

SOME MATHEMATICAL MODELS OF THE COSMOLOGICAL PROBLEMS IN ALTERNATIVE THEORIES OF GRAVITATION

Thesis Submitted For the Award of the Degree of

DOCTOR OF PHILOSOPHY

IN

MATHEMATICS

By

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PUNJAB (INDIA)

2022

Dedicated to
my family
specially to my beloved parents

Declaration of Authorship

I, hereby declared that the presented work in the thesis entitled “**Some mathematical models of the cosmological problems in alternative theories of gravitation**” in fulfilment of the degree of **Doctor of Philosophy (Ph.D.)** is the outcome of research work carried out by me under the supervision Dr. Binaya K. Bishi, working as Assistant Professor, in the Department of Mathematics of Lovely Professional University, Punjab, India. In keeping with general practice of reporting scientific observations, due acknowledgements have been made whenever work described here has been based on findings of other investigator. This work has not been submitted in part or full to any other University or Institute for the award of any degree.



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Certificate

This is to certify that the work reported in the Ph.D. thesis entitled “**Some mathematical models of the cosmological problems in alternative theories of gravitation**” submitted in fulfillment of the requirement for the reward of the degree of **Doctor of Philosophy (Ph.D.)** in the Department of Mathematics, is a research work carried out by Pratik Vijay Lapse (11919190), is a bonafide record of his original work carried out under my supervision and that no part of the thesis has been submitted for any other degree, diploma or equivalent course.



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Symbols

τ :	Stress tensor
$g_{\mu\nu}$:	Metric potential
$\Gamma_{\mu\nu}^{\lambda}$:	Christoffel symbol
$R_{\mu\nu}$:	Ricci tensor
R :	Ricci scalar
q :	Deceleration Parameter
$\mathcal{R}(t)$:	Scale factor
H :	Hubble parameter
H_0 :	Hubble constant at present epoch
$G_{\mu\nu}$:	Einstein tensor
L_m :	Matter lagrangian
s :	Einstein-Hilbert action
$T_{\mu\nu}$:	Stress energy-momentum tensor
T :	Track of stress energy-momentum tensor
p :	Pressure
p_c :	Particle creation
ρ :	Energy density
ω :	EoS parameter
t_0 :	Present time
Λ :	Cosmological constant
δ :	Skewness parameter
θ :	Scalar expansion
σ^2 :	Shear scalar
A_m :	Average anisotropy parameter

List of Publications and Conferences

Publications

1. Bishi, B.K. and Lepse, P.V., 2021. "Particle creation and quadratic deceleration parameter in Lyra geometry". New Astronomy, 85, p.101563.
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Abstract

This thesis is titled "**Some mathematical models of the cosmological problems in alternative theories of gravitation**". The main objective of the thesis is to provide a coherent description of the current accelerated expansion of the universe within alternative theories of gravity at an appropriate level of understanding. In the current cosmic scenario, various cosmological observations have reported the universe's accelerated expansion. This phenomenon can be addressed by invoking the exotic energy source which, dominates the entire energy content of the universe. Einstein's theory of gravity has been accepted as the principal theory of gravity. Despite that, it cannot be accountable for some exciting mysteries which motivate researchers to investigate and modify gravity theory that provides several interesting features. In the framework of modification of general relativity number of modified theories are developed, viz. Scalar-tensor theory, $f(T)$ gravity, $f(G)$ gravity, $f(R, G)$ gravity, $f(R)$ gravity, and $f(R, T)$ gravity etc. In this thesis, we studied Lyra geometry theory, Rastall gravity theory, $f(R, T)$ gravity theory. We have focused on some parameterization schemes to obtain the exact solutions of cosmological models.

The universe's evolution can be defined by parametrizing the Hubble parameter (H) and the deceleration parameter (q). On the other hand, the Scale factor $\mathcal{R}$ plays a crucial role in understanding the present evolution of the universe. We can determine the universe's rate of expansion through the scale factor. Parametrization of the deceleration parameter is one of the best approaches for obtaining the exact solution and exploring the physical behavior of models. Therefore, in this study, we have employed various types of deceleration parameters along with different matter sources. The chapterization details are discussed as following.

