 - 3\sqrt{33}}\right)}{3},$$

for Tribonacci sequence $\{\mathcal{U}_\mathfrak{y}\}_{\mathfrak{y}\geq 0}$ and Tribonacci-Lucas $\{\mathcal{V}_\mathfrak{y}\}_{\mathfrak{y}\geq 0}$ sequence described as

$$\mathcal{U}_{\mathfrak{y}+3} = \mathcal{U}_{\mathfrak{y}+2} + \mathcal{U}_{\mathfrak{y}+1} + \mathcal{U}_\mathfrak{y};\ \mathcal{V}_{\mathfrak{y}+3} = \mathcal{V}_{\mathfrak{y}+2} + \mathcal{V}_{\mathfrak{y}+1} + \mathcal{V}_\mathfrak{y};\ \mathfrak{y} = 0, 1, 2, \ldots,$$

with $\mathcal{U}_0 = 0, \mathcal{U}_1 = 1, \mathcal{U}_2 = 1, \mathcal{V}_0 = 3, \mathcal{V}_1 = 1$, and $\mathcal{V}_2 = 3$.

Similarly, authors in [45] determined the Binet's formula using the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

for Pentanacci sequences given by

$$\mathcal{A}_{j+5} = \mathcal{A}_{j+4} + \mathcal{A}_{j+3} + \mathcal{A}_{j+2} + \mathcal{A}_{j+1} + \mathcal{A}_j,$$

and

$$\mathcal{C}_{j+5} = \mathcal{C}_{j+4} + \mathcal{C}_{j+3} + \mathcal{C}_{j+2} + \mathcal{C}_{j+1} + \mathcal{C}_j\ ; j = 0, 1, 2, \ldots,$$

with $\mathcal{C}_0 = 0, \mathcal{C}_1 = 1, \mathcal{C}_2 = 1, \mathcal{C}_3 = 2, \mathcal{C}_4 = 4$, and $\mathcal{A}_0 = 5, \mathcal{A}_1 = 1, \mathcal{A}_2 = 3, \mathcal{A}_3 = 7$, and $\mathcal{A}_4 = 15$.

Authors [46] investigated the integral sum from the reciprocal of Fibonacci numbers and obtained the following identities:

For all natural numbers $n \geq 2$,

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k}\right)^{-1} \right\rfloor = \begin{cases} F_{n-2}, \text{if 2 divides } n \\ F_{n-2} - 1, \text{otherwise} \end{cases},$$

and for all $n \geq 1$,

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} -1 + F_{n-1}F_n & , \text{if } n \text{ is even} \\ F_{n-1}F_n & , \text{if } n \text{ is odd} \end{cases}$$

The following results were deduced by Holliday and Komastsu [47]:

$$\left\lfloor \left(\sum_{k=\mathfrak{u}}^{\infty} \frac{1}{G_k} \right)^{-1} \right\rfloor = \begin{cases} G_{\mathfrak{u}} - G_{\mathfrak{u}-1}, & \text{if } \mathfrak{u} \equiv 0 (mod\ 2) \\ G_{\mathfrak{u}} - G_{\mathfrak{u}-1} - 1, & \text{otherwise} \end{cases},$$

and

$$\left\lfloor \left(\sum_{k=\mathfrak{u}}^{\infty} \frac{1}{G_k^2} \right)^{-1} \right\rfloor = \begin{cases} aG_{\mathfrak{u}}G_{\mathfrak{u}-1}, & \text{if } \mathfrak{u} \text{ is odd} \\ aG_{\mathfrak{u}}G_{\mathfrak{u}-1} - 1, & \text{otherwise} \end{cases},$$

where $\{G_{\mathfrak{u}}\}_{u\geq 0}$ is the sequence of Generalized Fibonacci numbers [47] defined as,

$$G_{\mathfrak{u}+2} = aG_{\mathfrak{u}+1} + G_{\mathfrak{u}},$$

for $\mathfrak{u}, a \geq 0$, and $G_0 = 0, G_1 = 1$.

In 2013, the authors [48, 49] extended this work for the Fibonacci and Lucas polynomial. They derived:

$$\left\lfloor \left(\sum_{\hbar=n}^{\infty} \frac{1}{F_{a\hbar}(x)} \right)^{-1} \right\rfloor = F_{an}(x) - F_{an-a}(x) - 1,$$

$$\left\lfloor \left(\sum_{\hbar=n}^{\infty} \frac{1}{F_{a\hbar}^2(x)} \right)^{-1} \right\rfloor = F_{an}^2(x) - F_{an-a}^2(x) - 1,$$

for integers $x, n > 0$, and even $a \geq 2$.

And integer $x > 0$, and odd $b \geq 1$,

$$\left\lfloor \left(\sum_{\hbar=n}^{\infty} \frac{1}{F_{\ell \hbar}(x)} \right)^{-1} \right\rfloor = \begin{cases} F_{\ell n}(x) - F_{\ell n-\ell}(x), & \text{if } n \equiv 0 (mod\ 2) \\ F_{\ell n}(x) - F_{\ell n-\ell}(x) - 1, & \text{otherwise} \end{cases},$$

and

$$\left\lfloor \left(\sum_{\hbar=n}^{\infty} \frac{1}{F_{\ell \hbar}^2(x)} \right)^{-1} \right\rfloor = \begin{cases} F_{\ell n}^2(x) - F_{\ell n-\ell}^2(x), & \text{if } n \equiv 0 (mod\ 2) \\ F_{\ell n}^2(x) - F_{\ell n-\ell}^2(x) - 1, & \text{else} \end{cases}$$

Wang and Zhang [50] investigated the sum having even and odd indices for the Fibonacci sequence. They derived that:

For all $m \geq 3, n \geq 1$

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_{2k}} \right)^{-1} \right\rfloor = F_{2n-1} - 1,$$

and for all $m \geq 2, n \geq 1$

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_{2k-1}} \right)^{-1} \right\rfloor = F_{2n-2}$$

Wang and Wen [51], improves the results of Ohtsuka and Nakamura's [46] related to the reciprocal sum of Fibonacci numbers by deducting:

For all $n \geq 2$ and $m \geq 3$,

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2}, & \text{if } n \text{ is even} \\ F_{n-2} - 1, & \text{otherwise} \end{cases},$$

and for all $n \geq 1$ and $m \geq 0$

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} -1 + F_{n-1}F_n, \text{ if } n \text{ is even} \\ F_{n-1}F_n, \quad \text{otherwise} \end{cases}$$

In 2014, Yuan etal., [52] investigate the reciprocal sums of the following sequence

$$U_n = pU_{n-1} + qU_{n-2} \,;\, n = 2, 3, 4, \ldots,$$

with $U_0 = 0$ and $U_1 = 1$.

In 2017, the authors [53] deduced following results for the Lucas numbers

$$\left\lfloor \left(\sum_{k=n}^{m} \frac{1}{L_k} \right)^{-1} \right\rfloor = \begin{cases} L_{n-2} - 1, \text{ if } n \geq 4 \text{ and } m \geq 3n \\ L_{n-2}, \text{ if } n \geq 3 \text{ and } m \geq 2n \end{cases},$$

and

$$\left\lfloor \left(\sum_{k=n}^{m} \frac{1}{L_k^2} \right)^{-1} \right\rfloor = \begin{cases} L_{n-1}L_n + 1, \text{ if } n \geq 2 \text{ and } m \geq 2n+1 \\ L_{n-1}L_n - 2, \text{ if } n \geq 1 \text{ and } m \geq 2n \end{cases}$$

Fibonacci numbers and linear algebra are also having a lot of connections. Many researchers had been worked in this area. In 2006, Mike [54] utilized the sum property of determinant which is, “If A, B and C are matrices with indistinguishable entries except that one row(column) of C, say k^{th}, is sum of k^{th} rows(column) of A and B, then |A| + |B| = |C|” and by utilizing this property of determinant author validated the following property of the Fibonacci numbers:

$$F_m F_n - F_{m-r} F_{n+r} = (-1)^{m-r} F_r F_{n+r-m}$$

In 2009, Nalli & Haukkanen [10] introduced $h(x) -$ polynomials defined by (1.5) and (1.11) and introduced a matrix whose power generates the sequence of the Fibonacci numbers.

In 2013, Jishe Feng [55], utilized the technique of Laplace expansions to evaluate the determinant of D_n and constructed a type of 2×2 matrix determinant to approach a new method to substantiate the following identity

$$F_{m+n+1} = F_{m+1}F_{n+1} \; + \; F_m F_n \, ,$$

where

$$D_n = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 1 & 1 & -1 & \dots & 0 \\ 0 & 1 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

In 2017, Siimeyra [56] studied some new properties of, “Generalized Fibonacci and Lucas polynomials”, by using Laplace expansion of determinants and also described some new families of tridiagonal matrices given by

$$C(n) = \begin{pmatrix} p(t_1) & i\sqrt{g(t_1)} & 0 & \dots & 0 \\ i\sqrt{g(t_1)} & p(t_1) & i\sqrt{g(t_1)} & \dots & 0 \\ 0 & i\sqrt{g(t_1)} & p(t_1) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & i\sqrt{g(t_1)} \\ 0 & 0 & 0 & i\sqrt{g(t_1)} & p(t_1) \end{pmatrix},$$

and its successive determinants generate the following sequence:

$$F_{p,g,n+1}(t_1) = p(t_1)F_{p,g,n}(t_1) + g(t_1)F_{p,g,n-1}(t_1),$$

and

$$L_{p,g,n+1}(t_1) = p(t_1)L_{p,g,n}(t_1) + g(t_1)L_{p,g,n-1}(t_1),$$

with $n = 1, 2, 3, \dots, F_{p,g,0}(t_1) = 0,$ $F_{p,g,1}(t_1) = 1, L_{p,g,0}(t_1) = 2,$ and $L_{p,g,1}(t_1) = p(t_1)$.

“Both linear and non-linear ordinary and fractional differential equations can be solved by the operational matrices of derivatives [57]”. In [58], the author solves the following fractional differential equation by using an operational matrix of fractional order derivative of Fibonacci polynomials:

$$\begin{cases} D^{\alpha} y(x) + y^{(k)}(x) + y(x) = f(x) \\ y^{(r)}(0) = c_r \end{cases},$$

where $r = 0, 1, 2, \dots, m-1; k = 0, 1, 2, \dots, m; m - 1 < \alpha \leq m$.

In [59], the author solved the following Diophantine equation

$$P_{l} + P_m + P_n = 2^{c},$$

where P_k is the k^{th} term of Pell sequence described as

$$P_0 = 0, P_1 = 1, \text{and } P_{n+1} = 2P_n + P_{n-1}, \text{for all } n \geq 1.$$

In [60], authors work on the following Diophantine equation

$$L_k - L_l = 2^t,$$

where L_k and L_l are the Lucas numbers defined by recursive relation as follows:

$$L_0 = 2, L_1 = 1, \text{and } L_{n+1} = L_n + L_{n-1}, \text{for all } n = 1, 2, 3, \dots$$

In [61], authors deal with equation given by

$$u_n + u_m = w\, p_1^{z_1} \dots p_s^{z_s}$$

in non-negative integers $n, m, z_1, \dots, z_s$, where $\{u_n\}_{n\geq 0}$ is a binary recurrence sequence $p_1, \dots, p_s$ are the distinct primes and for integer $w \neq 0$, with $p_i \nmid w$, for all $1 \leq i \leq s$.

In [62], the authors solved the following equation

$$F_n - F_m = 3^a,$$

where F_n and F_m are n^{th} and m^{th} Fibonacci numbers respectively and deduced that its solutions are

$$\{(1,0,0),(2,0,0),(4,0,1),(3,1,0),(3,2,0),(4,3,0),(5,3,1),(6,5,1),(11,6,4)\}$$

In [63], authors solved the equation

$$F_n - F_m = 5^a,$$

And deducted that:

$$F_1 - F_0 = F_2 - F_0 = F_3 - F_2 = F_3 - F_1 = 5^0, F_5 - F_0 = F_4 - F_3 = F_6 - F_4 = F_7 - F_6 = 5$$

in non-negative integers m, n and a, where F_n and F_m are n^{th} and m^{th} Fibonacci numbers respectively.

Different behavior of the Diophantine equation involving of various kind recurrence sequences has been deduced in [64-74].

1.5 Proposed Objectives of the Research Work

Keeping in mind the above work done in the area of the Fibonacci numbers, the objective of the research work is:

1. The study will focus on finding the infinite sum of reciprocal of Jacobsthal polynomial and $h(x)$ −Fibonacci polynomial.
2. Some important properties of 2-Fibonacci sequence will be generalized. Also, work will be done on some properties involving Fibonacci polynomial in three variables by using matrices.
3. Some properties of derivatives of Fibonacci polynomials and Lucas polynomials will be obtained through matrices.

1.6 Proposed Methodology of the Research Work

To achieve the proposed objective, opted methodology is mentioned below:

1. By using multiplication modulo group, properties will be derived for 2-Fibonacci sequences. And by applying some particular conditions, the results

obtained will be verified by the results which was already proved for Fibonacci numbers and Lucas numbers.

2. The methodology for finding out the finite and infinite sum reciprocal of Jacobsthal polynomial and $h(x)$ −Fibonacci polynomials will be proceed on the base for finding the finite and infinite sum of reciprocal of Fibonacci numbers, Lucas numbers and their polynomials in one variable. By using some inequalities and properties of floor function, the infinite sum of reciprocal of Jacobsthal Polynomial and its square will be obtained.
3. By using some properties of matrices, determinants and eigen values, work will be done on Fibonacci polynomials in three variables.

1.7 Structure of Thesis

The proposed research work entitled "***Fibonacci Numbers, Polynomials and its Sequences***" is motivated by the sequence of the numbers, polynomials associated to the Fibonacci numbers. The roots of the subject matter lie in a series of our papers that are mentioned in the last part of the thesis. The thesis is framed in the following manner:

In the first chapter of this thesis, an introduction of Fibonacci numbers constituting history and its applications in various fields is given. Also, we briefly recall a few definitions and well-known results of the Fibonacci numbers which fulfill the minimum prerequisite for the subsequent chapters. The section of literature review lighting on the work in the field of the Fibonacci numbers, related polynomials by the various researcher is also included in this chapter. In the review, the research gap has been identified, also the objectives and methodology to achieve these gaps have been detailed in this chapter.

The remaining chapters are an attempt to discuss the behavior and different properties of sequences of polynomials based on the Fibonacci numbers.

Chapter 2, is categorizes into four sections. In the first section, various summation formulae have been derived for the Generalized Fibonacci polynomials described as

$$Q_{n+2}(x) = rxQ_{n+1}(x) + sQ_n(x); n = 0, 1, 2, \ldots, \quad (1.23)$$

with $Q_0(x) = a, Q_1(x) = b$, where a, b, r and s are any real numbers.

We derive various summation formulae for the above polynomials and its 1st order derivative having even indices, odd indices, and alternating summation formulae. Thereafter, we discuss some particular cases of the polynomial defined by (1.23) for the Fibonacci numbers, Pell numbers, Tetrabonacci numbers, etc., by giving different values to r, s, a and b.

Then, we discuss the convergence of the solutions of the following differential equation

$$y" - y' - y = 0 \tag{1.24}$$

After that, the study about the extremum values for the Fibonacci polynomials in one variable (1.4) have been done by using Descartes' Rule of sign.

Similarly, the observations about the extremum values for Fibonacci polynomials of two variables or Bivariate Fibonacci polynomials have been devised. In the last section of this chapter, these polynomials are represented in graphical form using MATLAB.

Chapter 3, is categorizes into two sections, that revolves around the topic of 2-Fibonacci sequences described as

$$\alpha_{n+2} = \beta_{n+1} + \beta_n, \quad \beta_{n+2} = \alpha_{n+1} + \alpha_n;\ n = 0, 1, 2, \ldots, \tag{1.25}$$

with initial conditions $\alpha_0 = 0, \alpha_1 = 1, \beta_0 = 2$, and $\beta_1 = 1$.

Various attributes like generating function, Binet's Formula to find out its n^{th} terms, generating function, relationship with the Fibonacci numbers, and multiple relations using the concept of congruences established. In the next section of this chapter, we introduce a matrix

$$E = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

called as generating matrix and derived the Binet's Formula for the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ by using the matrix E and concept of diagonalizability.

In Chapter 4, we work on 2-Fibonacci sequences (1.25). The above 2-Fibonacci sequences satisfy the following recurrence relation

$$\mathcal{C}_{n+4} = \mathcal{C}_{n+2} + 2\mathcal{C}_{n+1} + \mathcal{C}_n;\ n = 0, 1, 2, \ldots, \tag{1.26}$$

with real numbers $\mathcal{C}_0 = a_1$, $\mathcal{C}_1 = b_1$, $\mathcal{C}_2 = c_1$, and $\mathcal{C}_3 = d_1$. If we take in particular $a_1 = 0, b_1 = 1, c_1 = 3$, and $d_1 = 2$, then it becomes the sequence of 2-Fibonacci numbers $\{\alpha_n\}$ and if $a_1 = 2, b_1 = 1, c_1 = 1$, and $d_1 = 4$, then it arises the 2-Fibonacci sequence of numbers $\{\beta_n\}$. We work on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$. Here, the characteristics of these sequences along with their squares are developed and validated in the categories of closed forms of the summation formulae. In particular, we discuss the properties involving an alternating sum of 2-Fibonacci sequences and the product of 2-Fibonacci terms having different indices, even and odd indices. Obviously, above mentioned properties are demonstrated by induction. We derive multiple results involving the summation of these sequences.

Further, the Generalized Fibonacci polynomials defined as

$$F_{k,q,n+1}(g) = k(g)F_{k,q,n}(g) + q(g)F_{k,q,n-1}(g), \tag{1.27}$$

with initial conditions $F_{k,q,0}(g) = 0, F_{k,q,1}(g) = 1$.

And the Generalized Lucas polynomials defined as

$$L_{k,q,n+1}(g) = k(g)L_{k,q,n+1}(g) + q(g)L_{k,q,n+1}(g), \tag{1.28}$$

with initial conditions $L_{k,q,0}(g) = 2, L_{k,q,1}(g) = k(g)$, where $k(g)$ and $q(g)$ are polynomials with positive coefficients and $n = 1, 2, 3, \ldots$.

Chapter 5, mainly focuses on the sequences of Generalized Fibonacci polynomials (1.27) and Generalized Lucas polynomials (1.28). Further, we take in account that the polynomial $k(g)$ with only positive coefficients, so that it has no real roots. In this chapter, we deduce reciprocals sum formulae with even indices from Generalized Fibonacci and Lucas polynomials.

Now, the Jacobsthal polynomials [10] described by the recursive relation

$$J_{n+2}(x) = J_{n+1}(x) + xJ_n(x), \tag{1.29}$$

where $J_0(x) = 0$, and $J_1(x) = 1$ and the Jacobsthal Lucas polynomial [17] by

$$j_{n+2}(x) = j_{n+1}(x) + xj_n(x); n \geq 0, \tag{1.30}$$

where $j_0(x) = 2$, and $j_1(x) = 1$.

In Chapter 6, we derive the reciprocals sum formulae of polynomials (1.29) and (1.30) and also that of their respective squares.

The chapter 7, basically works on Trivariate Fibonacci polynomials defined by:

For any three variables quantities e', m' and f' and, for integer $n \geq 3$, Trivariate Fibonacci polynomials [23], $H_n(e', m', f')$ followed by:

$$H_n(e', m', f') = e'H_{n-1}(e', m', f') + m'H_{n-2}(e', m', f') + f'H_{n-3}(e', m', f'), \tag{1.31}$$

with $H_0(e', m', f') = 0, H_1(e', m', f') = 1$, and $H_2(e', m', f') = e'$.

For $e' = m' = 1$, and $f' = 0$, it becomes the sequence of the Fibonacci numbers.

In the first section of this chapter, we produce a $n \times n$ matrix given by

$$S_n(e', m', f') = \begin{pmatrix} e' & -m' & f' & 0 & \dots & 0 \\ 1 & e' & -m' & f' & \dots & 0 \\ 0 & 1 & e' & -m' & \ddots & \vdots \\ 0 & 0 & 1 & e' & \ddots & f' \\ \vdots & \vdots & \vdots & \ddots & \ddots & -m' \\ 0 & 0 & 0 & 0 & 1 & e' \end{pmatrix}_{n \times n},$$

whose successive determinant produces Trivariate Fibonacci polynomials. In the second section, we present Binet's formula for this sequence by using matrix methods and the concept of diagonalizability of matrix. And in the last section we address an observation related to n^{th} Generalized Lucas numbers described as

$$j_n(x) = j_{n-1}(x) + xj_{n-2}(x),$$

where $\mathfrak{n} = 2, 3, 4, \ldots$, and x is any positive integer with $j_0(x) = 2$, and $j_1(x) = 1$.

First part of Chapter 8, deals with originating an operational matrix for the fractional derivative of Lucas polynomials and Pell-Lucas polynomials to solve the following fractional differential equation:

$$\begin{cases} D^{\alpha}(x) + y^{(k)}(x) + y(x) = f(x) \\ y^{(r)}(0) = c_r \end{cases}, \quad (1.32)$$

where $r = 0, 1, 2, \ldots, m - 1; k = 0, 1, 2, \ldots, m; m - 1 < \alpha \leq m$.

Further, we construct a matrix that generates the sequence of derivatives of Fibonacci polynomials.

In Chapter 9, we solve the following Diophantine equation

$$Q_{\mathfrak{n}} - Q_{\mathfrak{m}} = 2^{\mathfrak{a}},$$

where $Q_{\mathfrak{n}}$ and $Q_{\mathfrak{m}}$ are Pell-Lucas numbers with $\mathfrak{n} > \mathfrak{m} \geq 0$.

Chapter 2

Summation Formulae of Generalized Sequence of Fibonacci Polynomials

2.1 Introduction

This chapter presents some summation formulae for the Generalized Fibonacci polynomials defined by (1.23) and their first-order derivatives. In the following section, the extremum values of Fibonacci polynomials of one and two variables are examined. In the next section, we discuss the solution convergence of a differential equation mentioned below:

$$y'' - y' - y = 0$$

And in the last section, we graphically represent these polynomials using MATLAB.

2.2 Summation Formulae of Generalized Fibonacci Polynomials

Here, we develop the summation formulae for the Generalized Fibonacci polynomials prescribed as follows:

$$Q_{n+2}(x) = rxQ_{n+1}(x) + sQ_n(x), Q_0(x) = a, Q_1(x) = b, n \geq 0$$

with real numbers a, b, r and, s.

By taking $Q_0(x) = 0, Q_1(x) = 1, r = 1,$ and $s = 1,$ it becomes the sequence of Fibonacci polynomials and if, $Q_0(x) = 2,$ $Q_1(x) = 1, r = s = x = 1,$ it becomes the sequence of Lucas numbers.

Now the results obtained for the sequence of Generalized Fibonacci polynomials are followed:

Theorem 2.2.1 For $n \geq 0$, Generalized Fibonacci polynomials satisfy:

(a) If $s + rx - 1 \neq 0,$ then

$$\sum_{k=0}^{n} Q_k(x) = \frac{\mu}{s + rx - 1}$$

where,

$$\mu = Q_{n+2}(x) + (1 - rx)Q_{n+1}(x) + Q_1(x) + (rx - 1)Q_0(x)$$

(b) If $s^2 - r^2x^2 - 2s + 1 \neq 0$, then

$$\sum_{k=0}^{n} Q_{2k}(x) = \frac{\mu}{s^2 - r^2x^2 - 2s + 1}$$

where,

$$\mu = (s - 1)Q_{2n+2}(x) - rxsQ_{2n+1}(x) + rxQ_1(x) - (r^2x^2 + s - 1)Q_0(x)$$

(c) If $s^2 - r^2x^2 - 2s + 1 \neq 0$, then

$$\sum_{k=0}^{n} Q_{2k+1}(x) = \frac{-\mu}{s^2 - r^2x^2 - 2s + 1}$$

where,

$$\mu = rxQ_{2n+2}(x) - s(s - 1)Q_{2n+1}(x) + (s - 1)Q_1(x) - rxQ_0(x)$$

Proof:

(a)

Using the recurrence relation

$$sQ_n(x) = Q_{n+2}(x) - rxQ_{n+1}(x)$$

As a result,

$$sQ_0(x) = Q_2(x) - rxQ_1(x)$$

$$sQ_1(x) = Q_3(x) - rxQ_2(x)$$

$$\vdots$$

$$sQ_n(x) = Q_{n+2}(x) - rxQ_{n+1}(x)$$

On solving this, we get our desired result.

(b) and *(c)*

Again, making the use of

$$rxQ_{n+1}(x) = Q_{n+2}(x) - sQ_n(x)$$

Therefore,

$$rxQ_1(x) = Q_2(x) - sQ_0(x)$$

$$rxQ_3(x) = Q_4(x) - sQ_2(x)$$

$$\vdots$$

$$rxQ_{2n+1}(x) = Q_{2n+2}(x) - sQ_{2n}(x)$$

On simplifying these system of equations, it yields

$$rx\sum_{k=0}^{n} Q_{2k+1}(x) = Q_{2n+2}(x) + \left(\sum_{k=0}^{n} Q_{2k}(x) - Q_0(x)\right) - s\left(\sum_{k=0}^{n} Q_{2k}(x)\right) \tag{2.1}$$

Now,

$$sQ_n(x) = Q_{n+2}(x) - rxQ_{n+1}(x)$$

By assigning even values to n, it becomes

$$rxQ_2(x) = Q_3(x) - sQ_1(x)$$

$$rxQ_4(x) = Q_5(x) - sQ_3(x)$$

$$\vdots$$

$$rxQ_{2n}(x) = Q_{2n+1}(x) - sQ_{2n-1}(x)$$

On adding these equations, we get

$$rx\left(-Q_0(x) + \sum_{k=0}^{n} Q_{2k}(x)\right)$$

$$= \left(\sum_{k=0}^{n} Q_{2k+1}(x) - Q_1(x)\right) - s\left(\sum_{k=0}^{n} Q_{2k+1}(x) - Q_{2n+1}(x)\right)$$

(2.2)

On solving equations (2.1) and (2.2), the proof of *(b)* and *(c)* follows.

Corollary 2.2.1: For $n \geq 0$, following results hold for Fibonacci numbers:

(a)

$$\sum_{k=0}^{n} Q_k = Q_{n+2} + Q_1$$

(b)

$$\sum_{k=0}^{n} Q_{2k} = Q_{2n+1} - Q_1 + Q_0$$

(c)

$$\sum_{k=0}^{n} Q_{2k+1} = Q_{n+2} - Q_0$$

Proof: On taking $Q_0(x) = 0, Q_1(x) = 1, r = 1, s = 1,$ and $x = 1$ in theorem 2.2.1, the result is simpler to prove.

Corollary 2.2.2: For $n \geq 0$, Pell numbers follows the results:

(a)

$$\sum_{k=0}^{n} Q_k = \frac{Q_{n+2} - Q_{n+1} + Q_1 + Q_0}{2}$$

(b)

$$\sum_{k=0}^{n} Q_{2k} = \frac{Q_{2n+1} - Q_1 + 2Q_0}{2}$$

(c)

$$\sum_{k=0}^{n} Q_{2k} = \frac{Q_{n+2} - Q_0}{2}$$

Proof: By assigning value $Q_0(x) = 0, Q_1(x) = 1, r = 2$, $s = 1$, and $x = 1$ in theorem 2.2.1, it can be straightforwardly achieved.

Theorem 2.2.2 For $n \geq 0$, the first-order derivatives of the Generalized Fibonacci polynomials follow:

(a) If $(s + rx - 1)^2 \neq 0$, then

$$\sum_{k=0}^{n} Q'_k(x) = \frac{\mu}{(s + rx - 1)^2}$$

where,

$$\mu = (s + rx - 1)Q'_{n+2}(x) - rQ_{n+2}(x) - (rx - 1)(s + rx - 1)Q'_{n+1}(x)$$
$$-r(1 - rx)Q_{n+1}(x) - (s + rx - 1)Q'_1(x) - rQ_1(x)$$
$$+ (rx - 1)(s + rx - 1)Q'_0(x) + rsQ_0(x)$$

(b) If $(s^2 - r^2x^2 - 2s + 1) \neq 0$, then

$$\sum_{k=0}^{n} Q'_{2k}(x) = \frac{\mu}{(s^2 - r^2x^2 - 2s + 1)^2}$$

where,

$$\mu = \rho(s - 1)Q'_{2n+2}(x) + r(2xrs - 2xr - s^3 - s + 2s^2 + r^2x^2s)Q_{2n+1}(x)$$
$$- rx\rho Q'_{2n+1}(x) + r(s^2 + 1 - 2s + r^2x^2)Q_1(x) + rx\rho Q'_1(x)$$
$$- 2xr^2s(s - 1)Q_0(x) - \rho(r^2x^2 + s - 1)Q'_0(x),$$

and $\rho = s^2 - r^2x^2 - 2s + 1$.

