NUMERICAL SIMULATION OF PARTIAL DIFFERENTIAL EQUATIONS USING RADIAL BASIS FUNCTION METHODS

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By

KIRAN BALA

Registration Number: 42000118

Supervised By

Dr. Geeta Arora (UID-18820)

Department of Mathematics

LOVELY PROFESSIONAL UNIVERSITY

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DECLARATION

I, Kiran Bala, declare that the work presented in the thesis entitled "**Numerical Simulation of Partial Differential Equations using Radial Basis Function Methods**" in fulfilment of degree of **Doctor of Philosophy (Ph.D.)** is outcome of research work carried out by me under the supervision of Dr. Geeta Arora, working as a Professor, in the Department of Mathematics of Lovely Professional University, Punjab, India. I confirm that no part of this research work has been submitted to any other University or Institute for the award of any degree.

(Signature of Scholar)

Name of the Scholar: Kiran Bala
Registration No.: 42000118
Department/School: Mathematics
Lovely Professional University,
Punjab, India

CERTIFICATE

This is to certify that the work reported in the Ph.D. thesis entitled "**Numerical Simulation of Partial Differential Equations using Radial Basis Function Methods**" submitted in fulfillment of the requirement for the award of degree of **Doctor of Philosophy (Ph.D.)** in the department of Mathematics, is a research work carried out by Kiran Bala, 42000118, is bonafide record of her original work carried out under my supervision and that no part of the thesis has been submitted for any other degree, diploma or equivalent course.

(Signature of Supervisor)

Name of Supervisor: Geeta Arora

Designation: Associate Professor

Department/School: Mathematics

Lovely Professional University

Punjab, India

ABSTRACT

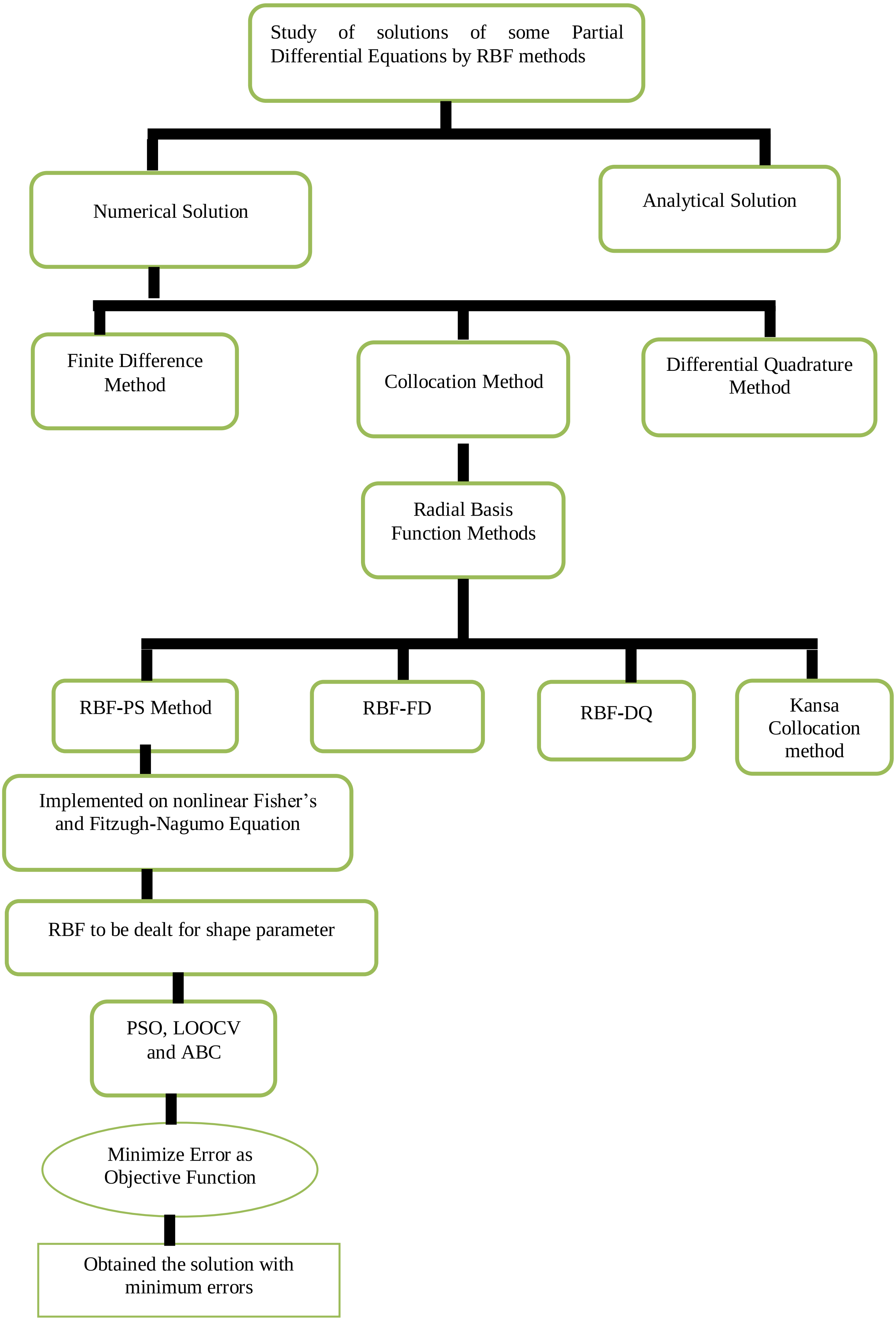
Partial differential equations (PDEs) are indeed a fundamental tool in mathematical modeling used to describe complex real-world phenomena. Their versatility and ability to capture complex relationships between variables make them valuable in scientific research and engineering applications. Numerical methods are the best practical approach for solving PDEs in the condition when the solutions are not available or too complex to calculate by analytical methods. Radial basis function (RBF) methods are such methods to find the numerical solutions of PDEs where traditional grid-based methods may be impractical.

RBF methods are a class of mesh less methods as they do not require a structured grid or mesh for discretizing the domain. Their advantages in handling complex geometrical domain, scattered data points, and adaptive meshing make them a preferred choice in certain scenarios. RBF methods use a set of basis functions that depend on the radial distance from a centre point. The method often starts with a set of scattered data points or nodes for discretizing the domain, and the radial basis functions are employed to approximate the solution.

The main focus of the present thesis is to find the numerical solutions of PDEs using one of the RBF methods by optimizing the shape parameter of RBF through optimization techniques and to analysis the solutions in order to comprehend the implications of the positive aspects.

In the present thesis, the RBF based pseudo spectral (RBF-PS) method is employed to solve PDEs numerically in which three techniques, namely Particle Swarm Optimization, LOOCV approach and Artificial Bee Colony algorithm are applied to obtain the involved shape parameter. RBF-PS method with optimization techniques forms a hybrid approach to overcome the limitation of flexibility in geometrical domain with maintaining higher accuracy. There are various RBFs which contain a shape parameter can be employed for finding the solution of PDEs. In this work, the proposed hybrid approach is employed to find the numerical solutions of various PDEs by approximating the derivatives included in PDEs with optimizing the shape parameter of RBF by minimizing the errors. This hybrid approach has been implemented to find the numerical solutions of two partial differential equations: the Fitzhugh- Nagumo equation and Fisher’s equation. The results are derived in the form of error norms and the shape parameter values with their graphical representation. An evaluation of the derived results is done to assess the efficacy and relevancy of the proposed approach in the form of comparison. The proposed method has potential that can be extended as an evaluation of mathematical problems in irregular domains.

Graphical Abstract



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LIST OF ABBREVIATIONS

RBF	Radial Basis Functions
PDEs	Partial Differential Equations
MQ	Multiquadric
GS	Gaussian Function
IMQ	Inverse Multiquadric
IQ	Inverse Quadric
TPS	Thin Plate Spline
LR	Linear Radial Function
ODEs	Ordinary Differential Equations
FDM	Finite Difference Method
FEM	Finite Element Method
FVM	Finite Volume Method
RBF-DQ	Radial Basis Function Differential Quadrature
RBF-FD	Radial Basis Function-Finite Difference
RBF-PUM	Radial Basis Function Partition of Unity Method
RBF-PS	Radial Basis Function Pseudospectral
LOOCV	Leave-One-Out-Cross-Validation

Chapter 1

Introduction

Part of this chapter is Published in a paper entitled " A review of radial basis function with applications explored”, in Journal of *the Egyptian Mathematical Society* (JOEMS), vol. 31, 6 (2023).

1.1. Overview

A differential equation is a mathematical equation that establishes a relationship between functions and their derivatives [1]. In the literature, differential equations (DEs) are employed for the modeling of complex systems such as mechanical system and electrical system, etc. Mechanical systems and electrical systems [2] are two types of physical systems that undergo changes over time. DEs are employed as a mathematical framework to depict the continuous evolution of physical systems. But in today's research, DEs have greater impact in various disciplines like biology, social sciences, engineering, and economics. Partial differential equations (PDEs) and ordinary differential equations (ODEs) are two distinct categories of differential equations that are employed in the modeling and characterization of the dynamics exhibited by physical systems. ODEs are mathematical equations that express the relationship between the change into a dependent variable and an independent variable. On the other hand, PDEs describe the variations of a dependent variable through multiple independent variables. In the eighteenth century, the contributions made by French mathematician Jean D'Alembert and Swiss mathematician Leonhard Euler were regarded as pioneers in the field of nonlinear differential equations. Various researchers conducted research to find the behaviour of nonlinear phenomena through the DEs, which established the fundamental basis for the examination of nonlinear dynamical models.

During the 19th century, significant progress in the field of nonlinear differential equations was made by a French mathematician, Henri Poincare, and a German mathematician, Georg Friedrich Bernhard Riemann. Their contributions further develop the fundamental theory of DEs. In the twentieth century, the emergence of chaos theory and nonlinear dynamics has had a significant impact on the field of nonlinear differential equations (NLDEs) which has implications in various areas based on biology, engineering, and physics [3, 4].

Partial differential equations (PDEs) are indeed a fundamental tool in modeling of various physical phenomena [5] in fields like physics, engineering, and applied mathematics. Linear partial differential equations (LPDEs) have the property of framing the copious physical and natural phenomena with higher accuracy. Due to this characteristic, LPDEs have various applications such as the study of heat distribution in objects and complex structures like buildings, electronic devices, and industrial processes; wave propagation in various Media like sonar, seismology, optics, and communication systems; electrostatics and electromagnetic fields; fluid dynamics; quantum mechanics; diffusion and mass transport; finance (Black-Scholes equation,) etc. These applications illustrate the broad impact of linear PDEs in diverse fields that enable the prediction and optimization of various natural and engineered systems.

NLPDEs are an extension of PDEs with nonlinear functions or terms that are particularly able to model a diverse array of physical phenomena. One significant advantage of NLPDEs is their ability to model complex systems with diverse behaviours, including chaos and unpredictability [6]. Their applications are also found in fields like turbulence, pattern formation, and self-organization, which are essential in physics, chemistry, and biology [7, 8]. Moreover, various models have been employed based on NLPDEs using spatiotemporal scales includes in weather patterns, ocean currents [9], stock market fluctuations [10], and various geophysical systems.

Numerical analysis of PDEs is a fundamental aspect of the mathematical modeling of various phenomena in science and engineering. In the present research field, numerical simulation of various PDEs is an approaching field. Analytical solutions are often challenging or impossible to obtain for complex PDEs where numerical methods and discretization techniques are required. In the literature, various approximation techniques have been employed for the solutions of various PDEs that are defined in a simple domain or whose perimeters are in a specific pattern. So, traditional methods are not suitable for such PDEs. Due to this, some numerical techniques have been used for solving PDEs, such as the finite element method (FEM), which divides the domain into smaller elements and approximates the solution over each element using basis functions. The finite difference method (FDM) approximates the derivatives of the PDE using difference equations on a discretized grid. Spectral methods use high-order approximations through the expansion of the basis functions. The boundary element method transforms the PDE into an integral equation on the boundary of the domain. Optimal control and variational methods seek to optimize a cost function that deals with the PDE and its boundary conditions. In addition to these methods, mesh-free methods have gained popularity as a powerful approach for finding the solutions to various PDEs. These methods avoid the need for a predefined mesh, making them advantageous for problems with complex or evolving geometries. They use point clouds or other spatial arrangements to define the solution, often with radial basis functions or other interpolants.

Some researchers proposed computational techniques that have been implemented for finding the numerical solutions of PDEs, including the Quadrature Technique [11], B-spline finite element methods [12], RBF methods [13], exponential B-spline with PSO [14], the Modified Cubic B-Spline Differential Quadrature Method [15], and a spline-based differential quadrature approach [16]. And there are various other approaches, like the finite difference method, Kansa's approach for solving parabolic, elliptic, and hyperbolic PDEs, and RBF-based methods [17-20]. The numerical solutions of fractional ordinary equations (FODEs) and PDEs are the most demanding field of today's research, for which various numerical methods have been proposed by various researchers. Maayah B. [21] proposed the multistep Laplace optimized decomposition method for fractional systems of ODEs based on the Runge-Kutta method, and Arora G. [22] presented the residual power series method for fractional relaxation–oscillation equations. Arqub in [23] presented the Dirichlet model using a computational approach based on the reproducing kernel in a time-fractional sense. There are some differential equations generated as a result of mathematical modeling when the domain is uniformly distributed.

In mesh-free methods, Radial Basis Function (RBF) methods use scattered nodes to approximate the solution of PDEs. In these mesh-free methods, the RBF finite difference (FD) method, the RBF differential quadrature (QR) method, the RBF pseudo-spectral (PS) method, the RBF Kansa collocation method, and the RBF partition of unity (PU) method are some numerical methods employed for finding the solutions of various PDEs. In the present thesis work, the RBF-PS method is employed for finding the numerical solutions of various PDEs. RBF methods have various applications in many fields, including simulation of PDEs, interpolation, function approximation, machine learning, geomodeling, price options, neural networks, data mining, image processing, etc.

An effective numerical strategy for solving PDEs is to approximate their behavior with RBFs, which make the process computationally efficient. The Radial Basis Function was later developed for finding the solution of interpolation matrices and was then implemented for solutions of partial differential equations. RBF is easy to use and works well in dynamic and irregular domains. In some of the generated PDEs, there is a need to find the solution in the presence of non-uniform data, which is not easy to handle and leads to complexity. To solve this problem, mesh-free methods are used. RBF methods are one of the useful techniques in the mesh- free methods. To approximate the multivariate functions in today's research, only the RBF methods are in practice especially in the non-appearance of the grid data. These methods have been analysed and experienced with various positive characteristics [24] for several years now. The implementation of RBF techniques in approaching multivariate scattered data has been highly appreciated.

Hardy [25] introduced the RBF method in the context of quadric surfaces dealing with the topological approach. Hardy was the first to develop the multi-quadric (MQ) approximation technique. Franke [26] experimented with scattered data interpolation. He evaluates methods in terms of time, storage, exactness, and ease of implementation, and he also considers multi-quadric (which is a type of RBF) to be one of the best. Micchelli [27] made a step forward by demonstrating that multi-quadraic surface interpolation is always solvable. The MQ approach has the benefit of obtaining the interpolant using a linear combination of basis functions that are only dependent on the distance from a specific node, which is known as the center.

In order to solve a PDE, Edward Kansa invented the Kansa method [28, 29] in 1990. It first used the multi-quadraic, a widely supported interpolant. Despite being used in many different applications, the Kansa technique has certain drawbacks, such as the asymmetrical traits of the interpolation matrix, which results in a poorly conditioned matrix for a large number of nodes. As an upgrade to the Kansa method, Fasshauer presented a hermite-based methodology in 1996 in which the obtained collocation matrices are typically more symmetric and have a lower condition number [30]. The

RBF collocation method based on symmetric properties does have some drawbacks, as it is more challenging to implement compared to an asymmetric technique. Larsson and Fornberg [31] and Power and Barraco [32] compared the proposed method based on symmetric and unsymmetrical approaches. There are some other methods, including pre- conditioning of the interpolation matrix [33], the domain decomposition method [34], etc., that have been suggested to deal with the aforesaid difficulties. These techniques can help, to some extent, by lessening the matrix's poor conditioning. The local approach is another really promising option to address these types of Kansa method issues. Only the local approximation should be taken into consideration for collocation in this strategy, rather than all the nodes throughout the entire domain.

There are various types of RBFs that contain a shape parameter to determine the structure of the RBF. Some RBFs have the best accuracy when the form parameter is set to small values, but this result in improper conditioning of the matrix. An approach for the RBF's stable computation for all values of the shape parameter was put forth by Fornberg and Wright [35] in 2004. In 2007, Fornberg and Piret significantly enhanced the method to create the new RBF-QR method, which completely eliminates the matrix's improper condition in cases of nearly flat basis functions [36]. In order to maximize the benefits of RBFs, the method continues to be improved by combining RBF with other widely used approaches. In 2003, Shu [37] proposed a hybrid method called the RBF-DQ method, which combines the exemplary accuracy and efficiency of the Differential Quadrature (DQ) method with the mesh-free aspect of RBF. Various researchers have employed this method to resolve PDEs in fluid dynamics (Navier-Stokes and shallow water challenges). Tolstykh [38] developed a finite difference method based on RBF called as RBF-FD in 2003 using a restricted domain. Its discretizations are completely mesh-free and very simple to use, even when local refinements are required [39]. The RBF-PUM method, which integrates RBF with the partition of unity method, is another effective strategy for solving PDEs [40]. Partitioning the domain into overlapped sub-domains is the principle underlying the RBF-PUM approach. To obtain the global approximation, one performs the local approximation on the sub domains and combines them. RBF-PUM keeps accuracy high while lowering computation costs. The approximation done on the points of the sub domain for finding the solution by the local approximation and whole is used for the global approximation.

1.2. Radial Basis Function

A function Φ: $\mathbb{R}^t \rightarrow \mathbb{R}$ is called radial if there exist a function of one variable $: [0, \infty) \rightarrow \mathbb{R}$ such that $\Phi(\mathrm{x}) = \varphi(\|\mathrm{x}\|)$, here Euclidean norm$\|.\|$ is used and $\mathrm{t} \in \mathbb{N}$. $\Phi(r)$ is a univariate continuous real valued radial basis function whose value based upon distance value that is measure from any fixed centre point or the origin [41].

From the definition, it is clear that Φ is a special function which is radially symmetric and only depends on the distance between points. The applications of RBF to the high dimensional problem are easy as the problem of the interpolation is heedless to the space dimension. In all space dimensions, one can work with the function φ that is univariate instead of using a multivariate function Φ. We centring on types of radial basis functions that are distinguished by the smoothness- piecewise smooth RBFs which are free from shape parameter ϵ and infinitely differentiable which have parameters called the shape parameter.

1.2.1. Types of RBFs

Different types of RBFs have been used by various researchers. Some recognized RBFs are as follows:

1. Infinitely smooth RBFs – These RBFs are based on a parameter $\epsilon > 0$ that controls the shape or outline of the RBF, is called shape parameter. If ϵ is tending to 0 then form of RBFs becomes flat. Table1.1 presents different types of infinitely smooth RBFs.

Table 1.1 Types of infinitely smooth RBFs with λ>0 and r∈||x||

RBFs	**Definition**
Gaussian Function (GS)	$\varphi(r) = e^{-(\epsilon r)^2}$
Multiquadric (MQ)	$\varphi(r) = \sqrt{1 + (\epsilon r)^2}$
Inverse Multiquadric (IMQ)	$\varphi(r) = \dfrac{1}{1 + (\epsilon r)^2}$
Inverse quadric (IQ)	$\varphi(r) = \dfrac{1}{\sqrt{1 + (\epsilon r)^2}}$

2. Piecewise smooth RBFs – these RBFs have no shape parameter. Different types of piecewise smooth RBFs are shown in Table 1.2.

Table 1.2 Some piecewise smooth RBFs with r ∈ ||x||

RBFs	**Definition**
Thin Plate Spline (TPS)	$\varphi(r) = r^2 \ln(r)$
Linear radial function (LR)	$\varphi(r) = r$
Cubic function	$\varphi(r) = r^3$
Monomial	$\varphi(r) = r^{2k-1}; k \in \mathbb{N}$

The behaviour of the Gaussian and inverse multi-quadratic RBFs with the different shape parameter are shown by the figure 1.1 and the figure 1.2. From the analysis of the figures 1.1 and 1.2, it seems that a change in the value of the shape parameter results in a change in the shape of the radial function.

