STUDY OF LIGHTLIKE SUBMANIFOLDS OF SEMI-RIEMANNIAN MANIFOLDS

Thesis Submitted for the Award of the Degree of

DOCTOR OF PHILOSOPHY

In

MATHEMATICS

By

EJAZ SABIR LONE

11900873

Supervised By

Dr. PANKAJ PANDEY

Lovely Professional University

PUNJAB

2022

Declaration of Authorship

I, EJAZ SABIR LONE, declare that this thesis entitled, " STUDY OF LIGHTLIKE SUBMANIFOLDS OF SEMI-RIEMANNIAN MANIFOLDS" and the work presented in it are my own. I confirm that:

- This work was done wholly while in candidature for a research degree at this University.
- Where any part of this project has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this project is entirely my own work.
- I have acknowledged all main sources of help.
- Where the project is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed:

Author: EJAZ SABIR LONE
Registration No.: 11900873
Date: July 2022

Certificate From Supervisor

This is to certify that **Mr. EJAZ SABIR LONE**, has completed the thesis entitled **"STUDY OF LIGHTLIKE SUBMANIFOLDS OF SEMI-RIEMANNIAN MANIFOLDS"** under my guidance and supervision. To the best of my knowledge, the present work is the result of his original investigation and study. No part of this thesis has ever been submitted for any other degree at any university and elsewhere.

The thesis is fit for the submission and the partial fulfilment of the conditions for the award of **DOCTOR OF PHILOSOPHY,** in Mathematics.

Pankaj Pandey

Signed:

Supervisor: Dr. PANKAJ PANDEY

Date: July 2022

Acknowledgements

First of all I bow in reverence to almighty "**God**" the cherisher and sustainer to benediction give me the required zeal for completion of my research work. The successful completion of any research work is a matter of great endeavourance but, because of the personal and practical support of numerous people makes the job entirely enjoyable and simple. It is a pleasant aspect that I have now the opportunity to express my gratitude for them.

First and foremost, I would like to express my deep sense of gratitude and everlasting indebtedness to my supervisor Dr. PANKAJ PANDEY, Assistant Professor, Department of Mathematics, School of Chemical Engineering and Physical Sciences, Lovely Professional University, Phagwara-Punjab, for his patience, continuous encouragement and guidance throughout my research work. Without his guidance it would not have been possible for me to complete the work. I could not wish a better friendly supervisor.

I feel privileged to my sincere regards and gratitude to Dr. Kulwinder Singh, Head of the Department of Mathematics, for his valuable guidance and constant encouragement throughout the course of my research work. I am also thankful to all the faculty members of the university for their generous support, encouragement and valuable suggestions to improve the present work .

I am greatly indebted my father late Mohammad Sabir Lone and my mother Smt. Sarwa Beagum for their invaluable support and blessings. I must record my sincere thanks to my esteemed sublings Mr. Mushtaq Ahmad Lone, Mr. Athiqullah Lone, Mr. Hilal Ahmad Lone, Mrs. Hussain Ara, Mrs. Aabida Sabir for their steady encouragement, timely support, unconditional love and moral support, without whom I would never have enjoyed so many opportunities.

I particularly thank my colleagues Er. Faicel Mohammad, Dr. Aijaz Afzal and Dr. Farooz Ahmad for their stimulating discussions and friendly atmosphere created during this work. I remember those sleepless nights, we were working together before deadlines and for all the fun we have during this research work. Without their precious support, it would not have been possible to conduct this research. I would like to express my gratitude towards their moral and emotional support that helped me in completion of this thesis and inspired me to work tirelessly to accomplish this task .

Last but not the least I would like to thank my friends Mr. Ajaz Ahmad Pir, Mr. Anees Akber Butt, Mr. Suhail Nisar Ganie and Mr. Ubaid Ahmad Pir who were always there for help, support and share their precious time during my stay in the university.

I also express my gratitude to all who knowingly and unknowingly contributed in completion of this thesis.

July, 2022 **EJAZ SABIR LONE**

List of Symbols and Abbreviations

$\mathbb{R}$- Set of real numbers.

$\mathbb{R}^{\mathbf{n}}$- n-tuples of real numbers.

$\mathbb{R}^{\mathbf{n}}_{\mathbf{q}}$- Semi-Euclidean of constant index q.

$Rad\bar{M}$- Radical subspace of $\bar{M}$.

$S\bar{M}$- Screen subspace of $\bar{M}$.

$W^{\perp}$- Perpendicular subspace of W.

$\bar{M}$- Manifold.

M- Hypersurface of $\bar{M}$.

$T_p\bar{M}$- Tangent space at point p.

$T\bar{M}$- Tangent bundle.

R- Riemannian curvature tensor.

$Ric, \quad S$- Ricci curvature tensor.

Q- Ricci operator.

r- Scaler curvature tensor.

$K(\pi)$- Sectional curvature tensor of plane π.

$grad\ g$- Gradiant of function g.

H^g- Hessian of g.

L_vF- Lie derivative of F with respect to v .

$[\ ,\ ]$- Lie bracket.

G- Distribution.

$\bar{M} \times \bar{N}$- Product manifold.

$RadT_pM$- Radical subspace of tangent space at p.

$Rad(TM)$- Radical distribution.

$S(TM)$- Screen distribution.

$Ltr(TM)$- Lightlike transversal bundle of M.

$\oplus_{orth}$- Orthogonal direct sum.

$B(\ ,\ )$- Local second fundamental form of M.

h^l- Lightlike second fundamental form.

h^s- Screen transversal second fundamental form.

$A_X F$- Shape operator of M.

Contents

Abstract

The current thesis, entitled "STUDY OF LIGHTLIKE SUBMANIFOLDS OF SEMI-RIEMANNIAN MANIFOLDS" is the result of research outcome conducted by me under the esteemed guidance and supervision of " Dr. PANKAJ PANDEY", Assistant Professor, Department of Mathematics, Lovely Professional University, Phagwara-Punjab. The research work is now being submitted to the Department of Mathematics, School of Chemical Engineering and Physical Sciences, Lovely Professional University, Phagwara-144411, Punjab, India, for the award of "Doctor of Philosophy" in Mathematics.

Differential geometry is the study of spaces equipped with some specified structures like metric, distance, geodesic, connection and curvature up to isometries. In 18th and 19th centuries, differential geometry laid its foundation on the geometry of plane curves, space curves, and surfaces in three dimensional Euclidean spaces. Differential geometry laid its foundation in the early twentieth century as a branch of mathematics concerned with the study of geometric structures on differentiable manifolds. The focus shifted to investigate the local and global properties of smooth manifolds equipped with a metric tensor g of fixed signature on the tangent spaces of manifolds which encodes the geometry.

In the analysis of geometrical structures on differentable manifolds, the difference in the signature of a metric tensor g has fundamental significance. If a differentiable manifold has a positive definite metric tensor, it is termed a Riemannian manifold and if it has a non-degenerate metric tensor g of arbitrary signature, it is called a semi-Riemannian manifold. Hence, the study of differential geometry of Riemannian and semi-Riemannian manifolds is fascinating tool and vital topic for researchers to address. Depending on the casual character of $(0, 2)$-type metric tensor g, semi-Riemannian manifolds have three sorts of curves like spacelike curves, timelike curves, and lightlike (null or degenerate) curves. The geometry of spacelike and timelike curves resembles that of Riemannian curves in many ways, but the geometry of lightlike curves differs from that of non-degenerate curves. In the geometry of Riemannian manifold submanifolds had got a lot of attention and is well explored, but in the geometry of semi-Riemannian manifolds, lightlike submanifolds are relatively recent field of study. In contrast to the classical theory of non-degenerate submanifolds, the tangent bundle and the normal vector bundle intersect non-trivially in the lightlike submanifolds. As a result, the geometry of lightlike submanifolds of semi-Riemannian manifolds becomes more complex

and distinct from the geometry of non-degenerate submanifolds, which in the case of lightlike submanifolds cannot be used to analyse induced geometrical objects.

Since these degenerate submanifolds provide models of several categories of horizons, such as event horizons of the Kruskal and Cauchy's horizons which are helpful in the study of black holes. The theory of lightlike submanifolds has extensive applications in the field of relativity. In addition to the black hole theory, electromagnetism, radiation fields, and asymptotically to spacetime, the geometry of lightlike submanifolds has major applications. We also require the geometry of lightlike submanifolds to close the gap in the general theory of submanifolds. As a result, the huge number of applications of lightlike submanifolds prompted us to study the subject.

The thesis is organized into seven chapters, each of which has multiple sections in which we acquire some new results on semi-Riemannian manifold. We have obtained results, perticularly on lightlike hypersurfaces of semi-Riemannian manifolds and symmetric lightlike hypersurfaces of semi-Riemannian manifolds. The numbers in square brackets [ ] correspond to the end-of-section references. In this thesis sections, subsections, theorems, lemmas, corollaries, and definitions are numbered by using bold decimal notation. The following paragraphs provide a chapter wise summary of the work provided in this thesis.

Chapter 1 covers the fundamental, yet essential definitions and findings of semi-Euclidean space, semi-Riemannian manifolds, lightlike submanifolds and lightlike hypersurfaces of semi-Riemannian manifolds. To build up terminology for chapters 2 to 7, the information gathered or collected from various research publications, books, and review articles is given in this chapter.

In chapter 2, we study the behaviour of parallel vector fields and new type of vector fields on lightlike hypersurface of Lorentzian manifolds. We have obtained some new results on lightlike hypersurfaces by using rigging vector fields. Totally umbalical, screen homothetic , screen conformal and totally geodesic lightlike hypersurfaces are obtained in this chapter. Also Ricci soliton lightlike hypersurfaces of Lorentzian manifolds equipped with parallel vector fields and new type of vector fields are discussed.

The theory of almost generalised weakly symmetric lightlike hypersurfaces of indefnite Kenmotsu manifolds are investigated in chapter 3. We study almost generalised weakly symmetric and almost generalised weakly Ricci symmetric lightlike hypersurfaces of indefinite Kenmotsu manifolds. We showed some sufficient conditions of cyclic parallelism and Codazzi type of Ricci tensor on an almost generalised weakly Ricci symmetric lightlike hypersurface of an indefinite Kenmotsu manifolds. We constructed an example that justifies the existence of almost generalised weakly symmetric lightlike hypersurface of an indefinite Kenmotsu manifolds.

In chapter 4, lightlike hypersurfaces of cosymplectic space form $\bar{M}(c)$ and weakly ϕ-Ricci symmetric lightlike hypersurfaces of cosymplectic manifolds are investigated. We have discussed screen semi-invariant, totally geodesic and totally umbalical lightlike hypersurface of cosymplectic space form $\bar{M}(c)$. Parallelism of local second fundamental form is also investigated. We have obtained η-Einstein lightlike hypersurface of weakly ϕ-symmetric lightlike hypersurface of cosymplectic manifolds. Finally, we study cyclic parallelism and Codazzi type of Ricci tensor on weakly ϕ-Ricci symmetric lightlike hypersurface of cosymplectic manifolds of constant curvature -1.

In chapter 5, the geometry of lightlike hypersurfaces of Sasaki-like statistical space form is discussed. We compute the Guass and Codazzi formulae of degenerate hypersurface of Sasaki-like statistical manifolds. In this chapter, it is shown that if the Sasaki-like statistical manifold is having parallel screen distribution and parallel second fundamental form then we can not obtain lightlike hypersurface of Sasaki-like statistical manifold. Finally, we prove that we can not obtain lightlike hypersurface of Sasaki-like statistical space form having constant curvature $c \neq 1$ and $c \neq 5$.

In chapter 6, we study lightlike hypersurface of semi-Euclidean space Z_q^n. We have discussed weakly semi-pseudo symmetric and weakly semi-pseudo Ricci symmetric lightlike hypersurface of semi-Euclidean spaces. We study totally geodesic weakly semi-pseudo Ricci symmetric lightlike hypersurface of semi-Euclidean space by satisfying cyclic parallism and Codazzi type of Ricci tensor.

In chapter 7, geometry of geodesic lightlike submanifolds of Lorentzian para-Sasakian manifolds is investigated. Invariant lightlike submanifolds, CR-lightlike submanifolds and screen CR-lightlike submanifolds of Lorentzian para-Sasakian manifolds are investigated in this chapter. We study mixed geodesic, totally geodesic and $\bar{G}$ geodesic lightlike submanifolds of LP-Sasakian manifolds.

A bibliography is included at the end of the thesis, which is by no means comprehensive, but does identify all of the research articles and books that were mentioned in the main text.

Chapter 1

Introduction

A brief and concise introduction to Riemannian manifolds, semi-Riemannian manifolds, and semi-Euclidean spaces is given in this chapter. Basic definitions and terminologies used in the following chapters are also presented to understand the study undertaken. The main theme of the thesis is presented along with a short history. Based on the review of literature, the research gap is identified and consequently the objectives of the present work are proposed. The primary goal of this chapter is to present previous knowledge of fundamental topics that will be required in the presentation of the thesis's succeeding chapters. The geometry of lightlike submanifolds and lightlike hypersurfaces of semi-Riemannian manifolds is the focus of this thesis.

1.1 Introduction

The conventional approach to differential geometry is to use differential calculus to investigate how the curves and surfaces are contained in three-dimensional Euclidean space. The careful analysis of the surfaces revealed that some of the characteristics were intrinsic, meaning that they relied only on the surface itself and not on how it was embedded in Euclidean space. Some of the features are dependent on the way the surface is embedded; these are extrinsic surface properties.

The analysis of curves and surfaces in three-dimensional Euclidean space in any dimension and co-dimension, as well as arbitrary ambient spaces is called submanifold theory. The possibility that the space we live in is curved has provided a major drive for geometric innovation. Since then manifolds have been regarded as the central objects because they allow more complicated structures to be defined in terms of the relatively well-understood features of Euclidean space. The concept of a manifold is essential to many disciplines of geometry, mathematics, and physics as solution sets of systems of equations and graphs of functions manifolds emerge spontaneously. The investigation of surfaces in Euclidean space was generalised into Riemannian geometry which inherits much of the intuitiveness of Euclidean geometry.

The Riemannian and semi-Riemannian geometries have been active research topics in the differential geometry of manifolds and have found their applications not only in mathematics but also in every field of science. The study of semi-Riemannian geometry and its submanifolds has yielded a lot of information about the similarities and differences between Riemannian and semi-Riemannian geometries. Many physicists regard the **cosmos** as a 4-dimensional submanifold embedded in a higher-dimensional spacetime manifold. Mathematicians then investigated not only the submanifolds of Riemannian manifolds, but also semi-Riemannian manifolds, inspired by Einstein's general theory of relativity, the Kaluza-Kleins theory, and the string theory.

There are two categories of semi-Riemannian submanifolds, namely the non-degenerate submanifolds and the lightlike (null) submanifolds. In the mid-seventies, attention shifted from the geometry of Riemannian and semi-Riemannian manifolds towards Loren-tzian geometry. Meanwhile, for any semi-Riemannian manifold, lightlike subspaces occur naturally. Therefore, Duggal and Bejancu worked on the geometry of lightlike submanifolds of semi-Riemannian manifolds to fill a central missing portion in the universal theory of submanifolds. The geometry of lightlike submanifolds is quite different and more interesting than that of their counterparts, non-degenerate submanifolds. In quantum physics, string theory, electromagnetism, radiation fields, and asymptotically at space times, the geometry of lightlike submanifolds is playing an increasing role. As a result of the growing importance of lightlike submanifolds in mathematical physics, particularly their general applications in relativity, and the scarcity of knowledge about the general theory of lightlike submanifolds, we decided to conduct research in this area.

1.2 A brief history of manifold

In differential geometry, the oldest geometry that is connected to manifolds is non-Euclidean geometry. The geometry where Euclid's parallel postulate fails is known as non-Euclidean geometry. Euclid's fifth parallel postulate is distinct from the first four postulates. The problem of independence of Euclid's fifth postulate from his first four postulates remained unanswered for over two centuries until it was solved by Bolyai, Lobachevsky, and Gauss in the nineteenth century, laid the foundation for non-Euclidean geometry. Hyperbolic and elliptic geometry are two new types of geometries that result from non-Euclidean geometry.

To begin with, **Gauss** regarded abstract spaces as mathematical objects in and of themselves, and he discussed a method for computing the curvature of a surface without taking into account the ambient space in which the surface lies. The **Theorema Egregium** outlined how to determine the curvature of a surface without taking into account the ambient space. This theorem proved and affirmed that a surface's curvature is an intrinsic feature. A surface like this is referred to as a manifold in current terminology. Manifolds are studied primarily for their intrinsic qualities, while their extrinsic properties are ignored.

The differential calculus was first used in the last quarter of the 18th century to better understand geometry problems, laying the groundwork for differential geometry. Riemann used the German word **"Mannigfaltigkeit"** to describe the manifold, which was later translated as **"manifoldness"** by William Kingdon Clifford. Manifolds are a generalisation of curves and surfaces that are closely related to linear algebra, topology, group theory, and relativity theory.

Bernhard Riemann gave a habilitation lecture in Göttingen in 1854, in which he addressed numerous ways to provide a space of any dimension with a metric. He also described the formal development of coordinate systems and charts. In 1857, Bernhard Riemann was the first who proposed the concept of a Riemannian surface. He also popularised the idea of a surface in higher dimensions. The natural and beneficial improvement of dierential geometry of surfaces in R^3 is Riemannian geometry. The special categories of Riemannian geometry are Euclidean and non-Euclidean geometry. In the field of dierential geometry, the theory of submanifolds is as deep-rooted as dierential geometry itself. Submanifold geometry was once the only a component of Riemannian geometry, but it is now one of many distinct parts of multi-dimensional expansions of the classical theory of surfaces. The geometry of submanifolds begins with the notion of extrinsic geometry of a surface, whereas Riemannian geometry is a natural and successful evolution of Gauss's idea of intrinsic geometry. Hermann Weyl was the first who gave the foundational definition of a differentiable manifold and later Whitney proved that the manifold naturally described by charts could be embedded in a Euclidean space.

1.3 Some Important Definitions

This section provides an overall overview of Riemannian geometry. We begin with some fundamental definitions and formulas on Riemannian manifold. Then we go on to look at semi-Euclidean spaces, lightlike submanifolds and lightlike hypersurfaces of semi-Riemannian manifolds and their basic fundamental features. Informally, an n-dimensional manifold is a "space" that looks like a flat space $\mathbb{R}^n$, when viewed through a microscope.

1.3.1. Manifold or Smooth Manifold ([19],[119])

"A topological space M is said to be a locally Euclidean space if each point of M has a neighbourhood homeomorhic to an open subset of $\mathbb{R}^n$. A topological space M is called an ndimensional smooth manifold if
(i) It is a Hausdorff space.
(ii) It is a second countable space.
(iii) It is a locally Euclidean space of dimension n".

A function $\varphi : U \subset \mathbb{R}^n \to \mathbb{R}$ defined on an open subset U of $\mathbb{R}^n$ is said to be a $\mathbb{C}^r$ function on U if all the partial derivatives of order k, $1 \leq k \leq r$, exist and are continuous. If φ is a $\mathbb{C}^r$ function for every positive integer r then φ is called a $\mathbb{C}^\infty$ function.

Two functions $\varphi : U \subseteq M \to \varphi(U) \subseteq \mathbb{R}^n$ and $\psi : V \subseteq M \to \varphi(V) \subseteq \mathbb{R}^n$ are said to be $\mathbb{C}^r$ related [119] if, for $U \cap V \neq \emptyset$, the maps

$$\begin{aligned} \varphi o \psi^{-1} : \ \psi(U \cap V) \ &\to \ \varphi(U \cap V), \\ and \quad \psi o \varphi^{-1} : \ \varphi(U \cap V) \ &\to \ \psi(U \cap V), \end{aligned} \tag{1.1}$$

are $\mathbb{C}^r$ functions, where U and V are open subsets of M.

1.3.2. Differentiable Manifold ([19],[119])

Let $\varphi : U \subseteq M \to \varphi(U) \subseteq \mathbb{R}^n$ be a homeomorphism from an open subset U of M to an open subset $\varphi(U)$ of $\mathbb{R}^n$ then U is called a coordinate neighbourhood, φ is called a coordinate map and the pair (U, φ) is called a coordinate chart or simply an open chart.

A collection $\zeta = \{(U_\alpha, \varphi_\alpha) : \alpha \in \wedge\}$ of $\mathbb{C}^\infty$ charts on a locally-Euclidean space M is called a $\mathbb{C}^\infty$ atlas, if the following conditions satisfy:
(i) $\cup U_\alpha = M$, $\alpha \in \wedge$ *i.e.* $\{U_\alpha : \alpha \in \wedge\}$ is an open covering of M.

(ii) Every pair of coordinate charts are $\mathbb{C}^\infty$ related *i.e.* the mapping

$$\begin{aligned} &\varphi_\alpha o {\varphi_\beta}^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \rightarrow \varphi_\alpha(U_\alpha \cap U_\beta), \\ &or \quad \varphi_\beta o {\varphi_\alpha}^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \rightarrow \varphi_\beta(U_\alpha \cap U_\beta), \end{aligned} \tag{1.2}$$

is a $\mathbb{C}^\infty$ mapping for all $\alpha, \beta \in \wedge$, where $\wedge$ is an index set.

A maximal C^r atlas is called a differentiable structure of class r or a C^r manifold structure. The pair (M, ζ) consisting of a topological manifold M of dimension n together with a differentiable structure ζ of class r is called an n-dimensional C^r manifold. If the maximal C^r atlas is of class C^∞ then (M, ζ) is called a smooth manifold.

1.3.3. Tangent Vector and Tangent Bundle ([19],[119])

A tangent vector F_p at a point $p \in M$ is a mapping $F_p : \mathbb{C}^\infty(M) \rightarrow \mathbb{R}$ such that $F_p f \in \mathbb{R}$ and satisfying

(i) $F_p(\alpha f + \beta g) = \alpha(F_p f) + \beta(F_p g)$,

(ii) $F_p(fg) = (F_p f)g(p) + f(p)(F_p g)$,

for all $\alpha, \beta \in \mathbb{R}$ and $f, g \in \mathbb{C}^\infty(M)$, the set of all smooth functions on M.

For $p \in M$, if $T_p(M)$ denotes the set of all tangent vectors at point p, then $T_p(M)$ becomes a vector space of dimension n over $\mathbb{R}$ and called a tangent space at $p \in M$. The collection of all tangent spaces $\{T_p(M) : p \in M\}$ is denoted by $T(M)$ and called a tangent bundle of the manifold M.

1.3.4. Cotangent Vector and Cotangent Bundle ([19],[119])

Let $T_p^*(M)$ denotes the dual space of $T_p(M)$ at $p \in M$ *i.e.* $T_p^*(M)$ is the set of all linear functional $\omega : T_p(M) \rightarrow \mathbb{R}$ then ω is called a cotangent vector at $p \in M$ and $T_p^*(M)$ is called a cotangent space at $p \in M$. The collection of all cotangent spaces $\{T_p^*(M) : p \in M\}$ is denoted by $T^*(M)$ and called a cotangent bundle of the manifold M.

1.3.5. Vector Field ([19],[119])

A vector field F on a smooth manifold M is a smooth assignment of a tangent vector $F_p \in T_p(M)$ at each point $p \in M$. A smooth assignment means for all $f \in \mathbb{C}^\infty(M)$, the function $p \mapsto (Ff)(p) = F_p(f)$ is a $\mathbb{C}^\infty$ function. The collection of all vector fields on a manifold M is denoted by $\chi(M)$. A smooth assignment of a cotangent vector field at a point $p \in M$ is known as 1-form *i.e.* a 1-form $\omega \in T_p^*(M)$ is a tensor field of type (0,1).

1.3.6. Lie Bracket ([19],[119])

"A Lie bracket [,] of vector fields F and J is defined by $[F, J](p)(f) = F_p(Jf) - J_p(Ff)$, for all $p \in M, f \in \mathbb{C}^\infty(M)$".

In general, the Lie bracket is represented by $[F, J] = FJ - JF$ and is a smooth tangent vector field on a manifold M. "For $F,\ J,\ L \in \chi(M)$ and $f,\ g \in \mathbb{C}^\infty(M)$ the Lie bracket satisfies the following relations

(i) $[F, J] = -[J, F]$,

(ii) $[F, F] = 0$,

(iii) $[F, J + L] = [F, J] + [F, L]$,

(iv) $[fF, gJ] = fg[F, J] + f(Fg)J - g(Jf)F$,

(v) $[F, [J, L]] + [J, [L, F]] + [L, [F, J]] = 0$, $\quad$ $(Jacobi\ \ identity)$,

(vi) $[F, J](f + g) = [F, J](f) + [F, J](g)$,

(vii) $[F, J](f.g) = f[F, J](g) + g[F, J](f)$."

The Lie bracket $[F, J]$ assigns a linear mapping $L : \chi(M) \times \chi(M) \to \chi(M)$ such that $L(F, J) = L_F J = [F, J]$ and known as Lie derivative of J with respect to F.

1.3.7. Connection ([19],[119])

Let M be an n-dimensional manifold then a bilinear map $D : \chi(M) \times \chi(M) \to \chi(M)$ is said to be an affine connection if it satisfies the following conditions:

(i) $D_{\alpha F+\beta J}L = \alpha D_F L + \beta D_J L$,

(ii) $D_{fF+gJ}L = fD_F L + gD_J L$,

(iii) $D_F(fJ + gL) = fD_F J + (Ff)J + gD_F L + (Fg)L$,

for all $\alpha, \beta \in \mathbb{R}, f, g \in \mathbb{C}^\infty(M)$ and $F, J, L \in \chi(M)$.

A linear connection or the Koszul connection are also reffered as an affine connection [119]. The operation of an affine connection D on F and J is denoted by $D(F, J) = D_F J$, where $D_F J \in \chi(M)$ is called covariant derivative of J with respect to F and D_F is called an operation of covariant differentiation with respect to F.

1.3.8. Covariant Derivative[119]

Let M be an n-dimensional smooth manifold and $K(F_1, ..., F_r, \omega_1, ..., \omega_s)$ is a tensor field of type (r, s) then covariant derivative of a tensor field K with respect to vector field F is a tensor field of type $(r, s + 1)$ and defined by

$$\begin{aligned}(D_F K)(F_1, ..., F_r, \omega_1, ..., \omega_s) &= D_F K(F_1, ..., F_r, \omega_1, ..., \omega_s)\\ &- \sum_{i=1}^{r} K(F_1, ..., F_{i-1}, D_F F_i, F_{i+1}, ..., F_r, \omega_1, ..., \omega_s)\\ &- \sum_{j=1}^{s} K(F_1, ..., F_r, \omega_1, ..., \omega_{j-1}, D_F \omega_j, \omega_{j+1}, ..., \omega_s),\end{aligned} \tag{1.3}$$

for all vector fields $F_i \in \chi(M), \omega_j \in \chi^*(M), i = 1, 2, ..., r$ and $j = 1, 2, ..., s$.
A tensor field K of type (r, s) is said to be covariant constant if

$$(D_F K)(F_1, ..., F_r, \omega_1, ..., \omega_s) = 0. \tag{1.4}$$

1.3.9. Riemannian Metric and Riemannian Manifold [119]

A $(0, 2)$-tensor field g, $g : T_pM \times T_pM \to \mathrm{R}$ is known to be as a Riemannian metric if it satisfies:

(i) $g(F, J) = g(J, F)$,

(ii) $g(F, J) \geq 0$ and $g(F, F) = 0$ if and only if $F = 0$,

for all $F, J \in \Gamma(TM)$. The above conditions (i) and (ii) signifies, that the Riemannian metric g is symmetric and positive definite.

"A connection D is said to be a metric compatible connection or a metric connection if the covariant derivative of Riemannian metric g vanishes otherwise it is called a non-metric connection".

A smooth manifold M equipped with a Riemannian metric g is called a Riemannian manifold. Every n-dimensional Riemannian manifold admits a unique torsion free metric compatible connection.

1.3.10. Torsion Tensor [119]

"Let M be a smooth manifold with an affine connection D. Then a mapping
$T : \chi(M) \times \chi(M) \to \chi(M)$ given by

$$T(F, J) = D_F J - D_J F - [F, J], \tag{1.5}$$

is called a torsion tensor with respect to D, where $F, J \in \chi(M)$".

1.3.11. Riemannian Connection ([19],[119])

Let $\{M, g\}$ be an n-dimensional Riemannian manifold with a Riemannian metric g then an affine connection D on M is called a Levi-Civita connection or a Riemannian connection if

(i) D is a symmetric or torsion free connection,

(ii) D is a metric compatible connection.

1.3.12. Curvature Tensor [119]

Let M be a smooth manifold with an affine connection D. Then a mapping
$R : \chi(M) \times \chi(M) \times \chi(M) \to \chi(M)$ defined by

$$R(F, J)L = D_F D_J L - D_J D_F L - D_{[F,J]} L, \tag{1.6}$$

is a tensor field of type (1,3) and called a Riemannian curvature tensor, for all $F, J, L \in \chi(M)$.

If g be a Riemannian metric then a tensor $'R(F, J, L, Y)$ of type (0,4) defined by

$$'R(F, J, L, Y) = g(R(F, J)L, Y), \tag{1.7}$$

is called an associated Riemannian curvature tensor [119] and satisfies the following properties:

$$'R(F, J, L, Y) = -'R(J, F, L, Y) = -'R(F, J, Y, L) =' R(L, Y, F, J), \tag{1.8}$$

$$'R(F, J, L, Y) +' R(J, L, F, Y) +' R(L, F, J, Y) = 0, \tag{1.9}$$

$$(D_F\, 'R)(J, L, Y, V) + (D_J\, 'R)(L, F, Y, V) + (D_L\, 'R)(F, J, Y, V) = 0. \tag{1.10}$$

1.3.13. Ricci and Scalar Curvature Tensor [119]

"A Ricci tensor Ric or S is a tensor field of type (0,2) and defined by

$$Ric(F, J) = S(F, J) = Trace\{L \to R(F, J)L\}, \tag{1.11}$$

for all $F, J, L \in \chi(M)$".

"A scalar curvature r is a scalar function [119] defined by

$$r = Trace(Q), \tag{1.12}$$

where Q is a tensor field of type (1,1) and called Ricci operator defined by $S(F, J) = g(QF, J)$".

Let $\{e_1, e_2, ..., e_n\}$ be an orthonormal basis of a tangent space $T_p(M), p \in M$ then
(i) The Ricci tensor S [119] is defined by

$$S(J, L) = \sum_{i=1}^{n} R(e_i, J, L, e_i). \tag{1.13}$$

(ii) "The scalar curvature [119] r is defined

$$r = \sum_{i=1}^{n} S(e_i, e_i). \tag{1.14}$$

(iii) The normalized scalar curvature [119] is defined by

$$\rho = \frac{2r}{n(n-1)}. \tag{1.15}$$

(iv) The mean curvature vector H is given by

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i). \tag{1.16}$$

(v) The squared norm of mean curvature vector [119] is defined by

$$||H||^2 = \frac{1}{n^2}\sum_{k=n+1}^{m}(\sum_{i=1}^{n} h_{ij}^k)^2. \tag{1.17}$$

In the above equation, for $i, j \in \{1, 2, ..., n\}$ and $k \in m+1, m+2, ...n$, $h_{ij}^k = g(h(e_1, e_j), e_k)$".

1.3.14. Manifold of Constant Curvature [119]

A Riemannian manifold M is said to be a manifold of constant curvature if the Riemannian curvature tensor R is of the form

$$R(F, J)L = \alpha[g(J, L)F - g(F, L)J], \tag{1.18}$$

where α is a non-zero constant.
If the curvature tensor R of type (0,4) has the form

$$\begin{aligned} R(F, J, L, Y) = {} & \alpha[g(J, L)g(F, Y) - g(F, L)g(J, Y)] \\ & + \beta[g(J, L)\omega(F)\omega(Y) - g(F, L)\omega(J)\omega(Y) \\ & + g(F, Y)\omega(J)\omega(L) - g(J, Y)\omega(F)\omega(L)], \end{aligned} \tag{1.19}$$

then M is called a manifold of quasi-constant curvature, where α, β are non-zero constant, ω is a 1-form and $R(F, J, L, Y) = g(R(F, J)L, Y)$.

1.3.15. Nijenhuis Tensor [119]

The Nijenhuis tensor $N(F, J)$ with respect to a (1,1)-type tensor field P is well-defined by

$$N(F, J) = [PF, PJ] - [F, J] - P[PF, J] - P[F, PJ]. \tag{1.20}$$

1.3.16. Complex Manifold [119]

A complex manifold is a manifold such that every neighbourhood looks like complex n-space in a coherent way. More precisely, a complex manifold has an atlas of charts to the open unit disc in $\mathbb{C}^n$ such that the change of coordinate between charts is holomorphic.

1.3.17. Almost Complex Manifold [119]

Let M be an even-dimensional differentiable manifold then an endomorphism P defined by $P : T_p(M) \to T_p(M)$, is said to be an almost complex structure if it satisfies

$$P(PF) = -F. \tag{1.21}$$

C. Ehresmann defined an almost complex manifold as an even dimensional differentiable manifold in 1947. An almost complex manifold M is a manifold with a fixed almost complex structure [119] and is represented by (M, P).

On an almost complex manifold (M, P) a bilinear function A is said to be pure, if

$$A(PF, PJ) = -A(F, J), \tag{1.22}$$

and hybrid, if

$$A(PF, PJ) = A(F, J). \tag{1.23}$$

1.3.18. Almost Hermitian Manifold [119]

Let (M, P) be an n-dimensional almost complex manifold with an almost complex structure P and Riemannian metric g satisfying

$$g(PF, PJ) = g(F, J), \tag{1.24}$$

then M is called an almost Hermitian manifold [119] and the structure $\{P, g\}$ is called an almost Hermitian structure.

An almost Hermitian manifold (M, P, g) is said to be a Hermitian manifold if the (1,1) type tensor field P is integrable *i.e.* $dP = 0$.

1.3.19. Kähler Manifold [119]

D.V. Dantzing, J.A. Schouten and E. Kähler revealed an important class of the Riemannian manifold which is known as Kähler manifold.

Let M be an n-dimensional almost Hermitian manifold with an almost Hermitian structure $\{P, g\}$ satisfying the condition

$$(D_F P)J = 0, \tag{1.25}$$

then M is called a Kähler manifold. The above condtion implies with respect to Levi-Civita connection D, if an almost complex structure P is parallel.

1.3.20. Almost Kähler Manifold [119]

Let M be an n-dimensional almost Hermitian manifold with almost Hermitian structure $\{P, g\}$. If M satisfies

$$(D_F\,{}'P)(J,L) + (D_J\,{}'P)(L,F) + (D_L\,{}'P)(F,J) = 0, \tag{1.26}$$

${}'P$ being second fundamental tensor defined by ${}'P(F,J) = g(PF,J)$ then M is called an almost Kähler Manifold.

1.3.21. Einstein Manifold and Quasi-Einstein Manifold[119]

Let M be an n-dimensional Riemannian manifold. If the Ricci tensor S is proportional to the metric tensor g *i.e.*

$$S(F,J) = \alpha g(F,J), \tag{1.27}$$

then M is called an Einstein manifold, where $F, J \in \chi(M)$ and α is a non-vanishing constant.

If the Ricci tensor S has the form

$$S(F,J) = \alpha g(F,J) + \beta\omega(F) \otimes \omega(J), \tag{1.28}$$

then M is called a quasi-Einstein manifold, where α and β are non-zero constants.

1.3.22. Koszul Formula [119]

$$\begin{aligned} 2g(D_F J, L) &= F\ g(J,L) + J\ g(L,F) - L\ g(F,J) \\ &\quad + g([F,J],L) - g([J,L],F) + g([L,F],Y). \end{aligned} \tag{1.29}$$

1.3.23. Almost Contact Metric Manifold[[14], [119]]

Let M be an odd dimensional differentiable manifold and (ϕ, ξ, η, g) be an almost contact metric structure. Then a manifold M equipped with almost contact metric structure is known as almost contact metric manifold and is denoted by (M, ϕ, ξ, η, g), such that

$$\begin{Bmatrix} \phi^2(F) = -F + \eta(F) \otimes \xi, \quad \eta(\xi) = 1, \eta(F) = g(F,\xi), \\ g(\phi F, J) = -g(F, \phi J), \quad g(\phi F, \phi J) = g(F,J) - \eta(F)\eta(J), \end{Bmatrix} \tag{1.30}$$

for any $F, J \in \Gamma(TM)$. In an almost contact metric structure

(i) ϕ represents a tensor field of type (1,1).

(ii) ξ represents an associated vector field.

(iii) η represents a 1-form,

(iv) g represents Riemannian metric.

The covariant derivative of a tensor field ϕ of type (1,1) is given by

$$(D_F\phi)J = D_F\phi F - \phi D_F J \tag{1.31}$$

1.3.24. Contact Manifolds ([14], [119])

An odd-dimensional differentiable manifold (M, ϕ, ξ, η, g) is said to have a contact structure (ϕ, ξ, η, g), if it carries a global 1-form η satisfying

$$\eta \wedge (d\eta)^n \neq 0. \tag{1.32}$$

Where η represents a contact 1-form and exponent represents a nth exterior power every where on M. If a manifold (M, ϕ, ξ, η, g) is equipped with a contact metric structure then the manifold is known to be a contact manifold.
It may be noted that if (M, ϕ, ξ, η, g) is contact manifold equipped with contact structure η, then for any $F, J \in \Gamma(TM)$ there exists an almost contact metric structure such that

$$g(F, \phi J) = d\eta(F, J). \tag{1.33}$$

1.3.25. Sasakian Manifolds ([4], [14], [119])

In 1960's, S. Sasaki introduced the concept of Sasakian manifolds as an odd-dimensional analogous of Kähler manifolds in contact geometry of differentiable manifolds.

Let (ϕ, ξ, η, g) be a contact metric structure and (M, ϕ, ξ, η, g) be an odd dimensional contact metric manifold. If the contact metric structure of (M, ϕ, ξ, η, g) is normal, then (M, ϕ, ξ, η, g) is said to be the Sasakian manifold. For (M, ϕ, ξ, η, g), we have

$$(D_F\phi)J = g(F, J)\xi - \eta(J)F, \quad D_F\xi = -\phi F, \tag{1.34}$$

for any $F, J \in \Gamma(TM)$.

1.3.26. Generalized Sasakian Space Form [4]

The concept of generalized sasakian space forms in the field of contact geometry of differentiable manifolds was first introduced by Alegre et al. in 2004.

Let (M, ϕ, ξ, η, g) be an $(2n+1)$-dimensional an almost contact metric manifold. If for any $F, J, L \in \Gamma(TM)$, the Riemannian curvature tensor $R(F, J)L$ of M satisfies

$$\begin{aligned} R(F, J)L =& k_1[g(Y, L)F - g(F, L)J] + k_2[g(F, \phi L)\phi J \\ & - g(J, \phi L)\phi J + 2g(F, \phi J)\phi L] + k_3[\eta(F)\eta(L)J \\ & - \eta(J)\eta(L)F + g(F, L)\eta(J)\xi - g(J, L)\eta(F)\xi], \end{aligned} \tag{1.35}$$

where k_1, k_2 and k_3 are differentiable functions on M. The notion of Sasakian, Kenmotsu and cosymplectic space forms are developed from the generalised Sasakian space form.

The notion of Sasakian, Kenmotsu and cosymplectic space forms are developed from the generalised Sasakian space form. If in a generalized Sasakian space form

(i) $k_1 = \frac{c+3}{4}$ and $k_2 = k_3 = \frac{c-1}{4}$, then Sasakian space form is obtained.

(ii) $k_1 = \frac{c-3}{4}$ and $k_2 = k_3 = \frac{c+1}{4}$, then Kenmotsu space form is obtained.

(iii) $k_1 = k_2 = k_3 = \frac{c}{4}$, then cosymplectic space form is obtained.

1.3.27. Nearly Trans-Sasakian Manifolds [49]

The concept of nearly trans-sasakian structure of type (α, β) in the field of contact geometry of differentiable manifolds was first introduced by C. Gherghe in 2000. If in a nearly trans-Sasakian manifold $\alpha = 0$ and $\beta = 0$ then it is said to be nearly $\alpha-$Sasakian and nearly $\beta-$Kenmotsu manifolds respectively.

A nearly trans-Sasakian structure of (α, β) is an almost contact metric structure (ϕ, ξ, η, g) on M if

$$\begin{aligned} (D_F\phi)J + (D_J\phi)F =& \alpha(2g(F, J)\xi - \eta(J)F - \eta(F)J) \\ & - \beta(\eta(J)\phi F + \eta(F)\phi J), \end{aligned} \tag{1.36}$$

is satisfied for any $F, J \in \Gamma(TM)$.