(c) If $s^2 - r^2x^2 - 2s + 1 \neq 0$, then

$$\sum_{k=0}^{n} Q'_{2k+1}(x) = \frac{\mu}{\rho^2}$$

where,

$$\mu = \rho r x Q'_{2n+2}(x) - r(r^2x^2 + (s-1)^2)Q_{2n+2}(x) + 2xsr^2(s-1)Q_0(x)Q_{2n+1}(x) - s(s-1)\rho Q'_{2n+1}(x) - 2r^2x(s-1)Q_1(x) + (s-1)\rho Q'_1(x) + r(r^2x^2 + (s-1)^2)Q_0(x) - \rho r x Q'_0(x),$$

and

$$\rho = -(s^2 - r^2x^2 - 2s + 1)$$

Proof: The results follow from the theorem 2.2.1 just by differentiation w.r.t. x.

Corollary 2.2.3: For $n \geq 0$, the following formulae hold:

(a)

$$\sum_{k=0}^{n} Q'_k(1) = Q'_{n+2}(1) - Q_{n+2}(1) - Q'_1(1) - Q_1(1) + Q_0(1)$$

(b)

$$\sum_{k=0}^{n} Q'_{2k}(1) = Q'_{2n+1}(1) + Q_{2n+1}(1) - Q'_1(1) + Q_1(1) + Q'_0(1)$$

(c)

$$\sum_{k=0}^{n} Q'_{2k+1}(1) = -Q_{2n+1}(1) - Q'_0(1) + Q_0(1)$$

Proof: By using $r = s = x = 1$ in the above theorem 2.2.2, the above results follow.

Furthermore, we express the solution of an ordinary differential equation in terms of Fibonacci numbers and then determine the extremum values of Fibonacci polynomials having one and two variables.

The Fibonacci polynomials of one variable is given as

$F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$, for integral values of $n \geq 2$, (2.3)

with $F_1(x) = 1, F_2(x) = x$.

The Fibonacci polynomials $F_n(x)$ stated above, satisfy the second-order linear [75] differential equation expressed as

$$(x^2+4)y''+3xy'-(n^2-1)y=0 \tag{2.4}$$

Also, the Fibonacci polynomials in two variables [11] are defined by the following

$$F_n(z',t')=z'F_{n-1}(z',t')+t'F_{n-2}(z',t'), \text{ for } n\geq 2, \tag{2.5}$$

with $F_0(z',t')=0, F_1(z',t')=1$.

2.3 Solution of a Differential Equation in Terms of Fibonacci Numbers

Consider the following second-order linear homogenous differential equation

$$y''-y'-y=0 \tag{2.6}$$

The solution of (2.6) is

$$y=c_1e^{\alpha x}+c_2e^{\beta x}, \tag{2.7}$$

where c_1 and c_2 are arbitrary constants and $\alpha=\frac{1+\sqrt{5}}{2}$, and $\beta=\frac{1-\sqrt{5}}{2}$.

If we take $c_1=1$ and $c_2=-1$, then the solution (2.7) becomes

$$y=e^{\alpha x}-e^{\beta x} \tag{2.8}$$

Next, using the Maclaurin series expansion of an exponential function, we get

$$y=(\alpha-\beta)x+\frac{(\alpha^2-\beta^2)x^2}{2!}+\frac{(\alpha^3-\beta^3)x^3}{3!}+\frac{(\alpha^4-\beta^4)x^4}{4!}+\cdots \tag{2.9}$$

Dividing both sides by $(\alpha-\beta)$, we have

$$\frac{y}{(\alpha-\beta)}=x+\frac{(\alpha^2-\beta^2)x^2}{2!\,(\alpha-\beta)}+\frac{(\alpha^3-\beta^3)x^3}{3!\,(\alpha-\beta)}+\frac{(\alpha^4-\beta^4)x^4}{4!\,(\alpha-\beta)}+\cdots$$

Further, the Binet formula of Fibonacci numbers [3] gives,

$$\frac{y}{(\alpha-\beta)}=F_0+F_1x+\frac{F_2x^2}{2!}+\frac{F_3x^3}{3!}+\frac{F_4x^4}{4!}+\cdots$$

$$=\sum_{n=0}^{\infty}\frac{F_nx^n}{n!}$$

Further, the radius of convergence of this series

$$\sum_{n=0}^{\infty} \frac{F_n x^n}{n!}$$

is infinite, that why the solution of differential equation (2.6) converges for all real values of x.

2.4 Extremum Values of Fibonacci Polynomials and Bivariate Fibonacci Polynomials

From the recurrence relation (2.3), we observe that for every odd values of n, sequence of the Fibonacci polynomials of one variable has no real roots by Descartes' rule of sign. And for every even values of n, it has only one real root equal to zero. Further, by Descartes' rule of sign, we observe that there is only one critical point equal to zero for the sequence of Fibonacci polynomials having odd indices and there is no critical point for a sequence of Fibonacci polynomials having even indices (2.3), therefore have no extreme values. On the other hand, Fibonacci polynomials having odd indices has a positive second derivative at $x = 0$, therefore attain minima at $x = 0$. Thus, we can conclude that Fibonacci polynomials defined by (2.3) have

1. no extreme value for every even values of n
2. Only one extreme value for every odd values of n.

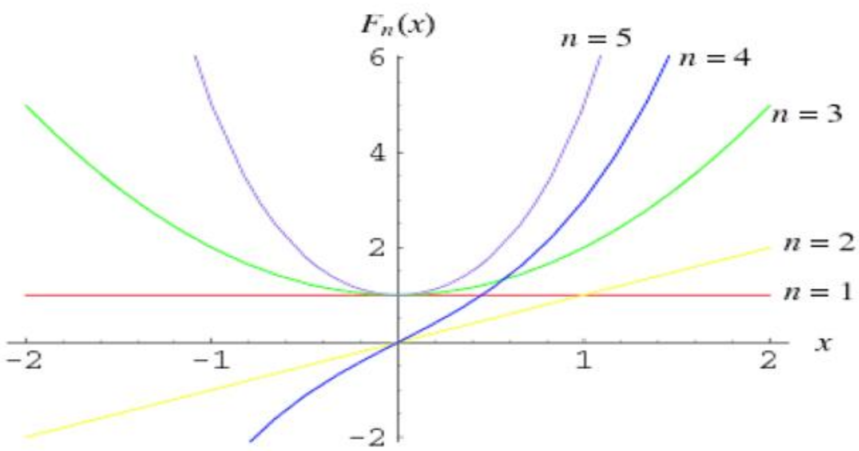


Figure 2.1: Graph for Fibonacci polynomials

Next, we find out the extremum values for the Fibonacci polynomials in two variables defined by (2.5) for $n = 1$ to 7. Now, the necessary condition for a function $f(x, y)$ of two variables to have an extremum at (a, b) is that

$$\frac{\partial f}{\partial x} = 0, \text{ and } \frac{\partial f}{\partial y} = 0$$

at (a, b).

We observe that there is no critical point for $n = 1, 2, 3$. And for $n = 4, 5, 6, 7$ there is only one critical point (0,0).

Now, we find the nature of critical point (0,0) for $n = 4, 5, 6, 7$

$$F_4(z', t') - F_4(0,0) < 0, \text{ for } z' = -0.0001, \text{ and } t' = 0.001$$

And $F_4(z', t') - F_4(0,0) > 0$, for $z' = 0.0001$, and $t' = 0.001$, therefore $F_4(z', t')$ has neither maxima nor minima at $(0,0)$.

Now, for $n = 5$

$$F_5(z', t') - F_5(0,0) < 0, \text{ for } z' = 0.01, \text{ and } t' = -0.001$$

And $F_5(z', t') - F_5(0,0) > 0$, for $z' = 0.01$, and $t' = 0.001$, therefore $F_5(z', t')$ has neither maxima nor minima at $(0,0)$.

Now, for $n = 6$

$$F_6(z', t') - F_6(0,0) < 0, \text{ for } z' = -0.01, \text{ and } t' = 0.001$$

And $F_6(z', t') - F_6(0,0) > 0$, for $z' = 0.01$, and $t' = 0.001$, therefore $F_6(z', t')$ has neither maxima nor minima at $(0,0)$.

Now, for $n = 7$

$$F_7(z', t') - F_7(0,0) < 0, \text{ for } z' = 0.001, \text{ and } t' = -0.00001$$

And $F_7(z', t') - F_7(0,0) > 0$, for $z' = 0.01$ and $t' = 0.001$, therefore $F_7(z', t')$ has neither maxima nor minima at $(0,0)$. Therefore, the Fibonacci polynomials defined by (2.5) has no extremum values for $n = 1$ to 7.

2.5 Graphical Representation of Bivariate Fibonacci Polynomials

Now, we plot the graph of the bivariate Fibonacci polynomials defined by (2.5), for $n = 3, 4, \dots, 7$ using MATLAB.

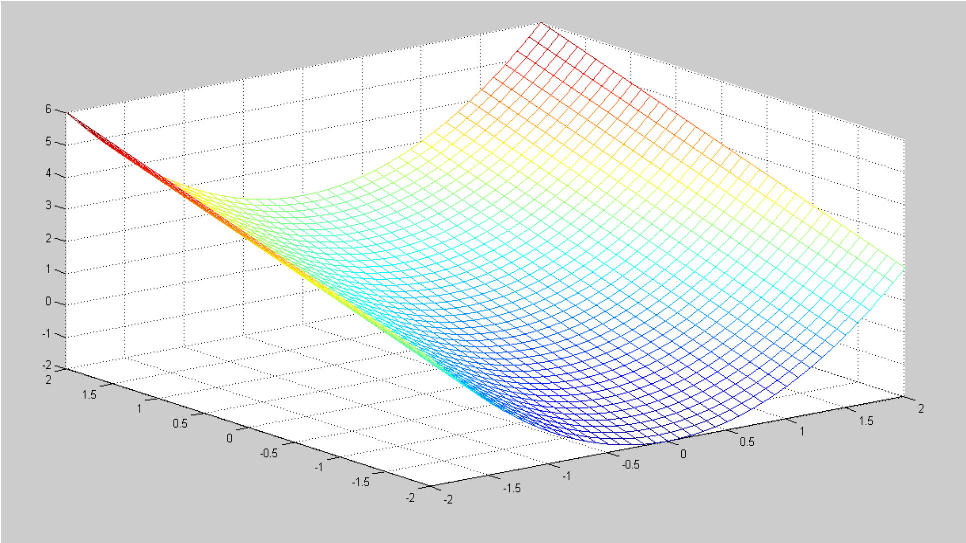


Figure 2.2: Graph for $F_3(z', t') = z'^2 + t'$

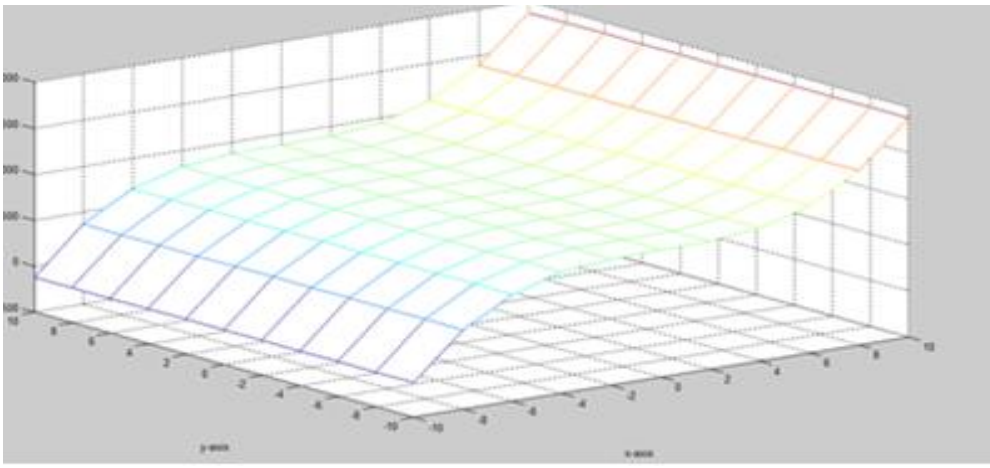


Figure 2.3: Graph for $F_4(z', t') = z'^3 + 2z't'$

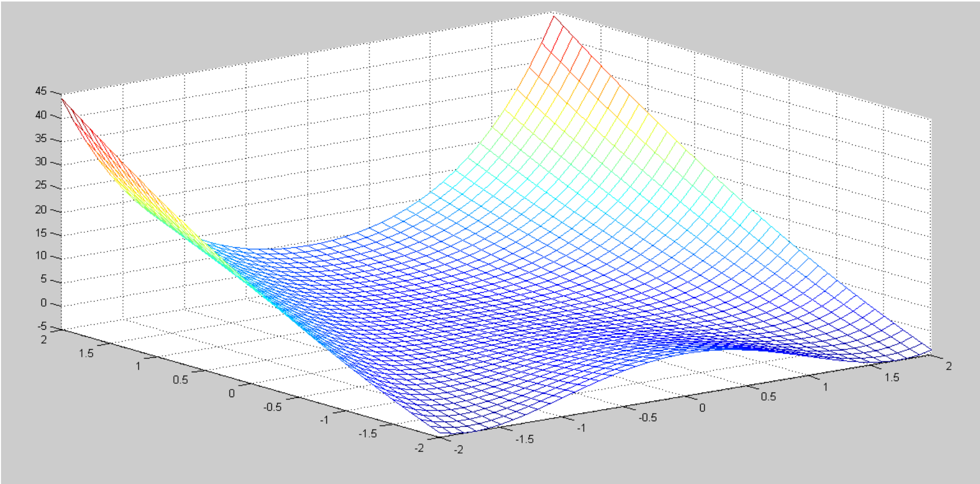


Figure 2.4: Graph for $F_5(z', t') = z'^4 + 3z'^2 t' + t'^2$

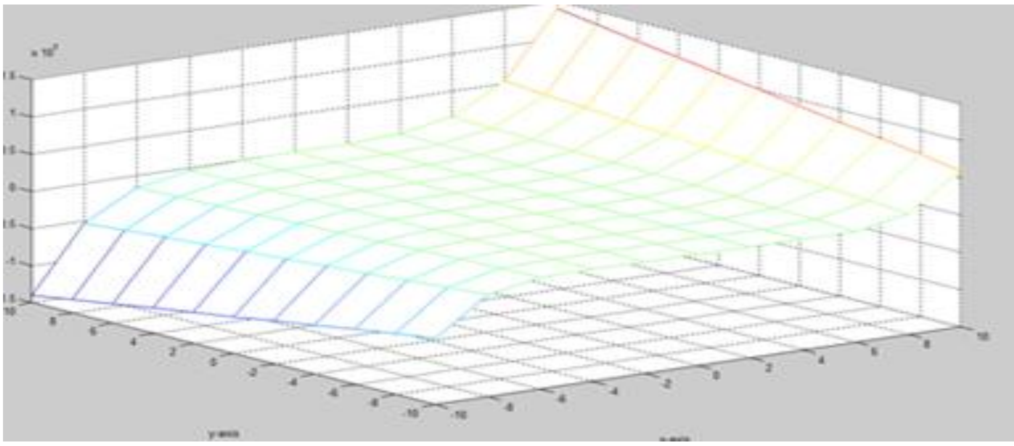


Figure 2.5: Graph for $F_6(z', t') = z'^5 + 4z'^3 t' + 3z' t'^2$

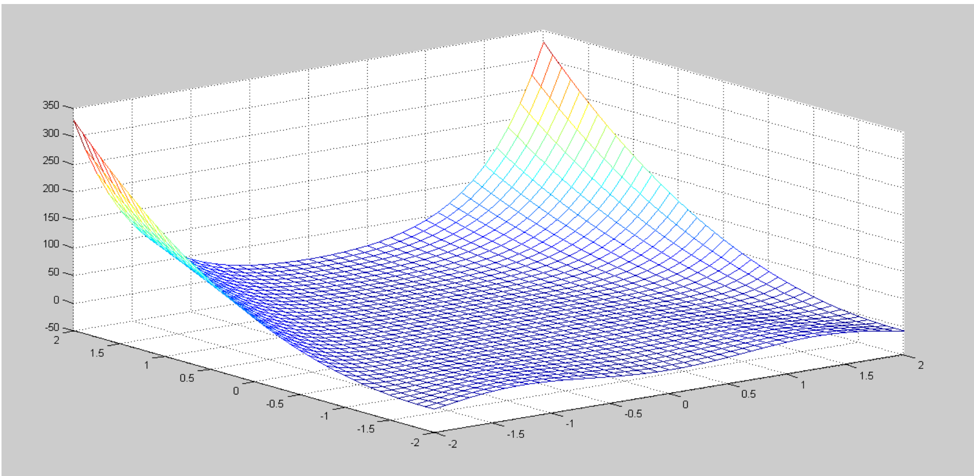


Figure 2.6: Graph for $F_7(z', t') = z'^6 + 5z'^3 t' + 6z'^2 t'^2 + t'^3$

Chapter 3

2-Fibonacci Sequences based on Congruences and Matrices

3.1 Introduction

Atanassov [19] introduced four different ways for constructing new sequences named as 2-Fibonacci sequences described by

$$\alpha_{n+2} = \beta_{n+1} + \beta_n, \beta_{n+2} = \alpha_{n+1} + \alpha_n, \quad (3.1)$$

$$\alpha_{n+2} = \alpha_{n+1} + \beta_n, \beta_{n+2} = \beta_{n+1} + \alpha_n, \quad (3.2)$$

$$\alpha_{n+2} = \beta_{n+1} + \alpha_n, \beta_{n+2} = \alpha_{n+1} + \beta_n, \quad (3.3)$$

$$\alpha_{n+2} = \alpha_{n+1} + \alpha_n, \beta_{n+2} = \beta_{n+1} + \beta_n, \quad (3.4)$$

with $\alpha_0 = a,\ \beta_0 = b\ ,\ \alpha_1 = c,\ \beta_1 = d$, and $n \geq 0$.

Here, we examine some of the characteristics of following 2-Fibonacci sequences:

$$\alpha_{n+2} = \beta_{n+1} + \beta_n, \beta_{n+2} = \alpha_{n+1} + \alpha_n, \quad (3.5)$$

where $\alpha_0 = 0, \alpha_1 = 1, \beta_0 = 2, \beta_1 = 1$, and $n = 0, 1, 2, \ldots$

Few terms of 2-Fibonacci sequences are listed as:

Table 3.1: 2-Fibonacci Sequences

n	α_n	β_n
2	3	1
3	2	4
4	5	5
5	9	7
6	12	14

This chapter is structured into mainly three sections. Numerous properties consisting of generating function, Binet's Formula, the relationship between α_n and β_n, correlation of 2-Fibonacci sequences with Fibonacci numbers, Cassini's Identity, etc., are described in the first two sections. Further, we introduce a matrix E called as generating matrix defined as

$$E = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

By using this matrix, we establish multiple relations for 2-Fibonacci sequences. After that, we derive Binet's formula by using the notion of diagonalizability of matrix in the third section.

3.2 Generating Function and Binet's Formulae for 2-Fibonacci Sequences

A Generating function $g_F(\mathfrak{v})$ of a sequence, say $\{c_n\}$ is a power series expansion having coefficients as the terms of a sequence $\{c_n\}$ and is given by

$$g_F\,(\mathfrak{v}) = \sum_{n=0}^{\infty} c_n\, \mathfrak{v}^n$$

Further, we aim to formulate the generating functions and Binet's formula for the 2-Fibonacci sequences α_n and β_n.

Theorem 3.2.1 Following results hold for 2-Fibonacci sequences $\{\alpha_n\}$ and $\{\beta_n\}$:

$$\sum_{n=0}^{\infty} \alpha_n\, \mathfrak{v}^n = \frac{\mathfrak{v} + 3\mathfrak{v}^2 + \mathfrak{v}^3}{1 - \mathfrak{v}^2 - 2\mathfrak{v}^3 - \mathfrak{v}^4}, \tag{3.6}$$

and

$$\sum_{n=0}^{\infty} \beta_n\, \mathfrak{v}^n = \frac{2 + \mathfrak{v} - \mathfrak{v}^2 - \mathfrak{v}^3}{1 - \mathfrak{v}^2 - 2\mathfrak{v}^3 - \mathfrak{v}^4} \tag{3.7}$$

Proof: Consider

$$g_F(\mathfrak{v}) = \sum_{n=0}^{\infty} \alpha_n \mathfrak{v}^n = \mathfrak{v} + 3\mathfrak{v}^2 + 2\mathfrak{v}^3 + \sum_{n=4}^{\infty} (\alpha_{n-2} + 2\alpha_{n-3} + \alpha_{n-4})\mathfrak{v}^n$$

$$= \mathfrak{v} + 3\mathfrak{v}^2 + \mathfrak{v}^3 + g_F(\mathfrak{v})(\mathfrak{v}^2 + 2\mathfrak{v}^3 + \mathfrak{v}^4)$$

Thus,

$$g_F(\mathfrak{v}) = \frac{\mathfrak{v} + 3\mathfrak{v}^2 + \mathfrak{v}^3}{1 - \mathfrak{v}^2 - 2\mathfrak{v}^3 - \mathfrak{v}^4}$$

Similarly, generating function $g_F(\mathfrak{v})$ for 2-Fibonacci sequence $\{\beta_n\}$ is

$$g_F(\mathfrak{v}) = \sum_{n=0}^{\infty} \beta_n \mathfrak{v}^n = 2 + \mathfrak{v} - \mathfrak{v}^2 - \mathfrak{v}^3 + g_F(\mathfrak{v})(\mathfrak{v}^2 + 2\mathfrak{v}^3 + \mathfrak{v}^4)$$

Thus,

$$\sum_{n=0}^{\infty} \beta_n \mathfrak{v}^n = \frac{2 + \mathfrak{v} - \mathfrak{v}^2 - \mathfrak{v}^3}{1 - \mathfrak{v}^2 - 2\mathfrak{v}^3 - \mathfrak{v}^4}$$

Theorem 3.2.2 For $n \geq 0$, the n^{th} term of 2-Fibonacci sequences can be expressed by the following formulae

$$\boldsymbol{\alpha_n} = \frac{(\gamma^{n+4} + \gamma^{n+1})}{(\gamma - \delta)(\gamma - \omega)(\gamma - \omega^2)} + \frac{(\delta^{n+4} + \delta^{n+1})}{(\delta - \gamma)(\delta - \omega)(\delta - \omega^2)}$$
$$+ \frac{2\omega^{n+1}}{(\omega - \gamma)(\omega - \delta)(\omega - \omega^2)} + \frac{2\omega^{2n+2}}{(\omega^2 - \gamma)(\omega^2 - \delta)(\omega^2 - \omega)},$$

and

$$\boldsymbol{\beta_n} = \frac{2\gamma^{n+3}}{(\gamma - \delta)(\gamma - \omega)(\gamma - \omega^2)} + \frac{2\delta^{n+3}}{(\delta - \gamma)(\delta - \omega)(\delta - \omega^2)}$$
$$- \frac{2\omega^{n+1}}{(\omega - \gamma)(\omega - \delta)(\omega - \omega^2)} - \frac{2(\omega^2)^{n+1}}{(\omega^2 - \gamma)(\omega^2 - \delta)(\omega^2 - \omega)},$$

where

$$\gamma = \frac{1 + \sqrt{5}}{2}, \quad \delta = \frac{1 - \sqrt{5}}{2}, \quad \omega = \frac{-1 + i\sqrt{3}}{2}, \quad \omega^2 = \frac{-1 - i\sqrt{3}}{2}$$

Proof: Let

$$\sum_{n=0}^{\infty} \alpha_n \mathfrak{v}^n = \frac{\mathfrak{v} + 3\mathfrak{v}^2 + \mathfrak{v}^3}{1 - \mathfrak{v}^2 - 2\mathfrak{v}^3 - \mathfrak{v}^4} = \frac{A}{(\gamma - \mathfrak{v})} + \frac{B}{(\delta - \mathfrak{v})} + \frac{C}{(\omega - \mathfrak{v})} + \frac{D}{(\omega^2 - \mathfrak{v})}$$

By using binomial expansion and comparing the coefficients of $\mathfrak{v}^n$, it becomes

$$\alpha_n = \frac{(\gamma^{n+4} + \gamma^{n+1})}{(\gamma - \delta)(\gamma - \omega)(\gamma - \omega^2)} + \frac{(\delta^{n+4} + \delta^{n+1})}{(\delta - \gamma)(\delta - \omega)(\delta - \omega^2)} + \frac{2\omega^{n+1}}{(\omega - \gamma)(\omega - \delta)(\omega - \omega^2)} + \frac{2\omega^{2n+2}}{(\omega^2 - \gamma)(\omega^2 - \delta)(\omega^2 - \omega)}$$

Similarly, by using the generating functions (3.7), we can easily prove that

$$\beta_n = \frac{2\gamma^{n+3}}{(\gamma - \delta)(\gamma - \omega)(\gamma - \omega^2)} + \frac{2\delta^{n+3}}{(\delta - \gamma)(\delta - \omega)(\delta - \omega^2)} - \frac{2\omega^{n+1}}{(\omega - \gamma)(\omega - \delta)(\omega - \omega^2)} - \frac{2(\omega^2)^{n+1}}{(\omega^2 - \gamma)(\omega^2 - \delta)(\omega^2 - \omega)}$$

3.3 Results of 2-Fibonacci Sequences based on Congruences

3.3.1 $\alpha_n \equiv \big(\beta_n + (1 - r)\big)(mod\ 3)$, where $n \equiv r(mod\ 3)$.

Proof: Evidently, the result is true for $n = 1$.

Assuming the induction hypothesis, we have

$$\beta_{k+1} + (1 - r) \equiv \beta_k + \beta_{k-1} - 3r + 6(mod\ 3) \equiv \beta_k + \beta_{k-1}(mod\ 3) \equiv \alpha_{k+1}(mod\ 3),$$

where $k \equiv r - 1(mod\ 3)$.

Corollary 3.3.1: $\alpha_n > \beta_n$, if $n \equiv 2(mod\ 3)$, and $\beta_n > \alpha_n$, if $n \equiv 0(mod\ 3)$

Proof: It follows directly from result 3.3.1.

3.3.2 Relation of 2-Fibonacci sequences with Fibonacci Numbers

$$F_{n+1} \equiv \alpha_n + (1 - r) \equiv \beta_n + (r - 1)(mod 3), \text{ where } n \equiv r(mod 3)$$

Proof: Undoubtedly, the result is true for $n = 1$.

Further, by induction hypothesis and result (3.3.1), we have

$$F_{n+2} = F_{n+1} + F_n \equiv \alpha_n + \alpha_{n-1} + r - 1 (mod\ 3) \equiv \beta_{n+1} + (r-1)(mod\ 3)$$
$$\equiv \alpha_{n+1} + (1-r)(mod\ 3),$$

where $n + 1 \equiv r(mod\ 3)$.

3.3.3 $\boldsymbol{\alpha_n}$, and $\boldsymbol{\beta_n}$ are even if $\boldsymbol{n \equiv 0(mod\ 3)}$ and odd if $\boldsymbol{n \equiv 1, 2(mod\ 3)}$

Proof: Since F_n is even if $n \equiv 0(mod\ 3)$, see [42], therefore from result 3.3.2, we obtain our required result.

3.3.4 Binet Formula for 2-Fibonacci sequences can also be expressed as

$$\boldsymbol{\frac{\gamma^{n+1} - \delta^{n+1}}{\gamma - \delta} = \alpha_n + (1-r) = \beta_n + (r-1),}$$

where $\boldsymbol{n \equiv r(mod\ 3)}$.

Proof: By using Binet's formula for Fibonacci sequence [40] is

$$\frac{\gamma^{n+1} - \delta^{n+1}}{\gamma - \delta} = F_{n+1},$$

where $\gamma = \frac{1+\sqrt{5}}{2}, \delta = \frac{1-\sqrt{5}}{2}$ and result 3.3.2, it can be easily proved.

3.3.5 If $\boldsymbol{n \equiv 0(mod\ 3)}$, then $\boldsymbol{{\beta_n}^2 \equiv 0(mod\ 4)}$, otherwise $\boldsymbol{{\beta_n}^2 \equiv 1(mod\ 4)}$

3.3.6 If $\boldsymbol{n \equiv 0(mod\ 3)}$, then $\boldsymbol{{\alpha_n}^2 \equiv 0(mod\ 4)}$, otherwise $\boldsymbol{{\alpha_n}^2 \equiv 1(mod\ 4)}$

3.3.7 $\boldsymbol{{\beta_n}^2 - {\beta_{n-3}}^2 \equiv {\alpha_n}^2 - {\alpha_{n-3}}^2 (mod\ 4)}$

Proof: These can be easily accomplished by using result 3.3.3.

3.3.8 (Cassini identity)

$$\boldsymbol{\beta_{n-1}\beta_{n+1} - {\beta_n}^2 \equiv 3(-1)^n(mod\ 4),}$$

and

$$\boldsymbol{\alpha_{n-1}\alpha_{n+1} - {\alpha_n}^2 \equiv 3(-1)^{n-1}(mod\ 4)}$$

Proof: Evidently, it is true for $n = 1$.