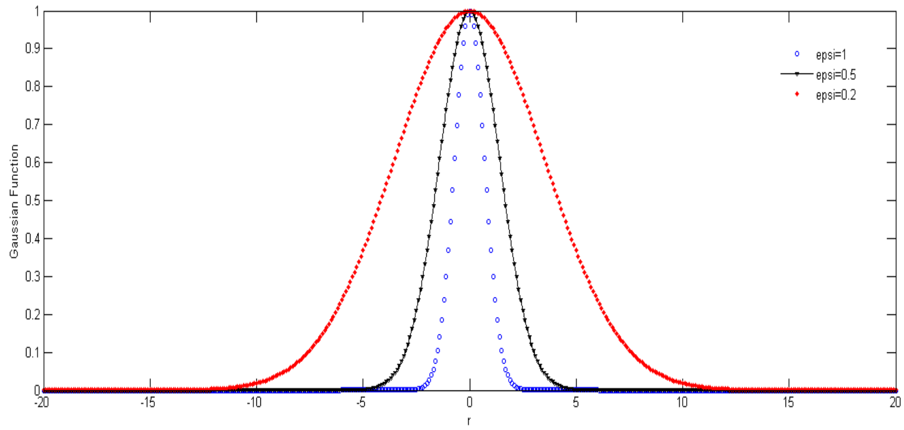

Figure 1.1. Gaussian RBF with different values of shape parameter

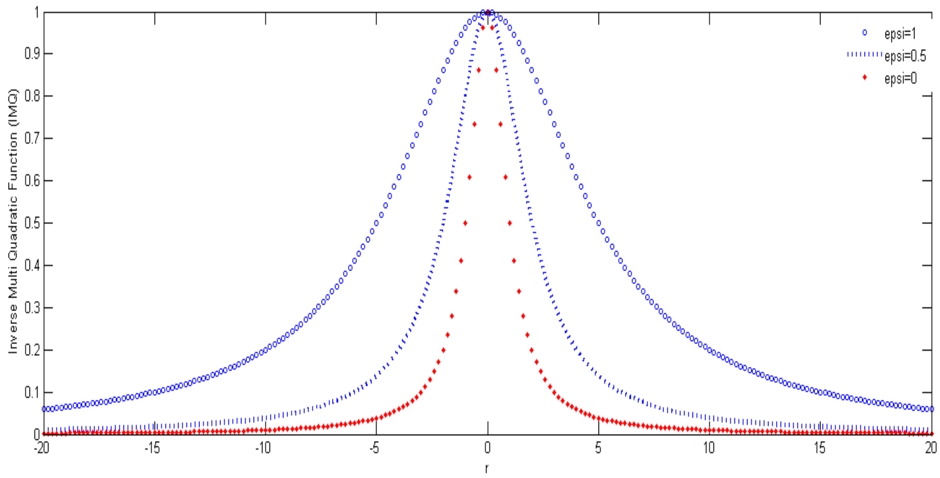

Figure 1.2. Inverse Multi Quadratic RBF with different values of shape parameter

1.3. RBF Methods for solving PDEs

RBF methods are known for their easy implementation due to their mesh-free nature, simplicity in the approximation of multivariate scattered data, and reduction in computational time in getting results with higher accuracy. For solving partial differential equations, a recent historic and chronologically developed strategy of RBF methods has been discussed as follows:

1.3.1. Kansa-Collocation Method

The Kansa method is one of the mesh free approaches, often known as the RBF collocation method. Compared to mesh methods, mesh-free methods have various advantages including less computational time and do not required domain or surface discretization. Kansa [28] introduced an asymmetric approach based on RBF for solving PDEs.

Mathematically, Consider $x \in \mathbb{R}^d$ and $d \in \mathbb{N}$; the norm $\| \cdot \|$ as Euclidian norm and the RBFs of the form $\emptyset(\|x - x_i\|)$ to be strictly positive definite. The approximation in the form of RBF can be written by using nodes that are spotted arbitrary in the domain $\Omega \subset \mathbb{R}^d$ and assigning a collection of neighbourhood nodes x_i that are integrated in the supportive domain with respect to every x as:

$$u(x) = \sum_{i=1}^{N} \alpha_i \emptyset(\|x - x_i\|) \quad (1.1)$$

where α_i is undetermined coefficients and the numbers of nodes are represented by N. By substituting this solution u(x) in PDE, gives the linear system of equations as

$$A\mathbb{X} = B \; where \; \mathbb{X} = [\alpha(x_1), \alpha(x_2), \alpha(x_3), \dots \dots \dots \dots \alpha(x_N)]^T \quad (1.2)$$

Various problems have been successfully solved by the Kansa collocation approach. Zhou et al. [41] also solved shallow water modelling problem based on Kansa collocation method, and Chen et al. [42] also proposed a method based on the Kansa collocation approach for the solutions of convection diffusion problems and fractional diffusion equation. Kovacevic et al. [43] solved Stefan problem; Chantasiriwan [44] time-dependent heat conduction problems; and Duan et al. [45] solved electrostatic problems using the Kansa method. The drawback of the Kansa method is its computational cost, which becomes very high due to the unsymmetric interpolation matrix. The accuracy of this method is lower in the domain closest to the boundary. To get better accuracy and hence reduce errors, the very simplest way is to raise the interpolation points that lead to a high condition number matrix. The obtained results indicate the need for modification in this method and hence give rise to the following three methods:

a) Symmetric collocation method (SCM)

b) Modified collocation method (MCM)

c) Local radial basis collocation method

While comparing them with the Kansa collocation method, these modified methods were found to be more stable for numerical problems.

To see the procedure and method employed for the elliptic partial differential equation with mathematical formulation at domain D given by

$$L[v(x)] = f(x), x \in \mathrm{D} \tag{1.3}$$

with boundary conditions

$$v(x) = g(x), x \in \partial D. \tag{1.4}$$

The approximation can be represented in the following manner with discrete N node points in the domain, which $\{x_j\} \in D, j = 1, 2, \dots, N_i \; and \; \{x_j\} \in \partial D, j = N_{i+1}, \dots,$ with the property of being a linear combination of RBF:

$$v_n(x,t) = \sum_{j=1}^{N} \delta_j(t)\, \emptyset(\|x - x_j\|) \tag{1.5}$$

Here the interpolation coefficients are$\delta_j(t)$, and the radial basis function is denoted by $\emptyset$ with N node points. Therefore, the approach is not suitable for problems with a large set of interpolation points. When we include all points of the domain, the resulting matrix turns complex and is hence unconditioned. The best shape parameter for different RBFs is still difficult to determine.

1.3.2. Differential Quadrature Method with RBF (RBF-DQ)

Bellmanet al. [46] proposed an approach-the Differential Quadrature that approximate derivative of the function rather than function itself. In this technique, a smooth function is consider whose partial derivative is estimated as the linear summation of the functional values at the different nodes which is similar to the concept of integral quadrature. The derivative at the node x_i can be written as

$$f^n(x_i) = \sum_{i=1}^{N} a_{ij}^n\, f(x_j);\;\; i = 1,2,3, \dots N \tag{1.6}$$

Instead of using Lagrange's interpolation, the RBF is used by Shu & Wu [47] in the differential quadrature approach for finding the value of weighting coefficients, and hence the method is named the RBF-DQ method. The RBF-DQM can be applicable in two forms for finding the solutions of the PDEs, which are given by Shu et al. [48, 49]

as global and local of RBF-DQ method. The ill conditioning problem occurs and the computational cost becomes high by using huge set of nodes.

2-D Navier-Stokes equations solved by Shu *et al.* [48] by the use of LRBFDQ method and then Shu et al. [50] apply it for compressible flows. This method is applied for the boundary level problems by Shen [51]. Two-dimensional transient heat conduction problems are also solved by this method in the work of Soleimani et al. [52]. Integrated radial basis function network used by Shu and Wu [53] with the concept of differential quadrature named as IRBF-DQ method applied on one-dimensional burger's equation. By using LRBFDQ, Dehghan and Nikopour [54] proposed the numerical simulation of the boundary value problems using Multi- quadric (MQ) radial basis function.

1.3.3. Partition of Unity Method (RBF-PUM)

Babuska and Melenk [55] proposed the partition of unity with finite element method in 1997 for finding the solutions of PDEs. By the proposal of the PUM, the region is fractionalized into intersecting local domains. This approach is important for the selection of a family of compactly supported, a continuous function, The PUM method based upon radial basis function is a best mode to decrease the computation cost with attaining the higher accuracy. The main advantage of this approach in high dimensional problems is to hold the geometrical flexibility, to overcome computation cost and to facilitate adaptive approximation.

In this method, local approximation are defined on sub-domains then merge to structure global approximation by using weight functions which figure out the PUM. The integrated form of PUM with RBF proposed by Wendland [40] to deal with the solutions of various mathematical problems on large extent.

Algorithm for spherical interpolation proposed by Cavoretto & Rossi [56] for finding the numerical solution of problem using basis function that further projected a method by the use of the partition of unity method. Further in the extension of this work, Cavoretto & Rossi [57] intended an algorithm of partition of unity method in which domain is partitioning into nodes or cell. Applications of PU method investigate by Safdari et al. [58] for the solutions of parabolic partial differential equation. This method can be applied to irregular shaped domains due to their restricted nature. Further improvement in PUM is done by Heryudono et al. [59].

1.4. Methods for the Optimal Shape Parameter

In the research field, optimizing the shape parameter ϵ of RBF is a major concern for the problematic model. In this regard, a number of studies have been conducted**.** There are some methods for finding the best shape parameter ϵ listed in Table 1.3**.**

Table 1.3: Approaches to find the best shape parameter

Approaches	Author	Shape parameter value
Trial and Error	Rolland L. Hardy (1971)	$0.815\mathrm{d}$; $\mathrm{d} = \frac{1}{N}\sum_{k=1}^{N} \mathrm{d_i}$; where $\mathrm{d_i}$ represents the distances between point and its neighbourhood
	Richard Franke (1982)	$\frac{1.25\ \mathrm{D}}{\sqrt{\mathrm{N}}}$; D represents the diameter.
	G. E. Fasshauer (2002)	$\frac{2}{\sqrt{\mathrm{N}}}$
The Power Function	Neyman and Pearson (1936)	-
Leave-One-Out Cross-Validation (LOOCV)	Rippa S. (1999)	-

Huang et al. [60] proposed a method based on arbitrary precision computing to determine the relationship between the value of RBF's shape parameter and the exactness of the solutions in the form of errors. Guo & Jung [61, 62] proposed a method for calculating the best value of ϵ by discretization of the domain with Taylor series. Homayoon et al. [63] used the RBF-based differential quadrature method (RBF-DQ) to find the results of shallow water and long waves' problems.

Leave-one-out-cross-validation (LOOCV) is also one of the best approaches for finding the optimal shape parameter. LOOCV is a technique commonly used to assess the performance and validate the accuracy of predictive models, particularly in the context of machine learning and statistical modeling. Its origins can be traced back to the early days of statistical analysis and cross-validation techniques. It gained prominence in the context of cross-validation as a robust validation technique in the 1970s and 1980s. However, LOOCV can also be adapted and applied to measure the output of the numerical solutions for PDEs. For evaluating the numerical solutions of PDEs, LOOCV with RBF is first used by Rippa [64] for optimizing the shape parameter by minimizing the error function. The use of RBF methods in combination with LOOCV for solving PDEs has been a topic of research and application in the computational mathematics and engineering communities. The integration of LOOCV and RBFs is a natural step to validate the predictive performance and generalization of the model. By leaving out each point and assessing the accuracy of the numerical solution at that point, one can gain insights into the overall performance and reliability of the solution method. Various researchers have implemented this technique for getting the optimal value of ϵ as represented by Table 1.4.

Table 1.4. A chronological scheme of the LOOCV technique

Researcher	**Year**	**Findings**
D. M. Allen [65]	1974	For ridge regression
Peter Craven and Grace Wahba [66]	1979	For smoothing splines
S. Rippa [64]	1999	Optimized RBF’s shape parameter ϵ.
G. E. Fasshauer and Jack G. Zhang [67]	2007	Extensions of LOOCV approach

Here's a step-by-step approach to implementing LOOCV with RBF for PDEs:

1. Generate the discretization of the domain
2. Initialize Metrics
3. Leave-One-Out Procedure
4. Aggregate the errors
5. Interpretation and Analysis
6. Iterative Refinement

Using LOOCV with RBF for the numerical simulation of various PDEs involves assessing the accuracy and performance of the RBF-based method across different spatial points in the domain.

In RBF kernel methods, there is no specific technique or process for calculating the shape parameter. The shape parameter can be chosen through numerical evaluations of RBFs to stabilize the solution, which is exceedingly hard and time-consuming. Timesli & Saffah [68] build an algorithm for measuring the optimal shape parameter ϵ based upon the idea of combining the RBF method, numerical continuation approach, and high-order algorithm Taylor expansion. Marko Urleb [69] proposed a strategy to find the optimized value of the shape parameter ϵ of multi-quadric RBF by using the Newton method to derive the solutions of the diffusion PDE with initial and boundary conditions. Nga Y. L. et al. [70] proposed a novel higher order RBF-FD schema with an optimal variable shape parameter of multi-quadric for the numerical results of various PDEs. In the research work of Kazeem et al. [71], the inverse multi quadric (IMQ) RBF function was included for implementing a technique to find the numerical solutions of PDEs. Preference is given to the selection of the shape parameter, which must be made carefully.

1.5. Stability and Rate of Convergence

In the present thesis work, infinite smooth RBFs have been applied for the numerical treatment of various PDEs. For these RBFs, stability totally depends upon the shape parameter value. The analysis of the stability of the numerical approach is based upon the set of eigen values as already derived in the literature [20]. So, the stability of the proposed scheme for PDEs lays on the transformed system of ODEs that employs a matrix by applying the RBF-PS method. If the real part of eigen values of that matrix are zero or negative real number then the numerical approach is stable.

In addition to the stability, rate of convergence (R.O.C) of the novel proposed approach is also analysed by obtaining the order of convergence (O.O.C) by the following formula:

$$R.\mathrm{O.C} = \frac{\log\left(\frac{\text{Error at N}}{\text{Error at 2N}}\right)}{\log\left(\frac{2\mathrm{N}}{\mathrm{N}}\right)} \tag{1.7}$$

here, L_∞ error norm is considering an error at N and error at 2N with respect to the number of the node points N and 2N respectively. The present thesis work focus on the optimization of the shape parameter using the optimization techniques with RBF-PS method. So, the rate of convergence of the proposed approach is not presented because the parameter value is changing in each iteration and hence the error.

1.6. Motivation and Objectives

The numerical simulation of PDEs through mesh-free RBF methods is essential due to its less computational cost, higher accuracy with good numerical stability and convergence, and flexibility in complex geometrical domains and the main characteristic is integrating with various optimization techniques, which is the main motivation behind the present research work. So, finding the numerical solutions of PDEs using RBF methods is a challenging aspect of mathematical modeling for various phenomena in science and engineering. The proposed objectives are as follows:

1) To understand the concept of Radial Basis function and its application.
2) To explore the possibility of using Particle Swarm Optimization (PSO) technique in the Radial basis function.
3) To explore the possibility of finding the shape parameter using various other optimization techniques for solving partial differential equations.
4) To extend the application of proposed method in fractional equations.

1.7. Organization of the Thesis

The first chapter starts with the fundamentals of partial differential equations. Also, discuss RBFs and their development for the methods of approximation existing in the literature to solve the partial differential equations. Various shape parameter strategies based on literature are also studied.

Chapter 2 is based on the optimization techniques, namely PSO and ABC, and the mathematical implementation of one of the RBF methods: the pseudo-spectral approach with RBF (RBF-PS method), which is useful for finding the solutions to various problems of PDEs numerically. The afore-mentioned optimization techniques are employed for evaluating the optimized shape parameter.

In chapter 3, the RBF-PS method is employed for the numerical treatment of one of the non-linear partial differential equations, the Fitzhugh-Nigumo equation. The techniques PSO and LOOCV are employed in this chapter for optimizing the unknown parameter of RBF while considering the error as an objective function. This unknown parameter is the shape parameter that forms the shape of different RBFs. The last section concludes with the process of the hybrid approach and the implementation of the RBF-PS method for the Fitzhugh-Nigumo equation and its numerical application.

Chapter 4 presents the implementation of the RBF-PS method for finding the numerical solutions of the Fisher's equation. In this chapter, two optimization techniques, namely PSO and ABC, are employed with the RBF-PS method for testing the exactness of the obtained numerical results with the optimized RBF shape parameter. The comparison of the obtained numerical results of the Fisher's equation using the present hybrid approach based on ABC and PSO is presented in the form of tables and figures.

Chapter 5 investigates the fundamentals of fractional differential equations (FDEs) with some basic definitions of fractional calculus, and the process of discretization of the fractional operators with RBFs is also done. The methods existing in the literature for solving the various fractional differential equations and their applications are also discussed. In this chapter, the layout of the above-mentioned proposed hybrid approach is employed with the RBF discretization for the numerical solutions of various FDEs.

Chapter 6 concludes the whole work based on the results presented in the above-mentioned chapters, and the future implications are suggested.

1.8. Summary and Conclusion

This chapter provides an overview of the radial basis function and the approaches based on RBFs for finding the solutions to the various PDEs. We make an attempt to highlight some of the current developments of the RBF methods and the approaches for finding the optimized value of the shape parameter. This work is intended to familiarize the reader with some of the RBF methods such as Kansa collocation methods, RBF-DQ and RBF-PUM methods. The RBF-PS method is also one of the RBF methods that will be discussed in chapter 2. These strategies aid in the lowering of computational costs and are particularly useful in the solution of large-scale problems. The minimal value of shape parameter leads to good accuracy for smooth RBFs, while the near flat radial basis leads to poor conditioning of the interpolation matrices. To overcome this issue, several algorithms were proposed that are listed in this work. The given approaches can be further enhanced by investigating the optimal value of the shape parameter for the better accuracy and steadiness of RBF approximations. The efficiency of RBF techniques for solving higher-order PDEs are still being investigated. The main motivation is the hybrid approach based on the integration of RBF with pseudo-spectral methods implemented with PSO, LOOCV and ABC techniques for finding the numerical solutions of various PDEs.