The notion of nearly Sasakian, nearly Kenmotsu and nearly cosymplectic structures are developed from the nearly trans-Sasakian structure of type (α, β). If in a nearly trans-Sasakian structure

(i) $\alpha = 1, \beta = 0$, then nearly Sasakian structure is obtained.

(ii) $\alpha = 0, \beta = 1$, then nearly Kenmotsu structure is obtained.

(iii) $\alpha = 0, \beta = 0$, then nearly cosymplectic structure is obtained.

1.3.28. Warped Product Manifold [5]

Warped product manifolds are a more comprehensive concepts than of product manifolds and may be found in dierential geometry. Bishop and O'Neill investigated these manifolds using instances of negative curvature manifolds. Geometers across the world may explore this unique and imaginative topic, and there are several studies accessible in the literature. A surface of revolution, for example is a warped product manifold, as is the best relativistic Schwarzschild space-time model characterising the neighbourhood of entities with a massive gravitational domain. There have been several major physical uses of warped product manifolds found. For decades, the study of warped product submanifolds in nearly Hermitian and almost contact metric manifolds has been a key field.

Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds and k be a differentiable function on Riemannian manifold M_1. Let $\rho = M_1 \times M_2 \to M_1$ and $\sigma = M_1 \times M_2 \to M_2$ be projection maps. Then $M = M_1 \times_k M_2$ the wraped product manifold [5] is the product manifold $M_1 \times M_2$ equipped with Riemannian metric g defined by

$$g(F, J) = g_1(\rho_a F, \sigma_a J) + (k \circ \rho) g_2(\rho_a F, \sigma_a J), \tag{1.37}$$

for any $F, J \in \Gamma(TM)$ and the function k is commonly known as the wraping function [5] on warped product manifold M.

If the warping function k is constant, then the warped product manifold is trivial and is generally a Riemannian product manifold. For $M = M_1 \times_k M_2$, [5] the submanifolds M_1 and M_2 are totally geodesic and totally umbalical of M respectively.

1.4 Semi-Euclidean Space

Let N be an $n-$dimensional vector space with $g : N \times N \to R$ a symmetric bilinear mapping. If there exists a vector $\xi \neq 0$ of N, satisfying

$$g(\xi, v) = 0, \qquad \forall v \in N, \tag{1.38}$$

then we say that N is degenerate ([42], [47]) . It is easy to see that g is non-degenerate if and only if the following condition is satisfied

$$g(u, v) = 0, \qquad \forall \ \ v \in N \quad implies \ \ u = 0. \tag{1.39}$$

It is essential to note that g, a symmetric bilinear form on N may induce either a degenerate or non-degenerate symmetric bilinear form on a subspace of N. The null space or radical space of N, with respect to g, is a subspace of N ([42], [47]) defined by

$$Rad(N) = \{\xi \in N; g(\xi, v) = 0, \quad v \in N\}. \tag{1.40}$$

The dimension of a radical subspace $Rad(V)$ is the nullity degree of g. If the nullity degree of g equals zero or is less than zero, then g is non-degenerate or degenerate respectively on vector space N.

For any $v \in N$, the symmetric bilinear mapping g is positive (negative) definite on a vector space N, if $g(v, v) > 0$ $(g(v, v) < 0)$. Therefore a positive or negative defnite symmetric bilinear form g, is non degenerate. A symmetric bilinear form g on N is positive (negative) semi-definite [[42], [47]], if $g(v, v) \geq 0$ $(g(v, v \leq 0))$ for any $v \in N$ and there is a non-zero $u \in V$ such that $g(u, u) = 0$.
Let us consider a subsapce W of N, then $g|W$ the restriction of g on $W \times W$ is also symmetric bilinear form on W. The dimension of largest subspace of vector space N, on which the restriction of g on W is negative definite is known as the index of g on N and is denoted by $indN$.

The semi-Euclidean metric or scaler product is a non-degenerate symmetric bilinear form g on N, and the space N equipped with semi-Euclidean metric is called as semi-Euclidean space ([42], [47]). Depending on the behavour of symmetric bilinear form g the following cases arise
(i) g is inner-product (Euclidean metric) and N is an Euclidean space, if g is positive definite.
(ii) g is Minkowski (Lorentz) metric and N is Minkowski (Lorentz) space, if the index of g is 1.
(iii) If g is degenerate metric then N is lightlike (degenerate) vector space.

1.5 Semi-Riemannian Manifolds

Let (M, g) be an n-dimensional differentiable manifold equipped with a $(0, 2)$-type metric tensor g. A semi-Riemannian metric tensor g on a differentiable manifold M is a bilinear, symmetric and non-degenerate $(0, 2)$-type tensor field of constant index on M, [42] satisfying
(i) $g(F, J) = g(J, F) \quad \forall \;\; F, J \in \Gamma(T_pM)$,
(ii) $g(\alpha F + \beta J, L) = \alpha g(F, L) + \beta g(J, L) \quad \forall \;\; F, J, L \in \Gamma(T_pM)$ and $\alpha, \beta \in \mathbb{R}$,
(iii) $g(F, J) = 0 \;\; \forall F \in T_pM$ implies $F = 0$.

Therefore an $n-$dimensional pseudo-Riemannian manifold is an $n-$dimensional differentiable manifold M furnished with a semi-Riemannian metric g. If the index of a semi-Riemannian manifold is zero, in this case g is positive definite inner product on tangent space and the manifold M is Riemannian manifold.

On an even dimensional manifold M a semi-Riemannian metric is called neutral metric, if the index of g is equal to the half of the dimension of M [42]. In the signature of a semi-Riemannian metric g, the number of negative signs represents the index of g and the manifold M is termed as Lorentz manifold and the accompanying metric is Lorentzian metric, if the index of semi-Riemannian metric g is one. Generally, $<,>$ is used to represent g. This means, $< F, J >= g(F, J)$.

A semi-Riemannian metric g is negative (positive) definite if $g(F, F) < 0$ $(g(F, F) > 0)$ for any $F \in T_pM$. It follows that g is non-degenerate if it is negative or positive definite ([42], [47]). A tangent vector F is called
(i) spacelike, if $g(F, F) > 0$ or $F = 0$.
(ii) timelike, if $g(F, F) < 0$.
(iii) lightlike, if $g(F, F) = 0$ and $F \neq 0$.

1.6 Lightlike Manifolds

With a symmetric $(0, 2)$-type tensor field g let (M, g) be a real smooth manifold of $n-$dimensional. For any $p \in M$ the index of symmetric tensor of type $(0, 2)$ is of constant index q on T_pM. On lightlike manifolds g is degenerate on T_pM, that is $g(\xi, X) = 0$ for vector $\xi \neq 0$, of T_pM. In accordance with respect to g, radical $RadT_pM$ or null space of tangent space T_pM [42] is defined by

$$RadT_pM = \{\xi \in T_pM; g(\xi, F) = 0, \quad \forall F \in T_pM.\} \tag{1.41}$$

The nullity of g is the dimension of radical space $RadT_pM$ and is represented by $null\ T_pM$. The associated quadratic form of $(0, 2)$-type symmetric tensor g is a mapping defined by $h(F) = g(F, F)$, for any tangent space T_pM and for all $F, J \in T_pM$, we have the following expression of g expressed in accordance of h [42]

$$g(F, J) = \frac{1}{2}\{h(F + J) - h(F) - h(J)\} \tag{1.42}$$

To each point, $p \in M$ the mapping $RadTM$ assigns a radical subspace $RadT_pM$ of T_pM and defines a smooth distribution of rank $r > 0$ on M. A smooth manifold (M, g) satisfying the above scheme is known as $r-$ lightlike manifold and a $(0, 2)-$type symmetric tensor g is called an r-degenerate metric on M [42]. On r-lightlike manifolds we have

$$g(\xi, F) = 0, \qquad \xi \in \Gamma(Rad\ TM), \quad F \in \Gamma(TM). \tag{1.43}$$

Moreover if symmetric tensor g has a constant rank $(n-r)$ on M, then (M,g) is an r-lightlike manifold. In lightlike manifolds if radical bundle $Rad\ TM$ is non-zero then the degenerate metric tensor is symmetric tensor field. For the existence of Levi-Civita connection on lightlike manifolds, we have

Theorem 1.6.1. *([42], [47]) Let (M,g) be the lightlike manifold. Then the following conditions are equivalent*

(i) (M,g) is Rienhart lghtlike manifold.

(ii) $Rad\ TM$ is a killing distribution.

(iii) There exists a Levi-civita connection on (M,g) with respect to g.

If a lightlike manifold (M,g) admits global null vector field then the lightlike manifold is called globally null manifold that is complete Riemannian hypersurface.

1.7 Lightlike Submanifolds

Let $(\bar{M},g)$ be $m+n$-dimensional pseudo-Riemannian manifold of constant index q and $(M,\bar{g})$ be an m-dimensional submanifold of pseudo-Riemannian manifold $(\bar{M},g)$. The symmetric, bilinear $(0,2)$-type tensor $\bar{g}$ is an induced metric of g on M. On tangent bundle TM of submanifold $(M,\bar{g})$, if the induced metric $\bar{g}$ is degenerate then the submanifold $(M,\bar{g})$ is called a lightlike or null submanifold ([42], [47]) of semi-Riemannian manifold. For degenerate metric $\bar{g}$ of lightlike submanifold $(M,\bar{g})$, $TM^{\perp}$ is degenerate n-dimensional subspace of tangent space $T_p\bar{M}$ defined by

$$TM^{\perp} = \cup\{F \in T_p\bar{M} : g(F,J) = 0 \ \ \forall\ J \in T_pM, \ \ p \in M\}, \tag{1.44}$$

and hence both tangent space and normal space of lightlike hypersurface are orthogonal degenerate but no longer complementary subspaces. In this situation there emerges a radical space $Rad\ T_pM = T_pM \cap T_pM\ \perp$. A mapping defined by

$$Rad\ TM : p \in M \to Rad\ T_pM, \tag{1.45}$$

on lightlike hypersurface defines a rank $r>0$ smooth distribution then $(M,\bar{g})$ is called a r-lightlike submanifold of semi-Riemannian manifold $(\bar{M},g)$. For constructing $tr(TM)$ of lightlike submanifold ([42], [47]) of $(\bar{M},g)$, we have the following possibilities

(i) Lightlike submanifold $(M,\bar{g})$ is $r-$lightlike for $0<r<min(m,n)$.

(ii) Lightlike submanifold $(M,\bar{g})$ is co-isotropic for $1<r=n<m$.

(iii) Lightlike submanifold $(M,\bar{g})$ is isotropic for $1<r=m<n$.

(iv) Lightlike submanifold $(M,\bar{g})$ is totally lightlike submanifold for $1<r=m=n$.

in accordance of the dimensions and co-dimensions of lightlike submanifold $(M,\bar{g})$ and radical bundle $Rad\ TM$. For condition (i), like submanifolds there lasts a screen distribution $S(TM)$, which is non-degenerate and complementry vector bundle to radical bundle $Rad\ TM$ in tangent bunble TM. Therefore we have the following decomposition

$$TM = RadTM \perp S(TM), \tag{1.46}$$

where $\perp$, indicates orthogonal direct sum. It is natural that $S(TM)$ screen bundle is not unique and is isomorphic to $TM/RadTM$. Let transversal bundle $tr(TM)$ and lightlike transversal bundle $ltr(TM)$ be complementry vector bundles to tangent bundle TM to $T\bar{M}|_M$ and radical bundle $RadTM$ in screen transversal bundle $S(TM^{\perp})$, respectively . Therefore we have the following decompositions

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}) \tag{1.47}$$

$$\begin{aligned} T\bar{M} &= TM \oplus tr(TM) \\ &= (RadTM \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp}) \end{aligned} \tag{1.48}$$

Since for any $\{\xi_i\}$ of radical bundle $RadTM$ ther is X_i a local null frame of sections in the orthogonal complement of screen transversal vector bundle, satisfying $g(\xi_i, X_j) = \delta_{ij}$ and $g(X_i, X_j) = 0$, it follows that there is $ltrTM$ locally spanned by local null frame of sections X_i.

Let $\bar{D}$ be the Levi-Civita connection on semi-Riemannian manifold, then according to decomposition(1.48), the Gauss and Weingarten formullas are given by

$$\bar{D}_F J = D_F J + h(F, J), \quad \forall\ F, J \in \Gamma(TM), \tag{1.49}$$

$$\bar{D}_F U = -A_U F + D^t_F U, \quad \forall\ F \in \Gamma(TM),\ U \in \Gamma(tr(TM)), \tag{1.50}$$

where $\{D_F J, A_U F\}$ and $h(F, J)$, $D^t_F U$ belongs to $\Gamma(TM)$ and $\Gamma(tr(TM))$. D and D^t are respectively linear connections on $r-$lightlike submanifold M and on vector bundle $tr(TM)$. From (1.49) and (1.50), for any $F, J \in \Gamma(TM), X \in \Gamma(ltr(TM))$ and $O \in \Gamma(S(TM^{\perp}))$, we acquire

$$\bar{D}_F J = D_F J + h^l(F, J) + h^s(F, J), \tag{1.51}$$

$$\bar{D}_F X = -A_X F + D^l_F X + D^s(F, X), \tag{1.52}$$

$$\bar{D}_F O = -A_O F + D^s_F O + D^l(F, O), \tag{1.53}$$

where $h^l(F, J) = L(h(F, J))$, $h^s(F, J) = S(h(F, J))$, $D^l(F, O) = L(D^t_F O)$, $D^s(F, X) = SD^t_F X$. L and S are the projection morphisms of transversal bundle $tr(TM)$ on lightlike transversal bundle $ltr(TM)$ and screen transversal vector bundle $S(TM\ \perp)$ respectively.

Now by using Levi-Civita connection $\bar{D}$ and equations (1.49), (1.51)(1.52) and (1.53), we obtain

$$g(h^s(F,J),O) + g(J, D^l(F,O)) = \bar{g}(A_O F, J) \tag{1.54}$$

$$g(D^s(F,X),O) = g(X, A_O F) \tag{1.55}$$

Let P be the projection morphism of tangent bundle TM on screen bundle $S(TM)$. By decomposition of TM ([42], [47]), for any $F, J \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$, we acquire

$$D_F PJ = D^*_F PJ + h^*(F, PJ), \tag{1.56}$$

$$D_F \xi = -A^*_\xi F + D^{*t}_F \xi, \tag{1.57}$$

$$g(D^s(F,X),O) = g(X, A_O F), \tag{1.58}$$

$$g(A_X F, N) + g(X, A_{X'} F) = 0, \tag{1.59}$$

$$g(A_X F, PJ) = g(X, \bar{D}_F PY), \tag{1.60}$$

where $\{D^*_F PJ, \quad A^*_\xi F\} \in \Gamma(S(TM))$ and $\{h^*(F, PJ), \; D^{*t}_F \xi \in \Gamma(RadTM).\}$ It may be noted that the linear connection D on lightlike submanifold M and transversal connection D^t on are not metric metric connections [[42], [47]]. Hence by making use of above equations, we derive

$$g(h^l(F, PJ), \xi) = \bar{g}(A^*_\xi F, PJ), \tag{1.61}$$

$$g(h^l(F, PJ), X) = \bar{g}(A_X F, PJ), \tag{1.62}$$

$$g(h^l(F, \xi), \xi) = 0, \qquad A^*_\xi \xi = 0, \tag{1.63}$$

By using the fact that $\bar{D}$ is metric connection and using equations (1.51)-(1.53), we get

$$(D_F \bar{g})(J, L) = g(h^l(F,J), L) + g(h^l(F,L), J) \tag{1.64}$$

and

$$(D^t_F g)(V, V^{"}) = -\{g(A_V F, V^{"}) + g(A^{"}_V F, V)\} \tag{1.65}$$

It may be noted that for any $F \in \Gamma(TM)$ and $J \in \Gamma(RadTM)$ if radical bundle $RadTM$ is parallel distribution with respect to linear connection D, then the induced connection D on lightlike submanifold M is metric connection [42].

1.8 Lightlike Hypersurfaces

Let $(\bar{M}, g)$ be a semi-Riemannian manifold with index $(g) = q \geq 1$ and let $(M, \bar{g})$ be the hypersurface of $(\bar{M}, g)$ such that $g = \bar{g}|_M$. If an induced metric $\bar{g}$ on hypersurface $(M, \bar{g})$ is degenerate, then the hypersurface $(M, \bar{g})$ is said to be lightlike (null or degenerate) hypersurface ([42], [47]). On lightlike hypersurfaces there exists null vector field $\xi \neq 0$ such that

$$\bar{g}(\xi, F) = 0, \qquad \forall F \ \in \Gamma(T\bar{M}). \tag{1.66}$$

In the field of degenerate geometry of manifolds $RadT_pM$ is a radical space or null space of a tangent space T_pM and is a subspace at each point $F \in (M, \bar{g})$ and is defined defined by

$$RadT_FM = \xi \in RadT_FM : g_F(\xi, F) = 0, F \in \Gamma(T\bar{M}. \tag{1.67}$$

The nullity degree of an induced metric $\bar{g}$ is the dimension of $RadT_FM$ ([42], [47]) and it is well known that for light-like hypersurface $(M, \bar{g})$ the nullity degree of $\bar{g}$ is 1. $RadTM$ is called radical distribution and is spanned by null vector field ξ.
$S(TM)$ is a complementary vector bundle of $RadTM$ in tangent bundle TM called screen bundle of degenerate hypersurface (M). We note that any $S(TM)$ is non degenerate and $S(TM)^{\perp}$ is the complementary vector bundle of $S(TM)$ in TM of rank 2 and is called screen transversal bundle. Since radical distribution $RadTM$ is null bundle of screen bundle $S(TM)^{\perp}$ there exists X as a unique local section of $S(TM)^{\perp}$ ([42], [47]), such that

$$g(X, W) = \quad g(X, X) = 0, \quad g(X, Z) = 1. \tag{1.68}$$

The pair (X, Z) is local frame field of $S(TM)^{\perp}$ and note that X is transversal to hypersurface M. Then there exists line bundle $ltrTM$ of tangent bundle of a manifold called as lightlike transversal bundle locally spanned by X. Hence we have

$$T\bar{M} = TM \oplus ltr(TM) = S(TM) \perp RadTM \oplus ltr(TM). \tag{1.69}$$

Where $\oplus$ represents direct sum but not orthogonal. In view of (1.69), the Gauss and Weingarten formulas are respectively given by

$$\bar{D}_F J = D_F J + h(F, J), \tag{1.70}$$

$$\bar{D}_F X = -A_X F + D^t_F X. \tag{1.71}$$

If we set

$$B(F, J) = g(h(F, J), Z), \tag{1.72}$$

$$\tau(F) = g(D^t_F X, Z), \tag{1.73}$$

then above equations (1.70) and (1.71), becomes

$$\bar{D}_F J = D_F J + B(F, J)X, \tag{1.74}$$

$$\bar{D}_F X = -A_X F + \tau(F)X. \tag{1.75}$$

For any $F, J \in \Gamma(TM)$. Where $X \in \Gamma ltr(TM), D_F J, -A_X F \in \Gamma(TM)$ and $h(F, J), D^t_F X \in \Gamma(ltr(TM))$. Here A_X, B, D and D^t represents shape operator, second fundamental form, torsion free linear connection on null hypersurface M and linear connection on $ltr(TM)$ respectively. From (1.56), for we can state that an induced connection on lightlike hypersurface M satisfies the following condition

$$(D_F g)(J, L) = B(F, J)\theta(L) + B(F, L)\theta(J). \tag{1.76}$$

For any F, J, $L \in \Gamma(TM)$ and θ is a differential 1-form on $\bar{M}$ locally defined by $\theta(F) = g(X, F)$ and D is an induced non-metric connection on smooth manifold M. It may be noted that B of $\bar{M}$ is independent of the choice of $S(T\bar{M})$ and may be represented as $B(-, Z) = 0$. With respect to projection P, the local Gauss and Weingarten formulas ([42], [47]) are given by

$$D_F PJ = D^*_F PJ + C(F, PJ)Z, \tag{1.77}$$

$$D_F Z = -A^*_Z F + \tau(F)Z. \tag{1.78}$$

Where $D_F PJ$ and $A^*_\zeta F \in S(TM)$. Here A^*, C and D^* represent local shape operator, local second fundamental form and induced connection on screen bundle $S(TM)$. On degenerate hypersurface shape operators B and C are related as

$$\bar{g}(A_X F, PJ) = C(F, PJ), \qquad \bar{g}(A_X F, X) = 0, \tag{1.79}$$

$$\bar{g}(A^*_\xi F, PJ) = B(F, PJ), \qquad \bar{g}(A^*_\xi F, \xi) = 0. \tag{1.80}$$

$\forall\ F, J \in \Gamma(TM)$. With respect to Levi-Civita connection $\bar{D}$ let R^1 be curvature tensors respectively then we have

$$\begin{aligned} \bar{R}(F, J)L = R(F, J)L + A_{h(F,L)}J - A_{h(J,L)}F \\ + (D_F h)(J, L) - (D_J h)(F, L). \end{aligned} \tag{1.81}$$

Let R, $\bar{R}$ and R^* represent curvature tensors with respect D, $\bar{D}$ and D respectively. For lightlike hypersurface $\bar{M}$ and screen bundle $S(TM)$ using Gauss formula and Weingarten formulae ([42], [47]), we have

$$g(\bar{R}(F,J)L,PO) = \bar{g}(R(F,J)L,PO) + B(F,L)C(J,PO) - B(J,L)C(F,PO), \tag{1.82}$$

$$g(\bar{R}(F,J)L,Z) = (D_F B)(J,L) - (D_J B)(F,L) + \tau(F)B(J,L) - \tau(J)B(F,L), \tag{1.83}$$

$$\bar{g}(R(F,J)PL,X) = (D_F C)(J,PL) - (D_J C)(F,PL) + \tau(J)C(F,PL) - \tau(F)B(J,PL), \tag{1.84}$$

$$g(\bar{R}(F,J)L,X) = \bar{g}(R(F,J)L,X), \tag{1.85}$$

$$\begin{aligned} \bar{R}(F,J)L = R(F,J)L + B(F,L)A_X J - B(J,L)A_X F \\ + \{(D_F B)(J,L) - (D_J B)(F,L) \\ + \tau(F)B(J,L) - \tau(J)B(F,L)\}X, \end{aligned} \tag{1.86}$$

for any $F, J, L \in \Gamma(T\bar{M})$.

Chapter 2

Vector Fields on Lightlike Hypersurfaces of Lorentzian Manifolds

In this chapter, we study parallel vector fields and new type of vector fields on lightlike hypersurfaces $(M, \bar{g})$ of the Lorentzian manifold $(\bar{M}, g)$. We study totally umbilical, totally umbilical screen homothetic, screen conformal and totally geodesic lightlike hypersurfaces of Lorentzian manifolds $(\bar{M}, g)$, admitting parallel vector fields and new type of vector fields. Furthermore, Ricci soliton lightlike hypersurfaces admitting parallel vector fields and new type of vector fields are studied and some characterizations for this frame of hypersurfaces are obtained.

The main purpose of this chapter is to examine the behavior of parallel vector fields or parallel fields on lightlike hypersurfaces and Ricci solitons lightlike hypersurfaces of a Lorentzian manifolds. However there are few problems to deal with while investigating parallel vector fields and Ricci solitons for these kinds of hypersurfaces. The principal difficulty is the existence of degenerate induced metric and hence not applicable for a lightlike hypersurface. Some noteworthy differential operators such as the divergence, Laplacian operators and gradient with respect to the degenerate metric cannot be explained. To overcome of this problem with the help of a rigging vector field we study the associated metric defined on such hypersurfaces. The subsequent main difficulty is that the Ricci tensor of any degenerate hypersurface is not symmetric and in this situation the equation of Ricci soliton loses its geometric and physical senses. To overcome this situation, we explored this Ricci soliton equation on degenerate hypersurfaces of Lorentzian manifolds whose Ricci tensor is symmetric.

This chapter is divided into three sections. In section-1, a brief introduction to parallel vector fields and new type of vector fields on lightlike hypersurfaces of Lorentzian manifolds is given. In section-2, we study the lightlike hypersurface of Lorentzian manifolds admitting parallel vector fields and in section-3, the lightlike hypersurface of Lorentzian manifolds admitting new type of vector fields is studied.

2.1 Introduction

In 1943, K. Yano [117] proved that there exists a smooth vector field v, so called concurrent on a smooth Riemannian manifold $(M, \bar{g})$ which satisfies the following condition

$$D_F v = F, \tag{2.1}$$

for every vector field F tangent to $\bar{M}$, where D is the Levi-Civita connection with respect to Riemannian metric g on $(\bar{M}, g)$. The Riemannian and semi-Riemannian manifolds equipped with concurrent vector fields have found applications in every field of differential geometry and have been intensely studied by various authors ([15], [31]). In 2020, E. Kilic et al. [67] studied concurrent vector fields on lightlike hypersurfaces of Lorentzian manifolds and showed that an induced connection on lightlike hypersurfaces is also concurrent vector field. Beside these facts, the notion of a Ricci soliton [59] was initially witnessed by Hamilton's Ricci flow and then the Ricci solitons drew attention after G . Perelman [91]. The same author [59] applied Ricci solitons to solve the problem of Poincare conjecture.

Definition 2.1.1. The Riemannian manifold $(\bar{M}, g)$ equipped with a symmetric Riemannian metric tensor g is called a Ricci soliton, [31] if there exists a smooth vector field v tangent to M satisfying equation (2.2)

$$\frac{1}{2} L_v g(F, J) + Ric(F, J) = \lambda g(F, J). \tag{2.2}$$

In the preceding twenty years, the assumption of geometric flow is the most vital mathematical invention to describe the geometrical configurations in the field of Riemannian geometry. A detailed fragment of description on which the metric develops, diffeomorphisms have substantial influence in the investigation of singularities of the flows as, they display up as promising singularity models. These models are commonly authorized soliton solutions.

On the other hand, the Yamabe flow was presented by Hamilton [59] at the same time at the emergence of Ricci flow. The Ricci soliton and Yamabe soliton are extraordinary solutions of the Ricci flow and Yamabe flow respectively. For dimension $n = 2$, the Yamabe soliton and the Ricci soliton are alike. For dimension, $n > 2$, the Yamabe soliton and the Ricci soliton are not same and in the enlargement of current existences, the imaginary conception of geometric flows are developed. For instance, the Ricci flow and the Yamabe flow has been the crucial matter of interest in the field for numerous geometers.

At the beginning of 2019, Guler and Crasmareanu [51] presented the investigation of alternative geometric flow namely Ricci-Yamabe map. This map is nothing, but a scalar combination of the Ricci flow and the Yamabe flow. This is moreover termed to be as (α, β) type Ricci-Yamabe flow. This category of flow is a development for metrics on the Riemannian manifolds [47] well defined by

$$\frac{\delta}{\delta t} = \beta r(t)g(t) - 2\alpha S(t), \quad g_0 = g(0), \tag{2.3}$$

where S denotes the Ricci tensor, r represents the scalar curvature, and $\alpha, \beta, \gamma \in R$. One can consider the Ricci-Yamabe flow as singular Riemannian or semi-Riemannian or Riemannian flow because of the representation of symbols α and β as involved scalars. This category of different selections can be respected in some physical or mathematical models such as relativistic theories.

Therefore typically the Ricci-Yamabe solitons are arising as the restriction of the soliton of the Ricci-Yamabe flow. One more solid encouragement that started the exploration of the Ricci-Yamabe solitons is that besides the fact that the Ricci solitons and the Yamabe solitons are equal in 2-dimensional spaces, they are mainly different in higher dimensions. The Ricci-Yamabe soliton on a differentiable manifold $(\bar{M}, g)$ [31] is a data $g, F, \alpha, \beta, \gamma$ accomplishing

$$L_F g(F, J) = -2\alpha S(F, J) - (2\lambda - \beta r)g(F, J), \tag{2.4}$$

where S specifies the Ricci tensor, L represents the Lie-derivative, r represents the scalar curvature and $\alpha, \beta, \gamma \in R$. If f symbolizes a smooth function and F is the gradient of f, then the above mentioned notion is termed as the gradient Ricci-Yamabe soliton [31] and the above equation (2.3) turns out to be

$$\bar{D}^2 f = \alpha S(F, J) - (\lambda - \frac{1}{2}\beta r)g(F, J), \tag{2.5}$$

where $\bar{D}^2 f$ denotes the hessian of f. The gradient Ricci-Yamabe soliton (or Ricci-Yamabe soliton) is expanding for $\lambda > 0$, steady for $\lambda = 0$ and shrinking when $\lambda < 0$. If λ, β and α are smooth functions on $\bar{M}$, then that the gradient Ricci-Yamabe soliton (or Ricci-Yamabe soliton) is called an almost gradient Ricci-Yamabe soliton (or almost

Ricci-Yamabe soliton). For $\beta = 0$, $\alpha = 1$, then gradient Ricci-Yamabe soliton (or Ricci-Yamabe soliton) turns into gradient Ricci soliton (or Ricci soliton) [51]. Similarly, for $\beta = 1$, $\alpha = 0$, then gradient Ricci-Yamabe soliton (or Ricci-Yamabe soliton) is gradient Yamabe soliton (or Yamabe soliton) [59]. On the other hand gradient Ricci-Yamabe soliton (or Ricci-Yamabe soliton) shrinks to gradient Einstein soliton (or Einstein soliton) [21] for $\beta = -1$ and $\alpha = 1$.

Duggal-Bejancu [42] introduced the concept of lightlike geometry of semi-Riemannian manifolds and is wholly different from Riemannian and semi-Riemannian manifolds. To overcome this difficulty ascended due to degenerate metric authors [39] obtained transversal bundle for such hypersurfaces. After [39] and [42] researchers across the globe studied lightlike hypersurface of manifolds by succeeding Duggal-Bejancu methodology. For degenerate hypersurfaces of manifolds, we refer ([39], [42], [44], [85]) and references therein.

For any lightlike hypersurface of semi-Riemannian manifolds, we have

$$(L_F\bar{g})(J, L) = B(F, J)\theta(L) + B(F, L)\theta(J) + \bar{g}(D_F J, L) + \bar{g}(D_F L, J), \tag{2.6}$$

equation (2.6), can also be written as

$$(L_F\bar{g})(J, L) = (D_F\bar{g})(J, L) + \bar{g}(D_F J, L) + \bar{g}(D_F L, J), \quad \forall\ F, J, L \in \Gamma(TM). \tag{2.7}$$

"In 1998, Petersen [94] showed there exists a vector field v so called parallel vector field on a Riemannian manifold M if for any F tangent to M, one has the following relation

$$D_F v = 0, \tag{2.8}$$

where D is the Levi-Civita connection and represents covariant derivative of parallel vector field v in the direction of F".

"A vector field v on a Riemannian manifold M is said to be a new type of vector field [35] if, for any F tangent to M one has the following equation

$$D_F v = B(F)v - F, \tag{2.9}$$

where the 1-form B is associated with the vector field v, such that $g(F, v) = B(v)$ and D denotes the Levi-Civita connection". In literature, such vector fields are also known as geodesic fields since integral curves of such vector fields are geodesics.

On the other hand, the notion of concurrent vector fields can naturally be extended to semi-Riemannian manifolds. In that situation, Chen [30] proved that a Lorentzian manifold is a generalized Robertson–Walker spacetime if and only if it admits a timelike concircular vector field. Concircular vector fields are also important in the study of concircular mappings that is conformal mappings are preserving geodesic circles [117]. In physics, particularly in general relativity concircular vector fields have fruitfull applications.

Definition 2.1.2. [67] "Let $(M, \bar{g})$ be a degenerate hypersurface of a semi-Riemannian manifold $(\bar{M}, g)$. A point p in M is umbilical, if $B_P(F_P, J_P) = k\bar{g}_p(F_P, J_P)$, for $F_p, J_p \ inM$ where B represents local second fundamental form. M is totally umbilical, if any point of hypersurface is umbilical".

Definition 2.1.3. [67] "A null hypersurface $(M, \bar{g})$ of a semi-Riemannian manifold is screen locally conformal, if the shape operators A_X and A^*_ξ of M and $S(TM)$ respectively are related by $A_X = \phi A^*_\xi$. If ϕ is non-zero constant then M is screen homothetic".

Definition 2.1.4. A lightlike hypersurface $(M, \bar{g}, S(TM))$ of $(\bar{M}, g)$ is of genus zero [35] with screen S(TM), if M admits a unique screen distribution $S(TM)$ that makes a unique section N and null hypersurface M admits an induced symmetric Ricci tensor.

Definition 2.1.5. "The smooth manifold is said to be as an Einstein degenerate hypersurface [44] if for any vector fields $F, J \in \Gamma(TM)$, the resulting relation is satisfied

$$R^((0,2))(F, J) = \gamma\bar{g}(F, J), \tag{2.10}$$

where γ is constant".

For any degenerate hypersurface $(M, \bar{g}, S(TM))$, some noteworthy differential operators such as the gradient could be defined by the assistance of its associated metric and a rigging vector field. Therefore, we shall firstly remember some basic truths linked to rigging vector fields and their some elementary properties before take in parallel vector fields on $(M, \bar{g}, S(TM))$.

2.2 Study of Parallel Vector Fields on Lightlike Hypersurfaces of Lorentzian Manifolds

Let $(M, \bar{g}, S(TM))$ be degenerate hypersurface of a Lorentzian manifold $(\bar{M}, g)$ and ξ be a vector field well-defined in an open set containing M. If there exists a 1-form η such that $\eta(F) = g(F, \xi)$ for any vector field $F \in \Gamma(TM)$, then ξ is a rigging vector field for M. Now let $X \in tr(TM)$ be a rigging vector field for M and η be a 1-form defined by

$$\eta(F) = g(F, X), \tag{2.11}$$

for any $F \in \Gamma(TM)$. In this situation, one can express a $(0,2)$ type metric tensor g as follows

$$g(F,J) = \bar{g}(F,J) + \eta(F)\eta(J), \tag{2.12}$$

for any $F, J \in TM$. Therefore from equations (2.11) and (2.12), we acquire

$$g(\xi,\xi) = 1, g(\xi,F) = \eta(F), \tag{2.13}$$

and

$$\bar{g}(F,J) = g(F,J), \eta(F) = 0, \tag{2.14}$$

forall $F, J \in \Gamma(S(TM))$. Now let v be the parallel vector field on $\Gamma(TM)$ then, we can describe parallel vector field v as the tangential and transversal components by

$$v = V^T + V^X, \tag{2.15}$$

where $V^T \in \Gamma(TM)$ and $V^X \in tr(TM)$. From equations (2.12) and (2.15), we acquire

$$g(V^T,\xi) = \bar{g}(V^T,\xi) + \eta(V^T)\eta(\xi)),$$

$$g(V^T,\xi) = \bar{g}(V^T,X). \tag{2.16}$$

For any $F \in \Gamma(TM)$, we write

$$F\bar{g}(V^T+\xi) = Fg(V^T+X), \tag{2.17}$$

$$Fg(V^T+\xi) = g(\bar{D}_F v + X) + g(v, \bar{D}_F X), \tag{2.18}$$

$$Fg(V^T,\xi) = \tau(F)\eta(v) - g(v, A_X F). \tag{2.19}$$

Now let us assume v lies in tangent bundle i,e. $v = V^T$. So for any $v^S \in \Gamma S(TM))$ and $\eta(v) = a$, we can write

$$v = v^S + a\xi. \tag{2.20}$$

Lemma 2.2.1. *Let v be the parallel vector field on $\Gamma(TM)$ and $(M, \bar{g}, S(TM))$ be null hypersurface of a Lorentzian manifold $(\bar{M}, g)$, then*

$$\tau(v) = \frac{1}{2} g(v^S, A_X v). \tag{2.21}$$

Proof. From equations (2.16), (2.19) and equation (2.20), we acquire

$$\begin{aligned}(v^S + a\xi)g(v^S + a\xi, \xi) =& g(v^S + a\xi, v)g(v^S + a\xi, X) \\ &- g(v^S + a\xi, X) - g(v^S + a\xi, A_X(v^S + a\xi)) \\ &+ \tau(v^S + a\xi)g(v^S + a\xi, X),\end{aligned} \tag{2.22}$$

$$(v^S + a\xi)[g(v^S, \xi) + a(v^S + a\xi)g(\xi, \xi)] = \tau(v^S + a\xi)[\eta(V^T) + a\eta(\xi)]. \tag{2.23}$$

By straightforward computation of the above equation, we acquire

$$a\tau(v) = g(v^S, A_X v) + ag(v^S, A_X\xi). \tag{2.24}$$

From equation (2.24), we obtain equation (2.21). □

Lemma 2.2.2. *Let v be the parallel vector field on $\Gamma(TM)$ and $(M, \bar{g}, S(TM))$ be null hypersurface of $(\bar{M}, g)$. Then for any $F \in (TM)$*

$$D_F v = 0, \qquad B(F, v) = 0. \tag{2.25}$$

Proof. From equations (2.8) and (1.72), we obtain

$$\bar{D}F_v = D_F v + B(F, v). \tag{2.26}$$

Using equation (1.72) in the above equation, we obtain

$$D_F v + B(F, v)X = 0. \tag{2.27}$$

Comparing tangential and transversal parts of equation (2.27), we obtain equations (2.25). □

Lemma 2.2.3. *Tangent to the structural vector field, let $(M, \bar{g}, S(TM))$ be screen conformal null hypersurface of Lorentzian manifold $(\bar{M}, g)$. Let v be the parallel vector field on $\Gamma(TM)$, then for any $F \in \Gamma(TM)$*

$$\tau(v) = 0. \tag{2.28}$$

Proof. From equations (1.72), (2.18) and lemma-2.2.1, we obtain

$$\tau(v) = \frac{1}{a}C(v, V^S). \tag{2.29}$$

Since $(M, \bar{g}, S(TM))$ is screen conformal null hypersurface, equation (2.29) acquire

$$\tau(v) = \frac{1}{a}\frac{1}{\phi}B(v, V^S). \tag{2.30}$$

Using equation (2.25) in equation (2.30), we acquire equation (2.28). □

Lemma 2.2.4. *Let v be the parallel vector field on $\Gamma(TM)$ and $(M, \bar{g}, S(TM))$ be screen conformal null hypersurface of Lorentzian manifold then*

$$D^*_\xi v = 0 \quad \textit{and} \quad C(\xi, v) = 0. \tag{2.31}$$

Proof. Since the second fundamental form B vanishes on $Rad(TM)$, we have

$$D_F J = D^*_F J + C(F, J)\xi, \tag{2.32}$$

Using equation (2.32) in the Gauss and Codazzi formulae, we obtain

$$\bar{D}_\xi v = D_\xi v, \tag{2.33}$$

$$D_\xi v = D^*_\xi v + C(\xi, v)\xi. \tag{2.34}$$

Use of equations (2.8) in equation (2.34), we acquire

$$D^*_\xi v + C(\xi, v) = 0. \tag{2.35}$$

Comparing tangential and transversal components of equation (2.35), we obtain (2.31).