Further, by induction hypothesis and result 3.3.7, we have

$$\beta_n\beta_{n+2} - {\beta_{n+1}}^2 = \beta_n(\beta_n + 2\beta_{n-1} + \beta_{n-2}) - (\beta_{n-1} + 2\beta_{n-2} + \beta_{n-3})^2$$

$$\equiv {\beta_n}^2 - {\beta_{n-3}}^2 + 2\beta_n\beta_{n-1} + \beta_n\beta_{n-2} - {\beta_{n-1}}^2 - 2\beta_{n-1}\beta_{n-3}(mod\ 4)$$

$$\equiv 2\beta_{n-1}(\beta_n - \beta_{n-3)} + 3(-1)^{n-1}(mod\ 4) \equiv 3(-1)^{n+1}(mod\ 4)$$

In the same way, it can be demonstrated that

$$\alpha_{n-1}\alpha_{n+1} - {\alpha_n}^2 \equiv 3(-1)^{n-1}(mod\ 4)$$

3.4 Binet's Formula for 2-Fibonacci Sequences using Matrices

In this section, we establish a formula named Binet's formula for finding out any n^{th} term of 2-Fibonacci sequences using the concept of diagonalizability of a matrix E called as generating matrix, defined by

$$E = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{3.8}$$

such that, $detE = -1$.

Some results on 2-Fibonacci sequences (3.5) by using the matrix E are given as:

Theorem 3.4.1 Following results hold:

(a)

$$\begin{pmatrix} \alpha_{n+3} \\ \alpha_{n+2} \\ \alpha_{n+1} \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_{n+2} \\ \alpha_{n+1} \\ \alpha_n \\ \alpha_{n-1} \end{pmatrix},$$

where $n \geq 0$

(b)

$$\begin{pmatrix}\boldsymbol{\beta_{n+3}}\\ \boldsymbol{\beta_{n+2}}\\ \boldsymbol{\beta_{n+1}}\\ \boldsymbol{\beta_{n}}\end{pmatrix}=\begin{pmatrix}\mathbf{0}&\mathbf{1}&\mathbf{2}&\mathbf{1}\\ \mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}\end{pmatrix}\begin{pmatrix}\boldsymbol{\beta_{n+2}}\\ \boldsymbol{\beta_{n+1}}\\ \boldsymbol{\beta_{n}}\\ \boldsymbol{\beta_{n-1}}\end{pmatrix},$$

where $\boldsymbol{n \geq 0}$

(c)

$$\begin{pmatrix}\boldsymbol{\alpha_{n+2}}\\ \boldsymbol{\alpha_{n+1}}\\ \boldsymbol{\alpha_{n}}\\ \boldsymbol{\alpha_{n-1}}\end{pmatrix}=\boldsymbol{E^{n-1}}\begin{pmatrix}\mathbf{2}\\ \mathbf{3}\\ \mathbf{1}\\ \mathbf{0}\end{pmatrix},$$

where $\boldsymbol{n \geq 1}$

(d)

$$\begin{pmatrix}\boldsymbol{\beta_{n+2}}\\ \boldsymbol{\beta_{n+1}}\\ \boldsymbol{\beta_{n}}\\ \boldsymbol{\beta_{n-1}}\end{pmatrix}=\boldsymbol{E^{n-1}}\begin{pmatrix}\mathbf{4}\\ \mathbf{1}\\ \mathbf{1}\\ \mathbf{2}\end{pmatrix},$$

where $\boldsymbol{n \geq 1}$.

Proof: By using induction, these results have a simple proof.

(a) Using the recursive relation of sequence $\{\alpha_n\}_{n\geq 0}$, we have

$$\begin{pmatrix}0&1&2&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}\begin{pmatrix}\alpha_{n+2}\\ \alpha_{n+1}\\ \alpha_{n}\\ \alpha_{n-1}\end{pmatrix}=\begin{pmatrix}\alpha_{n+1}+2\alpha_n+\alpha_{n-1}\\ \alpha_{n+2}\\ \alpha_{n+1}\\ \alpha_{n}\end{pmatrix}=\begin{pmatrix}\alpha_{n+3}\\ \alpha_{n+2}\\ \alpha_{n+1}\\ \alpha_{n}\end{pmatrix}$$

(b) Using the recursive relation of sequence $\{\alpha_n\}_{n\geq 0}$, we have

$$\begin{pmatrix}0&1&2&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}\begin{pmatrix}\beta_{n+2}\\ \beta_{n+1}\\ \beta_{n}\\ \beta_{n-1}\end{pmatrix}=\begin{pmatrix}\beta_{n+1}+2\beta_n+\beta_{n-1}\\ \beta_{n+2}\\ \beta_{n+1}\\ \beta_{n}\end{pmatrix}=\begin{pmatrix}\beta_{n+3}\\ \beta_{n+2}\\ \beta_{n+1}\\ \beta_{n}\end{pmatrix}$$

(c) We approach the induction methodology to prove our required result.

Evidently, the result holds for $n = 1$.

By using the induction hypothesis and above obtained, we have

$$\begin{pmatrix} \alpha_{n+3} \\ \alpha_{n+2} \\ \alpha_{n+1} \\ \alpha_n \end{pmatrix} = E \begin{pmatrix} \alpha_{n+2} \\ \alpha_{n+1} \\ \alpha_n \\ \alpha_{n-1} \end{pmatrix} = EE^{n-1} \begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \end{pmatrix} = E^n \begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \end{pmatrix}$$

(d) We approach induction methodology to prove our required result.

Evidently, the result holds for $n = 1$.

By induction, we get

$$\begin{pmatrix} \beta_{n+3} \\ \beta_{n+2} \\ \beta_{n+1} \\ \beta_n \end{pmatrix} = E \begin{pmatrix} \beta_{n+2} \\ \beta_{n+1} \\ \beta_n \\ \beta_{n-1} \end{pmatrix} = EE^{n-1} \begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \end{pmatrix} = E^n \begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \end{pmatrix}$$

Further, we proceed to derive the Binet formulae for the 2-Fibonacci (3.5) sequences using the matrix E.

Clearly, the eigenvalues of matrix E are given by

$$\gamma = \frac{1+\sqrt{5}}{2}, \qquad \delta = \frac{1-\sqrt{5}}{2}, \qquad \omega = \frac{-1+i\sqrt{3}}{2}, \qquad \omega^2 = \frac{-1-i\sqrt{3}}{2}$$

Further, the eigenvectors corresponding to eigenvalues γ, δ, ω and $\omega^2$are

$$\begin{pmatrix} \gamma^3 \\ \gamma^2 \\ \gamma \\ 1 \end{pmatrix}, \begin{pmatrix} \delta^3 \\ \delta^2 \\ \delta \\ 1 \end{pmatrix}, \begin{pmatrix} \omega^3 \\ \omega^2 \\ \omega \\ 1 \end{pmatrix}, \text{ and } \begin{pmatrix} (\omega^2)^3 \\ (\omega^2)^2 \\ \omega^2 \\ 1 \end{pmatrix}$$

respectively.

Next, consider a matrix M containing eigenvectors of matrix E

$$M = \begin{pmatrix} \gamma^3 & \delta^3 & \omega^3 & \omega^6 \\ \gamma^2 & \delta^2 & \omega^2 & \omega^4 \\ \gamma & \delta & \omega & \omega^2 \\ 1 & 1 & 1 & 1 \end{pmatrix}_{4\times 4}$$

Then its inverse is given by

$$M^{-1} = \frac{1}{det\ M}\begin{pmatrix} A & A(\omega^2+\omega+\delta) & -\omega A(\omega^2+\omega\delta+\delta) & \delta\omega\omega^2 A \\ -B & -B(\omega^2+\omega+\gamma) & \omega B(\omega^2+\omega\gamma+\gamma) & -\gamma\omega\omega^2 B \\ C & C(\omega^2+\gamma+\delta) & -C(\omega^2\gamma+\omega^2\gamma+\gamma\delta) & \gamma\delta\omega^2 C \\ -D & -D(\omega+\gamma+\delta) & D(\omega\gamma+\gamma\delta+\omega\delta) & -\gamma\delta\omega D \end{pmatrix}_{4\times 4},$$

where $A = (\omega-\delta)(\omega^2-\delta)(\omega^2-\omega)$, $B = (\omega-\gamma)(\omega^2-\gamma)(\omega^2-\omega)$, $C = (\delta-\gamma)(\omega^2-\gamma)(-\omega^2+\delta)$, $D = (\delta-\gamma)(\omega-\gamma)(\delta-\omega)$, and $det\ M = (\omega-\delta)(\omega^2-\delta)(\omega^2-\omega)(\omega-\gamma)(\omega^2-\gamma)(\gamma-\delta)$.

Let's consider a diagonal matrix D involving eigenvalues of matrix E. Then, using the diagonalizability of matrix E, we have

$$E = MDM^{-1}, \text{ or } E^n = MD^nM^{-1} \tag{3.9}$$

Then, by using theorem 3.4.1 in (3.9), we have

$$\alpha_n = \frac{(\gamma^{n+4}+\gamma^{n+1})}{(\gamma-\delta)(\gamma-\omega)(\gamma-\omega^2)} + \frac{(\delta^{n+4}+\delta^{n+1})}{(\delta-\gamma)(\delta-\omega)(\delta-\omega^2)} + \frac{2\omega^{n+1}}{(\omega-\gamma)(\omega-\delta)(\omega-\omega^2)} + \frac{2\omega^{2n+2}}{(\omega^2-\gamma)(\omega^2-\delta)(\omega^2-\omega)},$$

and

$$\beta_n = \frac{2\gamma^{n+3}}{(\gamma-\delta)(\gamma-\omega)(\gamma-\omega^2)} + \frac{2\delta^{n+3}}{(\delta-\gamma)(\delta-\omega)(\delta-\omega^2)} - \frac{2\omega^{n+1}}{(\omega-\gamma)(\omega-\delta)(\omega-\omega^2)} - \frac{2(\omega^2)^{n+1}}{(\omega^2-\gamma)(\omega^2-\delta)(\omega^2-\omega)},$$

which is the required result.

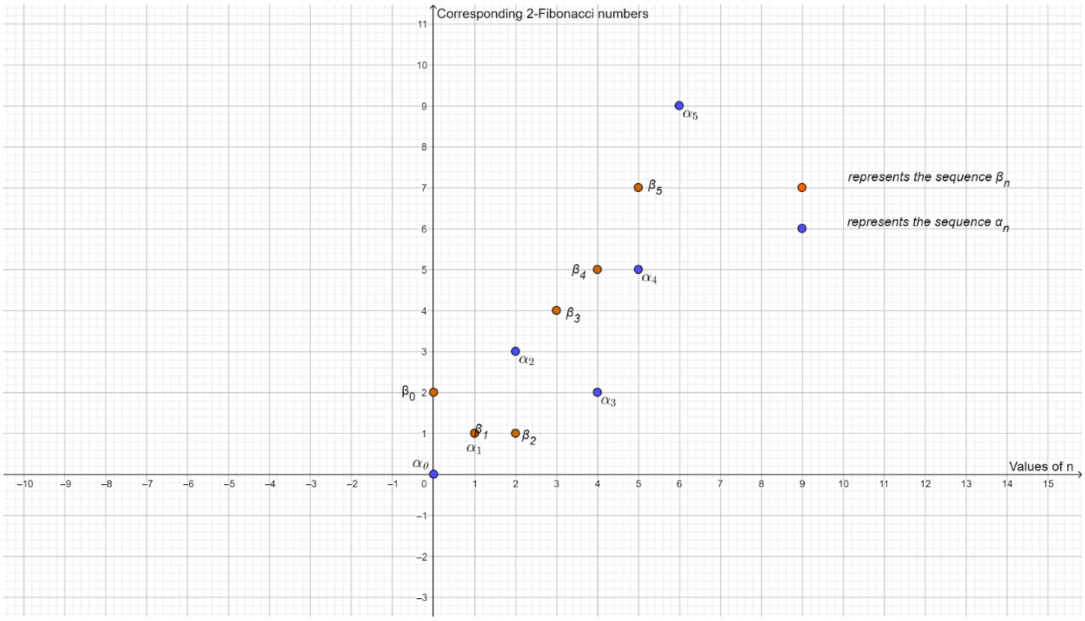


Figure 3.1: Comparison between 2-Fibonacci Sequences α_n and β_n

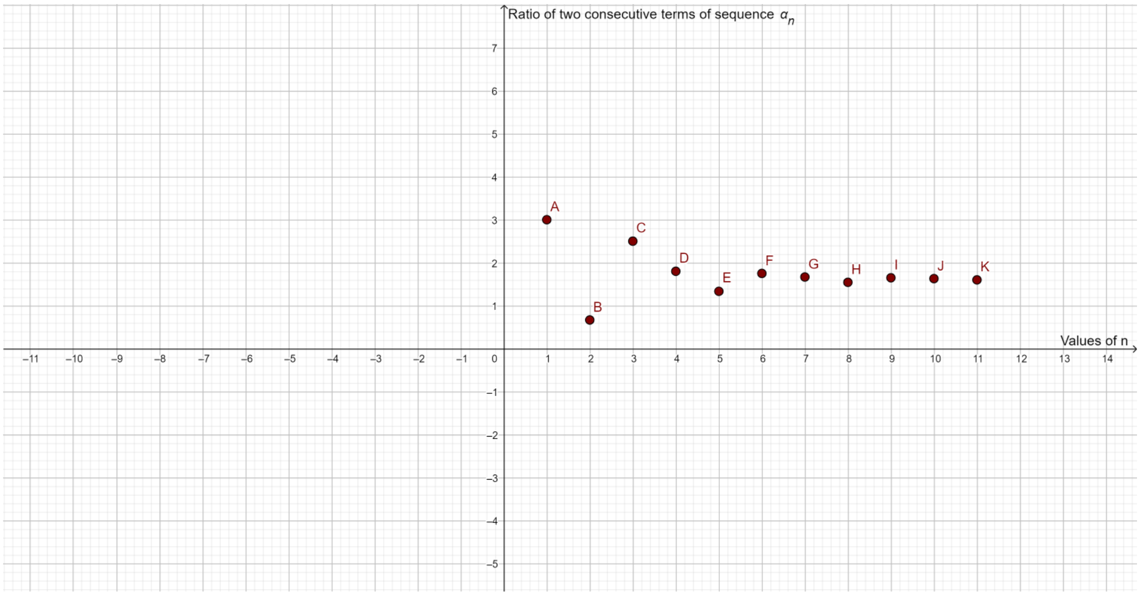


Figure 3.2: 2-Fibonacci Sequences α_n approaching the Golden Ratio

Chapter 4

Summation Formulae of 2-Fibonacci Sequences

4.1 Introduction

In this chapter, we derive various summation formulae for 2-Fibonacci sequences (3.5) defined by

$$\alpha_{n+2} = \beta_{n+1} + \beta_n,\ \beta_{n+2} = \alpha_{n+1} + \alpha_n,$$

where $\alpha_0 = 0$, $\alpha_1 = 1$, $\beta_0 = 2, \beta_1 = 1$, and n is a non-negative integer.

It can be clearly seen these sequences $\{\alpha_n\}_{n\geq 0}$ and $\{\beta_n\}_{n\geq 0}$ satisfy the following recursive relation

$$\mathcal{C}_{n+4} = \mathcal{C}_{n+2} + 2\mathcal{C}_{n+1} + \mathcal{C}_n, \tag{4.1}$$

with $\mathcal{C}_0 = a_1$, $\mathcal{C}_1 = b_1, \mathcal{C}_2 = c_1$, and $\mathcal{C}_3 = d_1$, are any real numbers. If we take $a_1 = 0$, $b_1 = 1, c_1 = 3$, and $d_1 = 2$, then it becomes the 2-Fibonacci sequence of numbers $\{\alpha_n\}$ and if $a_1 = 2, b_1 = 1, c_1 = 1$, and $d_1 = 4$, then it becomes the 2-Fibonacci sequence of numbers $\{\beta_n\}$.

Now, we develop summation formulae for 2- Fibonacci sequence of numbers listed as:

4.2 Sum Formulae of 2-Fibonacci Sequences

Theorem 4.2.1 For $m \geq 0$, 2-Fibonacci sequence satisfies:

(a)

$$\sum_{k=0}^{m} \mathcal{C}_k = \frac{1}{3}(\mathcal{C}_{m+4} + \mathcal{C}_{m+3} - 2\mathcal{C}_{m+1} - \mathcal{C}_3 - \mathcal{C}_2 + 2\mathcal{C}_0)$$

(b)

$$\sum_{k=0}^{m} \mathcal{C}_{2k} = -\frac{1}{3}(\mathcal{C}_{2m+2} - 2\mathcal{C}_{2m+3} + \mathcal{C}_{2m} + 2\mathcal{C}_3 - \mathcal{C}_2 - 4\mathcal{C}_0)$$

(c)

$$\sum_{k=0}^{m} \mathcal{C}_{2k+1} = \frac{1}{3}(2\mathcal{C}_{2m+2} + 3\mathcal{C}_{2m+1} + 2\mathcal{C}_{2m} - \mathcal{C}_{2m+3} + \mathcal{C}_3 - 2\mathcal{C}_2 - 2\mathcal{C}_0)$$

Proof:

(a) Using

$$\mathcal{C}_m = \mathcal{C}_{m+4} - \mathcal{C}_{m+2} - 2\mathcal{C}_{m+1}$$

We obtain

$$\mathcal{C}_0 = \mathcal{C}_4 - \mathcal{C}_2 - 2\mathcal{C}_1$$

$$\mathcal{C}_1 = \mathcal{C}_5 - \mathcal{C}_3 - 2\mathcal{C}_2$$

$$\vdots$$

$$\mathcal{C}_m = \mathcal{C}_{m+4} - \mathcal{C}_{m+2} - 2\mathcal{C}_{m+1}$$

By adding these equations, we get the result *(a)*.

(b) and *(c)*

Again using (4.1), we have

$$\mathcal{C}_{m+2} = \mathcal{C}_{m+4} - 2\mathcal{C}_{m+1} - \mathcal{C}_m$$

We obtain

$$\mathcal{C}_2 = \mathcal{C}_4 - 2\mathcal{C}_1 - \mathcal{C}_0$$

$$\mathcal{C}_4 = \mathcal{C}_6 - 2\mathcal{C}_3 - \mathcal{C}_2$$

$$\vdots$$

$$\mathcal{C}_{2m} = \mathcal{C}_{2m+2} - 2\mathcal{C}_{2m-1} - \mathcal{C}_{2m-2}$$

By adding these equations, we get

$$\begin{aligned} &\left(\sum_{k=0}^{m} \mathcal{C}_{2k} - \mathcal{C}_0\right) \\ &= \mathcal{C}_{2m+2} + \left(\sum_{k=0}^{m} \mathcal{C}_{2k} - \mathcal{C}_0 - \mathcal{C}_2\right) - 2\left(\sum_{k=0}^{m} \mathcal{C}_{2k+1} - \mathcal{C}_{2m+1}\right) \\ &- \left(\sum_{k=0}^{m} \mathcal{C}_{2k} - \mathcal{C}_{2m}\right) \end{aligned} \tag{4.2}$$

Again, using the recurrence relation

$$\mathcal{C}_{m+2} = \mathcal{C}_{m+4} - 2\mathcal{C}_{m+1} - \mathcal{C}_m$$

We write some following obvious equations

$$\mathcal{C}_3 = \mathcal{C}_5 - 2\mathcal{C}_2 - \mathcal{C}_1$$

$$\mathcal{C}_5 = \mathcal{C}_7 - 2\mathcal{C}_4 - \mathcal{C}_3$$

$$\vdots$$

$$\mathcal{C}_{2m+1} = \mathcal{C}_{2m+3} - 2\mathcal{C}_{2m} - \mathcal{C}_{2m-1}$$

Now, by adding these equations, we get

$$\left(\sum_{k=0}^{m} \mathcal{C}_{2k+1} - \mathcal{C}_1\right)$$

$$= \left(\sum_{k=0}^{m} \mathcal{C}_{2k+1} - \mathcal{C}_1 - \mathcal{C}_3 + \mathcal{C}_{2m+3}\right) - 2\left(\sum_{k=0}^{m} \mathcal{C}_{2k} - \mathcal{C}_0\right)$$

$$-\left(\sum_{k=0}^{m} \mathcal{C}_{2k+1} - \mathcal{C}_{2m+1}\right)$$

(4.3)

On solving (4.2) and (4.3), we get the desired results *(b)* and *(c)*.

Corollary 4.2.1: For $m \geq 0$ summation formulae for 2-Fibonacci sequence $\{\alpha_m\}$ holds:

(a)

$$\sum_{k=0}^{m} \alpha_k = \frac{1}{3}(\alpha_{m+4} + \alpha_{m+3} - 2\alpha_{m+1} - 5)$$

(b)

$$\sum_{k=0}^{m} \alpha_{2k} = -\frac{1}{3}(\alpha_{2m+2} - 2\alpha_{2m+3} + \alpha_{2m} + 1)$$

(c)

$$\sum_{k=0}^{m} \alpha_{2k+1} = \frac{1}{3}(2\alpha_{2m+2} + 3\alpha_{2m+1} + 2\alpha_{2m} - \alpha_{2m+3} - 4)$$

Proof:

The proof is obvious by taking $\mathcal{C}_0 = 0, \mathcal{C}_1 = 1, \mathcal{C}_2 = 3,$ and $\mathcal{C}_3 = 2$ in theorem 4.2.1.

Corollary 4.2.2: For $m \geq 0$, the following formulae holds for 2-Fibonacci sequence $\{\beta_m\}$:

(a)

$$\sum_{k=0}^{m} \beta_k = \frac{1}{3}(\beta_{m+4} + \beta_{m+3} - 2\beta_{m+1} - 1)$$

(b)

$$\sum_{k=0}^{m} \beta_{2k} = -\frac{1}{3}(\beta_{2m+2} - 2\beta_{2m+3} + \beta_{2m} - 1)$$

(c)

$$\sum_{k=0}^{m} \beta_{2k+1} = \frac{1}{3}(2\beta_{2m+2} + 3\beta_{2m+1} + 2\beta_{2m} - \beta_{2m+3} - 2)$$

Proof:

Solution is obvious using $\mathcal{C}_0 = 2, \mathcal{C}_1 = 1, \mathcal{C}_2 = 1$, and $\mathcal{C}_3 = 4$, in theorem 4.2.1.

Theorem 4.2.2 For $\mathfrak{m} \geq 0$, we have:

(a)

$$\sum_{\mathscr{k}=0}^{\mathfrak{m}} \alpha_{\mathscr{k}}\, \alpha_{\mathscr{k}+3} + 2\sum_{\mathscr{k}=0}^{\mathfrak{m}} \alpha_{\mathscr{k}}\, \alpha_{\mathscr{k}+2} = -6 + \alpha_{\mathfrak{m}+3}\alpha_{\mathfrak{m}+4} - 2\alpha_{\mathfrak{m}+1}\alpha_{\mathfrak{m}+3}$$

(b)

$$\sum_{\mathscr{k}=0}^{\mathfrak{m}} \alpha_{\mathscr{k}}^2 - \sum_{\mathscr{k}=0}^{\mathfrak{m}} \alpha_{\mathscr{k}}\, \alpha_{\mathscr{k}+3}$$

$$= 1 + \alpha_{\mathfrak{m}+1}\alpha_{\mathfrak{m}+4} + \alpha_{\mathfrak{m}+1}\alpha_{\mathfrak{m}+2} - \alpha_{\mathfrak{m}+1}\alpha_{\mathfrak{m}+3} - \alpha_{\mathfrak{m}+2}\alpha_{\mathfrak{m}+4} - \alpha_{\mathfrak{m}+1}^2 + \alpha_{\mathfrak{m}+2}^2$$

(c)

$$-2\sum_{\mathscr{k}=0}^{\mathfrak{m}} \alpha_{\mathscr{k}}\, \alpha_{\mathscr{k}+3}$$

$$= \alpha_{\mathfrak{m}+4}^2 + \alpha_{\mathfrak{m}+3}^2 + 3\alpha_{\mathfrak{m}+2}^2 + \alpha_{\mathfrak{m}+1}^2 - \alpha_{\mathfrak{m}+1}\alpha_{\mathfrak{m}+3} - 3\alpha_{\mathfrak{m}+2}\alpha_{\mathfrak{m}+4} - \alpha_{\mathfrak{m}+1}\alpha_{\mathfrak{m}+4} + 3\alpha_{\mathfrak{m}+1}\alpha_{\mathfrak{m}+2} - \alpha_{\mathfrak{m}+3}\alpha_{\mathfrak{m}+4} - 4$$

Proof:

(a) Now, using the relation

$$\alpha_{\mathfrak{m}}\alpha_{\mathfrak{m}+3} = \alpha_{\mathfrak{m}+4}\alpha_{\mathfrak{m}+3} - \alpha_{\mathfrak{m}+2}\alpha_{\mathfrak{m}+3} - 2\alpha_{\mathfrak{m}+1}\alpha_{\mathfrak{m}+3}$$

We have

$$\alpha_0\alpha_3 = \alpha_4\alpha_3 - \alpha_2\alpha_3 - 2\alpha_1\alpha_3$$

$$\alpha_1\alpha_4 = \alpha_5\alpha_4 - \alpha_3\alpha_4 - 2\alpha_2\alpha_4$$

$$\vdots$$

$$\alpha_{m}\alpha_{m+3} = \alpha_{m+4}\alpha_{m+3} - \alpha_{m+2}\alpha_{m+3} - 2\alpha_{m+1}\alpha_{m+3}$$

Thus, from the above system of equations, we obtain

$$\sum_{k=0}^{m} \alpha_k\, \alpha_{k+3} + 2\sum_{k=0}^{m} \alpha_k\, \alpha_{k+2} = -6 + \alpha_{m+3}\alpha_{m+4} - 2\alpha_{m+1}\alpha_{m+3} \tag{4.4}$$

(b) Using the relation

$$\alpha_m^2 = \alpha_{m+4}^2 + \alpha_{m+2}^2 + 4\alpha_{m+1}^2 - 2\alpha_{m+2}\alpha_{m+4} - 4\alpha_{m+1}\alpha_{m+4} + 4\alpha_{m+1}\alpha_{m+2}$$

We have

$$\alpha_0^2 = \alpha_4^2 + \alpha_2^2 + 4\alpha_1^2 - 2\alpha_2\alpha_4 - 4\alpha_1\alpha_4 + 4\alpha_1\alpha_2$$

$$\alpha_1^2 = \alpha_5^2 + \alpha_3^2 + 4\alpha_2^2 - 2\alpha_3\alpha_5 - 4\alpha_2\alpha_5 + 4\alpha_2\alpha_3$$

$$\vdots$$

$$\alpha_m^2 = \alpha_{m+4}^2 + \alpha_{m+2}^2 + 4\alpha_{m+1}^2 - 2\alpha_{m+2}\alpha_{m+4} - 4\alpha_{m+1}\alpha_{m+4} + 4\alpha_{m+1}\alpha_{m+2}$$

Thus, from the above system of equations, we obtain

$$\begin{aligned} -5\sum_{k=0}^{m} \alpha_k^2 + 2\sum_{k=0}^{m} \alpha_k\, \alpha_{k+2} + 4\sum_{k=0}^{m} \alpha_k\, \alpha_{k+3} - 4\sum_{k=0}^{m} \alpha_k\, \alpha_{k+1} \\ = \alpha_{m+4}^2 + \alpha_{m+3}^2 + 2\alpha_{m+2}^2 + 6\alpha_{m+1}^2 - 11 - 2\alpha_{m+1}\alpha_{m+3} - 2\alpha_{m+2}\alpha_{m+4} \\ - 4\alpha_{m+1}\alpha_{m+4} + 4\alpha_{m+1}\alpha_{m+2} \end{aligned} \tag{4.5}$$

Now, using the relation

$$\alpha_{m}\alpha_{m+1} = \alpha_{m+1}\alpha_{m+4} - \alpha_{m+1}\alpha_{m+2} - 2\alpha_{m+1}^{2}$$

We have

$$\alpha_{0}\alpha_{1} = \alpha_{1}\alpha_{4} - \alpha_{1}\alpha_{2} - 2\alpha_{1}^{2}$$

$$\alpha_{1}\alpha_{2} = \alpha_{2}\alpha_{5} - \alpha_{2}\alpha_{3} - 2\alpha_{2}^{2}$$

$$\vdots$$

$$\alpha_{m}\alpha_{m+1} = \alpha_{m+1}\alpha_{m+4} - \alpha_{m+1}\alpha_{m+2} - 2\alpha_{m+1}^{2}$$