Chapter 2

RBF-PS Method and Optimization Techniques for solving Partial Differential Equations

2.1. Introduction

Evaluation of the solution of PDEs is a fundamental aspect of mathematical modeling for various phenomena in science and engineering. Analytical solutions are often challenging or impossible to obtain for complex PDEs for which numerical methods and discretization techniques are required. Optimization techniques play a critical role in solving PDEs. The procedure of optimization is based on the process of optimizing the positive characteristics or minimizing the negative aspects of the problematic models. It has been useful in determining the parameters required to solve mathematical models in various ways. There are various types of optimization techniques that are employed for finding solutions to optimization problems based on their characteristics. The method for categorizing optimization issues is to consider the modality of the goal landscape, which may be split into multimodal and uni-modal issues, including convex optimization. The terminology employed in the literature on optimization, as well as the classification of optimization issues, may be complicated and varied. The classification of optimization is presented in the following figure:

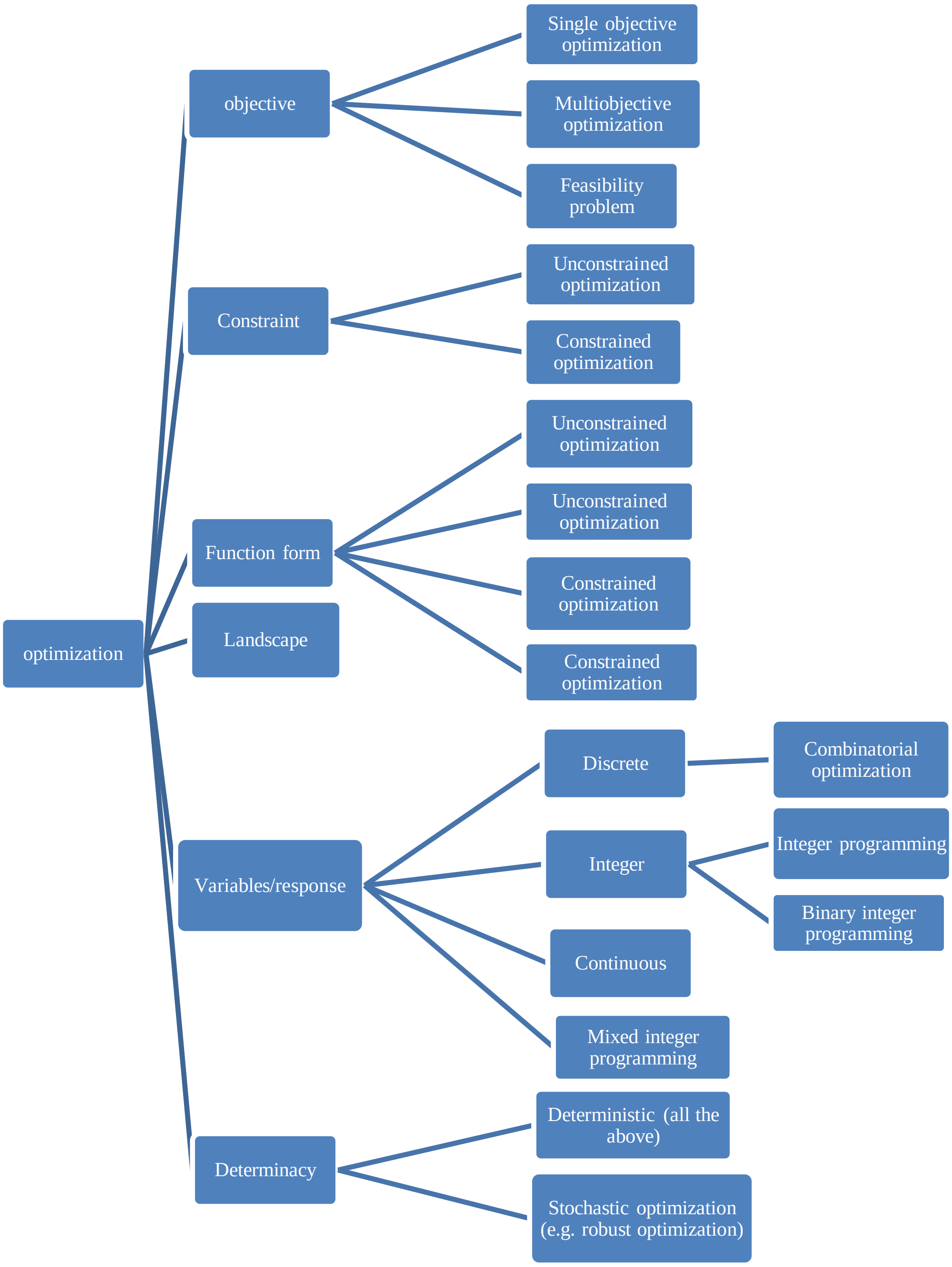

Figure 2.1. Various Types of Optimization Problems

To deal with the optimization problems, there are various nature-inspired optimization methods such as genetic algorithms (GA) [72], particle swarm optimization (PSO) [73], artificial bee colony algorithm (ABC) [74-75], ant colony algorithm (ACO) [76-77], differential evolution (DE) [78], and bacteria foraging optimization (BFO) [79]. These are some valuable tools in solving optimization problems. Optimization techniques are used by the researchers to find the optimum feasible solution to a problematic natural model. These techniques are applied in various fields to find the solution to mathematical models or physical phenomena that best exploits or minimizes their parameters. For example, minimizing the cost factor in the production of goods or services and maximizing profit are common needs that require optimization tools.

In this chapter, one of the mesh-free methods based on RBF is proposed along with the optimization techniques for obtaining the numerical simulation of various PDEs. Radial basis functions (RBFs) have numerous applications in areas such as statistics, machine learning, and finding the numerical solutions of various PDEs. This approach has the benefit of requiring no particular mesh or triangulation structure, which is one of its benefits. Regardless of the size of the space, it is easy to apply this approach to a multidimensional problem. Implementations of RBF have been done in various fields with applications in signal processing, surveying and mapping, geodesy, geophysics, and other fields. In the proposed approach, the RBF-PS method is used with optimization techniques for the numerical simulation of various PDEs. Various RBFs are enveloped with the shape parameter as studied in Chapter 1, which forms the accuracy for the numerical problems.

Madych [80] proposed a method in 1991 based on the study of the convergence of the indefinitely smooth RBFs (Gaussian and Inverse MQ) that confirmed the exponential convergence occurs wherever the shape parameter tending to zero. It has been analysed that various factor like domain points, the selection of one of the RBFs, the position of the centre points, and optimal the shape parameter, all affect the accuracy of the approximate solution. Additionally, there is consistently a conflict between precision and numerical stability. The interpolation matrix is ill-conditioned, or its condition number increases substantially by the use of the high shape parameter value. Results are significantly impacted by the ill-conditioning. Schaback [81] refers to this occurrence as the uncertainty principle or trade-off. The researchers are utilizing both static and dynamic interpolation for improving the stability and accuracy of the interpolation using RBF. In the first case, the shape parameter is changed to attain the desired accuracy with the fixed interpolation's points. On the other hand, Dynamic interpolation has a variable number of interpolation points with fixed shape parameter. It is necessary to choose such optimal shape parameter that maintains accuracy and stability. Therefore a proper shape parameter value is required because increasing the interpolation points enhances the cost of computing. Several articles [31, 35, 82, 83] explain how stability depends on the shape parameter.

Pseudo-spectral approaches are well known for higher accuracy. But these approaches are not flexible at geometrical domains. So integrating the pseudo-spectral approaches with RBF can get over this constraint. Fasshauer was the first to employ the radial basis function based pseudospectral (RBF-PS) approach, also known as the RBF collocation method in pseudo-spectral manner [84]. RBF-PS method has a best characteristic as solutions are approximated only on a limited set of grid points. Researchers have recently employed this method to resolve various PDEs. The RBF-PS method has been used by Uddin and Ali [85] to solve stiff nonlinear PDEs. The approach was further developed by Uddin et al. [86] by solving the Boussinesq equation. Uddin [87] developed a method to find the solution of the equation that modelled the waves produced in a shallow water channel of equal width. Krowiak [88] adopted the RBF-PS approach to analyse static isotropic plates with numerous boundary conditions.

The aim of this chapter to provides the details of two optimization techniques and the mathematical formulation of the RBF-PS method which forms the hybrid approach for finding the numerical solution of different PDEs by minimizing the errors and optimized the parameter of the RBF. The numerical simulation of a PDE using the RBF requires the value of the shape parameter that is responsible for defining the shapes of RBFs and needs to be evaluated precisely. This chapter concludes the optimization techniques with complete process and their pseudo-codes for the best understanding of the readers.

2.2. Radial Basis Function Pseudo-spectral (RBF-PS) Method

To implement the RBF-PS method, the collocation process based on a global radial basis function employed for dealing with the approximate solution to construct a distinct structure of the differential operator. As RBFs are of mesh- free nature due to which the method attains higher accuracy. To apply the RBF-PS method, firstly the domain discretized by using N randomly placed nodes for which the RBF approximation can be written as follows:

$$u_n(x,t) = \sum_{j=1}^{N} \delta_j(t)\, \emptyset(\|x - x_j\|) \tag{2.1}$$

where the interpolation coefficients are $\delta_j(t)$ and the RBF is denoted by $\emptyset$. Based on the nature of the mathematical problem, different type of RBFs can be applied. The equation (2.1) can be transformed by the interpolation technique as follows:

$$u_n(x_i,t) = \sum_{j=1}^{N} \delta_j(t)\, \emptyset(\|x_i - x_j\|),\ 1 \le i \le N \tag{2.2}$$

The equation (2.2) can also be represented in the matrix form as

$$u_N = AY \tag{2.3}$$

Here interpolation coefficients are $Y = [\delta_{1,}\delta_2, \delta_3, \dots \delta_n]^t$ and $A = [A_{ij}]$ is an evaluation matrix whose elements are of the form$A_{ij} = \emptyset(\|x_i - x_j\|)$. Derivative of u_n is derived by the differentiation of the basis function as follows:

$$\frac{\partial}{\partial x} u_n(x_i,t) = \sum_{j=1}^{N} \delta_j(t)\, \frac{d}{dx_i} \emptyset(\|x_i - x_j\|) \tag{2.4}$$

The matrix representation of the equation (2.4) with all node points of the domain can be written as

$$u_N{}' = A_x Y \text{ where } A_x = \frac{d}{dx} \emptyset\big(\|x_i - x_j\|\big). \tag{2.5}$$

It is essential for the assurance of the inverse of the interpolation matrix before using the RBF-PS approach. It is based on the selection of RBF and the node points. The positive definite RBF results the invertible interpolation matrix. By inverting the interpolation matrix of equation (2.3), we can calculate

$$A^{-1} u_N = Y \tag{2.6}$$

By putting interpolation coefficients (2.6) in equation (2.5), we get first derivative as

$$u_N{}' = A_x A^{-1} u_N \tag{2.7}$$

Where $A_x A^{-1}$ is matrix of differentiation and$A_x = \frac{d}{dx} \emptyset(\|x_i - x_j\|)$.

Similarly, we can evaluate second derivative as

$$u_N'' = A_{xx}A^{-1}u_N \text{ ; where } A_{xx} = \frac{d}{dx}(A_{xx}) \quad (2.8)$$

Using all these results in given problem of PDEs that transforms into a system of ODEs at scattered nodes of the domain.

So, system of ordinary differential equations is obtained by the discretization of the PDE at scattered nodes. The generated ODEs can be discretized in time using an ODE45 solver that is employed for solving the differential equation by MATLAB.

2.3. Optimization Techniques

2.3.1. Particle Swarm Optimization (PSO) Algorithm

PSO is an admired meta-heuristic optimization technique that was stimulated by the social behaviour of swarms, bird flocking or movement of the insects. It was initially developed by James Kennedy and Russell Eberhart in 1995 [89]. This algorithm simulates the behaviour of a group of particles that represents the optimal solution to the problematic models. Each particle iteratively sets their position and velocities for finding the optimal or near-optimal solutions in the search space.

In the present optimization technique, population particles are represented by swarms which share the information with others to enhance the search solution and find the global optimal value. Each particle has their best experience. A particle changes its optimal position when it locates a better location that represents the new optimal solution. After iteration, each n-particle has a new optimal solution. By comparing all potential solutions, this approach selects the best one. This cycle repeats until the specified iterations or the objective is not achieved. Procedure of PSO can be defined in the following steps.

1. Initialization of the parameters:
    - Initializing the parameters is the most important step to start the PSO algorithm in the problem space.
2. Position and Velocity of particles:
    - Assign the position and velocity to all the particles in the optimization problem.
3. Evaluation:

- Evaluate the personal best and the fitness values of each particle. Particle population are modified by PSO technique based on fitness values.

4. Update velocity and position:
   - The velocity V_i^{k+1} and the position X_i^{k+1} are updated by the help of following expression:

$$V_i^{k+1} = V_i^k + a_1 * random * \left(P_{best_i} - X_i^k\right) + a_2 * random * \left(G_{best} - X_i^k\right), \tag{2.10}$$

$$X_i^{k+1} = X_i^k + V_i^{k+1} \tag{2.11}$$

Where, V_i^k - is velocity of particle i at kth iteration,

a_1, a_2- are real parameters,

$random$ -is a random number, whose value lies between 0 & 1,

X_i^k = is ith- Particle position at kth iteration,

P_{best_i} - Personal best position of particle i,

G_{best} - Global best position of whole search space.

5. Repeat until the stopping criteria (no. of iterations or objective achieved) not met.
6. Return the best optimized solution.

2.3.1.1. Pseudo-code and flow chart for PSO Algorithm

Enter the parameters- fitness, lb, ub, Np, T.

1. Initialize population (P) and velocity U_i of particle i
2. Evaluate objective function (f)
3. Assign P_{best_i} = P and f_{best}= f
4. Estimation of best fitness solution and locate G_{best}= solution and fitness=f_{best}

   For t=1 to T (iteration)

   For i=1 to Np (population size)

   Find the velocity U_i and new position (X_i)

Bound X_i and find objective function value

Update the population by including $X_i \& f_i$

Update $P_{best}, f_{pbest}, G_{best}$ and f_{gbest}

End

End

PSO is an iterative method for computation of optimization problem by minimizing the errors. It is a statistical process for figuring out parameter values. The particles cooperate, share information and adhere to a straightforward rule to determine the best solution to an optimization problem. It is a global search based mechanism and has a wide range of parameter-space properties. It is a novel approach for finding the numerical solutions of nonlinear PDEs. The following figure represents the complete process of PSO algorithm and link of Yarpiz* video lecture is give below:

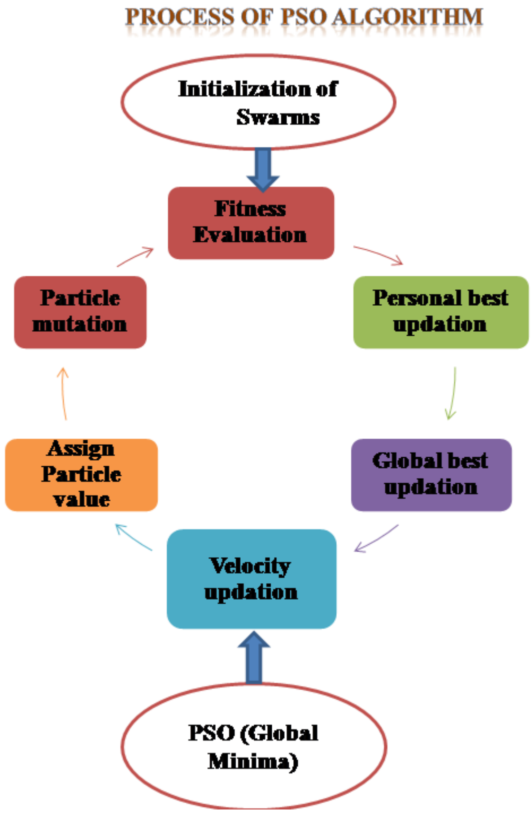

Figure 2.2. Entire Process of PSO

https://www.google.com/search?client=firefoxe&q=yarpiz+by+PSO#fpstate=ive&vld=cid:96742440,vid:sB1n9a9yxJk,st:0 *

2.3.2. Artificial Bee Colony (ABC) Algorithm

ABC optimization technique originated by Karaboga [90] in 2005 and this technique is based on the concept of swarm intelligence to simulate the foraging behaviour of honeybees. ABC has the advantages of being reliable, efficient, and simple to implement. In this algorithm, particles are in the form of bees like employed bees (workers), onlooker bees (observers), and scout bees. The ABC algorithm starts with the deployment of bees and their dispersion in various regions. Bees that are actively searching for nectar are all employed bees which convey the facts about the food to other bees. Some observer bees are waiting for the scout bees for finding the information about the fresh food sources. Employed bees dance in a designated area to exchange information about the food availability. Observer bees keep an eye on the dance and decide which food source to use based on its reliability. Better food sources are selected by more bees. All working bees turn into scouts when a food source is depleted.

Each food source in this technique represents a potential solution to the problematic model, and the amount of nectar represents the assessment of the fitness value. There is exactly one busy bee for every potential food source, and the entire number of food sources is equal to the total number of bees at work. An observer bee chooses a food source based on the associated probability factor defined below.

$$P_j = \frac{Fit_j}{\sum_{t=1}^{Np} Fit_t} \tag{2.12}$$

Here, the employed bees are relative to the ideal solution that evaluates the fitness Fit_j parameter of the jth solution, and N_p represents the number of food sources. Fitness of a solution is derived for $f \geq 0$ as $\frac{1}{1+f}$ and for $f < 0$ as $1 + |f|$ where f is objective function value. The observer bee uses (2.13) to travel to new spot with the assistance of the selected employed bee.

$$v_j = x_j + \emptyset_j(x_j - x_k) \tag{2.13}$$

Here, the current position is indicated byx_j, selection of the employed bees is indicated byx_k, and $\emptyset_j$ is chosen at randomly -1 to 1 to locate the best food sources. After several iterations, every bee that is incapable of discovering a superior food source gets replaced by a scout bee. Using (2.14), the scout bees sent to substitute for the failed bee hovers around any random or unexplored location to explore its surroundings.

$$x_t = Lb + \emptyset_j(UB - LB) \quad (2.14)$$

Here UB and LB represents the upper bound and the lower bound, respectively. A position that cannot be improved during the specified cycles is utilized to determine the parameter limit (number of cycles). The three phases of ABC are shown in the next section with their pseudo codes.

a) Employed Phase

The following considerations should be kept in mind for the production of new solutions during this stage:

- The number of food sources and employed bees are equal.
- For the creation of a novel solution, an associate partner is chosen at random but both should not be same.
- For the formulation of a new solution, modification of a randomly chosen variable is essential.

$$\mathrm{x}_{\mathrm{new}}^{\mathrm{j}} = \mathrm{x}^{j} + \vartheta(\mathrm{x}^{\mathrm{j}} - \mathrm{x}_{\mathrm{p}}^{\mathrm{j}}) \quad (2.15)$$

Where $\mathrm{x}_{\mathrm{p}}^{j}$ = j^{th} variable of p^{th} solution

$\mathrm{x}_{\mathrm{new}}^{\mathrm{j}}$ = j^{th} variable of new solution

ϑ = Random variable lies between -1 to 1

x^{j} = j^{th} variable of current solution

- Bounds of generated solution.

$\mathrm{x}_{\mathrm{new}}^{j} = \mathrm{lb}$ if $\mathrm{x}_{\mathrm{new}}^{j} < lb$

$\mathrm{x}_{\mathrm{new}}^{j} = \mathrm{ub}$ if $\mathrm{x}_{\mathrm{new}}^{j} > ub$

Here, evaluates the value of the objective function and determine the fitness value of the new solution using the conditions of bounds. Choose the newly generated solution to update the current one. Keep track of trial of every solution's failures. Reset it if the new solution is better; if not then increase the trial by one. The updated solution is considered for further. Pseudo-code of employed bee phase is:

Enter the initial parameters- population size N, trial, lb, ub, Np=s/2= no. of food source.

For k=1 to Np

Select a partner (p) randomly s. t. $k \neq p$.

Selected a variable (j) randomly and update j^{th} variable

$$x^j_{new} = x^j + \vartheta(x^j - x^j_p)$$

Bound modified j (x^j_{new})

Evaluate objective function (f_{new}) and fitness (fit_{new})

Accept x^{new}, if fit_{new} >fitness and Set trial=0. Else increase trial by 1.

End.

b) Onlooker Bees Phase

There is a condition of the probability of bees to exploit a particular food source to enter into this phase. As by the fitness of each food source calculate the probability determined by (2.12). Pseudo-code for onlooker bee phase:

Input the parameters: Obj. function, fit, trial, lb, ub, Np=s/2, prob.

Set m=0 & n= 1(m & n correspond to food source & onlooker bees respectively)

While $m < N_p$

Generate random no. r

If $r < prob$ of n

Randomly select a partner (p) s. t. $n \neq p$.

Selected a variable (j) randomly and update j^{th} variable.

$$x^j_{new} = x^j + \vartheta(x^j - x^j_p)$$

Bound modified j (x^j_{new})

Evaluate objective function (f_{new}) and fitness (fit_{new})

Accept x^{new}, if fit_{new} > fitness of n^{th} solution and Set trial=0.

Else increase trial by 1

m=m+1

end

n = n + 1

Modify n=1($if\ n > Np$)

End.

c) Scout Bees Phase

Every solution is associated with an individual trial for which need to specify a parameter limit that is a user specified integer value. To enter into this phase, value of trial should be greater than limit and trial of abandoned solution is reset to zero. Not every solution passes through the scout phase. Limit can be calculated as Np*d with d-dimensional problem space. This phase can occurred only when trial counter of atleast one solution is greater than limit. The pseudo-code for scout bee phase is given as follows:

Input= Obj. function, fit, trial, lb, ub, limit, P.

Identify food source (t) whose $trial > limit$.