□

Theorem 2.2.1. *Let $(M, \bar{g}, S(TM))$ be screen conformal null hypersurface and let v be the parallel vector field lying on $\Gamma(TM)$. Then $\tau(\xi)$=0.*

Proof. From lemma 2.2. 3 and the Guass Weingarten formulas, we obtain

$$D_v \xi = -A^*_\xi v \tag{2.36}$$

$$\bar{D}_\xi X = -A_X v \tag{2.37}$$

Since v is a parallel vector field, therefore using lemma 2.2.4 in the above equation, we achieve

$$D_\xi v = 0. \tag{2.38}$$

From equations (2.36) and (2.38), we acquire

$$D_\xi v - D_v \xi = -A^*_\xi v, \tag{2.39}$$

$$[\xi, v] = -A_\xi^* v. \tag{2.40}$$

Letting $v = v^S + a\xi$ then equation (2.40), becomes

$$[\xi, v^S + a\xi] = -A_\xi^*(v^S + a\xi), \tag{2.41}$$

$$[\xi, v^S] = -A_\xi^* v^S - aA_\xi^*\xi, \tag{2.42}$$

$$D_\xi v^S - D_(v^S)\xi = -A_\xi^* v^S - aA_\xi^*\xi. \tag{2.43}$$

Using the Guass and Weingarten formulas in above equation, we attain

$$D_\xi^* v^S + C(\xi, v^S)\xi - A_\xi^* v^S + \tau(v^S)\xi = -A_\xi^* v^S - aA_\xi^*\xi. \tag{2.44}$$

Since $A_\xi^* \in S(TM)$, therefore from equation (2.44), we acquire

$$\tau(v^S)\xi + C(\xi, v^S)\xi = 0. \tag{2.45}$$

Using lemma 2.2.4 in (2.45), we attain

$$\tau(v^S) = 0. \tag{2.46}$$

Using $\tau(v) = 0$ in equation (2.46), we obtain

$$\tau(v) = \tau(v^S + a\xi) = 0, \tag{2.47}$$

or

$$a\tau(\xi) = 0. \tag{2.48}$$

From the equation (2.48), we conclude that either $a = 0$ or $\tau(\xi) = 0$. But $a = \eta(v) \neq 0$, hence $\tau(\xi) = 0$. □

Theorem 2.2.2. *Let v be parallel vector field with respect to Levi-Civita connection $\bar{D}$. Then correspondingly v is parallel vector field with respect to induced Riemannian connection D^1.*

Proof. From equation (2.12), for any $F \in \Gamma(TM)$, we have

$$Fg(v, v) = Fg(v, v) + F(g(v, N)^2)$$

$$Fg(v, v) = 2g(D_F v, v) + 2\eta(v)[-g(v, A_X F) + \tau(F)\eta(v)]. \tag{2.49}$$

Setting $Fg(v,v) = 2\bar{g}(D^1_F v, v)$, in the (2.49), we acquire

$$Fg(v,v) = 2\bar{g}(D^1_F v, v) + 2\eta(D^1_F v)\eta(v). \tag{2.50}$$

From equations (2.49) and (2.50), we acquire

$$D^1_F v + \eta(D^1_F v)X = [\tau(F)\eta(v) - g(v, A_X F)]X. \tag{2.51}$$

Comparing the tangential and transversal parts on both sides of equation (2.51), we obtain

$$D^1_F v = 0, \tag{2.52}$$

and

$$\eta(D^1_F v) = \tau(F)\tau(v) - g(v, A_X F). \tag{2.53}$$

From equation (2.52), it is clear that v is also a parallel vector field with respect to D^1. From equation (2.53), we may state the following corollaries □

Corollary 2.2.1. *Let v be a parallel vector field with respect to D^1. Then v is also parallel with respect to D if and only if the following condition*

$$\eta(D^1_F v) = \tau(F)\eta(v) - g(v, A_X F), \tag{2.54}$$

is satisfied for all $X \in \Gamma(TM)$.

Corollary 2.2.2. *Let v be the parallel vector field with respect to D^1 lying on $\Gamma(TM)$ and $(M, \bar{g}, S(TM))$ be screen conformal null hypersurface with respect to D. Then v is also parallel with respect to D if and only if the following condition*

$$\eta(D^1_V v) = -g(v, A_X F). \tag{2.55}$$

is satisfied.

Theorem 2.2.3. *Let $(M, \bar{g}, S(TM))$ be totally umbilical screen-homothetic null hypersurface of $\bar{M}(c)$ admitting parallel vector field v on $\Gamma(T\bar{M})$, then $(M, \bar{g}, S(TM))$ is semi-Euclidean space.*

Proof. From theorem 2.2.1, we know that $\tau(\xi) = 0$, therefore the Gauss and Codazzi formula leads equation (2.56)

$$D_\xi \xi = A^*_\xi \xi, \tag{2.56}$$

but

$$B(\xi, J) = \bar{g}(A^*_\xi \xi, J). \tag{2.57}$$

From equation (2.57), it is clear that $A^*_\xi \xi = 0$ and $D_\xi \xi = 0$.

Now the Riemannian curvature tensor R, equipped with parallel vector field v is given by

$$R(\xi, v)\xi = D_\xi D_v \xi - D_v D_\xi \xi - D_([\xi, v])\xi. \tag{2.58}$$

Using equation (2.40) in equation (2.58), we acquire

$$R(\xi, v)\xi = D_{A^*_\xi v}\xi. \tag{2.59}$$

Since $(M, \bar{g}, S(TM))$ is totally umbilical, equation (2.59) becomes

$$R(\xi, v)\xi = D_{\lambda v}\xi, \tag{2.60}$$

$$R(\xi, v)\xi = \lambda D_v \xi. \tag{2.61}$$

Using equation (2.36) in equation (2.61), we obtain

$$R(\xi, v)\xi = -\lambda(A^*_\xi v). \tag{2.62}$$

Using equation (1.83) in equation (2.62), we obtain

$$\bar{g}(R(\xi, v)\xi, X) = \lambda \bar{g}(D_v \xi, X), \tag{2.63}$$

and

$$\bar{g}(R(\xi, v)\xi, X) = 0. \tag{2.64}$$

From equation (2.64), it is clear that lightlike hypersurface $(M, \bar{g}, S(TM))$ is semi-Euclidean space. □

2.2.1 Ricci Soliton Lightlike Hypersurfaces of Lorentzian Manifolds admitting Parallel vector Fields

Let $(M, \bar{g}, S(TM))$ be the null hypersurface of $(\bar{M}, g)$ and v be the parallel vector field on tangent bundle of $(\bar{M}, g)$, then v can be written as

$$v = V^T + fX, \tag{2.65}$$

where V^T and fX are tangential and transversal components of v and $f = g(v, \xi)$.

Theorem 2.2.4. *Let v be the parallel vector field lying on $\Gamma(TM)$ and $(M, \bar{g}, S(TM))$ be the null hypersurface of $(\bar{M}, g)$, then*

$$D_F v^T = f A_X F, \tag{2.66}$$

$$B(F, v^T) = f\tau(F) - Ff. \tag{2.67}$$

Proof. From equation (2.65), we acquire

$$\tau(v) = \tau(V^T + fX). \tag{2.68}$$

Since v is a parallel vector field, we have

$$\bar{D}_F v = 0. \tag{2.69}$$

Using equation (2.65) in equation (2.69), we acquire

$$\bar{D}_F v^T + \bar{D}_F(fX) = 0. \tag{2.70}$$

Using Guass and Weingarten formulas in equation (2.70), we achieve

$$D_F v^T + B(F, v^T)X + F(f)X - fA_X F - f\tau(F)X = 0. \tag{2.71}$$

From equations (2.69) and (2.71), we obtain equations (2.66) and (2.67). □

Theorem 2.2.5. *Let v be the parallel vector field lying on $\Gamma(TM)$ and $(M, \bar{g}, S(TM))$ be a totally geodesic null hypersurface of Lorentzian manifold $(\bar{M}, g)$. Then either the function f is constant or the Lie bracket is trivial.*

Proof. Since degenerate (lightlike) hypersurface $(M, \bar{g}, S(TM))$ is totally geodesic, therefore from equation (2.67), we acquire

$$f\tau(F) = Ff, \tag{2.72}$$

similarly

$$f\tau(J) = Jf. \tag{2.73}$$

Subtracting equations (2.72) and (2.73), we obtain

$$[F, J](f) = f[J(\tau(F)) - F(\tau(J))], \tag{2.74}$$

for any $F, J \in \Gamma(TM)$. Using Corollary 2.2.1 and proposition 2 of [60] in equation (2.74), we achieve

$$[F, J](f) = 0. \tag{2.75}$$

From equation (2.75), it is clear that either the function f is constant or the lie bracket is trivial. □

Lemma 2.2.5. *Let v be the parallel vector field lying on $\Gamma(TM)$ and $(M, \bar{g}, S(TM))$ be null hypersurface of $(\bar{M}, g)$ then for any $F, J \in \Gamma(TM)$, we have*

$$(L_{v^T}\bar{g})(F, J) = B(v^T, F)\theta(J) + B(v^T, J)\theta(F) + f\bar{g}(A_X F, J) + f\bar{g}(A_X J, F), \quad (2.76)$$

or equivalently

$$(L_{v^T}\bar{g})(F, J) = (D_{v^T}\bar{g})(F, J) + f[\bar{g}(A_X F, J) + \bar{g}(A_X J, F)]. \quad (2.77)$$

Proof. From equations (2.6) and (2.7), we obtain

$$(L_{v^T}\bar{g})(F, J) = B(v^T, F)\theta(J) + B(v^T, J)\theta(F) - \bar{g}(D_F v^T, J) - \bar{g}(D_J v^T, F), \quad (2.78)$$

or equivalently

$$(L_{v^T}\bar{g})(F, J) = (D_{v^T}\bar{g})(F, J) + \bar{g}(D_F v^T, J) + \bar{g}(D_J v^T, F). \quad (2.79)$$

Using equations (2.65) and (2.66) in equations (2.78) and (2.79), we acquire (2.76) and (2.77) respectively. □

If the parallel vector field v lies on $\Gamma(T\bar{M})$, that is if $f = 0$, that is $v = V^T$, then we may state

Lemma 2.2.6. *Let v be the parallel vector field lying on $\Gamma(TM)$ and $(M, \bar{g}, S(TM))$ be null hypersurface of $(\bar{M}, g)$, then for any $X, Y \in \Gamma(TM)$, we have*

$$(L_{v^T}\bar{g})(F, J) = (D_{v^T}\bar{g})(F, J). \quad (2.80)$$

Proposition 2.2.1. *Let v^T be a potential vector field and $(M, \bar{g}, S(TM))$ be null hypersurface of $(\bar{M}, g)$. Then for any $F, J \in \Gamma(TM)$, we have*

$$2R^{(0,2)}(F, J) = 2\lambda\bar{g}(F, J) - f[\bar{g}(A_X F, J) + \bar{g}(A_X J, F)](D_{v^T}\bar{g})(F, J), \quad (2.81)$$

or equivalently

$$R^{(0,2)}(F, J) = \lambda\bar{g}(F, J) - \frac{f}{2}[\bar{g}(A_X F, J) + \bar{g}(A_X Y, F)] - \frac{1}{2}(D_{v^T}\bar{g})(F, J). \quad (2.82)$$

Proof. Since, $(M, \bar{g}, S(TM))$ is Ricci soliton lightlike hypersurface of $(\bar{M}, g)$ then, we have

$$(L_{v^T}\bar{g})(F, J) + 2R^{(0,2)}(F, J) = 2\lambda\bar{g}(F, J). \quad (2.83)$$

Using lemma 2.2.5 in equation (2.83) we obtain, equations (2.81) and (2.82). □

Proposition 2.2.2. *Let v be a potential vector field and $(M, \bar{g}, S(TM))$ be a Ricci soliton null (lightlike) hypersurface of a semi-Euclidean space of (n-1)-dimensional. Then for any $F \in \Gamma(S(TM))$, we have*

$$nB(F,F)traceA_X - B(A_X F, F) = \lambda - f\bar{g}(A_X F, F), \tag{2.84}$$

where F is a unit vector field.

Proof. Let us assume that $(a_1, a_2, a_3, ...a_n)$ be orthonormal basis. Therefore from equation (83) of [60], we obtain

$$R^{(0,2)}(a_i, a_i) = nB(a_i, a_i)traceA_X - B(A_N a_i, a_i). \tag{2.85}$$

From equation (2.82) equation (2.85), becomes

$$nB(a_i, a_i)traceA_X - B(A_X a_i, a_i) = \lambda\bar{g}(a_i, a_i) - f\bar{g}(A_X a_i, a_i) - \frac{1}{2}(D_{v^T} g)(a_i, a_i). \tag{2.86}$$

Replacing a_i with F in equation (2.86), we obtain

$$nB(F,F)traceA_X - B(A_X F, F) = \lambda - f\bar{g}(A_X F, F). \tag{2.87}$$

□

Corollary 2.2.3. *Let v be a parallel vector field and $(M, \bar{g}, S(TM))$ be Ricci soliton null (lightlike) hypersurface of semi-Euclidean space. Then for any $F, J \in \Gamma S((TM))$, if $(M, \bar{g}, S(TM))$ is screen conformal and totally umbilical, then for any $F \in (S(T\bar{M}))$*

$$n^2[B(F,F)]^2\phi - [B(F,F)]^2 = \lambda - 2fB(F,F), \tag{2.88}$$

is satisfied, where F is a unit vector field.

Proposition 2.2.3. *Let v be a parallel vector field on $\Gamma(TM)$ and $(M, \bar{g}, S(TM))$ be Ricci soliton null (lightlike) hypersurface of a Lorentzian manifold. Then for any $F, J \in \Gamma(S(TM))$, we have*

$$R^{(0,2)}(F,J) = \lambda\bar{g}(F,J) - \frac{1}{2}[B(v,F)\theta(J) + B(v,J)\theta(F)]. \tag{2.89}$$

Proof. Since $(M, \bar{g}, S(TM))$ is a Ricci soliton and v is a parallel vector field lying on $\Gamma(TM)$ tangent bundle. Therefore equation (2.83), can be written as

$$(L_v\bar{g})(F,J) + 2R^{(0,2)}(F,J) = \lambda\bar{g}(F,J). \tag{2.90}$$

For any $F, J \in \Gamma(TM)$, from (2.83), we acquire

$$(L_v\bar{g})(F,J) = (D_v\bar{g})(F,J) + 2[B(F)B(J) - \bar{g}(F,J). \tag{2.91}$$

Using equations (2.90) in (2.91), we obtain

$$R^{(0,2)}(F,J) = \lambda\bar{g}(F,J) - \frac{1}{2}(L_v\bar{g})(F,J). \tag{2.92}$$

From equations (1.76) and (2.92), the proof is straightforward. □

Corollary 2.2.4. *Let v be a parallel vector field on $\Gamma(TM)$ and $(M, \bar{g}, S(TM))$ be Ricci soliton null (lightlike) hypersurface of a Lorentzian manifold. Then for any $F, J \in \Gamma(TM)$ if $(M, \bar{g}, S(TM))$ is totally geodesic then $(M, \bar{g}, S(TM))$ is Einstein Ricci soliton degenerate hypersurface.*

Proof. Let us assume that $(a_1, a_2, a_3, \ldots\ldots\ldots\ldots a_n, \xi)$ and $(a_1, a_2, a_3, \ldots\ldots\ldots\ldots a_n)$ be basis on $\Gamma(TM)$ and $\Gamma(S(TM))$ respectively, then we have

$$R^{(0,2)}(a_i,a_j) = \lambda\bar{g}(a_i,a_j) - \frac{1}{2}[B(v,a_i)\theta(a_j) + B(v,a_j)\theta(a_i)]. \tag{2.93}$$

$$R^{(0,2)}(a_i,a_j) = \lambda\delta_{ij} \tag{2.94}$$

where $i, j = 1, 2, 3, \ldots\ldots\ldots\ldots n$ and δ_{ij} represents the Kronecker delta.
Putting $F = J = \xi$ in equation (2.89), we acquire

$$R^{(0,2)}(\xi,\xi) = \lambda\bar{g}(xi,\xi) - \frac{1}{2}[B(v,\xi)\theta(a_j) + B(v,\xi)\theta(a_i)]. \tag{2.95}$$

$$R^{(0,2)}(\xi,\xi) = \lambda. \tag{2.96}$$

From equations (2.93) and (2.96), we obtain our required result. □

2.3 New Type of Vector Fields on Lightlike Hypersurface of Lorentzian Manifolds

Let $(M, \bar{g}, S(TM))$ be degenerate hypersurface of a Lorentzian manifold $(\bar{M}, g)$ and ξ be a vector field well-defined in an open set containing M. If there exists a 1-form η such that $\eta(F) = g(F, \xi)$ for any vector field $F \in \Gamma(TM)$, then ξ is a rigging vector field for M. Now let $X \in tr(TM)$ be a rigging vector field for M and η be a 1-form defined by

$$\eta(F) = g(F, X), \tag{2.97}$$

for any $F \in \Gamma(TM)$, a $(0, 2)$-type metric tensor g is defined by

$$g(F, J) = \bar{g}(F, J) + \eta(F)\eta(J) \tag{2.98}$$

It may be noted that the associated metric $\bar{g}$ is non-degenerate.
From equations (2.97) and (2.98), we acquire

$$g(\xi, \xi) = 1, g(\xi, F) = \eta(F), \tag{2.99}$$

and

$$\bar{g}(F, J) = g(F, J), \eta(F) = 0, \tag{2.100}$$

forall $F, J \in \Gamma(S(TM))$. Let v be the new type of vector field on $\Gamma(TM)$ then, we can define new type of vector field v as the tangential and transversal constituents by

$$v = V^T + V^X, \tag{2.101}$$

From equations (2.98) and (2.101), we have

$$g(V^T, \xi) = \bar{g}(V^T, \xi) + \eta(V^T)\eta(\xi)),$$

$$g(V^T, \xi) = \bar{g}(V^T, X) \tag{2.102}$$

for any $F \in \Gamma(TM)$, equation (2.102) becomes

$$Fg(V^T + \xi) = Fg(V^T + X), \tag{2.103}$$

$$Fg(V^T + \xi) = g(\bar{D}_F v + X) + g(v, \bar{D}_F X), \tag{2.104}$$

$$Fg(V^T + \xi) = B(F)\eta(v) - \eta(F)\tau(F)\eta(v) - g(v, A_X F). \tag{2.105}$$

Now let us assume v lies in tangent bundle i,e. $v = V^T$. So for any $v^S \in \Gamma S(TM))$ and $\eta(v) = a$, we can write

$$v = v^S + a\xi. \tag{2.106}$$

Lemma 2.3.1. *Let v be the new type of vector field on $\Gamma(TM)$ and $(M, \bar{g}, S(TM))$ be null hypersurface of a Lorentzian manifold $(\bar{M}, g)$, then*

$$\tau(v) = 1 + \frac{1}{a} g(v^S, A_X v). \tag{2.107}$$

Proof. From equations (2.102),(2.105) and (2.106), we acquire

$$\begin{aligned}(v^S + a\xi)g(v^S + a\xi, \xi) =& g(v^S + a\xi, v)g(v^S + a\xi, X) \\ &- g(v^S + a\xi, X) - g(v^S + a\xi, A_X(v^S + a\xi)) \\ &+ \tau(v^S + a\xi)g(v^S + a\xi, X),\end{aligned} \tag{2.108}$$

$$(v^S + a\xi)[g(v^S, \xi) + a(v^S + a\xi)g(\xi, \xi)] = \tau(v^S + a\xi)[\eta(V^T) + a\eta(\xi)]. \tag{2.109}$$

By straightforward computation of equation (2.109), we acquire

$$a\tau(v) = a + g(v^S, A_X v) + ag(v^S, A_X \xi). \tag{2.110}$$

From equation (2.110), we obtain equation (2.107). □

Lemma 2.3.2. *Let v be the new type of vector field on $\Gamma(TM)$ and $(M, \bar{g}, S(TM))$ be null hypersurface of $(\bar{M}, g)$. Then for any $F \in (TM)$*

$$D_F v = B(F)v - F, \qquad B(F, v) = 0. \tag{2.111}$$

Proof. From equations (2.9) and (1.72), we obtain

$$\bar{D}F_v = D_F v + B(F, v). \tag{2.112}$$

Using equation (1.72) in the above equation, we acquire

$$D_F v + B(F, v)X = B(F)v - F. \tag{2.113}$$

Comparing tangential and transversal parts of equation (2.113), we obtain equations (2.111). □

Lemma 2.3.3. *Tangent to the structural vector field let $(M, \bar{g}, S(TM))$ be screen conformal null hypersurface of Lorentzian manifold $(\bar{M}, g)$. Let v be the new type of vector field on $\Gamma(TM)$, then*

$$\tau(v) = 1, \tag{2.114}$$

for any $F \in \Gamma(TM)$.

Proof. From equations (1.72), (2.104) and lemma-2.3.1, we obtain

$$\tau(v) = 1 + \frac{1}{a} C(v, V^S). \tag{2.115}$$

Since, $(M, \bar{g}, S(TM))$ is screen conformal null hypersurface, then equation (2.115) becomes

$$\tau(v) = 1 + \frac{1}{a}\frac{1}{\phi} B(v, V^S). \tag{2.116}$$

Using equation (2.111) in equation (2.27), we obtain equation (2.114). □

Lemma 2.3.4. *Let v be the new type of vector field on $\Gamma(TM)$ and $(M, \bar{g}, S(TM))$ be screen conformal null hypersurface of Lorentzian manifold then*

$$D^*_\xi v = B(\xi)v \quad \text{and} \quad C(\xi, v) = -1. \tag{2.117}$$

Proof. As second fundamental form B vanishes on $Rad(TM)$, then using the Guass and Weingarten formulae, we obtain

$$D_F J = D^*_F J + C(F, J)\xi. \tag{2.118}$$

Setting $F = \xi$ in equation (2.118), we obtain

$$\bar{D}_\xi v = D_\xi v, \tag{2.119}$$

$$D_\xi v = D^*_\xi v + C(\xi, v)\xi. \tag{2.120}$$

Uuse of equations (2.9) in equation (2.120), we acquire

$$B(\xi)v - \xi = D^*_\xi v + C(\xi, v). \tag{2.121}$$

Comparing tangential and transversal terms of equation (2.121), we obtain (2.117). □

Theorem 2.3.1. *Let* $(M, \bar{g}, S(TM))$ *be screen conformal null hypersurface and let* v *be the new type of vector field lying on* $\Gamma(TM)$. *Then* $\tau(\xi)$=0.

Proof. From lemma 2.3.3 and the Guass Weingarten formulas, we obtain

$$D_v\xi = -A^*_\xi v - \xi, \tag{2.122}$$

$$\bar{D}_\xi X = -A_X v - \xi. \tag{2.123}$$

Since v is a new type of vector field, therefore using lemma 2.3.4, we achieve

$$D_\xi v = B(\xi)v - \xi \tag{2.124}$$

From equations (2.122) and (2.124), we acquire

$$D_\xi v - D_v\xi = B(\xi)v - \xi + A^*_\xi v - \xi, \tag{2.125}$$

$$[\xi, v] = B(\xi)v - 2\xi + A^*_\xi v. \tag{2.126}$$

Putting $v = v^S + a\xi$ in equation (2.126), we obtain

$$[\xi, v^S + a\xi] = B(\xi)(v^S + a\xi) - 2\xi + A^*_\xi(v^S + a\xi), \tag{2.127}$$

$$[\xi, v^S] = B(\xi)v^S + B(\xi)a\xi + A^*_\xi v^S + aA^*_\xi\xi - 2\xi, \tag{2.128}$$

$$D_\xi v^S - D_(v^S)\xi = B(\xi)v^S + B(\xi)a\xi + A^*_\xi v^S + aA^*_\xi\xi - 2\xi. \tag{2.129}$$

Using Guass and Weingarten formulas in equation (2.129), we attain

$$D^*_\xi v^S + C(\xi, v^S)\xi - A^*_\xi v^S + \tau(v^S)\xi = B(\xi)v^S + aB(\xi)\xi + A^*_\xi v^S + aA^*_\xi\xi - 2\xi \tag{2.130}$$

Since $A^*_\xi \in S(TM)$, therefore equation (2.130) becomes

$$\tau(v^S)\xi + C(\xi, v^S)\xi = -2\xi. \tag{2.131}$$

Using lemma 2.3.4 in equation (2.131), we attain

$$\tau(v^S) = -1. \tag{2.132}$$

Using $\tau(v) = 1$ in equation (2.132), we obtain

$$\tau(v) = \tau(v^S + a\xi) = 1, \tag{2.133}$$

or

$$a\tau(\xi) = 2. \tag{2.134}$$

Therefore our result follows from equation (2.134). □

Theorem 2.3.2. *Let v be new type of vector field with respect to Levi-Civita connection $\bar{D}$. Then correspondingly v is new type of vector field with respect to induced Riemannian connection D^1.*

Proof. From equation (2.98), for any $F \in \Gamma(TM)$, we have

$$Fg(v,v) = F\bar{g}(v,v) + F(g(v,N)^2)$$

$$Fg(v,v) = 2g(D_F v, v) + 2\eta(v)[\eta(v)B(F) - \eta(F) - g(v, A_X F) + \tau(F)\eta(v)]. \tag{2.135}$$

Since $Fg(v,v) = 2\bar{g}(D^1_F v, v)$ equation (2.135) turns into

$$Fg(v,v) = 2\bar{g}(D^1_F v, v) + 2\eta(D^1_F v)\eta(v). \tag{2.136}$$

From equations (2.135) and (2.136), we acquire

$$D^1_F v + \eta(D^1_F v)X = B(F)v - F + [\tau(F)\eta(v) - \eta F - g(v, A_X F)]X. \tag{2.137}$$

Comparing the corresponding parts on both sides, we obtain

$$D^1_F v = B(F)v - F, \tag{2.138}$$

and

$$\eta(D^1_F v) = \tau(F)\eta(v) - \eta F - g(v, A_X F). \tag{2.139}$$

From equation (2.138), it is clear that v is also a new type of vector field with respect to D^1. □

From equation (2.139), we may state the following corollaries

Corollary 2.3.1. *Let v be a new type of vector field with respect to D^1, then v is also new type of with respect to D, if and only if the following condition*

$$\eta(D^1_F v) = B(F)\eta(v) + \tau(F)\eta(v) - g(v, A_X F), \tag{2.140}$$

is satisfied for all $F \in \Gamma(TM)$.

Corollary 2.3.2. *Let v be the new type of vector field with respect to D^1 lying on $\Gamma(TM)$ and $(M, \bar{g}, S(TM))$ be screen conformal null hypersurface with respect to D. Then v is* *new type of with respect to D, if and only if the following condition*

$$\eta(D^1_V v) = \eta(v) - g(v, A_X F). \tag{2.141}$$

is satisfied.

2.3.1 Ricci Solitons Lightlike Hypersurfaces of Lorentzian Manifolds Admitting New Type of Vector Fields

Let $(M, \bar{g}, S(TM))$ be the null hypersurface of $(\bar{M}, g)$ and v be the new type of vector field on tangent bundle of $(\bar{M}, g)$. Then vector field v, can be written as

$$v = V^T + fX, \tag{2.142}$$

where V^T and fX are tangential and transversal components of v respectively, and $f = g(v, \xi)$.

Theorem 2.3.3. *Let v be the new type of vector field lying on $\Gamma(TM)$ and $(M, \bar{g}, S(TM))$, be null hypersurface of $(\bar{M}, g)$, then*

$$D_F v^T = B(F)v - F + fA_X F, \tag{2.143}$$

$$B(F, v^T) = f\tau(F) - Ff. \tag{2.144}$$

Proof. From equation (2.142), we acquire

$$\tau(v) = \tau(V^T + fX). \tag{2.145}$$

Since v is a new type of vector field with respect to $\bar{D}$, we acheive

$$\bar{D}_F v = B(F)v - F, \tag{2.146}$$

$$\bar{D}_F v^T + \bar{D}_F(fX) = B(F)v - F. \tag{2.147}$$

Using Guass and Weingarten formulas in equation (1.50.27), we achieve

$$\bar{D}_F v = D_F v^T + B(F, v^T)X + F(f)X - fA_X F - f\tau(F)X. \tag{2.148}$$

From equations (2.146) and (2.148), we obtain equations (2.143) and (2.144) respectively.

□

Theorem 2.3.4. *Let v be the new type of vector field lying on $\Gamma(TM)$ and $(M, \bar{g}, S(TM))$ be a totally geodesic null hypersurface of Lorentzian manifold $(\bar{M}, g)$. Then either the function f is constant or the Lie bracket is trivial.*

Proof. The proof follows directly from **Theorem 2.2.5**, so we omit it. □

Lemma 2.3.5. *Let v be the new type of vector field lying on $\Gamma(TM)$ and $(M, \bar{g}, S(TM))$ be null hypersurface of $(\bar{M}, g)$, then for any $F, J \in \Gamma(TM)$, we have*

$$\begin{aligned}(L_{v^T}\bar{g})(F,J) =& B(v^T,F)\theta(J) + B(v^T,J)\theta(F) + f\bar{g}(A_X F, J) + \\ & f\bar{g}(A_X J, F) + 2B(F)B(J) - 2\bar{g}(F,J),\end{aligned} \tag{2.149}$$

or equivalently

$$\begin{aligned}(L_{v^T}\bar{g})(F,J) =& B(v^T,F)\theta(J) + B(v^T,J)\theta(F) + f[\bar{g}(A_X F, J) + \\ & \bar{g}(A_X J, F)] + 2[B(F)B(J) - \bar{g}(F,J)].\end{aligned} \tag{2.150}$$

Proof. From equations (2.6) and (2.7), we obtain

$$(L_{v^T}\bar{g})(F,J) = B(v^T,F)\theta(J) + B(v^T,J)\theta(F) - \bar{g}(D_F v^T, J) - \bar{g}(D_J v^T, F), \tag{2.151}$$

or equivalently

$$(L_{v^T}\bar{g})(F,J) = (D_{v^T}\bar{g})(F,J) + \bar{g}(D_F v^T, J) + \bar{g}(D_J v^T, F). \tag{2.152}$$

Using equations (2.142) and (2.143) in equations (2.151) and (2.152), we obtain (2.149) and (2.150) respectively. □

If the new type of vector field v lies on $\Gamma(T\bar{M})$, that is $v = V^T$ then, we may state

Lemma 2.3.6. *Let v be the new type of vector field lying on $\Gamma(TM)$ and $(M, \bar{g}, S(TM))$ be null hypersurface of $(\bar{M}, g)$, then for any $X, Y \in \Gamma(TM)$, we have*

$$(L_{v^T}\bar{g})(F,J) = (D_{v^T}\bar{g})(F,J) + 2[B(F)B(J) - \bar{g}(F,J)]. \tag{2.153}$$

Proposition 2.3.1. *Let v^T be a potential vector field and $(M, \bar{g}, S(TM))$ be null hypersurface of $(\bar{M}, g)$. Then for any $F, J \in \Gamma(TM)$, we have*

$$2R^{(0,2)}(F,J) = 2\lambda\bar{g}(F,J) - f[\bar{g}(A_X F, J) + \bar{g}(A_X J, F)] - (D_{v^T}\bar{g})(F,J), \tag{2.154}$$

or equivalently

$$\begin{aligned} R^{(0,2)}(F,J) =&(\lambda-1)\bar{g}(F,J) - \frac{f}{2}[\bar{g}(A_X F,J)+\bar{g}(A_X J,F)]- \\ &\frac{1}{2}(D_{v^T}\bar{g})(F,J) - B(F)B(J). \end{aligned} \tag{2.155}$$

Proof. Since $(M,\bar{g},S(TM))$ is Ricci soliton lightlike hypersurface then it satisfies the following relation

$$(L_{v^T}\bar{g})(F,J) + 2R^{(0,2)}(F,J) = 2\lambda\bar{g}(F,J). \tag{2.156}$$

Using lemma 2.3.1 in equation (2.156), we obtain (2.154) and (2.155) respectively. □

Proposition 2.3.2. *Let v be a potential vector field and $(M,\bar{g},S(TM))$ be a Ricci soliton null (lightlike) hypersurface of a semi-Euclidean space of (n-1)-dimensional. Then for any $F \in \Gamma(S(TM))$, we have*

$$nB(F,F)traceA_X - B(A_X F,F) = \lambda - 1 - f\bar{g}(A_X F,F) - (B(F))^2, \tag{2.157}$$

where F is a unit vector field.

Proof. Let us assume that $(a_1,a_2,a_3,\ldots\ldots a_n)$ be orthonormal basis. Therefore from equation (2.156) of [60], we obtain

$$R^{(0,2)}(a_i,a_i) = nB(a_i,a_i)traceA_X - B(A_N a_i,a_i). \tag{2.158}$$

Using equation (2.155) in equation (2.158), we obtain

$$\begin{aligned} nB(a_i,a_i)traceA_X - B(A_X a_i,a_i) &= (\lambda-1)\bar{g}(a_i,a_i) - f\bar{g}(A_X a_i,a_i) \\ &\quad - (B(a_i))^2 - \frac{1}{2}(D_{v^T}\bar{g})(a_i,a_i). \end{aligned} \tag{2.159}$$

Replacing a_i with F, we obtain

$$nB(F,F)traceA_X - B(A_X F,F) = \lambda - 1 - f\bar{g}(A_X F,F) - (B(F))^2. \tag{2.160}$$

□

Corollary 2.3.3. *Let v be a new type of vector field and $(M,\bar{g},S(TM))$ be Ricci soliton null (lightlike) hypersurface of semi-Euclidean space. Then for any $F,J \in \Gamma S((TM))$, if $(M,\bar{g},S(TM))$ is screen conformal and totally umbilical, then*

$$n^2[B(F,F)]^2\phi - [B(F,F)]^2 = \lambda - 1 - 2fB(F,F) - (B(F))^2, \tag{2.161}$$

for any $F \in (S(T\bar{M}))$ is a unit vector field.

Proposition 2.3.3. *Let v be a new type of vector field on $\Gamma(TM)$ and $(M, \bar{g}, S(TM))$ be Ricci soliton null (lightlike) hypersurface of a Lorentzian manifold. Then for any $F, J \in \Gamma(S(TM))$, we have*

$$R^{(0,2)}(F, J) = (\lambda - 1)\bar{g}(F, J) - \frac{1}{2}[B(v, F)\theta(J) + B(v, J)\theta(F)] - B(F)B(J). \tag{2.162}$$

Proof. Since $(M, \bar{g}, S(TM))$ is a Ricci soliton and v is a new type of vector field lying on tangent bundle. Therefore equation (2.156), can be written as

$$(L_v\bar{g})(F, J) + 2R^{(0,2)}(F, J) = 2\lambda\bar{g}(F, J). \tag{2.163}$$

For any $F, J \in \Gamma(TM)$ and from lemma 2.3.5, we have

$$(L_v\bar{g})(F, J) = (D_v\bar{g})(F, J) + 2[B(F)B(J) - \bar{g}(F, J)]. \tag{2.164}$$

Using equation (2.163) in equation (2.164), we obtain

$$R^{(0,2)}(F, J) = (\lambda - 1)\bar{g}(F, J) - \frac{1}{2}(D_v\bar{g})(F, J) - B(F)B(J). \tag{2.165}$$

Hence, from equations (1.76) and (2.165), the proof is straightforward. □

Corollary 2.3.4. *Let v be a new type of vector field on $\Gamma(TM)$ and $(M, \bar{g}, S(TM))$ be Ricci soliton null (lightlike) hypersurface of a Lorentzian manifold. Then for any $F, J \in \Gamma(TM)$ if $(M, \bar{g}, S(TM))$ is totally geodesic then $(M, \bar{g}, S(TM))$ is Einstein Ricci soliton degenerate hypersurface.*

Proof. Assume that $(a_1, a_2, a_3, ...a_n, \xi)$ and $(a_1, a_2, a_3, ...a_n)$ be basis on $\Gamma(TM)$ and $\Gamma(S(TM))$ respectively. Then from equation (2.162), we acquire

$$R^{(0,2)}(a_i, a_j) = (\lambda - 1)\bar{g}(a_i, a_j) - \frac{1}{2}[B(v, a_i)\theta(a_j) + B(v, a_j)\theta(a_i)]. \tag{2.166}$$

$$R^{(0,2)}(a_i, a_j) = (\lambda - 1)\delta_{ij}, \tag{2.167}$$

where $i, j = 1, 2, 3, \ldots\ldots\ldots\ldots.n$ and δ_{ij} represents the Kronecker delta. Putting $F = J = \xi$ in equation (2.166), we acquire

$$R^{(0,2)}(\xi, \xi) = (\lambda - 1)\bar{g}(xi, \xi) - \frac{1}{2}[B(v, \xi)\theta(a_j) + B(v, \xi)\theta(a_i)]. \tag{2.168}$$

$$R^{(0,2)}(\xi, \xi) = \lambda - 1. \tag{2.169}$$

Now from equations (2.166) and (2.169), we obtain our required result. □

Chapter 3

Almost Generalised Weakly Symmetric Lightlike Hypersurfaces $A(GWS) - LH$ of Indefinite Kenmotsu Manifolds

The objective of this chapter is to study almost generalised weakly symmetric lightlike hypersurface $A(GWS) - LH$ and almost generalised weakly Ricci-symmetric lightlike hypersurface $A(GWRS) - LH$ of an indefinite Kenmotsu manifolds $(\bar{M}, g)$. We obtain some sharp relations between the 1-forms of an almost generalised weakly symmetric lightlike hypersurface and an almost generalised weakly Ricci-symmetric lightlike hypersurface of indefnite Kenmotsu manifold. It is shown that $A(GWRS) - LH$ of an indefnite Kenmotsu manifold admits cyclic parallelism and Codazzi type of Ricci tensor S. The existence of an almost generalised weakly symmetric lightlike hypersurface $A(GWS) - LH$ of $(\bar{M}, g)$ is been shown by a non-trivial example.

3.1 Introduction

In differential geometry the concept of a weakly symmetric Riemannian manifolds has been introduced by Tamassy and Binh [110]. An n-dimensional Riemannian manifold

M is said to be a weakly symmetric manifold [110], if the following condition is satisfied

$$\begin{aligned}(D_W\bar{R})(F,J,L,I) = {} & A_1(W)R(F,J,L,I) + B_1(F)R(W,J,L,I)+ \\ & C_1(Z)R(F,W,L,I) + D_1(L)R(F,J,W,I)+ \\ & E_1(I)R(F,J,J,W),\end{aligned} \tag{3.1}$$

and if the Ricci tensor S satisfies

$$(D_FS)(J,L) = A_1(F)S(J,L) + B_1(J)S(F,L) + C_1(L)S(J,F), \tag{3.2}$$

then M is called a weakly Ricci symmetric manifold, where A_1, B_1, C_1, D_1, E_1 are simultaneously non-vanishing 1-forms and W, F, J, L, I are vector fields.

In 1995, M. Prvanovic [97] proved that, if M is a weakly symmetric manifold satisfying equation (3.1), then $B_1 = C_1 = D_1 = E_1 = \rho$ (say) and then equations (3.1) and (3.2) becomes

$$\begin{aligned}(D_W\bar{R})(F,J,L,I) = {} & A_1(W)R(F,J,L,I) + \rho(F)R(W,J,L,I)+ \\ & \rho(J)R(F,W,L,I) + \rho(L)R(F,J,W,I)+ \\ & \rho(I)R(F,J,J,W),\end{aligned} \tag{3.3}$$

and

$$(D_FS)(J,L) = A_1(F)S(J,L) + \rho(J)S(F,L) + \rho(L)S(J,F). \tag{3.4}$$

According to [110], If the curvature tensor $\bar{R}$ of type $(0,4)$ is not identically zero and fullfilis the identity

$$\begin{aligned}(D_W\bar{R})(F,J,L,I) = {} & [A_1(W) + B_1(W)]\bar{R}(F,J,L,I) \\ & + C_1(F)\bar{R}(W,J,L,I) + C_1(J)\bar{R}(F,W,L,I) \\ & + D_1(L)\bar{R}(F,J,W,I) + D_1(I)\bar{R}(F,J,L,W)\end{aligned} \tag{3.5}$$

then weakly symmetric manifold is known to be an almost weakly- pseudo symmertic manifold, wherein A_1, B_1, C_1 and D_1 represent non-zero 1-forms characterized by $A_1(W) = g(W,\sigma_1)$, $B_1(W) = g(W,\Sigma_1)$, $C_1(W) = g(W,\pi_1)$, and $D_1(W) = g(W,\Pi_1)$, for all W, and $R(F,J,L,I) = g(R(F,J)L,I)$.

By following Dubey, we will call an n-dimensional Riemannian manifold an $A(GWS)$ manifold [37], if it satisfies the following relationship

$$\begin{aligned}(D_W\bar{R})(F,J,L,I) =&[A_1(W)+B_1(W)]\bar{R}(F,J,L,I)\\&+C_1(F)\bar{R}(W,J,L,I)+C_1(J)\bar{R}(F,W,L,I)\\&+D_1(L)\bar{R}(F,J,W,I)+D_1(I)\bar{R}(F,J,L,W)\\&+[A_2(W)+B_2(W)]\bar{G}(F,J,L,I)\\&+C_2(F)\bar{G}(W,J,L,I)+C_2(J)\bar{G}(F,W,L,I)\\&+D_2(L)\bar{G}(F,J,W,I)+D_2(I)\bar{G}(F,J,L,W)\end{aligned} \tag{3.6}$$

where

$$\bar{G}(F,J,L,I) = g(J,L)g(F,I) - g(F,L)g(J,I), \tag{3.7}$$

where A_i, B_i, C_i and D_i are non-zero 1-forms that are characterised by $A_i(W) = g(W, \sigma_i)$, $B_i(W) = g(W, \Sigma_i)$, $C_i(W) = g(W, \pi_i)$, and $D_i(W) = g(W, \Pi_i)$, for all W.