Thus, from the above system of equations, we obtain

$$2\sum_{k=0}^{m}\alpha_{k}\,\alpha_{k+1} - \sum_{k=0}^{m}\alpha_{k}\,\alpha_{k+3} + 2\sum_{k=0}^{m}\alpha_{k}^{2} = -2\alpha_{m+1}^{2} + \alpha_{m+1}\alpha_{m+4} - \alpha_{m+1}\alpha_{m+2} \tag{4.6}$$

Now, using the relation

$$\alpha_{m}\alpha_{m+2} = \alpha_{m+4}\alpha_{m+2} - \alpha_{m+2}^{2} - 2\alpha_{m+1}\alpha_{m+2}$$

We have

$$\alpha_{0}\alpha_{2} = \alpha_{4}\alpha_{2} - \alpha_{2}^{2} - 2\alpha_{1}\alpha_{2}$$

$$\alpha_{1}\alpha_{3} = \alpha_{5}\alpha_{3} - \alpha_{3}^{2} - 2\alpha_{2}\alpha_{3}$$

$$\vdots$$

$$\alpha_{m}\alpha_{m+2} = \alpha_{m+4}\alpha_{m+2} - \alpha_{m+2}^{2} - 2\alpha_{m+1}\alpha_{m+2}$$

Thus, from the above system of equations, we obtain

$$\sum_{\mathcal{k}=0}^{\mathfrak{m}} \alpha_{\mathcal{k}}^2 + 2\sum_{\mathcal{k}=0}^{\mathfrak{m}} \alpha_{\mathcal{k}}\, \alpha_{\mathcal{k}+1}$$
$$= -1 + \alpha_{\mathfrak{m}+1}\alpha_{\mathfrak{m}+3} + \alpha_{\mathfrak{m}+2}\alpha_{\mathfrak{m}+4} - \alpha_{\mathfrak{m}+1}^2 - \alpha_{\mathfrak{m}+2}^2 - 2\alpha_{\mathfrak{m}+1}\alpha_{\mathfrak{m}+2}$$

(4.7)

Solving (4.6) and (4.7), we get

$$\sum_{\mathcal{k}=0}^{\mathfrak{m}} \alpha_{\mathcal{k}}^2 - \sum_{\mathcal{k}=0}^{\mathfrak{m}} \alpha_{\mathcal{k}}\, \alpha_{\mathcal{k}+3}$$
$$= 1 + \alpha_{\mathfrak{m}+1}\alpha_{\mathfrak{m}+4} + \alpha_{\mathfrak{m}+1}\alpha_{\mathfrak{m}+2} - \alpha_{\mathfrak{m}+1}\alpha_{\mathfrak{m}+3} - \alpha_{\mathfrak{m}+2}\alpha_{\mathfrak{m}+4} - \alpha_{\mathfrak{m}+1}^2$$
$$+ \alpha_{\mathfrak{m}+2}^2$$

(4.8)

(c) Solving (4.5) and (4.6), we get

$$-\sum_{\mathcal{k}=0}^{\mathfrak{m}} \alpha_{\mathcal{k}}^2 + 2\sum_{\mathcal{k}=0}^{\mathfrak{m}} \alpha_{\mathcal{k}}\, \alpha_{\mathcal{k}+2} + 2\sum_{\mathcal{k}=0}^{\mathfrak{m}} \alpha_{\mathcal{k}}\, \alpha_{\mathcal{k}+3}$$
$$= \alpha_{\mathfrak{m}+4}^2 + \alpha_{\mathfrak{m}+3}^2 + 2\alpha_{\mathfrak{m}+2}^2 + 2\alpha_{\mathfrak{m}+1}^2 - 2\alpha_{\mathfrak{m}+1}\alpha_{\mathfrak{m}+3} - 2\alpha_{\mathfrak{m}+2}\alpha_{\mathfrak{m}+4}$$
$$- 2\alpha_{\mathfrak{m}+1}\alpha_{\mathfrak{m}+4} + 2\alpha_{\mathfrak{m}+1}\alpha_{\mathfrak{m}+2} - 11$$

(4.9)

Solving (4.4) and (4.9), we get

$$-\sum_{\mathcal{k}=0}^{\mathfrak{m}} \alpha_{\mathcal{k}}^2 + \sum_{\mathcal{k}=0}^{\mathfrak{m}} \alpha_{\mathcal{k}}\, \alpha_{\mathcal{k}+3}$$
$$= \alpha_{\mathfrak{m}+4}^2 + \alpha_{\mathfrak{m}+3}^2 + 2\alpha_{\mathfrak{m}+2}^2 + 2\alpha_{\mathfrak{m}+1}^2 - 2\alpha_{\mathfrak{m}+2}\alpha_{\mathfrak{m}+4} - 2\alpha_{\mathfrak{m}+1}\alpha_{\mathfrak{m}+4}$$
$$+ 2\alpha_{\mathfrak{m}+1}\alpha_{\mathfrak{m}+2} - 5 - \alpha_{\mathfrak{m}+3}\alpha_{\mathfrak{m}+4}$$

(4.10)

Solving (4.8) and (4.9), we get

$$2\sum_{k=0}^{m} \alpha_k \alpha_{k+2} - \sum_{k=0}^{m} \alpha_k \alpha_{k+3}$$
$$= \alpha_{m+4}^2 + \alpha_{m+3}^2 + 3\alpha_{m+2}^2 + \alpha_{m+1}^2 - 3\alpha_{m+1}\alpha_{m+3} - 3\alpha_{m+2}\alpha_{m+4} + \alpha_{m+1}\alpha_{m+4} - 3\alpha_{m+1}\alpha_{m+2} - 10 \quad (4.11)$$

Solving (4.4) and (4.11), we get

$$-2\sum_{k=0}^{m} \alpha_k \alpha_{k+3}$$
$$= \alpha_{m+4}^2 + \alpha_{m+3}^2 + 3\alpha_{m+2}^2 + \alpha_{m+1}^2 - \alpha_{m+1}\alpha_{m+3} - 3\alpha_{m+2}\alpha_{m+4} - \alpha_{m+1}\alpha_{m+4} + 3\alpha_{m+1}\alpha_{m+2} - \alpha_{m+3}\alpha_{m+4} - 4$$

Theorem 4.2.3 For $m \geq 0$, the following formulae holds for 2-Fibonacci sequence of numbers:

(a)

$$\sum_{k=0}^{m} k\mathcal{C}_k = \frac{1}{3}(m\mathcal{C}_{m+4} + (m-1)\mathcal{C}_{m+3} - 2\mathcal{C}_{m+2} - 2(m+1)\mathcal{C}_{m+1} + \mathcal{C}_3 + 2\mathcal{C}_2 + 2\mathcal{C}_1)$$

(b)

$$\sum_{k=0}^{m} k\mathcal{C}_{2k+1} = \frac{1}{3}(2m\mathcal{C}_{2m+2} + (3m-1)\mathcal{C}_{2m+1} - m\mathcal{C}_{2m+3} + 2(m-1)\mathcal{C}_{2m} - 2\mathcal{C}_{2m-1} + 2\mathcal{C}_3 - \mathcal{C}_1 - 2\mathcal{C}_0)$$

Proof:

(a)

Using recursive relation

$$\mathcal{C}_m = \mathcal{C}_{m+4} - \mathcal{C}_{m+2} - 2\mathcal{C}_{m+1}$$

We obtain

$$1.\mathcal{C}_1 = 1.\mathcal{C}_5 - 1.\mathcal{C}_3 - 2.1\mathcal{C}_2$$
$$2.\mathcal{C}_2 = 2.\mathcal{C}_6 - 2.\mathcal{C}_4 - 2.2\mathcal{C}_3$$

$$\vdots$$

$$m.\mathcal{C}_m = m.\mathcal{C}_{m+4} - m.\mathcal{C}_{m+2} - 2.m\mathcal{C}_{m+1}$$

by adding these equations, we get *(a)*.

(b)

Again, on using (4.2), we have

$$\mathcal{C}_{m+2} = \mathcal{C}_{m+4} - 2\mathcal{C}_{m+1} - \mathcal{C}_m$$

Now by taking even values of m and approaching the same procedure, we obtain

$$\left(\sum_{k=0}^{m} k\mathcal{C}_{2k}\right) = m\mathcal{C}_{2m+2} + \left(\sum_{k=0}^{m} (k-1)\mathcal{C}_{2k} + \mathcal{C}_0\right) - 2\left(\sum_{k=0}^{m} (k+1)\mathcal{C}_{2k+1} - (m+1)\mathcal{C}_{2m+1}\right) - \left(\sum_{k=0}^{m} (k+1)\mathcal{C}_{2k+1} - (m+1)\mathcal{C}_{2m}\right) \quad (4.12)$$

Making the use of the recurrence relation

$$\mathcal{C}_{m+2} = \mathcal{C}_{m+4} - 2\mathcal{C}_{m+1} - \mathcal{C}_m$$

Further, by taking odd values of m and approaching the similar procedure used in (a), following equation generated:

$$\left(\sum_{k=0}^{m} k\mathcal{C}_{2k+1}\right) = \left(\sum_{k=0}^{m} (k-1)\mathcal{C}_{2k+1} + \mathcal{C}_1 + m\mathcal{C}_{2m+3}\right) - 2\left(\sum_{k=0}^{m} k\mathcal{C}_{2k}\right) - \left(\sum_{k=0}^{m} (k+1)\mathcal{C}_{2k+1} - (m+1)\mathcal{C}_{2m+1}\right) \quad (4.13)$$

On solving (4.12) and (4.13), proof of *(b)* follows.

Corollary 4.2.3: For $m \geq 0$, the following formulae holds for the 2-Fibonacci sequence $\{\alpha_m\}$:

(a)

$$\sum_{k=0}^{m} k\alpha_k = \frac{1}{3}(m\alpha_{m+4} + (m-1)\alpha_{m+3} - 2\alpha_{m+2} - 2(m+1)\alpha_{m+1} + 10)$$

(b)

$$\sum_{k=0}^{m} k\alpha_{2k+1} = \frac{1}{3}(2m\alpha_{2m+2} + (3m-1)\alpha_{2m+1} - m\alpha_{2m+3} + 2(m-1)\alpha_{2m} - 2\alpha_{2m-1} + 3)$$

Proof

(a)

Using recursive relation

$$\alpha_m = \alpha_{m+4} - \alpha_{m+2} - 2\alpha_{m+1}$$

We obtain

$$1.\alpha_1 = 1.\alpha_5 - 1.\alpha_3 - 2.1\alpha_2$$

$$2.\alpha_2 = 2.\alpha_6 - 2.\alpha_4 - 2.2\alpha_3$$

$$\vdots$$

$$m.\alpha_m = m.\alpha_{m+4} - m.\alpha_{m+2} - 2.m\alpha_{m+1}$$

Solving these equations, we get

$$\sum_{k=0}^{m} k\alpha_k = \frac{1}{3}(m\alpha_{m+4} + (m-1)\alpha_{m+3} - 2\alpha_{m+2} - 2(m+1)\alpha_{m+1} + \alpha_3 + 2\alpha_2 + 2\alpha_1)$$

$$= \frac{1}{3}(m\alpha_{m+4} + (m-1)\alpha_{m+3} - 2\alpha_{m+2} - 2(m+1)\alpha_{m+1} + 10)$$

(b)

Again, on using (4.2), we have

$$\alpha_{m+2} = \alpha_{m+4} - 2\alpha_{m+1} - \alpha_m$$

Now by taking even values of m and approaching the same procedure, we obtain

$$\left(\sum_{k=0}^{m} k\alpha_{2k}\right) = m\alpha_{2m+2} + \left(\sum_{k=0}^{m}(k-1)\alpha_{2k} + \alpha_0\right) - 2\left(\sum_{k=0}^{m}(k+1)\alpha_{2k+1} - (m+1)\alpha_{2m+1}\right) - \left(\sum_{k=0}^{m}(k+1)\alpha_{2k} - (m+1)\alpha_{2m}\right)$$

(4.14)

Making the use of the recurrence relation

$$\alpha_{m+2} = \alpha_{m+4} - 2\alpha_{m+1} - \alpha_m$$

Further, on taking odd values of m, we have

$$\left(\sum_{k=0}^{m} k\alpha_{2k+1}\right) = \left(\sum_{k=0}^{m}(k-1)\alpha_{2k+1} + \alpha_1 + m\alpha_{2m+3}\right) - 2\left(\sum_{k=0}^{m} k\alpha_{2k}\right) - \left(\sum_{k=0}^{m}(k+1)\alpha_{2k+1} - (m+1)\alpha_{2m+1}\right)$$

(4.15)

On solving (4.14) and (4.15), proof of *(b)* follows.

Corollary 4.2.4: For $m \geq 0$, the following formulae holds for 2-Fibonacci sequence $\{\beta_m\}$:

(a)

$$\sum_{k=0}^{m} k\beta_k = \frac{1}{3}(m\beta_{m+4} + (m-1)\beta_{m+3} - 2\beta_{m+2} - 2(m+1)\beta_{m+1} + 8)$$

(b)

$$\sum_{k=0}^{m} k\beta_{2k+1} = \frac{1}{3}(2m\beta_{2m+2} + (3m-1)\beta_{2m+1} - m\beta_{2m+3} + 2(m-1)\beta_{2m} - 2\beta_{2m-1} + 3)$$

Proof

(a)

Using recursive relation

$$\beta_m = \beta_{m+4} - \beta_{m+2} - 2\beta_{m+1}$$

We obtain

$$1.\beta_1 = 1.\beta_5 - 1.\beta_3 - 2.1\beta_2$$
$$2.\beta_2 = 2.\beta_6 - 2.\beta_4 - 2.2\beta_3$$
$$\vdots$$
$$m.\beta_m = m.\beta_{m+4} - m.\beta_{m+2} - 2.m\beta_{m+1}$$

by adding these equations, we get

$$\sum_{k=0}^{m} k\beta_k = \frac{1}{3}(m\beta_{m+4} + (m-1)\beta_{m+3} - 2\beta_{m+2} - 2(m+1)\beta_{m+1} + 8)$$

(b)

Again, on using (4.2), we have

$$\beta_{m+2} = \beta_{m+4} - 2\beta_{m+1} - \beta_m$$

Taking even values of m and approaching the same procedure, it results

$$\left(\sum_{k=0}^{m} k\beta_{2k}\right) = m\beta_{2m+2} + \left(\sum_{k=0}^{m}(k-1)\beta_{2k} + \beta_0\right) - 2\left(\sum_{k=0}^{m}(k+1)\beta_{2k+1} - (m+1)\beta_{2m+1}\right) - \left(\sum_{k=0}^{m}(k+1)\beta_{2k} - (n+1)\beta_{2m}\right) \tag{4.16}$$

Making the use of the recurrence relation

$$\beta_{m+2} = \beta_{m+4} - 2\beta_{m+1} - \beta_m$$

On taking odd values of m, it yields

$$\left(\sum_{k=0}^{m} k\beta_{2k+1}\right) = \left(\sum_{k=0}^{m}(k-1)\beta_{2k+1} + \beta_1 + m\beta_{2m+3}\right) - 2\left(\sum_{k=0}^{m} k\beta_{2k}\right) - \left(\sum_{k=0}^{m}(k+1)\beta_{2k+1} - (m+1)\beta_{2m+1}\right)$$

(4.17)

On solving (4.16) and (4.17), proof of *(b)* follows.

Theorem 4.2.4 For $m \geq 0$, we have:

(a)

$$\sum_{v=0}^{m} \beta_v \, \beta_{v+3} + 2\sum_{k=0}^{m} \beta_v \, \beta_{v+2} = \beta_{m+3}\beta_{m+4} - 2\beta_{m+1}\beta_{m+3}$$

(b)

$$\sum_{v=0}^{m} \beta_v^2 \; - \sum_{v=0}^{m} \beta_v \, \beta_{v+3}$$

$$= -1 + \beta_{m+1}\beta_{m+4} + \beta_{m+1}\beta_{m+2} - \beta_{m+1}\beta_{m+3} - \beta_{m+2}\beta_{m+4} - \beta_{m+1}^2 + \beta_{m+2}^2$$

(c)

$$-2\sum_{v=0}^{m} \beta_v \, \beta_{v+3}$$

$$= \beta_{m+4}^2 + \beta_{m+3}^2 + 3\beta_{m+2}^2 + \beta_{m+1}^2 - \beta_{m+1}\beta_{m+3} - 3\beta_{m+2}\beta_{m+4} - \beta_{m+1}\beta_{m+4} + 3\beta_{m+1}\beta_{m+2} - \beta_{m+3}\beta_{m+4} - 4$$

Proof:

(a) Now, using the relation

$$\beta_m\beta_{m+3} = \beta_{m+4}\beta_{m+3} - \beta_{m+2}\beta_{m+3} - 2\beta_{m+1}\beta_{m+3}$$

We have

$$\beta_0\beta_3 = \beta_4\beta_3 - \beta_2\beta_3 - 2\beta_1\beta_3$$

$$\beta_1\beta_4 = \beta_5\beta_4 - \beta_3\beta_4 - 2\beta_2\beta_4$$

$$\vdots$$

$$\beta_m\beta_{m+3} = \beta_{m+4}\beta_{m+3} - \beta_{m+2}\beta_{m+3} - 2\beta_{m+1}\beta_{m+3}$$

Thus, from the above system of equations, we obtain

$$\sum_{v=0}^{m} \beta_v \, \beta_{v+3} + 2\sum_{v=0}^{m} \beta_v \, \beta_{v+2} = \beta_{m+3}\beta_{m+4} - 2\beta_{m+1}\beta_{m+3}$$

(4.18)

(b) Using the relation

$$\beta_m^2 = \beta_{m+4}^2 + \beta_{m+2}^2 + 4\beta_{m+1}^2 - 2\beta_{m+2}\beta_{m+4} - 4\beta_{m+1}\beta_{m+4} + 4\beta_{m+1}\beta_{m+2}$$

We have

$$-5\sum_{v=0}^{m} \beta_v^2 + 2\sum_{v=0}^{m} \beta_v\beta_{v+2} + 4\sum_{v=0}^{m} \beta_v\beta_{v+3} - 4\sum_{v=0}^{m} \beta_v\beta_{v+1}$$
$$= \beta_{m+4}^2 + \beta_{m+3}^2 + 2\beta_{m+2}^2 + 6\beta_{m+1}^2 - 7 - 2\beta_{m+1}\beta_{m+3} - 2\beta_{m+2}\beta_{m+4}$$
$$- 4\beta_{m+1}\beta_{m+4} + 4\beta_{m+1}\beta_{m+2}$$

(4.19)

Now, using the relation

$$\beta_m\beta_{m+1} = \beta_{m+1}\beta_{m+4} - \beta_{m+1}\beta_{m+2} - 2\beta_{m+1}^2$$

We have

$$2\sum_{v=0}^{m} \beta_v\beta_{v+1} - \sum_{v=0}^{m} \beta_v\beta_{v+3} + 2\sum_{v=0}^{m} \beta_v^2 = 2 + \beta_{m+1}\beta_{m+4} - \beta_{m+1}\beta_{m+2} - 2\beta_{m+1}^2$$

(4.20)

Now, using the relation

$$\beta_m\beta_{m+2} = \beta_{m+4}\beta_{m+2} - \beta_{m+2}^2 - 2\beta_{m+1}\beta_{m+2}$$

We have

$$\sum_{v=0}^{m} \beta_v^2 - 2\sum_{v=0}^{m} \beta_v\, \beta_{v+1}$$
$$= 3 + \beta_{m+1}\beta_{m+3} + \beta_{m+2}\beta_{m+4} - \beta_{m+1}^2 - \beta_{m+2}^2 - 2\beta_{m+1}\beta_{m+2}$$

(4.21)

Solving (4.20) and (4.21), we get

$$\sum_{v=0}^{m} \beta_v^2 - \sum_{v=0}^{m} \beta_v\, \beta_{v+3}$$
$$= -1 + \beta_{m+1}\beta_{m+4} + \beta_{m+1}\beta_{m+2} - \beta_{m+1}\beta_{m+3}$$
$$- \beta_{m+2}\beta_{m+4} - \beta_{m+1}^2 + \beta_{m+2}^2$$

(4.22)

(c) Solving (4.19) and (4.20), we get

$$-\sum_{v=0}^{m} \beta_v^2 + 2\sum_{v=0}^{m} \beta_v\, \beta_{v+2} + 2\sum_{v=0}^{m} \beta_v\, \beta_{v+3}$$
$$= \beta_{m+4}^2 + \beta_{m+3}^2 + 2\beta_{m+2}^2 + 2\beta_{m+1}^2 - 2\beta_{m+1}\beta_{m+3} - 2\beta_{m+2}\beta_{m+4}$$
$$- 2\beta_{m+1}\beta_{m+4} + 2\beta_{m+1}\beta_{m+2} - 3$$

(4.23)

Solving (4.18) and (4.23), we get

$$-\sum_{v=0}^{m} \beta_v^2 + \sum_{v=0}^{m} \beta_v\, \beta_{v+3}$$
$$= \beta_{m+4}^2 + \beta_{m+3}^2 + 2\beta_{m+2}^2 + 2\beta_{m+1}^2 - 2\beta_{m+2}\beta_{m+4} - 2\beta_{m+1}\beta_{m+4}$$
$$+ 2\beta_{m+1}\beta_{m+2} - 3 - \beta_{m+3}\beta_{m+4}$$

(4.24)

Solving (4.22) and (4.23), we get

$$2\sum_{v=0}^{m}\beta_v\,\beta_{v+2}-\sum_{v=0}^{m}\beta_v\,\beta_{v+3}$$
$$=\beta_{m+4}^2+\beta_{m+3}^2+3\beta_{m+2}^2+\beta_{m+1}^2-3\beta_{m+1}\beta_{m+3}-3\beta_{m+2}\beta_{m+4}$$
$$-\beta_{m+1}\beta_{m+4}+3\beta_{m+1}\beta_{m+2}-4 \tag{4.25}$$

On solving (4.18) and (4.25), we get

$$-2\sum_{v=0}^{m}\beta_v\,\beta_{v+3}=\beta_{m+4}^2+\beta_{m+3}^2+3\beta_{m+2}^2+\beta_{m+1}^2-\beta_{m+1}\beta_{m+3}-3\beta_{m+2}\beta_{m+4}$$
$$-\beta_{m+1}\beta_{m+4}+3\beta_{m+1}\beta_{m+2}-\beta_{m+3}\beta_{m+4}-4=0$$

Chapter 5

Infinite Sum of Reciprocals of Generalized Fibonacci and Lucas Polynomials

5.1 Introduction

This chapter is based upon the Generalized Fibonacci polynomials (1.7) defined as

$$F_{k,q,n+1}(g) = k(g)F_{k,q,n}(g) + q(g)F_{k,q,n-1}(g); n \geq 1,$$

with $F_{k,q,0}(g) = 0, F_{k,q,1}(g) = 1$, and the Generalized Lucas polynomials (1.12) defined as

$$L_{k,q,n+1}(g) = k(g)L_{k,q,n}(g) + q(g)L_{k,q,n-1}(g); n \geq 1,$$

with initial conditions $L_{k,q,0}(g) = 2, L_{k,q,1}(g) = k(g)$, where $k(g)$ and $q(g)$ are polynomials having real coefficients. Here, we consider that $q(g) = 1$ or in other words, we study the sequences of polynomials $F_{k,1,n}(g)$ and $L_{k,1,n}(g)$. We notate these polynomials sequences by $F_{k,n}(g)$ and $L_{k,n}(g)$ respectively. Further, $k(g)$ is a polynomial having positive coefficients, so having no real roots. In this chapter, we mainly calculate the infinite the reciprocals sum of Generalized Fibonacci and Lucas polynomials defined by

$$F_{k,n+1}(g) = k(g)F_{k,n}(g) + F_{k,n-1}(g), \text{ for } n \geq 1, \text{ with } F_{k,0}(g) = 0,\ F_{k,1}(g) = 1, \tag{5.1}$$

and

$$L_{k,n+1}(g) = k(g)L_{k,n}(g) + L_{k,n-1}(g), \text{ for } n \geq 1, \text{ with } L_{k,0}(g) = 2,\ L_{k,1}(g) = k(g), \tag{5.2}$$

having even indices along with their squares generalizing the respective results of Fibonacci and Lucas polynomials.

5.2 Results on Reciprocal Sum of Generalized Fibonacci and Lucas Polynomials

Lemma 5.2.1 For $n \geq 0$, we have

(a)

$$\boldsymbol{F_{k,n}(g) = \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2}} \tag{5.3}$$

(b)

$$\boldsymbol{L_{k,n}(g) = \alpha_1^n + \alpha_2^n} \tag{5.4}$$

with $\alpha_1 = \frac{k(g)+\sqrt{k^2(g)+4}}{2}$, and $\alpha_2 = \frac{k(g)-\sqrt{k^2(g)+4}}{2}$.

Proof:

(a) Cleary, the result holds for $n = 1$.

By induction hypothesis, we have

$$\frac{\alpha_1^{n+1} - \alpha_2^{n+1}}{\alpha_1 - \alpha_2} = \frac{(\alpha_1 + \alpha_2)(\alpha_1^n - \alpha_2^n) - \alpha_1\alpha_2(\alpha_1^{n-1} - \alpha_2^{n-1})}{\alpha_1 - \alpha_2}$$

$$= k(g)F_{k,n}(g) + F_{k,n-1}(g) = F_{k,n+1}(g)$$

(b) Cleary, the result is true for $n = 1$.

Further,

$$\alpha_1^{n+1} + \alpha_2^{n+1} = (\alpha_1 + \alpha_2)(\alpha_1^n + \alpha_2^n) - \alpha_1\alpha_2(\alpha_1^{n-1} + \alpha_2^{n-1})$$

$$= k(g)L_{k,n}(g) + L_{k,n-1}(g) = L_{k,n+1}(g)$$

Lemma 5.2.2 The following results hold:

(a)

$$F_{\Bbbk,m}(g)F_{\Bbbk,n}(g) = \begin{cases} \dfrac{1}{\Bbbk^2(g)+4}\left(L_{\Bbbk,m+n}(g) - (-1)^n L_{\Bbbk,m-n}(g)\right), \text{if } m \geq n \\ \dfrac{1}{\Bbbk^2(g)+4}\left(L_{\Bbbk,m+n}(g) - (-1)^m L_{\Bbbk,n-m}(g)\right), \text{if } n \geq m \end{cases}$$

(5.5)

(b)

$$L_{\Bbbk,m}(g)L_{\Bbbk,n}(g) = \begin{cases} L_{\Bbbk,m+n}(g) + (-1)^n L_{\Bbbk,m-n}(g), \text{if } m \geq n \\ L_{\Bbbk,m+n}(g) + (-1)^m L_{\Bbbk,n-m}(g), \text{if } n \geq m \end{cases}$$

(5.6)

(c)

$$F_{\Bbbk,m}(g)L_{\Bbbk,n}(g) = \begin{cases} F_{\Bbbk,m+n}(g) + (-1)^n F_{\Bbbk,m-n}(g), \text{if } m \geq n \\ F_{\Bbbk,m+n}(g) - (-1)^m F_{\Bbbk,n-m}(g), \text{if } n \geq m \end{cases}$$

(5.7)

(d)

$$L_{\Bbbk,n}(g) = F_{\Bbbk,n+1}(g) + F_{\Bbbk,n-1}(g),$$

(5.8)

with positive integers g, m and n.

Proof:

(a) Now, using (5.3) and (5.4), we get

$$F_{k,m}(g)F_{k,n}(g) = \frac{\alpha_1^{m+n} + \alpha_2^{m+n} - \alpha_1^n\alpha_2^m - \alpha_1^m\alpha_2^n}{\Bbbk^2(g)+4}$$

For $n \geq m$,

$$F_{\Bbbk,m}(g)F_{\Bbbk,n}(g) = \frac{\alpha_1^{m+n} + \alpha_2^{m+n} - \alpha_1^m(\alpha_2^{n-m} + \alpha_1^{n-m})}{\Bbbk^2(g)+4}$$

$$= \frac{1}{\mathscr{k}^2(\mathscr{g}) + 4}\Big(L_{\mathscr{k},m+n}(\mathscr{g}) - (-1)^m L_{\mathscr{k},n-m}(\mathscr{g})\Big)$$

If $m \geq n$, then

$$F_{\mathscr{k},m}(\mathscr{g})F_{\mathscr{k},n}(\mathscr{g}) = \frac{\alpha_1^{m+n} + \alpha_2^{m+n} - \alpha_1^n(\alpha_2^{m-n} + \alpha_1^{m-n})}{\mathscr{k}^2(\mathscr{g}) + 4}$$

$$= \frac{1}{\mathscr{k}^2(\mathscr{g}) + 4}\Big(L_{\mathscr{k},m+n}(\mathscr{g}) - (-1)^n L_{\mathscr{k},m-n}(\mathscr{g})\Big)$$

(b)

Using the same procedure, we can prove our desired result.