In chapter 1, we have discussed modern cosmology's history and basic ideas. Also, we have briefly discussed various gravity theories. In chapter 2, We presented the FRW cosmological model and particle creation in the Lyra geometry framework. We derived field equations and obtained the exact solutions for Dust filled models (DFM), Stiff matter model (SFM), Radiation dominated model (RDM), and Vacuum energy model (VEM) in the context of Lyra's geometry. In chapter 3, We investigated Bianchi type-I dark energy cosmological models in the framework of Lyra's geometry. In this chapter, we have developed four models in which the first two models are investigated by using volumetric expansion law, i.e., Exponential and Power-law expansion. Furthermore, we studied two models based on variable deceleration parameters. In chapter 4, we have discussed the Gödel universe in light of $f(R, T)$ gravity. This chapter focuses on the first class of functional form $f(R, T) = R + 2h(T)$, where R is the Ricci scalar and $h(T)$ is an arbitrary function of the energy-momentum tensor (T). We demonstrated the Gödel solution for three different trace-dependent functions, $h(T)$. In chapter 5, we developed the cosmological model in the framework of Rastall gravity. We established the relationship between the particle creation mechanism and Rastall gravity and set the field equation. Also, by using various deceleration parameters, we developed the models and discussed the physical behavior.

Chapter 1

Introduction

This chapter deals with an overview of modern cosmology. Cosmology is a branch of science that explores the universe's ultimate fate, which reveals about origin, evolution, dimensions, nature, and Laws of the universe.

1.1 Cosmology and History

Presently the "Big-Bang Theory" (BBT) is the most accepted theory about the origin and ultimate fate of the universe. The central theme of the theory is about universe's evolution. So it implies that in the distant past, there were substantial dense and hot cosmology deals with the problems of understanding the formulation of galaxies and clusters of galaxies together with the determination of nature of their masses. Before the BBT, many cosmologists from their studies and investigations gave an idea about the shape, size, nature, and formations of the universe. In the early days, Humans believed that the earth was not spherical rather than flat in the beginning. That thought may be expected because the curvature of the surface of our planet is not perceptible at once. However, this idea of assuming the earth to be flat creates a problem that the earth must end somewhere unless we imagine it to extend infinitely. Astronomers and Philosophers from various parts of the world tried to explain their views about cosmologies. The Greek philosopher Aristotle suggested that our planet was not like a flat plate but relatively spherical. He was the first to perceive that eclipses of the moon occur when the earth comes in between the moon and the sun. He observed that the

earth's shadow on the moon was always round. That would be true only if the earth were spherical. There are some cosmological developments from the very late time mentioned below.

From 500 BC to 300 BC, Greek philosopher Pythagoras believed that the earth was in motion. Aristotle measured the circumference of the earth as around 4,00,000 stadia approximately. From 300 BC to 210 BC, Greek mathematician Aristarchus proposed the solar system by the scientific heliocentric model and placed the sun at the center of the universe. Moreover, he presented the orders of planets from the sun. In 200 AD, Ptolemy proposed an Earth-centered model of universe including Sun and Planets, which stated that they revolve around the earth. In 1500 AD, many astronomers like European mathematician Copernicus and Indian mathematicians, the great Aryabhata and Bhaskaracharya, proposed a model of the universe having the sun positioned at the center. In 1576 AD, Thomas Digges modified the Copernican system and proposed a universe model containing many stars extending to infinity. He was the one who postulated the "Dark night sky paradox." In 1584 AD, Giordano Bruno, an Italian cosmologist, proposed a non-hierarchical cosmology that the universe is infinite and could have no "center." In 1600 AD, Tycho was a Danish astronomer who delivered a theory that stated that the planets other than earth orbited around the sun while the sun orbited around the earth. In 1609 AD, Johannes Kepler, a German Astronomer, used the dark night sky to argue for a finite universe. He stated that the planets moved in an elliptical orbit, not in a circular orbit about the sun, famous as the law of planetary motion.

In 1687 AD, one of the greatest mathematicians and physicists, Sir Isaac Newton, established the Laws of motion. Also, he stated his heliocentric view of the solar system. In 1791 AD, Erasmus Darwin, an English physician, declared the first description of a universe's cyclic expansion and contraction. In 1905 AD, the great Mathematician Albert Einstein published the Special Theory of Relativity, pointing out that space and time are not separated continuums. In 1916 AD, Albert Einstein published the General Theory of Relativity (GTR) which turned the study of cosmology differently. In 1927 AD, Georges Lemaitre, a Belgian Mathematician, presented his theory of the universe's expansion. In 1929 AD, The American Astronomer Hubble delivered the concept of the universe's expansion and the cosmological constant. 1965 AD, Penzias, and Wilson, a pair of American astronomers, discovered cosmic microwave background radiation. In

1970 AD, cosmologists accepted the *Hot Big Bang model* of the universe. Moreover, this continues the research in cosmology at a very extended level.