In 1920s, E. Carten [20] studied locally symmetric as an induction of the space forms. Such spaces are defined by $DR = 0$, where D and R represents Levi-Civata connection and (1,3)-type Riemannian curvature tensor. All locally symmetric spaces satisfies $\mathbf{R}.R = 0$. Where $\mathbf{R}$ acts as derivation over R. [78] investigated semi-parallel lightlike hypersurfaces of an indefinite Kenmotsu manifold and obtained geometrical configuration of such lightlike hypersurfaces. In 2009 [78] investigated semi-parallel lightlike hypersurfaces an indefinite Kenmotsu manifold, obtained certain results on parallel and semi-parallel lightlike hypersurfaces of an indefnite Kenmotsu manifolds. In 2013, [80] investigated degenerate hypersurfaces of an infinite Kenmotsu space forms and showed that locally symmetric and semi-symmetric null hypersurfaces are totally geodesic and parallel. In 2012, D.H. Jin [61] studied symmetric and semi-symmetric lightlike hypersurfaces of indefinite Kenmotsu manifolds. Y. S Balkan [11] studied non existence of lightlike hypersurface of f-Kenmotsu manifolds. In 2012, [58] studied geodesic lightlike submanifolds of indefinite Kenmots authors. R. S. Gupta and A. Sharifuddin [56] in 2010, investgated invariant, contact CR, contact SCR lightlike submanifolds of an indefinite Kenmotsu manifold.In 2014, [74] obtained some conditions under which lightlike hypersurfaces of an indefinite Kenmotsu space form, semi-parallel, local symmetry, semi-symmetry and Ricci semi-symmetry notions are equivalent. In 2017, P. Moile [84] investigated special weakly Ricci symmetric screen locally conformal lightlike hypersurface of an indefinite Kenmotsu space form.

It is well known fact that when the induced metric is degenerate, the non-degenerate screen distribution $S(TM)$ is not unique in the case of lightlike submanifolds. Since 1996, a lot of effort has gone into finding unique screens distribution. As a result, Duggal and Sahin [46] discovered a unique screen distribution for a class of semi-Riemannian manifolds as screen conformal lightlike submanifolds.

In mathematical physics and general relativity, the general theory of lightlike submanifolds have been studied from time to time and found its applications in event horizons of black horizons, the Kruskal and black holes. In (1966), Duggal and Bejancu [42] introduced the concept of degenerate particularly null-geometry of semi-Riemannian manifolds, and they gave extrinsic approach in the field of differential geometry. This created widespread interest among researchers to study lightlike geometry of semi-Riemannian manifolds. With the passage of time several authors studied null hypersurfaces of psedo-Riemannian manifolds ([7], [77], [46]). N. Aktan [2] investigated lightlike hypersurfaces of indefnite Sasakian space form and indefnite Kenmotsu space form [3], and proved non existence of null hypersurfaces of these manifolds.

Thi chapter is divided into four sections. Several known results of indefnite Kenmotsu manifolds are discussed in Section 2. In section 3, we investigate lightlike hypersurfaces of indefnite Kenmotsu manifolds. Throughout sections 4 and 5 $A(GWS) - LH$ and $A(GWRS) - LH$ of an indefnite Kenmotsu manifolds are discussed respectively.

3.2 Indefnite Kenmotsu Manifolds

Let $(\bar{M}, g)$ be an $(2n+1)$ dimensional almost contact metric manifold endowed with $(\phi, \xi, \eta, \bar{g})$ as an almost contact metric structure, where ϕ, ξ, η and g represent a $(1,1)-$tensor field, an associated vector field, a $1-$form and the Riemannian metric respectively satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \ \phi(\xi) = 0, \ \eta \circ \phi = 0, \tag{3.8}$$

$$g(\phi F, \phi J) = g(F, J) - \eta(F)\eta(J); \quad \eta(F) = g(F, \xi), \tag{3.9}$$

where I represents identity tensor field and $F,\ J \in \Gamma(TM)$. A normal contact metric manifold $(\bar{M}, g)$ is said to be a indefinite Kenmotsu manifold, if it satisfies [105]

$$(\bar{D}_F \phi)(J) = -g(\phi F, J)\xi + \eta(J)\phi F \tag{3.10}$$

and

$$\bar{D}_F \xi = -F + \eta(F)\xi. \tag{3.11}$$

For any vector fields F, J on M and $\bar{D}$ represents the Levi-Civita connection on $\bar{M}$. The plane section $\bar{\sigma}$ in tangent space $T_P M$ is called $\phi-$ section if it is spanned by F and ϕF, where F is a unit vector field and is orthogonal to ξ. $\phi-$ sectional curvature is the sectional curvature of a $\phi-$section $\bar{\sigma}$. If a indefnite Kenmotsu manifold $(\bar{M}, g)$

has constant ϕ-sectional curvature c at a point then $(\bar{M}, g)$ is said to be as indefnite Kenmotsu space form and is denoted as $\bar{M}(c)$.

The curvature tensor $\bar{R}$ of indefinite Kenmotsu space form is given ([93], [105]) by

$$\begin{aligned}\bar{R}(F,J)L &= \frac{c-3}{4}\{g(J,L)F - g(F,L)J\} \\ &+ \frac{c+1}{4}\{\eta(F)\eta(L)J - \eta(J)\eta(L)F + g(F,L)\eta(j)\xi \\ &- g(J,L)\eta(F)\xi + g(\phi J,L)\phi F - g(\phi F,L)\phi J - 2g(\phi F,J)\phi L\},\end{aligned} \tag{3.12}$$

for any $F, J, L \in \Gamma(TM)$.

3.3 Lightlike Hypersurfaces of Indefnite Kenmotsu Manifolds

Tangent to the structure vector field ξ i,e. $\xi \in \Gamma(TM)$, let $(M,\ \bar{g})$ be null hypersurface of indefnite Kenmotsu manifold $(\bar{M}, g)$. Then $(M,\ \bar{g})$ is called lightlike (degenerate) hypersurface if the induced metric $\bar{g}$ is degenerate. The manifold $\bar{M}$ having $ind(g) = 1$ and $ind(g) > 1$ are spacelike and timelike indefnite Kenmotsu manifold. If Z is the local section of radical distribution i,e $Z \in Rad(TM)$, then ϕZ is tangent to M and $g(\phi Z, Z) = 0$. Therefore $\phi(Rad(TM))$ is 1-dimensional distribution on lightlike hypersurface M.

Definition 3.3.1. Let $(\bar{M}, \phi, \xi, \eta, g)$ be a $(2n+1)$ dimensional indefinite Kenmotsu manifold and $(M, \bar{g})$ be the lightlike hypersurface of $(\bar{M}, \phi, \xi, \eta, g)$, then the hypersurface $(M, \bar{g})$ is screen semi invariant lightlike (degenerate) hypersurface if following conditions are satisfied

$$\phi(Rad(TM)) \subset S(TM) \ \ and \ \ \phi(ltr(TM)) \subset S(TM).$$

From equations (3.5) and (3.6), we have

$$g(\phi X, Z) = -g(X, \phi Z) = 0, \quad g(\phi X, X) = 0, \tag{3.13}$$

$$g(\phi X, \phi Z) = 1. \tag{3.14}$$

Therefore $\phi(Rad(TM)) \oplus \phi(ltr(TM))$ is non degenerate vector sub-bundle of S(TM) having rank 2. If lightlike hypersurface M is tangent to ξ then $\xi \in S(TM)$. Since

$g(\phi Z, \xi) = g(\phi X, \xi) = 0$, there exists non degenerate distribution D_0 on lightlike hypersurface M of rank $2n-4$ such that

$$S(TM) = \{\phi(TM^{\perp}) \oplus \phi(X(TM))\} \perp D_0 \perp < \xi > . \tag{3.15}$$

where $< \xi >= span\xi$.

The second fundamental form B is independent on screen distribution so we have the following equation.

$$B(.,Z) = 0. \tag{3.16}$$

From 1st equation of (3.6), we see that $\phi Z \quad and \quad \phi X$ are degenerate vector fields. Let us assume that the local degenerate vector fields $Y = -\phi X \in \phi(ltr(TM))$ and $V = -\phi Z \in D$ and suppose P be the projection of screen bundle $S(TM)$ on degenerate hypersurface M. For any $F \in \Gamma(TM)$ can be written as

$$F = SF + QF, \quad QF = u(F)Y. \tag{3.17}$$

Applying ϕ to the above equation, we acheive

$$\phi F = \phi(SF) + u(F)\phi Y, \tag{3.18}$$

where Q and S are the projection morphisms of tangent bundle into distributions G and G^1 respectively and $u(F) = g(V, F)$ is the differential 1-form locally defined on degenerate hypersurface. Putting $\bar{\phi}F = \phi(Sx)$, for any $F \in \Gamma(TM)$ in the last equation, we obtain

$$\phi F = \bar{\phi}F - u(F)X, \tag{3.19}$$

where ϕ and $\bar{\phi}$ are tensor fields on $\bar{M}$ and M respectively. Applying ϕ again to equation (3.19), we aquire

$$\phi^2 F = \phi\bar{\phi}F - u(F)\phi X, \tag{3.20}$$

$$-F + \eta(F)\xi = \phi\bar{\phi}F - u(F)Y. \tag{3.21}$$

In the above equations, we note that if for any $F \in \Gamma(TM)$, $SF \in D$, $\phi F = \bar{\phi}(SF) \in D$, then $S(\bar{\phi}F) = \bar{\phi}F$ and forgoing equation leads us to

$$\phi^2 F = -F + \eta(F)\xi + u(F)Y, \tag{3.22}$$

$$D_F\xi = F - \eta(F)\xi. \tag{3.23}$$

By using equations (1.74), (1.75) and (3.14), we acquire

$$(D_F\phi)J = g(\phi F, J)\xi - \eta(J)\phi F - B(F, J)X - u(J)A_X F, \tag{3.24}$$

and

$$(\bar{D}_F u)J = -B(F, \phi J) - u(J)\tau(F) - \eta(J)u(F). \tag{3.25}$$

For lightlike hypersurface of indefnite Kenmotsu manifold [61], we have

$$v(F) = g(X, F), \tag{3.26}$$

and

$$R(F, J)L = c\{g(J, L)F - g(F, L)J\}, \tag{3.27}$$

forall $F, J, L \in \Gamma(TM)$.

$$R^{(0,2)}(F, J) = trace\ \{L \to R(L, F)J\} \tag{3.28}$$

forall $F, J \in \Gamma(TM)$. Equations (3.27) and (3.28), represent curvature tensor R of constant curvature c and non-symmetric induced Ricci tensor $R^{(0,2)}$ and is symmetric if and only if $d\tau = 0$ [61] . Lightlike hypersurface M is said to be of constant negative curvature -1, if

$$R(F, J)L = g(F, L)J - g(J, L)F, \tag{3.29}$$

forall $F, J, L \in \Gamma(TM)$ [61].

3.4 A(GWS)-LH of Indefnite Kenmotsu Manifolds

A lightlike hypersurface M of an indefnite Kenmotsu manifold is said to be an $A(GWS)-LH$ of an indefnite Kenmotsu manifold, if it satisfies

$$\begin{aligned}
(D_W R)(F, J, L, I) = {} & [A_1(W) + B_1(W)]R(F, J, L, I) \\
& + C_1(F)R(W, J, L, I) + C_1(J)R(F, W, L, I) \\
& + D_1(L)R(F, J, W, I) + D_1(I)R(F, J, L, W) \\
& + [A_2(W) + B_2(W)]G(F, J, L, I) \\
& + C_2(F)G(W, J, L, I) + C_2(J)G(F, W, L, I) \\
& + D_2(L)G(F, J, W, I) + D_2(I)G(F, J, L, W),
\end{aligned} \tag{3.30}$$

where A_i, B_i, C_i and D_i are non-zero 1-forms that are characterised by $A_i(W) = g(W, \sigma_i)$, $B_i(W) = g(W, \Sigma_i)$, $C_i(W) = g(W, \pi_i)$, and $D_i(W) = g(W, \Pi_i)$, for all W.

Theorem 3.4.1. *Let $(M, \bar{g})$ be an $A(GWS) - LH$ of an indefnite Kenmotsu manifold $(\bar{M}, g)$, then for all $W,\ F,\ J,\ L,\ I \in \Gamma(TM)$ the sum of non-zero $1-$forms are related as*

$$\begin{aligned}&(2n-2)[A_1(\xi)+B_1(\xi)+C_1(\xi)+D_1(\xi)]\\&=(2n-1)[A_2(\xi)+B_2(\xi)+C_2(\xi)+D_2(\xi)]\end{aligned} \tag{3.31}$$

Proof. Contracting equation (3.30) on both sides, we get

$$\begin{aligned}(D_XS)(J,L)=&[A_1(W)+B_1(W)]S(J,L)+C_1(R(W,J)L)\\&+C_1(J)S(W,L,)+D_1(L)S(J,W)+D_1(R(W,L)J)\\&+(2n-1)[\{A_2(W)+B_2(W)\}g(J,L)+C_2(J)g(W,L)]\\&+D_2(L)(2n-1)g(J,W)+C_2(G(W,J)L)+D_2(G(W,L)J),\end{aligned} \tag{3.32}$$

Putting $L=\xi$ in equation (3.32), we obtain

$$\begin{aligned}(D_XS)(J,\xi)=&[A_1(W)+B_1(W)]S(J,\xi)+C_1(R(W,J)\xi)\\&+C_1(J)S(W,\xi)+D_1(\xi)S(J,W)+D_1(R(W,\xi)J)\\&+(2n-1)[\{A_2(W)+B_2(W)\}g(J,\xi)+C_2(J)g(W,\xi)]\\&+D_2(\xi)(2n-1)g(J,W)+C_2(G(W,J)\xi)+D_2(G(W,\xi)J).\end{aligned} \tag{3.33}$$

From equation (3.29) and using $S(W,\xi)=-(2n-2)\eta(W)$ in equation (3.33), we acquire

$$\begin{aligned}(D_XS)(J,\xi)=&-(2n-2)[A_1(W)+B_1(W)]\eta(J)-(2n-2)C_1(J)\eta(W)\\&+C_1(J)\eta(W)-C_1(W)\eta(J)+D_1(\xi)S(J,W)\\&+D_1(\xi)g(W,J)-\eta(J)D_1(W)+(2n-1)[\{A_2(W)\\&+B_2(W)\}\eta(J)+C_2(J)\eta(W)+D_2(\xi)g(J,W)]\\&+C_2(W)\eta(J)-C_2(J)\eta(W)+D_2(W)\eta(J)-D_2(\xi)g(J,W).\end{aligned} \tag{3.34}$$

Since we know that

$$(D_WS)(J,L)=D_WS(J,L)-S(D_WJ,L)-S(J,D_WL). \tag{3.35}$$

Replacing L by ξ in equation (3.35) equation, we acheive

$$(D_WS)(J,\xi)=D_WS(J,\xi)-S(D_WJ,\xi)-S(J,D_W\xi). \tag{3.36}$$

Using equation (3.23), $(D_W\eta)(F) = g(F,W) - \eta(W)\eta(F)$ and $S(W,\xi) = -(2n-2)\eta(W)$ in equation (3.36), we acquire

$$(D_W S)(J,\xi) = -(2n-2)g(W,J) - S(J,W). \tag{3.37}$$

From equation (3.34) and equation (3.37), we obtain

$$\begin{aligned}-(2n-2)g(W,J) - S(J,W) = &- (2n-2)[A_1(W) + B_1(W)]\eta(J)\\ &- (2n-2)C_1(J)\eta(W) + C_1(J)\eta(W) - C_1(W)\eta(J)\\ &+ D_1(\xi)S(J,W) + D_1(\xi)g(W,J) - \eta(J)D_1(W)\\ &+ (2n-1)[\{A_2(W) + B_2(W)\}\eta(J) + C_2(J)\eta(W)\\ &+ D_2(\xi)g(J,W)] + C_2(W)\eta(J) - C_2(J)\eta(W)\\ &+ D_2(W)\eta(J) - D_2(\xi)g(J,W).\end{aligned} \tag{3.38}$$

By setting $W = J = \xi$ in equation (3.38), we obtain

$$(2n-2)[A_1(\xi)+B_1(\xi)+C_1(\xi)+D_1(\xi)] = (2n+1)[A_2(\xi)+B_2(\xi)+C_2(\xi)+D_2(\xi)]. \tag{3.39}$$

Hence from equation (3.39), we may state □

Corollary 3.4.1. *For all $W, F, J, L, I \in \Gamma(TM)$, let $(M, \bar{g})$ be an $A(GWS) - LH$ of $(\bar{M}, g)$. If in perticular $A_2(\xi) = B_2(\xi) = C_2(\xi) = D_2(\xi) = 0$, then*

$$A_1(\xi) + B_1(\xi) + C_1(\xi) + D_1(\xi) = 0. \tag{3.40}$$

Theorem 3.4.2. *Let $(M, \bar{g})$ be an $A(GWS) - LH$ of $(\bar{M}, g)$, then for all $W, F, J, L, I \in \Gamma(TM)$, the sum of non-zero $1-$forms are related by*

$$\begin{aligned}&(2n-2)[A_1(W) + B_1(W) + C_1(W) + D_1(W)]\\ &= (2n-1)[A_2(W) + B_2(W) + C_2(W) + D_2(W)]\\ &+ (2n-2)C_2(W).\end{aligned} \tag{3.41}$$

Proof. Using equation (3.23), $(D_W\eta)(F) = g(F,W) - \eta(W)\eta(F)$ and $S(W,\xi) = -(2n-2)\eta(W)$ in equation (3.32) and also for $J = \xi$, we obtain

$$\begin{aligned}(D_WS)(\xi,L) =& -(2n-2)[A_1(W)+B_1(W)]\eta(L) + C_1(\xi)g(W,L)\\ &- C_1(W)\eta(L) + C_1(\xi)S(W,L) + D_1(L)\eta(W)\\ &- \eta(L)D_1(W) - (2n-2)D_1(L)\eta(W) + (2n-1)[A_2(W)\\ &+ B_2(W)]\eta(L) + C_2(\xi)g(W,L) + (2n-1)D_2(L)\eta(W)\\ &+ C_2(W)\eta(L) - C_2(\xi)g(W,L) + D_2(W)\eta(L) - D_2(L)\eta(W).\end{aligned} \tag{3.42}$$

Putting $J = \xi$ in equation (3.35) and using (3.23), $(D_W\eta)(F) = g(F,W) - \eta(W)\eta(F)$ and $S(W,\xi) = -(2n-2)\eta(W)$, we acquire

$$(D_WS)(\xi,L) = -(2n-2)g(W,L) - S(L,W). \tag{3.43}$$

From equation (3.42) and equation (3.43), we get

$$\begin{aligned}-(2n-2)g(W,L) - S(L,W) =& -(2n-2)[A_1(W)+B_1(W)]\eta(L)\\ &+ C_1(\xi)g(W,L) - C_1(W)\eta(L)\\ &+ C_1(\xi)S(W,L) + D_1(L)\eta(W) - \eta(L)D_1(W)\\ &- (2n-2)D_1(L)\eta(W) + (2n-1)[A_2(W)+B_2(W)]\eta(L)\\ &+ C_2(\xi)g(W,L) + (2n-1)D_2(L)\eta(W) + C_2(W)\eta(L)\\ &- C_2(\xi)g(W,L) + D_2(W)\eta(L) - D_2(L)\eta(W).\end{aligned} \tag{3.44}$$

Putting $L = \xi$ in (3.44) and using equation (3.23), $(D_W\eta)(F) = g(F,W) - \eta(W)\eta(F)$ and $S(W,\xi) = -(2n-2)\eta(W)$ in equation (3.44), we obtain

$$\begin{aligned}&-(2n-2)[A_1(W)+B_1(W)] - (2n-2)C_1(\xi)\eta(W)\\ &+ C_1(\xi)\eta(W) - C_1(W) - (2n-2)D_(\xi)\eta(W) + D_(\xi)\eta(W)\\ &- D_1(W) + (2n-1)[A_2(W)+B_2(W)] + (2n-1)D_2(\xi)\eta(W)\\ &+ C_2(W) + D_2(\xi)\eta(W) = 0.\end{aligned} \tag{3.45}$$

Putting $W = \xi$ in (3.44) and by making use of and using (3.23), $(D_W\eta)(F) = g(F,W) - \eta(W)\eta(F)$ and $S(W,\xi) = -(2n-2)\eta(W)$ in equation (3.45), we obtain

$$\begin{aligned}&-(2n-2)[A_1(\xi)+B_1(\xi)\eta(L)] - (2n-2)C_1(\xi)\eta(L)\\ &+ D_1(L) - D_1(\xi)\eta(L) - (2n-2)D_1(L) + (2n-1)[A_2(\xi)\\ &+ B_2(\xi)\eta(L)] + (2n-1)D_2(L) - C_2(\xi)\eta(L) + D_2(\xi)\eta(L) - D_2(L) = 0.\end{aligned} \tag{3.46}$$

After replacing L by W in equation (3.46), we get

$$\begin{aligned}
&-(2n-2)[A_1(\xi)+B_1(\xi)]\eta(W)-(2n-2)C_1(\xi)\eta(W)\\
&-(2n-3)D_1(W)-D_1(\xi)\eta(W)+(2n-1)[A_2(\xi)+B_2(\xi)\eta(W)]\\
&+(2n-2)D_2(W)-C_2(\xi)\eta(W)+D_2(\xi)\eta(W)=0.
\end{aligned}\tag{3.47}$$

Adding equations (3.45) and (3.47), we acquire

$$\begin{aligned}
&-(2n-2)[A_1(W)+B_1(W)+C_1(W)+D_1(W)]+(2n-3)C_1(W)\\
&+(2n-1)[A_2(W)+B_2(W)+C_2(W)+D_2(W)]+(2n-2)C_2(W)\\
&-(4n-5)C_1(\xi)\eta(W)-(2n-2)D_1(\xi)\eta(W)+2nD_2(\xi)\eta(W)-C_2(\xi)\eta(W)\\
&-(2n-2)[A_1(\xi)+B_1(\xi)]\eta(W)+(2n-1)[A_2(\xi)+B_2(\xi)]\eta(W)=0.
\end{aligned}\tag{3.48}$$

Putting $W=\xi$ in (3.38), we acheive

$$\begin{aligned}
&-(2n-2)[A_1(\xi)+B_1(\xi)+C_1(\xi)+D_1(\xi)]\eta(W)\\
&-(2n-3)C_1(W)+(2n-3)C_1(\xi)\eta(W)+(2n-1)[A_2(\xi)+B_2(\xi)\\
&+C_2(\xi)+D_2(\xi)]\eta(W)-(2n-2)C_2(\xi)\eta(W)-(2n-2)D_2(\xi)\eta(W).
\end{aligned}\tag{3.49}$$

In view of equation (3.39), equation (3.49) becomes

$$\begin{aligned}
&-(2n-3)C_1(W)+(2n-3)C_1(\xi)\eta(W)\\
&-(2n-2)C_2(\xi)\eta(W)-(2n-2)D_2(\xi)\eta(W)=0.
\end{aligned}\tag{3.50}$$

Adding equations (3.48) and (3.50), we obtain

$$\begin{aligned}
&-(2n-2)[A_1(W)+B_1(W)+C_1(W)+D_1(W)]\\
&+(2n-1)[A_2(W)+B_2(W)+C_2(W)+D_2(W)]\\
&-(2n-2)[A_1(\xi)+B_1(\xi)+C_1(\xi)+D_1(\xi)]\eta(W)\\
&+(2n-1)[A_2(\xi)+B_2(\xi)+C_2(\xi)+D_2(\xi)]\eta(W)\\
&+(2n-2)C_2(W)-(4n-2)C_2(\xi)\eta(W)-(6n-3)D_2(\xi)\eta(W)=0.
\end{aligned}\tag{3.51}$$

Using corollary 3.4.1 in equation, we get

$$\begin{aligned}
&(2n-2)[A_1(W)+B_1(W)+C_1(W)+D_1(W)]\\
&=(2n-1)[A_2(W)+B_2(W)+C_2(W)+D_2(W)]+(2n-2)C_2(W).
\end{aligned}\tag{3.52}$$

Now if in equation (3.52), $A_2=B_2=C_2=D_2=0$ then, we may state □

Theorem 3.4.3. *Let $(M, \bar{g})$ be $(WS) - LH$ of $(\bar{M}, g)$ tangent to ξ, then*

$$A_1(W) + B_1(W) + C_1(W) + C_1(W) = 0, \tag{3.53}$$

where A_1, B_1, C_1 and D_1 are the associated $1-$ forms.

3.5 A(GWRS)-LH of Indefnite Kenmotsu Manifolds

A lightlike hypersurface $(M, \bar{g})$ of an indefnite Kenmotsu manifold $(\bar{M}, g)$ is an $A(GWRS)-LH$, if satisfying

$$\begin{aligned}(D_W S)(J, L) =& [A_1(W) + B_1(W)]S(J, L) + C_1(J)S(W, L)\\ &+ D_1(L)S(J, W) + [A_2(W) + B_2(W)]g(J, L)\\ &+ C_2(J)g(W, L) + D_2(L)g(J, W),\end{aligned} \tag{3.54}$$

where A_i, B_i, C_i and D_i are non-zero 1-forms and are characterised by $A_i(W) = g(W, \sigma_i)$, $B_i(W) = g(W, \Sigma_i)$, $C_i(W) = g(W, \pi_i)$, and $D_i(W) = g(W, \Pi_i)$, for all W.

Theorem 3.5.1. *Let $(M, \bar{g})$ be an almost generilised weakly symmetric lightlike hypersurface $A(GWRS) - LH$ of an indefinite Kenmotsu manifold $(\bar{M}, g)$, tangent to ξ then*

$$\begin{aligned}&(2n-2)[A_1(W) + B_1(W) + C_1(W) + D_1(W)]\\ &= A_2(W) + B_2(W) + C_2(W) + D_2(W),\end{aligned} \tag{3.55}$$

where A_i, B_i, C_i and D_i are the associated $1-$ forms but non zero.

Proof. Putting $L = \xi$ in equation(3.54), we get

$$\begin{aligned}(D_W S)(J, \xi) =& [A_1(W) + B_1(W)]S(J, \xi) + C_1(J)S(W, \xi)\\ &+ D_1(\xi)S(J, W) + [A_2(W) + B_2(W)]g(J, \xi)\\ &+ C_2(J)\eta(W) + D_2(\xi)g(J, W).\end{aligned} \tag{3.56}$$

Using equation (3.37) in equation (3.56), we obtain

$$\begin{aligned}-(2n-2)g(W, J) - S(J, W) =& -(2n-2)[A_1(W) + B_1(W)]\eta(J)\\ &- (2n-2)C_1(J)\eta(W) + D_1(\xi)S(J, W)\\ &+ [A_2(W) + B_2(W)]\eta(J) + C_2(J)\eta(W)\\ &+ D_2(\xi)g(J, W).\end{aligned} \tag{3.57}$$

Putting $W = J = \xi$ in in equation (3.57) and using equation (3.23), $(D_W\eta)(F) = g(F,W) - \eta(W)\eta(F)$ and $S(\xi,\xi) = -(2n-2)$ in the resulting equation, we acquire

$$\begin{aligned} &-(2n-2)[A_1(\xi) + B_1(\xi) + C_1(\xi) + D_1(\xi)] \\ &+ [A_2(\xi) + B_2(\xi) + C_2(\xi) + D_2(\xi)] = 0. \end{aligned} \tag{3.58}$$

Replacing ξ for W in equation (3.57), we get

$$\begin{aligned} &-(2n-2)[A_1(\xi) + B_1(\xi) + D_1(\xi)]\eta(J) - (2n-2)C_(J) \\ &+ [A_2(\xi) + B_2(\xi) + D_2(\xi)]\eta(J) + C_2(J) = 0. \end{aligned} \tag{3.59}$$

Again setting $J = \xi$ in equation (3.57) and using equation (3.23), $(D_W\eta)(F) = g(F,W) - \eta(W)\eta(F)$ and $S(\xi,\xi) = -(2n-2)$ in the resulting equation, we obtain

$$\begin{aligned} &-(2n-2)[A_1(W) + B_1(W)] - (2n-2)C_1(\xi) - (2n-2)D_1(\xi)\eta(W) \\ &+ [A_2(W) + B_2(W)] + C_2(\xi)\eta(W) + D_2(\xi)\eta(W) = 0. \end{aligned} \tag{3.60}$$

Letting $J = W$ in equation (3.59) and adding the resulting equation with equation (3.60), we obtain

$$\begin{aligned} &-(2n-2)[A_1(W) + B_1(W) + C_1(W)] + [A_2(W) + B_2(W) + C_2(W)] \\ &-(2n-2)[A_1(\xi) + B_1(\xi) + C_1(\xi) + D_1(\xi)]\eta(W) \\ &+ [A_2(\xi) + B_2(\xi) + C_2(\xi) + D_2(\xi)]\eta(W) \\ &-(2n-2)D_1(\xi)\eta(W) + D_2(\xi)\eta(W). \end{aligned} \tag{3.61}$$

Using equation (3.58) in equation (3.61), we get

$$\begin{aligned} &-(2n-2)[A_1(W) + B_1(W) + C_1(W)] + [A_2(W) + B_2(W) + C_2(W)] \\ &-(2n-2)D_1(\xi)\eta(W) + D_2(\xi)\eta(W). \end{aligned} \tag{3.62}$$

Setting $W = J = \xi$ in equation (3.54), we obtain

$$\begin{aligned} &-(2n-2)[A_1(\xi) + B_1(\xi) + C_1(\xi)]\eta(L) + [A_2(\xi) + B_2(\xi) + C_2(\xi)]\eta(L) \\ &-(2n-2)D_1(L) + D_2(L) = 0. \end{aligned} \tag{3.63}$$

Setting $L = W$ in equation (3.63) and adding the resulting equation with equation (3.62), we acquire

$$\begin{aligned}
&-(2n-2)[A_1(W) + B_1(W) + C_1(W) + D_1(W)] \\
&+ [A_2(W) + B_2(W) + C_2(W) + D_2(W)] \\
&-(2n-2)[A_1(\xi) + B_1(\xi) + C_1(\xi) + D_1(\xi)]\eta(W) \\
&+ [A_2(\xi) + B_2(\xi) + C_2(\xi) + D_2(\xi)]\eta(W) = 0.
\end{aligned} \tag{3.64}$$

Using equation (3.58) in equation (3.64), we get

$$\begin{aligned}
&(2n-2)[A_1(W) + B_1(W) + C_1(W) + D_1(W)] \\
&= [A_2(W) + B_2(W) + C_2(W) + D_2(W)].
\end{aligned} \tag{3.65}$$

□

Theorem 3.5.2. *Let $(M, \bar{g})$ be an A(GWRS)-LH of $(\bar{M}, g)$ of constant curcature -1 satisfying Einestein hypersurface properties. Then for any $F = Z \in \Gamma(Rad(TM))$, $\lambda = -(2n-2)$.*

Proof. From equations (3.43) and (3.56), we obtain

$$\begin{aligned}
&-(2n-2)g(W, L) - S(L, W) \\
&= [A_1(W) + B_1(W)]S(J, \xi) + C_1(J)S(W, \xi) + D_1(\xi)S(J, W) \\
&+ [A_2(W) + B_2(W)]g(J, \xi) + C_2(J)\eta(W) + D_2(\xi)g(J, W).
\end{aligned} \tag{3.66}$$

Since $(M, \bar{g})$ is Einestein lightlike hypersurface then equation (3.66), leads us to (3.67).

$$\begin{aligned}
&-(2n-2)g(W, L) - \lambda g(L, W) \\
&= [A_1(W) + B_1(W)]\lambda g(J, \xi) + C_1(J)\lambda g(W, \xi) + D_1(\xi)\lambda g(J, W) \\
&+ [A_2(W) + B_2(W)]g(J, \xi) + C_2(J)\eta(W) + D_2(\xi)g(J, W).
\end{aligned} \tag{3.67}$$

Putting $W = L = \xi$ in equation (3.67), we acquire

$$\begin{aligned}
-(2n-2) - \lambda =& [A_1(\xi) + B_1(\xi) + C_1(\xi) + D_1(\xi)]\lambda \\
&+ [A_2(\xi) + B_2(\xi) + C_2(\xi) + D_2(\xi)].
\end{aligned} \tag{3.68}$$

Using corollary 4.2 in equation (3.68), we get equation (3.69)

$$\lambda = -(2n-2). \tag{3.69}$$

□

Theorem 3.5.3. *Let $(M, \bar{g})$ be an almost generalised weakly Ricci symmetric lightlike hypersurface $A(GWRS) - LH$ of an indefinite Kenmotsu manifold $(\bar{M}, g)$ of negative curvature -1, then $A(GWRS) - LH$ admits a cyclic parallel Ricci tensor S.*

Proof. For any vector fields $W, J, L \in \Gamma(TM)$, if the cyclic sum of covariant derivative of Ricci tensor equals zero then a non-zero Ricci Tensor S of $(M, \bar{g})$ is said to be cyclic parallel [77] and satisfies equation (3.70)

$$(D_W S)(J, L) + (D_J S)(L, W) + (D_L S)(W, J) = 0. \tag{3.70}$$

From equation (3.70), we have

$$\begin{aligned}&(D_W S)(J, L) = \\ &[A_1(W) + B_1(W)]S(J, L) + C_1(J)S(W, L) + D_1(L)S(J, W) \\ &+ [A_2(W) + B_2(W)]g(J, L) + C_2(J)g(W, L) + D_2(L)g(J, W),\end{aligned} \tag{3.71}$$

$$\begin{aligned}&(D_J S)(L, W) = \\ &[A_1(J) + B_1(J)]S(L, W) + C_1(L)S(J, W) + D_1(W)S(L, J) \\ &+ [A_2(J) + B_2(J)]g(L, W) + C_2(J)g(L, W) + D_2(W)g(L, J),\end{aligned} \tag{3.72}$$

and

$$\begin{aligned}&(D_L S)(W, J) = \\ &[A_1(L) + B_1(L)]S(W, J) + C_1(W)S(L, J) + D_1(J)S(W, L) \\ &+ [A_2(L) + B_2(L)]g(W, J) + C_2(W)g(L, J) + D_2(J)g(W, L).\end{aligned} \tag{3.73}$$

Substituting equations (3.71), (3.72) and (3.73) in equation (3.70), we derive

$$\begin{aligned}&[A_1(W) + B_1(W)]S(J, L) + C_1(J)S(W, L) + D_1(L)S(J, W) \\ &+ [A_2(W) + B_2(W)]g(J, L) + C_2(J)g(W, L) + D_2(L)g(J, W) \\ &+ [A_1(J) + B_1(J)]S(L, W) + C_1(L)S(J, W) + D_1(W)S(L, J) \\ &+ [A_2(J) + B_2(J)]g(L, W) + C_2(J)g(L, W) + D_2(W)g(L, J) \\ &+ [A_1(L) + B_1(L)]S(W, J) + C_1(W)S(L, J) + D_1(J)S(W, L) \\ &+ [A_2(L) + B_2(L)]g(W, J) + C_2(W)g(L, J) + D_2(J)g(W, L) = 0.\end{aligned} \tag{3.74}$$

Setting $W = Z \in \Gamma Rad(TM)$ in equation (3.74), we obtain (3.70). $\square$

Example of an A(GWS)-LH of Indefnite Kenmotsu Manifolds

Let $\bar{M}$ be 3−dimensional almost generakised weakly Ricci symmetric lightkike hypersurface $A(GWRS) - LH$ of an indefnite Kenmotsu manifold M with a $\phi-$ basis, such that

$$\ddot{E}_1 = e^z(\frac{\partial}{\partial f}), \quad \ddot{E}_2 = \phi\ddot{E}_1 = e^{z-\alpha f}\frac{\partial}{\partial j}, \; \ddot{E}_3 = \xi = \frac{\partial}{\partial z}, \tag{3.75}$$

where $\{\ddot{E}_1, \ddot{E}_2, \ddot{E}_3\}$ and $\alpha \neq 0$ are linearly independent global frame on $\bar{M}$ and non zero constant respectively. The induced metric $\bar{g}$, is defined by

$$\bar{g}(\ddot{E}_1, \ddot{E}_2) = \bar{g}(\ddot{E}_2, \ddot{E}_3) = \bar{g}(\ddot{E}_3, \ddot{E}_1) = 0, \tag{3.76}$$

$$\bar{g}(\ddot{E}_1, \ddot{E}_1) = \bar{g}(\ddot{E}_2, \ddot{E}_2) = 1, \bar{g}(\ddot{E}_3, \ddot{E}_3) = -1. \tag{3.77}$$

With respect to metric $\bar{g}$, let D and R be the Levi-Civita connection and Riemannian curvature, then

$$[\ddot{E}_1, \ddot{E}_2] = -\alpha e^z\ddot{E}_2, \; [\ddot{E}_1, \ddot{E}_3] = -\ddot{E}_1, \; [\ddot{E}_2, \ddot{E}_3] = -\ddot{E}_2. \tag{3.78}$$

Using linearity property of $\bar{g}$, ϕ and the above results it is easily seen that $(\phi, \xi, \eta, \bar{g})$ is an almost contact metric structure.