Replace X_t with P as $X_t = lb + \varphi_i(ub - lb)$

Evaluate objective function (f_t) and assign fitness (fit_t)

2.3.2.1. Flow Chart and Pseudo-code of ABC

ABC initializes bee swarms and repeats until stopping criteria are met. ABC optimizes iteratively. Employed and onlooker bees agree on exploitation, while the process of exploration is performed with scout bees. Pseudo-code of the algorithm is given below with the process of basic ABC algorithm that is represented by fig. 2.3

1. Enter initial parameters= Obj. function, fit, lb, ub, limit, Np, T= iteration.
2. Evaluate the value of objective function (f).
3. Evaluate fitness value (fit) and set trial = 0.
4. For i =1 to T

    Enter into employed bee phase

    Calculate the probability

    Enter into onlooker bee phase for generating Np food source

    Locate the best food source value

    If $trial > limit$

        Analyse the scout bee phase

    End

End.

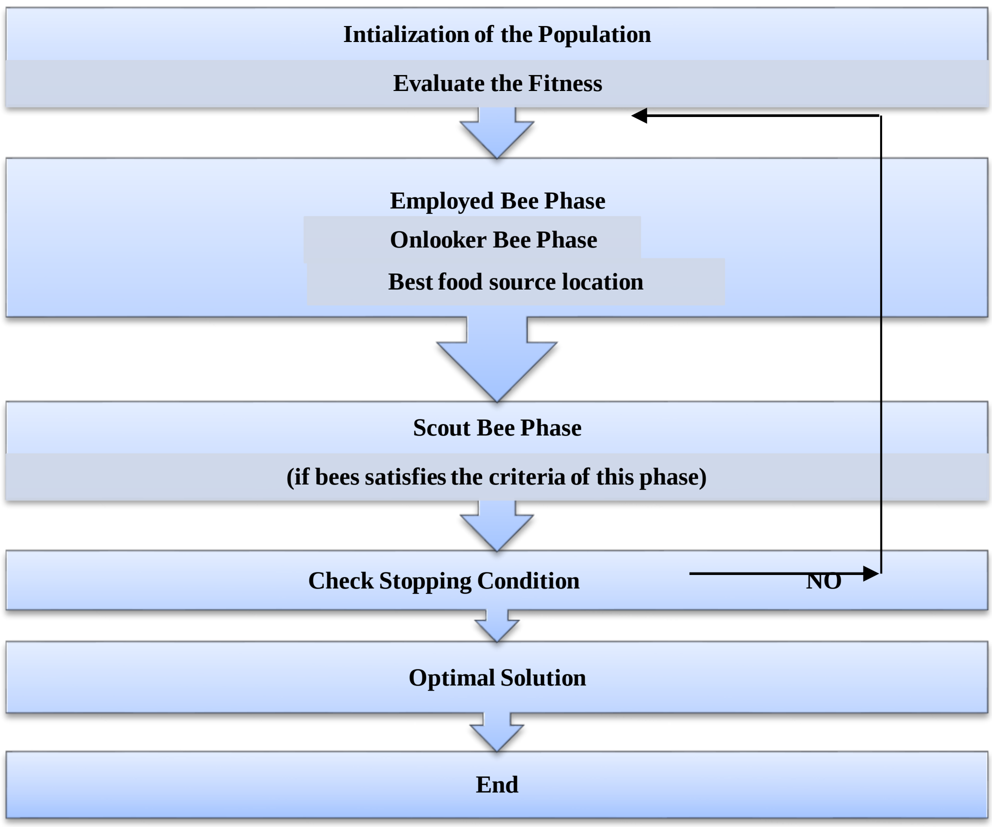

Figure 2.3. Process of ABC algorithm

The process of ABC iteratively improves the solutions through the exploration of the employed bee, selection of the onlooker bee, and scout bees' replacement of stagnant solutions. The best solution obtained by the optimization represents the solution to the PDE problem. The ABC algorithm involves the bees exploring the search space for finding the best solutions which minimizes the errors associated with solving the PDE. Refer the link of video lectures* for detail explanation of ABC algorithm.

2.4. Summary and Conclusion

This chapter represents the mathematical explanation of the RBF-PS method and the various optimization techniques for treating the mathematical models. PSO and ABC optimization algorithms are used in our proposed hybrid approach based on the combination with RBF-PS method for finding the numerical solutions of various PDEs. These algorithms have gained popularity due their characteristics of less computation time, ease of implementation, and obtaining the optimized parameter values. In this chapter, we also discussed the mathematical formulation of proposed hybrid approach for finding the numerical solutions of various PDEs. A step by step algorithm for PSO and ABC optimization techniques are discussed and the implementation is done component-wise with their pseudo-codes. The complete process of the ABC algorithm and the PSO algorithm are shown by the figures for deep understanding of the readers. Over the years, optimization approaches have continuously evolved for optimizing the objective function with respect to the some conditions. So, hybrid form employs best approximated solutions of the optimization model that helps in minimize the cost of computation and are particularly successful in solving large-scale problems.

* https://yarpiz.com/297/ypea114-artificial-bee-colony

Chapter 3

Numerical Treatment of Fitzhugh- Nagumo Equation using RBF-PS Method

3.1. Introduction

Mesh free methods have indeed become a valuable approach for solving various problems in engineering and sciences, especially when dealing with high-dimensional scenarios. These methods offer advantages in terms of handling complex geometrical domain and reducing the computational burden associated with mesh-based discretization. Meshfree methods can effectively transform high-dimensional problems into lower-dimensional by the use of univariate basis functions and norms, making them suitable for a wide range of applications. As studied in the previous chapters, the RBF method is indeed a powerful mesh free technique widely used for multivariate scattered data approximation and solving various multidimensional real-world problems in engineering and sciences. RBF pseudo-spectral technique does not require any meshes. RBF has been demonstrated as a successful basis function in RBF-PS method for simulating the ODEs and the PDEs numerically. Approaches based on RBF are suitable for multimodal approximations, and geometrical adjustability in the relevant region.

Hardy [25] firstly proposed the concept of RBF with topological quadric surface and created the multi- quadric piecewise RBF for interpolation purpose for a cartographic application. Franke [26] experimented with scattered data interpolation. He evaluates methods in terms of time, storage, exactness, and ease of implementation, and he also considers multi-quadric (which is a type of RBF) to be one of the best. Micchelli [27] made a step forward by demonstrating that multi-quadraic surface interpolation is always solvable. The MQ approach has the benefit of obtaining the interpolant using a linear combination of basis functions that are only dependent on the distance from a specific node, which is known as the center. In order to solve a PDE, Edward Kansa invented the Kansa method [28, 29] in 1990. It first used the multi quadraic, a widely supported interpolant to analyze the behaviour of PDEs. The numerical simulation of a PDE using RBF requires the value of the shape parameter that evaluates the shapes of RBFs and needs to be evaluated precisely. RBFs offer flexibility and efficiency over the complex domain for solving various PDEs. There are various types of RBFs based on their smoothness which contains a shape parameter. The shape parameter of RBFs is the main component for controlling the shape and behaviour of the basis functions. The selection of the shape parameter is crucial for achieving accurate results in various applications. Depending on the problem defined on the specific domain, selection of the RBF parameter can be necessary for the accuracy in the solutions of PDEs with RBF methods. So for optimizing the shape parameter with good errors, optimization techniques (discussed in the chapter 2) have been introduced with RBF methods.

In this chapter, PSO and LOOCV approach are employed for optimizing the error norms with good shape parameter value of RBF using RBF-PS method for solving a non-linear PDE. Optimization techniques can be customized for a particular class of problems. It can be evaluated as either a local approach or a global approach. Local optimization techniques provide the locally optimal result with a lower computing cost. While the global optimization techniques are more applicable because of the optimum value obtained globally. PSO is one such global optimization technique, a population-based meta-heuristic method, and an evolutionary algorithm [91] proposed by researchers for finding out the solutions of optimization problems in a wider range of applications. Applications of PSO exist in various diverse fields, including healthcare, the environment, industry, commerce, smart cities, and other general aspects [92], as it is used with reinforcement learning [93], the prediction of pollution, categorization of plants, the management and navigation of floods, the assessment of water, etc. It draws inspiration from the cooperation of mammals, such as the swarming of fish, the hovering of birds, and the theory of ants.

PSO is an efficient optimization algorithm introduced by Kennedy and Russell in 1995 [89]. In the present optimization approach, swarms are viewed as population particles that communicate information to enhance the efficiency of the search solution and locate the global optimum. Each particle of the swarm has its own position and velocity in the search domain. The particles interact, learn from each other, and obey a simple rule for finding the best solution to an optimization problem.

In the field of optimizing the shape parameter, numerous optimization techniques have been implemented with numerical techniques in the literature. The solution can be stabilized by the selection of the shape parameter and the process to obtain the same is extremely challenging and time-consuming. The power function, leave-one-out-cross validation, particle swarm optimization and trial-and-error are some methods for figuring out the ideal shape parameter. Firstly Hardy [25] and Fasshauer [19] used the trial-and-error method for optimizing the shape parameter's. Rippa [64] established the LOOCV methodology to optimize the parameter which was based on a cross-validation method. Arora and Bhatia [94] applied the LOOCV optimization technique by minimizing the RMS error function to fit the data based on interpolation. Hybrid RBF with the PSO technique was proposed for time series prediction problems by Hassan and Abdullah [95] in 2014, and in 2018, Koupaei et al. [96] integrated the PSO algorithm with one of the collocation method for determining the best value of parameter associated with RBF in the numerical simulation for PDEs. Salleh [97] projected the PSO optimization technique for optimizing the RBF’s shape parameter value for solving Fokker-Planck equations. In 2022, Abdulrazzaq [98] used PSO based on the Padé approximant for solving ordinary differential equations. Recently, Tsoulos and Charilogis [99] used a hybrid PSO method in 2023 to locate the parameter for RBF networks. Researchers proposed a novel approach based on PSO with Gaussian RBF [100] for testing the effectiveness and accuracy of surrogate model in which PSO is used for obtaining the optimal value of the parameter of Gaussian radial basis function. Ghalichi et al. [101] projected the method based on the algorithm for optimizing the RBF’s parameter within the image processing study. In 2023, authors used RBF as an activation function based on the Bayesian regularization deep neural network for finding the solutions of a fluid model whose data set was prepared by the Runge-Kutta method [102]. The meta-heuristic algorithms, especially PSO, were implemented by the researchers in [103–104] for the numerical treatment of the SEIR-NDC model based on COVID-19 and for a nonlinear second-order coupled Emden-Fowler model, in which artificial neural networks (ANNs) computational techniques with PSO were implemented.

In mathematical research, non-linear PDEs are intensively practiced to emulate the physical phenomena of the natural model. Numerical solutions of PDEs provide valuable facts and an improved form of mathematical models. The Fitzhugh Nagumo (FN) equation is a type of non- linear partial differential equation represent by:

$$u_t = u_{xx} + u(u - \zeta)(1 - u), \quad 0 < \zeta < 1 \tag{3.1}$$

Here, u(x,t) is an anonymous function that depends on the arbitrary constant. Nagumo *et al.* [105] and Fitzhugh [106] proposed the FN equation, which is used in population genetics, autocatalytic chemical reactions, circuit theory, and a neurophysiology [107-109].

In this chapter, the Fitzhugh-Nagumo equation is solved numerically using hybrid approach based upon RBF-PS method with the PSO algorithm and LOOCV approach. The associated parameter of RBFs is optimized by the PSO strategy and LOOCV with the RBF-PS method in the process of finding the numerical solutions of the one of the PDEs known as FN equation. The numerical solution is obtained for the problems of FN equation that highlight the efficacy of the proposed hybrid method. The analysis of the results obtained by the use of cubic matern (CM) RBF and multi-quadric (MQ) RBF with different node points is done and the results obtained in the form of errors L_∞, L_2 and RMS error presented by the figures and tables that are in better accuracy compared to the available results in the literature.

3.2. Description of the Proposed Approach

The proposed approach for finding the RBF shape parameter is based on the hybrid form of RBF-PS method with optimization techniques for numerical simulation of the partial differential equation. Here, the proposed approach is employed for finding the numerical solutions of FN equation (3.1) using RBF-PS method with PSO algorithm and LOOCV technique.

Since the PSO method is a global search optimization technique and has various advantages in the search space. So it is a novel approach for optimizing the shape parameter of RBF. The step-by-step computational technique based on the PSO algorithm is as follows:

I Select one of the RBFs.
Enter the exact solution and the conditions for optimizing theshape parameter.

II Discretization of the domain.

III Finding the optimum shape parameter.
Here, the PSO algorithm starts with the initial node and assigns a random value to each particle in the form of velocity.

IV Evaluation of the particles.

Calculate the fitness values of each particle. So, the particle position is updated based on the fitness values.

V Update position and velocity.

VI Repeat the above process from step 3 to step 6 until you get the desired results.

The numerical simulation of a PDE using RBF requires the value of the shape parameter, a factor that is responsible for defining the shapes of RBFs and needs to be evaluated precisely. The cubic matern and multi-qudaric (MQ) functions are considered RBF for the numerical approximation of the FN equation, and these RBF has a shape parameter ε that has been optimized by the PSO and LOOCV algorithm. RBF has an important role for approximating the derivatives of the FN equation and then obtained system is solved by MATLAB software for getting the optimal solutions.

3.3. Implementation of RBF-PS Method for Fitzhugh- Nagumo Equation

In this section, the RBF-PS method is projected in the numerical treatment of the FN equation (3.1) with the following conditions.

$$u(x,0) = f(x); x \in [a,b] \tag{3.2}$$

$$u(a,t) = g_1(t),\ u(b,t) = g_2(t), t \in [0,T] \tag{3.3}$$

Initial and boundary conditions are presented by (3.2) and (3.3). Numerous researchers proposed various techniques for solving the FN numerically in the literature. The Rayleigh Benard convection of binary fluid mixtures is described by the real Newell-Whitehead equation that reduced in the form of FN equation [110]. This equation captures the dynamical behaviour in the neighborhood of the bifurcation point. Nucci and Clarkson [111] had discovered the analytical solution of the FN problem using the Jacobi elliptic function. Shih et al. [112] applied the approximate conditional symmetric technique to explore the perturbed FN equation. The author looked into how different electric potentials affected cell membranes. Olmos and Shizgal [113] introduced the Chebyshev multi-domain approach integrated with pseudo-spectral method for finding the solutions of FN equation using

Chebyshev Gauss Lobatto nodes.

The RBF-PS method is an accurate and efficient method to find the solution of FN equation for which two stages are required. In the first stage, RBF approximates the derivatives present in the equation (3.1) and converted into the system of ODEs. After that obtained converted system of equations is solved by the software MATLAB. By discretizing the domain with N node points, it is possible to write the RBF approximation with the property of the linear combination of RBFs as given below:

$$u_n(x,t) = \sum_{j=1}^{N} \delta_j(t)\, \emptyset(\|x - x_j\|) \tag{3.4}$$

Here, the interpolation coefficients are $\delta_j(t)$ and the RBF is denoted by$\emptyset$. Based on the nature of the mathematical problem, different type of RBFs can be applied. The equation (3.4) can be transformed by the interpolation technique as follows:

$$u_n(x_i,t) = \sum_{j=1}^{N} \delta_j(t)\, \emptyset(\|x_i - x_j\|),\ 1 \le i \le N \tag{3.5}$$

The equation (3.5) can also be represented in the matrix form as

$$u_N = AY \tag{3.6}$$

Here interpolation coefficients are $Y = [\delta_{1,}\delta_2, \delta_3, \dots \delta_n]^t$ and $A = [A_{ij}]$ is an evaluation matrix whose elements are of the form$A_{ij} = \emptyset(\|x_i - x_j\|)$. Derivative of u_n is derived by the differentiation of the basis function as follows:

$$\frac{\partial}{\partial x} u_n(x_i,t) = \sum_{j=1}^{N} \delta_j(t) \frac{d}{dx_i} \emptyset(\|x_i - x_j\|) \tag{3.7}$$

The matrix representation of the above equation with all node points of the domain can be written as

$$u_N{}' = A_x Y \text{ where } A_x = \frac{d}{dx}\emptyset(\|x_i - x_j\|). \tag{3.8}$$

It is essential for the assurance of the inverse of the interpolation matrix before using the RBF-PS approach. It is based on the selection of RBF and the node points. The positive definite RBF results the invertible interpolation matrix. By inverting the interpolation matrix of equation (3.6), we can calculate

$$A^{-1}u_N = Y \tag{3.9}$$

By putting interpolation coefficients (3.9) in equation (3.8), we get first derivative as

$$u_N' = A_x A^{-1} u_N \tag{3.10}$$

So by the RBF-PS process, equation (3.1) is reduced into following equation.

$$\frac{du_N}{dx} - A_{xx}A^{-1}u_N = u_N(u_N - \zeta)(1 - u_N) \tag{3.11}$$

There are two ways to deal with the behavior of the boundary conditions of FN equation in the method RBF-PS. Firstly, check whether the radial basis functions satisfied the boundary conditions, otherwise to compel them uniquely some other conditions. Hence, the boundary conditions can be applied by changing the right side of the derived equation and then obtaining the resultant using an ODE solver. So, the solutions of the obtained system of ODEs are found by software MATLAB with ODE 45.

3.4. Stability of the RBF-PS method

The analysis of the stability of the numerical approach is based upon set of eigen values. So, the stability of the proposed scheme for FN equation (3.1) relies on the stability of the converted set of ODEs (3.4) which employs the matrixA_{xx}. So, eigen values of A_{xx} established the nature of the stability for equation (3.1). If the real part of eigen values of the matrix A_{xx} are zero or negative real number then the numerical approach is stable as discussed in chapter 1.

3.5. Discussion with Numerical Experiments

This part analyzes the numerical applications of the present hybrid approach which divides the numerical simulation into two parts. In the first part, the derivatives are approximated using RBF and in second part, the solution of the obtained equation is calculated with MATLAB using PSO algorithm and LOOCV approach. In this chapter, the multi-quadric (MQ) and the cubic matern radial basis function (CMRBF) is considered as a main basis function for numerical approximation of the FN equation and then compares the obtained results in the form of error norms. The following error norms are used for the numerical simulation of Fitzhugh-Nagumo equation.

Absolute error $= \left|u_{exact}(x_{i,}t) - u(x_{i,}t)\right|$;

$L_\infty = \max\left(\left|u_{exact}(x_{i,}t) - u(x_{i,}t)\right|\right)$;

$L_2 = \sqrt{h\sum_{i=1}^{N}\left|u_{exact}(x_{i,}t) - u(x_{i,}t)\right|^2}$;

$L_{rms} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left|u_{exact}(x_{i,}t) - u(x_{i,}t)\right|^2}$; $1 \leq i \leq N$.

Problem 3.1. The non-linear general Fitzhugh-Nagumo equation (3.1) of dimension one in the domain [-10,10] with exact solution [114] and the boundary conditions derived by the exact solution that are represented in the following manner with the initial conditions:

$$u(x,t) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{1}{2\sqrt{2}}\left(x - \frac{2\zeta-1}{\sqrt{2}}t\right)\right); \qquad x\epsilon[-10,10] \,\&\, \mathrm{t} \geq 0 \tag{3.12}$$

$$u(x,0) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{x}{2\sqrt{2}}\right); \tag{3.13}$$

A numerical solution of the general FN equation (3.1) is obtained at a = −10, b = 10 $and\zeta$=0.75 at various time intervals, and the results are compared with the derived results available in the literature. TheL_∞, L_2 and RMS errors are calculated using hybrid approach based PSO represented by Table 3.1 for time intervals 0.2; 0.5; 1; 1.5; 2; 3; 5 & N= 71 with CMRBF. The derived results are in good accuracy and comparable to the results available in Bhatia & Arora [114] and Jiwari *et al.* [115]. The calculated value of ϵ is 0.145094 with the proposed approach. Table 3.2 presents the comparison of results of L_∞ and $Lrms$ calculated by the proposed approach using MQ RBF at N=101, Δt=0.0001with the results available in Ahmad *et al.* [116] and optimal shape parameter is calculated as 0.132874 with PSO algorithm and 0.100050 by LOOCV approach. It is conclude that the results derived by the RBF-PS method with PSO algorithm at higher node points with MQ RBF are better than the results obtained by the proposed approach based on LOOCV and the available method in [116].