(c)

Using (5.3) and (5.4), we have

$$F_{\mathscr{k},m}(\mathscr{g})L_{\mathscr{k},n}(\mathscr{g}) = \left(\frac{\alpha_1^m - \alpha_2^m}{\alpha_1 - \alpha_2}\right)(\alpha_1^n + \alpha_2^n)$$

$$= \frac{1}{\alpha_1 - \alpha_2}(\alpha_1^{m+n} - \alpha_2^{m+n} + \alpha_1^m\alpha_2^n - \alpha_1^n\alpha_2^m)$$

If $m \geq n$, then

$$F_{\mathscr{k},m}(\mathscr{g})L_{\mathscr{k},n}(\mathscr{g}) = \frac{1}{\alpha_1 - \alpha_2}\big(\alpha_1^{m+n} - \alpha_2^{m+n} + \alpha_1^n\alpha_2^n(\alpha_1^{m-n} - \alpha_2^{m-n})\big)$$

$$= F_{\mathscr{k},m+n}(\mathscr{g}) + (-1)^n F_{\mathscr{k},m-n}(\mathscr{g})$$

If $n \geq m$, then

$$F_{\mathscr{k},m}(\mathscr{g})L_{\mathscr{k},n}(\mathscr{g}) = \frac{1}{\alpha_1 - \alpha_2}\big(\alpha_1^{m+n} - \alpha_2^{m+n} - \alpha_1^m\alpha_2^m(\alpha_1^{n-m} - \alpha_2^{n-m})\big)$$

$$= F_{\mathscr{k},m+n}(\mathscr{g}) - (-1)^m F_{\mathscr{k},n-m}(\mathscr{g})$$

(d) By approaching same methodology, we can accomplish the goal.

After this, we move to the results on reciprocals sum of Generalized Fibonacci and Lucas polynomials.

Theorem 5.2.3 The following results hold for any positive integers $\mathscr{g}, n$ and even integer $l \geq 2$:

(a)

$$\left\lfloor \left(\sum_{r=n}^{\infty} \frac{1}{F_{\mathscr{k},lr}(\mathscr{g})} \right)^{-1} \right\rfloor = F_{\mathscr{k},ln}(\mathscr{g}) - F_{\mathscr{k},ln-l}(\mathscr{g}) - 1 \qquad (n \geq 1)$$

(b)

$$\left\lfloor \left(\sum_{r=n}^{\infty} \frac{1}{L_{\mathscr{k},lr}(\mathscr{g})} \right)^{-1} \right\rfloor = L_{\mathscr{k},ln}(\mathscr{g}) - L_{\mathscr{k},ln-l}(\mathscr{g}) \qquad (n \geq 1)$$

(c)

$$\left\lfloor \left(\sum_{r=n}^{\infty} \frac{1}{F^2_{\mathscr{k},lr}(\mathscr{g})} \right)^{-1} \right\rfloor = F^2_{\mathscr{k},ln}(\mathscr{g}) - F^2_{\mathscr{k},ln-l}(\mathscr{g}) - 1 \qquad (n \geq 1)$$

(d)

$$\left\lfloor \left(\sum_{r=n}^{\infty} \frac{1}{L^2_{\mathscr{k},lr}(\mathscr{g})} \right)^{-1} \right\rfloor = L^2_{\mathscr{k},ln}(\mathscr{g}) - L^2_{\mathscr{k},ln-l}(\mathscr{g}) + 1 \qquad (n \geq 2)$$

Proof:

(a) Using the identity (5.5), we have

$$\frac{1}{F_{\mathscr{k},lr}(\mathscr{g})} - \frac{1}{F_{\mathscr{k},lr}(\mathscr{g}) - F_{\mathscr{k},lr-l}(\mathscr{g})} + \frac{1}{F_{\mathscr{k},lr+l}(\mathscr{g}) - F_{\mathscr{k},lr}(\mathscr{g})}$$

$$= \frac{F^2_{\mathscr{k},lr}(\mathscr{g}) - F_{\mathscr{k},lr+l}(\mathscr{g})F_{\mathscr{k},lr-l}(\mathscr{g})}{F_{\mathscr{k},lr}(\mathscr{g})\left(F_{\mathscr{k},lr}(\mathscr{g}) - F_{\mathscr{k},lr-l}(\mathscr{g})\right)\left(F_{\mathscr{k},lr+l}(\mathscr{g}) - F_{\mathscr{k},lr}(\mathscr{g})\right)}$$

$$= \frac{\left(L_{\mathscr{k},2l}(\mathscr{g}) - 2\right)}{F_{\mathscr{k},lr}(\mathscr{g})\left(F_{\mathscr{k},lr}(\mathscr{g}) - F_{\mathscr{k},lr-l}(\mathscr{g})\right)\left(F_{\mathscr{k},lr+l}(\mathscr{g}) - F_{\mathscr{k},lr}(\mathscr{g})\right)(\mathscr{k}^2(\mathscr{g}) + 4)}$$

(5.9)

Since, $F_{\hbar,n}(g)$ is a monotonic increasing sequence for positive integers g and, n therefore

$$\left(F_{\hbar,lr}(g) - F_{\hbar,lr-l}(g)\right) > 0 \text{ , and } \left(F_{\hbar,lr+l}(g) - F_{\hbar,lr}(g)\right) > 0$$

Also,

$$\left(L_{\hbar,2l}(g) - 2\right) > 0$$

Thus,

$$\frac{1}{F_{\hbar,lr}(g)} > \frac{1}{F_{\hbar,lr}(g) - F_{\hbar,lr-l}(g)} - \frac{1}{F_{\hbar,lr+l}(g) - F_{\hbar,lr}(g)} \tag{5.10}$$

Thus, we have

$$\sum_{r=n}^{\infty} \frac{1}{F_{\hbar,lr}(g)} > \sum_{r=n}^{\infty} \left(\frac{1}{F_{\hbar,lr}(g) - F_{\hbar,lr-l}(g)} - \frac{1}{F_{\hbar,lr+l}(g) - F_{\hbar,lr}(g)}\right)$$
$$= \frac{1}{F_{\hbar,ln}(g) - F_{\hbar,ln-l}(g)} \tag{5.11}$$

Next, we prove that

$$\frac{1}{F_{\hbar,lr}(g)} < \frac{1}{F_{\hbar,lr}(g) - F_{\hbar,lr-l}(g) - 1} - \frac{1}{F_{\hbar,lr+l}(g) - F_{\hbar,lr}(g) - 1} \tag{5.12}$$

This will be true iff

$$F_{\hbar,lr+l}(g)F_{\hbar,lr-l}(g) - F_{\hbar,lr-l}(g) + F_{\hbar,lr+l}(g) - 1 > F^2_{\hbar,lr}(g)$$

Using identity (5.5) and (5.7), the above inequality becomes true iff

$$\left(F_{k,lr+l+2}(g) - L_{k,2l}(g)\right) + \left(F_{k,lr+l}(g) - F_{k,lr-l+2}(g)\right)$$
$$+ \left(F_{k,lr+l-2}(g) - F_{k,lr-l-2}(g) - L_{k,2}(g)\right)$$
$$+ \left(F_{k,lr+l}(g) - 2F_{k,lr-l}(g)\right) > 0$$

(5.13)

Moreover,

$$F_{k,lr+l}(g) - F_{k,lr-l+2}(g) > 0,$$

and

$$\left(F_{k,lr+l-2}(g) - F_{k,lr-l-2}(g) - L_{k,2}(g)\right) > 0, \left(F_{k,lr+l}(g) - 2F_{k,lr-l}(g)\right) > 0$$

for any positive integers g, k and even integer $l \geq 2$.

Further,

$$F_{k,lr+l+2}(g) - L_{k,2l}(g) > F_{k,2l+2}(g) - L_{k,2l}(g)$$
$$= k(g)\, F_{k,2l+1}(g) + F_{k,2l}(g) - F_{k,2l+1}(g) - F_{k,2l-1}(g) > 0$$

Hence, inequality (5.12) holds for any positive integers g, k and even integer $l \geq 2$.

From (5.12), we get

$$\sum_{r=n}^{\infty} \frac{1}{F_{k,lr}(g)} < \sum_{r=n}^{\infty} \left(\frac{1}{F_{k,lr}(g) - F_{k,lr-l}(g) - 1} - \frac{1}{F_{k,lr+l}(g) - F_{k,lr}(g) - 1} \right)$$
$$= \frac{1}{F_{k,ln}(g) - F_{k,ln-l}(g) - 1}$$

(5.14)

On solving (5.13) and (5.14), we get

$$\left\lfloor \left(\sum_{r=n}^{\infty} \frac{1}{F_{k,lr}(g)} \right)^{-1} \right\rfloor = F_{k,ln}(g) - F_{k,ln-l}(g) - 1$$

(b) Using the identity (5.6), consider

$$\frac{1}{L_{k,lr}(g)} - \frac{1}{L_{k,lr}(g) - L_{k,lr-l}(g)} + \frac{1}{L_{k,lr+l}(g) - L_{k,lr}(g)}$$

$$= \frac{\left(2 - L_{k,2l}(g)\right)}{L_{k,lr}(g)\left(L_{k,lr+l}(g) - L_{k,lr}(g)\right)\left(L_{k,lr}(g) - L_{k,lr-l}(g)\right)}$$

(5.15)

For $g, n > 0$, $L_{k,n}(g)$ is a monotonically increasing sequence, therefore,

$$\left(L_{k,lr}(g) - L_{k,lr-l}(g)\right) > 0, \text{ and } \left(L_{k,lr+l}(g) - L_{k,lr}(g)\right) > 0$$

for even $l \geq 2$.

Further,

$$\left(L_{k,2l}(g) - 2\right) > 0$$

Thus, we have

$$\frac{1}{L_{k,lr}(g)} < \frac{1}{L_{k,lr}(g) - L_{k,lr-l}(g)} - \frac{1}{L_{k,lr+l}(g) - L_{k,lr}(g)}$$

(5.16)

Using repeatedly (5.16), we have

$$\sum_{r=n}^{\infty} \frac{1}{L_{k,lr}(g)} < \frac{1}{L_{k,ln}(g) - L_{k,ln-l}(g)}$$

(5.17)

Next, we prove that

$$\frac{1}{L_{k,lr}(g)} > \frac{1}{L_{k,lr}(g) - L_{k,lr-l}(g) + 1} - \frac{1}{L_{k,lr+l}(g) - L_{k,lr}(g) + 1}$$

(5.18)

By inequality (5.18), we obtain

$$L_{k,lr+l}(g)L_{k,lr-l}(g) + L_{k,lr-l}(g) - L_{k,lr+l}(g) - 1 < L_{k,lr}^2(g)$$

Using identity (5.6), the above inequality becomes

$$L_{k,lr+l}(g) - L_{k,lr-l}(g) - L_{k,2l}(g) + 3 > 0,$$

which is true for $g, k > 0$, and even $l \geq 2$ as $L_{k,n}(g)$ is a monotonic increasing sequence.

Hence, inequality (5.18) holds for any positive integers g, k and even $l \geq 2$.

By (5.18), we have

$$\sum_{r=n}^{\infty} \frac{1}{L_{k,lr}(g)} > \sum_{r=n}^{\infty} \left(\frac{1}{L_{k,lr}(g) - L_{k,lr-l}(g) - 1} - \frac{1}{L_{k,lr+l}(g) - L_{k,lr}(g) - 1} \right)$$
$$= \frac{1}{L_{k,ln}(g) - L_{k,ln-l}(g) - 1} \tag{5.19}$$

From (5.18) and (5.19),

$$\left\lfloor \left(\sum_{r=n}^{\infty} \frac{1}{L_{k,lr}(g)} \right)^{-1} \right\rfloor = L_{k,ln}(g) - L_{k,ln-l}(g)$$

(c) Now,

$$\frac{1}{F^2_{k,lr}(g)} - \frac{1}{F^2_{k,lr}(g) - F^2_{k,lr-l}(g)} + \frac{1}{F^2_{k,lr+l}(g) - F^2_{k,lr}(g)}$$
$$= \frac{\left(L_{k,2l}(g) - 2\right)\left(F^2_{k,lr}(g) + F_{k,lr+l}(g) F_{k,lr-l}(g)\right)}{F^2_{k,lr}(g)\left(F^2_{k,lr}(g) - F^2_{k,lr-l}(g)\right)\left(F^2_{k,lr+l}(g) - F^2_{k,lr}(g)\right)(k^2(g) + 4)} \tag{5.20}$$

Further, $F_{k,n}(g)$ is a monotonic increasing sequence $g, n > 0$, therefore

$$\left(F^2_{k,lr}(g) - F^2_{k,lr-l}(g)\right) > 0, \text{ and } \left(F^2_{k,lr+l}(g) - F^2_{k,lr}(g)\right) > 0,$$

with even $l \geq 2$.

Moreover,

$$\left(L_{\Bbbk,2l}(g)-2\right)>0\,,$$

that makes the numerator of (5.20) to be positive. Therefore, (5.20) becomes

$$\frac{1}{F^2_{\Bbbk,lr}(g)}>\frac{1}{F^2_{\Bbbk,lr}(g)-F^2_{\Bbbk,lr-l}(g)}-\frac{1}{F^2_{\Bbbk,lr+l}(g)-F^2_{\Bbbk,lr}(g)} \tag{5.21}$$

Using repeatedly (5.21), we get

$$\sum_{r=n}^{\infty}\frac{1}{F^2_{\Bbbk,lr}(g)}>\frac{1}{F^2_{\Bbbk,ln}(g)-F^2_{\Bbbk,ln-l}(g)} \tag{5.22}$$

Next, we show that

$$\frac{1}{F^2_{\Bbbk,lr}(g)}<\frac{1}{F^2_{\Bbbk,lr}(g)-F^2_{\Bbbk,lr-l}(g)-1}-\frac{1}{F^2_{\Bbbk,lr+l}(g)-F^2_{\Bbbk,lr}(g)-1} \tag{5.23}$$

Inequality (5.23) is equivalent to

$$-F^4_{\Bbbk,lr}(g)+F^2_{\Bbbk,lr+l}(g)F^2_{\Bbbk,lr-l}(g)+F^2_{\Bbbk,lr+l}(g)-F^2_{\Bbbk,lr-l}(g)-1>0$$

Using (5.5) and (5.6), we have

$$(\Bbbk^2(g)+2)\,L_{\Bbbk,2lr+2l}(g)+4\,L_{\Bbbk,2lr}(g)-(\Bbbk^2(g)+6)\,L_{\Bbbk,2lr-2l}(g)-4+\,L_{\Bbbk,4l}(g)$$
$$-\,L_{\Bbbk,4}(g)-4-4\,L_{\Bbbk,2}(g)>0$$

Since, $L_{\Bbbk,n}(g)$ is a monotonic increasing sequence for fixed positive integers g and n, so by induction, we have

$$L_{\Bbbk,4l}(g)-\,L_{\Bbbk,4}(g)-4-4\,L_{\Bbbk,2}(g)>0$$

Now, consider

$$(\Bbbk^2(g)+2)\,L_{\Bbbk,2lr+2l}(g)+4\,F_{\Bbbk,2lr}(g)-(\Bbbk^2(g)+6)\,L_{\Bbbk,2lr-2l}(g)-4$$
$$>(\Bbbk^2(g)+6)\,L_{\Bbbk,2lr}(g)-(\Bbbk^2(g)+6)\,L_{\Bbbk,2lr-2l}(g)-4>0$$

Hence inequality (5.23) holds.

Repeating (5.23), we have

$$\sum_{r=n}^{\infty}\frac{1}{F_{k,lr}^2(g)} < \sum_{r=n}^{\infty}\left(\frac{1}{F_{k,lr}^2(g)-F_{k,lr-l}^2(g)-1}-\frac{1}{F_{k,lr+l}^2(g)-F_{k,lr}^2(g)-1}\right)$$

$$=\frac{1}{F_{k,ln}^2(g)-F_{k,ln-l}^2(g)-1}$$

(5.24)

From (5.23) and (5.24), we have

$$\left\lfloor\left(\sum_{r=n}^{\infty}\frac{1}{F_{k,lr}^2(g)}\right)^{-1}\right\rfloor = F_{k,ln}^2(g)-F_{k,ln-l}^2(g)-1$$

(d) By approaching in the same manner, we get the results we want.

Chapter 6

Infinite Sum of Reciprocal of Jacobsthal Polynomials

6.1 Introduction

In this chapter, we derived the sum of reciprocal of the Jacobsthal polynomials (1.14) defined as

$$J_{n+2}(x) = J_{n+1}(x) + xJ_n(x),$$

with $J_0(x) = 0$, and $J_1(x) = 1$, and the Jacobsthal-Lucas polynomial (1.15) described by

$$j_{n+2}(x) = j_{n+1}(x) + xj_n(x),$$

with $j_0(x) = 2$, and $j_1(x) = 1$, and n is a non-negative integer.

6.2 Sum of Reciprocal of Jacobsthal and Jacobsthal-Lucas Polynomials

To accomplish our main results, we derive some results regarding Jacobsthal and Jacobsthal-Lucas polynomials given as below:

Lemma 6.2.1 For $n \geq 0$, we have

(a)

$$\boldsymbol{J_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta}} \qquad \textbf{(6.1)}$$

(b)

$$\boldsymbol{j_n(x) = \alpha^n + \beta^n} \qquad \textbf{(6.2)}$$

where $\boldsymbol{\alpha = \frac{1+\sqrt{1+4x}}{2}}$, and $\boldsymbol{\beta = \frac{1-\sqrt{1+4x}}{2}}$.

Proof:

(a) We approach the induction methodology to prove our results.

Evidently, the result holds for $n = 1$.

Using induction hypothesis, we get

$$\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}=\frac{1}{\alpha-\beta}\left((\alpha^n-\beta^n)+x(\alpha^{n-1}-\beta^{n-1})\right)$$

$$=J_n(x)+xJ_{n-1}(x)=J_{n+1}(x)$$

(b) The result is evidently valid for $n=1$.

Using induction, we get

$$\alpha^{n+1}+\beta^{n+1}=(\alpha^n+\beta^n)(\alpha+\beta)-\alpha\beta(\alpha^{n-1}+\beta^{n-1})$$

$$=j_n(x)+xj_{n-1}(x)=j_{n+1}(x)$$

Lemma 6.2.2 For $x,m,n>0$, the following results hold:

(a)

$$J_m(x)J_n(x)=\begin{cases}\dfrac{1}{1+4x}(j_{m+n}(x)-(-x)^n j_{m-n}(x)), \text{if } m\geq n\\ \dfrac{1}{1+4x}(j_{m+n}(x)-(-x)^m j_{n-m}(x)), \text{if } n\geq m\end{cases} \tag{6.3}$$

(b)

$$j_m(x)j_n(x)=\begin{cases} j_{m+n}(x)+(-x)^n j_{m-n}(x), \text{if } m\geq n\\ j_{m+n}(x)+(-x)^m j_{n-m}(x), \text{if } n\geq m\end{cases} \tag{6.4}$$

(c)

$$J_m(x)j_n(x)=\begin{cases} J_{m+n}(x)+(-x)^n J_{m-n}(x), \text{if } m\geq n\\ J_{m+n}(x)-(-x)^m J_{n-m}(x), \text{if } n\geq m\end{cases} \tag{6.5}$$

(d)

$$j_n(x)=J_{n+1}(x)+xJ_{n-1}(x) \tag{6.6}$$

Proof:

(a) For any positive integers x,m and n, using (6.1) and (6.2), we get

$$J_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

and

$$j_n(x) = \alpha^n + \beta^n$$

Now,

$$J_m(x)J_n(x) = \frac{\alpha^{m+n} + \beta^{m+n} - \alpha^n\beta^m - \alpha^m\beta^n}{1+4x}$$

If $m \geq n$, then,

$$J_m(x)J_n(x) = \frac{j_{m+n}(x) - \alpha^n\beta^n(\beta^{m-n} + \alpha^{m-n})}{1+4x}$$

$$= \frac{1}{1+4x}(j_{m+n}(x) - (-x)^n j_{m-n}(x))$$

If $n \geq m$, then, we have

$$J_m(x)J_n(x) = \frac{j_{m+n}(x) - \alpha^m\beta^m(\beta^{n-m} + \alpha^{n-m})}{1+4x}$$

$$= \frac{1}{1+4x}(j_{m+n}(x) - (-x)^m j_{n-m}(x))$$

(b) Using (6.1) and (6.2), we have

$$j_m(x)j_n(x) = (\alpha^{m+n} + \beta^{m+n} + \alpha^n\beta^m + \alpha^m\beta^n)$$

If $m \geq n$, then,

$$j_m(x)j_n(x) = (\alpha^{m+n} + \beta^{m+n}) + \alpha^n\beta^n(\alpha^{m-n} + \beta^{m-n})$$
$$= j_{m+n}(x) + (-x)^n j_{m-n}(x)$$

and, if $n \geq m$, then, we get

$$j_m(x)j_n(x) = (\alpha^{m+n} + \beta^{m+n}) + \alpha^m\beta^m(\alpha^{n-m} + \beta^{n-m}) = j_{m+n}(x) + (-x)^m j_{n-m}$$

(c) By following the same procedure, we can achieve our objective

(d) Consider

$$J_{n+1}(x) + xJ_{n-1}(x) = \frac{\alpha^n(\alpha + x\alpha^{-1}) - \beta^n(\beta + x\beta^{-1})}{\sqrt{1+4x}} = \frac{(\alpha^n + \beta^n)\left(\sqrt{1+4x}\right)}{\sqrt{1+4x}}$$
$$= \alpha^n + \beta^n = j_n(x)$$

Now, we represent our main results:

Theorem 6.2.1 For any positive integers x, n and for positive even number a, following inequalities resulted:

(a)

$$\sum_{k=n}^{\infty} \frac{1}{J_{ak}(x)} > \frac{1}{J_{an}(x) - J_{an-a}(x)}$$

(b)

$$\sum_{k=n}^{\infty} \frac{1}{j_{ak}(x)} < \frac{1}{j_{an}(x) - j_{an-a}(x)}$$

(c)

$$\sum_{k=n}^{\infty} \frac{1}{J_{ak}^2(x)} > \frac{1}{J_{an}^2(x) - J_{an-a}^2(x)}$$

(d)

$$\sum_{k=n}^{\infty} \frac{1}{j_{ak}^2(x)} < \frac{1}{j_{an}^2(x) - j_{an-a}^2(x)}$$

Proof:

(a) For $x, k > 0$, and positive even number a, by using (6.3), consider

$$\frac{1}{J_{ak}(x)} - \frac{1}{J_{ak}(x) - J_{ak-a}(x)} + \frac{1}{J_{ak+a}(x) - J_{ak}(x)}$$
$$= \frac{x^{ak-a}(j_{2a}(x) - 2x^a)}{J_{ak}(x)\big(J_{ak}(x) - J_{ak-a}(x)\big)\big(J_{ak+a}(x) - J_{ak}(x)\big)} \tag{6.7}$$

Further, $J_{ak}(x)$ is a monotonic increasing sequence for $x, n > 0$, and for positive even number a, therefore

$$\left(J_{ak}(x) - J_{ak-a}(x)\right) > 0, \text{ and } \left(J_{ak+a}(x) - J_{ak}(x)\right) > 0$$

Also,

$$(j_{2a}(x) - 2x^a) > 0$$

that makes numerator of (6.7) to be positive. Therefore, we get

$$\frac{1}{J_{ak}(x)} > \frac{1}{J_{ak}(x) - J_{ak-a}(x)} - \frac{1}{J_{ak+a}(x) - J_{ak}(x)} \tag{6.8}$$

Thus, we have

$$\sum_{k=n}^{\infty} \frac{1}{J_{ak}(x)} > \frac{1}{J_{an}(x) - J_{an-a}(x)}$$

(b)

For $x, k > 0$, and even positive integer a, using (6.4), consider

$$\frac{1}{j_{ak}(x)} - \frac{1}{j_{ak}(x) - j_{ak-a}(x)} + \frac{1}{j_{ak+a}(x) - j_{ak}(x)} = \frac{x^{ak-a}(-j_{2a}(x) + 2x^a)}{j_{ak}(x)\left(j_{ak}(x) - j_{ak-a}(x)\right)\left(j_{ak+a}(x) - j_{ak}(x)\right)} \tag{6.9}$$

Further, $j_{ak}(x)$ is a monotonic increasing sequence for positive integers $x, n > 0$, and for positive even integer a, therefore

$$\left(j_{ak}(x) - j_{ak-a}(x)\right) > 0, \text{ and } \left(j_{ak+a}(x) - j_{ak}(x)\right) > 0$$

Moreover,

$$(j_{2a}(x) - 2x^a) > 0$$

that makes the numerator of (6.9) to be negative. Therefore, we get

$$\frac{1}{j_{ak}(x)} < \frac{1}{j_{ak}(x) - j_{ak-a}(x)} - \frac{1}{j_{ak+a}(x) - j_{ak}(x)} \tag{6.10}$$

Thus, we have

$$\sum_{k=n}^{\infty} \frac{1}{j_{ak}(x)} < \frac{1}{j_{an}(x) - j_{an-a}(x)}$$

(c)

Now, consider

$$\frac{1}{J_{ak}^2(x)} - \frac{1}{J_{ak}^2(x) - J_{ak-a}^2(x)} + \frac{1}{J_{ak+a}^2(x) - J_{ak}^2(x)}$$

$$= \frac{\left(J_{ak-a}^2(x) - J_{ak+a}(x)J_{ak-a}(x)\right)\left(J_{ak-a}^2(x) + J_{ak+a}(x)J_{ak-a}(x)\right)}{J_{ak}^2(x)\left(J_{ak}^2(x) - J_{ak-a}^2(x)\right)\left(J_{ak+a}^2(x) - J_{ak}^2(x)\right)}$$

$$= \frac{x^{ak-a}(j_{2a}(x) - 2x^a)(J_{ak-a}^2(x) + J_{ak+a}(x)J_{ak-a}(x))}{J_{ak}^2(x)\left(J_{ak}^2(x) - J_{ak-a}^2(x)\right)\left(J_{ak+a}^2(x) - J_{ak}^2(x)\right)(1+4x)}$$

(6.11)

Since $J_{ak}(x)$ is a monotonic increasing sequence for positive integers $x, n > 0$ and even integer a, hence

$$\left(J_{ak}^2(x) - J_{ak-a}^2(x)\right) > 0,$$

and

$$\left(J_{ak+a}^2(x) - J_{ak}^2(x)\right) > 0$$

for even a. Further,

$$(j_{2a}(x) - 2x^a) > 0$$

that makes the numerator of (6.11) to be positive. Therefore,

$$\frac{1}{J_{ak}^2(x)} > \frac{1}{J_{ak}^2(x) - J_{ak-a}^2(x)} - \frac{1}{J_{ak+a}^2(x) - J_{ak}^2(x)}$$

(6.12)

Thus, we have

$$\sum_{k=n}^{\infty} \frac{1}{J_{ak}^2(x)} > \frac{1}{J_{an}^2(x) - J_{an-a}^2(x)}$$

(d) Now, consider

$$\frac{1}{j_{ak}^2(x)} - \frac{1}{j_{ak}^2(x) - j_{ak-a}^2(x)} + \frac{1}{j_{ak+a}^2(x) - j_{ak}^2(x)}$$

$$= \frac{x^{ak-a}(-j_{2a}(x) + 2x^a)(j_{ak-a}^2(x) + j_{ak+a}(x)j_{ak-a}(x))}{j_{ak}^2(x)\left(j_{ak}^2(x) - j_{ak-a}^2(x)\right)\left(j_{ak+a}^2(x) - j_{ak}^2(x)\right)}$$

(6.13)

Since $j_{ak}(x)$ is monotonic increasing sequence for positive integers x and n and positive even integer a, therefore

$$\left(j_{ak}^2(x) - j_{ak-a}^2(x)\right) > 0,$$

and

$$\left(j_{ak+a}^2(x) - j_{ak}^2(x)\right) > 0$$

for even a. Further,

$$(j_{2a}(x) - 2x^a) > 0$$

that makes the numerator of (6.13) to be negative. Therefore,

$$\frac{1}{j_{ak}^2(x)} < \frac{1}{j_{ak}^2(x) - j_{ak-a}^2(x)} - \frac{1}{j_{ak+a}^2(x) - j_{ak}^2(x)}$$

(6.14)

Using repeatedly (6.14), it results

$$\sum_{k=n}^{\infty} \frac{1}{j_{ak}^2(x)} < \frac{1}{j_{an}^2(x) - j_{an-a}^2(x)}$$