Using $Koszul's$ formula we will calculate Levi-Civita connection and non vanishing covariant derivatives of R respectively by

$$\begin{aligned}
&D_{\ddot{E}_1}\ddot{E}_3 = \ddot{E}_2, \; D_{\ddot{E}_1}\ddot{E}_2 = 0, \; D_{\ddot{E}_1}\ddot{E}_1 = -\ddot{E}_3, \\
&D_{\ddot{E}_2}\ddot{E}_3 = \ddot{E}_1 D_{\ddot{E}_2}\ddot{E}_2 = \alpha e^z\ddot{E}_3, \; D_{\ddot{E}_2}\ddot{E}_1 = \alpha e^z\ddot{E}_2, \\
&D_{\ddot{E}_3}\ddot{E}_3 = 0, \quad D_{\ddot{E}_3}\ddot{E}_2 = 0, \; D_{\ddot{E}_3}\ddot{E}_1 = 0,
\end{aligned} \tag{3.79}$$

$$\begin{aligned}
&R(\ddot{E}_1, \ddot{E}_2, \ddot{E}_1, \ddot{E}_2) = -(1 - \alpha^2 e^{2z}) \\
&R(\ddot{E}_1, \ddot{E}_3, \ddot{E}_1, \ddot{E}_3) = 1R(\ddot{E}_2, \ddot{E}_3, \ddot{E}_2, \ddot{E}_3) = 1.
\end{aligned} \tag{3.80}$$

Therefore for any vector fields $F, J, Y, L \in \chi(\bar{M})$, we have

$$F = \sum_1^3 m_i\ddot{E}_i, \; J = \sum_1^3 n_i\ddot{E}_i, \; Y = \sum_1^3 o_i\ddot{E}_i, \; L = \sum_1^3 p_i\ddot{E}_i, \tag{3.81}$$

for all $\{\ddot{E}_1, \ddot{E}_2, \ddot{E}_3\}$ forms $\phi-$ basis.

$$\begin{aligned}
R(F, J, Y, L) = &-(m_1n_2 - m_2n_1)(o_1p_2 - o_2p_1)(1 - \alpha^2 e^{2z}) \\
&+ (m_2n_3 - (m_3n_2)(o_2p_3 - o_3p_2) = T_1(say).
\end{aligned} \tag{3.82}$$

Following T_1 , we get

$$
\begin{aligned}
&R(\ddot{E}_1, J, Y, L) = \Gamma_1\ (say),\ R(\ddot{E}_2, J, Y, L) = \Gamma_2\ (say),\\
&R(\ddot{E}_3, J, Y, L) = \Gamma_3\ (say),\ R(F, \ddot{E}_1, Y, L) = \Gamma_4\ (say),\\
&R(F, \ddot{E}_2, Y, L) = \Gamma_5\ (say),\ R(F, \ddot{E}_3, Y, L) = \Gamma_6\ (say)\\
&R(F, J, \ddot{E}_1, L) = \Gamma_7\ (say),\ R(F, J, \ddot{E}_2, L) = \Gamma_8\ (say),\\
&R(F, J, \ddot{E}_3, L) = \Gamma_9\ (say),\ R(F, J, Y, \ddot{E}_1) = \Gamma_{10}\ (say),\\
&R(F, J, Y, \ddot{E}_2) = \Gamma_{11}\ (say), R(F, J, Y, \ddot{E}_3) = \Gamma_{12}\ (say),
\end{aligned}
\tag{3.83}
$$

$$
\begin{aligned}
G(F, J, Y, L) =&(n_1o_1 + n_2o_2 - n_3o_3)(m_1p_1 + m_2p_2 - m_3p_3)\\
&- (m_1o_1 + m_2o_2 - m_3o_3)(n_1p_1 + n_2p_2 - n_3p_3) = T_2\ (say).
\end{aligned}
\tag{3.84}
$$

Keeping the tone of T_2, we obtain following results

$$
\begin{aligned}
&G(\ddot{E}_1, J, Y, L) =\ \bar{\omega}_1\ (say),\ G(\ddot{E}_2, J, Y, L) =\ \bar{\omega}_2\ (say)\\
&G(\ddot{E}_3, J, Y, L) =\ \bar{\omega}_3\ (say),\ G(F, \ddot{E}_1, Y, L) =\ \bar{\omega}_4\ (say)\\
&G(F, \ddot{E}_2, Y, L) =\ \bar{\omega}_5\ (say),\ G(F, \ddot{E}_3, Y, L) =\ \bar{\omega}_6\ (say),\\
&G(F, J, \ddot{E}_1, L) =\ \bar{\omega}_7\ (say),\ G(F, J, \ddot{E}_2, L) =\ \bar{\omega}_8\ (say),\\
&G(F, J, \ddot{E}_3, L) =\ \bar{\omega}_9\ (say),\ G(F, J, Y, \ddot{E}_1) =\ \bar{\omega}_{10}\ (say)\\
&G(F, J, Y, \ddot{E}_2) =\ \bar{\omega}_{11}\ (say),\ G(F, J, Y, \ddot{E}_3) =\ \bar{\omega}_{12}\ (say).
\end{aligned}
\tag{3.85}
$$

Now, we will obtain the covariant derivative of non vanishing components of R by using Kosozul's forrmula as

$$
\begin{aligned}
(D_{\ddot{E}_1}R)(F, J, Y, L) =&m_1\Gamma_3 - m_3\Gamma_2 + n_1\Gamma_6 - n_3\Gamma_5 + o_1\Gamma_9\\
&- o_3\Gamma_8 + p_1\Gamma_{12} - p_3\Gamma_1 1,
\end{aligned}
\tag{3.86}
$$

$$
\begin{aligned}
&(D_{\ddot{E}_2}R)(F, J, Y, L) = -\alpha e^z m_1\Gamma_2 - \alpha e^z m_2\Gamma_3 - m_3\Gamma_1 - \alpha e^z n_1\Gamma_5 - \alpha e^z n_2\Gamma_6\\
&- n_3\Gamma_4 - \alpha e^z o_1\Gamma_8 - \alpha e^z o_2\Gamma_9 - o_3\Gamma_7 - \alpha e^z p_1\Gamma_{11} - \alpha e^z p_2\Gamma_{12} - zp_3\Gamma_{10},
\end{aligned}
\tag{3.87}
$$

and

$$
(D_{\ddot{E}_1}R)(F, J, Y, L) = 2\alpha^2 e^{2z}(n_1 - m_2n_1)(o_1p_2 - o_2p_1). \tag{3.88}
$$

One may easily verify the relationships

$$
\ddot{A}_1(\ddot{E}_1) = \frac{m_1\Gamma_3 - m_3\Gamma_2}{T_1},\quad \ddot{A}_2(\ddot{E}_1) = \frac{o_1\Gamma_9 - o_3\Gamma_8}{T_2}, \tag{3.89}
$$

$$
\ddot{B}_1(\ddot{E}_1) = \frac{n_1\Gamma_6 - n_3\Gamma_5}{T_1},\quad \ddot{B}_2(\ddot{E}_1) = \frac{p_1\Gamma_{12} - p_3\Gamma_{11}}{T_2}, \tag{3.90}
$$

$$\ddot{A}_1(\ddot{E}_2) = -\frac{\alpha e^z m_1\Gamma_2 - \alpha e^z m_2\Gamma_3 + m_3\Gamma_1}{T_1}, \quad \ddot{A}_2(\ddot{E}_2) = -\frac{\alpha e^z o_1\Gamma_8 + \alpha e^z o_2\Gamma_9 + o_3\Gamma_7}{T_2}, \tag{3.91}$$

$$\ddot{B}_1(\ddot{E}_2) = -\frac{\alpha e^z n_1\Gamma_5 + \alpha e^z n_2\Gamma_6 + n_3\Gamma_4}{T_1}, \quad \ddot{B}_2(\ddot{E}_2) = -\frac{\alpha e^z p_1\Gamma_{11} + \alpha e^z p_2\Gamma_{12} + p_3\Gamma_{10}}{T_2}, \tag{3.92}$$

$$\ddot{A}_1(\ddot{E}_3) = \frac{\alpha^2 e^{2z}(m_1n_2 - m_2n_1)o_1p_2}{T_1}, \quad \ddot{B}_1(\ddot{E}_3) = -\frac{-\alpha^2 e^{2z}(m_1n_2 - m_2n_1)o_2p_1}{T_1}, \tag{3.93}$$

$$\ddot{C}_1(\ddot{E}_3) = \frac{1}{m_3\Gamma_3 + m_3\Gamma_6}, \quad \ddot{C}_2(\ddot{E}_3) = \frac{1}{m_3\bar{\omega}_3 + n_3\bar{\omega}_6}, \tag{3.94}$$

$$\ddot{D}_1(\ddot{E}_3) = -\frac{1}{o_3\Gamma_9 + p_3\Gamma_{12}}, \quad \ddot{D}_2(\ddot{E}_3) = -\frac{1}{o_3\bar{\omega}_9 + p_3\bar{\omega}_{12}}, \tag{3.95}$$

$$\ddot{A}_2(\ddot{E}_3) = \frac{\alpha^2 e^{2z}(m_1n_2 - m_2n_1)o_1p_2}{T_2}, \ddot{B}_2(\ddot{E}_3) = -\frac{\alpha^2 e^{2z}(m_1n_2 - m_2n_1)o_2p_1}{T_2}. \tag{3.96}$$

For $i = 1, 2, 3$, one can easily verify the following relation

$$\begin{aligned}
&(D_{\ddot{E}_i}R)(F, J, Y, L) = [\ddot{A}_1(\ddot{E}_i) + \ddot{B}_1(\ddot{E}_i)]R(F, J, Y, L) + \ddot{C}_1(F)R(\ddot{E}_i, J, Y, L)\\
&\ddot{C}_1(J)R(F, \ddot{E}_i, Y, L) + \ddot{D}_1(Y)R(F, J\ddot{E}_i, L) + \ddot{D}_1(L)R(F, J, Y, \ddot{E}_i)\\
&+ [\ddot{A}_2(\ddot{E}_i) + \ddot{B}_2(\ddot{E}_i)]G(F, J, Y, L) + \ddot{C}_2(F)G(\ddot{E}_i, J, Y, L)\\
&\ddot{C}_2(J)G(F, \ddot{E}_i, Y, L) + \ddot{D}_2(Y)G(F, J\ddot{E}_i, L) + \ddot{D}_2(L)G(F, J, Y, \ddot{E}_i).
\end{aligned} \tag{3.97}$$

Therefore from (3.97) we may state

Theorem 3.5.4. *There exists a null hypersurface of an indefnite Kenmotsu manifold which is A(GWS).*

Chapter 4

Lightlike Hypersurface of Cosymplectic Manifolds

This chapter is divided into two parts. In first part, certain results on lightlike hypersurfaces of cosymplectic space are obtained. We study screen semi-invariant lightlike hypersurface, parallelism of local second fundamental form, totally geodesic lightlike hypersurface of cosymplectic manifolds. We obtained totally umbalical lightlike hypersurface of cosymplectic space form $\bar{M}(c)$.

In section-2, we study weakly $\phi-$ Ricci symmetric lightlike hypersurface of cosymplectic manifolds. We study $\eta-$Einstein lightlike hypersurface, cyclic parallelism and Codazzi type properties of weakly $\phi-$ symmetric lightlike hypersurface of cosymplectic manifolds of constant curvature -1.

4.1 Introduction

There are different types of sub-manifolds of psedo-Riemannian manifolds like degenerate, timelike and spacelike submanifolds depending on the structure and behaviour of an induced metric $\bar{g}$ on tangent space $(T_P\bar{M})$ of lightlike hypersurface. In lightlike hypersurfaces as an induced metric is degenerate, the study becomes quite contrasting and more complicated from non-degenerate submanifolds of semi-Riemannian manifolds. One of the main difference or contrast between null hypersurfaces and non-degenerate is that, in case of null hypersurface the normal vector and tangent vector bundles has non-trivial intersections. The geometry of lightlike submanifolds of indefinite Kaehler manifolds was presented in a book by Duggal and Bejancu [42]. Riemannian geometry of contact and symplectic manifolds was studied by D. E. Blair in [14]. The slant submanifolds of a cosymplectic manifold was studied in [54] in 2006 by R.S.Gupta et.al.. lightlike submanifolds of indefinite cosymplectic manifolds was studied in [64] 2007, by T. H. Kang and S.K. Kim and [57] R.S.Gupta et.al studied slant lightlike submanifolds of indefinite cosymplictic manifolds. In 2012, D. H. Jin [62] and R.S, Gupat [52] studied lightlike hypersurfaces of indefinite cosymplectic manifolds. Screen conformal lightlike hypersurfaces of an indefinite cosymplectic manifold have been studied by A. Upadhyay. et.al.[115] in 2017. In 2011, semi-parallel lightlike hypersurfaces Of indefinite cosymplectic space forms have been studied by Abhitosh Upadhyay and Ram Shankar Gupta [113]. Ram Shankar Gupta [53] in 2013, investigated Ricci semi-ymmetric lightlike hypersurfaces of an indefinite cosymplectic space form and obtained some results justifying the existence of Ricci semi-symmetric lightlike hypersurfaces of indefinite cosymplectic manifolds. S. B. Kazan and B. Sahin [65] in 2016, studied pseudosymmetric lightlike hypersurfaces in indefinite sasakian space forms and obtained some sufficient conditions for lightlike hypersurfaces of indefinite cosymplictic to be semi-parallel, pseudo-symmetric and Ricci-pseudo symmetric.

Moreover in case of degenerate metric on hypersurfaces, the normal vector bundle is contained in tangent vector bundle. Duggal and Bejancu [42] gave extrinsic approach in the field of differential geometry. This created widespread interest among researchers to study lightlike geometry of semi-Riemannian manifolds. With the passage of time several authors studied null hypersurfaces of psedo-Riemannian manifolds. See also ([7], [66], [46]) and many more references there in. Motivated by these, in this chapter we study lightlike hypersurfaces of cosymplectic manifolds and weakly ϕ-Ricci symmetric lightlike hypersurfaces of cosymplectic manifolds.

Recent years have witnessed a surge in interest in the fields of contact and almost contact geometry, as well as related topics such as super gravity and M. theory. An odd dimensional cosymplectic manifold, proved to be a very valuable kind of almost contact manifold. In (1958), [70] Libermann proposed the concept of cosymplectic manifold, which, according to [105] is a smooth odd-dimensional manifolds endowed with closed 1-form η and 2-form ω. The odd dimensional differentiable manifold equivalent of a Kaehlar manifold is a cosymplectic manifold and is locally the product of a line or a circle with a Kaehlar manifold. In fact, a cosymplectic structure on an odd dimensional smooth manifold is a normal almost contact metric structure such that the $1-$ form and fundamental $2-$ form are closed [105] and many more references therein.

4.2 Cosymplectic Manifolds

This section is devoted to give brief introduction of cosymplectic manifolds of constant ϕ- sectional curvature c. A complete discussion on the contents used in this section can be found in ([46], [77], [14], [16], [105]). Let $(\bar{M}, g)$ be an n-dimensional almost contact metric manifold endowed with an almost contact metric structure $(\phi, \xi, \eta, \bar{g})$, where ϕ, ξ, η, g represent a $(1,1)-$tensor field ϕ, an associated vector field ξ, a $1-$form η and the Riemannian metric $\bar{g}$ respectively, satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \ \phi(\xi) = 0, \ \eta \circ \phi = 0, \tag{4.1}$$

$$g(\phi F, \phi J) = g(F, J) + \eta(F)\eta(J); \quad \eta(F) = g(F, \xi), \tag{4.2}$$

where I represents identity tensor field and $F,\ J \in \chi(M)$ are vector fields. A normal contact metric manifold $(\bar{M}, g)$ is said to be a cosymplectic manifold, if it satisfies [105] the following relations

$$(\bar{D}_F \phi)(J) = 0, \tag{4.3}$$

and

$$\bar{D}_F \xi = 0. \tag{4.4}$$

For any vector fields F, J on M and $\bar{D}$ represents the Levi-Civita connection on M. It is well known fact that $(\phi, \xi, \eta, \bar{g})$ is cosymplectic, if and only if $\bar{D}\eta = 0$ and the fundamental 2- form $\bar{D}\Phi$ vanishes, where $\bar{D}$ represents covariant derivative with respect to Riemannian metric g. The fundamental $2-$form is given by

$$g(\phi F, J) = g(F, \phi J) = \Phi(F, J). \tag{4.5}$$

The plane section $\bar{\sigma}$ in tangent space $T_P M$ is called $\phi-$ section if spanned by F and ϕF, where F is a unit vector field and orthogonal to ξ. $\phi-$ sectional curvature is the sectional curvature of a $\phi-$section $\bar{\sigma}$. If a cosymplectic manifold $(\bar{M}, g)$ has constant ϕ-sectional curvature c at a point, then $(\bar{M}, g)$ is said to be as cosymplectic space form and is denoted as $\bar{M}(c)$. The curvature tensor R^1 of cosymplectic space form is given ([93], [105]) by

$$\begin{aligned} R^1(F, J)L = \frac{c}{4}\{ & g(\phi J, \phi L)F - g(\phi F, \phi L)J \\ & + \eta(J)g(F, L)\xi - \eta(F)g(J, L)\xi + g(\phi J, L)\phi F \\ & - g(\phi F, L)\phi J + 2g(F, \phi J)\phi L\}, \end{aligned} \tag{4.6}$$

for any $F, J, L \in \Gamma(TM)$.

4.3 Lightlike Hypersurfaces of Cosymplectic Space Form

Tangent to the structure vector field ξ, let $(M,\ \bar{g})$ be a lightlike hypersurface of cosymplectic space form $\bar{M}(c)$. A cosymplectic manifold $\bar{M}$, if having $ind(g) = 1$ and $ind(g) > 1$ then $\bar{M}$ is said to be spacelike and timelike cosymplectic manifolds. respectively If Z is the local section of radical distribution i,e $Z \in Rad(TM)$, then ϕZ is tangent to M and $g(\phi Z, Z) = 0$. Hence $\phi(Rad(TM))$ is 1-dimensional distribution on lightlike hypersurface M.

Definition 4.3.1. Let $(\bar{M}, \phi, \xi, \eta, g)$ be a $(2n+1)$-dimensional cosyplectic manifold and $(M, \bar{g})$ be the lightlike hypersurface of $(\bar{M}, \phi, \xi, \eta, g)$. The hypersurface $(M, \bar{g})$ is screen semi invariant lightlike (degenerate) hypersurface, if following condition is satisfied

$$\phi(Rad(TM)) \subset S(TM) \ \ and \ \ \phi(ltr(TM)) \subset S(TM).$$

From equations (4.1) and (4.2), we have

$$g(\phi X, Z) = -g(X, \phi Z) = 0, \quad g(\phi X, X) = 0, \tag{4.7}$$

$$g(\phi X, \phi Z) = 1. \tag{4.8}$$

Therefore $\phi(Rad(TM)) \oplus \phi(ltr(TM))$ is non degenerate vector sub-bundle of S(TM) having rank 2. If lightlike hypersurface M is tangent to ξ then $\xi \in S(TM)$. Since $g(\phi Z, \xi) = g(\phi X, \xi) = 0$, there exists non degenerate distribution D_0 on lightlike hypersurface M of rank $2n - 4$ such that

$$S(TM) = \{\phi(TM^{\perp}) \oplus \phi(X(TM))\} \perp D_0 \perp < \xi >, \tag{4.9}$$

where $< \xi >= span\xi$. The second fundamental form B is independent on screen distribution, we have the following relation

$$B(., Z) = 0. \tag{4.10}$$

From equation ist of (4.2), we see that $\phi Z \ \ and \ \ \phi X$ are degenerate vector fields and

$$\phi^2 Z = -Z \ \ and \ \ \phi^2 X = -X. \tag{4.11}$$

Let us assume the local degenerate vector fields $Y = \phi X \in \phi(ltr(TM))$ and $V = \phi Z \in D$. Let P be the projection morphism of screen bundle $S(TM)$ on degenerate hypersurface M. For any $F \in \Gamma(TM)$, can be written as

$$F = SF + QF, \quad QF = u(F)Y. \tag{4.12}$$

Applying ϕ to equation (4.12), we get

$$\phi F = \phi(SF) + u(F)\phi Y, \tag{4.13}$$

where Q and S are the projection morphisms of tangent bundle into distributions G and G^1 respectively and $u(F) = g(V, F)$ is differential 1-form locally defined on degenerate hypersurface. Putting $\bar{\phi}F = \phi(Sx)$ for any $F \in \Gamma(TM)$ in equation (4.13), we acquire

$$\phi F = \bar{\phi}F - u(F)X, \tag{4.14}$$

Where ϕ and $\bar{\phi}$ are tensor fields of type $(1,1)$ on cosymplectic manifold $\bar{M}$ and lightlike hypersurface M respectively. We note that $u(Y) = -1$, for all $J \in \Gamma(TM)$ $u(J) = 0$. Now Applying ϕ again, we obtain

$$\phi^2 F = \phi\bar{\phi}F - u(F)\phi X, \tag{4.15}$$

$$-F + \eta(F)\xi = \phi\bar{\phi}F - u(F)Y. \tag{4.16}$$

In equations (4.15) and (4.16) if for any $F \in \Gamma(TM)$, $SF \in D$, $\phi F = \bar{\phi}(SF) \in D$, then $S(\bar{\phi}F) = \bar{\phi}F$ we obtain $\phi\bar{\phi}F = \bar{\phi}^2 F$ and then we can write

$$\bar{\phi}^2 F = -F + \eta(F)\xi + u(F)Y. \tag{4.17}$$

By using equations (1.85), (1.86) and (4.8), we acquire

$$(D_F\bar{\phi})J = -u(J)A_N F - B(F, J)Y, \tag{4.18}$$

and

$$(\bar{D}_F u)J = -B(F, \bar{\phi}J) - u(J)\tau(F). \tag{4.19}$$

Also for lightlike hypersurface of cosymplectic manifold, we have

$$v(F) = g(X, F). \tag{4.20}$$

Theorem 4.3.1. *Let $(M, \bar{g})$ be an invariant null hypersurface of cosymplectic manifold $(\bar{M}, \phi, \xi, \eta, g)$ tangent to the structural vector field ξ. Then $(M, \bar{g}, \xi, \eta, \bar{\phi})$ defines an almost para-contact metric manifold.*

Proof. Since $(M, \bar{g})$ is an invariant lightlike hypersurface of cosymplectic manifold $(\bar{M}, g)$, then from equation (4.14), we obtain

$$\bar{\phi}F = \phi F. \tag{4.21}$$

for any $F, J \in \Gamma(TM)$. Using 1st equation of equation (4.2) and equation (4.21), we get

$$\bar{\phi}^2 F = -F + \eta(F)\xi. \tag{4.22}$$

From equation (4.21), we have

$$\bar{\phi}\xi = 0. \tag{4.23}$$

From equation (4.22) and equation (4.23), we get

$$\eta \circ \bar{\phi} = 0 \qquad\qquad \eta(\xi) = 1.$$

From above equation and equations (4.21), (4.22), (4.23) and (4.2), we get

$$\bar{g}(\bar{\phi}F, \bar{\phi}J) = \bar{g}(F, J) - \eta(F)\eta(J). \tag{4.24}$$

Hence from equations (4.21), (4.22), (4.23) and (4.24), we conclude that $(M, \bar{g}, \xi, \eta, \bar{\phi})$ is an almost para-contact metric manifold. □

Theorem 4.3.2. *Let $(M, \bar{g})$ be an invariant (lightlike) null hypersurface of a cosymplectic manifold $(\bar{M}, \phi, \xi, \eta, g)$, tangent to the structural vector field ξ then*

$$(D_F\bar{\phi})J = 0, \qquad B(F, \phi J)X = B(F, \phi J)\phi X, \quad \bar{\phi}(A_X F) = A_{\phi X} F \tag{4.25}$$

for all $F, J, L \in \Gamma(TM)$.

Proof. We know that the covariant derivative of (1,1)-tensor field ϕ is given by

$$\begin{aligned}(\bar{D}_F\phi)J &= \bar{D}_F\phi J - \phi(\bar{D}_F J)\\ &= (D_F\phi)J + B(F, \bar{\phi J})X - B(F, J)\phi X.\end{aligned} \tag{4.26}$$

Now using equation (4.3) in equation (4.26), we obtain

$$(D_F\phi)J + B(F, \bar{\phi J})X - B(F, J)\phi X = 0. \tag{4.27}$$

Equating tangential and transversal components of equation equation (4.27), we get 1st two equations of (4.25), also

$$(\bar{D}_F\phi)X = \bar{D}_F\phi X - \phi(\bar{D}_F X). \tag{4.28}$$

From equation (4.3) and equation (1.75), we obtain 3rd equation of equation (4.25). □

Proposition 4.3.1. *Let $(M, \bar{g})$ be (lightlike) the lightlike hypersurface of cosymplectic manifold $(\bar{M}, \phi, \xi, \eta, g)$, tangent to the structural vector field ξ then*

$$\bar{\phi}^2 F = -F + \eta(F)\xi + u(\bar{\phi}F)X + u(F)\phi X, \tag{4.29}$$

$$B(F, \xi) = -u(F), \qquad C(F, \xi) = -v(F) \tag{4.30}$$

for any vector field $F \in \Gamma(TM)$.

Proof. From 1st equation of (4.1) and equation (4.14), we get (4.29). By using equations (4.3), (1.81), (4.14) and then comparing tangential and transversal parts we get (4.26). □

Definition 4.3.2. A degenerate (lightlike) hypersurface $(M, \bar{g}, \xi, \eta, \bar{\phi})$ of cosymplectic manifold is D^0 totally geodesic if $B(F, J) = 0$, for any $F, J \in D^0$.

Theorem 4.3.3. *Let $(M, \bar{g}, \xi, \eta, \bar{\phi})$ be a degenerate (lightlike) hypersurface of $(\bar{M}, \phi, \xi, \eta, g)$, tangent to the structural vector field ξ. Then for any vector fields $F, J \in \Gamma(TM)$ a degenerate hypersurface $(M, \bar{g}, \xi, \eta, \bar{\phi})$ is totally geodesic, if h is parallel.*

Proof. Let us pretend that h is parallel and we know that the covariant derivative of second fundamental form h is given by

$$(\bar{D}_F h)(J, \xi) = D_F h(J, \xi) - h(D_F J, \xi) - h(J, D_F \xi). \tag{4.31}$$

Using equation (4.3) in equation (4.27), we get equation (4.31). □

Theorem 4.3.4. *let $(M, \bar{g}, \xi, \eta, \bar{\phi})$ be a screen semi-invariant lightlike hypersurface of cosymplectic space form $\bar{M}(c)$, tangent to the structural vector field ξ and the local second fundamental form B is parallel. If $\tau(Z) \neq 0$, then $c = 0$, if and only if it is totally geodesic.*

Proof. Putting $J = Z$ in equation (1.83) and equation (4.6), we obtain

$$\begin{aligned} g(R^1(F, Z)L, Z) = {} & (\nabla_F B)(Z, L) - (\nabla_Z B)(F, L) \\ & + \tau(F)B(Z, L) - \tau(Z)B(F, L), \end{aligned} \tag{4.32}$$

and

$$\begin{aligned} g(R^1(F,Z)L,Z)) = \frac{c}{4}\{&g(\phi Z,\phi L)g(F,Z) - g(\phi F,\phi L)g(Z,Z) \\ &+ \eta(Z)g(F,L)g(\xi,Z) - \eta(F)g(Z,L)g(\xi,Z) \\ &+ g(\phi Z,L)g(\phi F,Z) - g(\phi F,L)g(\phi Z,Z) \\ &+ 2g(F,\phi Z)g(\phi L,Z)\}, \end{aligned} \tag{4.33}$$

respectively. Since B is parallel, then from equation (4.28) and equation (4.29), we acquire equation (4.34)

$$\frac{c}{4}u(F)u(L) = \tau(Z)B(F,L). \tag{4.34}$$

By replacing F and L by Y in equation (4.30), we get

$$\frac{c}{4} = \tau(Z)B(Y,Y). \tag{4.35}$$

If $\tau(Z) \neq 0$, then the proof is complete. □

Definition 4.3.3. If in a degenerate (lightlike) hypersurface $(M,\bar{g},\xi,\eta,\bar{\phi})$ of cosymplectic manifold $(\bar{M},g)$ equation (4.36) is satisfied, then $(M,\bar{g},\xi,\eta,\bar{\phi})$ is totally umbilical lightlike hypersurface i,e.

$$B(F,J) = \lambda\bar{g}(F,J). \tag{4.36}$$

Where λ represents a smooth function.

Theorem 4.3.5. *Let $(M,\bar{g},\xi,\eta,\bar{\phi})$ be a screen semi-invariant lightlike hypersurface $(SCI-LH)$ of cosymplectic space form $\bar{M}(c)$, tangent to the structural vector field ξ. Then a lightlike hypersurface $(M,\bar{g},\xi,\eta,\bar{\phi})$ is flat, If M is totally umbilical.*

Proof. From equation (4.6), we acquire

$$\begin{aligned} g(R^1(F,J)L,Z) = \frac{c}{4}\{&g(\phi J,\phi L)g(F,Z) - g(\phi F,\phi L)g(J,Z) \\ &+ \eta(J)g(F,L)g(\xi,Z) - \eta(F)g(J,L)g(\xi,Z) \\ &+ g(\phi J,L)g(\phi F,Z) - g(\phi F,L)g(\phi J,Z) \\ &+ 2g(F,\phi J)g(\phi L,Z)\}. \end{aligned} \tag{4.37}$$

Putting $\phi Z = V$, $\phi X = Y$ and $g(F,Z) = 0$ in equation (4.37), we obtain

$$\begin{aligned} g(R^1(F,J)L,Z) = \frac{c}{4}\{&g(\phi F,L)g(J,V) - g(\phi J,L)g(F,V) \\ &- 2g(\phi F,J)g(L,V)\}. \end{aligned} \tag{4.38}$$

forall $F, J, L \in \Gamma(TM)$. Replacing F, J and L by PF, Z and PL respectively in equation (4.38) and equation (4.6), we get

$$g(R^1(PF,Z)PL,Z) = \frac{c}{4}\{g(\phi Z,PL)g(Z,V) - g(\phi Z,PL)g(PF,V) - 2g(\phi PF,Z)g(PL,V)\}, \tag{4.39}$$

$$g(R^1(PF,Z)PL,Z) = \frac{-3c}{4}\{u(PF)u(PL)\}, \tag{4.40}$$

and

$$\begin{aligned} g(R^1(PF,Z)PL,Z) = &-B(D_{PF}Z,PL) - ZB(PF,PL) \\ &+ B(D_Z PF,PL) + B(PF,D_Z PL) \\ &- \tau(Z)B(PF,PL). \end{aligned} \tag{4.41}$$

From equations (4.40) and (4.41), and using equation (4.25), the Gauss and Weingarten formulas in the resulting equation, we obtain

$$\frac{-3c}{4}\{u(PF)u(PL) = \{\lambda^2 - Z(\lambda) - \lambda(\tau Z)\}\bar{g}(\phi F,PL). \tag{4.42}$$

Setting $F = L = Y \in S(TM)$ in equation (4.42), we get $PF = PL = Y, u(Y) = 1$ and $g(Y,Y) = 0$.
From (4.42), we have

$$c = 0 \tag{4.43}$$

From equation (4.33) as $c = 0$, we can say that the lightlike hypersurface $(M, \bar{g}, \xi, \eta, \bar{\phi})$ is flat. □

Lemma 4.3.1. *Let $(M, \bar{g}, \xi, \eta, \bar{\phi})$ be a screen semi-invarient lightlike hypersurface $SCI-LH$ of cosymplectic space form $\bar{M}(c)$, tangent to the structural vector field ξ. Then the Gauss and Codazzi formulas are respectively given by*

$$\begin{aligned} R(F,J)L = \frac{c}{4}\{&g(\phi J,\phi L)F - g(\phi F,\phi L)J \\ &+ \eta(J)g(F,L)\xi - \eta(F)g(J,L)\xi + g(\phi J,L)\bar{\phi}F \\ &- g(\phi F,L)\bar{\phi}J + 2g(F,\phi J)\bar{\phi}L\} \\ &- B(F,L)A_X J + B(J,L)A_X F, \end{aligned} \tag{4.44}$$

and

$$(D_F h)(J,L) - (D_J h)(F,L) = \frac{c}{4}\{g(\phi F,L)u(J) - g(\phi J,L)u(F) - 2g(F,\phi J)u(L)\}X. \tag{4.45}$$

for any $F, J, L \in \Gamma(TM)$.

Proof. Since $(M, \bar{g}, \xi, \eta, \bar{\phi})$ is a screen semi-invariant lightlike hypersurface of $\bar{M}(c)$, therefore from equations (4.6) and (4.33), we derive

$$\begin{aligned} R(F,J)L = \frac{c}{4}\{ & g(\phi J, \phi L)F - g(\phi F, \phi L)J \\ & + \eta(J)g(F,L)\xi - \eta(F)g(J,L)\xi + g(\phi J, L)\phi F \\ & - g(\phi F, L)\phi J + 2g(F, \phi J)\phi L\} - A_{h(F,L)}J + A_{h(J,L)}F \\ & - (D_F h)(J,L) + (D_J h)(F,L)). \end{aligned} \tag{4.46}$$

Using equation (4.14) in equation (4.46), we obtain

$$\begin{aligned} R(F,J)L = \frac{c}{4}\{ & g(\phi J, \phi L)F - g(\phi F, \phi L)J + \eta(J)g(F,L)\xi \\ & - \eta(F)g(J,L)\xi + g(\phi J, L)\bar{\phi}F - g(\phi J, L)u(F)X \\ & - g(\phi F, L)\bar{\phi}J + g(\phi F, L)u(J)X + 2g(F, \phi J)\bar{\phi}L \\ & - 2g(F, \phi J)u(L)X\} - A_{h(F,L)}J + A_{h(J,L)}F \\ & - (D_F h)(J,L) + (D_J h)(F,L)). \end{aligned} \tag{4.47}$$

Hence, we derive equation (4.44) and equation (4.45) correspondingly by comparing the tangential and transversal vector bundle components of equation (4.47). □

Lemma 4.3.2. *Let $(M, \bar{g}, \xi, \eta, \bar{\phi})$ be a screen semi-invariant lightlike hypersurface $SCI-LH$ of cosymplectic space form $\bar{M}(c)$, tangent to the structural vector field ξ then*

$$g(R^1(F,Z)L, X) = \frac{-c}{4}\{g(\phi F, \phi L) - u(L)\theta(\phi F) - 2u(F)\theta(\phi L)\}. \tag{4.48}$$

for any vector fields $F, J, L \in \Gamma(TM)$.

Proof. Putting Z for J in equation (4.9) and taking inner product of resulting equation with X, we obtain our result. □

Theorem 4.3.6. *There does not exist any screen semi-invariant lightlike hypersurface $(M, \bar{g}, \xi, \eta, \bar{\phi})$ of cosymplectic space form $\bar{M}(c)$, if h is parallel and $c \neq 0$.*

Proof. Suppose there exists a screen semi-invariant lightlike hypersurface $(M, \bar{g}, \xi, \eta, \bar{\phi})$ of cosymplectic space form $\bar{M}(c)$ with $c \neq 0$ and h to be parallel. Putting $J = Z$ and $L = \phi X$ in *equation*(4.45), we get

$$\frac{c}{4}u(F)X = 0. \tag{4.49}$$

Replacing F by ϕX, we acquire

$$c = 0.$$

Which is contradicting our supposition, hence there does not exist any screen semi-invariant lightlike hypersurface $(M, \bar{g}, \xi, \eta, \bar{\phi})$ of cosymplectic space form $\bar{M}(c)$, if h is parallel and $c \neq 0$. □

Theorem 4.3.7. *Let $(M, \bar{g}, \xi, \eta, \bar{\phi})$ be the lightlike hypersurface of cosymplectic space form $\bar{M}(c)$, if $(c \neq 0)$ and screen distribution is parallel then there does not exist any $SCI - LH$ $(M, \bar{g}, \xi, \eta, \bar{\phi})$ of $\bar{M}(c)$.*

Proof. Assume $c \neq 0$ and screen distribution is parallel then replacing F, J, and L by $Z, \phi X$ and ϕZ in equation (4.6), we obtain

$$g(R^1(Z, \phi X)\phi Z, X) = \frac{c}{4}. \tag{4.50}$$

From equation (1.85), we get

$$\begin{aligned} g(R^1(Z, \phi X)\phi Z, X) = (D_E C)(\phi X, P\phi Z) - (D_{\phi X} C)(\phi Z, P\phi Z) \\ + \tau(\phi X)C(Z, P\phi Z) - \tau(Z)C(\phi X, P\phi Z). \end{aligned} \tag{4.51}$$

As assumed $S(TM)$ is parallel, we obtain

$$g(R^1(Z, \phi X)\phi Z, X) = 0 \tag{4.52}$$

From equations (4.50) and (4.52), we obtain $c = 0$ which is a contradiction. Hence there does not any $SCI - LH$ $(M, \bar{g}, \xi, \eta, \bar{\phi})$ of $\bar{M}(c)$ with $c \neq 0$ and parallel screen distribution. □

4.4 Weakly $\bar{\varphi}$-Ricci Symmetric Lightlike Hypersurface of Indefinite Cosymplectic Manifolds

Many authors have explored the idea of $\phi-$symmetry on both complex and contact geometry of manifolds. Locally $\phi-$symmetric Sasakian manifolds were introduced by [109] as a weaker form of locally symmetric manifolds. Some examples on $\phi-$ symmetric Kenmotsu manifolds were examined by [32]. Later [34] proposed the concept of $\phi-$Ricci symmetric Sasakian manifolds and traced out some intersesting results. Authors in [106] studied $\phi-$Ricci symmetric on Kenmotsu manifolds and verified its existence with some examples. In addition, ([110], [111]), introduced weakly symmetric and weakly Ricci symmetric manifolds as generalizations of Chaki's pseudo-symmetric and pseudo-Ricchi symmetric manifolds. Motivated by above, we are interested to study weakly ϕ-Ricci symmetric lightlike submanifolds of cosymplectic manifolds.

Definition 4.4.1. For Levi-Civita connection $\bar{D}$, Riemannian metric g and an associated 1-forms $\ddot{\alpha}$, $\ddot{\beta}$, and $\ddot{\gamma}$, if the Ricci tensor S of $(M, \bar{g})$ satisfies

$$(\bar{D}_F S)(J, L) = \ddot{\alpha}(F)S(J, L) + \ddot{\beta}(J)S(F, L) + \ddot{\gamma}(L)S(J, F), \tag{4.53}$$

forall F, J, $L \in \bar{M}$, then the non-flat Riemannian manifold is called weakly Ricci symmetric [76] . Where $\ddot{\alpha}(F) = g(F, \ddot{\rho})$, $\ddot{\beta}(F) = g(F, \ddot{\delta})$ and $\ddot{\gamma}(F) = g(F, \ddot{\kappa})$, corresponding to 1-forms $(\ddot{\alpha}, \ddot{\beta}, \ddot{\gamma})$, $\ddot{\rho}, \ddot{\delta}, \ddot{\kappa}$ are associated vector fields respectively.

Also on Kenmotsu manifolds $\bar{M}$ $(n \geq 3)$, authors G. D.Prakasha et.al [96] generally introduced representation of $\phi-$Ricci symmetries on Kenmotsu manifolds. Then forall F, J, L the Riemannian manifold $\bar{M}$ satisfying

$$\phi^2(\bar{D}_F Q)(J) = A(F)Q(J) + B(J)Q(F) + g(QF, J)\ddot{\rho}, \tag{4.54}$$

is known to be as $\phi-$Ricci symmetric, where Q is Ricci operator and A, B not simultaneously zero $1-$forms, such that $g(F, \rho) = D(F)$. From equation (4.54), if

$$\phi^2(\bar{D}_F Q)(J) = 0, \tag{4.55}$$

is satisfied then, $\bar{M}$ $(n \geq 3)$ is locally ϕ-Ricci symmetric [96].

Tangent to ξ, let $(M, \bar{g})$ be degenerate (lightlike) hypersurface of cosymplectic manifold $(\bar{M}, g)$, such that $g(\xi, \xi) = \epsilon = \pm 1$. If $\bar{g}(\bar{\phi}Z, Z) = 0$, means ϕZ is tangent to hypersurface $(M, \bar{g})$. Here Z is the local section of $TM^{\perp}$. Therefore we can select a screen distribution $S(TM)$ in such a manner that it contains $\phi(TM^{\perp})$ as a vector subbundle.

Let us consider X a local section of transversal bundle $tr(TM)$. Therefore $\bar{g}(\bar{\phi}X, Z) = -\bar{g}(X, \bar{\phi}Z) = 0$, we found that $\bar{\phi}X$ is also tangential to $(M, \bar{g})$. But $\bar{g}(\bar{\phi}X, X) = 0$, implies with respect to Z the components of $\bar{\phi}X$ vanishes and hence $\bar{\phi}X \in \Gamma(S(TM))$. From (1.68), $g(\bar{\phi}X, \bar{\phi}Z) = 1$. [17] If $\xi \in M$, then $\xi \in S(TM)$ implies $\bar{g}(\bar{\phi}Z, \xi) = \bar{g}(\bar{\phi}X, \xi) = 0$, then there exists D_0, a rank $2n - 4$ distribution on $(M, \bar{g})$ such that $\bar{\phi}(D_0) = D_0$, then we have the following decomposition

$$TM = (D_1 \oplus D_2) \perp < \xi >, \tag{4.56}$$

where D_1 and D_2 are distributions on $(M, \bar{g})$. Now let us assume Y and V to be local null vector fields such that $Y = -\bar{\phi}X$ and $V = -\bar{\phi}Z$. Let R_1 and Q_1 be projection morphisms of tangent bundle TM into D_1 and D_2 respectively, then equation (4.56), yields

$$F = R_1F + Q_1F + \eta(F)\xi, \quad Q_1F = u(F)Y, \tag{4.57}$$

for any $F \in \Gamma(TM)$ and $u(F) = g(F, V)$, is a differential 1-form. Applying $\bar{\phi}$ to equation (6.26), we obtain

$$\bar{\varphi}F = \bar{\phi}F + u(F)X, \tag{4.58}$$

where $\bar{\varphi}$ represents $(1, 1)$ type tensor field on $(M, \bar{g})$ and is defined as $\bar{\phi}F = \bar{\varphi}R_1F$. In addition of this, we have following relations

$$B(F, \xi) = 0,\ C(F, \xi) = \theta(F),\ \bar{\varphi}^2F = -F + \eta(F)\xi + u(F)\xi, \quad D_F\xi = 0. \tag{4.59}$$

Assume that $(\bar{M}, g)$ is an indefnite cosymplectic manifold of constant curvature -1, such that

$$\bar{R}(F, J)L = g(J, L)F - g(F, L)J, \tag{4.60}$$

for any $F,\ J,\ L\ \in \Gamma(TM)$.