And Table 3.3 shows the comparative analysis of the absolute errors of problem 3.1for N= 51 and T = 1, 3, 5 with that of Bhatia & Arora [114] that represents absolute errors are comparable and the results derived by the present proposed approach are in good accuracy. So, PSO is best optimizer to give the numerical solution with minimum error.

Table 3.1 Comparison of error norms obtained by PSO with CMRBF at N=71

T **Errors**	**0.2**	**0.5**	**1.0**	**1.5**	**2.0**	**3.0**	**5.0**
L_{rms}	5.4717e-08	2.3756e-07	6.5064e-07	1.0534e-06	1.3731e-06	1.7020e-06	1.7264e-06
L_∞	3.8849e-06	1.6867e-05	4.6196e-05	7.4791e-05	9.7487e-05	1.2084e-04	1.2258e-04
L_2	4.6105e-07	2.0017e-06	5.4824e-06	8.8760e-06	1.1570e-05	1.4341e-05	1.4547e-05
L_{rms}**[114]**	1.5880e-05	3.8433e-05	8.1870e-05	1.3387e-04	1.9433e-04	3.4320e-04	7.8638e-04
L_∞**[114]**	4.7416e-05	1.2312e-04	2.6261e-04	4.2096e-04	5.9999e-04	1.0324e-03	2.3050e-03
L_2 **[114]**	2.3012e-06	5.5695e-06	1.1864e-05	1.9400e-05	2.8162e-05	4.9735e-05	1.1395e-04
L_{rms}**[115]**	1.3586e-07	5.2263e-07	1.4107e-06	2.4594e-06	3.6827e-06	7.0478e-06	2.0075e-05
L_∞ **[115]**	1.7959e-06	7.9698e-06	2.6954e-05	5.7054e-05	9.8852e-05	2.2214e-04	6.5934e-04
L_2 **[115]**	9.1137e-07	3.5059e-06	9.4635e-06	1.6498e-05	2.4704e-05	4.7278e-05	1.3467e-04

Figures 3.1, 3.2, 3.3 represent the comparison of exact solution and numerical solution with CMRBF at N=51, Δt=0.0001, iteration=30 with time interval T=0.2, 0.4, and 1; respectively which seems to be more accurate at more node points. Figures 3.4, 3.5, 3.6 represents the comparison of numerical solution with exact solution at N=21, Δt=0.0001, iteration=30 with time interval T=0.2, 0.4, and 1; respectively that are calculated by MQ RBF. And the figure 3.7 presents the numerical simulation of the solution of FN equation obtained with CMRBF at different time intervals with N= 51 andΔt = 0.01. Table 3.4 presents the variations in the value of shape parameter with respect to the number of iterations using CMRBF.

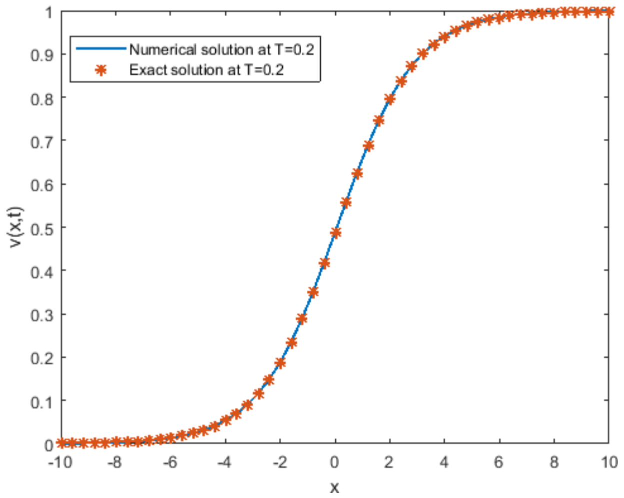

Figure 3.1. Comparison of numerical and exact solutions of problem 3.1 at N=51, Δt=0.0001 with PSO using CMRBF

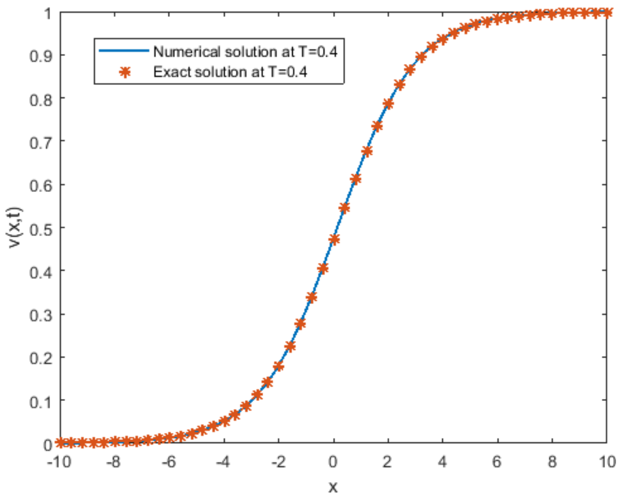

Figure 3.2. Comparison of numerical and exact solutions of problem 3.1 at T=0.4, N=51, Δt=0.0001 with PSO using CMRBF

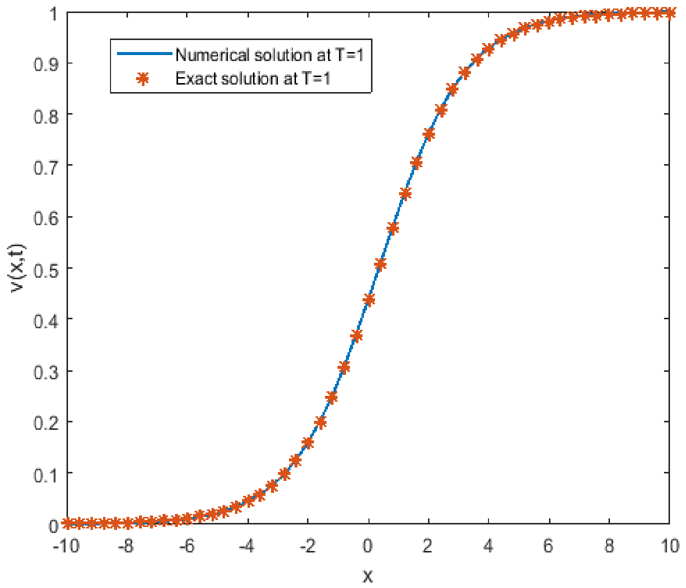

Figure 3.3. Comparison of numerical and exact solutions of problem 3.1 at N=51, T=1, Δt=0.0001 with PSO using CMRBF

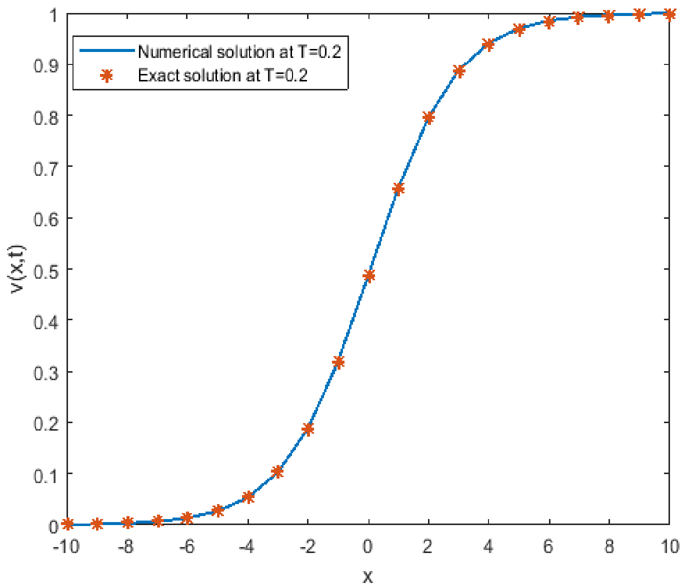

Figure 3.4. Comparison of numerical and exact solutions of problem 3.1 at N=21, $\Delta t = 0.0001$, T=0.2 with PSO using MQ RBF; iteration 30

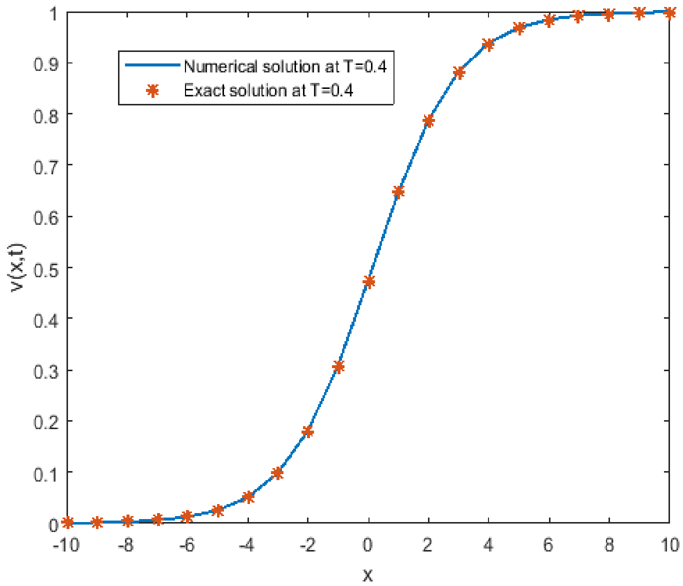

Figure 3.5. Comparison of numerical and exact solutions of problem 3.1 at N=21, $\Delta t = 0.0001$, T=0.4 with PSO using MQ RBF; iteration 30

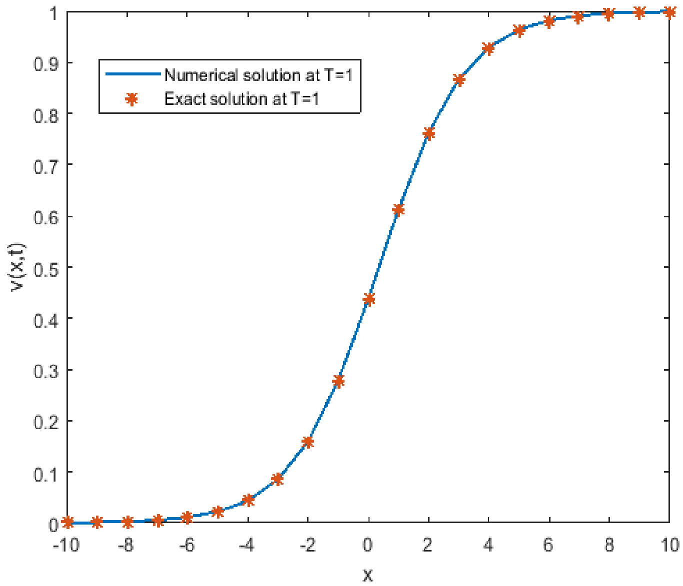

Figure 3.6. Comparison of numerical and exact solutions of problem 3.1 at N=21, Δt=0.0001, T=1 with PSO using MQ RBF; iteration 30

Table 3.2. Comparison of error norms using PSO and LOOCV with MQ at Δt=0.0001 with shape parameter values

Errors **T**	**0.2**	**0.5**	**1.0**	**1.5**	**2.0**	**3.0**
L_∞(PSO)	1.9497e-07	1.9497e-07	1.9497e-07	1.9497e-07	1.9497e-07	1.9497e-07
L_{rms}(PSO)	1.9304e-09	1.9304e-09	1.9304e-09	1.9304e-09	1.9304e-09	1.9304e-09
L_∞[LOOCV]	7.3508e-04	2.2319e-03	3.0503e-03	7.8870e-04	1.33359e-04	5.9699e-06
L_{rms}[LOOCV]	5.8391e-05	2.4955e-04	8.8015e-04	1.9422e-03	3.3972e-03	6.8676e-03
L_∞ [116]	4.7416e-05	4.7416e-05	4.7416e-05	4.7416e-05	4.7416e-05	4.7416e-05
L_{rms}[116]	2.1960e-07	2.1960e-07	2.1960e-07	2.1960e-07	2.1960e-07	2.1960e-07
ϵ [PSO]	0.151415	0.145865	0.142986	0.146731	0.132874	0.149728

Table 3.3. Comparison of Absolute errors obtained by PSO with CMRBF for problem 3.1 at N=51with Δt=0.01

X	**T= 1**	**T=3**	**T=5**	**T=1[114]**	**T=3[114]**	**T=5[114]**
-10	4.0769e-05	8.6880e-05	7.0745e-05	4.48071e-05	1.00337e-04	8.83064e-05
-8	1.3216e-06	2.7232e-05	1.3646e-05	1.51021e-06	1.37322e-05	1.70647e-05
-6	4.4200e-08	5.8937e-07	2.0265e-06	1.81798e-07	8.18689e-07	2.20027e-06
-4	6.6391e-08	7.8740e-07	5.8374e-07	2.24975e-07	7.21170e-07	1.12395e-06
-2	5.8187e-08	7.1326e-07	2.4114e-07	1.88839e-07	6.54491e-07	7.42538e-07
0	4.0492e-08	5.9973e-07	2.7734e-07	1.20045e-07	5.71240e-07	6.71247e-07
2	2.9207e-08	4.7270e-07	1.3389e-07	6.34033e-08	4.83999e-07	8.57893e-07
4	1.8830e-08	4.6145e-07	1.9818e-06	1.99497e-08	2.59708e-07	1.92061e-06
6	8.0430e-09	6.7002e-07	7.8785e-06	2.59799e-09	2.40972e-06	1.67456e-05
8	2.4613e-07	5.3948e-06	2.1770e-05	6.87074e-07	2.41819e-05	1.06304e-04
10	0	0	0	2.38731e-05	2.04314e-04	6.16341e-04

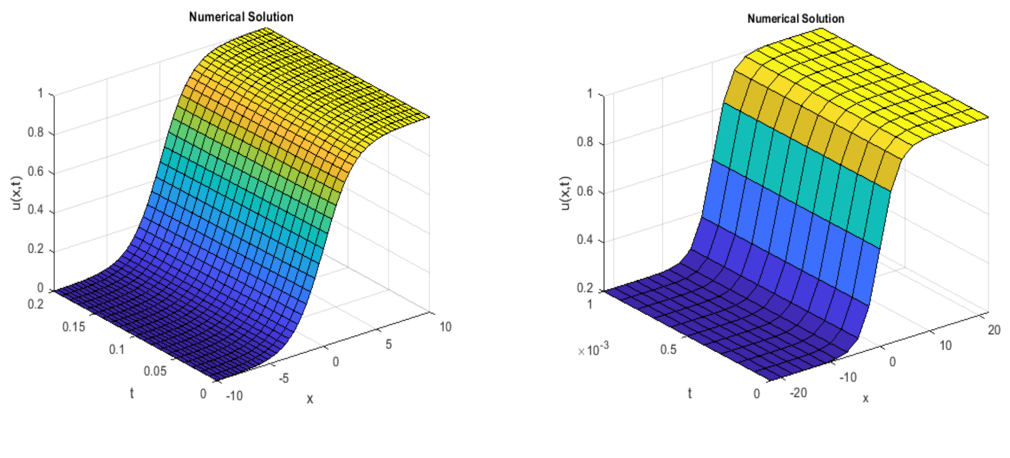

Figure 3.7. Numerical simulation of problem 3.1 with the square domain using (a) PSO and (b) LOOCV with CMRBF at N=51 and Δt=0.01

Table 3.4. Change in ϵ w.r.t. iterations at Δt =0.001, N= 51 and Δt = 0.01

No. of iterations	ϵ [PSO]
50	0.109641
40	0.109685
30	0.109705
25	0.109803
20	0.110075
15	0.110000
10	0.109246

Problem 3.2: Consider Newell-whitehead equation *i.e.*, special type of Fitzhugh-Nagumo equation (1) with ζ=−1.Its exact solution consider from [114] as

$$u(x,t) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{-1}{2\sqrt{2}}\left(x - \frac{3}{\sqrt{2}}t\right)\right); \tag{3.14}$$

With initial conditions as follows:

$$u(x,0) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{-x}{2\sqrt{2}}\right); \tag{3.15}$$

and boundary conditions can be extracted from the exact solution. For problem 3.2, Table 3.5 represents different errors which are comparable at different time intervals as 0.001, 0.002 and 0.003. The comparison of L_∞, L_2 and L_{RMS} errors are calculated at different time intervals with the results available in Arora & Bhatia [114] and also the numerical solution at N=21 are presented graphically by figure 3.8. The optimal shape parameter is 0.430749 calculated by PSO approach using CMRBF.

Table 3.6 represents the absolute errors that are better in comparison to the results available in [114] with different time intervals at 30 iterations and table 3.7 represented the change in parameter with change in the number of iterations.

Table 3.5. Error norms for problem 3.2 by CMRBF with PSO at $\zeta=-1$ and N=21.

T	L_∞	L_2	L_{RMS}	L_∞	L_2	L_{RMS}
		Present			**[114]**	
0.001	3.4700e-08	7.5722e-09	1.6524e-09	8.7088e-09	2.5076e-08	1.9004e-09
0.002	9.5593e-08	2.0860e-08	4.5521e-09	6.7952e-08	2.3994e-08	5.2359e-09
0.003	1.8119e-07	3.9540e-08	8.6283e-09	4.5460e-08	1.2648e-07	9.9202e-09

Table 3.6 Comparison of Absolute errors of problem 3.2 with PSO using CMRBF at N=21, Δt =0.001 and different time intervalwith iterations=30

X	**T=0.001**	**T=0.003**	**T=0.005**	**T=0.001**	**T=0.003**	**T=0.005**
					[114]	
0.1	2.3453e-12	1.3128e-10	1.0719e-09	1.03163e-10	6.85246e-09	3.14338e-08
0.2	7.8523e-12	3.8308e-12	6.2074e-11	2.22230e-11	1.34916e-10	1.86795e-09
0.3	1.0841e-12	3.4382e-12	6.1728e-13	1.45560e-11	7.38998e-12	5.84540e-11
0.4	9.2548e-13	2.9533e-12	8.3944e-13	9.02201e-12	4.85301e-12	4.81504e-12
0.5	1.8135e-13	7.9059e-13	6.6802e-13	4.64904e-12	3.17024e-13	5.10036e-13
0.6	8.9817e-13	3.4044e-12	9.1288e-13	5.31902e-12	1.55098e-12	2.19197e-12
0.7	4.5323e-12	3.0034e-12	7.9333e-11	2.97230e-11	1.38730e-11	7.51030e-11
0.8	1.8814e-11	1.7699e-10	2.5874e-09	3.07350e-11	1.67773e-10	2.30897e-09
0.9	1.5875e-10	9.5357e-09	4.3737e-08	1.32494e-10	8.45538e-09	3.89404e-08

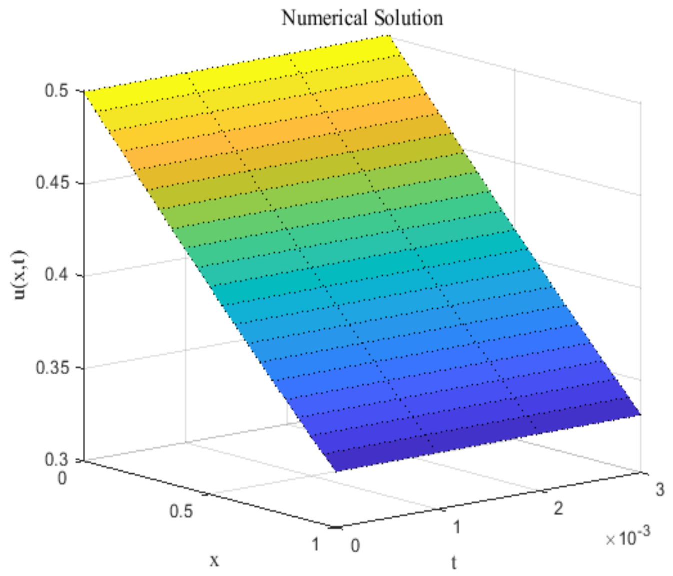

Figure 3.8. Numerical simulation of problem 3.2 with PSO using CMRBF at N=21

Table 3.7 Change in shape parameter values *w.r.t.* no. of iterations at N=21, $\Delta \boldsymbol{t}$ = 0.001, t= 0.002 with MQ-RBF.