1.2 Einstein's General Theory of Relativity

Newton, in 1687 established the laws of motion in the inverse square for the gravitational forces, and he formulated a mathematical description about the movement of bodies in space, which is also known as the Newtonian theory of gravitation. Later in 1905, Einstein developed the special theory of relativity. He discussed the relativity of uniform translatory motion in a free space region where the gravitational effects are neglected. The special theory of relativity is applicable only to study the relative motion in the inertial frame of reference. Still, in an accelerated frame of reference, the special theory of relativity fails to examine the relative motion. Einstein made a considerable effort to resolve the limitations of the special theory of relativity in 1915. Einstein generalized the special theory of relativity known as the ***General Theory of Relativity*** (GTR). The GTR represents a theory of space-time, describing a theory of the universe's dynamics. Einstein's theory of gravitation also supports the observational data and is treated as the foundation of other geometric theories of gravitation. The GTR has two postulates: (i) *the principle of covariance* and (ii) *the principle of equivalence*

1.2.1 Principle of Covariance

According to the general theory, we use general ideas of the relativity of all kinds of motion. Here the way must be described in the form which is independent of the particular space-time coordinate, or the laws of nature must remain invariant concerning any space-time coordinate system, i.e., the assumption that the laws of physics can be expressed in the form which is independent of the coordinate system is called as ***the principle of covariance***.

1.2.2 Principle of Equivalence

According to Weinberg, *the principle of equivalence* states that at every space-time point in an arbitrary gravitational field, it is plausible to select a locally inertial coordinate system such that, within a small region of the matter, the laws of nature takes same form as in accelerated cartesian system in the absence of gravitation, i.e., in the neighborhood of any given point we can distinguish between the gravitational field produced by accelerating uniformly in the inertial frame of reference. The principle of equivalence can be classified into

- i) *strong principle of equivalence*
- ii) *weak principle of equivalence*

The strong principle of equivalence is where the laws of nature include all the fundamental forces. In contrast, the weak principle of equivalence replaces the laws of nature with the laws of motion of freely falling particles.

1.2.3 Mach's Principle

In the late 1880, Mach proposed a new perspective on the origin of inertia which states *that the inertia of matter depends on the average mass of the universe*, which is further known as the ***Mach principle***. It concludes that the distribution of matter uniquely resolves the space-time geometry. According to Mach's hypothesis, the material particle will not move in a non-accelerated motion relative to the space but relative to the center of all masses in the universe. It is found that the equivalence principle and Mach principle are oppositely directed. It implies a slight variation in the inertial mass when a particle accelerates towards or away from the galactic center.

1.2.4 Einstein's field equation from action principle

let us consider the action $\mathcal{S}$ of the system

$$\mathcal{S} = \frac{1}{16\pi} \mathcal{S}_g + \mathcal{S}_F \quad (1.1)$$

where $\mathcal{S}_g$ is the action of the gravitational field and $\mathcal{S}_F$ is the action of others field. Here $\frac{1}{16\pi}$ is the constant for convenience. By the principle of least action we have,

$$\delta\mathcal{S} = \frac{1}{16\pi}\delta\mathcal{S}_g + \delta\mathcal{S}_F = 0 \quad (1.2)$$

Now we can write

$$\delta s = \int L_g d^4x \quad (1.3)$$

where L_g is the lagrangian density in the form

$$L_g = \sqrt{-g}R \quad \text{where } R \text{ is Ricci scalar}$$

therefore, (1.3) $\Rightarrow \delta\mathcal{S}_g = \delta \int \sqrt{-g}R d^4x$

$$\delta\mathcal{S} = \int d^4x \left((\delta\sqrt{-g})g^{\mu\nu}R_{\mu\nu} + \sqrt{-g}(\delta g^{\mu\nu})R_{\mu\nu} + \sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} \right) \quad (1.4)$$

$$\delta\mathcal{S} = I_1 + I_2 + I_3$$

Now,

$$I_1 = \int \delta(\sqrt{-g})R d^4x \quad (1.5)$$

$$= \int \left(-\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} \right) \quad (1.6)$$

further,

$$I_1 + I_2 = \int \sqrt{-g} \left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \right) \delta g^{\mu\nu} d^4x \quad (1.7)$$

we know that

$$R_{\mu\nu} = -\Gamma_{\mu\nu,r}^r + \Gamma_{\mu r,\nu}^r - \Gamma_{\mu\nu}^r \Gamma_{rs}^s + \Gamma_{\mu s}^r \Gamma_{\nu r}^s. \quad (1.8)$$