Now to define a non-symmetric indudeд Ricci-tensor $R^{(0,2)}$ on $(M, \bar{g})$, it is noted that D is not the Levi-Civita connection and $R^{(0,2)}$ has no physical meaning like that of symmetric Ricci tensor Ric on $(\bar{M}, g)$. Therefore by direct calculations an induced Ricci tensor $R^{(0,2)}(F, J)$ of lightlike hypersurface of cosymplectic manifold is given by

$$R^{(0,2)}(F, J) = S(F, J) = (2n - 1)\bar{g}(F, J) + B(F, J)trA_X - B(A_XF, J). \tag{4.61}$$

From equation (4.61), we obtain

$$R^{(0,2)}(F, \xi) = S(F, \xi) = (2n - 1)\eta(F) \tag{4.62}$$

$$Q\xi = (2n - 1)\xi. \tag{4.63}$$

Theorem 4.4.1. *Suppose $(\bar{M}, g)$ be an indefinite cosymplectic manifold of constant curvature -1 and let $(M, \bar{g})$ be weakly $\bar{\varphi}-$Ricci symmetric null hypersurface of $(\bar{M}, g)$, tangent to the structural vector field ξ, then the sum of non-zero 1-forms is zero i.e.*

$$A(\xi) + B(\xi) + D(\xi) = 0. \tag{4.64}$$

Proof. We know that a lightlike hypersurface $(M, \bar{g})$ of cosymplectic manifold $(\bar{M}, g)$ is called weakly $\bar{\varphi}-$Ricci symmetric lightlike hypersurface if

$$\bar{\varphi}^2(D_FQ)J = A(F)QJ + B(J)QF + S(F, J)\rho \tag{4.65}$$

is satisfied for all $F, J\Gamma(TM)$. Here $A(F) = g(F, \delta)$, $B(J) = g(J, \kappa)$ are non zero 1-forms and δ, κ, ρ are associated vector fields. From equation (4.59), we obtain

$$\begin{aligned} &- \bar{g}((D_F Q)(J) + \eta(D_F Q)(J)\xi + u(D_F Q(J))Y \\ &= A(F)QJ + B(J)QF + S(F, J)\rho. \end{aligned} \tag{4.66}$$

We know that the covariant derivative of Ricci operator Q is defined by

$$(D_F Q)(J) = D_F QJ - QD_F J. \tag{4.67}$$

Taking inner product of equation (4.65) with L, we acquire

$$\begin{aligned} A(F)S(J, L) + B(J)S(F, L) + D(L)S(F, J) = &-\bar{g}((D_F QJ, L) + \bar{g}(QD_F J, L) \\ &+ \eta(D_Q J)\eta(L) - \eta(QD_F J)\eta(L) \\ &+ u(D_F QJ)\bar{g}(Y, L) - u(QD_F J)\bar{g}(Y, L) \end{aligned} \tag{4.68}$$

Replacing J by ξ in equation (4.68), and using equation (4.59) and equation (**??**) in the resulting equation, we acquire

$$(2n - 1)A(F)\eta(L) + B(\xi)S(F, L) + (2n - 1)D(L)\eta(F) = 0. \tag{4.69}$$

Again putting $F = L = \xi$ in equation (4.69), we acquire

$$A(\xi) + B(\xi) + D(\xi) = 0. \tag{4.70}$$

□

Theorem 4.4.2. *Let $(\bar{M}, g)$ be an indefinite cosymplectic manifold of constant curvature -1 and tangent to the structure vector field ξ, let $(M, \bar{g})$ be weakly $\bar{\varphi}-$Ricci symmetric null hypersurface of $(\bar{M}, g)$. Let $(M, \bar{g})$ and screen bundle $S(TM)$ are totally umbalical, then $(M, \bar{g})$ is locally $\bar{\varphi}-$Ricci symmetric null hypersurface if $\bar{\alpha}\bar{\beta} = (2n-1)+\bar{\alpha}trA_X$.*

Proof. Assume $(M, \bar{g})$ to be weakly $\bar{\varphi}-$Ricci symmetric null hypersurface of $(\bar{M}, g)$ such that $\xi \in \Gamma(TM)$. Taking inner product of eqution (4.8) with L, we obtain

$$\bar{g}(\bar{\varphi}^2(D_F Q)J, L) = A(F)S(J, L) + B(J)S(F, L) + D(L)S(F, J). \tag{4.71}$$

Using equation (4.61) in equation (4.71), we acquire

$$\begin{aligned} \bar{g}(\bar{\varphi}^2(D_F Q)J, L) = &A(F)(2n - 1)\bar{g}(J, L) + B(J, L)trA_X - B(A_X J, L) \\ &+ B(J)(2n - 1)\bar{g}(F, L) + B(F, L)trA_X - B(A_X F, L) \\ &+ D(L)(2n - 1)\bar{g}(F, J) + B(F, J)trA_X - B(A_X F, J) \end{aligned} \tag{4.72}$$

Since lightlike hypersurface $(M, \bar{g})$ and screen bundle $S(TM)$ are totally umbalical. Then substituting $B(F, J) = \bar{\alpha}\bar{g}(F, J)$ and $C(F, J) = \bar{\beta}\bar{g}(F, J)$, in equation (4.72), we get

$$\begin{aligned} &\bar{g}(\bar{\varphi}^2(D_F Q)J, L) \\ &= [(2n-1) + \bar{\alpha} tr A_X - \bar{\alpha}\bar{\beta}]\bar{g}(A(F)J + B(J)F + \bar{g}(F, J)\rho, L), \end{aligned} \tag{4.73}$$

Where $\bar{\alpha}$ and $\bar{\beta}$ are smooth functions. Again using the given hypothesis $\bar{\alpha}\bar{\beta} = (2n-1) + \bar{\alpha} tr A_X$ in equation (4.73), we acquire

$$\bar{\varphi}^2(D_F Q)J = 0. \tag{4.74}$$

Hence from equation (4.74) it is clear that $(M, \bar{g})$ is locally $\bar{\varphi}-$Ricci symmetric null hypersurface of cosymplectic manifolds $(\bar{M}, g)$. □

Theorem 4.4.3. *Suppose $(\bar{M}, g)$ be an indefinite cosymplectic manifold of constant curvature -1 and let $(M, \bar{g})$ be weakly $\bar{\varphi}-$Ricci symmetric $\eta-$Einstein null hypersurface of $(\bar{M}, g)$ with $\xi \in \Gamma(TM)$. If $(M, \bar{g})$ is locally $\bar{\varphi}-$Ricci symmetric null hypersurface of $(\bar{M}, g)$, then either $\bar{\alpha} = -\bar{\beta}$ or sum of non zero 1-forms is zero every where.*

Proof. As assumed $(M, \bar{g})$ is locally $\bar{\varphi}-$Ricci symmetric null hypersurface of $(\bar{M}, g)$, then from equation (4.65), we obtain

$$\bar{\varphi}^2(D_F Q)J = 0, \tag{4.75}$$

that is

$$A(F)QJ + B(J)QF + S(F, J)\rho = 0. \tag{4.76}$$

Taking inner product of equation (4.76) with L, we get

$$A(F)S(J, L) + B(J)S(F, L) + D(L)S(F, J) = 0. \tag{4.77}$$

A lightlike hypersurface $(M, \bar{g})$ is called is an $\eta-$Einstein null hypersurface of $(\bar{M}, g)$, if it satisfies

$$S(F, J) = \bar{\alpha}\bar{g}(F, J) + \bar{\beta}\eta(F)\eta(J). \tag{4.78}$$

From equation (4.77), we have

$$\begin{aligned} A(F)[\bar{\alpha}\bar{g}(J, L) + \bar{\beta}\eta(J)\eta(L)] + B(J)[\bar{\alpha}\bar{g}(F, L) + \bar{\beta}\eta(F)\eta(L)] \\ + D(L)[\bar{\alpha}\bar{g}(F, J) + \bar{\beta}\eta(F)\eta(J)] = 0 \end{aligned} \tag{4.79}$$

Putting $F = J = \xi$, $F = L = \xi$ and $J = L = \xi$ in equation (4.79), by turns and then adding the resulting equations, we have

$$\begin{aligned} &A(\xi)[\bar{\alpha}\eta(L) + \bar{\beta}\eta(L) + \bar{\alpha}\eta(J) + \bar{\beta}\eta(J)] + A(F)[\bar{\alpha} + \bar{\beta}] \\ &+ B(\xi)[\bar{\alpha}\eta(L) + \bar{\beta}\eta(L) + \bar{\alpha}\eta(F) + \bar{\beta}\eta(F)] + B(J)[\bar{\alpha} + \bar{\beta}] \\ &+ D(\xi)[\bar{\alpha}\eta(J) + \bar{\beta}\eta(J) + \bar{\alpha}\eta(F) + \bar{\beta}\eta(F)] + D(L)[\bar{\alpha} + \bar{\beta}] = 0. \end{aligned} \tag{4.80}$$

Setting $F = J = L$ in equation (4.80), we acquire

$$A(\xi) + B(\xi) + D(\xi)[2\bar{\alpha}\eta(F) + 2\bar{\beta}\eta(F)] + A(F) + B(F) + D(F)[\bar{\alpha} + \bar{\beta}] = 0. \tag{4.81}$$

Using theorem 4.3.1 in equation (4.81), we obtain our result. □

Corollary 4.4.1. *Suppose $(\bar{M}, g)$ be an indefinite cosymplectic manifold of constant curvature -1 and let $(M, \bar{g})$ be weakly $\bar{\varphi}-$Ricci symmetric Einstein null hypersurface of $(\bar{M}, g)$ with $\xi \in \Gamma(TM)$. If $(M, \bar{g})$ is locally $\bar{\varphi}-$Ricci symmetric null hypersurface of $(\bar{M}, g)$, then sum of non zero 1-forms is zero every where.*

Theorem 4.4.4. *Let $(M, \bar{g})$ be weakly $\bar{\varphi}$-Ricci symmetric degenerate hypersurface of $(\bar{M}, g)$ of constant curvature -1 with $\xi \in \Gamma(TM)$. If $(M, \bar{g})$ admits Codazzi type of Ricci tensor, then $\bar{g}([F, J], \xi) = 0$ i,e. $F = J$.*

Proof. We know that the Ricci tensor $R^{(0,2)} = S$ satisfies Codazzi type, if

$$(D_F S)(J, L) = (D_J S)(F, L) \tag{4.82}$$

is satisfied for all $F, J, L \in T_P M$. From equation (4.61) and equation (4.53), we obtain

$$\begin{aligned} &(2n-1)[B(F,J)\theta(L) + B(F,L)\theta(J) - \bar{g}(D_F J, L) - \bar{g}(J, D_F L)] \\ &+ D_F B(J,L) tr A_X - D_F B(A_X J, L) + B(A_X D_F J, L) + B(A_X J, D_F L) \\ &= (2n-1)[B(J,F)\theta(L) + B(J,L)\theta(F) - \bar{g}(D_J F, L) - \bar{g}(F, D_J L)] \\ &+ D_J B(F,L) tr A_X - D_J B(A_X F, L) + B(A_X D_J F, L) + B(A_X F, D_J L) \end{aligned} \tag{4.83}$$

Replacing L by ξ in equation (4.83) and using $B(F, \xi) = 0$, we acquire

$$\bar{g}(D_F J, \xi) - \bar{g}(D_J F, \xi) = 0 \tag{4.84}$$

or

$$\bar{g}([F, J], \xi) = 0. \tag{4.85}$$

Hence equation (4.85) completes the proof. □

Theorem 4.4.5. *Let $(M, \bar{g})$ be weakly $\bar{\varphi}$-Ricci symmetric degenerate hypersurface of $(\bar{M}, g)$ of constant curvature -1 with $\xi \in \Gamma(TM)$. If $(M, \bar{g})$ is totally geodesic and admits cyclic paralle of Ricci tensor, then L is parallel vector field.*

Proof. From equation (4.83), we acquire

$$\begin{aligned}(D_J S)(L, F) =&(2n-1)[B(J, L)\theta(F) + B(J, F)\theta(L) - \bar{g}(D_J L, F) \\ &- \bar{g}(L, D_J F)] + D_J B(L, F) tr A_X - D_J B(A_X L, F) \\ &+ B(A_X D_J L, F) + B(A_X L, D_J F),\end{aligned} \tag{4.86}$$

and

$$\begin{aligned}(D_L S)(F, J) =&(2n-1)[B(L, F)\theta(J) + B(L, J)\theta(F) - \bar{g}(D_L F, J) \\ &- \bar{g}(F, D_L J)] + D_L B(F, J) tr A_X - D_L B(A_X F, J). \\ &+ B(A_X D_L F, J) + B(A_X F, D_L J)\end{aligned} \tag{4.87}$$

As assumed $(M, \bar{g})$, admits cyclic parallel of Ricci tensor S, we have

$$(D_F S)(J, L) + (D_J S)(L, F) + (D_L S)(F, J) = 0. \tag{4.88}$$

Putting equations (4.83), (4.86) and equation (4.87) in equation (4.88) and using $B(F, J) = 0$ in the resulting equation, we get

$$\begin{aligned}&\bar{g}(D_F J), L + \bar{g}(D_F L, J) + \bar{g}(D_J L, F) \\ &+ \bar{g}(D_J F, L) + \bar{g}(D_L F, J) + \bar{g}(D_L J, F) = 0.\end{aligned} \tag{4.89}$$

Replacing $F = \ J \ = \xi$ in equation (4.89), we obtain

$$\bar{g}(D_\xi L, \xi) = 0. \tag{4.90}$$

□

Chapter 5

Lightlike Hypersurfaces of Sasaki-Like Statistical Space Form

The main aim of this chapter is to study the lightlike hypersurface $(M, \bar{g})$ of the Sasaki-like statistical space form $(\bar{M}(c))$. In this chapter, we computed the Gauss and Codazzi formulae of lightlike hypersurface $(M, \bar{g})$ of the Sasaki-like statistical manifold $(\bar{M}, g)$. We showed, that we can not obtain null hypersurface of Sasaki-like space form $\bar{M}(c)$ with parallel screen distribution $S(TM)$ and parallel second fundamental form B. Finally in this chapter, we showed that we cannot obtain lightlike hypersurfaces of Sasaki-like statistical $\bar{M}(c)$ with $c \neq 1$ and $c \neq 5$.

5.1 Introduction

The geometry of statistical manifolds lies at the junction of some research areas like Information geometry (IG), Affine differential geometry and Hessaian geometry. Deeper under-standings and geometrical approaches to families of statistical models are provided by IG and is related to study the statistical approach in this field differential geometry. Information geometry (IG), found large number of applications like in finance, chemistry, physics and biology. The main aim of IG is to use different types of geometrical tools, to acquire information from different types of statistical models. Event Horizon Telescope (EHT), in April (2019) by taking use of "deep learning algorithms" released the shadow of black hole and is the direct resulting evidence of existence of theory of general relativity and black holes.

Amari in (1985), introduced the concept of statistical manifolds [6] and proved that in terms of geometrical properties of Riemannian manifolds, there exists statistical relationship between families of probability densities. This study shows intrinsic properties of Riemannian manifolds. In (1989), Vos [116] gave the notation and introduced the concept of statistical submanifolds in differntial geometry. Oguzhan Bahadir and Mukut Mani Tripathi [9] studied lightlike hypersurfaces of statistical manifolds and showed that $S(TM)$ has a canonical statistical structure. They [9] showed that lightlike hypersurface of a statistical manifold is not a statistical with regard to the induced connections. Later Oguzhan Bahadir [8] discussed lightlike hypersurfaces of an indefinite sasakian statistical manifold and find certain relationships between induced geometrical objects and dual connections in a lightlike hypersurface of an indefinite sasakian manifold. Vandana Rani and Jasleen Kaur [98] introduced the geometry of lightlike hypersurfaces of an indefinite Kaehler statistical manifold. Same authors [99] looked at the lightlike geometry of an indefinite Kaehler statistical manifold and establish features based on the structure of its Cauchy Riemannian lightlike submanifold. M. B. K. Balgeshir and S. Salahvarzi [10] in 2020, studied lightlike submanifolds of semi-Riemannian statistical manifolds and proved in spite of the Riemannian case an induced connections from statistical connections on a lightlike submanifold are not statistical.

This chapter is devoted to give a brief introduction of the Sasaki-like statistical manifolds of constant sectional curvature c. A complete discussion on content used in this chapter can be seen in ([46], [77], [108], [9], [116], [6]).

5.2 Sasaki-like Statistical Manifolds

Let $(\bar{M}, g)$ be the psedo-Riemannian manifold, D^0 and D^1 are affine torsion-free connections with torsion tensors T^{D^0} and T^{D^1} respectively. The pairs (D^0, g) and (D^0, g) are said to be the statistical structure on a smooth manifold, if it satisfies the following conditions conditions

$$(D^o_F g)(J, L) - (D^o_J g)(F, L) = g(T^{D^0}(F, J), L), \tag{5.1}$$

$$T^{D^0} = 0, \tag{5.2}$$

forall $F, J, L \in \Gamma(T\bar{M})$. Then the Riemannian manifold $(\bar{M}, g)$ endowed with statistical structure (D^0, g) (respectively (D^0, g)) is called a statistical manifold, if it satisfies

$$Lg(F, J) = g(D^0_L F, J) + g(F, D^1_L J), \tag{5.3}$$

for all vector fields $F, J, L \in \Gamma(T\bar{M})$. In statistical manifolds, we have [9]

(i) The connections D^0 and D^1 are known to be as dual or conjugate connections.

(ii) If structure (D^0,g) is statistical , so is (D^1,g) on Riemannian manifold $\bar{M}$.

(iii) For conjugate or dual connections D^0 and D^1, we have

$$D^{0*} = \frac{1}{2}(D^0 + D^1), \tag{5.4}$$

where D^{0*} denotes Levi-Civita connections on the Riemannian manifold $(\bar{M}, g)$.

(iv) R^0 and R^1 are curvature tensor fields of conjugate connections D^0 and D^1 respectively, satisfy the following condition

$$g(R^1(F, J)L, O) = -g(L, R^0(F, J)O). \tag{5.5}$$

Let $(\bar{M}, g)$ be $(2n+1)$ dimensional Riemannian manifold and $(M, \bar{g})$ be an $n-$ lightlike hypersurface respectively, then the Gauss formula are given by [116]

$$D^0_F J = D_F J + \tau(F, J), \tag{5.6}$$

$$D^1_F J = D^*_F J + \tau^*(F, J), \tag{5.7}$$

where τ^* and τ represents bilinear, symmetric and imbedding 1-forms of lightlike hypersurface $(M, \bar{g})$ inRiemannian manifold $(\bar{M}, g)$. The corresponding Gauss equations, in relation to connections D^0 and D^1 respectively are [116] given by

$$\begin{aligned} g(R^0(F, J)L, O) =& g(R(F, J)L, O) + g(\tau(F, L), \tau^1(J, O)) - \\ & g(\tau^1(F, O), \tau(J, L)), \end{aligned} \tag{5.8}$$

and

$$\begin{aligned} g(R^1(F, J)L, O) =& g(R(F, J)L, O) + g(\tau^*(F, L), \tau(J, O)) - \\ & g(\tau(F, O), \tau^*(J, L)). \end{aligned} \tag{5.9}$$

On an odd-dimensional manifold $(\bar{M}, g)$, let us represent $(1, 1)$-tensor field as ϕ, ξ as the associated vector field and η as a 1-form, satisfying the following relations

$$\eta(\xi) = 1, \tag{5.10}$$

$$\phi^2 F = -F + \eta(F)\xi, \tag{5.11}$$

for any $F \in \Gamma(T\bar{M})$ and $\eta(F) = g(F, \xi)$. An odd-dimensional differentiable manifold $(\bar{M}, g)$ in contact geometry, is almost contact metric-like manifold [108], if it has an almost contact metric structure (ϕ, ξ, g) on $(\bar{M}, g)$ satisfying

$$g(\phi F, J) + g(F, \phi^* J) = 0, \tag{5.12}$$

for any $F, J \in \Gamma(T\bar{M})$ and ϕ^* is $(1,1)$-type tensor field. An odd dimensional almost contact metric like manifold $(\bar{M}, g)$ satisfying [108] the following conditions is known to be as the Sasaki-like statistical manifold $(\bar{M}, D^0, g, \phi, \xi)$.

$$D^0_F \xi = -\phi F, \tag{5.13}$$

$$(D^0_F \phi)J = g(F, J)\xi - \eta(J)F, \tag{5.14}$$

where $F, J \in \Gamma(T\bar{M})$. The sectional curvature σ, in any space is the $\phi-$sectional curvature. where σ denotes sectional curvature of a $\phi-$section. Let us assume $(\bar{M}, D^0, g, \phi, \xi)$ has constant ϕ - sectional curvature c, then by virtue of [108], the Riemannian curvature tensors $\bar{R}^0$ and $(\bar{R}^1)$ of Sasaki-likr statistical manifold $(\bar{M}, g)$ [108] is given by

$$\begin{aligned} R^0(F,J)L =& \frac{c+3}{4}[g(J,L)F - g(F,L)J] \\ &+ \frac{c-1}{4}[g(\phi J, L)\phi F - g(\phi F, L)\phi J \\ &- g(\phi F, J)\phi L + g(F, \phi J)\phi L - g(J,\xi)g(L,\xi)F \\ &+ g(F,\xi)g(L,\xi)J + g(J,\xi)g(L,F)\xi - g(F,\xi)g(J,L)\xi], \end{aligned} \tag{5.15}$$

where $F, J, L \in T\bar{M}$. Interchanging ϕ for ϕ^* in the above equation, we can obtain Riemannian curvature tensor R^1 given by

$$\begin{aligned} R^1(F,J)L =& \frac{c+3}{4}[g(J,L)F - g(F,L)J] \\ &+ \frac{c-1}{4}[g(\phi^* J, L)\phi^* F - g(\phi^* F, L)\phi^* J \\ &- g(\phi^* F, J)\phi^* L + g(F, \phi^* J)\phi^* L - g(J,\xi)g(L,\xi)F \\ &+ g(F,\xi)g(L,\xi)J + g(J,\xi)g(L,F)\xi - g(F,\xi)g(J,L)\xi]. \end{aligned} \tag{5.16}$$

5.3 Lightlike Hypersurfaces of Sasaki-Like Statistical Space Form

Let $(M,\ \bar{g},\ D,\ D^{*})$ be a lightlike hypersurface of Sasaki-like statistical space form $\bar{M}(c)$. If $Z \in Rad(TM)$ then $g(\phi Z, Z) = 0$, we obtain

$$\begin{aligned} &g(\phi X, Z) = -g(X, \phi Z) = 0, \\ &g(\phi X, X) = 0, \\ &(\phi X, \phi Z) = 1 \end{aligned} \tag{5.17}$$

Since B, the second fundamental form is selfreliant on screen distribution, we have have

$$B(.,Z) = 0 \tag{5.18}$$

On screen bundle $S(TM)$ of degenerate hypersurface M, let us assume that P^0 is the projection and concedering $Y = \phi X \in \phi(ltr(TM))$ as degenerate vector field. Then for any vector field $F \in \Gamma(TM)$ can be represented by

$$F = SF + QF, \quad Qx = Yu(F), \tag{5.19}$$

$$\phi F = \phi(SF) + \phi Y u(F). \tag{5.20}$$

Putting $\phi^{*}F = \phi(SF)$ in equation (5.20), we obtain

$$\phi F = \phi^{*}F - Xu(F), \tag{5.21}$$

where ϕ and ϕ^{*} are $(1-1)-$ tensor fields on Sasaki-like statistical manifold $(\bar{M}, g)$ and lightlike hypersurface $(M, \bar{g})$ respectively. Applying ϕ to equation (3.7), we get

$$\phi^2 F = \phi\phi^{*}F - u(F)\phi X, \tag{5.22}$$

$$\phi^2 F = -F + \eta(F)\xi - u(F)Y. \tag{5.23}$$

Let S denote projection morphisms of tangent bundle TM into distribution G and Q denote projection morphisms of tangent bundle in distribution G^{*}. Also u and v are differential $1-$forms defined locally on lightlike hypersurface $(M, \bar{g})$ by

$$v(F) = g(F, V), \tag{5.24}$$

$$u(F) = g(F, X). \tag{5.25}$$

From equations (5.24) and (5.25), we acquire

$$u(Y) = -1, \tag{5.26}$$

$$u(J) = 0 \quad \forall J \in \Gamma(G), \tag{5.27}$$

$$\phi^2 X = \phi Y = -X, \tag{5.28}$$

$$\phi^2 Z = \phi V = -Z. \tag{5.29}$$

Let $(M, \bar{g})$ be null-hypersurface of Sasaki-Like Statistical Space Form $\bar{M}(c)$, then the Gauss and Weingarten formulas with respect to dual connections D^0 and D^1 are given by

$$D^0_F J = D_F J + B(F, J)X, \tag{5.30}$$

$$D^0_F X = -A_X F + \tau(F)X, \tag{5.31}$$

$$D^1_F J = D^*_F J + B^*(F, J)X, \tag{5.32}$$

$$D^1_F X = -A^*_X F + \tau^*(F)X, \tag{5.33}$$

for all $X \in \Gamma ltr(TM)$, $F,\ J\ \in \Gamma(TM)$. Where $D_F J$, $D^*_F J$, $A_X F$ and $A^*_N F\ \in \Gamma(TM)$. From eqautions (5.30), (5.30),(5.30) and (5.30), we know that

$$B(F, J) = g(D^0_F J, \tau), \qquad \tau(F) = g(D_F X, \tau), \tag{5.34}$$

$$B^*(F, J) = g(D^1_F J, \tau), \qquad \tau^*(F) = g(D^1_F X, \tau), \tag{5.35}$$

where A_X and A^*_X represent Weingarten mappings commonly known as shape operator, B and B^* represents second fundamental forms and D and D^* are induced connections in relation to congugate connections D^0 and D^1 respectively. By using equation (5.1) and the Gauss formula, we get [9]

$$\begin{aligned} F\bar{g}(J, L) &= \bar{g}(D^0_F J, L) + \bar{g}(J, D^1_F L) \\ &= \bar{g}(D_F J, L) + \bar{g}(J, D^*_F L) + B(F, J)\eta(L) + B^*(F, L)\eta(J), \end{aligned} \tag{5.36}$$

for any $F,\ J,\ L \in \Gamma(TM)$, where η is differential $1-$form locally defined on null hypersurface $(M, \bar{g})$.

Lemma 5.3.1. *Let $(M, \bar{g}, D, D^*)$ be a lightlike hypersurface of Sasaki-like statistical space form $\bar{M}(c)$. Then Gauss formula and Codazzi formula of Sasakilike statistitical manifold $(\bar{M}, D^0, g, \phi, \tau)$ are respectively, given by*

$$\begin{aligned} R(F,J)L =& \frac{c+3}{4}[g(J,L)F - g(F,L)J] + \frac{c-1}{4}[g(\phi J, L)\phi^* F \\ & - g(\phi F, L)\phi^* J - g(\phi F, J)\phi^* L + g(F, \phi J)\phi^* L \\ & - \eta(J)\eta(L)F + \eta(F)\eta(L)J + \eta(J)g(L,F)\xi - \eta(F)g(L,J)\xi] \\ & - B(F,L)A_X J + B(J,L)A_X F, \end{aligned} \tag{5.37}$$

and

$$\begin{aligned} (D_F h)(J,L) - (D_J h)(F,L) =& -\frac{c-1}{4}[g(\phi J, L)u(F) - g(\phi F, L)u(J) \\ & + g(\phi F, J)u(L) - g(F, \phi J)u(L)]X. \end{aligned} \tag{5.38}$$

Proof. Suppose $(M, \bar{g})$ be the lightlike hypersurface of $(\bar{M}, g)$ of constant $\phi-$sectional curvature c. From equation (5.12) and equation (1.79), we obtain

$$\begin{aligned} R(F,J)L =& \frac{c+3}{4}[g(J,L)F - g(F,L)J] \\ & + \frac{c-1}{4}[g(\phi J, L)\phi F - g(\phi F, L)\phi J \\ & - g(\phi F, J)\phi L + g(F, \phi J)\phi L - g(J,\xi)g(L,\xi)F \\ & + g(F,\xi)g(L,\xi)J + g(J,\xi)g(L,F)\xi - g(F,\xi)g(J,L)\xi] \\ & - A_{h(F,L)}J + A_{h(J,L)}F - (D_F h)(J,L) + (D_J h)(F,L), \end{aligned} \tag{5.39}$$

$$\begin{aligned} R(F,J)L =& \frac{c+3}{4}[g(J,L)F - g(F,L)J] \\ & + \frac{c-1}{4}[g(\phi J, L)\phi F - g(\phi F, L)\phi J \\ & - g(\phi F, J)\phi L + g(F, \phi J)\phi L - \eta(J)\eta(L)F \\ & + \eta(F)\eta(L)J + \eta(J)g(L,F)\xi - \eta(F)g(J,L)\xi] \\ & - A_{h(F,L)}J + A_{h(J,L)}F - (D_F h)(J,L) + (D_J h)(F,L). \end{aligned} \tag{5.40}$$

Using equation (5.21) in equation (5.38), we get equation (5.41)

$$\begin{aligned} R(F,J)L =& \frac{c+3}{4}[g(J,L)F - g(F,L)J] \\ &+ \frac{c-1}{4}[g(\phi J,L)\phi^* F - g(\phi F,L)\phi^* J \\ &- g(\phi F,J)\phi^* L + g(F,\phi J)\phi^* L - g(\phi J,L)u(F)X - g(\phi F,L)u(J)X \\ &- g(\phi F,J)u(L)X + g(F,\phi J)u(L)X - \eta(J)\eta(L)F \\ &+ \eta(F)\eta(L)J + \eta(J)g(L,F)\xi - \eta(F)g(J,L)\xi] \\ &- B(F,L)A_X J + B(J,L)A_X F - (D_F h)(J,L) + (D_J h)(F,L) \end{aligned} \tag{5.41}$$

Comparing tangential and transversal components of equation (5.39), we acquire

$$\begin{aligned} R(F,J)L =& \frac{c+3}{4}[g(J,L)F - g(F,L)J] + \frac{c-1}{4}[g(\phi J,L)\phi^* F \\ &- g(\phi F,L)\phi^* J - g(\phi F,J)\phi^* L + g(F,\phi J)\phi^* L \\ &- \eta(J)\eta(L)F + \eta(F)\eta(L)J + \eta(J)g(L,F)\xi - \eta(F)g(J,L)\xi] \\ &- B(F,L)A_X J + B(J,L)A_X F, \end{aligned} \tag{5.42}$$

$$\begin{aligned} (D_F h)(J,L) - (D_J h)(F,L) =& -\frac{c-1}{4}[g(\phi J,L)u(F) - g(\phi F,L)u(J) \\ &- g(\phi F,J)u(L) + g(F,\phi J)u(L)]X. \end{aligned} \tag{5.43}$$

Equation (5.42) and (5.43) completes the proof. Similarly, we can compute Gauss formula and Codazzi formula with respect to connection D^1. □

Lemma 5.3.2. *Suppose $(M,\bar{g})$, be a (null) lightlike hypersurface of Sasaki-like statistical space form $\bar{M}(c)$, then*

$$\begin{aligned} g(R(F,Z)L,X) =& -\frac{c+3}{4}g(F,L) - \frac{c-1}{4}[v(L)u(\phi F) \\ &- 2v(F)u(\phi L) - \eta(L)\eta(F)]. \end{aligned} \tag{5.44}$$

Proof. Our desired result directly follows from (5.37) by using Z in place of J in Gauss equation (5.42) and then taking inner product of resulting equation with X as

$$\begin{aligned} R(F,Z)L =& \frac{c+3}{4}[g(Z,L)F - g(F,L)Z] \\ &+ \frac{c-1}{4}[g(\phi Z,L)\phi^* F - g(\phi F,L)\phi^* Z - 2g(\phi F,Z)\phi^* L \\ &- \eta(Z)\eta(L)F + \eta(F)\eta(L)Z + \eta(Z)g(L,F)\xi - \eta(F)g(Z,L)\xi] \\ &- B(F,L)A_X Z + B(Z,L)A_X F, \end{aligned} \tag{5.45}$$

$$\begin{aligned}
g(R(F,Z)L,X) =& \frac{c+3}{4}[g(Z,L)g(F,X) - g(F,L)g(Z,X)] \\
&+ \frac{c-1}{4}[g(\phi Z,L)g(\phi^* F,X) - g(\phi F,L)g(\phi^* Z,X) \\
&- 2g(\phi F,Z)g(\phi^* L,X) - \eta(Z)\eta(L)g(F,X) \\
&+ \eta(F)\eta(L)g(Z,X) + \eta(Z)g(L,F)g(\xi,X) \\
&- \eta(F)g(Z,L)g(\xi,X)] \\
&- B(F,L)g(A_X Z,X) + B(Z,L)g(A_X F,X),
\end{aligned} \tag{5.46}$$

From equations (5.15), (5.17), (5.21), (5.24) and (5.25), we get

$$\begin{aligned}
g(R(F,Z)L,X) =& -\frac{c+3}{4}g(F,L) - \frac{c-1}{4}[v(L)u(\phi F) \\
&- 2v(F)u(\phi L) - \eta(F)\eta(L)].
\end{aligned} \tag{5.47}$$

Equation (5.47) completes the proof. □

Lemma 5.3.3. *Let $(M,\bar{g})$ be a lightlike hypersurface of Sasaki-like statistical manifold $(\bar{M},g)$, constant ϕ sectional curvature (c), then*

$$B(J,Y) = C(J,V), \tag{5.48}$$

for any vector field $J \in \Gamma(TM)$.

Proof. By knowing the properties of fundamental form B, we acquire

$$\begin{aligned}
B(J,\phi X) &= g(h(J,\phi X),Z), \\
&= g(D^0_J \phi X, Z), \\
&= -g(D^0_J X, \phi Z) + g((D^0_J \phi)X, Z),
\end{aligned} \tag{5.49}$$

from equations (5.9) and (1.77), we get

$$B(J,\phi X) = -g(D^0_J X, \phi Z), \tag{5.50}$$

$$B(J,\phi X) = -g(A_X J, \phi Z), \tag{5.51}$$

$$B(J,\phi X) = C(J,\phi Z), \tag{5.52}$$

Setting $U = \phi X$ and $V = \phi Z$ in equation (5.52), we obtain equation (5.48). □

Theorem 5.3.1. *For lightlike hypersurface $(M, \bar{g})$ of the Sasaki-like statistical manifold $(\bar{M}, g)$ of constant $\phi-$ sectional curvature c. If local fundamental form B is parallel on $(M, \bar{g})$, then there does not exist any lightlike hypersurface of Sasaki-like statistical space form $\bar{M}(c)$ for $c \neq 1$.*

Proof. Let us suppose there exists a (lightlike) null-hypersurface $(M, \bar{g})$ of Sasaki-like statistical space form $\bar{M}(c)$ for $c \neq 1$. By putting vector fields $J = Z$ and $L = \phi X$ in equation (5.38), we get

$$\begin{aligned}(D_F h)(Z, \phi X) - (D_Z h)(F, \phi X) = &-\frac{c-1}{4}[g(\phi Z, \phi X)u(F) - g(\phi F, \phi X)u(Z) \\ &- g(\phi F, Z)u(\phi X) + g(F, \phi Z)u(\phi X)]X.\end{aligned} \tag{5.53}$$

As assumed that the local second fundamental form B of lightlike hypersurface $(M, \bar{g})$ is parallel, then from equation (5.53), we obtain

$$c = 1 \tag{5.54}$$

which is a contradiction, hence there does not exist any lightlike hypersurface $(M, \bar{g})$ for $c \neq 1$ of Sasaki-like statistical space form $\bar{M}$(c). □

Theorem 5.3.2. *For $c \neq 5$, we can not find lightlike hypersurface $(M, \bar{g})$ of Sasaki-like statistical space form $(\bar{M}(C))$, having parallel screen distribution for all values on $\bar{M}$.*

Proof. We 1st suppose that M is a lightlike hypersurface of $\bar{M}(c))$ with parallel screen distribution. Replacing F by Z, J by ϕX and L by ϕZ in equation (5.15), we obtain

$$\begin{aligned}R^0(Z, \phi X)\phi Z =& \frac{c+3}{4}[g(\phi X, \phi Z)Z - g(Z, \phi Z)\phi X] \\ &+ \frac{c-1}{4}[g(\phi^2 X, \phi Z)\phi Z - g(\phi Z, \phi Z)\phi^2 N \\ &- g(\phi Z, \phi X)\phi^2 Z + g(Z, \phi^2 N)\phi^2 E \\ &- g(\phi X, \xi)g(\phi Z, \xi)Z + g(Z, \xi)g(\phi Z, \xi)\phi X \\ &+ g(\phi X, \xi)g(\phi Z, Z)\xi - g(Z, \xi)g(\phi Z, \phi X)\xi],\end{aligned} \tag{5.55}$$

where vector fields $F, J, L \in T\bar{M}$. Now taking the inner product of equation (5.55) with X, we acquire

$$g(R^0(Z, \phi X)\phi Z, X) = \frac{c-5}{4}. \tag{5.56}$$

From [77], we have

$$g(R^0(F, J)P^0 L, X) = 0, \tag{5.57}$$

From equation (5.57), we obtain

$$g(R^0(Z,\phi X)\phi Z, X) = 0. \tag{5.58}$$

From equation (5.57) and equation (5.55) we get

$$c = 5. \tag{5.59}$$

Which is contradicting, Hence, for $c \neq 5$, we can not find lightlike hypersurface $(M, \bar{g})$ of Sasaki-like statistical space form $(\bar{M}(C))$. □

Lemma 5.3.4. *Let $(M, \bar{g})$ be the (lightlike) null-hypersurface of Sasaki-like statistical manifold $(\bar{M}, g)$, then*

$$B(V,Y) = 0, \qquad C(V,V) = 0, \tag{5.60}$$

where V acts as the principle vector field.

Proof. Using equations (5.14) and (1.74), we acquire

$$D^0_F Y = D^0_F \phi X, \tag{5.61}$$

$$D_F Y + B(F,Y)X = \phi D^0_F X + (D^0_F \phi)X, \tag{5.62}$$

$$D_F Y + B(F,U)X = -\phi A_X F + \tau(F)\phi X + g(F,X)\xi. \tag{5.63}$$

Using (5.21) in equation (5.63), we obtain

$$D_F Y + B(F,Y)X = -\phi^* A_X F + u(A_X F)X + \tau(F)\phi X + g(F,X)\xi. \tag{5.64}$$

Now comparing tangential and transversal components of equation (5.64), we get

$$D_F Y = \tau(F)\phi X + g(F,X)\xi - \phi^* A_X F, \tag{5.65}$$

$$B(F,U) = u(A_X F) + g(A_X F, \phi Z) = C(F,V). \tag{5.66}$$

This completes our required result. □

Lemma 5.3.5. *Suppose $(M, \bar{g})$ be the lightlike hypersurface of Sasaki-like statistical space form $\bar{M}(c)$ then the Codazzi formula of lightlike hypersurface is given by*

$$\begin{aligned}(D_F A_X)J - (D_J A_X)F =& \frac{c+3}{4}[u(J)F - u(F)J] \\ &+ \frac{c-1}{4}[g(F,Y)\phi J - g(J,Y)\phi F + g(\phi F, J)Y \\ &- g(F,\phi J)Y + \eta(F)u(F)\xi] + \tau(F)A_X J - \tau(J)A_X F.\end{aligned} \tag{5.67}$$

Proof. In lemma-5.3.1, replacing L for X and after making some straight forward computations (5.67) is obtained. □

Now let us assume that $a_1, ...a_{n-2}, ..., a_{(2n-4)},\ \xi, ...Z,\ \phi X, \phi Z$ are orthonormal basis of tangent bundle $\Gamma(TM)$, then

$$\phi a_i = a_{n-2+i}, \quad \phi a_{n-2+i} = -a_i \tag{5.68}$$

and

$$\phi\xi = 0 \tag{5.69}$$

for each $i = 1, 2, 3, ..., m-2$.

Lemma 5.3.6. *Let $M, \bar{g}$ be the lightlike hypersurface of Sasaki-like statistical manifold $(\bar{M}, g)$, then*

$$A_X Y = \sum_{i=1}^{2n-4} \frac{C(Y, a_i)}{e_i} a_i + C(Y, \xi)\xi + C(Y, Y)V + C(Y, V)Y, \tag{5.70}$$

and

$$A_X Z = \sum_{i=1}^{2n-4} \frac{C(Z, a_i)}{e_i} a_i + C(Z, \xi)\xi + C(Z, Y)V, \tag{5.71}$$

where the signature of basis (a_i) is represented by (e_i).

Proof. For lightlike hypersurfaces of Sasaki-like statistical manifolds, we have

$$A_X Y = \sum_{i=1}^{2n-4} \gamma_i a_i + \lambda\xi + \beta_1 Z + \beta_\phi Z + \beta_3 \phi X. \tag{5.72}$$

From (6.17), we have $\gamma_i = \frac{C(Z, a_i)}{e_i}$, $\lambda = C(Y, \xi)$, $\beta_1 = 0$, $\beta_2 = C(Y, Y)$, and $\beta_3 = -C(Y, V)$. Using these values in equation (5.72), it yields (5.70), similarly, we can obtain (5.71). □

Theorem 5.3.3. *There does not exist any lightlike-hypersurface $(M, \bar{g})$, of the Sasaki-like statistical manifold $(\bar{M}, g)$ with $c \neq 5$, satisfying*

$$B(Y, Y) = 0, \tag{5.73}$$

and

$$\bar{g}((D_Z A_X)Y, V) = \bar{g}((D_Y A_X)Z, V). \tag{5.74}$$

Proof. Putting $J = U$ and $F = Z$ in equation (5.67), we get

$$(D_Z A_N)Y - (D_Y A_N)Z = \frac{c-5}{4}Y + \tau(Y)A_X Z - \tau(Z)A_X Y. \tag{5.75}$$

Using equations (5.70) and (5.71) in equation (5.75), we acquire

$$\begin{aligned}(D_Z A_X)Y - (D_Y A_N)Z = \frac{c-5}{4}Y + \tau(Y)[\frac{C(Z,a_i)}{e_i}a_i \\ + C(Z,\xi)\xi + C(Z,Y)V] - \tau(Z)[\frac{C(Y,a_i)}{e_i}a_i \\ + C(Y,\xi)\xi + C(Y,Y)V + C(Y,V)Y].\end{aligned} \tag{5.76}$$

Taking inner product of equation (5.76) with V of equation and in the resulting equation using lemma-5.3.4, we obtain

$$\bar{g}((D_Z A_X)Y - (D_Y A_X)Z, V) = \frac{c-5}{4} - \tau(Z)B(Y,Y). \tag{5.77}$$

Hence equation (5.77) completes the proof. □

Chapter 6

Lightlike Hypersurfaces of Semi-Euclidean Spaces

The main objective of this chapter is to study lightlike hypersurfaces (M, g) of semi-Euclidean space Z_q^n. As proposed by Tarafder et. al. [112] in this chapter, we study weakly semi-pseudo symmetric lightlike hypersurface (M, g) and the weakly semi-pseudo Ricci symmetric lightlike hypersurface of semi-Euclidean space Z_q^n. Finally, in this chapter, we showed that the Ricci tensor of a weakly semi-pseudo Ricci symmetric lightlike hypersurface (M, g) of semi-Euclidean spaces Z_q^n is totally geodesic, if it admits cyclic parallel of Ricci tensor and satisfies Codazzi type of Ricci tensor S.