Iterations	**ϵ[PSO]**
50	0.165351
40	0.169200
30	0.167671
25	0.167829
20	0.164215
15	0.173928
10	0.167713

Problem 3.3 The non-linear general Fitzhugh-Nagumo equation (3.1) of dimension one in the domain [-22, 22] with exact solution [114]

$$u(x,t) = \frac{1}{2}(1+\zeta) + \frac{1}{2}(1-\zeta)\tanh\left(\sqrt{2}(1-\zeta)\frac{x}{4} + \left(\frac{1-\zeta^2}{4}\right)t\right);\ \mathrm{t} \geq 0 \quad (3.16)$$

and boundary conditions extracted from the exact solution. Initial condition is as follows:

$$u(x,0) = \frac{1}{2}(1+\zeta) + \frac{1}{2}(1-\zeta)\tanh\left(\sqrt{2}(1-\zeta)\frac{x}{4}\right) \quad (3.17)$$

A numerical solution of the general FN equation is obtained at ζ=0.2, Δt = 0.0001 at various time intervals. TheL_∞, L_2 and RMS errors are calculated by the LOOCV approach with CMRBF. The obtained results in the form of L_∞, L_2 and L_{rms} are represented by Table 3.8 with computational time at ϵ = 0.100041. The stability of the obtained matrix is derived for N=21 having the real part of eigen values are obtained as zero which is represented by figure 3.9. Hence, the proposed numerical approach is stable as the derived results are accurate and reliable with minimum error. The numerical approximation of the problem 3.3 is demonstrated by figure 10 at N=21, $\Delta t = 0.0001$.

Table 3.8. Error norms with LOOCV at $\Delta \mathbf{t} = \mathbf{0.0001}$, a=0.2, N=71

T	L_∞	L_2	L_{RMS}	Elapsed time
0.001	2.0073e-09	2.1485e-09	2.5498e-10	0.201111
0.002	3.9593e-09	4.3141e-09	5.1200e-10	0.2524459
0.003	5.8559e-09	6.4969e-09	7.7104e-10	0.264105

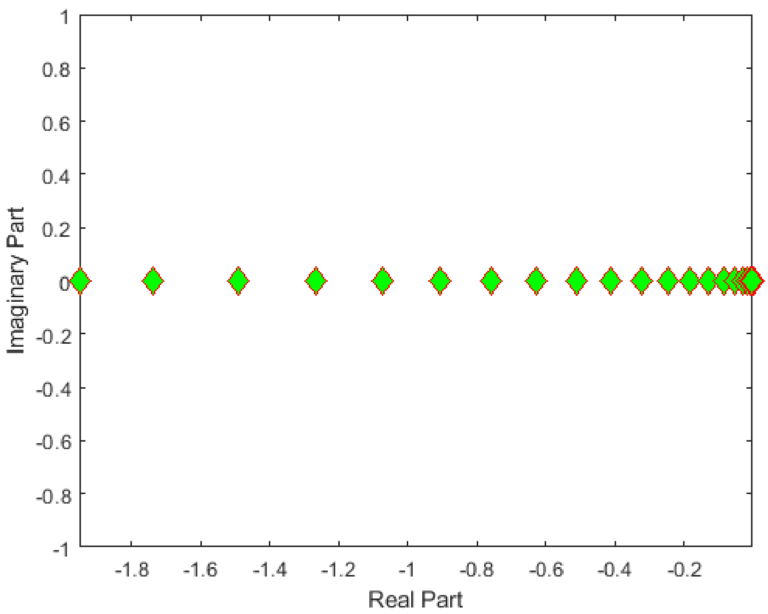

Figure 3.9. Eigen values of obtained matrix by LOOCV at N=21, $\Delta \mathbf{t} = \mathbf{0.0001}$

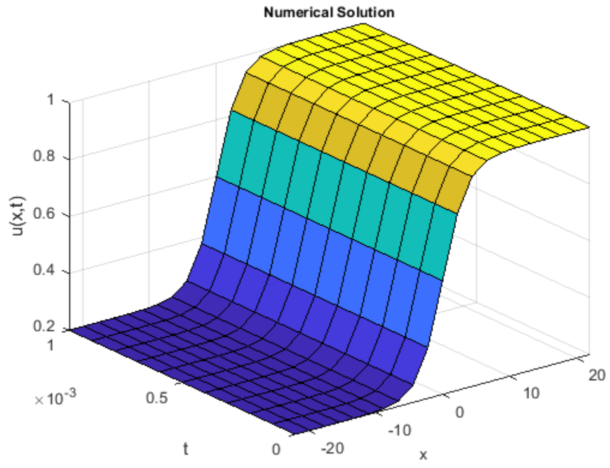

Figure 3.10. Numerical solution of problem 3.3 with N=21 at $\Delta t=0.0001$

3.6 Summary and Conclusion

In this chapter, an inventive hybrid approach is proposed for finding the numerical simulation of one of the partial differential equations by integrating the RBF with a pseudo-spectral method. The numerical implementation of RBF-PS with PSO and LOOCV approach for FN equation is also presented and optimizing the RBF's parameter by taking the errors as an objective function. PSO optimization algorithm and LOOCV approach are the most demanding technique used to find the numerical solutions of the differential equations.

In the proposed hybrid approach, the derivatives of the equation are evaluated, transforming the problem equation into a series of ODEs. Then the converted equations are solved using the ODE solver 45 in Matlab R2021a at Dell PC with processor Intel Core i5. The problems of FN equation are numerically analyzed based on the solutions obtained by CMRBF and MQ RBF. The obtained results by CMRBF are in good comparison with respect to the less node points and the results derived by MQ RBF are in good accuracy than the results available in the literature which demonstrates the correctness and efficacy of the present unique procedure. Results attained are good in comparison to other strategies that are available in the literature and are also equivalent to other strategies. By the analysis of the derived results, PSO is the best optimizer to give higher accuracy compared to the available numerical approaches in the literature.

Chapter 4

Numerical Solution of Fisher’s Equation using RBF-PS Method with ABC and PSO

Part of this chapter is

1. Published in a paper entitled "A Comparative Study of Particle Swarm Optimization and Artificial Bee Colony Algorithm for Numerical Analysis of Fisher’s Equation" in *Discrete Dynamics in Nature and Society*, vol. 2023, Article ID 9964744, 10 pages, 2023. https://doi.org/10.1155/2023/9964744.
2. Published in a paper: Bala, Kiran, Geeta Arora, Homan Emadifar, and Masoumeh Khademi. "Applications of particle swarm optimization for numerical simulation of Fisher’s equation using RBF." *Alexandria Engineering Journal* 84 (2023): 316-322.

4.1. Introduction

Partial differential equations are implemented to model most of the physical phenomena in science and engineering. Various researchers have developed different methods to obtain the numerical solutions of the PDEs with the optimized results in various disciplines. According to some researchers, Yagmahan and Yenisey [117], Kennedy and Eberhart [73], Price et al. [118], and Vesterstrom and Thomsen [119] considered that algorithms based on swarm intelligence have great potential in the field of numerical optimization. Swarm intelligence-based algorithms [120] and evolution [121] are two significant categories of population-based algorithms in the field of optimization. The primary goal of optimization is to find an optimal or nearly ideal solution with the least amount of computing work. The key to find the solution is to optimize parameters connected to a mathematical model. Several research fields adopt optimization approaches for numerical simulation of various linear and non-linear partial differential equations (PDEs) and also for optimizing the parameters related to problematic models. There are some well-known approaches of optimization such as Ant Colony Optimization (ACO) [112] is one of the optimization algorithms that are a meta-heuristic algorithm inspired by the foraging behaviour of ants and how they find the shortest path between their nest and a food source. While it is commonly used for combinatorial optimization problems, it can also be adapted for numerical solutions of PDEs. Particle Swarm Optimization (PSO) [73] is also a heuristic algorithm inspired by the social behaviour of birds and fish, where individuals (particles) in a group cooperate and communicate to find optimal solutions to a problem. Bacteria Foraging Optimization (BFO) [79] is stimulated by the foraging behaviour of Escherichia coli (E. coli) bacteria that mimics the way bacteria forage for nutrients in their environment to find the optimal solution for a given optimization problem. When adapting BFO for the numerical solution of PDEs, it can effectively explore the solution space for parameter settings that yield accurate and efficient numerical solutions to PDEs. These nature-inspired meta-heuristic optimization algorithms have recently gained popularity for developing an effective search algorithm.

Exploration and exploitation are two major determinants for the development of successful optimization algorithms for search mechanisms. A meta-heuristic optimization algorithm effectively explores the solution space, balancing between exploration (global exploration) and exploitation (local refinement) to find near-optimal or optimal solutions for a variety of optimization problems. Exploration involves the search for new, unexplored regions of the solution space. It aims to discover potential solutions that might be superior to the current ones. Exploitation involves focusing on known promising regions of the solution space to improve the quality of solutions. Its aim to refine and optimize the current solutions based on the information available. Researchers are motivated to develop such population-based optimization algorithms because of the abundance of natural resources. These population-based optimization methods assess fitness and provide almost perfect solutions to complex optimization problems.

Swarm intelligence (SI) is a field of study inspired by the collective behaviour of social insect colonies and other animal societies. It explores the principles and models of behaviour that emerge from the interactions of simple individuals within a group. The connection between SI and optimization lies in leveraging the collective behaviour observed in natural swarms to create effective optimization algorithms and strategies. According to Bonabeau et al. [123], SI uses social insect behaviour to create algorithms or distributed problem-solving tools. Bonabeau studied only social insects like termites, bees, wasps, and ants. Social species first developed swarm intelligence through trial and error. It simulates self-organizing swarms of interacting agents. An immune system, ant colony, or bird flock are swarm systems. Bees swarming around their hives illustrate swarm intelligence. Based on honey bee swarm social intelligence, the Artificial Bee Colony (ABC) algorithm first described by Karaboga [90] in 2005, and in 1995 [89], Kennedy and Russell proposed PSO for solving numerical optimization problems. The search mechanism based on the search for nutritious food influenced the procedure of ABC, and the process of PSO was attracted by the behaviour of social animals. These population-based stochastic search methods are simple and fast to compute the optimized results. These methods also solve complex, continuous, and unbounded optimization problems with multi-modal or uni-modal issues.

SI-based meta-heuristic algorithms are popular for solving various optimization models as [124] employed a novel adaptive artificial bee colony (A-ABC) algorithm for selecting the best search equation based on the current situation in order to more precisely predict the transport energy demand (TED), [125] four different meta heuristic algorithms used for natural gas demand forecasting based on meteorological indicators in Turkey, [126] proposed a new modified artificial bee colony (M-ABC) method that can more precisely calculate Turkey's energy usage by adaptively choosing an optimal search equation and many more examples are there in biology, physics, evolution, and human behaviour that inspire nature-inspired algorithms including ant colony optimization (ACO), artificial bee colony (abc), the firefly method, particle swarm optimization (PSO), brain storm optimization, sine and cosine algorithms, and genetic algorithms. With inspiration from SI, many researchers applied these meta-heuristic algorithms for the numerical simulation of various PDEs by optimizing the solution space.

RBF has been proven to be a useful basis function for numerically simulating the ODEs and the PDEs with optimization techniques. Numerical solutions to a nonlinear reaction diffusion equation are found in this study by employing a mesh-free method based on RBF with ABC and PSO optimization techniques. Both optimization strategies are used to determine the shape parameter (ϵ) related to RBF.

The reaction-diffusion equation is one of the most intriguing equations in physical phenomena. The present work concentrates on the form of reaction-diffusion, which is known as Fisher’s equation defines below on the domain $(-\infty, \infty)$ with initial and boundary conditions.

$$v_t - s v_{xx} = F(v) \tag{4.1}$$

$$v(x,0) = v_0(x), \tag{4.2}$$

Where $v_0(x) \in [0,1]$

$$\lim_{x\to-\infty} v(x,t) = 1,\ \lim_{x\to\infty} v(x,t) = 0 \tag{4.3}$$

$$\lim_{x\to\pm\infty} v(x,t) = 0 \tag{4.4}$$

Here, conditions (4.2) and (4.3) together are generally known as non-local conditions and the combined form of conditions (4.2) and (4.4) are recognised as local conditions. Many chemical and biological processes use $F(v) = v(1 - v)$. Fisher [127] introduced this equation to demonstrate a beneficent gene's kinetic advance rate. Fisher's equation shows population evolution through opposing physical phenomena. Fisher's equation dominates genetics, tissue engineering, growth models, and more in science and engineering. Fisher's equation was first simulated using the pseudo-spectral method developed by Gazdag and Canosa in 1974 [128]. Since then, many different approaches have been developed to solve the Fisher’s equation, such as the Petrov- Galerkin finite element method processed by Tang and Weber [129], the Tanh method by Wazwaz [130], the homotopy analysis method proposed by Tan et al. [131], etc. Other methods that have been used to solve this problem include the alternating iterative method by Evans and Sahimi [132], the central finite difference algorithm by Hagstrom and Keller [133], the explicit and implicit finite difference algorithms by Parekh and Puri [134], the collocation of cubic B-splines by Mittal and Arora [135], and the differential quadrature method by Bhatia and Arora [136]. Some researchers have applied various computational techniques for finding the numerical solutions of FE, which include fourth-order collocation method using function of B-spline, the hyperbolic B-spline-based differential quadrature method, the finite difference method, collocation methods and Galerkin methods with quintic B-spline [11, 12, 137-142].

In this chapter, the ABC and PSO algorithms with RBF-PS method are implemented to solve the Fisher’s equation by finding the best shape parameter of RBF to minimizing the error. The RBF-PS method is used for numerical simulation of Fisher’s equation by converting it into an ordinary differential equations (ODEs) system. MATLAB is used for optimizing the parameter ε and for numerical approximation of the Fisher’s equation. In this work, the results are obtained by the present hybrid approach in the form of error norms-L_∞, L_2, L_{rms} and shape parameter values at different time intervals, which are more comparable to the results available in the literature. Last section represents the description of proposed method with numerical applications of the Fisher's equation and the chapter end with the conclusion of the details of key findings.

4.2. Artificial Bee Colony (ABC) Algorithm for the Numerical Scheme

The artificial bee colony (ABC) algorithm was invented by Karaboga [90] in 2005.The algorithm seeks the optimal solution from the search space. In the ABC algorithm, particles are categorised as: employed bees (busy), onlookers, and scouts. The swarm that finds the best food supply (optimum solution) is more likely to be followed by the others. The complete process of the ABC optimization technique has been discussed in chapter 2.

Integrating the ABC optimization technique with RBF for the numerical simulation of PDEs is an effective hybrid approach in the research field. The RBF model is employed for approximating the solutions of PDEs based upon the chosen parameters. The effectiveness of this proposed approach depends upon appropriately defining the objective function, selection of the suitable RBF, and employed the parameters for both ABC and RBF. The ABC algorithm explores the search space for finding the best solution that minimizes the error associated with the solution of the PDE. A general outline of the proposed hybrid approach for numerical simulation of PDEs is as follows:

- Initialize a set of parameters for the RBF model.
- Evaluate the objective function which is the error function related to the PDE using the RBF with the current parameters.
- Update the best solution, considering the fitness associated with the objective function.
- Enter into employed bees phase in which evaluation of a new solution is done with respect to the current solution and the objective function.
- In onlooker bee phase, a new solution is chosen through the ideas of employed bee phase and evaluates the objective function for the new solutions.
- In scout bee phase, a random search is done to discover a new solution and then evaluates the objective function for the new solution.
- Update the global best solution.

- Repeat the process until a stopping criterion is met or for a specified iterations.
- Return the best solution as a final output.

The RBF based method with ABC optimization technique is fruitful for approximating the solutions of PDEs based on the chosen parameters. The effectiveness of this proposed approach depends upon appropriately defining the objective function, selection of the suitable RBF, and employed the parameters for both ABC and RBF.

4.3. Particle Swarm Algorithm (PSO) for Numerical Scheme

Kennedy and Eberhart [89] invented PSO as a popular swarm intelligence technique in 1995. PSO is effective technique in solving optimization problems by changing the paths of particles. Particle movement is mostly stochastic and deterministic. This method finds the best solution among all possible solutions. This process continues until the set iteration or the goal is not met. As the whole process of this algorithm has studied in the chapter 2 and procedure with proposed method is discussed in chapter 3. PSO is a computational technique that iteratively optimizes a problem to reduce the error. It is a statistical method used to determine parameter values. To find the optimal answer to an optimization problem, the particles communicate, share their knowledge, and follow a simple rule. It is an innovative method for evaluating the best shape parameter value of RBF using the non-linear Fisher's equation and a global search optimization strategy which offers numerous characteristics in the parameter space.

4.4. Description of the Proposed Approach with Numerical Applications

In this section the numerical solution of Fisher's equation using the RBF- PS method is obtained for the applications of the above novel approach. Two problems of the Fisher's equation are solved numerically using the present approach, and their results are calculated using the different error norms: L_∞, L_2, L_{rms}, and absolute errors along with the shape parameter values. A comparison of the obtained results is presented for the effectiveness and applicability of the proposed method. Firstly, derivatives are approximated using RBF, and then solutions are determined by MATLAB software with version R2022a using both the algorithms with the intel (R) Pentium processor and the window 7 operating system. The cubic radial matern basis is taken as a basis function for the numerical simulation of the equation. The initial parameters for both the algorithms are as follows: number of decision variables= 1, values of lower and upper bounds of decision variables are 0 and 1, respectively. Same set of values are used for obtaining shape parameter that minimize the errors in both the problems of Fisher's equation. The process of the proposed approach is shown graphically in the Figure 1.

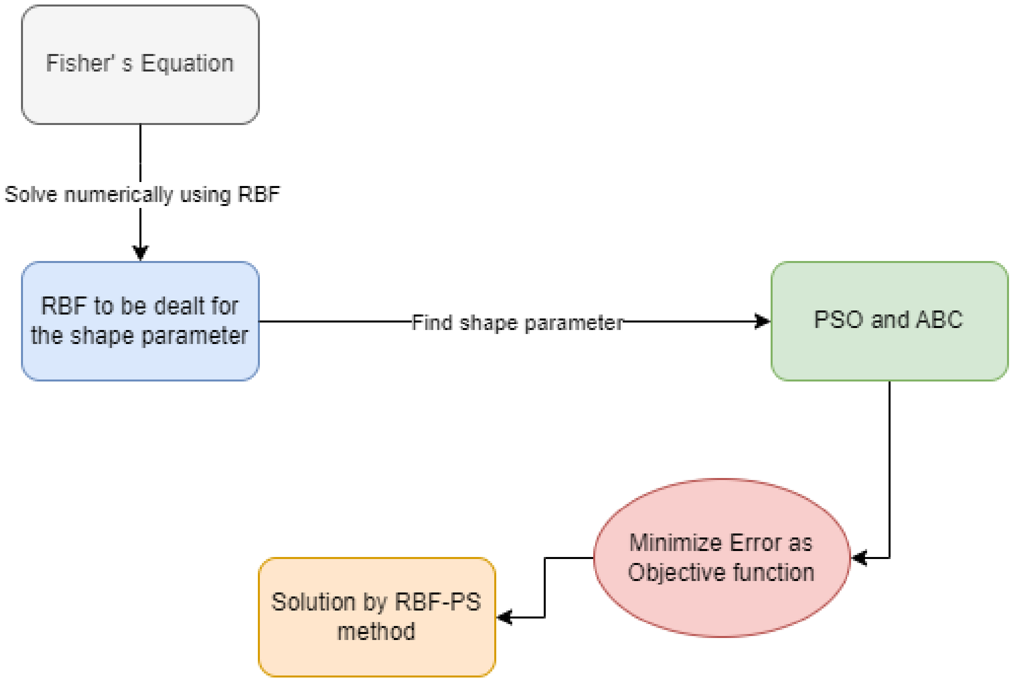

Figure 4.1. Graphical representation of the proposed approach

Problem 4.1: Taking one dimensional Fisher's equation (4.1) with$F(v) = v^2(1 - v)$; on the domain [0,1] with exact solution is considered from [143]

$$v(x,t) = \frac{1}{2} - \frac{1}{2}\tanh\left(a + \frac{\sqrt{2}}{4}\left(x - \frac{1}{\sqrt{2}}t\right)\right); \qquad (4.5)$$

Numerical simulation of the problem 4.1 is done by taking Δt=0.0001, N=21 at time interval 0.2, 0.5, 1 with boundary and initial conditions that are derived from (4.5). Table 4.1 represents the computed error normsL_∞, $L_2, L_{rms}$by ABC and PSO algorithm and comparison is done with results in literature [143, 144]. It can be seen from the results that the errors are less by PSO algorithm comparative to the ABC's approach. The obtained results in Table 4.1 are compared to the results calculated by other numerical methods available in literature. By optimizing the errors, shape parameter resulted in a value0.018062 using PSO and 0.065821 using ABC at which the best error occurs. Table 4.2 presents the values of shape parameter at different T for Δt=0.0001 and N=21. Table 4.3 shows the comparative analysis of the absolute errors using both the algorithms at N = 21 and Δt = 0.0001 with time 0.01, 0.02, 0.03 which concluded the errors are less by PSO algorithm in comparison to the results derived by ABC technique. Figure 4.2 demonstrate the graphical solution for 21 node points with Δt=0.0001 on domain [0, 1].