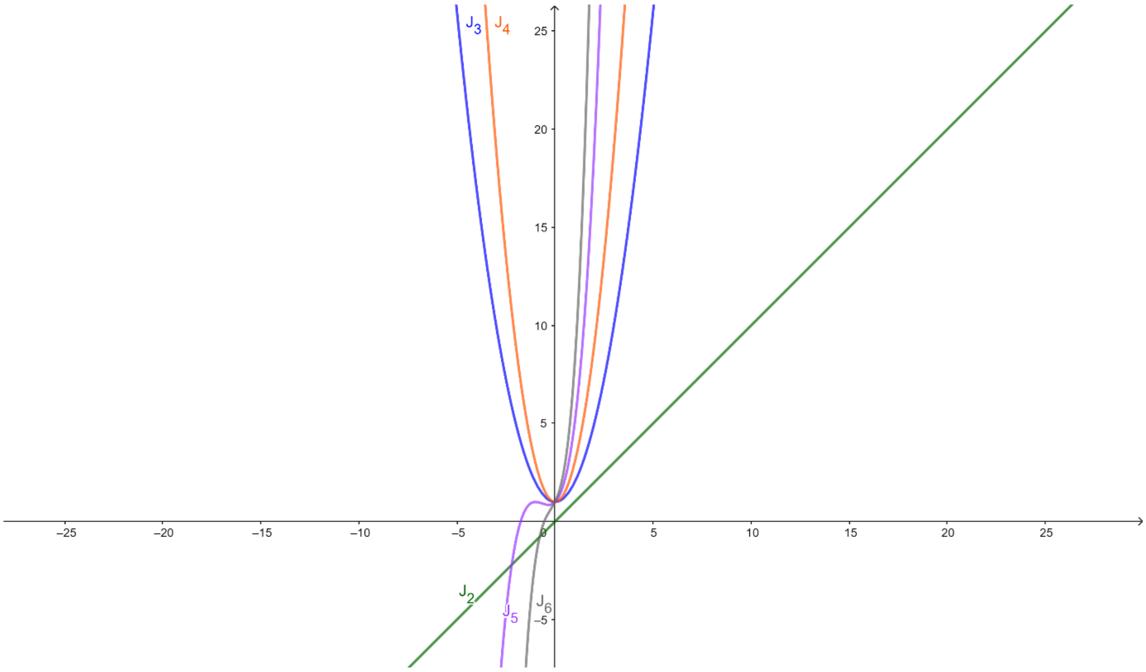


Figure 6.1: Graphical Representation of the Jacobsthal Polynomials

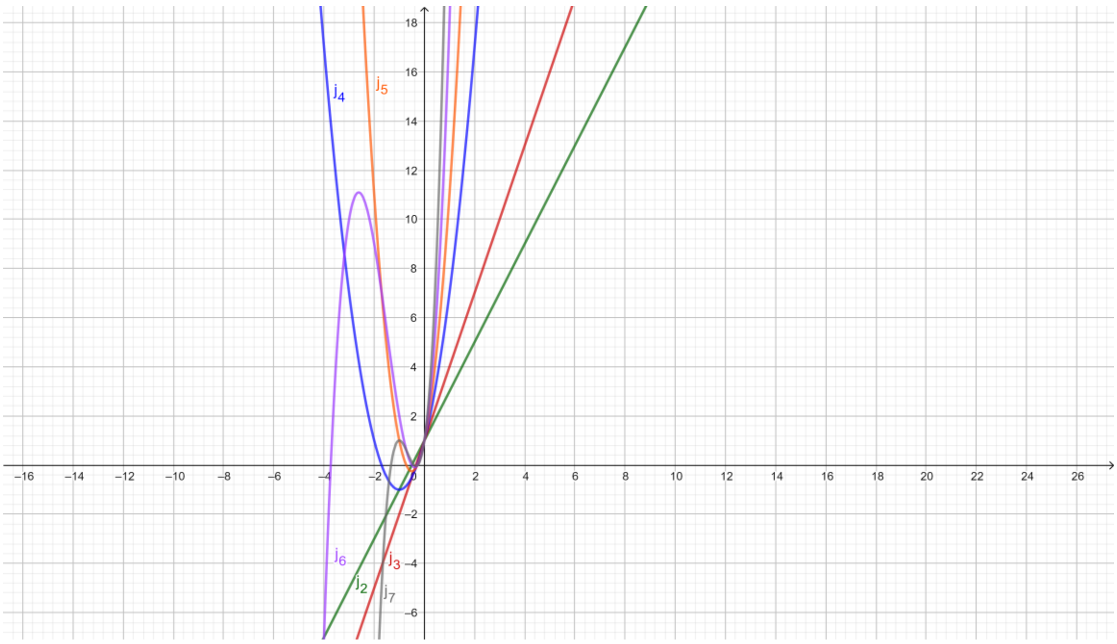


Figure 6.2: Graphical Representation of the Jacobsthal-Lucas Polynomials

Chapter 7

Trivariate Fibonacci Polynomials

7.1 Introduction

In the first section of this chapter, we present a $n \times n$ matrix whose successive determinant generates the following sequence of Trivariate Fibonacci polynomials (1.22):

$$H_n(e',m',f') = e'H_{n-1}(e',m',f') + m'H_{n-2}(e',m',f') + f'H_{n-3}(e',m',f'); n \geq 3,$$

with $H_0(e',m',f') = 0, H_1(e',m',f') = 1$, and $H_2(e',m',f') = e'$, where e',m',f' are any real numbers. For $e' = m' = 1$, and $f' = 0,$ it becomes the sequence of the Fibonacci numbers. In the second section, we represent Binet's formula for this sequence by using matrix methods and the concept of diagonalizability of matrix.

And in the last section, we address an observation related to n^{th} Generalized Lucas numbers (1.9) followed by

$$j_n(x) = j_{n-1}(x) + xj_{n-2}(x);\ n \geq 2,$$

where x is any positive integer with, $j_0(x) = 2$, and, $j_1(x) = 1$. For $x = 1$, it becomes the sequence of Lucas numbers.

7.2 Trivariate Fibonacci Polynomials using Determinants

In 2004, Strang [76, 77] presented a family of tridiagonal matrices given by

$$M(n) = \begin{pmatrix} 3 & 1 & 0 & \dots & 0 \\ 1 & 3 & 1 & \dots & 0 \\ 0 & 1 & 3 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix}_{n\times n},$$

whose determinant is a Fibonacci number F_{2n+2}. Another example of tridiagonal matrix is:

$$H(n) = \begin{pmatrix} 1 & i & 0 & \dots & 0 \\ i & 1 & i & \dots & 0 \\ 0 & i & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & i \\ 0 & 0 & 0 & i & 1 \end{pmatrix}_{n\times n},$$

and its determinant was found to be a Fibonacci number F_{n+1}.

Now, we construct a square matrix whose successive determinant generates the sequence of Trivariate Fibonacci polynomials.

Theorem 7.2.1 Let $S_n(e',m',f')$ denotes a $n \times n$ matrix described by

$$\begin{pmatrix} e' & -m' & f' & 0 & \dots & 0 \\ 1 & e' & -m' & f' & \dots & 0 \\ 0 & 1 & e' & -m' & \ddots & \vdots \\ 0 & 0 & 1 & e' & \ddots & f' \\ \vdots & \vdots & \vdots & \ddots & \ddots & -m' \\ 0 & 0 & 0 & 0 & 1 & e' \end{pmatrix}_{n\times n}$$

Then, $det(S_n(e',m',f')) = H_{n+1}(e',m',f')$.

Proof: Induction will be used on n.

For $n = 1$, $n = 2$ and, $n = 3$, it can be seen that

$$det(S_1(e',m',f')) = e' = H_2(e',m',f'),$$

$$det(S_2(e',m',f')) = (e')^2 + m' = H_3(e',m',f'),$$

and, $det(S_3(e',m',f')) = (e')^3 + 2m'f' + f' = H_4(e',m',f')$

Assume by induction that,

$$det(S_{n-1}(e',m',f')) = H_n(e',m',f'), det(S_{n-2}(e',m',f')) = H_{n-1}(e',m',f'), det(S_{n-3}(e',m',f')) = H_{n-2}(e',m',f')$$

Then, by using the induction hypothesis, we obtain

$$det(S_n(e',m',f')) = e'\,det(S_{n-1}(e',m',f')) + m'det(S_{n-2}(e',m',f')) + f'det(S_{n-3}(e',m',f')) = H_{n+1}(e',m',f')$$

7.3 Binet's Formula Using Matrix Method for Trivariate Fibonacci Polynomials

In this section, we derived a formula to determine a general term of the sequence of Trivariate Fibonacci polynomials by using matrix methods.

Let a, b and c be the characteristic roots of equation

$$k^3 - e'k^2 - m'k - f' = 0$$

To reach our goal, let's introduce a square matrix E of order 3 given as

$$E = \begin{pmatrix} e' & m' & f' \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}_{3\times 3},$$

such that $det(E) = f'$. Here, matrix E is known as a generating matrix for the sequence of Trivariate Fibonacci polynomials. To accomplish our objective, we first describe some results obtained for Trivariate Fibonacci polynomials:

Theorem 7.3.1 For $n \geq 1$, following identities hold:

(a)

$$\begin{pmatrix} H_{n+2} \\ H_{n+1} \\ H_n \end{pmatrix} = \begin{pmatrix} e' & m' & f' \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} H_{n+1} \\ H_n \\ H_{n-1} \end{pmatrix} \quad \textbf{(7.1)}$$

(b)

$$\begin{pmatrix} H_{n+2} \\ H_{n+1} \\ H_n \end{pmatrix} = E^n \begin{pmatrix} H_2 \\ H_1 \\ H_0 \end{pmatrix} \quad \textbf{(7.2)}$$

Proof:

(a) By using the recurrence relation for Trivariate Fibonacci polynomials, we have

$$\begin{pmatrix} e' & m' & f' \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} H_{n+1} \\ H_n \\ H_{n-1} \end{pmatrix} = \begin{pmatrix} e'H_{n+1} + m'H_n + f'H_{n-1} \\ H_{n+1} \\ H_n \end{pmatrix} = \begin{pmatrix} H_{n+2} \\ H_{n+1} \\ H_n \end{pmatrix}$$

(b) We approach the induction methodology to prove our result.

Evidently, the result holds for $n = 1$.

Using induction and result (7.1), we have

$$\begin{pmatrix} H_{n+2} \\ H_{n+1} \\ H_n \end{pmatrix} = E \begin{pmatrix} H_{n+1} \\ H_n \\ H_{n-1} \end{pmatrix} = E^n \begin{pmatrix} H_2 \\ H_1 \\ H_0 \end{pmatrix}$$

The characteristics equation of generating matrix E is

$$0 = det(E - kI_3) = \begin{vmatrix} e' - k & m' & f' \\ 1 & -k & 0 \\ 0 & 1 & -k \end{vmatrix} = k^3 - e'k^2 - m'k - f',$$

where k is characteristics root of matrix E. Note that, it is also the characteristics equation of the sequence of polynomials (1.22).Therefore, characteristics root of matrix E are a, b and c.

Clearly, the characteristics vectors are $\begin{pmatrix} a^2 \\ a \\ 1 \end{pmatrix}, \begin{pmatrix} b^2 \\ b \\ 1 \end{pmatrix}$ and $\begin{pmatrix} c^2 \\ c \\ 1 \end{pmatrix}$ corresponding to the characteristics roots a,b and c respectively.

Let

$$P = \begin{pmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{pmatrix}$$

Then, its inverse is given by

$$P^{-1} = \frac{1}{M} \begin{pmatrix} b - c & (b + c)(c - a) & bc(b - c) \\ c - a & (a + c)(a - c) & ac(c - a) \\ a - b & (b + a)(a - b) & ab(a - b) \end{pmatrix},$$

where $M = (a - b)(b - c)(c - a)$.

Further, consider the following diagonal matrix

$$D = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

Then, we have

$$E = PDP^{-1}$$

therefore, we get

$$E^n = PD^nP^{-1}$$

By using the results (7.1) and (7.2), we have

$$H_n(e', m', f') = \frac{a^{n+1}}{(a-b)(a-c)} + \frac{b^{n+1}}{(b-a)(b-c)} + \frac{c^{n+1}}{(c-b)(c-a)},$$

which is required Binet's formula for the sequence of Trivariate Fibonacci polynomials.

After that, we address an observation related to n^{th} Generalized Lucas numbers (1.9) followed as

$$j_n(x) = j_{n-1}(x) + xj_{n-2}(x); n \geq 2,$$

with x is any positive integer with $j_0(x) = 2,$ and $j_1(x) = 1.$

7.4 Generalized n^{th} Lucas Numbers

In [14], author counted those Lucas numbers in modulo n with respect to which n is a Lucas pseudo prime.

We observed following results related to Generalized n^{th} Lucas numbers:

1) For a prime p and a positive integer $x,$ by using the recurrence relation for Generalized n^{th} Lucas numbers, we observe that

$$j_p(x) \equiv 1 mod\ (p)$$

2) Generalized n^{th} Lucas polynomials are of degree $\left[\frac{n}{2}\right]$, where $[.]$ denotes the greatest integer.

3) For even integer $n,$ highest degree coefficient is 2.

Chapter 8

Solution of a Fractional Differential Equation

8.1 Introduction

Here firstly, we deal with originating an operational matrix for fractional derivative of Lucas polynomials and Pell-Lucas polynomials. After that, we discus numerous results of the sequence obtained by differentiating the Fibonacci polynomials.

8.2 Fractional Differential Equation

We solve a fractional differential equation given below:

$$\begin{cases} D^{\alpha} y(x) + y^{(k)}(x) + y(x) = f(x) \\ y^{(r)}(0) = c_r \end{cases}, \tag{8.1}$$

where $r = 0, 1, 2, \dots, m-1;\ k = 0, 1, 2, \dots, m;\ m-1 < \alpha \leq m.$

The fractional derivative of $y(t)$ in the Caputo sense is given by [78] and satisfies that: For $n - 1 < \alpha < n$,

$$D^{\alpha} t^{k} = \begin{cases} \dfrac{k!}{\Gamma(k-\alpha+1)} t^{k-\alpha}, & k \geq \alpha \\ 0, & k < \alpha \end{cases}$$

(8.2)

Now, we discuss about some properties of Lucas polynomials (1.10) given as

$$L_{n+2}(x) = xL_{n+1}(x) + L_n(x);\ n \geq 0,$$

with $L_0(x) = 2, L_1(x) = x$, and are generated [79] by the following form representation:

$$L_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \left(\frac{n}{n-k}\right) \binom{n-k}{k} x^{n-2k}$$

(8.3)

And the Pell-Lucas polynomials [80] is:

$$Q_{n+1}(x) = 2xQ_n(x) + Q_{n-1}(x),$$

where n is a natural number with $Q_0(x) = 2, Q_1(x) = 2x$, and represented [80] as:

$$Q_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \left(\frac{n}{n-k}\right)\binom{n-k}{k}\,(2x)^{n-2k}$$

(8.4)

Theorem 8.2.1 The following inversion formula holds:

(a)

$$\boldsymbol{x^n = \begin{cases} \displaystyle\sum_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i \binom{n}{i}\, L_{n-2i}(x)\,, & \text{if } n \text{ is odd} \\ \displaystyle\sum_{i=0}^{\left[\frac{n-1}{2}\right]} (-1)^i \binom{n}{i}\, L_{n-2i}(x) + (-1)^{\frac{n}{2}} \binom{n-1}{\frac{n}{2}} L_0(x)\,, & \text{otherwise} \end{cases}}$$

(8.5)

(b)

$$\boldsymbol{(2x)^n = \begin{cases} \displaystyle\sum_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i \binom{n}{i}\, Q_{n-2i}(x), & \text{if } n \text{ is odd} \\ \displaystyle\sum_{i=0}^{\left[\frac{n-1}{2}\right]} (-1)^i \binom{n}{i}\, Q_{n-2}(x) + (-1)^{\frac{n}{2}} \binom{n-1}{\frac{n}{2}} Q_0(x), & \text{if } n \text{ is even} \end{cases}}$$

(8.6)

Proof:

(a)

Case I: When n is odd

Evidently, the result holds for $n = 1$.

By induction hypothesis, we get

$$x^n = \sum_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i \binom{n}{i} L_{n-2i}(x)$$

Therefore,

$$x^{n+1} = \sum_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i \binom{n}{i} xL_{n-2i}(x)$$

$$= \sum_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i \binom{n}{i} L_{n-2i+1}(x) - \sum_{i=1}^{\left[\frac{n}{2}\right]+1} (-1)^{i-1} \binom{n}{i-1} L_{n-2i-1}(x)$$

(8.7)

It is required to show that

$$x^{n+1} = \sum_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i \binom{n+1}{i} L_{n-2i+1}(x) + (-1)^{\frac{n+1}{2}} \binom{n}{\frac{n+1}{2}} L_0(x)$$

To prove this, we have from (8.7),

$$x^{n+1} = L_{n+1}(x) + \sum_{i=1}^{\left[\frac{n}{2}\right]} (-1)^i \binom{n}{i} L_{n-2i+1}(x) - \sum_{i=1}^{\left[\frac{n}{2}\right]} (-1)^{i-1} \binom{n}{i-1} L_{n-2i+1}(x)$$

$$- (-1)^{\frac{n-1}{2}} \binom{n}{\frac{n-1}{2}} L_0(x)$$

(8.8)

Now, the coefficient of $L_{n-2i+1}(x) = \binom{n}{i} + \binom{n}{i-1} = \binom{n+1}{i}$, therefore equation (8.8) becomes

$$x^{n+1} = \sum_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i \binom{n+1}{i} L_{n-2i+1}(x) + (-1)^{\frac{n+1}{2}} \binom{n}{\frac{n+1}{2}} L_0(x)$$

Case II: When n is even

By induction hypothesis, we ger

$$x^n = \sum_{i=0}^{\left[\frac{n-1}{2}\right]} (-1)^i \binom{n}{i} L_{n-2i}(x) + (-1)^{\frac{n}{2}} \binom{n-1}{\frac{n}{2}} L_0(x)$$

Therefore,

$$x^{n+1} = \sum_{i=0}^{\left[\frac{n-1}{2}\right]} (-1)^i \binom{n}{i} xL_{n-2i}(x) + (-1)^{\frac{n}{2}} \binom{n-1}{\frac{n}{2}} xL_0(x)$$

$$= \sum_{i=0}^{\left[\frac{n-1}{2}\right]} (-1)^i \binom{n}{i} L_{n-2i+1}(x) - \sum_{i=0}^{\left[\frac{n-1}{2}\right]} (-1)^i \binom{n}{i} L_{n-2i-1}(x)$$

$$+ 2(-1)^{\frac{n}{2}} \binom{n-1}{\frac{n}{2}} L_1(x)$$

Our end goal is to combine these two summations into one. To achieve this, shifting the indices of the second summation up by 1, we have

$$x^{n+1} = \sum_{i=0}^{\left[\frac{n-1}{2}\right]} (-1)^i \binom{n}{i} L_{n-2i+1}(x) - \sum_{i=1}^{\left[\frac{n-1}{2}\right]+1} (-1)^i \binom{n}{i-1} L_{n-2i+1}(x)$$

$$+ 2(-1)^{\frac{n}{2}} \binom{n-1}{\frac{n}{2}} L_1(x)$$

Since n is even, therefore above becomes

$$x^{n+1} = L_{n+1}(x) + \sum_{i=1}^{\left[\frac{n-1}{2}\right]} (-1)^i \binom{n+1}{i} L_{n-2i+1}(x) + (-1)^{\left[\frac{n}{2}\right]} \binom{n}{\left[\frac{n}{2}\right] - 1} L_1(x)$$

$$+ 2(-1)^{\frac{n}{2}} \binom{n-1}{\frac{n}{2}} L_1(x) = \sum_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i \binom{n+1}{i} L_{n-2i+1}(x)$$

(b)

Case I: When n is odd.

Clearly, results hold for $n = 1$.

By induction hypothesis, we have

$$(2x)^n = \sum_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i \binom{n}{i} Q_{n-2i}(x)$$

Multiplying both sides by $2x$, we get

$$(2x)^{n+1} = \sum_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i \binom{n}{i} 2xQ_{n-2i}(x)$$

$$= \sum_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i \binom{n}{i} Q_{n-2i+1}(x) - \sum_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i \binom{n}{i} Q_{n-2i-1}(x)$$

Our end goal is to combine these two summations into one. To accomplish this, note that shifting the indices of the second summation up by 1, we have

$$(2x)^{n+1} = \sum_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i \binom{n}{i} Q_{n-2i+1}(x) - \sum_{i=1}^{\left[\frac{n}{2}\right]+1} (-1)^{i-1} \binom{n}{i-1} Q_{n-2i+1}(x) \tag{8.9}$$

In order for our hypothesis to be correct, it suffices to show that

$$(2x)^{n+1} = \sum_{i=0}^{\left[\frac{n}{2}\right]} (-1)^{i} \binom{n+1}{i} \, Q_{n-2i+1}(x) + (-1)^{\frac{n+1}{2}} \binom{n}{\frac{n+1}{2}} \, Q_0(x)$$

To prove this, we have from (8.9),

$$(2x)^{n+1} = Q_{n+1}(x) + \sum_{i=1}^{\left[\frac{n}{2}\right]} (-1)^{i} \binom{n}{i} \, Q_{n-2i+1}(x) - \sum_{i=1}^{\left[\frac{n}{2}\right]} (-1)^{i-1} \binom{n}{i-1} \, Q_{n-2i+1}(x) - (-1)^{\frac{n-1}{2}} \binom{n}{\frac{n-1}{2}} \, Q_0(x) \tag{8.10}$$

Now, the coefficient of $Q_{n-2i+1}(x) = \binom{n}{i} + \binom{n}{i-1} = \binom{n+1}{i}$, therefore equation (8.10) becomes

$$(2x)^{n+1} = \sum_{i=0}^{\left[\frac{n}{2}\right]} (-1)^{i} \binom{n+1}{i} \, Q_{n-2i+1}(x) + (-1)^{\frac{n+1}{2}} \binom{n}{\frac{n+1}{2}} \, Q_0(x)$$

Case II When n is even

Let the result holds for n i.e.,

$$(2x)^{n} = \sum_{i=0}^{\left[\frac{n-1}{2}\right]} (-1)^{i} \, Q_{n-2i}(x) + (-1)^{\frac{n}{2}} \binom{n-1}{\frac{n}{2}} Q_0(x)$$

Multiplying both sides by $2x$, we get

$$(2x)^{n+1} = \sum_{i=0}^{\left[\frac{n-1}{2}\right]} (-1)^{i} \binom{n}{i} \, 2xQ_{n-2i}(x) + (-1)^{\frac{n}{2}} \binom{n-1}{\frac{n}{2}} 2xQ_0(x)$$

$$= \sum_{i=0}^{\left[\frac{n-1}{2}\right]} (-1)^i \binom{n}{i} Q_{n-2i+1}(x) - \sum_{i=0}^{\left[\frac{n-1}{2}\right]} (-1)^i \binom{n}{i} Q_{n-2i-1}(x)$$
$$+ 2(-1)^{\frac{n}{2}} \binom{n-1}{\frac{n}{2}} Q_1(x)$$

Our end goal is to combine these two summations into one. To accomplish this, note that shifting the indices of the second summation up by 1, we have

$$(2x)^{n+1} = \sum_{i=0}^{\left[\frac{n-1}{2}\right]} (-1)^i \binom{n}{i} Q_{n-2i+1}(x) - \sum_{i=1}^{\left[\frac{n-1}{2}\right]+1} (-1)^{i-1} \binom{n}{i-1} Q_{n-2i+1}(x)$$
$$+ 2(-1)^{\frac{n}{2}} \binom{n-1}{\frac{n}{2}} Q_1(x)$$

Since n is even, therefore above becomes

$$(2x)^{n+1} = Q_{n+1}(x) + \sum_{i=1}^{\left[\frac{n-1}{2}\right]} (-1)^i \binom{n+1}{i} Q_{n-2i+1}(x) + (-1)^{\left[\frac{n}{2}\right]} \binom{n}{\left[\frac{n}{2}\right]-1} Q_1(x)$$
$$+ 2(-1)^{\frac{n}{2}} \binom{n-1}{\frac{n}{2}} Q_1(x) = \sum_{i=0}^{\left[\frac{n-1}{2}\right]} (-1)^i \binom{n+1}{i} Q_{n-2i+1}(x)$$

which is what we desire.

8.3 Solution of Fractional Differential Equation using Lucas Polynomials

Next, we take step to forward our work.

Let (8.2) has a continuous solution which a linear combination of Lucas polynomials

$$y(x) = \sum_{k=1}^{\infty} a_k L_k(x),$$

and let $y_n(x)$ be n^{th} approximation of $y(x)$, therefore

$$y_n(x) = \sum_{\mathcal{s}=1}^{n+1} a_{\mathcal{s}} L_{\mathcal{s}}(x) = A^T L(x),$$

where $A^T = [a_1\ a_2 \dots a_{n+1}], L(x) = [L_1(x)\ \ L_2(x)\ \ \dots L_{n+1}(x)]^T$

Further, the r^{th} order derivative [81] of (8.2) is:

$$y^{(r)}(x) = A^T D^r L(x), D = [d_{ij}] = \begin{cases} i \sin\dfrac{(j-i)\pi}{2}, & j > i \\ 0, & \text{else} \end{cases}, \tag{8.11}$$

with $r = 0, 1, 2, \dots n$, and D denotes a $n \times n$ operational matrix for the derivative.

The caputo derivative of the vector $L(x)$ written as

$$D^{\alpha} L(x) = B^{(\alpha)} L(x), \tag{8.12}$$

where $B^{(\alpha)}$ denotes a $(n+1) \times (n+1)$ operational matrix of fractional derivative. Next, we generate the matrix $B^{(\alpha)}$.

Using (8.2) and (8.5) for $i = 0, 1, 2, , \dots, n$, we have

$$D^{\alpha} L_{\mathcal{s}}(x) = \sum_{j=0}^{\left[\frac{\mathcal{s}}{2}\right]} \left(\frac{\mathcal{s}}{\mathcal{s}-j}\right) \binom{\mathcal{s}-j}{j} D^{\alpha} x^{\mathcal{s}-2j}$$

$$= \begin{cases} \displaystyle\sum_{j=0}^{\left[\frac{\mathcal{s}}{2}\right]} \left(\frac{\mathcal{s}}{\mathcal{s}-j}\right) \binom{\mathcal{s}-j}{j} \frac{\Gamma(\mathcal{s}+1-2j)}{\Gamma(\mathcal{s}+1-2j-\alpha)} x^{\mathcal{s}-2j-\alpha}, & \mathcal{s}-2j \geq \alpha \\ 0, & \mathcal{s}-2j < \alpha \end{cases}$$

If $\mathcal{s} - 2j \geq \alpha$, and $\mathcal{s} - 2j$ is odd, then using (8.5), we have

$$D^{\alpha}L_{s}(x) = x^{-\alpha}\sum_{j=0}^{\left[\frac{s}{2}\right]}\left(\frac{s}{s-j}\right)\binom{s-j}{j}\frac{\Gamma(s+1-2j)}{\Gamma(s+1-2j-\alpha)}x^{s-2j}$$

$$= x^{-\alpha}\sum_{j=0}^{\left[\frac{s}{2}\right]}\left(\frac{s}{s-j}\right)\binom{s-j}{j}\frac{\Gamma(s+1-2j)}{\Gamma(s+1-2j-\alpha)}A_{sjk}L_{s-2j-2k}(x)$$

$$= x^{-\alpha}\sum_{j=0}^{\left[\frac{i_u}{2}\right]}B_{sjk}L_{s-2j-2k}\,(x)$$

where,

$$A_{sjk} = \sum_{k=0}^{\left[\frac{s-2j}{2}\right]}(-1)^{k}\binom{s-2j}{k}, B_{ijk} = \binom{s-j}{j}\left(\frac{s}{i-j}\right)\frac{\Gamma(s+1-2j)}{\Gamma(s+1-2j-\alpha)}A_{sjk}$$

If $s-2j \geq \alpha$, and $s-2j$ is even, then then using (8.5), we have

$$D^{\alpha}L_{s}(x) = x^{-\alpha}\sum_{j=0}^{\left[\frac{s}{2}\right]}\left(\frac{s}{s-j}\right)\left(\frac{s}{s-j}\right)\frac{\Gamma(s+1-2j)}{\Gamma(s+1-2j-\alpha)}x^{s-2j}$$

$$= x^{-\alpha}\sum_{j=0}^{\left[\frac{s}{2}\right]}\left(\frac{s}{s-j}\right)\left(\frac{s}{s-j}\right)\frac{\Gamma(s+1-2j)}{\Gamma(s+1-2j-\alpha)}A_{sjk}L_{s-2j-2k}(x)$$

$$+ x^{-\alpha}\sum_{j=0}^{\left[\frac{s}{2}\right]}\left(\frac{s}{s-j}\right)\left(\frac{s}{s-j}\right)\frac{\Gamma(s+1-2j)}{\Gamma(s+1-2j-\alpha)}(-1)^{\frac{s-2j}{2}}\binom{s-2j-1}{\frac{s-2j}{2}}L_0(x)$$

$$= x^{-\alpha}\sum_{j=0}^{\left[\frac{s}{2}\right]}B_{sjk}L_{i-2j-2k}(x)$$

$$+ x^{-\alpha}\sum_{j=0}^{\left[\frac{s}{2}\right]}\left(\frac{s}{s-j}\right)\binom{s-j}{j}\frac{\Gamma(s+1-2j)}{\Gamma(s+1-2j-\alpha)}(-1)^{\frac{s-2j}{2}}\binom{s-2j-1}{\frac{s-2j}{2}}L_0(x)$$

where,

$$A_{sjk} = \sum_{k=0}^{\left[\frac{s-2j-1}{2}\right]} (-1)^k \binom{s-2j}{k}, B_{sjk} = \left(\frac{s}{s-j}\right)\binom{s-j}{j}\frac{\Gamma(s+1-2j)}{\Gamma(s+1-2j-\alpha)}A_{sjk}$$

Substituting approximation obtained into (8.1), we find that

$$\begin{cases} B^{(\alpha)}L(x) + A^T D^k L(x) + A^T L(x) = f(x) \\ A^T D^r L(0) = c_r, r_1 = 0,1, \dots, m-1 \end{cases} \tag{8.13}$$

8.4 Solution of Fractional Differential Equation using Pell-Lucas Polynomials

Suppose that the equation (8.1) has a continuous function solution that can be expressed in the Pell-Lucas polynomials

$$y(x) = \sum_{k=1}^{\infty} a_k Q_k(x),$$

and let $y_n(x)$ be an approximation to $y(x)$, that is

$$y_n(x) = \sum_{k=1}^{n+1} a_k Q_k(x) = A^T Q(x),$$

where $A^T = [a_1\ a_2 \dots a_{n+1}], Q(x) = [Q_1(x)\ \ Q_2(x)\ \ \dots Q_{n+1}(x)]^T$

The caputo derivative of the vector $Q(x)$ can be expressed by

$$D^\alpha Q(x) = B^{(\alpha)} Q(x), \tag{8.14}$$

where $B^{(\alpha)}$ is the $(n+1) \times (n+1)$ operational matrix of fractional derivative. In this section, we derive matrix $B^{(\alpha)}$.