6.1 Introduction

In 1920s, E. Carten [20] studied locally symmetric as an induction of the space forms. Such spaces are defined by $DR = 0$, where D and R represents Levi-Civita connection and (1,3) type Riemannian curvature tensor. All locally symmetric spaces satisfies $\mathbf{R}.R = 0$. Where $\mathbf{R}$ acts as derivation over R. Tamassy and Bin [110] acquainted the general annotation of weakly-symmetric Riemannian manifolds. After Tamassy and Bin [110] alot of work has been carried on this research problem. In the sprit of [110], a non-flat n-dimensional Riemannian manifold is known to be as weakly symmetric manifold, if R satisfies

$$\begin{aligned}(D_W R)(F, J, L, H) =&(A_1(W) + B_1(W))R(F, J, L, H)\\ &+ C_1(F)R(W, J, L, H) + C_1(J)R(F, W, L, H)\\ &+ D_1(L)R(F, J, W, H) + D_1(H)R(F, J, L, W),\end{aligned} \tag{6.1}$$

where R is the Riemannian curvature tensor of type $(0,4)$, and $A_1, B_1, C_1 and\ D_1$ represent associated 1-forms defined by $A_1(F) = g(F,\theta_1)$, $B_1(F) = g(F,\sigma_1)$, $C_1(F) = g(F,\alpha_1)$ and $D_1(F) = g(F,\Phi_1)$.

In Riemannian geometry of differentiable manifods, by keeping in tune with Dubey, [38] introduced the concept and notation of generalised weakly symmetric properties in Riemannian manifolds as

$$\begin{aligned}(D_W R)(F,J,L,H) =&(A_1(W)+B_1(W))R(F,J,L,H)\\&+C_1(F)R(W,J,L,H)+C_1(J)R(F,W,L,H)\\&+D_1(L)R(F,J,W,H)+D_1(H)R(F,J,L,W)\\&+(A_2(W)+B_2(W))G(F,J,L,H)\\&+C_2(F)G(W,J,L,H)+C_2(J)G(F,W,L,H)\\&+D_2(L)G(F,J,W,H)+D_2(H)G(F,J,L,W),\end{aligned} \tag{6.2}$$

where $G(F,J,L,H) = g(J,L)g(F,H) - g(F,L)g(J,H)$.
By following equation (6.2) and the concept of Dubey [38], we will define weakly semi-psedo symmetric property of Riemannian manifolds as an n-dimensional Riemannian manifold, if the associated 1-forms $A_1 = -B_1$, $C_1 = D_1 = \alpha$ and $A_2 = C_2 = D_2 = 0$ and admits the following equation

$$\begin{aligned}(D_W R)(F,J,L,H) =&\alpha(F)R(W,J,L,H)+\alpha(J)R(F,W,L,H)\\&+\alpha(L)R(F,J,W,H)+\alpha(H)R(F,J,L,W),\end{aligned} \tag{6.3}$$

where α is a differential 1-form defined as $\alpha(F) = g(F,\rho)$.

In mathematical physics and general relativity, the widespread concept of lightlike submanifolds has been premeditated from time to time and had found its applications in the event horizons of black horizons, the Kruskal and Kerr black holes. In Authors (1966), Duggal and Bejancu [42] introduced the idea of degenerate or lightlike geometry of semi-Riemannian manifolds, and the authors [42] gave an extrinsic way in the field of differential geometry. A. Bejancu et.al [13] studied geometry of a lightlike hypersurface in a semi-Euclidean space and proved the Bernstein theorem for lightlike hypersurfaces. Bayram Sahin in 2007, [100] studied semi-symmetric properties of lightlike hypersurfaces of semi-Euclidean spaces and showed that screen conformal lightlike hypersurface of spaces are semisymmetric lightlike hypersurfaces. D. Saglam et.al [102] showed that lightlike hypersurfaces satisfying minimal translation are hyperplanes and established a relation that justifies that all lightlike hypersurfaces of semi-Riemannian spaces can be computed as a sum of functions of one variables. In 2009, Mehmet Erdogan and

Ignace Van de Woestyne [50] examined translation and homothetical lightlike hypersurfaces of semi-Euclidean space and proved taht lightlike hypersuraces of this space are hyperplane and minimal. B. Sahin [100] investigate lightlike hypersurfaces which are semi-symmetric, Ricci semi-symmetric, parallel or semi-parallel in a semi-Euclidean space and showed that every screen conformal lightlike hypersurface of the Minkowski spacetime is semi-symmetric. In 2017, N. O. Poyraz and E. Yasar [95] obtained results on screen semi-invariant lightlike hypersurfaces of a golden semi-Riemannian manifold and screen conformal semi-invariant lightlike hypersurfaces. A Upadhyay et.al. [114] investigated semi-symmetric, Ricci semi-symmetric lightlike hypersurfaces of an indefinite generalized Sasakian space and obtained some necessary conditions for Ricci tensor of lightlike hypersurface of indefinite generalized Sasakian space form to be symmetric and parallel. Motivated by these in this chapter, we study weakly semi-pseudo symmetric and weakly semi-pseudo Ricci symmetric light like hypersurface of semi-Euclidean spaces.

6.2 Preliminaries

This section is devoted to give a brief introduction on lightlike geometry of differential geometry of manifolds and semi-Euclidean spaces. Let an $(n+2)$-dimensional space Z_q^{n+2} be the semi-Euclidean and let (M, g) be its lightlike hypersurface. The metric used in null hypersurface acts as an induced metric. In this we consider weakly semi-pseudo symmetric properties on degenerate hypersurfaces of semi-Euclidean spaces. Sufficient conditions for degenerate hypersurfaces to be weakly semi-pseudo symmetric degenerate hypersurfaces are checked and some characterization conditions for such hypersurfaces are obtained.

If an induced-metric g on (M, g) is degenerate then (M, g) is said to be lightlike hypersurface, see also ([100], [46], [77], [40]). On lightlike hypersurfaces there exists null but non zero vector field $\xi \neq 0$, such that

$$g(\xi, F) = 0, \tag{6.4}$$

for all $F \in \Gamma(T\bar{M})$. In degenerate geometry of manifolds, $RadT_FM$ is a radical space or null space of a tangent space T_FM and is a subspace at every $F \in (M, g)$ defined by [42]

$$RadT_FM = \{\xi \in RadT_FM : g_F(\xi, F) = 0, F \in \Gamma(T\bar{M})\}. \tag{6.5}$$

The nullity degree of an induced metric g is the dimension of $RadT_FM$ and for null hypersurfaces the nullity degree of an induced metric g is 1. $RadTM$ is called radical distribution, spanned by null vector field ξ. $S(TM)$ is a known to be as screen-bundle of null-hypersurface of Z_q^{n+2} and it is a complementary vector bundle of $RadTM$ in TM. We note that any $S(TM)$ is non degenerate and $S(TM)^{\perp}$ is the complementary vector bundle of $S(TM)$ in TM of rank 2 and is known to as screen transversal bundle. Since radical distribution $RadTM$ is null bundle of screen bundle $S(TM)^{\perp}$, then there exists a local section of $S(TM)^{\perp}$, satisfying

$$g(F,X) = \quad g(X,X) = 0, \quad g(Z,X) = 1. \tag{6.6}$$

The pair (Z,X) acts as the local frame field of screen transversal bundle $S(TM)^{\perp}$ and it should be noted that X is transversal to degenerate hypersurface M then there exists line bundle $ltrTM$ of tangent bundle of a manifold called as null-transversal bundle spanned by X. Hence the following decomposition emerges

$$T\bar{M} = TM \oplus ltr(TM) = S(TM) \perp RadTM \oplus ltr(TM). \tag{6.7}$$

In equation (6.7) $\oplus$ represents direct sum but not orthogonal ([42],[46]). Thus in view of equation (6.4), the corresponding Gauss formula and Weingarten formulae are given by

$$\bar{D}_F J = D_F J + h(F,J), \tag{6.8}$$

$$\bar{D}_F X = -A_X F + D^t_F X, \tag{6.9}$$

putting

$$B(F,J) = g(h(F,J),Z), \tag{6.10}$$

and

$$\tau(F) = g(D^t_F X, Z), \tag{6.11}$$

Using equations (2.108) and (2.109) in equations (2.106) and (2.107), we acquire

$$\bar{D}_F J = D_F J + B(F,J)X, \tag{6.12}$$

and

$$\bar{D}_F X = -A_X F + \tau(F)X, \tag{6.13}$$

for any vector fields $F,J \in \Gamma(TM)$. Where $X \in \Gamma ltr(TM), -A_X F, D_F J \in \Gamma(TM)$ and $h(F,J), D^t_F X \in \Gamma(ltr(TM))$. Here A_X and B represent shape operator and second fundamental form respectively. D and D^t represents linear connections on null hypersurface M and on $ltr(TM)$ respectively [42]. From equation (2.18), we can state that D as an induced connection on null-hypersurface M as

$$(D_F g)(J,L) = B(F,J)\eta(L) + B(F,L)\eta(J), \tag{6.14}$$

for all $F,\, J,\, L \in \Gamma(TM)$ and D is an induced non-metric connection on smooth manifold M. Let P be the projection morphism, the local Gauss and Weingarten formulas are given by

$$\nabla_F PJ = \nabla^0_P J + C(F, PJ)\xi, \tag{6.15}$$

$$D_F\xi = -A^0_\xi F + \tau(F)\xi, \tag{6.16}$$

where $\nabla_F PJ$ and $A^0_\zeta F \in S(TM)$. Also A^0_X, C and ∇^0 represents local shape operator, local second fundamental form and induced connection on screen bundle $S(TM)$ respectively. On lightlike hypersurface shape operators and local fundamental forms B and C are related by

$$g(XF, PJ) = C(F, PJ), \qquad g(XF, X) = 0, \tag{6.17}$$

$$g(A^0_\xi F, PJ) = B(F, PJ), \qquad g(A^0_\xi F, \xi) = 0, \tag{6.18}$$

for all $F, J \in \Gamma(TM)$. Since $g(D_F\xi, \xi) = 0$, we have

$$B(F, \xi) = 0. \tag{6.19}$$

Let R be curvature tensors of lightlike hypersurface, we have (see [42])

$$R(F, J)L = R(F, J)L + A_{h(F,L)}J - A_{h(J,L)}F + (D_F h)(J, L) - (D_J h)(F, L), \tag{6.20}$$

where D is linear connections on R.

6.3 Weakly semi-pseudo symmetric lightlike hypersurfaces of semi-Euclidean spaces

Since semi-Euclidean spaces are flat, then for any $F, J, L \in \Gamma(TM)$ and $N \in \Gamma(tr(TM))$, the Gauss equation of degenerate hypersurface of Z^n_q is given by [100]

$$R(F, J)L = B(J, L)AF - B(F, L)AJ. \tag{6.21}$$

Proposition 6.3.1. *The lightlike hypersurface (M, g) of a semi-Euclidean space Z^n_q is weakly semi-pseudo symmetric, if*

$$(D_W B)(F, L) = \alpha(F)B(W, L) + \alpha(L)B(F, W), \tag{6.22}$$

$$(D_W A)F = \alpha(F)AW + g(AF, W)\rho, \tag{6.23}$$

are satisfied for vector fields $F, J, L \in \Gamma(TM)$, where $\alpha(F) = g(F, \rho)$.

Proof. Since Z_q^n is semi-Euclidean it means Z_q^n is flat space. It follows $\bar{R} = 0$, then by virtue of equation (6.21), we obtain

$$\begin{aligned}(D_W R)(F, J)L = {} & (D_W B)(J, L)AF - (D_W B)(F, L)AJ \\ & + B(J, L)(D_W A)F - B(F, L)(D_W A)J.\end{aligned} \tag{6.24}$$

Substituting (6.21) in (6.3), we obtain

$$\begin{aligned}&\alpha(F)R(W, J, L) + \alpha(J)R(F, W, L) + \alpha(L)R(F, J, W) \\ &+ g(R(F, J, L)W)\rho = \alpha(F)\{B(J, L)AW - B(W, L)AJ\} \\ &+ \alpha(J)\{B(W, L)AF - B(F, L)AW\} + \alpha(L)\{B(J, W)AF \\ &- B(F, W)AJ\} + B(J, L)g(AF, W)\rho - B(F, L)g(AJ, W)\rho,\end{aligned} \tag{6.25}$$

for all $F, J, L, W \in \Gamma(TM)$. Using equations (6.22) and (6.23) in equation (6.25), lightlike hypersurface (M, g) is weakly semi-pseudo symmetric lightlike hypersurface of semi-Euclidean space Z_q^n. □

Theorem 6.3.1. *Let M be weakly semi-pseudo symmetric degenerate (lightlike) hypersurface of semi-Euclidean space, if $\rho \in S(TM)$ then either M is totally geodesic or $A_X\xi$ has zero eigenvalue.*

Proof. Since (M, g) is weakly semi-pseudo symmetric lightlike hypersurface then from equation (6.24) and equation (6.25), we obtain

$$\begin{aligned}&(D_W B)(J, L)AF - (D_W B)(F, L)AJ + B(J, L)(D_W A)F \\ &- B(F, L)(D_W A)J = \alpha(F)\{B(J, L)AW - B(W, L)AJ\} \\ &+ \alpha(J)\{B(W, L)AF - B(F, L)AW\} + \alpha(L)\{B(J, W)AF \\ &- B(F, W)AJ\} + B(J, L)g(AF, W)\rho - B(F, L)g(AJ, W)\rho,\end{aligned} \tag{6.26}$$

for all $F, J, L, W \in \Gamma(TM)$. Putting $L = \xi$ in equation (6.26), we acquire

$$(D_W B)(J, \xi)AF - (D_W B)(F, \xi)AJ = \alpha(\xi)\{B(J, W)AF - B(F, W)AJ\}, \tag{6.27}$$

$$B(F, D_W^\xi)AJ - B(J, D_W^\xi)AF = \alpha(\xi)\{B(J, W)AF - B(F, W)AJ\}. \tag{6.28}$$

Putting equation (6.16) in equation (6.28), we get

$$B(J, A_\xi^0 W)AF - B(F, A_\xi^0 W)AJ = \alpha(\xi)\{B(J, W)AF - B(F, W)AJ\}. \tag{6.29}$$

Putting $F = \xi$ in equation (6.29), we acquire

$$B(J, A_\xi^0 W)A\xi - \alpha(\xi)B(J, W)A\xi = 0. \tag{6.30}$$

$$g(A_\xi^0 J, A_x^0 iW - \alpha(\xi)W) = 0. \tag{6.31}$$

From equation (6.31), it follows that either $A_\xi^0 J = 0$ or $g(A_\xi^0 J, A_\xi^0 W - \alpha(\xi)W) = 0$. □

6.4 Weakly semi-psedo Ricci Symmetric Lightlike Hypersurfaces of Semi-Euclidean Spaces

In this section , we will examine weakly semi-pseudo Ricci symmetric lightlike hypersurface of semi-Eculidean space Z_q^n and proved under certain conditions, that weakly semi-pseudo Ricci symmetric lightlike hyper surfaces are totally geodesic.

For any vector fields $F, J \in \Gamma(TM)$, the induced Ricci tensor S of lightlike hypersurface is given by

$$S(F, J) = -\sum_{i=1}^{n} \epsilon_i \{B(F, J)C(w_i, w_i) - g(A_\xi^0 J, AF)\}, \tag{6.32}$$

where $\epsilon = \pm 1$ and $\{w_i\}_{i=1}^n$ represents orthonormal basis of screen bundle $S(TM)$. For vector fields $F, J, W \in \Gamma(TM)$, a lightlike hypersurface is weakly semi-psedo Ricci symmetric lightlike hypersurface, if

$$(D_W S)(F, J) = \alpha(F)S(W, J) + \alpha(J)S(F, W). \tag{6.33}$$

Proposition 6.4.1. *Let (M, g) be the lightlike hypersurface of semi-Euclidean space Z_q^n, then the lightlike hypersurface (M, g) of a semi-Euclidean space Z_q^n is weakly semi-pseudo Ricci-symmetric, if*

$$(D_W B)(F, J) = \alpha(W)B(F, J) - \alpha(F)B(W, J) - \alpha(J)B(F, W), \tag{6.34}$$

and

$$B(J, AD_W F) + B(D_W J, AF) = -\alpha(F)B(J, AW) - \alpha(J)B(W, AF), \tag{6.35}$$

are satisfied for vector fields $F, J, L \in \Gamma(TM)$, where $\alpha(F) = g(F, \rho)$.

Proof. Suppose that the lightlike hypersurface (M, g) is weakly semi-pseudo Ricci symmetric lightlike hypersurface of Z_q^n. Therefore from equations (6.32) and (6.33), we get

$$\begin{aligned}(D_W S)(F,J) = -\beta\{B(D_W F,J)+B(F,D_W J)\}\\ +g(A^0_\xi J, AD_W F)+g(A^0_\xi J, AF),\end{aligned} \tag{6.36}$$

$$\begin{aligned}\alpha(F)B(W,J)-\alpha(J)B(F,W) = \beta\{\alpha(F)B(W,J)\\ +\alpha(J)B(F,W)\}-\alpha(F)g(A^0_\xi J, AW)-\alpha(J)g(A^0_\xi W, AF),\end{aligned} \tag{6.37}$$

where $\beta = \sum_{i=1}^{n}\epsilon_i C(w_i,w_i)$. Now if equation (6.34) and (6.35), equation (6.36) and equation (6.37) are satisfied then the lightlike hypersurface (M,g) is weakly semi-pseudo Ricci symmetric lightlike hypersurface of semi-Euclidean space Z^n_q. □

Theorem 6.4.1. *Let M be screen conformal weakly semi-psedo Ricci symmetric null-hypersurface of Z^n_q. Then for A^0_ξ, $((\beta+\phi)A^0_\xi-\phi(A^0_\xi)^2-\alpha(\xi)\beta)F$ is an eigen-vector field corresponding to zero eigenvalue for any $F\in\Gamma(S(TM))$.*

Proof. Let (M,g) be a weakly semi-psedo Ricci symmetric lightlike hypersurface of semi-Euclidean space Z^n_q, then from equation (6.35) and equation (6.37), we obtain

$$\begin{aligned}&-\beta\{B(D_W F,J)+B(F,D_W J)\}+g(A^0_\xi J, AD_W F)+g(A^0_\xi J, AF)\\ &=\alpha(F)B(W,J)-\alpha(J)B(F,W)=\beta\{\alpha(F)B(W,J)+\alpha(J)B(F,W)\}\\ &-\alpha(F)g(A^0_\xi J, AW)-\alpha(J)g(A^0_\xi W, AF).\end{aligned} \tag{6.38}$$

Putting $J=\xi$ in equation (6.38) and then using $B(F,\xi)=0$ in the resulting equation, we acquire

$$-\beta B(F,D_W\xi)+B(D_W\xi, AF)=\beta\alpha(\xi)B(F,W)-\alpha(\xi)B(W,AF). \tag{6.39}$$

Using equation (6.16) in equation (6.39), we get

$$\beta g(A^0_\xi F, A^0_\xi W)-g((A^0_\xi)^2 O, AF)=\alpha(\xi)\{\beta g(A^0_\xi F, W)-g(A^0_\xi W, AF)\}. \tag{6.40}$$

By given assumption (M,g) is screen conformal, therefore the above equation becomes

$$g(A^0_\xi[(\beta+\phi)A^0_\xi-\phi(A^0_\xi)^2-\alpha(\xi)\beta]F,W)=0. \tag{6.41}$$

As screen distribution $S(TM)$ is semi-Riemannian, from equation (6.41), our required result is obtained. □

Theorem 6.4.2. *Let (M, g) be a lightlike hypersurface of semi-Euclidean space Z_q^n, then weakly semi-pseudo Ricci-symmetric lightlike hypersurface (M, g) of a semi-Euclidean space Z_q^n is totally geodesic, if the Ricci tensor S of (M, g) admits cyclic parallelism.*

Proof. Let us assume that the weakly semi-psedo Ricci symmetric lightlike hypersurface (M, g) of semi-Euclidean space Z_q^n admits cyclic parallel of Ricci tensor S, satisfying

$$(D_W S)(F, J) + (D_F S)(J, W) + (D_J S)(W, F) = 0. \tag{6.42}$$

From equations (6.33), we obatain

$$(D_F S)(J, W) = \alpha(J)S(F, W) + \alpha(W)S(J, F), \tag{6.43}$$

$$(D_J S)(W, F) = \alpha(W)S(J, F) + \alpha(F)S(W, J), \tag{6.44}$$

Using equations (6.33), (6.43) and (6.44) in equation (6.42), we acquire

$$\begin{aligned} &\alpha(F)S(W, J) + \alpha(J)S(F, W) + \alpha(J)S(F, W)+ \\ &\alpha(W)S(J, F) + \alpha(W)S(J, F) + \alpha(F)S(W, J) = 0. \end{aligned} \tag{6.45}$$

Putting $F = Z \in Rad(TM)$, in equation (6.45), we get

$$\alpha(J)B(AE, W) = 0. \tag{6.46}$$

Replacing J by ξ in equation (6.46), we obtain

$$\alpha(\xi)B(AE, W) = 0. \tag{6.47}$$

From equation (6.47) it is clear that either $\alpha(\xi) = 0$ or $B(AE, W) = 0$. But $\alpha(\xi) \neq 0$ as $\alpha(\xi)$ is a 1-form. Therefore $B(AE, W) = 0$, i,e. weakly semi-pseudo Ricci symmetric lightlike hypersurface (M, g) is totally geodesic. □

Theorem 6.4.3. *The weakly semi-pseudo Ricci-symmetric lightlike hypersurface (M, g) of the semi-Euclidean space Z_q^n is totally geodesic, if it satisfies the Codazzi type of Ricci tensor S.*

Proof. A weakly semi-pseudo Ricci symmetric lightlike hypersurface satisfies Codazzi type of Ricci tensor S, if

$$(D_W S)(F, J) = (D_F S)(W, J). \tag{6.48}$$

From equations (6.33), we obtain

$$(D_F S)(W, J) = \alpha(W)S(F, J) + \alpha(J)S(W, F). \tag{6.49}$$

Using equation (6.33) and equation (6.49) in equation (6.48), we acquire

$$\alpha(F)S(W, J) + \alpha(J)S(F, W) = \alpha(W)S(F, J) + \alpha(J)S(W, F). \tag{6.50}$$

Putting $F = Z \in Rad(TM)$ in equation (6.50) , we get

$$\alpha(J)B(Z, W) = \alpha(W)B(AE, J). \tag{6.51}$$

Putting $W = \xi$ in equation (6.51), we acquire

$$\alpha(\xi)B(AE, J) = 0. \tag{6.52}$$

From equation (6.52) it ia clear that either $\alpha(\xi) = 0$ or $B(AE, J) = 0$ but $\alpha(\xi) \neq 0$. Therfore $B(AE, J) = 0$, we conclude that weakly semi-pseudo Ricci-symmetric lightlike hypersurface (M, g) is totally geodesic. □

Chapter 7

Geodesic Lightlike Submanifolds of Lorentzian Para-Sasakian Manifolds

In this chapter, we study geodesic lightlike submanifolds of Lorentzian para-Sasakian manifolds. We study invariant lightlike submanifolds of Lorentzian para-Sasakian manifolds. We investigated geodesic CR-lightlike submanifolds of Lorentzian para-Sasakian manifolds. We study screen CR-lightlike submanifolds of Lorentzian para-Sasakian manifolds. Some necessary and sufficient conditions for mixed geodesic, totally geodesic, $\bar{G}$-geodesic and G'-geodesic contact CR-lightlike submanifolds and SCR- lightlike submanifolds are obtained.

7.1 Introduction

D. N. Kupeli [69], K. L. Duggal and A. Bejancu [45] introduced and developed the concept of lightlike submanifolds in the field of differential geometry of manifolds. K. L. Duggal and A. Bejancu [41] constructed the principal vector bundle in lightlike hypersurface of semi-Riemannian manifolds and obtained the Gauss and Weingarten formulae. Since then authors studied and investigated lightlike submanifolds of paracontact and paracontact metric manifolds especially of para-Sasakian manifolds [82], [41], and [42]. If the induced metric on a submanifold of a semi-Riemannian manifold is degenerate, we call such a submanifold as a lightlike submanifold.

In 2001, Bayram Sahin, et al. [101] obtained some necessary and sufficient conditions on totally geodesic, D-geodesic, D'-geodesic and mixed geodesic CR-lightlike submanifolds. In 2005, Volker Perlick [92] introduced the notion of being totally umbilic lightlike submanifolds for non-degenerate and degenerate submanifolds of semi-Riemannian manifolds. The notions of photon surfaces and of null strings were linked to mathematical notion of totally umbilic submanifolds [92]. In 2017, N. O. Poyraz and E. Yasar [95] introduced lightlike hypersurfaces of a golden semi-Riemannian manifold and proved there is no radical anti-invariant lightlike hypersurface of a golden semi-Riemannian manifolds. Bilal Eftal Acet [1] in 2018, introduced and studied lightlike hypersurfaces of a metallic semi-Riemannian manifold. In this [1] author showed that the induced structure on an invariant lightlike hypersurface is also metallic. In 2018, F.E. Erdogan [48] studied the geometry of transversal lightlike submanifolds and radical transversal lightlike submanifolds of metallic semi-Riemannian manifolds. Garima Gupta et.al [55] in 2016, obtained some necessary and sufficient condition for a screen conformal lightlike submanifold of constant curvature to be a semi-Euclidean space and prove that for an irrotational screen conformal lightlike submanifold of a semi-Riemannian space form, the induced Ricci tensor is symmetric and the null sectional curvature vanishes. The intersection of normal vector bundle and tangent bundle of lightlike submanifolds is non-empty [41]. This feature of null (lightlike) submanifold is unique to study and is different from the study of non- degenerate submanifolds. A. Bejancu initiated CR-lightlike submanifolds of Kaehlar manifolds. Further, the CR lightlike submanifolds are developed by several authors ([12], [118], [26], [29]). S Kaneyuki and M. Konzai [63] initiated paracontact geometry and defined an almost paracontact structure. S. Zamkovoy [120] recently studied para-contact metric manifolds and its subclasses. B. Sahin and R. Gunes [101] studied CR-geodesic lightlike submanifolds in Kaehler manifolds. K Haider, M. Thakur and Advin studied geodesic lightlike submanifolds of indefinite Kenmotsu manifolds. Dong and Liu [36] investigated geodesic lightlike submanifolds of indefinite Sasakian manifolds.

Motivated by these authors in this chapter, we study geodesic lightlike submanifolds of Lorentzian para-Sasakian manifolds and invariant geodesic lightlike submanifolds of Lorentzian para-Sasakian manifolds. The chapter is arranged in five sections. In section 2, we remind some basic facts and definitions about almost paracontact metric manifolds, Lorentzian para-Sasakian manifolds. In section 3, we study invariant lightlike submanifolds of semi-Riemannian manifolds. In section 4, we study geodesic paracontact CR-lightlike submanifolds of Lorentzian para-Sasakian manifolds. In section 5, we study paracontact screen CR-lightlike submanifolds of Lorentzian para-Sasakian manifolds.

7.2 Lorentzian Para-Sasakian Manifolds

An n- dimensional differentiable manifold $(\bar{M}, g)$ endowed with a (1,1) tensor field ϕ, a vector field ξ, a 1-form η and a Lorentzian metric g satisfying

$$\eta(\xi) = -1, \quad \phi^2 F = F + \eta(F)\xi, \tag{7.1}$$

$$g(\phi F, \phi J) = g(F, J) + \eta(F)\eta(J), \tag{7.2}$$

$$\bar{D}_F \xi = \phi F, \quad g(F, \xi) = \eta(\xi), \tag{7.3}$$

$$(\bar{D}_F \phi)(J) = g(F, J)\xi + \eta(F)J + 2\eta(F)\eta(J)\xi, \tag{7.4}$$

for all vector fields F, J on $\bar{M}$. With respect to Lorentzian metric g, and $\bar{D}$ the Levi-Civita connection, such a manifold is said to be Lorentzian para-Sasakian (shortly LP-Sasakian) manifold and denoted simply as $(\bar{M}, \phi, \xi, \eta, g)$ [81], [83]. The following conditions are also provided for LP-Sasakian manifolds

$$\phi(\xi) = 0, \tag{7.5}$$

$$\eta(\phi F) = 0. \tag{7.6}$$

For any vector fields F, J on Lorentzian para-Sasakian manifold $(\bar{M}, g)$, a (0,2)-type symmetric tensor field Φ [82] is defined by

$$\Phi(F, J) = g(\phi F, J). \tag{7.7}$$

In addition, if on an LP-Sasakian manifold $(\bar{M}, g)$ a 1-form η is closed [81] then, we have

$$(\bar{D}_F \eta)(J) = \Phi(F, J), \quad \Phi(F, \xi) = 0, \tag{7.8}$$

for any vector fields F, J on Lorentzian para-Sasakian manifold $(\bar{M}, g)$ we have following relations [28], [29]

$$R(\xi, F)J = g(F, J)\xi - \eta F, \tag{7.9}$$

$$R(F, J)\xi = \eta(J)F - \eta(F), \tag{7.10}$$

$$S(F, \xi) = (n-1)\eta(F), \tag{7.11}$$

$$S(\phi F, \phi J) = S(F, J) + (n-1)\eta(F)\eta(J), \tag{7.12}$$

where R and S represents Riemannian curvature tensor and Ricci curvature tensor of LP-Sasakian manifolds. For every $F, J \in \Gamma(T\bar{M})$, if the non-vanishing Ricci tensor S of a manifold $(\bar{M}, \phi, \xi, \eta, g)$, satisfies the following relations

$$S(F, J) = \lambda g(F, J), \tag{7.13}$$

$$S(F, J) = ag(F, J) + b\eta(F)\eta(J) + cg(\phi F, J), \tag{7.14}$$

where λ, a, b and c are scalar functions, then LP-Sasakian manifold $(\bar{M}, \phi, \xi, \eta, g)$ is said to be as Einstein manifold and η-Einstein manifold respectively. The Gauss and Weingarten formulas of $(\bar{M}, \phi, \xi, \eta, g)$ are given by

$$\bar{D}_F J = D_F J + h(F, J), \qquad \forall F, J \in \Gamma(T\bar{M}), \tag{7.15}$$

$$\bar{D}_F X = -A_X F + D_F^{\perp} X, \qquad \forall X \in \Gamma(T^{\perp}\bar{M}), \tag{7.16}$$

where $D_F J$, $A_X F$ belong to $\Gamma(T\bar{M})$ and $h(F, J)$, $D_F^{\perp} X$ belong to $\Gamma(T^{\perp}\bar{M})$ respectively.

7.3 Invariant Lightlike Submanifolds

Definition 7.3.1. [73] "Tangent to the structural vector field ξ, let $(M, \bar{g})$ be a lightlike submanifold of an LP- Sasakian manifold $(\bar{M}, \phi, \xi, \eta, g)$. Then $(M, \bar{g})$ is said to be invariant lightlike submanifold, if it satisfies the following relations

$$\phi(RadTM) = RadTM, \quad \phi(G) = G, \tag{7.17}$$

where $S(TM) = G \perp \xi$ and G is complementary and non-degenerate distribution to ξ in $S(TM)$".

From equations (1.51), (7.2), (7.3) and equation (7.17), we acquire

$$h^l(F, \xi) = 0, \tag{7.18}$$

$$h^s(X, \xi) = 0, \tag{7.19}$$

$$D_F \xi = \phi F \tag{7.20}$$

$$h(F, \phi J) = \phi h(F, J) = h(\phi F, J) \quad \forall\ F,\ J \in \Gamma(TM). \tag{7.21}$$

Theorem 7.3.1. *Tangent to the structural vector field ξ, let $(M, \bar{g}, S(TM), S(TM)^{\perp})$ be a lightlike submanifold of an LP- Sasakian manifold $(\bar{M}, g)$. If the second fundamental forms h^l and h^s are parallel then lightlike hypersurface $(M, \bar{g}, S(TM), S(TM)^{\perp})$ is totally geodesic.*

Proof. We first suppose that h^l is parallel, then

$$(D_F h^l)(J, \xi) = 0, \tag{7.22}$$

$$D_F h^l(J, \xi) - h^l(D_F J, \xi) - h^l(J, D_F \xi) = 0, \tag{7.23}$$

for all $F, J \in \Gamma(TM)$. From equation (7.18) and equation (7.23), we acqure

$$h^l(J, D_F \xi) = 0, \tag{7.24}$$

for all $F, J \in \Gamma(TM)$. Using equation (7.2) in equation (7.24), we get

$$h^l(J, \phi F) = 0 \quad or \quad h^l = 0 \quad \forall\ F,\ J \in \Gamma(TM). \tag{7.25}$$

Now to show that h^s is parallel, we follow as

$$(D_F h^s)(J, \xi) = 0, \tag{7.26}$$

$$D_F h^s(J, \xi) - h^s(D_F J, \xi) - h^s(J, D_F \xi) = 0, \tag{7.27}$$

for all $F, J \in \Gamma(TM)$. From equation (7.19) and equation (7.27), we obtain

$$h^s(J, D_F \xi) = 0, \tag{7.28}$$

for all $F, J \in \Gamma(TM)$. Using equation (7.2) in equation (7.28), we get

$$h^s(J, \phi F) = 0 \quad or \quad h^s = 0 \ \ \forall\ F,\ J \in \Gamma(TM). \tag{7.29}$$

□

Theorem 7.3.2. *Tangent to the structural vector field ξ, let $(M, \bar{g}, S(TM), S(TM)^{\perp})$ be a lightlike submanifold of a Lorentzian para-Sasakian manifold $(\bar{M}, g)$. If $(M, \bar{g}, S(TM)$ $, S(TM)^{\perp})$ is totally umbilical then $(M, \bar{g}, S(TM), S(TM)^{\perp})$ is totally geodesic lightlike sub-manifold of Lorentzian para-Sasakian manifold.*

Proof. Let $(M, \bar{g}, S(TM), S(TM)^{\perp})$ be the totally umbilical lightlike submanifold of Lorentzian para-Sasakian manifold $(\bar{M}, g)$. Then from equation (1.51), we obtain

$$\bar{D}_F\xi = D_F\xi + h^l(F,\xi) + h^s(F,\xi), \forall F \in \Gamma(TM), \tag{7.30}$$

$$\phi F = D_F\xi + h^l(F,\xi) + h^s(F,\xi), \forall F \in \Gamma(TM). \tag{7.31}$$

By equating transversal parts of the above equation (7.31), we get

$$h^l(F,\xi) + h^s(F,\xi) = CF. \tag{7.32}$$

Replacing F by ξ in equation (7.32), we have

$$h^l(\xi,\xi) + h^s(\xi,\xi) = C\xi. \tag{7.33}$$

Now from $\phi F = PF + CF$, equation (7.5) and equation (7.33), we acquire

$$h^l(\xi,\xi) = 0, \quad h^s(\xi,\xi) = 0. \tag{7.34}$$

From equation (7.34) and by the definition of totally umbilical submanifolds, we get

$$h^l g(\xi,\xi) = 0, \quad h^s g(\xi,\xi) = 0. \tag{7.35}$$

Since ξ is non-null, therefore $h^l = h^s = 0$ and therefore from equation (7.35), we obtain

$$h^l(F,J) = 0, \quad h^s(F,J) = 0. \tag{7.36}$$

From equation (7.36) it is clear that totally umbalical submanifold $(M, \bar{g}, S(TM), S(TM)^{\perp})$ is totally. □

Theorem 7.3.3. *Let $(M, \bar{g}, S(TM), S(TM)^{\perp})$ be a lightlike submanifold of nullity degree 2 of a Lorentzian para-Sasakian manifold $(\bar{M}, g)$, then $RadTM$ defines a totally geodesic foliation on $(M, \bar{g}, S(TM), S(TM)^{\perp})$.*

Proof. Let $(M, \bar{g}, S(TM), S(TM)^{\perp})$ be a lightlike submanifold of an LP-Sasakian manifold $(\bar{M}, g)$, then by the definition $(M, \bar{g}, S(TM), S(TM)^{\perp})$, $RadTM$ defines totally geodesic foliation, if it stisfies

$$g(D_F J, L) = 0, \quad \forall F, J \in \Gamma(TM), L \in \Gamma(S(TM)). \tag{7.37}$$

Since rank of $RadTM = 2$, one can write $F,\ J \in \Gamma(RadTM)$ as a linear combination of ξ and $\phi\xi$ as

$$X = A_1\xi + B_1\phi\xi, \quad Y = A_2\xi + B_2\phi\xi. \tag{7.38}$$

Since the linear connection $\bar{D}$ is a metric connection, using equation (1.51), we get

$$\begin{aligned} g(\bar{D}_F J, L) &= Xg(J, L) - g(J, \bar{D}_F L), \\ &= -g(J, \bar{D}_F L), \\ &= -g((A_2\xi + B_2\phi\xi), h^l(A_1\xi + B_1\phi\xi, Z)), \\ &= -A_1A_2 g(\xi, h^l(\xi, L)) - B_1A_2 g(\xi, h^l(\phi\xi, L)) \\ &- B_2A_1 g(\phi\xi, h^l(\phi\xi, L)) - B_2A_2 g(\phi\xi, h^l(\phi\xi, L)). \end{aligned} \tag{7.39}$$

From [42], we have

$$g(\xi_i, X_j) = \delta_{ij}, \tag{7.40}$$

$$.g(X_i, X_j) = 0 \tag{7.41}$$

From equations (7.40), (7.41), (7.21) and (7.39), we obtain

$$g(D_F J, L) = 0 \tag{7.42}$$

Hence $RadTM$ define totally geodesic foliation on lightlike hypersurface $(M, \bar{g}, S(TM), S(TM)^{\perp})$ of Lorentzian manifold $(\bar{M}, g)$. □

7.4 Geodesic Paracontact CR-lightlike Submanifolds

Definition 7.4.1. [73] "Let $(M, \bar{g}, S(TM), S(TM)^{\perp})$ be a lightlike submanifold of Lorentzian para-Sasakian manifold $(\bar{M}, g)$, such that $\xi \in \Gamma(TM)$. If the following conditions

(i) $RadTM$ is a distribution on $(M, \bar{g}, S(TM), S(TM)^{\perp})$, such that

$$RadTM \cap \phi RadTM = 0. \tag{7.43}$$

(ii) There exists vector bundles G^0 and G^* on $(M, \bar{g}, S(TM), S(TM)^{\perp})$, such that

$$S(TM) = \{\phi RadTM \oplus G^*\} \perp G^0 \perp \xi, \tag{7.44}$$

$$\phi G^0 = G^0, \quad \phi G^* = Ltr(TM) + S(TM)^{\perp}, \tag{7.45}$$

are satisfied on $(M, \bar{g}, S(TM), S(TM)^{\perp})$, then $(M, \bar{g}, S(TM), S(TM)^{\perp})$ is called para-contact CR-lightlike submanifolds. Where G^0 is non-degenerate distribution".