Table 4.1. Comparison of error norms of problem 4.1with Δt=0.0001 & N=21 on varied T and iteration=71

Time (T)	**Methods**	$\mathbf{L_\infty}$	$\mathbf{L_2}$	$\mathbf{L_{rms}}$
0.2	ABC	8.4869e-05	1.1884e-05	1.6441e-06
	PSO	1.2639e-05	2.7580e-06	6.0185e-07
	[143]	2.1050e–05	4.4750e–03	3.6330e–03
	[144]	1.0500e-06	7.0220e-07	1.6040e-07
0.5	ABC	7.4960e-03	1.6358e-03	3.5695e-04
	PSO	1.2542e-04	2.7369e-05	5.9725e-06
	[143]	1.7360e–05	4.4440e–03	3.8210e–03
	[144]	1.1240e-06	8.4700e-07	1.8480e-07
1.0	ABC	3.0199e-03	6.5900e-04	1.4381e-04
	PSO	1.3587e-04	2.9649e-05	6.4700e-06
	[143]	1.1030e–05	2.6900e–03	2.4640e–03
	[144]	1.2030e-06	9.4990e-07	2.1540e-07

Table 4.2. $\boldsymbol{\epsilon}$ values of problem 4.1 with different T at Δt=0.0001 and N=21

Time (T)	ϵ	
	ABC	PSO
0.2	0.065821	0.065599
0.5	0.077052	0.291291
1.0	0.094752	0.018062

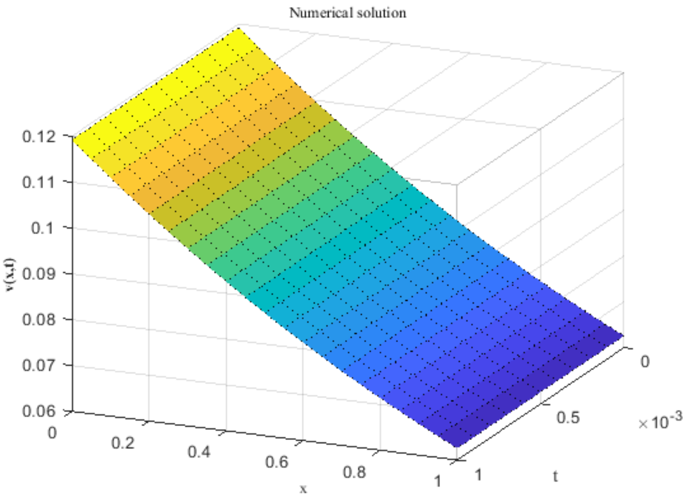

Figure 4.2. Numerical simulation of problem 4.1 with N=21, Δt = 0.0001 and Δt≤0.001

Table 4.3. Comparison of absolute errors of problem 4.1at N = 11, Δt = 0.0001

X	T 0.01	0.02	0.03	0.01	0.02	0.03
		ABC			PSO	
0.1	7.9581e-07	7.1076e-06	1.2974e-05	1.0359e-08	7.9851e-09	3.3186e-10
0.2	7.2611e-07	1.8189e-06	2.9827e-06	3.2868e-09	1.4958e-08	7.5032e-08
0.3	1.8184e-07	1.5313e-06	2.2719e-06	6.3746e-09	4.0134e-09	4.8178e-09
0.4	5.9365e-08	3.5776e-06	6.3239e-06	3.0546e-09	1.3058e-08	5.8104e-08
0.5	8.8167e-09	5.1369e-06	8.6968e-06	7.1879e-09	1.8792e-09	6.1799e-08
0.6	2.5911e-08	5.4938e-06	8.5364e-06	3.4203e-09	1.3243e-08	5.1268e-08
0.7	4.1140e-08	4.9160e-06	8.4974e-06	1.1938e-10	1.1625e-08	2.1215e-08
0.8	4.3687e-07	2.9209e-06	6.4768e-06	6.5021e-09	3.6290e-08	8.7129e-08
0.9	1.4408e-06	6.2425e-07	1.8456e-06	1.6344e-08	5.7144e-08	8.0607e-08

Problem 4.2: Consider the general Fisher's equation (4.1) with$F(v) = v(1 - v^a)$ with domain [0, 1], whose derived exact solution [145] is as follows:

$$v(x,t) = \left(\frac{1}{2} + \frac{1}{2}\tanh\left(-\frac{a}{2\sqrt{2a+4}}\left(x - \frac{a+4}{\sqrt{2a+4}}t\right)\right)\right)^{\frac{2}{a}}; \tag{4.6}$$

Initial and boundary conditions are derived from equation (4.6). Table 4.4 represents the comparison of the results obtained by ABC and PSO algorithm with results in literature [145] for a =1 with Δt= 0.0001 at N=21 along with the exact solution. Table 4.5 shows the comparison of different error norms calculated by proposed approach using both the algorithms atΔt=0.00001, N=21with various time 0.001, 0.002, 0.003, 0.004and iteration=71 that are similar to the available results in the literature [20]. Table 4.6 represents the analysis of the shape parameter values which shows PSO algorithm as best optimizer with minimum error at optimized shape parameter value 0.251490.Table 4.7 presents the comparative analysis of absolute errors of the problem using ABC algorithm with [20] at different time levels which seems better than the available results in the literature and the optimized shape parameter value is 0.246477. Numerical solution is presented in figure 4.3 for N=21, Δt = 0.0001 at various time intervals.

Table 4.4 Comparison of numerical solution of problem 4.2 at N=21 &Δt=0.0001 with time T.

X	T	Numerical Solution			Exact Solution
		PSO	ABC	[145]	
0.25	0.5	0.3341	0.3301	0.3341	0.3340
	1	0.4559	0.4554	0.4557	0.4557
	2	0.6842	0.6786	0.6839	0.6839
0.5	0.5	0.3057	0.3034	0.3057	0.3057
	1	0.4255	0.4248	0.4255	0.4255
	2	0.6582	0.6536	0.6592	0.6592
0.75	0.5	0.2782	0.2769	0.2783	0.2783
	1	0.3949	0.3944	0.3954	0.3954
	2	0.6308	0.6271	0.6333	0.6333

Table 4.5. Comparative study of error norms atΔt=0.00001 and N=21 with time interval at iteration=71

Methods	T	$\mathbf{L_\infty}$	$\mathbf{L_2}$	$\mathbf{L_{rms}}$
ABC	0.001	3.54e-10	1.06e-10	3.22e-11
PSO	0.001	1.44e-11	3.15e-12	6.87e-13
[20]	0.001	4.13e-11	1.45e-08	3.16e-09
ABC	0.002	1.23e-09	3.72e-10	1.12e-10
PSO	0.002	4.18e-11	9.13e-12	1.99e-12
[20]	0.002	2.97e-11	3.98e-08	8.68e-09
ABC	0.003	2.26e-09	6.81e-10	2.05e-10
PSO	0.003	8.01e-11	1.74e-11	3.81e-12
[20]	0.003	1.00e-11	7.53e-08	1.64e-08
ABC	0.004	2.80e-09	8.44e-10	2.54e-10
PSO	0.004	1.01e-10	2.20e-11	4.81e-12
[20]	0.004	9.43e-12	1.21e-07	2.63e-08

Table 4.6. Values of shape parameter values (ε) with time intervals at Δt=0.00001 and N=21

T	**Parameter (ε) values**	
	ABC	PSO
0.001	0.253516	0.253874
0.002	0.267221	0.251490
0.003	0.263530	0.253324
0.004	0.272190	0.253231

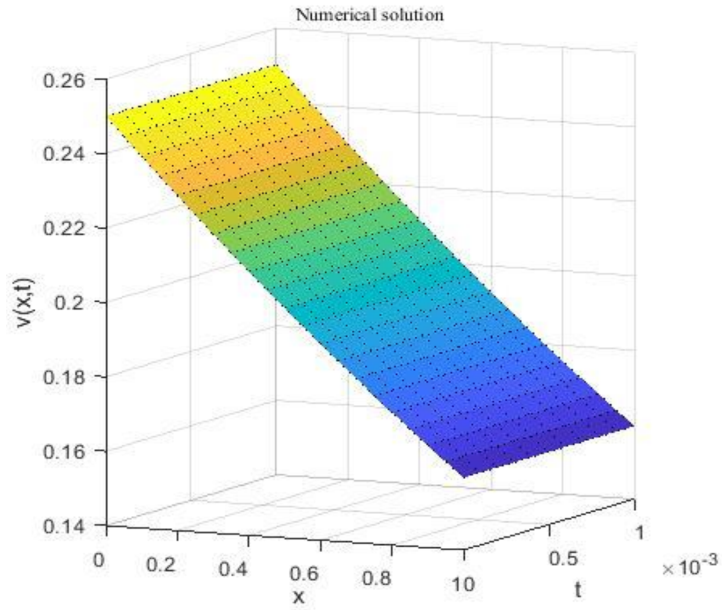

Figure 4.3 Numerical simulation of problem 4.2 with N=21, Δt = 0.0001 and $\Delta t \leq 0.001$

Table 4.7.Comparison of absolute errors of problem 4.2 with N = 11, Δt = 0.0001 and a = 1 at different T and iteration=20.

X	ABC			[20]		
	T 0.001	0.002	0.003	0.001	0.002	0.003
0.1	6.1583e-11	1.7297e-09	2.3772e-09	2.684e-10	4.563e-09	1.798e-08
0.2	1.4237e-11	1.0123e-09	1.4069e-10	2.930e-11	3.683e-11	3.384e-10
0.3	7.1848e-12	4.8205e-10	5.2623e-11	3.477e-11	2.240e-11	1.830e-11
0.4	6.4626e-12	2.3021e-10	1.9678e-11	6.720e-12	2.199e-12	2.511e-12
0.5	7.7925e-12	1.1580e-10	2.5276e-12	4.654e-12	3.591e-12	2.555e-12
0.6	1.1981e-11	7.5733e-11	2.4497e-12	1.134e-11	1.051e-11	1.001e-11
0.7	2.2177e-11	8.3356e-11	1.3942e-12	1.974e-12	2.707e-12	3.189e-12
0.8	5.0164e-11	1.4345e-10	6.0104e-12	2.685e-11	1.646e-11	1.819e-11
0.9	1.8208e-10	1.6411e-10	2.3064e-10	2.758e-11	4.511e-10	1.749e-09

Using the current hybrid approach, two Fisher's equation is solved numerically, and the results are derived in terms of various error norms, including absolute errors and shape parameter values. The comparison of the obtained results is performed and presented to test the efficacy and application of the present novel approach.

4.5. Summary and Conclusion

In this chapter, a novel hybrid technique is proposed for computing the numerical solution of the Fisher's equation using PSO and ABC optimization algorithms with the RBF-PS method. Using PSO and ABC optimization algorithms, the concept of ideal shape parameters is proposed because there is a discrepancy between numerical stability and accuracy when using different radial basis functions. To show the accuracy and efficiency of the specific method, two problems are solved numerically. Based on the error norms and the shape parameter values, the obtained results are compared with the results available in the literature and the comparison of two optimization techniques is presented. The errors obtained by PSO are lower compared to the errors obtained by the ABC algorithm, thus PSO values of shape parameter are good in comparison to the ABC's results. Also the obtained results are more accurate in comparison to the results available in the literature. Form the results it can be concluded that PSO gives more accurate results compared to the ABC algorithm in terms of less error.

Chapter 5

An Overview of Fractional Differential Equation and Description with RBF

5.1. Introduction

Fractional calculus is currently quite popular, and it has a wide area of applications in various fields since the differential and integral is an operator that contains both the integer-order derivatives and the integrals as special instances. When the order of the integration is ambiguous, the fractional integral can be employed for the better accumulation of a quantity. Similarly, the fractional order derivatives come into force to describe the various physical phenomena at a large scale. Its implication also in the disciplines of viscous-elasticity fluid flow, diffusion-like diffusive transport, control theory of dynamical systems, probability, electrical networks, statistics, dynamical phenomena in conscience and hollow formations, ophthalmology, image analysis. Fractional calculus has become an increasingly useful technique for the description of non-classical events in applied science and the engineering over the past ten years. Many mathematical models in surface and subsurface hydrology [146, 147], finance [148, 149], epidemiology [150, 151], plasma turbulence [152, 153] and ecology [154, 155], have been simulated using fractional-order differential operators.

The study of fractional derivatives and integrals, often known is a subfield of calculus that focuses on the properties of non-integer order derivatives and integrals. Techniques for solving differential equations with fractional order derivatives often known as fractional differential equation (FDEs), is the main focus of the research field. About at the same time as classical calculus was developed, the historical development of fractional calculus commenced. The concept of semi derivative was originally raised in Leibniz's letter to l' Hospital in 1695. Many well-known mathematicians as Laplace (1812), Lacroix (1819) and Fourier (1822), Lagrange (1772), Fourier, Euler (1730) etc. over the years established fractional calculus on theoretical grounds. Even higher orders of fractional derivatives and integrals, left and right derivatives in the fractional form and the fractional integrals are also included in the system of fractional theory.

Since FD models have grown more by the scientific field, numerical approaches for fractional differential equations (FDEs) are being studied. Differential equations of various kinds are frequently used in mathematical representations of real-world issues. Scientists and mathematicians have recently analysed that FDEs are more effective than the classical counterparts at modelling in which any natural or complicated processes found in practical applications. Fractional order derivative of a function requires knowledge of that function from its initial state to its current state but the classical derivatives only need the fact about the function at the certain points and their surroundings. The non-locality of the derivatives in the fractional form makes them a highly effective tool for characterizing the inherited characteristics of numerous materials and processes. A FDE is a differential equation with at least one operator of order 'a' satisfying a<n, where n is a positive integer defined as $\frac{d^a}{dx^a}$.

In the context of nonlinear studies, non-linear PDEs have gained popularity due to address various issues in the fields of ecology, epidemiology and money-making systems, quantum study and image processing. Many physical applications including wave dispersion and propagation, supersonic and turbulent flow, magneto-hydrodynamic movement through pipes, computational fluid dynamics, population modelling, magnetic resonance imaging, medical imaging, electrically signalling nerves, and many more frequently make use of PDEs [156–158]. Take a look at the source referenced in [159] to learn more. An accurate assessment of COVID patients in [160, 161] had proved prevalent nature of PDEs. As shown in [162], PDE can be used to estimate the resultant form of COVID-19. By this analysis, the FPDE gives more accuracy comparatively the results of the integer-order PDE for a number of exigent problems in these fields. So this seems to be very crucial for developing computational techniques for finding the results of various fractional partial differential equations (FPDEs).

Symmetry, a central notion in the context of mathematics and physic, is also one of the ways to find out the solutions of PDEs. An accurate or approximate solution can be obtained by adopting symmetry for finding the solutions of FPDE solutions, which are widely used to simplify mathematical models [163, 164]. It can be used in the fractional PDE to reduce the number of factors that are independent, which will facilitate solution. Additionally, it can also be applicable for locating the results that are unchanged under particular transformations like translations, scaling, and rotations. Resultant directed the conclusions which are easier to comprehend and have more tangible effects. Symmetry is a useful tool to deal with fractional PDEs on a wider scale and can be highly useful in describing the underlying physical processes. Some of the references [165,164] are mentioned here for providing the general idea of FDEs with uses. [167, 168] provide a thorough study of fractional calculus, FDEs and their uses in several fields. The fundamental concept of FDEs and its practical implications have been studied in relation to visco-elasticity [169, 170].

A large number of FDEs are challenging to solve analytically because of the intricacy of fractional orders. Therefore, numerical simulation for various FDEs is investigating by various researchers. Numerous algorithms for numerical computation have been developed for this purpose [171, 172]. Spectrum methods are numerical techniques for approximating the outcome of differential equations. There are several spectrum approaches that have been applied to the solution of FDEs, including the tau, collocation, and Galerkin methods. The tau technique is a spectral approach for approximating the solution of FDEs that makes use of a shifted Legendre polynomial basis. This collocation method, which is spectral in nature, approximates the results of FDEs with collocation regions using a basis of orthogonal polynomials. The efficiency of spectral approaches for solving FDEs has been shown in numerous publications [173–179]. In general, time-fractional differential equations and FDEs can be solved using spectrum techniques. These techniques are excellent tools for modelling and simulation in a variety of domains because they deliver precise solutions with high efficiency.

The applications of FPDEs to simulate complicated multi-scale phenomena with overlapped microscopic and macroscopic aspects has become becoming more and more apparent. PDEs of fractional order can be a function of time as well as space or smooth distribution with respect to the PDEs of integer order. Many well-known academicians have made contributions in the field of FPDEs because of its significance in engineering and research. Numerous approaches have been addressed for finding numerical results of FPDEs which includes numerical method for time fractional phi four equations [180], the reduced differential-transform method for coupled time fractional non-linear evolution equations [181], the Yang transform decomposition method for diffusion equation of fractional order [182], the natural transform decomposition method for the solution of fractional Caudrey Dodd Gibbon equations [183] and fractional Kuramoto Sivashinsky equations [184], fractional homotopy-analysis method for solving the fractional epidemic model [185] and fractional KdV Burgers Kuramoto equation [186], Homotopy perturbation transform method for solving fractional Noyes Field model [187] and time fractional Fisher's equation [188], variational iteration transform method for fractional ordered Newell Whitehead Segel equations [189] and for fractional ordered Boussinesq equation [190], approximate analytical method for the solution of time fractional telegraph equations [191] and the Adams Bashforth method to study the time fractional Tricomi equation with non-local and non-singular kernel [192].

This chapter focus on the literature review of fractional differential equations and their applicability with various RBFs. Therefore, for a class of fractional differential equations, naturally a mesh-free radial function based global techniques are provided in our study. Global RBF techniques offer a greater level of convergence and accuracy than traditional spectral methods. Furthermore, it is typically easy to expand RBF schemes to larger dimensions. The next sections give some basic calculus theory like formulas of the fractional integrals and derivatives which is useful for the derivation of PDEs and presents the methods available in literature for solving FDEs. The last section concludes with the analysis of RBF discretization with FDEs.

5.2. Basics of Fractional Calculus

Differential equations are solved by some rules that are based on some special functions. Here we mentioned some basic functions for fractional calculus.

5.2.1. Gamma functions

Base function in the fractional theory is an Euler's gamma function which is defined

i) For natural numbers as $\Gamma(n) = (n-1)!$
ii) For a complex no. X, $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt$ that is absolutely convergent and also known Euler's integral equation of second kind. Also

$$\Gamma(\mathfrak{z}+1) = \mathfrak{z}\Gamma(\mathfrak{z})$$

iii) For $-r < Re(\mathfrak{z}) \leq -r+1$,

$\Gamma(\mathfrak{z}) = \frac{\Gamma(\mathfrak{z}+r)}{\mathfrak{z}(\mathfrak{z}+1)\ldots(\mathfrak{z}+r-1)}$

Its derivative calculated as

$$\frac{d^n}{dx^n}\Gamma(x) = \int_0^\infty e^{-t}\,(\ln t)^n\, t^{x-1}\,dt, x > 0$$

5.2.2. Beta function

$$\mathcal{B}(\mathrm{x},\mathrm{m}) = \int_0^1 (1-t)^{m-1} t^{x-1}\,dt = \frac{(m-1)!\,(x-1)!}{(m+x-1)!} = \mathcal{B}(\mathrm{m},\mathrm{x});\ x, m \in \mathbb{R}^+$$

Relation of Beta function and Gamma function defined by $\mathcal{B}(x,m) = \frac{\Gamma(\mathrm{x})\Gamma(\mathrm{m})}{\Gamma(\mathrm{x}+\mathrm{m})}$

5.2.3. Mittag Lefler (ML) Function

Exponential function has a vital role in the field of fractional calculus. So, ML function containing one and two parameter that are totally based on exponential function (e^x or $\exp(x)$) defined by

$$E_\beta(x) = \sum_{r=0}^{\infty} \frac{x^r}{\Gamma(\beta r+1)}, \beta > 0$$

And

$E_{\beta,\gamma}(x) = \sum_{r=0}^{\infty} \frac{x^r}{\Gamma(\beta r+\gamma)};\ \beta > 0, \gamma > 0$ respectively.