Using (8.2) and (8.6) for $i = 0, 1, 2, , \dots, n$

$$D^{\alpha}Q_{\mathfrak{v}}(x) = \sum_{j=0}^{\left[\frac{\mathfrak{v}}{2}\right]} \left(\frac{\mathfrak{v}}{\mathfrak{v}-j}\right)\binom{\mathfrak{v}-j}{j} D^{\alpha}(2x)^{\mathfrak{v}-2j}$$

$$= \begin{cases} \sum_{j=0}^{\left[\frac{\mathfrak{v}}{2}\right]} 2^{\mathfrak{v}-2j}\left(\frac{\mathfrak{v}}{\mathfrak{v}-j}\right)\binom{\mathfrak{v}-j}{j}\frac{\Gamma(\mathfrak{v}+1-2j)}{\Gamma(\mathfrak{v}+1-2j-\alpha)}x^{\mathfrak{v}-2j-\alpha}, & \mathfrak{v}-2j \geq \alpha \\ 0, & \mathfrak{v}-2j < \alpha \end{cases}$$

If $\mathfrak{v}-2j \geq \alpha$, and $\mathfrak{v}-2j$ is odd, then using (8.6), we have

$$D^{\alpha}Q_{\mathfrak{v}}(x) = x^{-\alpha}\sum_{j=0}^{\left[\frac{\mathfrak{v}}{2}\right]} 2^{\mathfrak{v}-2j}\left(\frac{\mathfrak{v}}{\mathfrak{v}-j}\right)\binom{\mathfrak{v}-j}{j}\frac{\Gamma(\mathfrak{v}+1-2j)}{\Gamma(\mathfrak{v}+1-2j-\alpha)}x^{\mathfrak{v}-2j}$$

$$= x^{-\alpha}\sum_{j=0}^{\left[\frac{\mathfrak{v}}{2}\right]} 2^{\mathfrak{v}-2j}\left(\frac{\mathfrak{v}}{\mathfrak{v}-j}\right)\binom{\mathfrak{v}-j}{j}\frac{\Gamma(\mathfrak{v}+1-2j)}{\Gamma(\mathfrak{v}+1-2j-\alpha)}Q_{\mathfrak{v}jk}L_{\mathfrak{v}-2j-2k}$$

$$= x^{-\alpha}\sum_{j=0}^{\left[\frac{i}{2}\right]} B_{\mathfrak{v}jk}Q_{\mathfrak{v}-2j-2k}$$

where,

$$A_{\mathfrak{v}jk} = \sum_{k=0}^{\left[\frac{\mathfrak{v}-2j}{2}\right]} (-1)^{k}\binom{\mathfrak{v}-2j}{k}$$

and

$$B_{\mathfrak{v}jk} = \left(\frac{\mathfrak{v}}{\mathfrak{v}-j}\right)\binom{\mathfrak{v}-j}{j}\frac{\Gamma(\mathfrak{v}+1-2j)}{\Gamma(\mathfrak{v}+1-2j-\alpha)}A_{\mathfrak{v}jk}$$

If $\mathfrak{v}-2j \geq \alpha$, and $\mathfrak{v}-2j$ is even, then then using (8.6), we have

$$D^{\alpha}Q_{\mathfrak{v}}(x) = x^{-\alpha}\sum_{j=0}^{\left[\frac{\mathfrak{v}}{2}\right]} 2^{\mathfrak{v}-2j}\left(\frac{\mathfrak{v}}{\mathfrak{v}-j}\right)\binom{\mathfrak{v}-j}{j}\frac{\Gamma(\mathfrak{v}+1-2j)}{\Gamma(\mathfrak{v}+1-2j-\alpha)}x^{\mathfrak{v}-2j}$$

$$= x^{-\alpha} \sum_{j=0}^{\left[\frac{\mathfrak{v}}{2}\right]} \left(\frac{\mathfrak{v}}{\mathfrak{v}-j}\right) \binom{\mathfrak{v}-j}{j} \frac{\Gamma(\mathfrak{v}+1-2j)}{\Gamma(\mathfrak{v}+1-2j-\alpha)} A_{\mathfrak{v}jk} Q_{\mathfrak{v}-2j-2k}$$

$$+ x^{-\alpha} \sum_{j=0}^{\left[\frac{\mathfrak{v}}{2}\right]} \left(\frac{\mathfrak{v}}{\mathfrak{v}-j}\right) \binom{\mathfrak{v}-j}{j} \frac{\Gamma(\mathfrak{v}+1-2j)}{\Gamma(\mathfrak{v}+1-2j-\alpha)} (-1)^{\frac{\mathfrak{v}-2j}{2}} \binom{\mathfrak{v}-2j-1}{\frac{\mathfrak{v}-2j}{2}} Q_0(x)$$

$$= x^{-\alpha} \sum_{j=0}^{\left[\frac{\mathfrak{v}}{2}\right]} B_{\mathfrak{v}jk} Q_{\mathfrak{v}-2j-2k}$$

$$+ x^{-\alpha} \sum_{j=0}^{\left[\frac{\mathfrak{v}}{2}\right]} \left(\frac{\mathfrak{v}}{\mathfrak{v}-j}\right) \binom{\mathfrak{v}-j}{j} \frac{\Gamma(\mathfrak{v}+1-2j)}{\Gamma(\mathfrak{v}+1-2j-\alpha)} (-1)^{\frac{\mathfrak{v}-2j}{2}} \binom{\mathfrak{v}-2j-1}{\frac{\mathfrak{v}-2j}{2}} Q_0(x)$$

where,

$$A_{\mathfrak{v}jk} = \sum_{j=0}^{\left[\frac{\mathfrak{v}-2j-1}{2}\right]} \left(\frac{\mathfrak{v}}{\mathfrak{v}-j}\right) (-1)^k \binom{\mathfrak{v}-2j}{k}, B_{\mathfrak{v}jk}$$

$$= \left(\frac{\mathfrak{v}}{\mathfrak{v}-j}\right) \binom{\mathfrak{v}-j}{j} \frac{\Gamma(\mathfrak{v}+1-2j)}{\Gamma(\mathfrak{v}+1-2j-\alpha)} A_{\mathfrak{v}jk}$$

8.5 Numerical Problem

Example 8.5.1 Consider the following equation

$$\begin{cases} \boldsymbol{z^{(3)}(x) + D^{(\frac{3}{2})} z(x) + z(x) = 1 + x} \\ \boldsymbol{z(0) = 1} \\ \boldsymbol{z^{(1)}(0) = 1} \\ \boldsymbol{z^{(2)}(0) = 0} \end{cases}$$

We applied the Lucas polynomial approach to solve (8.12) with $n = 2$. In this case we have

$$L = [L_0(x), L_1(x), L_2(x)] = [2\ , x\ , x^2 + 2], \qquad A = [a_1, a_2\ , a_3]$$

$$D^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B^{\frac{3}{2}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{\Gamma(3)}{\Gamma\left(\frac{3}{2}\right)} x^{-\frac{3}{2}} & 0 & \frac{\Gamma(3)}{\Gamma\left(\frac{3}{2}\right)} x^{-\frac{3}{2}} \end{bmatrix}$$

and

$$A^T D^3 L + A^T B^{\left(\frac{3}{2}\right)} L + A^T L = f(x),$$

$$z(0) = \sum_{k=0}^{3} a_k L_k(0) = 1, z^{(1)}(0)$$

$$= \sum_{k=0}^{3} a_k D L_k(0) = 1, z^{(2)}(0) = \sum_{k=0}^{3} a_k D^2 L_k(0) = 0$$

On solving this, we obtain $a_i{}'s$ and finds that $z(x) = x + 1$ is the exact solution.

8.6 Derivative of the Fibonacci polynomials

Here, we construct a matrix that produces the sequence obtained on differentiating the Fibonacci polynomials. In [82], the author deduced a lot of results based on the derivative of the polynomials given by

$$F_k(x) = xF_{k-1}(x) + F_{k-2}(x), \text{ for } k \geq 2,$$

with $F_1(x) = 1$, $F_2(x) = x$.

By deriving the Fibonacci polynomials, few terms are mentioned below:

$F_1'(x) = 0$

$F_2'(x) = 1$

$F_3'(x) = 2x$

$F_4'(x) = 3x^2 + 2$

$F_5'(x) = 4x^3 + 6x$

$F_6'(x) = 5x^4 + 12x^2 + 3$

$F_7'(x) = 6x^5 + 20x^3 + 12x$

$F_8(x) = 7x^6 + 30x^4 + 30x^2 + 4$

$F_9'(x) = 8x^7 + 42x^5 + 60x^3 + 20x$

$F_{10}'(x) = 9x^8 + 56x^6 + 105x^4 + 60x^2 + 5$

Now,

$$F' = BX,$$

where $F' = [F_1'(x), F_2'(x), F_2'(x), \ldots]^t, X = [1, x, x^2, \ldots]^t$

and B is invertible lower triangular matrix having elements as the coefficients of the derivatives of Fibonacci polynomials in increasing powers of x:

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 12 & 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 0 & 20 & 0 & 6 & 0 & 0 & 0 & 0 \\ 4 & 0 & 30 & 0 & 30 & 0 & 7 & 0 & 0 & 0 \\ 0 & 20 & 0 & 60 & 0 & 42 & 0 & 8 & 0 & 0 \\ 5 & 0 & 60 & 0 & 105 & 0 & 56 & 0 & 9 & 0 \\ 0 & 30 & 0 & 140 & 0 & 168 & 0 & 72 & 0 & 10 \end{bmatrix}$$

Now, the following equation

$$r^4 - 2xr^3 + (x^2 - 2)r^2 + 2xr + 1 = 0 \tag{8.15}$$

is satisfied by the sequence of derivative of Fibonacci polynomials.

On solving equation (8.15), roots are:

$$\gamma = \frac{x + \sqrt{x^2 + 4}}{2}, \delta = \frac{x + \sqrt{x^2 + 4}}{2}$$

Now, we develop the generating function, Binet's formula for the sequence obtained by differentiation of Fibonacci polynomials.

8.7 Properties of Sequence of Derivatives of Fibonacci Polynomials

Theorem 8.7.1 Then the generating function for sequence of first-order derivatives of Fibonacci sequences $\{F'_n(x)\}$ is

$$\sum_{n=0}^{\infty} F'_n(x)\,u^n = \frac{u^2}{1-\ 2xu+(x^2-2)u^2+2xu^3+u^4}$$

Proof:

Consider

$$g_F\,(u) = \sum_{n=0}^{\infty} F'_n(x)\,u^n = u^2 + g_F\,(u)(2xu-(x^2-2)u^2-2xu^3-u^4)$$

Thus,

$$g_F\,(u) = \frac{u^2}{1-\ 2xu+(x^2-2)u^2+2xu^3+u^4}$$

Theorem 8.7.2 For $n \geq 0$:

$$F'_n(x) = \frac{(-1)^n\big(-2F_{n+1}(x)+(n+1)L_n(x)\big)}{(\gamma-\delta)^2}$$

Proof: This theorem has a simpler proof on using theorem 8.7.1.

Theorem 8.7.3 For a natural number m, following identity hold:

(a)

$$\begin{pmatrix} F'_{m+4}(x) \\ F'_{m+3}(x) \\ F'_{m+2}(x) \\ F'_{m+1}(x) \end{pmatrix} = \begin{pmatrix} 2x & -(x^2-2) & -2x & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} F'_{m+3}(x) \\ F'_{m+2}(x) \\ F'_{m+1}(x) \\ F'_{m}(x) \end{pmatrix}$$

Proof: By using recursive relation for the sequence of derivative of Fibonacci polynomials, we obtain

$$\begin{pmatrix} 2x & -(x^2-2) & -2x & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} F'_{m+3}(x) \\ F'_{m+2}(x) \\ F'_{m+1}(x) \\ F'_m(x) \end{pmatrix}$$

$$= \begin{pmatrix} 2xF'_{m+3}(x) - (x^2-2)F'_{m+2}(x) - 2xF'_{m+1}(x) - F'_m(x) \\ F'_{m+2}(x) \\ F'_{m+1}(x) \\ F'_m(x) \end{pmatrix}$$

$$= \begin{pmatrix} F'_{m+4}(x) \\ F'_{m+3}(x) \\ F'_{m+2}(x) \\ F'_{m+1}(x) \end{pmatrix}$$

Chapter 9

Solutions of the Diophantine equation $Q_{\mathfrak{n}} - Q_{\mathfrak{m}} = 2^{\mathfrak{a}}$

9.1Introduction

In this chapter, we solve the following Diophantine equation

$$Q_{\mathfrak{n}} - Q_{\mathfrak{m}} = 2^{\mathfrak{a}}, \tag{9.1}$$

where $Q_{\mathfrak{n}}$ and $Q_{\mathfrak{m}}$ are Pell-Lucas numbers given by

$$Q_{\mathfrak{n}+1} = 2Q_{\mathfrak{n}} + Q_{\mathfrak{n}-1}, \forall\, \mathfrak{n} \geq 1,$$

with initial conditions $Q_0 = Q_1 = 2$. By using the theory of linear forms in logarithms of algebraic numbers, we derived an explicit upper bound for $\mathfrak{n}$, and then, by reduction method based on continued fraction algorithm, we achieved our objective.

Pell-Lucas sequence are generated by the following formula:

$$Q_{\mathfrak{n}} = \alpha_1^{\mathfrak{n}} + \beta_1^{\mathfrak{n}},$$

where $\alpha_1 = 1 + \sqrt{2}$ and $\beta_1 = 1 - \sqrt{2}$.

Specifically,

$$\alpha_1^{\mathscr{k}-1} \leq Q_{\mathscr{k}} \leq 2\alpha_1^{\mathscr{k}}, \tag{9.2}$$

holds for all $\mathscr{k} \geq 1$.

A few useful findings related to achieve our objective are summarized below:

9.2 Related Results

The minimal primitive polynomial over the set of integers $\mathbb{Z}$ of an algebraic number ϑ of degree p over $\mathbb{Q}$ is:

$$c_0x^p + c_1x^{p-1} + \cdots + c_p = c_0 \prod_{\mathrm{i}=1}^{p} \left(x - \vartheta^{\mathrm{i}}\right) \in \mathbb{Z}[x],$$

where c_{i}'s are co-primes in pairs with $c_0 > 0$ and ϑ^{i} 's are conjugates of ϑ. Now, logarithmic height [60] of ϑ is

$$\mathfrak{h}\,(\vartheta) = \frac{1}{p}\left(\log|c_0| + \sum_{\mathrm{i}=1}^{p} \log\left(\max\{|\vartheta^{\mathrm{i}}|, 1\}\right)\right)$$

If $\vartheta = \frac{a_1}{b_1}$ is a rational number with $\gcd(a_1, b_1) = 1$ and $b_1 \geq 1$, then

$$\mathfrak{h}\,(\vartheta) = \log(\max\{|a_1|, b_1\})$$

Logarithmic height satisfies:

$$\mathfrak{h}\left(\vartheta \pm \gamma_p\right) \leq \mathfrak{h}(\vartheta) + \mathfrak{h}\left(\gamma_p\right) + \log 2, \tag{9.3}$$

$$\mathfrak{h}\left(\vartheta\gamma_p^{\pm 1}\right) \leq \mathfrak{h}\,(\vartheta) + \mathfrak{h}\left(\gamma_p\right), \tag{9.4}$$

$$\mathfrak{h}(\vartheta^s) = |s|\mathfrak{h}(\vartheta), \tag{9.5}$$

where ϑ and γ_p are any algebraic numbers over $\mathbb{Q}$ and s is any integer.

Theorem 9.2.1. [64] Let γ_1, γ_2 be non-zero algebraic numbers. Let

$$D = [\mathbb{Q}(\gamma_1, \gamma_2) : \mathbb{Q}]/[\mathbb{R}(\gamma_1, \gamma_2) : \mathbb{R}],$$

and

$$\Gamma_1 = \mathfrak{b}_2 \log\gamma_2 - \mathfrak{b}_1 \log\gamma_1,$$

where $\mathfrak{b}_1$ and $\mathfrak{b}_2$ are positive integers. Further, let $\mathcal{B}_1, \mathcal{B}_2$ be any real numbers > 1 with

$$\log\mathcal{B}_i \geq \max\left\{\mathfrak{h}(\gamma_i), \frac{|\log(\gamma_i)|}{D}, \frac{1}{D}\right\}, i = 1, 2$$

If γ_1 and γ_2 are multiplicatively independent, then

$$\log|\Gamma_1| > (-30.9)(D^4)\left(\max\left\{\log\mathfrak{b}, \frac{21}{D}, \frac{1}{2}\right\}\right)^2 \log\mathcal{B}_1 \log\mathcal{B}_2\,,$$

where,

$$\mathfrak{b} = \frac{\mathfrak{b}_1}{D\log\mathcal{B}_2} + \frac{\mathfrak{b}_2}{D\log\mathcal{B}_1}$$

We can find the least upper bound for $\mathfrak{n}$, by approaching as the same procedure in [65] and [66].

Theorem 9.2.2 [65, 66] *"In a real algebraic number field L of degree D over* $\mathbb{Q}$*, let* $\gamma_1, \gamma_2, \dots, \gamma_\mathfrak{n}$ *be any positive real algebraic numbers and suppose* $\mathfrak{b}_1, \mathfrak{b}_2, \dots, \mathfrak{b}_\mathfrak{n}$ *be rational numbers with*

$$0 \neq \Lambda_1 + 1 = (\gamma_1)^{\mathfrak{b}_1} \cdots (\gamma_\mathfrak{n})^{\mathfrak{b}_\mathfrak{n}}, \mathcal{B} = max\{|\mathfrak{b}_1|, \dots, |\mathfrak{b}_\mathfrak{n}|\},$$

and $\mathcal{B}_1, \dots, \mathcal{B}_\mathfrak{n}$ *be the real numbers such that*

$$\mathcal{B}_j \geq max\{ D\mathfrak{h}(\gamma_j), |log\, \gamma_j|, 0.16\}, \qquad for\ all\ 1 \leq j \leq \mathfrak{n}$$

If $\Lambda_1 \neq 0$*, then*

$$log\, |\Lambda_1| > (-1.4)(30^{\mathfrak{n}+3}\mathfrak{n}^{4.5})D^2(1 + log\, D)(1 + log\, \mathcal{B})\mathcal{B}_1 \cdots \mathcal{B}_\mathfrak{n} \text{ "}$$

Theorem 9.2.3 [68] "*Let* $\frac{p}{q}$ *be a convergent of the continued fraction of any irrational number* γ *and* $\mathcal{M}$ *be any positive integer such that* $q > 6\mathcal{M}$*. Let* $\mathcal{A}, \mathcal{B}, \mu_1$ *denotes any real numbers with* $\mathcal{A} > 0$ *and* $\mathcal{B} > 1$ *such that:*

$$\varepsilon = \| \mu_1 q \| - \mathcal{M} \| \gamma q \|,$$

where distance from the nearest integer is denoted by $\|.\|$*. Then, there is no solution of the following inequality if* $\varepsilon > 0$*:*

$$0 < |\mathfrak{u}\gamma - \mathfrak{v} + \mu_1| < \mathcal{A}\mathcal{B}^{-\mathfrak{w}},$$

in positive integers $\mathfrak{u}, \mathfrak{v}$ *and* $\mathfrak{w}$ *with*

$$\mathfrak{u} \leq \mathcal{M} \text{ and } \mathfrak{w} \geq \frac{log(\mathcal{A}q/\varepsilon)}{log\, \mathcal{B}} \text{".}$$

Theorem 9.2.4 [59] "The only positive integral solutions $(\mathfrak{n}, \mathfrak{q}, \mathfrak{x})$ with $\mathfrak{q} \geq 2$ of the diophantine equation:

$$P_{\mathfrak{n}} = \mathfrak{x}^{\mathfrak{q}},$$

are $(\mathfrak{n}, \mathfrak{q}, \mathfrak{x}) = (1, \mathfrak{q}, 1)$ and $(7,2,13)$, where $P_{\mathfrak{n}}$ denotes the $\mathfrak{n}^{th}$ term of Pell-number. Henceforth, the only perfect powers of exponent larger than 1 in the Pell-numbers are"

$$P_1 = 1 \text{ and } P_7 = 13^2.$$

9.3 Proof of the Main Result

We mainly prove the following result:

Theorem 9.3.1 The possible non-negative integral solutions in $(\mathfrak{n}, \mathfrak{m}, \mathfrak{a})$ with $0 \leq \mathfrak{m} < \mathfrak{n}$, of the Diophantine equation given by

$$\boldsymbol{Q_{\mathfrak{n}} - Q_{\mathfrak{m}} = 2^{\mathfrak{a}}},$$

are

$$\{(\mathbf{2}, \mathbf{0}, \mathbf{2}), (\mathbf{2}, \mathbf{1}, \mathbf{2}), (\mathbf{3}, \mathbf{2}, \mathbf{3}), (\mathbf{4}, \mathbf{0}, \mathbf{5}), (\mathbf{4}, \mathbf{1}, \mathbf{5})\}.$$

Proof: Assume that equation (9.1) holds. Firstly, we discuss the case for $\mathfrak{n} - \mathfrak{m} = 1$, then by using the recurrence relation of Pell-Lucas numbers in (9.1), we have

$$Q_{\mathfrak{m}+1} - Q_{\mathfrak{m}} = Q_{\mathfrak{m}} + Q_{\mathfrak{m}-1} = 2^{\mathfrak{a}}$$

Using $Q_{\mathfrak{m}} + Q_{\mathfrak{m}-1} = 4P_{\mathfrak{m}}$, where $P_{\mathfrak{m}}$ denotes the $\mathfrak{m}^{th}$ term of Pell numbers, we get

$$P_{\mathfrak{m}} = 2^{\mathfrak{a}-2},$$

which implies from theorem 9.2.4 that $(\mathfrak{n}, \mathfrak{m}, \mathfrak{a}) \in \{(2,1,2), (3,2,3)\}$

If $\mathfrak{n} - \mathfrak{m} = 2$, then (9.1) becomes,

$$Q_{\mathfrak{m}+2} - Q_{\mathfrak{m}} = 2^{\mathfrak{a}}$$

Since, $Q_{\mathfrak{m}+2} - Q_{\mathfrak{m}} = 2Q_{\mathfrak{m}+1}$, and $Q_{\mathfrak{m}+1} = 2(P_{\mathfrak{m}+1} + P_{\mathfrak{m}})$, therefore, from above, we get

$$P_{\mathfrak{m}+1} + P_{\mathfrak{m}} = 2^{\mathfrak{a}-2},$$

Therefore, using the results obtained from [59], we have $(\mathfrak{n}, \mathfrak{m}, \mathfrak{a}) \in \{(2,0,2)\}$. Therefore, we discuss the case for $\mathfrak{n} - \mathfrak{m} \geq 3$.

For $0 \leq \mathfrak{m} < \mathfrak{n} \leq 43$, by a easier calculation it can be easily seen that,

$$Q_2 - Q_0 = 2^2, Q_2 - Q_1 = 2^2, Q_3 - Q_2 = 2^3, Q_4 - Q_1 = 2^5, Q_4 - Q_0 = 2^5$$

Next, we will discuss the case for $\mathfrak{n} > 43$ by adopting the methodology as in [59, 60, 62]. For this, we need the following Lemma given by:

Lemma 9.3.1 If $(\mathfrak{n}, \mathfrak{m}, \mathfrak{a})$ is any positive integral solution of equation (9.1) with $\mathfrak{n} > \mathfrak{m}$, and $\mathfrak{n} > 43$, then

$$\mathfrak{a} \leq 2 * \mathfrak{n} < 2 * 5.59831 * 10^{23}.$$

Proof:

Firstly, we formulate a connection between $\mathfrak{n}$ and $\mathfrak{a}$.

By (9.1) and (9.2), we get

$$2^{\mathfrak{a}} \leq 2\alpha_1^{\mathfrak{n}} - \alpha_1^{\mathfrak{m}-1} < 2^{2\mathfrak{n}+1} - \alpha_1^{\mathfrak{m}-1} = 2^{2\mathfrak{n}+1}\left(1 - 2^{-(2\mathfrak{n}+1)}\alpha_1^{\mathfrak{m}-1}\right) < 2^{2\mathfrak{n}+1},$$

as $\alpha_1 < 2^2$. Thus, $\mathfrak{a} \leq 2\mathfrak{n}$

From (9.1), we have

$$\alpha_1^{\mathfrak{n}} - 2^{\mathfrak{a}} = -\beta_1^{\mathfrak{n}} + Q_{\mathfrak{m}}$$

Therefore,

$$|\alpha_1^{\mathfrak{n}} - 2^{\mathfrak{a}}| = |-\beta_1^{\mathfrak{n}} + Q_{\mathfrak{m}}| \leq |\beta_1|^{\mathfrak{n}} + Q_{\mathfrak{m}} < \frac{1}{2} + 2\alpha_1^{\mathfrak{m}}, \tag{9.6}$$

or

$$|1 - 2^{\mathfrak{a}}\alpha_1^{-\mathfrak{n}}| < \frac{3}{\alpha_1^{\mathfrak{n}-\mathfrak{m}}}$$

(9.7)

Now, take parameters:

$$\gamma_1 = 2, \gamma_2 = \alpha_1, \mathfrak{b}_1 = \mathfrak{a}, \mathfrak{b}_2 = -\mathfrak{n}$$

The algebraic number field containing γ_1 and γ_2 is $Q(\sqrt{2})$, therefore $D = 2$.

Further, it is essential to show that $\Lambda_1 = 2^{\mathfrak{a}}\alpha_1^{-\mathfrak{n}} - 1$ is non-zero. Suppose, $\Lambda_1 = 0$, then, from (9.1) and Binet's formula for Pell-Lucas sequence, we get

$$2^{\mathfrak{a}} = \alpha_1^{\mathfrak{n}} = Q_{\mathfrak{n}} - \beta_1^{\mathfrak{n}} > Q_{\mathfrak{n}} - 1 \geq Q_{\mathfrak{n}} - Q_{\mathfrak{m}} = 2^{a}$$

which is impossible.