Here we can select locally Riemannian co-ordinate system where all the first derivative of $g_{\mu\nu} = 0$ and therefore all $\Gamma = 0$ at the origin of the system.

$$R_{\mu\nu} = -\Gamma_{\mu\nu,r}^r + \Gamma_{\mu r,\nu}^r \quad (1.9)$$

$$\begin{aligned}
\therefore g^{\mu\nu} \delta R_{\mu\nu} &= g^{\mu\nu} (-\delta \Gamma_{\mu\nu,r}^r + \delta \Gamma_{\mu r,\nu}^r) \\
&= -g^{\mu\nu} \frac{\partial}{\partial x^r} \delta \Gamma_{\mu\nu}^r + g^{\mu\nu} \frac{\partial}{\partial x^\nu} \delta \Gamma_{\mu r}^r \quad \because \delta \frac{\partial}{\partial x^r} = \frac{\partial}{\partial x^r} \delta \\
&= -g^{\mu\nu} \frac{\partial}{\partial x^r} \delta \Gamma_{\mu\nu}^r + g^{\mu r} \frac{\partial}{\partial x^r} \delta \Gamma_{\mu\nu}^\nu \quad \text{by } \nu \leftrightarrow r \text{ in second term} \\
&= -\frac{\partial}{\partial x^r} (g^{\mu\nu} \delta \Gamma_{\mu\nu}^r) + \frac{\partial}{\partial x^r} (g^{\mu r} \delta \Gamma_\nu^\nu), \quad \because \text{all } g^{\mu\nu}, r = 0 \\
&= \frac{\partial}{\partial x^r} (-g^{\mu\nu} \delta \Gamma_{\mu\nu}^r) + (g^{\mu r} \delta \Gamma_\nu^\nu)
\end{aligned} \tag{1.10}$$

Hence (1.10) takes the form

$$g^{\mu\nu} \delta R_{\mu\nu} = \frac{\partial}{\partial x^r} a^r, \text{ where, } a^r = B^r + C^r \tag{1.11}$$

$$\begin{aligned}
\Rightarrow g^{\mu\nu} \delta R_{\mu\nu} &= a^r; r \quad \because \text{partial derivative is a covariant derivative.} \\
&= \frac{1}{\sqrt{-g} \frac{\partial}{\partial x^r}} (\sqrt{-g} a^r), \text{ by the definition of divergence of } a^r
\end{aligned}$$

$$\begin{aligned}
\Rightarrow I_3 &= \int \frac{\partial}{\partial x^r} (\sqrt{-g} a^r) d^4x \\
&= \int_{\Sigma} \sqrt{-g} a^r d\Sigma_r, \text{ by Gauss theorem} \\
&= 0 \quad \because \text{variation on } \Sigma_r \text{ vanishes}
\end{aligned}$$

Then we get

$$\delta \mathcal{S}_g = (I_1 + I_2) = \int \sqrt{-g} (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}) \delta g^{\mu\nu} d^4x$$

with this equation (1.2) we get

$$\frac{1}{16\pi} \int \sqrt{-g} (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}) \delta g^{\mu\nu} d^4x + \delta \mathcal{S}_F = 0 \tag{1.12}$$

as we know that

$$\delta \mathcal{S}_F = \frac{1}{2} \int \sqrt{-g} T_{\mu\nu} \delta g^{\mu\nu} d^4x, \quad \text{where } T_{\mu\nu} \text{ is energy momentum tensor}$$

Now from equation (1.12) we get,

$$\begin{aligned} \Rightarrow \int \sqrt{-g} \left((R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}) + 8\pi T_{\mu\nu} \right) \delta g^{\mu\nu} d^4x &= 0 \\ \Rightarrow R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + 8\pi T_{\mu\nu} &= 0 \quad \because \delta g^{\mu\nu} \text{ are arbitrary} \end{aligned}$$

So, we get

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -8\pi T_{\mu\nu} \quad (1.13)$$

The equation (1.13) is Einstein's field equation.