5.2.4. Caputo Fractional Derivative (CFD)

CFD of a function with an order $\beta > 0$ in [a, b] is defined with $m \in Z^+; m-1 < \beta \leq m$

i) Left hand sided CFD

${}_{\mathrm{C}}D^{\beta}_{a^+} f(x) = \frac{1}{\Gamma(m-\beta)} \int_0^x (x-t)^{m-\beta-1} f^m(t)dt$

ii) Right hand sided CFD

${}_{C}D_{b^-}^{\beta} f(x) = \frac{1}{\Gamma(m-\beta)} \int_0^x (t-x)^{m-\beta-1} f^m(t)dt,$ respectively.

Interpolation in Caputo operator is analysed by

$$\lim_{\beta \to m} D_a^{\beta} f(x) = f^m(x)$$

$$\lim_{\beta \to m-1} D_a^{\beta} f(x) = f^{m-1}(x) - f^{m-1}(a)$$

Caputo operator satisfied the condition of Linearity but not the commutative law which are defined by

$$D_a^{\beta}[pf(x) + qg(x)] = pD_a^{\beta} f(x) + qD_a^{\beta} f(x)$$

$$D_a^{\beta} D^m f(x) = D_a^{m+\beta} f(x) \neq D^m D_a^{\beta}$$

5.2.5. Riemann- Liouville (RL) Operators

a) Riemann- Liouville Fractional Integral (RLFI)

RLFI of left hand side and right hand sided of order β in [a, b] are defined below:

$$\left(I_{a^+}^{\beta} f\right)(x) = \frac{1}{\Gamma(\beta)} \int_a^x (x-t)^{\beta-1} f(t)dt$$

$$\left(I_{b^-}^{\beta} f\right)(x) = \frac{1}{\Gamma(\beta)} \int_a^x (t-x)^{\beta-1} f(t)dt$$

Following properties satisfied by the fractional integration:

1. $I_{a^+}^{\beta} I_{a^+}^{\gamma} f = I_{a^+}^{\beta+\gamma} f, \quad I_{b^-}^{\beta} I_{b^-}^{\gamma} f = I_{b^-}^{\beta+\gamma} f;\ \beta > 0, \gamma > 0$
2. $I_{a^+}^{\beta}[pf(x) + qg(x)] = pI_{a^+}^{\beta} f(x) + qI_{a^+}^{\beta} g(x);$
3. $I_{b^-}^{\beta}[pf(x) + qg(x)] = pI_{b^-}^{\beta} f(x) + qI_{b^-}^{\beta} g(x)$

b) Riemann-Liouville Fractional Derivative (RLFD)

Left and right hand side RLFD of a function f of order β is defined on [a, b] as

$$D_{a^+}^{\beta} f(x) = \frac{1}{\Gamma(m-\beta)} \frac{d^m}{dx^m} \int_a^x (x-t)^{m-\beta-1} f(t)dt$$

and

$$D_{b^-}^{\beta} f(x) = \frac{(-1)^m}{\Gamma(m-\beta)} \frac{d^m}{dx^m} \int_x^b (t-x)^{m-\beta-1} f(t)dt$$

Or we can say that

$$D_{a^+}^{\beta} f = D^m I_a^{m-\beta} f$$

1. Interpolation in RLFD is defined by $\lim\limits_{\beta \to m} D_a^{\beta} f(x) = D^m I_a^{m-\beta} f$
2. RLFD satisfies the property of Linearity but not hold commutative law.
3. Following properties satisfied by the fractional integration

1. $D_{a^+}^{\beta} I_{a^+}^{\beta} f(x) = f(x)$ *and* $D_{b^-}^{\beta} I_{b^-}^{\beta} f(x) = f(x);\ \beta > 0$
2. $D^m D_{a^+}^{\beta} f(x) = D_{a^+}^{m+\beta} f(x)$ *and* $D^m D_{b^-}^{\beta} f(x) = D_{b^-}^{m+\beta} f(x)$ $(if\ FDs\ exist)$

Relation between RLFD and CFD

- ${}_{C}D_{a^+}^{\beta} f(x) = D_{a^+}^{\beta} f(x) - \sum_{r=0}^{m-1} \frac{f^r(a)}{\Gamma(r+1-\beta)} (x-a)^{r-\beta}$
- ${}_{C}D_{b^-}^{\beta} f(x) = D_{b^-}^{\beta} f(x) - \sum_{r=0}^{m-1} \frac{(-1)^r f^r(b)}{\Gamma(r+1-\beta)} (b-x)^{r-\beta}$

5.2.6. Riesz-space fractional derivative (RSFD)

RSFD of order β of function f(x,t) on [a, b] is defined by

$\frac{\partial^{\beta}}{\partial |X|^{\beta}} f(x,t) = -C_{\beta}(left\ sided\ RLFD + Right\ sided\ RLFD),$ where

$$C_{\beta} = \frac{1}{2\cos\left(\frac{\pi\beta}{2}\right)}\ , \beta \neq odd\ numbers$$

The entire operators (defined above) applicable on function with some properties (according to the requirement) that are defined below.

1. $D^{\beta}_{a^+}f(x) = D^m[\, I^{m-\beta}_{a^+} f(x)]$
2. $D^{\beta}_{b^-}f(x) = (-1)^m D^m[\, I^{m-\beta}_{b^-} f(x)]$
3. Left sided CFD $= \ I^{m-\beta}_{a^+}[f^m(x)]$
4. Right sided CFD $= \ (-1)^m I^{m-\beta}_{b^-}[f^m(x)]$

5.3. Methods and Applications of Fractional Differential Equation

Since fractional derivatives are clearly non local operators, even basic models or interconnected theories have higher computational cost in the forms of space, processing period, and general convolution of numerical techniques. As a result, these local schemes lose their efficacy as the coefficient matrices are no longer sparse. Therefore, due to their high precision and use of fewer discretization points, global approaches seem to offer a number of advantages for finding the numerical solutions of the fractional models. Even though many of global approaches are precise and have higher orders of convergence, but expansion to high dimensional systems can be time-consuming and can necessitate proper method modification. There are two different categories of numerical approaches for PDE models. The first is the mesh-free approach, and the second is the mesh-grid method. Researchers are currently paying more attention to mesh-free approaches for numerical simulation of various PDEs, particularly of the fractional orders.

Mesh-free methods need evenly distributed nodes in the domain because meshing is not necessary for them. Mesh-free methods are more accurate than mesh-grid methods and take less computational time to complete the process. Several well-known mesh-free techniques include homotopy analysis [193], variational iteration method [194], spectral method [195], adomian decomposition method [196], and RBF approaches [197, 198]. With initial and boundary value problems, meshfree methods are simple to implement, and there are a variety of generating functions available in the literature. Both local and global forms of RBF functions are employed. The basis functions for implementing the RBF method includes Gaussian, multi-quadric, inverse multi-quadric, quadric, and so on. The primary challenge with RBF functions is to select the appropriate shape parameter for finding the better numerical solution. Finding the numerical solutions can usually be difficult in high dimensional model.

Here, we mentioned some mesh-less approaches available in the literature. Certain techniques with RBF also demonstrate that one can drive the numerical solutions to various FPDEs by employing collocation methods with RBF with unevenly distributed domain points. Also time-space dependent PDE [199] of fraction order is solved by mesh-less RBF method with Gaussian function has applications in physical sciences and chemical engineering. Authors prove that these approaches are highly accurate in high dimensional models in Caputo sense for time dependent derivatives and Riemann-Liouville, Grunwald-Letnikov and Riesz derivatives for space dependent derivatives.

A numerical approach with radial functions is proposed in [200] using Coimbra theory of fractional derivatives. In [201] reveal a scheme for solving time fractional diffusion equations numerically by mesh free method locally with RBF using Laplace transform that was used for dealing with fractional derivative. If the error decreases or stays constant over the computational domain, a time-stepping technique is stable.

5.4. RBF Discretization of Fractional Operators

RBF is a univariate real valued continuous function that completely lies on the concept of distance, which is measured from any fixed center point or the origin. It is a special function which is radially symmetric and only depends on the distance between node points. The applications of RBF to the high dimensional problem are easy to interpolate compared to the interpolation problems that are insensitive to the space dimension. In all space dimensions, one can work with the univariate function instead of multivariate function. This part demonstrates the types of RBFs that are distinguished by the smoothness- piecewise smooth RBFs which are free from shape parameter ϵ and infinitely differentiable which have parameters called the shape parameterϵ. Some of the piecewise smooth and infinitely smooth RBFs are Cubic function, linear radial function monomial and Gaussian function, multi- quadric, inverse multi-quadric etc. as discussed in the chapter 1.

5.4.1. Discretization of Fractional Operators

Interpolation of RBF in 1-dim on the centre node points$\mathbb{x}_j$, j varies from 1 to m, of the form is

$$S_{\sigma}(x) = \sum_{j=1}^{m} \gamma_j \varphi\left(\frac{|\mathbb{x}-\mathbb{x}_j|}{\epsilon}\right) \tag{5.1}$$

with interpolation coefficients γ_j. Here, $\varphi$is one of the available radial basis functions and ϵ is the shape parameter connected to RBF. These interpolation coefficients $\gamma_j$are unknown and their value can be found by interpolating the following conditions at the collocation points $\mathbb{x}_\mathrm{i}$.

$$S_{\sigma}(\mathbb{x}_i) = \zeta_i, i = 1,2, \dots, m \tag{5.2}$$

This leads to a representation of a system of linear equations as follows:

$$\mathbb{A}\mathcal{X} = \mathfrak{B}; \text{ where } \mathbb{A} \text{ is non-singular} \tag{5.3}$$

$$\mathbb{A} = \left(\varphi\left(\frac{|\mathbb{x}_i - \mathbb{x}_j|}{\epsilon}\right)\right)_{1\le i,j \le m}$$

$$\mathcal{X} = [\gamma_{1,}\ \gamma_2\ , \dots, \gamma_m\ ]^T$$

$$\mathfrak{B} = [\zeta_1, \zeta_2, \dots, \zeta_m]^T$$

By taking a differential operator D^{α}(one of the operators defined above) whose discretization is done with expansion of radial basis function. Resultant is

$$\mathbb{g}_i = \sum_{j=1}^{m} \gamma_j \left(D^{\alpha}\varphi\left(\frac{|\mathbb{x}-\mathbb{x}_j|}{\epsilon}\right)\right)(\mathbb{x}_i),\ 1 \le i \le m \tag{5.4}$$

where $\mathbb{g}_i$ are fractional operator of function at each point. After applying D^{α}, obtained results gives new system of equations as follows:

$$\mathbb{A}'\mathcal{X} = \mathfrak{B}' \tag{5.5}$$

Where $\mathbb{A}' = \left(D^{\alpha}\varphi\left(\frac{|\mathbb{x}-\mathbb{x}_j|}{\epsilon}\right)\right)$

$\mathfrak{B}' = [\mathbb{g}_1, \mathbb{g}_2, \dots, \mathbb{g}_m]^T$.

Equation (5.3) gives $\mathcal{X} = \mathbb{A}^{-1}\mathcal{B}$ by the non singularity of $\mathbb{A}$. So, eliminating $\mathcal{X}$ from equation (5.5), we get

$$\mathfrak{B}' = \mathbb{A}'\mathbb{A}^{-1}\mathfrak{B} \tag{5.6}$$

Here, $\mathbb{A}'\mathbb{A}^{-1}$ is a matrix which discretize D^{α}.

Hence for finding the expression $\left(D^{\alpha}\varphi\left(\frac{|\cdot - \mathbb{x}_t|}{\epsilon}\right)\right)$; Define a function for arbitrary no. t, $\varphi_t : \mathbb{R} \to \mathbb{R}$ such that

$$\varphi_t(x) = \varphi\left(\frac{|\mathbb{x} - t|}{\epsilon}\right)$$

Then evaluate $(D^{\alpha}\varphi_t)(x)$ for getting desire results by (5.6).

5.5. Summary and Conclusion

This chapter provides the depth analysis of the fundamentals of fractional differential equations with RBFs. Also, discretization of fractional operators with RBF is proposed and the numerical methods are proposed future research work. Also, computational approach with RBF is also defined for the successful implementation of scientific and engineering applications in the real world.

Chapter 6

Summary and Future Scope

6.1. Summary

Partial differential equations (PDEs) have significant applications in solving nonlinear phenomena in various fields, including the mathematical sciences and engineering. Developing numerical methods for solving these PDEs is a fundamental part of computational mathematics and scientific research. In the present study, the methods based on radial basis function (RBFs) are employed for the numerical simulation of the various PDEs. RBF approaches have appealing characteristics such as mesh free nature, simplicity in implementation, high convergence rate of the approximated solutions, adaptability to irregular geometry, and integration with existing methods. Due to the aforementioned characteristics, RBF approaches have several advantages over the existing meshed based method. RBF was initially employed as a successful method for multivariate approximation of scattered data. PDEs can be solved by using various RBF techniques that have been developed over time. RBF techniques are coupled with the existing techniques for obtaining the best numerical results with optimal shape parameter values.

The main objectives of the thesis work is to explore the numerical advancement of the RBF methods and the execution of RBF-PS method combining with the optimization techniques for finding the numerical solutions of various PDEs. This combination leverages the strengths of the method, achieving high accuracy and reducing computational cost with potentially providing greater geometric flexibility. RBFs are known for ability to approximate functions with high accuracy, and they can be particularly useful for scattered or irregular data. The shape parameter has a significant place in forming the shapes of the RBFs and for the effectiveness of the RBF methods. The accuracy of the RBF-PS method depends on selection of the node points, type of radial basis function, position of the centre points and the shape parameter. Additionally, it has been noted that accuracy and stability are consistently at conflict when the RBF is put into practice. The interpolating matrix becomes ill-conditioned when the shape parameter is used with a greater value i.e., increasing the interpolation matrix's condition number exponentially. It is important to build the balance between accuracy and stability during the selection of the shape parameter's value. Therefore, determining the optimal value of shape parameter is a demanding area of interest to researchers. The chapters' summaries are:

The first chapter starts with the fundamentals of PDEs and RBFs. Also, discussed about the behaviour of various types of RBFs and the development of the approximation methods using RBF those are available in the literature to solve the various PDEs. Various shape parameter strategies based on literature are also studied.

The chapter 2 based on the various optimization techniques and mathematical implementation of the one of the RBF methods: RBF-PS method that is useful for finding the optimized shape parameter value and also for finding the numerical solutions of various problems of PDEs in the thesis work.

In the chapter 3, RBF-PS method is employed for finding the approximate solutions of the one of the non-linear partial differential equation naming Fitzhugh- Nagumo equation. Optimization technique PSO and LOOCV approach are employed for optimizing the unknown parameter of RBF with considering the error as an objective function. This unknown parameter is shape parameter that forms the shape of different RBF. The last section is concluded with the process of hybrid approach based on the combination of RBF-PS method with particle swarm optimization technique and LOOCV approach and with the implementation of RBF-PS method for Fitzhugh- Nigumo equation with their numerical application.

The chapter 4 presented the numerical treatment of the Fisher’s equation by employing RBF-PS method with PSO and ABC optimization techniques that are employed for testing the accuracy of the obtained results by the optimized RBF shape parameter. And the comparison of the derived results of the Fisher’s equation using ABC and PSO is presented in the form of error norms.

The chapter 5 investigates the fundamentals of fractional differential equation (FDE) with some basic definition of fractional calculus theory and the process of discretization of the fractional operators with RBFs is also done. The methods available in the literature for finding the numerical solutions of the various fractional differential equation and their applications is also discussed. In the present work, proposed hybrid approach is employed for the numerical solutions of various FDEs.

6.2. Future Scope

The following recommendations will be put into practice in future work:

1) Explore the possibility of the RBF-PS method with other well-known optimization techniques for finding the numerical solutions of various PDEs with optimized shape parameter.

2) Examine the applicability of RBF-PS method in irregular domains.

3) Explore the possibility of the RBF-PS method with two or more basis functions.

4) Expanding the proposed method with real world applications.

5) Explore the possibility of the proposed method for finding the solutions of various fractional partial differential equations.

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LIST OF PUBLICATIONS

1. Geeta Arora, Kiran Bala, Homan Emadifar, Masoumeh Khademi, "A Comparative Study of Particle Swarm Optimization and Artificial Bee Colony Algorithm for Numerical Analysis of Fisher's Equation", *Discrete Dynamics in Nature and Society*, vol. 2023, Article ID 9964744, 10 pages, 2023. https://doi.org/10.1155/2023/9964744.

   (SCOPUS, Impact Factor- 1.4, SJR 2022-0.26, Q3, CiteScore- 2, ISSN: 1026-0226 (Print); 1607-887X (Online)

2. Bala, Kiran, Geeta Arora, Homan Emadifar, and Masoumeh Khademi. "Applications of particle swarm optimization for numerical simulation of Fisher's equation using RBF." *Alexandria Engineering Journal* 84 (2023): 316-322. DOI: 10.1016/j.aej.2023.11.024

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3. Arora, Geeta, Kiran Bala, Homan Emadifar, and Masoumeh Khademi. "A review of radial basis function with applications explored." *Journal of the Egyptian Mathematical Society* 31, no. 1 (2023): 6.

4. "Radial Basis Function with Artificial Bee Colony Algorithm for Numerical Simulation of Fisher's Equation" **accepted** for AIP conference proceedings.

5. A Construction Factor-based PSO form the RBF shape parameter for Fitzhugh-Nagumo Simulation (communicated in Scopus indexed journal).

6. Exploring applications of Fractional Differential Equations with Radial Basis Function (Communicated in Afrika Mathematica- Scopus Indexed).

7. Exploring Optimization Technique: Genetic Algorithm (Accepted by **ICCS-23** conference held in LPU).

LIST of BOOK CHAPTERS

1. Geeta A. & Kiran B. (2023). “Radial Basis Function: A Meshless tool for the Numerical Simulation” IJRTS Takshila Foundation; I S B N : 9 7 8 - 8 1 - 9 6 2 1 3 4 - 0 - 4

2. Kiran Bala, and Geeta Arora (2023) “Unleashing the Power of Nature-Inspired Optimization: Exploring Particle Swarm Optimization and Artificial Bee Colony Algorithms” – **Accepted** for Book chapter by CRC publication.

LIST of ATTENDED CONFERENCES

1. Presented a paper titled with “A Review of Radial Basis Function with Application Explored” in 4^{th} International conference on Frontiers in Industrial & Applied Mathematics organized by Dept. of Mathematics, SLIET Longowal, during 21-22 Dec. 2021.

2. Presented a paper “Applications of PSO for Numerical Simulation of Fisher’s Equation using Radial Basis Function” in the International Conference on Evolution in Pure & Applied Mathematics (ICEPAM-2022), 16-18 November 2022, Akal University Talwandi Sabo, Bathinda, Punjab.

3. Presented a paper titled with “Radial Basis Function: A Meshless tool for the Numerical Simulation” in National Seminar on Recent Development in Mathematics organized by Dept. of Mathematics, Govt. College, Hisar on February 20, 2023.

4. Presented a paper “A Construction Factor-based PSO form the RBF shape parameter for Fitzhugh-Nagumo Simulation” in Recent Advances in Fundamental and Applied sciences (RAFAS-23) on 20 April,2023.

5. Presented a paper entitled “Exploring Optimization Technique: Genetic Algorithm” in the 7th International Joint Conference on Computing Sciences (ICCS-2023) “KILBY100” held on 5th May, 2023 organized by School of Computer Science and Engineering, LPU in association with Southern Federal University, Russia and Mizan Tepi University, Ethiopia at Lovely Professional University, Punjab, India.