Thus,

$$\mathfrak{h}(\gamma_1) = \log 2, \mathfrak{h}(\gamma_2) = \mathfrak{h}(\alpha_1) = \frac{\log\alpha_1}{2}$$

Take

$$\mathcal{B}_1 = 1.4 \geq \max\{2\log 2, |\log\gamma_1|, 0.16\}, \mathcal{B}_2 = 0.88$$

As $\mathfrak{a} < 2\mathfrak{n} + 1$, therefore

$$\mathcal{B} = \max\{|\mathfrak{a}|, |-\mathfrak{n}|, 1\} = 2\mathfrak{n} + 1$$

Using theorem 9.2.2, we get,

$$\frac{3}{\alpha_1^{\mathfrak{n}-\mathfrak{m}}} > |2^{\mathfrak{a}}\alpha_1^{-\mathfrak{n}} - 1|$$
$$> \exp\{(-1.4) * 30^5 * 2^{4.5} * 4 * (1 + \log 2)(1 + \log(2\mathfrak{n} + 1)) * 1.4 * 0.88\}$$

(9.8)

Taking logarithm on both sides and using the result:

$$1 + \log(2\mathfrak{n} + 1) < 2\log\mathfrak{n}, \text{ for all } \mathfrak{n} > 5,$$

we have,

$$(\mathfrak{n}-\mathfrak{m})\log\alpha_1 < 1.28459 * 10^{10}\log\mathfrak{n} \tag{9.9}$$

From (9.1), we have,

$$|\alpha_1^{\mathfrak{n}}(1-\alpha_1^{\mathfrak{m}-\mathfrak{n}}) - 2^{\mathfrak{a}}| = |-\beta_1^{\mathfrak{n}} + \beta^{\mathfrak{m}}| < 1 \tag{9.10}$$

Dividing both sides by $\alpha_1^{\mathfrak{n}}(1-\alpha_1^{\mathfrak{m}-\mathfrak{n}})$, we obtain,

$$|1 - 2^{\mathfrak{a}}\alpha_1^{-\mathfrak{n}}(1-\alpha_1^{\mathfrak{m}-\mathfrak{n}})^{-1}| < \frac{1}{\alpha_1^{\mathfrak{n}}(1-\alpha_1^{\mathfrak{m}-\mathfrak{n}})} \tag{9.11}$$

Since,

$$\alpha_1^{\mathfrak{m}-\mathfrak{n}} = \frac{1}{\alpha_1^{\mathfrak{n}-\mathfrak{m}}} < \frac{1}{\alpha_1} = 0.414,$$

therefore

$$\frac{1}{1-\alpha_1^{\mathfrak{m}-\mathfrak{n}}} < 1.71 \tag{9.12}$$

Therefore, (9.11) becomes

$$|1 - 2^{\mathfrak{a}}\alpha_1^{-\mathfrak{n}}(1-\alpha_1^{\mathfrak{m}-\mathfrak{n}})^{-1}| < \frac{1.71}{\alpha_1^{\mathfrak{n}}} \tag{9.13}$$

Let

$$\Lambda_1 = 2^{\mathfrak{a}}\alpha_1^{-\mathfrak{n}}(1-\alpha_1^{\mathfrak{m}-\mathfrak{n}})^{-1} - 1$$

Take

$$\gamma_1 = 2, \gamma_2 = \alpha_1, \gamma_3 = (1 - \alpha_1^{\mathfrak{m}-\mathfrak{n}})^{-1}, \mathfrak{b}_1 = \mathfrak{a}, \mathfrak{b}_2 = -\mathfrak{n}, \mathfrak{b}_3 = 1$$

As $D = \left[Q(\sqrt{2}): Q\right] = 2$. Now, it is obvious that $\Lambda_1 \neq 0$ as if $\Lambda_1 = 0$, then

$$2^{\mathfrak{a}} = \alpha_1^{\mathfrak{n}}(1 - \alpha_1^{\mathfrak{m}-\mathfrak{n}}) = \alpha_1^{\mathfrak{n}} - \alpha_1^{\mathfrak{m}} = Q_{\mathfrak{n}} - \beta_1^{\mathfrak{n}} - Q_{\mathfrak{m}} + \beta_1^{\mathfrak{m}} \neq 2^{\mathfrak{a}}$$

Therefore, $\mathcal{B}_1 = 1.4, \mathcal{B}_2 = 0.88, \mathcal{B} = 2\mathfrak{n} + 1$

Further, we find the value of $\mathcal{B}_3$. For all $n_p - m_u \geq 3$, we have

$$|\log(\gamma_3)| < 1$$

Taking (9.3), (9.4)and (9.5) into consideration, we have

$$\mathfrak{h}(\gamma_3) = \mathfrak{h}((1 - \alpha_1^{\mathfrak{m}-\mathfrak{n}})^{-1}) \leq \mathfrak{h}_1(1) + \mathfrak{h}(\alpha_1^{\mathfrak{m}-\mathfrak{n}}) + \log 2$$

$$\leq |\mathfrak{m} - \mathfrak{n}|\mathfrak{h}(\alpha_1) + \log 2 = \frac{(\mathfrak{n} - \mathfrak{m})\log\alpha_1}{2} + \log 2 \quad (9.14)$$

Thus,

$$\mathcal{B}_3 \geq \max\{2\mathfrak{h}(\gamma_3), |2\log(\gamma_3)|, 0.16\}$$

So, we can take

$$\mathcal{B}_3 = (\mathfrak{n} - \mathfrak{m})\log\alpha_1 + 2\log 2 = (\mathfrak{n} - \mathfrak{m})\log\alpha_1 + \log 4$$

Now, by theorem 9.2.2 and equation (9.13), we get

$$\frac{1.71}{\alpha_1^{\mathfrak{n}}} > |\Lambda| > \exp\{(-1.4) * 30^5 * 2^{4.5} * 4 * (1 + \log 2)(1 + \log(2\mathfrak{n} + 1)) * 1.4 * 0.88 * ((\mathfrak{n} - \mathfrak{m})\log\alpha_1 + \log 4)\} \quad (9.15)$$

Using

$$1 + \log(2n + 1) < 2\log n, \text{ for all } n > 5,$$

and by Mathematica software, we get

$$n < 5.59831 * 10^{23}$$

(9.16)

After determining upper bound of n, the next step is to reduce it. Let

$$z = a\log 2 - n\log\alpha_1$$

Now, from (9.7),

$$|1 - e^{z}| < \frac{3}{\alpha_1^{n-m}}$$

(9.17)

Now, from (9.1) and Binet's formula for Pell-Lucas sequence, we have

$$\alpha_1^{n} = Q_n - \beta^{n} > Q_n - 1 \geq Q_n - Q_m = 2^{a}$$

Consequently, $1 > 2^{a}\alpha_1^{-n}$, and hence, $z < 0$.

Since, for $n - m \geq 3$, we have

$$\frac{3}{\alpha_1^{n-m}} \leq 0.213203,$$

it follows that $e^{|z|} \leq 1.27098$.

Hence, since $x < e^{x} - 1$ for $x > 0$, we get

$$0 < |z| < e^{|z|} - 1 = e^{|z|}|1 - e^{z}| < \frac{4}{\alpha_1^{n-m}},$$

$$0 < |a\log 2 - n\log\alpha_1| < \frac{4}{\alpha_1^{n-m}}$$

(9.18)

Now, the algebraic number field containing γ_1 and γ_2 is $Q(\sqrt{2})$, therefore $D = 2$.

Dividing both sides of (9.18) by $\log\alpha_1$, we have

$$|\mathfrak{a}\gamma - \mathfrak{n}| < \frac{5}{\alpha_1^{\mathfrak{n}-\mathfrak{m}}}$$

(9.19)

where,

$$\gamma = \frac{\log 2}{\log\alpha_1}$$

Let $\frac{p_{\mathfrak{n}}}{q_{\mathfrak{n}}}$ be $\mathfrak{n}^{th}$ convergent of γ. Using Mathematica software, we have,

$$558371393446601568385 9930 = q_{48} < 5.59831 * 10^{23}$$
$$< 666036315191163261927801 = q_{49}$$

Furthermore, $a_N = \max\{a_i : i = 0, 1, \dots, 49\} = a_{27} = 100$.

So, by using results of continued fraction, we have

$$|\mathfrak{a}\gamma - \mathfrak{n}| > \frac{1}{(a_N + 2)\mathfrak{a}}$$

(9.20)

Comparing (9.19) and (9.20), we get

$$\alpha_1^{\mathfrak{n}-\mathfrak{m}} < 5 * 102 * 5.59831 * 10^{23} * 2 = 5.71028 * 10^{26}, \quad (9.21)$$

which leads to $\mathfrak{n} - \mathfrak{m} \leq 69$.

Using (9.15), we get

$$\mathfrak{n} < 2.80728 * 10^{13}$$

Take

$$z_2 = \mathfrak{a}\log 2 - \mathfrak{n}\log\alpha_1 + \log(1-\alpha_1^{\mathfrak{m}-\mathfrak{n}})^{-1}$$

Thus, (9.13) implies that

$$|1-e^{z_2}| < \frac{1.71}{\alpha_1^{\mathfrak{n}}}$$

For $\mathfrak{n} > 1$, we have

$$\frac{1.71}{\alpha_1^{\mathfrak{n}}} < \frac{1}{2}$$

If $z_2 > 0$, then

$$0 < z_2 < e^{z_2} - 1 < \frac{1.71}{\alpha_1^{\mathfrak{n}}}$$

If $z_2 < 0$, then

$$|1-e^{z_2}| = 1 - e^{z_2} < \frac{1.71}{\alpha_1^{\mathfrak{n}}} < \frac{1}{2}$$

Thus,

$$e^{|z_2|} < 2$$

Therefore,

$$0 < |z_2| < e^{|z_2|} - 1 \leq e^{|z_2|}|1-e^{z_2}| < \frac{3.42}{\alpha_1^{\mathfrak{n}}}$$

Thus,

$$0 < \left|\mathfrak{a}\frac{\log 2}{\log\alpha_1} - \mathfrak{n} + \frac{\log(1-\alpha_1^{\mathfrak{m}-\mathfrak{n}})^{-1}}{\log\alpha_1}\right| < \frac{3.42}{\log\alpha_1}\ \alpha_1^{-\mathfrak{n}}$$

(9.22)

Considering theorem 9.2.3, we get

$$\gamma = \frac{\log 2}{\log\alpha_1}, \mu_1 = \frac{\log(1-\alpha_1^{\mathfrak{m}-\mathfrak{n}})^{-1}}{\log\alpha_1}, \mathcal{A} = \frac{3.42}{\log\alpha_1}, \mathcal{B} = \alpha_1, \mathfrak{w} = \mathfrak{n}$$

Clearly, γ is irrational.

Also,

$$3 \leq \mathfrak{n} - \mathfrak{m} \leq 69$$

Next, we determine the denominator q of continued fraction of γ.

Further, $\mathcal{M} = 2.80728 * 10^{13}$, choose the 30th denominator $q = 283712258661163$,

such that $q > 6\mathcal{M}$.

For using theorem 9.2.3 to equation (9.21) with $3 \leq \mathfrak{n} - \mathfrak{m} \leq 69$, we obtain

$$\varepsilon = \| \mu_1 q_{30} \| - \mathcal{M} \| \gamma\, q_{30} \| \geq 0.0475574$$

by Mathematica software. Also, from theorem 9.2.3, there is no solution of equation (9.21) for the values of $\mathfrak{n}$ with

$$\mathfrak{n} \geq \frac{\log\left(\frac{\mathcal{A}q}{\varepsilon}\right)}{\log\mathcal{B}} = 42.7523.$$

Thus, $\mathfrak{n} \leq 42$, that contradicts to our supposition that $\mathfrak{n} > 43$.

Summary and Conclusions

In Chapter 2, we developed various summation formulae for the Generalized Fibonacci polynomials defined by (1.23) and its 1st order derivative having even indices, odd indices, and alternating summation formulae. Thereafter, we discussed some particular cases of the polynomial defined by (1.23) for Fibonacci numbers, Pell numbers, Tetrabonacci numbers, etc., by giving the different values to r, s, a and b. Following results are developed:

1) If $s + rx - 1 \neq 0$, then

$$\sum_{k=0}^{n} Q_k\,(x) = \frac{\mu}{s + rx - 1}$$

where,

$$\mu = Q_{n+2}(x) + (1 - rx)Q_{n+1}(x) + Q_1(x) + (rx - 1)Q_0(x)$$

2) If $s^2 - r^2x^2 - 2s + 1 \neq 0$, then

$$\sum_{k=0}^{n} Q_{2k}(x) = \frac{\mu}{s^2 - r^2x^2 - 2s + 1}$$

where,

$$\mu = (s - 1)Q_{2n+2}(x) - rxsQ_{2n+1}(x) + rxQ_1(x) - (r^2x^2 + s - 1)Q_0(x)$$

3) If $s^2 - r^2x^2 - 2s + 1 \neq 0$, then

$$\sum_{k=0}^{n} Q_{2k+1}\,(x) = \frac{-\mu}{s^2 - r^2x^2 - 2s + 1,}$$

where,

$$\mu = rxQ_{2n+2}(x) - s(s - 1)Q_{2n+1}(x) + (s - 1)Q_1(x) - rxQ_0(x)$$

4)

$$\sum_{k=0}^{n} Q_k = Q_{n+2} + Q_1$$

5)

$$\sum_{k=0}^{n} Q_{2k} = Q_{2n+1} - Q_1 + Q_0$$

6)

$$\sum_{k=0}^{n} Q_{2k+1} = Q_{n+2} - Q_0$$

7)

$$\sum_{k=0}^{n} Q_k = \frac{Q_{n+2} - Q_{n+1} + Q_1 + Q_0}{2}$$

8)

$$\sum_{k=0}^{n} Q_{2k} = \frac{Q_{2n+1} - Q_1 + 2Q_0}{2}$$

9)

$$\sum_{k=0}^{n} Q_{2k} = \frac{Q_{n+2} - Q_0}{2}$$

10) If $(s + rx - 1)^2 \neq 0$, then

$$\sum_{k=0}^{n} Q'_k(x) = \frac{\mu}{(s + rx - 1)^2}$$

where,

$$\mu = (s + rx - 1)Q'_{n+2}(x) - rQ_{n+2}(x) - (rx - 1)(s + rx - 1)Q'_{n+1}(x)$$

$$-r(1 - rx)Q_{\boldsymbol{n}+1}(x) - (s + rx - 1)Q'_1(x) - rQ_1(x)$$

$$+ (rx - 1)(s + rx - 1)Q'_0(x) + rsQ_0(x)$$

11) If $(s^2 - r^2x^2 - 2s + 1) \neq 0$, then

$$\sum_{k=0}^{n} Q'_{2k}(x) = \frac{\mu}{(s^2 - r^2x^2 - 2s + 1)^2}$$

where,

$$\mu = \rho(s-1)Q'_{2n+2}(x) + r(2xrs - 2xr - s^3 - s + 2s^2 + r^2x^2s)Q_{2n+1}(x) - rx\rho Q'_{2n+1}(x) + r(s^2 + 1 - 2s + r^2x^2)Q_1(x) + rx\rho Q'_1(x) - 2xr^2s(s-1)Q_0(x) - \rho(r^2x^2 + s - 1)Q'_0(x),$$

and $\rho = s^2 - r^2x^2 - 2s + 1$

12) For $s^2 - r^2x^2 - 2s + 1 \neq 0$,

$$\sum_{k=0}^{n} Q'_{2k+1}(x) = \frac{\mu}{\rho^2}$$

where,

$$\mu = \rho rxQ'_{2n+2}(x) - r(r^2x^2 + (s-1)^2)Q_{2n+2}(x) + 2xsr^2(s-1)Q_0Q_{2n+1}(x) - s(s-1)\rho Q'_{2n+1}(x) - 2r^2x(s-1)Q_1(x) + (s-1)\rho Q'_1(x) + r(r^2x^2 + (s-1)^2)Q_0(x) - \rho rxQ'_0(x)$$

and,

$$\rho = -(s^2 - r^2x^2 - 2s + 1)$$

13)

$$\sum_{k=0}^{n} Q'_k(1) = Q'_{n+2}(1) - Q_{n+2}(1) - Q'_1(1) - Q_1(1) + Q_0(1)$$

14)

$$\sum_{k=0}^{n} Q'_{2k}(1) = Q'_{2n+1}(1) + Q_{2n+1}(1) - Q'_1(1) + Q_1(1) + Q'_0(1)$$

15)

$$\sum_{k=0}^{n} Q'_{2k+1}(1) = -Q_{2n+1}(1) - Q'_0(1) + Q_0(1)$$

After that, we discussed the convergence of the solution linear differential equation (1.24) and concluded that it converges for all real values of x.Thereafter, the study about the extremum values for the Fibonacci polynomials in one variable (1.4) has been done by using Descartes' Rule of sign and concluded that

a) No extremum values for even values of n.

b) Only one extremum value for odd values of n.

Similarly, the observations about the extremum values for Fibonacci polynomials of two variables have been devised. In the last section of this chapter, these polynomials are represented in graphical form using MATLAB.

In chapter 3, various attributes of 2-Fibonacci sequence like generating function, Binet's Formula to find out its n^{th} terms, generating function, relationship with Fibonacci numbers, and multiple relations using the concept of congruences were established. We developed following relations:

1)

$$\sum_{n=0}^{\infty} \alpha_n \mathfrak{v}^n = \frac{\mathfrak{v} + 3\mathfrak{v}^2 + \mathfrak{v}^3}{1 - \mathfrak{v}^2 - 2\mathfrak{v}^3 - \mathfrak{v}^4}$$

2)

$$\sum_{n=0}^{\infty} \beta_n \mathfrak{v}^n = \frac{2 + \mathfrak{v} - \mathfrak{v}^2 - \mathfrak{v}^3}{1 - \mathfrak{v}^2 - 2\mathfrak{v}^3 - \mathfrak{v}^4}$$

3)

$$\alpha_n = \frac{(\gamma^{n+4} + \gamma^{n+1})}{(\gamma - \delta)(\gamma - \omega)(\gamma - \omega^2)} + \frac{(\delta^{n+4} + \delta^{n+1})}{(\delta - \gamma)(\delta - \omega)(\delta - \omega^2)} + \frac{2\omega^{n+1}}{(\omega - \gamma)(\omega - \delta)(\omega - \omega^2)} + \frac{2\omega^{2n+2}}{(\omega^2 - \gamma)(\omega^2 - \delta)(\omega^2 - \omega)}$$

4)

$$\beta_n = \frac{2\gamma^{n+3}}{(\gamma-\delta)(\gamma-\omega)(\gamma-\omega^2)} + \frac{2\delta^{n+3}}{(\delta-\gamma)(\delta-\omega)(\delta-\omega^2)} - \frac{2\omega^{n+1}}{(\omega-\gamma)(\omega-\delta)(\omega-\omega^2)} - \frac{2(\omega^2)^{n+1}}{(\omega^2-\gamma)(\omega^2-\delta)(\omega^2-\omega)}$$

where $\gamma = \frac{1+\sqrt{5}}{2}$, $\delta = \frac{1-\sqrt{5}}{2}$, $\omega = \frac{-1+i\sqrt{3}}{2}$, $\omega^2 = \frac{-1-i\sqrt{3}}{2}$

5)

$$\alpha_n \equiv \big(\beta_n + (1-r)\big)(mod\ 3), \text{where } n \equiv r(mod\ 3)$$

6)

$$F_{n+1} \equiv \alpha_n + (1-r) \equiv \beta_n + (r-1)(mod 3) \text{where } n \equiv r(mod 3)$$

7)

$$\frac{\gamma^{n+1}-\delta^{n+1}}{\gamma-\delta} = \alpha_n + (1-r) = \beta_n + (r-1), \text{where } n \equiv r(mod\ 3)$$

8)

$$\beta_{n-1}\beta_{n+1} - {\beta_n}^2 \equiv 3(-1)^n (mod\ 4)$$

9)

$$\alpha_{n-1}\alpha_{n+1} - {\alpha_n}^2 \equiv 3(-1)^{n-1} (mod\ 4)$$

10)

$$\begin{pmatrix}\alpha_{n+3}\\ \alpha_{n+2}\\ \alpha_{n+1}\\ \alpha_n\end{pmatrix} = \begin{pmatrix}0 & 1 & 2 & 1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}\begin{pmatrix}\alpha_{n+2}\\ \alpha_{n+1}\\ \alpha_n\\ \alpha_{n-1}\end{pmatrix}$$

11)

$$\begin{pmatrix}\beta_{n+3}\\ \beta_{n+2}\\ \beta_{n+1}\\ \beta_n\end{pmatrix} = \begin{pmatrix}0 & 1 & 2 & 1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}\begin{pmatrix}\beta_{n+2}\\ \beta_{n+1}\\ \beta_n\\ \beta_{n-1}\end{pmatrix}$$

12)

$$\begin{pmatrix}\alpha_{n+2}\\ \alpha_{n+1}\\ \alpha_n\\ \alpha_{n-1}\end{pmatrix} = E^{n-1}\begin{pmatrix}2\\ 3\\ 1\\ 0\end{pmatrix}$$

13)

$$\begin{pmatrix}\beta_{n+2}\\ \beta_{n+1}\\ \beta_n\\ \beta_{n-1}\end{pmatrix} = E^{n-1}\begin{pmatrix}4\\1\\1\\2\end{pmatrix}$$

In Chapter 4, the characteristics of 2-Fibonacci sequences (1.24) along with their squares are developed and validated in the categories of closed forms of the summation formulae. In particular, the properties involving an alternating sum of 2-Fibonacci sequences and the product of 2-Fibonacci terms having different indices, even and odd indices were discussed. We developed following summation formulae:

1)

$$\sum_{k=0}^{m} \mathcal{C}_k = \frac{1}{3}(\mathcal{C}_{m+4} + \mathcal{C}_{m+3} - 2\mathcal{C}_{m+1} - \mathcal{C}_3 - \mathcal{C}_2 + 2\mathcal{C}_0)$$

2)

$$\sum_{k=0}^{m} \mathcal{C}_{2k} = -\frac{1}{3}(\mathcal{C}_{2m+2} - 2\mathcal{C}_{2m+3} + \mathcal{C}_{2m} + 2\mathcal{C}_3 - \mathcal{C}_2 - 4\mathcal{C}_0)$$

3)

$$\sum_{k=0}^{m} \mathcal{C}_{2k+1} = \frac{1}{3}(2\mathcal{C}_{2m+2} + 3\mathcal{C}_{2m+1} + 2\mathcal{C}_{2m} - \mathcal{C}_{2m+3} + \mathcal{C}_3 - 2\mathcal{C}_2 - 2\mathcal{C}_0)$$

4)

$$\sum_{k=0}^{m} \alpha_k\, \alpha_{k+3} + 2\sum_{k=0}^{m} \alpha_k\, \alpha_{k+2} = -6 + \alpha_{m+3}\alpha_{m+4} - 2\alpha_{m+1}\alpha_{m+3}$$

5)

$$\sum_{k=0}^{m} \alpha_k^2 - \sum_{k=0}^{m} \alpha_k\, \alpha_{k+3}$$
$$= 1 + \alpha_{m+1}\alpha_{m+4} + \alpha_{m+1}\alpha_{m+2} - \alpha_{m+1}\alpha_{m+3} - \alpha_{m+2}\alpha_{m+4} - \alpha_{m+1}^2 + \alpha_{m+2}^2$$

6)

$$-2\sum_{k=0}^{m}\alpha_{k}\,\alpha_{k+3}$$

$$= \alpha_{m+4}^2 + \alpha_{m+3}^2 + 3\alpha_{m+2}^2 + \alpha_{m+1}^2 - \alpha_{m+1}\alpha_{m+3} - 3\alpha_{m+2}\alpha_{m+4} - \alpha_{m+1}\alpha_{m+4} + 3\alpha_{m+1}\alpha_{m+2} - \alpha_{m+3}\alpha_{m+4} - 4$$

7)

$$\sum_{k=0}^{m} k\mathcal{C}_{k} = \frac{1}{3}(m\mathcal{C}_{m+4} + (m-1)\mathcal{C}_{m+3} - 2\mathcal{C}_{m+2} - 2(m+1)\mathcal{C}_{m+1} + \mathcal{C}_3 + 2\mathcal{C}_2 + 2\mathcal{C}_1)$$

8)

$$\sum_{v=0}^{m}\beta_{v}\,\beta_{v+3} + 2\sum_{k=0}^{m}\beta_{v}\,\beta_{v+2} = \beta_{m+3}\beta_{m+4} - 2\beta_{m+1}\beta_{m+3}$$

9)

$$\sum_{v=0}^{m}\beta_{v}^2 - \sum_{v=0}^{m}\beta_{v}\,\beta_{v+3}$$

$$= -1 + \beta_{m+1}\beta_{m+4} + \beta_{m+1}\beta_{m+2} - \beta_{m+1}\beta_{m+3} - \beta_{m+2}\beta_{m+4} - \beta_{m+1}^2 + \beta_{m+2}^2$$

10)

$$-2\sum_{v=0}^{m}\beta_{v}\,\beta_{v+3}$$

$$= \beta_{m+4}^2 + \beta_{m+3}^2 + 3\beta_{m+2}^2 + \beta_{m+1}^2 - \beta_{m+1}\beta_{m+3} - 3\beta_{m+2}\beta_{m+4} - \beta_{m+1}\beta_{m+4} + 3\beta_{m+1}\beta_{m+2} - \beta_{m+3}\beta_{m+4} - 4$$

In Chapter 5, derived the reciprocals sum of Generalized Fibonacci (1.27) and Generalized Lucas polynomials (1.28) with even indices described below:

1)

$$\left\lfloor\left(\sum_{r=n}^{\infty}\frac{1}{F_{k,lr}(g)}\right)^{-1}\right\rfloor = F_{k,ln}(g) - F_{k,ln-l}(g) - 1 \qquad (n \geq 1)$$

2)

$$\left\lfloor \left(\sum_{r=n}^{\infty} \frac{1}{L_{\mathscr{k},lr}(\mathscr{g})} \right)^{-1} \right\rfloor = L_{\mathscr{k},ln}(\mathscr{g}) - L_{\mathscr{k},ln-l}(\mathscr{g}) \qquad (n \geq 1)$$

3)

$$\left\lfloor \left(\sum_{r=n}^{\infty} \frac{1}{F^2_{\mathscr{k},lr}(\mathscr{g})} \right)^{-1} \right\rfloor = F^2_{\mathscr{k},ln}(\mathscr{g}) - F^2_{\mathscr{k},ln-l}(\mathscr{g}) - 1 \qquad (n \geq 1)$$

4)

$$\left\lfloor \left(\sum_{r=n}^{\infty} \frac{1}{L^2_{\mathscr{k},lr}(\mathscr{g})} \right)^{-1} \right\rfloor = L^2_{\mathscr{k},ln}(\mathscr{g}) - L^2_{\mathscr{k},ln-l}(\mathscr{g}) + 1 \qquad (n \geq 2)$$

for positive integer $\mathscr{g}$, n and even integer $l \geq 2$.

In Chapter 6, we derived the following results for Jacobsthal polynomials (1.29) and Jacobsthal-Lucas polynomials (1.30):

(a)

$$\sum_{k=n}^{\infty} \frac{1}{J_{ak}(x)} > \frac{1}{J_{an}(x) - J_{an-a}(x)}$$

(b)

$$\sum_{k=n}^{\infty} \frac{1}{j_{ak}(x)} < \frac{1}{j_{an}(x) - j_{an-a}(x)}$$

(c)

$$\sum_{k=n}^{\infty} \frac{1}{J^2_{ak}(x)} > \frac{1}{J^2_{an}(x) - J^2_{an-a}(x)}$$

(d)

$$\sum_{k=n}^{\infty}\frac{1}{j_{ak}^{2}(x)}<\frac{1}{j_{an}^{2}(x)-j_{an-a}^{2}(x)}$$

with positive integers x, n and for positive even integer a.

In chapter 7, we introduced a $n \times n$ matrix given by

$$S_n(e',m',f')=\begin{pmatrix} e' & -m' & f' & 0 & \dots & 0 \\ 1 & e' & -m' & f' & \dots & 0 \\ 0 & 1 & e' & -m' & \ddots & \vdots \\ 0 & 0 & 1 & e' & \ddots & f' \\ \vdots & \vdots & \vdots & \ddots & \ddots & -m' \\ 0 & 0 & 0 & 0 & 1 & e' \end{pmatrix}_{n\times n}$$

and concluded that its successive determinant generates Trivariate Fibonacci (1.31) polynomials.

After that we presented Binet's formula for this sequence by using matrix methods and the concept of diagonalizability of matrix. And then, we addressed an observation related to n^{th} Generalized Lucas numbers defined by (1.9).

Next, Chapter 8, we solved the fractional differential equation (1.32) by using an operational matrix for Lucas polynomials. Thereafter, a matrix is developed generating the derivatives of Fibonacci polynomials.

In Chapter 9, we completely solved the following Diophantine equation

$$Q_{\mathfrak{n}}-Q_{\mathfrak{m}}=2^{\mathfrak{a}},$$

where $Q_{\mathfrak{n}}$ and $Q_{\mathfrak{m}}$ are Pell-Lucas numbers and deduced that its solutions are:

$$\{(2,0,2),(2,1,2),(3,2,3),(4,0,5),(4,1,5)\}$$

Future and Scope

1. Infinite reciprocal sum of Generalized Jacobsthal and Generalized Jacobsthal-Lucas polynomials can be developed.
2. One can possibly develop summation formulae for the sequence of 2-Fibonacci polynomials.
3. Numerous properties such as Binet's formula and Generating function can be deduced for the sequence of 2-Fibonacci polynomials.
4. One can solve the following Diophantine equation

$$Q_n - Q_m = y^a,$$

   where Q_n and Q_m are Pell-Lucas numbers and y is a positive integer. In place of Pell-Lucas numbers, one can take other sequence like Pell-numbers, Jacobsthal numbers, 2-Fibonacci numbers etc.