1.3 Particle Creation

The concept of particles is delicate, and the exact meaning of particles in curved space-time is more complex than in flat space-time. In the general curved space-time, we do not have the Lorentz symmetry, and more precisely, the Lorentz symmetry permits us to identify the vacuum state in the flat space-time. The Lorentz symmetry signifies that the vacuum state is identical for all inertial observers. There are regions where a particular choice of physical vacuum is quite natural that occurs due to the time-dependence of the field in that region being almost harmonic. The background metric and the eventual boundaries which do not depend on time is the primary condition to have a field with harmonic time dependence. In some cases, the background metric accepts this condition only in the asymptotic time period. In such cases, there is a possibility of physical interpretation of particles at that asymptotical time but not between them which results in a non-static curved space-time background that may lead to the phenomenon of *particle creation*. The theory of particle creation in the universe is one of the most unsolved theories in cosmology. The particle creation mechanism depends on the interpretation of matter energy-stress tensor in an open thermodynamical system. The particle creation infers an irreversible flow of energy from the gravitational field to the constituents of the created particles. Due to this, particle creation pressure is included in the energy-momentum tensor. The negative value of the particle creation pressure may lead to the creation of matter. The average particle creation rate is determined by the expansion rate and the universe's density. At the initial time, the particle creation rate is so low that the back reaction of the particle creation on the gravitational field gets neglected. The

particle creation process will take place everywhere, and mainly the particles will be formed in the vast intergalactic space. The universe evolves like an adiabatic open thermodynamical system due to the gravitationally induced particle creation mechanism. The first law of thermodynamics which may be generalized for an open system stated as

$$d(eV) + pdV - \left(\frac{\rho + p}{n}\right)dN = 0 \quad (1.14)$$

where V is the volume of the system, p is the pressure, ρ is the density and $n = \frac{N}{V}$ is the particle number density and N is the particle number. The above conservation equation is expressed as

$$d(\rho V) + (p + p_c)dV = 0 \quad (1.15)$$

The supplementary pressure which corresponds to matter creation, denoted by p_c is written as

$$p_c = -\left(\frac{\rho + p}{n}\right)\frac{dN}{dV} \quad (1.16)$$

The particle creation problem arises due to the curved space-time initially discussed by Schrodinger [1]. Various cosmologists studied the phenomenological concept of particle creation and its effects on the universe's evolution. Hoyle [2], Hoyle and Narlikar [3], Bondi and Gold [4] investigated particle creation during the universe's expansion. Prigogine [5] pointed out that particle creation and bulk viscosity are the two independent mechanisms that may lead to the different cosmic theories of universe's evolution. In 1968 Parker [6] investigated the results of particle creation in expanding universe. Ford [7] concluded that particle creation arise due to changes in space-time metric at the time of cosmic inflation in the early universe. Mottola [8] constructed a cosmological model in which he introduced particle creation in de sitter space-time. Berger [9] reconstructed the problem of quantized scalar field particle creation in the background of an anisotropic spatially homogeneous universe. Wald [10] explored Hawking's analysis of particle creation due to black holes and formulated the expression for the quantum mechanical state vector in the presence of particle creation. Khriplovich [11] investigated a model by using the Schwinger formula for particle creation due to charged black holes. Celani et al. [12] investigated the bouncing cosmological model where the reason for bounce is due to quantum cosmological effects. Firouzjaee et al. [13] investigated a dynamic cosmological model to examine the backreaction issue by finding the quantum stress tensor in the presence of particle creation. Anderson [14] proposed that the

particle creation in the declining phase gives large energy densities and the global de sitter space is unstable for particle creation. Good et al. [15] constructed a model in which they investigated two-dimensional flat space-time with an accelerated mirror in the presence of particle creation due to quantized scalar field. Lyth et al. [16] discussed consequences in a cosmological model of particle creation during inflation. Dissipation of the quantum fields due to the presence of particle creation was discussed by Calzetta et al. [17].