For lightlike submanifolds, we have the following decompositions

$$TM = \{G \oplus G^*\} \perp \xi, \tag{7.46}$$

and

$$G = RadTM \perp \phi RadTM \perp G^0. \tag{7.47}$$

Using $G^* = G \perp \xi$ in equation (7.46), we acquire

$$TM = G^* \perp G, \qquad \phi G^* = G^*. \tag{7.48}$$

Theorem 7.4.1. *Let $(M, \bar{g}, S(TM), S(TM)^{\perp})$ be a paracontact CR-lightlike submanifold of an LP-Sasakian manifold $(\bar{M}, g)$. Then $(M, \bar{g}, S(TM), S(TM)^{\perp})$ is totally geodesic, if and only if $g(J, A_O F) = g(J, D^l(F, O))$, and $D_X \phi J$ has no component in ϕL^1.*

Proof. To show $(M, \bar{g}, S(TM), S(TM)^{\perp})$ is totally geodesic, it suffices to prove $h(F, J) = 0$ for any $F, J \in \Gamma(TM)$, that is to show that

$$g(h^l(F, J), Z) = 0 \qquad Z \in RadTM, \tag{7.49}$$

$$g(h^s(F, J), O) = 0, \qquad O \in S(TM)^{\perp}, \tag{7.50}$$

In view of equation (1.51), equation (1.53) and equation (7.2), we acquire

$$g(h^l(F, J), Z) = g(\bar{D}_F J, Z), \tag{7.51}$$

$$g(h^l(F, J), Z) = g(\phi \bar{D}_F J, \phi Z) - \eta(\bar{D}_F J)\eta(Z), \tag{7.52}$$

$$g(h^l(F, J), Z) = g(\phi \bar{D}_F J, \phi Z), \tag{7.53}$$

$$g(h^l(F, J), Z) = -g(F, \phi Z)\eta(J) + g(\bar{D}_F \phi J, \phi Z). \tag{7.54}$$

Similarly

$$g(h^s(F, J), O) = g(\bar{D}_F J, O), \tag{7.55}$$

$$g(h^s(F, J), O) = Fg(J, O) - g(J, \bar{D}_F O), \tag{7.56}$$

$$g(h^s(F, J), O) = -g(Y, \bar{D}_F O), \tag{7.57}$$

$$g(h^s(F, J), O) = g(J, A_O F) - g(J, D^l(F, O)). \tag{7.58}$$

From equation (7.54) and equation (7.58), our proof is complete. □

Definition 7.4.2. [73] "A paracontact CR-lightlike submanifold $(M, \bar{g}, S(TM), S(TM)^{\perp})$ is called G^* geodesic paracontact CR-lightlike submanifold of an LP-Sasakian manifold, if the second fundamental form h satisfies the following condition

$$h(F, J) = 0, \quad \forall \;\; J \in \Gamma(D^*)\text{''}. \tag{7.59}$$

Theorem 7.4.2. *Let $(M, \bar{g}, S(TM), S(TM)^{\perp})$ be a paracontact CR-lightlike submanifold of an LP-Sasakian manifold $(\bar{M}, g)$. Then $(M, \bar{g}, S(TM), S(TM)^{\perp})$ is G^*geodesic, if and only if $D^*_F\phi Z \in \Gamma(\phi RadTM \perp L^2)$, and $D_F Y$has no component in ϕL^2 for any $F, J \in \Gamma(D^*)$.*

Proof. We know that $(M, \bar{g}, S(TM), S(TM)^{\perp})$ is G^*-geodesic, if and only if

$$g(h^l(F, J), Z) = 0 \quad F, J \in \Gamma(D^*), \quad Z \in RadTM, \tag{7.60}$$

and

$$g(h^s(F, J), O) = 0, \quad F, J \in \Gamma(D^*), \quad O \in S(TM)^{\perp}. \tag{7.61}$$

From equations (1.51), (1.56), and equation (7.2), we acquire

$$g(h^l(F, J), Z) = g(\bar{D}_F J, Z), \tag{7.62}$$

$$g(h^l(F, J), Z) = g(\phi\bar{D}_F J, \phi Z) - \eta(\bar{D}_F J)\eta(Z), \tag{7.63}$$

$$g(h^l(F, J), Z) = g(\bar{D}_F \phi J, \phi Z), \tag{7.64}$$

$$g(h^l(F, J), Z) = -g(\phi J, \bar{D}_F \phi Z), \tag{7.65}$$

$$g(h^l(F, J), Z) = -g(\phi J, D^*_F \phi Z). \tag{7.66}$$

Similarly

$$g(h^s(F, J), O) = g(\bar{D}_F J, O), \tag{7.67}$$

$$g(h^s(F, J), O) = g(\phi\bar{D}_F J, \phi O) - \eta(\bar{D}_F J)\eta(O), \tag{7.68}$$

$$g(h^s(F, J), O) = g(\bar{D}_F \phi J, \phi O), \tag{7.69}$$

$$g(h^s(F, J), O) = -g(\phi J, \bar{D}_F \phi O), \tag{7.70}$$

$$g(h^s(F, J), O) = g(D_F J, \phi O). \tag{7.71}$$

Equation (7.66) and equation (7.71) completes the proof. □

Definition 7.4.3. A paracontact CR-lightlike submanifold $(M, \bar{g}, S(TM), S(TM)^{\perp})$ is called G^0 geodesic paracontact CR lightlike submanifold of an LP-Sasakian manifold, if the second fundamental form h satisfies the following condition

$$h(F, J) = 0, \quad \forall F, J \in \Gamma(G^0) \tag{7.72}$$

Theorem 7.4.3. *Let $(M, \bar{g}, S(TM), S(TM)^{\perp})$ be a paracontact CR lightlike submanifold of an LP-Sasakian manifold $(\bar{M}, g)$. Then $(M, \bar{g}, S(TM), S(TM)^{\perp})$ is G^0 geodesic, if and only if $A^*_Z F$, $A_O F$ has no component in $\phi L^2 \perp \phi RadTM$, for any $F, J \in \Gamma(D^0)$.*

Proof. We know that $(M, \bar{g}, S(TM), S(TM)^{\perp})$ is G^0-geodesic, if and only if

$$g(h^l(F,J),Z) = 0 \quad \forall \;\; F,J \in \Gamma(D^0), \quad Z \in \Gamma RadTM, \tag{7.73}$$

and

$$g(h^s(F,J),O) = 0, \;\; \forall \;\; F,J \in \Gamma(D^0), \quad O \in (TM)^{\perp}. \tag{7.74}$$

From equation (1.51) and equation (1.54), we obtain

$$g(h^l(F,J),Z) = g(\bar{D}_F, J, Z), \tag{7.75}$$

$$g(h^l(F,J),Z) = Fg(J,Z) - g(J, \bar{D}_F, Z), \tag{7.76}$$

$$g(h^l(F,J),Z) = -g(J, \bar{D}_F Z), \tag{7.77}$$

$$g(h^l(F,J),Z) = -g(J, D^*_Z F). \tag{7.78}$$

Similarly

$$g(h^s(F,J),O) = g(\bar{D}_F J, O), \tag{7.79}$$

$$g(h^s(F,J),O) = Fg(J,O) - g(J, \bar{D}_F O), \tag{7.80}$$

$$g(h^s(F,J),O) = -g(J, \bar{D}_F O), \tag{7.81}$$

$$g(h^s(F,J),O) = g(J, A_F O). \tag{7.82}$$

Equations (7.78) and equation (7.82) completes the required results. □

Definition 7.4.4. [73] "A paracontact CR-lightlike submanifold $(M, \bar{g}, S(TM), S(TM)^{\perp})$ is called mixed geodesic paracontact CR-lightlike submanifold of the LP-Sasakian manifold, if the second fundamental form h satisfies the following condition relation

$$h(F,J) = 0, \tag{7.83}$$

for any $F \in \Gamma(D^*)$ and $\quad J \in \Gamma(D^0)$".

Theorem 7.4.4. *Let $(M, \bar{g}, S(TM), S(TM)^{\perp})$ be a paracontact CR lightlike submanifold of an LP-Sasakian manifold $(\bar{M}, g)$. Then $(M, \bar{g}, S(TM), S(TM)^{\perp})$ is totally geodesic, if and only if $A_{\phi V}U$ has no component in $\phi RadTM \perp \phi L^2$, for any $U \in \Gamma(D^*)$, $V \in \Gamma(D^0)$.*

Proof. Let us assume that $(M, \bar{g}, S(TM), S(TM)^{\perp})$ is mixed geodesic paracontact CR-lightlike submanifold of the LP-Sasakian manifold $(\bar{M}, g)$, then we have

$$g(h^l(U,V),Z) = 0, \quad U \in \Gamma(D^*), \quad V \in \Gamma(D^0) \;\; \textit{and} \;\; E \in RadTM, \tag{7.84}$$

$$g(h^s(U,V),O) = 0, U \in \Gamma(D^*), \quad V \in \Gamma(D^0) \quad and \quad O \in S(TM)^{\perp}, \tag{7.85}$$

Using equations (1.51), (1.56) and equation (7.2), we acquire

$$g(h^l(U,V),Z) = g(\bar{D}_U V, Z), \tag{7.86}$$

$$g(h^l(U,V),Z) = g(\phi\bar{D}_U V, \phi Z) - \eta(\bar{D}_U V)\eta(Z), \tag{7.87}$$

$$g(h^l(U,V),Z) = g(\phi\bar{D}_U V, \phi Z), \tag{7.88}$$

$$g(h^l(U,V),Z) = -g(U,\phi Z)\eta(V) + g(\bar{D}_U \phi V, \phi Z), \tag{7.89}$$

$$g(h^l(U,V),Z) = g(\bar{D}_U \phi V, \phi Z), \tag{7.90}$$

$$g(h^l(U,V),Z) = -g(\bar{D}_{\phi V} U, \phi Z), \tag{7.91}$$

and

$$g(h^s V, O) = g(\bar{D}_U V, O), \tag{7.92}$$

$$g(h^s V, O) = g(\phi\bar{D}_U V, \phi W) - \eta(\bar{D}_U V)\eta(O), \tag{7.93}$$

$$g(h^s V, O) = g(\phi\bar{D}_U V, \phi O), \tag{7.94}$$

$$g(h^s V, O) = g(\bar{D}_U \phi V, \phi O), \tag{7.95}$$

$$g(h^s V, O) = g(A_{\phi V} U, \phi O). \tag{7.96}$$

Hence equation (7.91) and equation (7.96) completes our proof. □

7.5 Geodesic Paracontact Screen CR-Lightlike submanifolds

Definition 7.5.1. [73] "Let $(M, \bar{g}, S(TM), S(TM)^{\perp})$ be a lightlike submanifold of Lorentzian para-Sasakian manifold $(\bar{M}, g)$ in such a way that $\xi \in (TM)$. If the following conditions are satisfied then, $(M, \bar{g}, S(TM), S(TM)^{\perp})$ is called paracontact screen CR-lightlike submanifold

(i) There exists G and $G^{\perp}$ as real non-null distributions, such that

$$S(TM) = D \perp G^{\perp} \perp \xi, \phi(D^{\perp}) \subset S(TM^{\perp}), G \cap D^{\perp} = 0, \tag{7.97}$$

(ii)

$$\phi RadTM = RadTM, \quad \phi ltr(TM) = ltr(TM), \phi(G) = G, \tag{7.98}$$

where $D^{\perp}$is orthogonal complementary to $G \perp \xi$ in $S(TM)$".

From equations (7.97) and equation (7.98), we have the following decompositions

$$TM = \bar{G} \perp D^{\perp} \perp \xi, \quad \bar{D} = D \perp RadTM. \tag{7.99}$$

Let $\bar{D} = D \perp \xi$, for any F tangent to $(M, \bar{g}, S(TM), S(TM)^{\perp})$, we put

$$\phi F = PF + CF, \tag{7.100}$$

$$\phi O = BO + EO, \tag{7.101}$$

where $PF,\ BO \in \Gamma(D^{\perp})$, $CF \in \phi D^{\perp},\ O \in \Gamma S(TM^{\perp})$ and $EO \in \Gamma S(TM^{\perp}) \quad - \phi D^{\perp}$.

Theorem 7.5.1. *Let $(M, \bar{g}, S(TM), S(TM)^{\perp})$ be a paracontact CR-lightlike submanifold of Lorentzian para-Sasakian manifold $(\bar{M}, g)$. Then $(M, \bar{g}, S(TM), S(TM)^{\perp})$ is totally geodesic, if and only if*

$$(L_Z g)(F, J) = 0, \tag{7.102}$$

and

$$(L_O g)(F, J) = 0, \tag{7.103}$$

for any $F, J \in \Gamma(TM), \quad Z \in \Gamma(RadTM)$ and $O \in \Gamma(S(TM^{\perp}))$.

Proof. Let us assume that $(M, \bar{g}, S(TM), S(TM)^{\perp})$ is totally geodesic, we have

$$g(h^l(F, J), Z) = g(\bar{D}_F J, Z), \tag{7.104}$$

$$g(h^l(F, J), Z) = fg(J, Z) - g(J, \bar{D}_F Z), \tag{7.105}$$

$$g(h^l(F, J), Z) = g(Y, [Z, F]) - g(J, \bar{D}_F Z), \tag{7.106}$$

$$g(h^l(F, J), Z) = -g(Z, \bar{D}_J F) - (L_Z g)(F, J) = 0, \tag{7.107}$$

$$g(h^l(F, J), Z) = -g(\bar{D}_J F, Z) - (L_Z g)(F, J) = 0, \tag{7.108}$$

$$g(h^l(F, J), Z) = -g(h(F, J), Z) - (L_Z g)(F, J) = 0, \tag{7.109}$$

$$(L_E g)(F, J) = 0. \tag{7.110}$$

Similarly

$$(L_O g)(F, J) = 0. \tag{7.111}$$

Hence equations (7.110) and (7.112), completes our proof. □

Definition 7.5.2. [73] "A paracontact screen CR-lightlike submanifold $(M, \bar{g}, S(TM), S(TM)^{\perp})$ of the LP-Sasakian manifold $(\bar{M}, g)$ is called the mixed geodesic, if its second fundamental form h satisfies $h(F, J) = 0$, for any $F \in \Gamma(G)$ and $F \in \Gamma(G^{\perp})$".

Theorem 7.5.2. *Let $(M, \bar{g}, S(TM), S(TM)^{\perp})$ be a paracontact screen CR-lightlike submanifold of the Lorentzian para-Sasakian manifold $(\bar{M}, g)$. Then $(M, \bar{g}, S(TM), S(TM)^{\perp})$ is mixed geodesic, if and only if $D^t_U \phi V \in \Gamma(G^{\perp})$ and $A_{\phi V} U \in \Gamma(G)$ for any $U \in \Gamma(G)$ and $V \in \Gamma(G^{\phi})$.*

Proof. By definition a paracontact screen CR-lightlike submanifold $(M, \bar{g}, S(TM), S(TM)^{\perp})$ is called mixed geodesic, if and only if $h(U, J) = \bar{D}_U V - D_U V = 0$. From definition there exists $O \in S(TM^{\perp})$, such that

$$\phi W = V. \tag{7.112}$$

Thus from equations (7.3), (7.100) and equation (7.101), we acquire

$$\bar{D}_U \phi O - D_U V = 0, \tag{7.113}$$

$$\phi \bar{D}_U O - D_U V = 0, \tag{7.114}$$

$$\phi(A_O U + D^t_U O) - D_U V = 0, \tag{7.115}$$

$$-PA_O U - FA_O U + B^t_U O + ED^t_U O - D_U V = 0. \tag{7.116}$$

From equations (7.100) and (7.101), we obtain

$$FA_O U = 0, \quad ED^t_U O = 0. \tag{7.117}$$

Hence from equations (7.112) and (7.1), our proved is complete. □

Theorem 7.5.3. *Let $(M, \bar{g}, S(TM), S(TM)^{\perp})$ be a paracontact screen CR-lightlike submanifold of Lorentzian para-Sasakian manifold $(\bar{M}, g)$. Then $G^{\perp}$ defines a totally geodesic foliation if and only if, $h^s(S, \phi N)$ and c has no components in $\Gamma(\phi G^{\perp})$, for any $X \in (G)$ and $S \in \Gamma(G^{\perp})$.*

Proof. Le us assume that $D^{\perp}$ defines a totally geodesic foliation, we have

$$D_S T \in \Gamma(G^{\perp}), \quad \forall S, T \in (G^{\perp}), \tag{7.118}$$

From equation (7.118), we acquire

$$\bar{g}(D_S T, L) = 0, \quad L \in \Gamma(G), \tag{7.119}$$

$$\bar{g}(D_S T, X) = 0, \quad X \in \Gamma(ltr(TM)). \tag{7.120}$$

By using equation (7.4), in equation (7.120), we get

$$\bar{g}(D_S T, L) = g(\bar{D}_S T, L), \tag{7.121}$$

$$\bar{g}(D_S T, L) = S\bar{g}(T, L) - g(T, \bar{D}_S L), \tag{7.122}$$

$$\bar{g}(D_S T, L) = -\bar{g}(T, \bar{D}_S L), \tag{7.123}$$

$$\bar{g}(D_S T, L) = -\bar{g}(\phi T, \phi(\bar{D}_S L)), \tag{7.124}$$

$$\bar{g}(D_S T, L) = -\bar{g}(\phi T, \phi(\bar{D}_S L)), \tag{7.125}$$

$$\bar{g}(D_S T, L) = -\bar{g}(\phi T, \bar{D}_S \phi L), \tag{7.126}$$

$$\bar{g}(D_S T, L) = -\bar{g}(\phi T, h^S(S, \phi L)), \tag{7.127}$$

Similarly

$$\bar{g}(D_S T, X) = \bar{g}(\bar{D}_S T, X) \tag{7.128}$$

$$\bar{g}(D_S T, X) = \bar{g}(\phi \bar{D}_S T, \phi X) - \eta(\bar{D}_S T)\eta(X), \tag{7.129}$$

$$\bar{g}(D_S T, X) = \bar{g}(\phi \bar{D}_S T, \phi X), \tag{7.130}$$

$$\bar{g}(D_S T, X) = -\bar{g}((\bar{D}_S \phi)T, \phi X) + g((\bar{D}_S \phi T, \phi X), \tag{7.131}$$

$$\bar{g}(D_S T, X) = S\bar{g}(\phi T, \phi X) - g(\phi T, \bar{D}_S \phi X), \tag{7.132}$$

$$\bar{g}(D_S T, X) = -\bar{g}(\phi T, h^s(S, \phi X)). \tag{7.133}$$

Equations (7.127) and (7.133), the proof is complete. □

Future Scope

Despite the fact that our research met its objectives, there are substantial unresolved problems that we hope to address in future work. More specifically, we propose to look at the following research problems:

1. To obtain optimal inequalities on lightlike hypersurface of semi-Riemannian manifolds.
2. To establish the relationship between a warping function and the squared norm of mean curvature for lightlike submanifolds of semi-Riemannian manifolds.
3. To acquire Wintgen inequality for lightlike hypersurfaces of semi-Riemannian manifolds of constant holomorphic sectional curvature.
4. To investigate the geometry of screen transversal lightlike submanifolds by using the Norden metric on almost-complex manifolds.
5. To obtain results on statistical submanifolds of semi-Riemannian manifolds.
6. To obtain optimal inequalities on statistical submanifolds of semi-Riemannian manifolds.
7. To obtain Ricci-soliton submanifolds of semi-Riemannian manifolds.
8. To investigate Ricci-Yambe solitons on semi-Riemannian manifolds.

Despite the study of statistical manifolds, lightlike manifolds and related submanifolds is a relatively new field, there are still a number of issues that need to be addressed.

Paper Publications

Papers Published from the Thesis

1. Ejaz Sabir Lone and Pankaj Pandey, On weakly ϕ Ricci symmetric lightlike hypersurfaces of indefinite cosymplectic manifolds, Turkish Journal of Computer and Mathematics Education, **11(3)**, (2020), 1053-1060. (**Scopus**)

2. Pankaj Pandey and Ejaz Sabir Lone, Study of parallel vector fields on lightlike hypersurfaces of Lorentzian manifolds, International Journal of Mechanical Engineering , **6(2)**, (2021) 133-142. (**Scopus**)

3. Ejaz Sabir Lone and Pankaj Pandey, Study of new type of vector fields on lightlike hypersurfaces of Lorentzian manifolds , International Journal of Mechanical Engineering, **6(3)**,(2021) 3851-3858. (**Scopus**)

4. Ejaz Sabir Lone and Pankaj Pandey, Geodesic lightlike submanifolds of Lorentzian para-Sasakian manifolds, Journal of Physics: Conference Series, **2267** (2022) 012074. (**Scopus**)

Papers Accepted from the Thesis

1. Pankaj Pandey and Ejaz Sabir Lone, On lightlike hypersurfaces of cosymplectic space form, Communications of the Korean Mathematical Society, (2022). (**Scopus**)

Papers Communicated from the Thesis

1. Ejaz Sabir Lone, Pankaj Pandey and Aliya Naaz Siddiqui, Study of almost generalised weakly symmetric lightlike hypersurfaces $A(GWS)$-LH of indefinite Kenmotsu manifolds, Balkan Journal of Geometry and Its Applications, (2022). (**SCIE**)

2. Pankaj Pandey and Ejaz Sabir Lone, On weakly semi-pseudo symmetric lightlike hypersurfaces of semi-Euclidean Space, Stochastic Modeling & Applications (2022). (**UGC**)

3. Pankaj Pandey and Ejaz Sabir Lone, On lightlike hypersurfaces of Sasaki-like statistical space form, Honam Mathematical Journal, (2022). (**ESCI**)

Bibliography

[1] B. E. Acet, Lightlike hypersurfaces of metallic semi-Riemannian manifolds, International Journal of Geometric Methods in Modern Physics, 1850201, 15(12) (2018).

[2] N. Aktan, On non exixtance of lightlike hypersurfaces of indefnite Kenmotsu space form, Inter. j. Math. Statistic, 3(A08) (2008), 12-21.

[3] N. Aktan, On non exixtance of lightlike hypersurfaces of indefnite Sasakian space form, Turk. j. Math. Statistic, 32 (2008), 127-139.

[4] P. Alegre, D.E. Blair and A. Carriazo, Generalized Sasakian space forms, Israel J. Math., 141 (2004), 157-183.

[5] L.S. Alqahtani, M.S. Stankovic and S. Uddin, Warped product bi-slant submanifolds of cosymplectic manifolds, Filomat, 31(16) (2017), 5065-5071.

[6] S. Amari, Differential geometrical methods in statistics, Lecture notes in statistics, Springer, New York., 28 (1985).

[7] C. Atindogbe, J. P. Ezin and J. Tossa, Lightlike Einstein hypersurfaces in Lorentzian manifolds with constant curvature, Kodai Math. J., 29(1) (2006), 58-71.

[8] O. Bahadir, On lightlike geometry of indefnite Sasakian statistical manifolds, https:arxiv.org/abs/2004.01512.

[9] O. Bahadir and M.M. Tripathi, Geometry of Lightlike hypersurfaces of Statistical manifolds, https:arxiv.org/abs/1901.09251

[10] M.K. Balgeshir, S. Salahvarzi, Lightlike submanifolds of semi-Riemannian statistical manifolds, Balkan Journal of Geometry and Its Applications, 25(2) 2020, 52-65.

[11] Y.S. Balkan, On Lightlike Hypersurfaces of An Indefinite f-Kenmotsu Space Form, Mathematical Sciences and Applications E-Notes, 5.2 10-16.

[12] A. Bejancu, CR-submanifolds of Kaehler Manifold II, Trans. Amer. Math. Soc., 250 (1979), 333-345.

[13] A. Bejancu, A. Ferrández and P. Lucas, A new view point on geometry of a lightlike hypersurface in a semi-Euclidean space, Saitama Math. J. 16 (1998), 31-38.

[14] D. E. Blair, Contact manifolds in Riemannian geometry, Lecture Notes in Math. no., Springer, 509 (1976).

[15] F. Brickell, K. Yano, Concurrent vector fields and Minkowski structures, Kodai Mathematical Seminar Reports, Department of Mathematics, Tokyo Institute of Technology, Tokyo, Japan, 26 (1974), 22–28.

[16] A. Cabras, A. Ianus and G. H. Piitis, Extrinsic spheres and parallel submanifolds in cosymplectic manifolds, Math. J.Toyama univ., 17 (1974), 31-53.

[17] C. Calin, Contribution to geometry of CR-submanifolds, Ph.D. thesis, University of Iasi, Romania, (1998).

[18] O. Calin and C. Udriste, Geometric modeling in probability and statistics, Springer, (2014), MR3308142.

[19] M. P. D. Carmo, Riemannian Geometry, Birkhauser Boston, (2013), pp. 300.

[20] E. Cartan, Sur une classes remarquable de spaces de Riemannian, Bull. Soc. Math. France, 54 (1926), 214-264.

[21] G. Catino and L. Mazzieri, Gradient Einstein solitons, Nonlinear Analysis, 132 (2016), 66-94.

[22] B. B. Chaturvedi and P. Pandey, Study on special type of a weakly symmetric Kaehler manifold, Diff. Geom. Dyn. Syst., 17 (2015) 32-37.

[23] B. B. Chaturvedi and P. Pandey, Some examples on weak symmetries, Gen. Math. Notes, 29(1) (2015),61-66.

[24] B. B. Chaturvedi and P. Pandey, Weak symmetric manifolds equipped with semi-symmetric metric connection, Palestine Journal of Mathematics, 6(2) (2017), 507-514.

[25] B. B. Chaturvedi and P. Pandey, A note on semi-symmetric metric connection in Riemannian manifolds, Thai Journal of Mathematics, 19(4) (2021), 1199-1207.

[26] S. K. Chaubey and U. C. De, Lorentzian para-Sasakian manifolds admitting a new type of quarter-symmetric non-metric connection, 2019 Int. Electron, J. Geom., 12(2) 250-259.

[27] S. K. Chaubey and A. Yildiz, Riemannian manifolds admitting a new type of semi-symmetric non-metric connection, Turk J Math., 43 (2019), 1887-1904.

[28] B. Y. Chen, CR-submanifolds of a Kaehler Manifold I, J. Differential Geom., 16 (1981), 305-323.

[29] B. Y. Chen, CR-submanifolds of a Kaehler Manifold II, J. Differential Geom., 16 (1981), 493-509.

[30] B. Y. Chen, A simple characterization of generalized Robertson-Walker spacetime, Gen. Relat. Gravitation, 46 (12), (2014).

[31] B. Y. Chen, S. Deshmukh, Ricci solitons and concurrent vector fields, Balk. J. Geom. Appl., , 20 (2015), 14–25.

[32] U. C. De, On ϕ- symmetric Kenmotsu manifolds, int. Electronic J. Geom, 1(1) (2008), 33-38.

[33] U. C. De and S. Bandyopadhyay, On weakly symmetric spaces, Acta Mathematics Hungarica, 83 (2000), 205212.

[34] U. C. De and A. Sarkar,, On Ricci $\phi-$symmetric Sasakian manifolds, Proceedings of the jangjeon Math. Soc., 11(1) (2008), 47-52.

[35] U. C. De, A. Sarkar and K. DE, Ricci-Yambe solitons and 3-dimensional manifolds, Turk. J. Math. 46 (2022), 1078-1088.

[36] J. Dong and X. Liu, Geodesic lightlike submanifolds of indefinite Sasakian manifolds, Advances in Pure Mathematics, 1 (2011) 378–383.

[37] R. S. D. Dubey, Generalized recurrent spaces, Indian J. Pure Appl. Math., 10 (1979), 1508-1513.

[38] R. S. D. Dubey, Generalized recurrent spaces, Indian J. Pure Appl. Math., 10 (1979), 1508-1513.

[39] K. L. Duggal, On scalar curvature in lightlike geometry, J. Geom. Phys., (2007), 57, 473–481.

[40] K. L. Duggal, Foliations of lightlike hypersurfaces and their physical interpretation, Cent. Eur. J. Math., 10(5) (2012), 1789-1800.

[41] K. L. Duggal and A. Bejancu, Lightlike submanifolds of semi-Riemannian manifolds, Acta Appl. Math., 38 (1995).

[42] K. L. Duggal and A. Bejancu, Lightlike Submanifolds of semi-Riemannian Manifolds and Applications, Mathematics and its Applications, Kluwer Academic Publishers, Dordrecht, The Netherlands, 364 (1996).

[43] K. L. Duggal and D. H. Jin . Null curves and hypersurfaces of semi-Riemannian manifolds. World Scientific Publishing Company, (2007).

[44] K. L. Duggal, D. H. Jin, Totally umbilical lightlike submanifolds. Kodai Math. J., 26 (2003), 49–68.

[45] Duggal K. L. and B. Sahin, Lightlike submanifolds of indefinite Sasakian manifolds, 2007 Int. J. Math. Sci., 57585 1-21.

[46] K. L. Duggal and B. Sahin, Differential geometry of lightlike submanifolds, Frontiers in Mathematics Birkhauser Verlag, Basel, (2010), MR2598375.

[47] K. L. Duggal and B. Sahin, Differential Geometry of Lightlike Submanifolds, Frontiers in Mathematics Birkhauser Boston, (2010).

[48] F. E. Erdogan, Transversal lightlikesubmanifolds of metallic semi-Riemannian manifolds, Turkish Journal of Mathematics, 42 (6) (2018), 3133-3148.

[49] C. Gherghe, Harmonicity of nearly trans-Sasakian manifolds, Demonstratio. Math., 33 (2000), 151-157.

[50] W. Goemans and V. D. Woestyne, Translation and homothetical lightlike hypersurfaces of semi-Euclidean space. Kuwait J. Sci. Eng., 38(2A) (2011), 35-42.

[51] S. M. Guler and Crasmareanu, Ricci-Yamabe maps for Riemannian flow and their volume variation and volume entropy, Turkish Journal of Mathematics, 43 (2019), 2631-2641.

[52] R.S. Gupta, Geometry of lightlike hypersurfaces of an indefinite cosymplectic manifolds, In AIP Conference Proceedings, American Institute of Physics, 1479(1) (2012), 1651-1654.

[53] R. S. Gupta, Ricci Semi-Symmetric Lightlike Hypersurfaces of an Indefinite Cosymplectic Space Form, Kyungpook Mathematical Journal, 53(4) (2013), 593-602.

[54] R.S. Gupta, M. K. Haider and A. Sharfuddin.: Slant submanifolds in Cosymplectic Manifolds, Colloquim Mathematicum, 105 (2006), 207-219.

[55] G. Gupta, R. Kumar and R. K. Nagaich, Screen conformal lightlike submanifolds of semi-Riemannian manifolds, J. Geom., 107(2016), 635–655 .

[56] R. S. Gupta, and A. Sharfuddin, Lightlike submanifolds of indefinite Kenmotsu manifold. Int. J. Contemp. Math. Sciences, 5(10) 2010, 475-496.

[57] R. S. Gupta, A. Upadhyayand and A. Sharfuddin.:Slant lightlike submanifolds of indefinite Cosymplectic manifolds, Mediterr. J. Math., 8 (2011), 215-227.

[58] S. M. Haider, and M. Thakur. Geodesic Lightlike Submanifolds of Indefinite Kenmotsu Manifolds. International Scholarly Research Notices, (2012).

[59] R. S. Hamilton, The Ricci flow on surfaces, Mathematics and general relativity, 71 (1998), 237-262.

[60] R. S. Hamilton, Lecture on geometric flows, (1989).

[61] D. H. Jin, The curvatures of lightlike hypersurfaces of an indefinite Kenmotsu manifold, Balikin journal of geometry and its applications, 17 (2012), 49-57.

[62] D. H. Jin, geometry of lightlike hypersurfaces of an indefinite cosymplectic manifold, Communications of the Korean Mathematical Society, 27(1) (2012), 185–195.

[63] S.Kaneyuki and M. Konzai, Paracomplex structure and affine symmetric spaces, Tokyo J. Math., 8 (1985), 301-308.

[64] T. H. Kang and S.K. Kim, Lightlike hypersurfaces of indefinite Cosymplectic manifolds, International Mathematical Forum, 2 (2007), 3303-3316.

[65] S. B. Kazan and B. Sahin, Pseudosymmetric lightlike hypersurfaces in indefinite Sasakian space forms, Journal of Applied Analysis and Computation, 6(3) (2016), 699-719.

[66] E. Kilic and O. Bahadur, Lightlike hypersurfaces of a semi-Riemannian product manifold and quarter-symmetric non-metric connections, Int. J.Math. Sci., (2012), ID 178390.

[67] E. Kılıc, M. Gulbahar, & E. Kavuk, Concurrent Vector Fields on Lightlike Hypersurfaces. Mathematics, 9 1 (2020), 59.

[68] J. Kowolik, On some Riemannian manifolds admitting torse-forming vector fields, Dem. Math, 18(3) (1985), 885-891.

[69] D. N. Kupeli, Singular semi-Riemannian geometry, Kluwer Dordrecht, (1996).

[70] P. Libermann and Sur Les, Automorphismes in nitesimaux des structures symplectiques et des structures de contact, Colloque Geom. Diff. Globale Bruxelles, Centre Belge Rech. Math., Louvain, (1959), 37-59.

[71] E.S. Lone and P. Pandey, On weakly ϕ Ricci symmetric lightlike hypersurfaces of indefinite cosymplectic manifolds, Turkish Journal of Computer and Mathematics Education, 11(3) (2020), 1053-1060.

[72] E.S. Lone and P. Pandey, Study of new type of vector fields on lightlike hypersurfaces of Lorentzian manifolds , International Journal of Mechanical Engineering, 6(3) (2021) 3851-3858.

[73] E.S. Lone and P. Pandey, Geodesic Lightlike Submanifolds of Lorentzian Para-Sasakian Manifolds, Journal of Physics: Conference Series, 2267(1) (2022), 012074).

[74] O. Lungiambudila, F. Massamba, and J. Tossa, Equivalence conditions of symmetry properties in lightlike hypersurfaces of indefinite Kenmotsu manifolds, Bull. of the Korean Mathematical Society, 53(4) (2016), 1259-1280.

[75] F. Malek and M. Samawaki, On weakly symmetric Riemannian manifolds, Differential Geometry-Dynamical System, 10 (2008), 215-220.

[76] F. Massamba, On weakly symmetric Lightlike hypersurfaces of indefnite Sasakian manifolds, STU J. Math, 42(2) (2008), 165-185.

[77] F. Massamba, On weakly Ricci symmetric Lightlike hypersurfaces of indefnite Sasakian manifolds, STU J. Math., 44(2) (2008), 165-185.

[78] F. Massamba, On semi-parallel lightlike hypersurfaces of indefinite Kenmotsu manifolds, J. Geom., 95 (2009), 73–89

[79] F. Massamba, On lightlike geometry in indefinite Kenmotsu manifolds, Math. Slovaca, 62 (2012), 315–344.

[80] F. Massamba, Symmetries of null geometry in indefinite Kenmotsu manifolds, Mediterranean journal of mathematics, 10(2) (2013), 1079-1099.

[81] K. Matsumoto, On Lorentzian Paracontact manifolds, Bull. Yamagata Univ., 12(2) (1989), 151-156.

[82] K. Matsumoto and I. Mihai , On a certain transformation in a Lorentzian para-Sasakian manifold, Tensor N. S., 47 (1988), 189-197.

[83] I. Mihai and R. Rosca, On Lorentzian P-Sasakian manifolds, Classical Analysis, World Scientific. Publ. (1992), 155-169.

[84] P. Moile, Symmetries in Lightlike Hypersufaces of Indefinite Kenmotsu Manifolds, International Journal of Contemporary Mathematical Sciences, 12(3), 117-132.

[85] B. Olea, Null Hypersurfaces on Lorentzian Manifolds and Rigging Techniques, Springer Proceedings in Mathematics and Statistics Cham, Switzerland, 211 (2016), 237–251.

[86] F. Ozen and S. Altay, On weakly and pseudo symmetric riemannian spaces, Indian Journal of Pure and Applied Mathematics, 33(10) (2001), 14771488.

[87] P. Pandey, On weakly cyclic generalised Z-symmetric manifolds, National academy science letters, 43(4) (2020), 347-350.

[88] P. Pandey and B.B. Chaturvedi, On a Lorentzian complex space form, National academy science letters, 43(4) (2020), 351-353.

[89] P. Pandey, B.B. Chaturvedi and E. S Lone, Study of hypersurface of semi-almost Hermitian manifold equipped with quarter-symmetric non metric connection, Journal of Physics: Conference series, 1531 (2020), 012051.

[90] P. Pandey, and E. S Lone, Study of parallel vector fields on lightlike hypersurfaces of Lorentzian manifolds, International Journal of Mechanical Engineering , 6(2) (2021), 133-142.

[91] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, arXiv (2002), arXiv:math/0211159.

[92] V. Perlick, On totally umbilic submanifolds of semi-Riemannian manifolds, Nonlinear Analysis, Theory, Methods & Applications, 63(5-7) (2005), e511-e518.

[93] S. Y. Perktas, E. Kilic and B. E. Acet, Lightlike hypersurfaces of a Para-Sasakian space form, Gulf journal of mathematics, 2 (2014), 7-18.

[94] P. Petersen, Riemannian geometry.

[95] N. O Poyraz, E. Yasar, Lightlike Hypersurfaces of a Golden Semi-Riemannian Manifold, Mediterr. J. Math., 14(5) (2017), 1-20.

[96] D. G. Prakasha, S.K. Hui, and K. Vikas On Weaklyϕ-Ricci symmetric Kenmotsu manifolds, International Journal of Pure and Applied Mathematics, 95(4) (2004), 515-521.

[97] M. Prvanovic, On weakly symmetric Riemannian manifolds, Publ. Math. Debre., 46(1) (1995), 19-25.

[98] V. Rani, and J. Kaur, On the structure of lightlike hypersurfaces of an indefinite kaehler statistical manifold, Differential Geometry-Dynamical Systems, 23 (2021), 229-242.

[99] V. Rani, and J. Kaur, Cauchy Riemann-lightlike submanifolds in the aspect of an indefinite Kaehler statistical manifold, Malaya Journal of Matematik (MJM), 9(1) 2021, 136-143.

[100] B. Sahin, Lightlike hypersurfaces of semi-Euclidean spaces satisfying curvature conditions of semisymmetry type, Turkish J. Math. 31 (2007), 139–162.

[101] B. Sahin and R. Gunes. Geodesic CR-lightlike submanifolds, Contributions to Algebra and Geometry, 42 (2001), 583–594.

[102] D. Saglam, E. Soytürk, and A. Sabuncuoglu, Minimal translation lightlike hypersurfaces of semi-euclidean spaces, Department of Mathematics Northwest University 7(1) (2011), 107-113.

[103] A. A. Shaikh, M. Ali and Z. Ahsan, Curvature properties of a special type of pure radiation metrics, Journal of Geometry and Physics, 136 (2019), 195–206.

[104] A. A. Shaikh, H Kundu and Z. Ahsan, Curvature properties of Robinson–Trautman metric, J. Geom., 109(2) (2018), 1-20.

[105] M. Shoeb, M. H. Shahid, and A. Sharffuddin, On submanifolds of cosymplectic manifolds, Soochow journal of mathematics, 27(2) (2001), 161-174.

[106] S. S. Shukla and M. K. Shukla, On ϕ-Ricci symmetric Kenmotsu manifolds, Novi Sad J. Math, 39(2) (2009), 89-95.

[107] H. Takeno, Concircular scalar field in spherically symmetric space-times, I. Tensor 20(2) (1967), 167–176.

[108] K. Takano, statistical manifolds with almost contact structures and their statistical submersions, j. Geom., 85 (2006), 171-187.

[109] T. Takahashi, Sasakian $\phi-$symmetric spaces, Tohoku Math. j., 29 (1977), 91-113.

[110] L.Tamassy, and T. Q. Binh, On weakly symmetric and weakly projective symmetric Riemannian manifolds, Coll. Math. Soc., J. Bolyai, 56 (1989), 663-670.

[111] L. Tamassy and T. Q. Binh, On weakl symmetries of Einstein and Sasakian manifolds, Tensor N. S, 53 (1993), 140-148.

[112] M. Tarafdar, A. A, Musa and Jawarneh, Semi-pseudo Ricci symmetric manifold, j. Indian inst. of Science, 73 (1993), 140-148.

[113] A. Upadhyay and R.S. Gupta, Semi-parallel lightlike hypersurfaces of indefinite cosymplectic space forms, Publications de lInstitut Mathematique, 89(103) (2011), 69-75.

[114] A. Upadhyay, R. S .Gupta, and A. Sharfuddin, Semi-symmetric and Ricci semi-symmetric lightlike hypersurfaces of an indefinite generalized Sasakian space form, International Electronic Journal of Geometry, 5(1) (2012), 140-50.

[115] A. Upadhyay, R. S. Gupta, and A. Sharfuddin, Screen conformal lightlike hypersurfaces of an indefinite cosymplectic manifold, Georgian Mathematical Journal, 24(4) (2017), 609-620.

[116] P. W. Vos, Fundamental equations for statistical submanifolds with applications to the Bartlett corrections, Ann. inst. stst. Math., 14(3) (1999), 99-110.

[117] K. Yano, Concircular geometry. I. Concircular transformations, Proc. Imp. Acad. Tokyo, 16 (1940), 195–200.

[118] K. Yano and M. Kon, On CR –Submanifolds of a complex projective space, Differential Geom., 16 (1981), 431-444.

[119] K. Yano and M. Kon, Structures on Manifolds, Series in Pure Mathematics, 3 (1984).

[120] S. Zamkovoy, Canonical connection on paracontact manifolds, Ann. Glob. Anal. Geo., 36 (2009), 37-60.