1.4 The Gödel Universe and Gödel Metric

The *Gödel universe* is an exact solution to the Einstein field equation. In the Gödel universe the stress-energy tensor contains two factors, the first factor is about the matter density of a homogeneously distributed swirling dust particles, and the second factor is associated with a nonzero cosmological constant. The Gödel universe is also known as the Gödel solution or Gödel metric. The Gödel space-time is a rare example of a singularity-free solution of the Einstein field equation. The Gödel universe was discovered in 1949 by sir Kurt Gödel [18]. The Gödel universe demonstrates the exact and stationary solution of Einstein's field equation in the presence of cosmological constant and homogeneous pressureless matter. The rotating and non-rotating black holes were investigated by Eric et.al. [19] and they presented the exact solutions for the model in the framework of Gödel Universe. In 1977 Hiscock et al. [20] constructed a model which provides an exact solution to the classical scalar wave equation in the context of the Gödel universe. Reboucas et al. [21] examined the Gödel -type space-time under different conditions based on the structured algebraic geometry of the energy-momentum tensor. Bampi et al. [22] delivered a note on the Gödel universe in which they described Einstein's field equation for perfect fluid, which is necessarily isometric to the Gödel universe. Many researcher's developed cosmological models of the Gödel universe in the framework of various modified gravities. Furtado et al. [23] discussed the modified gravity by considering Einstein-Hilbert action, which declared that the Gödel metric solved the modified equation of motion. Santos et al. [24] demonstrated a path of understanding the solutions of the Gödel universe by using the structure of $f(R, T)$ modified theories of gravity. Barrow et.al. [25] discussed the exact solutions

for the Gödel metric in the framework of string theory. Raychaudhary et al. [26] investigated the condition for homogeneity of cylindrically symmetric metric in the cosmological models based on the Gödel universe. Desszcz et al. [27] discussed the curvature properties of the Gödel metric by investigating the geometric structure of Gödel space-time. Gama et al. [28] constructed a cosmological model by investigating the Gödel type solutions within the framework of $f(R, Q)$ modified theories of gravity.

The Gödel metric is given by

$$ds^2 = \mathcal{R}^2(dt^2 - dx^2 - dz^2 + \frac{1}{2}e^{2x}dy^2 + 2e^x dydt) \quad (1.17)$$

where t, x, y, z are co-moving coordinates. Further, Gödel himself transformed the metric (1.17) as

$$ds^2 = dt^2 - d\gamma^2 - dz^2 + 2\sqrt{2}\sinh^2\gamma d\phi dt + (\sinh^4\gamma \sinh^2\gamma)d\phi^2 \quad (1.18)$$

where the coordinate ϕ is an angle coordinate.

1.5 Bianchi Type Models

The cosmological model of the observable universe is very well represented by Friedmann–Lemaître–Robertson–Walker (FLRW) models, which are both isotropic and spatially homogeneous. These models are considered best in the sense of global approximation of the universe. Observations provided by the Differential Microwave Radiometers (DMR) on Cosmic Background Explorer (CBE) registered anisotropy in distant angle scales. On a large scale, the observed universe looks isotropic, while the early and late time universe could be anisotropic, was pointed out by Ellis [29]. Moreover, the interest in Bianchi type models has been encouraged in recent years depending on the analytical results and interpretation of Wilkinson Microwave Anisotropy Probe (WMAP) data. The observable universe is the best expressed with the FLRW (Friedmann–Lemaître–Robertson–Walker) space-time. It is exciting to know the facts about the universe’s early-stage evolution. In the present scenario, the universe is described by homogeneous and isotropic cosmological models. However, at the same

time, we cannot expect that they are equally described in the early stages of the universe. Thus Bianchi spaces are crucial for constructing cosmological models in GR and alternative gravity theories that are homogeneous and anisotropic.

1.6 Homogeneity and Anisotropy

Homogeneity and isotropy are two important concepts in cosmology. On large scale, the universe is assumed to be uniform. This basic concept is treated as the cosmological principle. The cosmological principle is based on two basic aspects: (i) The Universe is homogeneous, which means that if you look at the universe from any positions, no matter where you are in the universe, then it will look the same. (ii) The Universe is isotropic, which means that from any direction, the universe looks structurally the same. Mathematically it can be expressed as: A physical system is homogeneous if it is invariant to transformation $x^\mu = x'^\mu + x^\nu$, where $x^\nu = \text{constant}$ (look the same everywhere) and isotropic if invariant to rotation (looks the same in all directions). The standard model of cosmology is constructed on the basis of the existence of homogeneous surfaces as the background field for structural formation. At the same time, homogeneity acts as a basis for general relativistic and modified gravity models. Also, considering the concept of homogeneity, we can interpret the observation of cosmic microwave background (CMB) and the galaxy distribution. By assuming the Copernican principle, We can directly observe and test for isotropy. Researchers believed

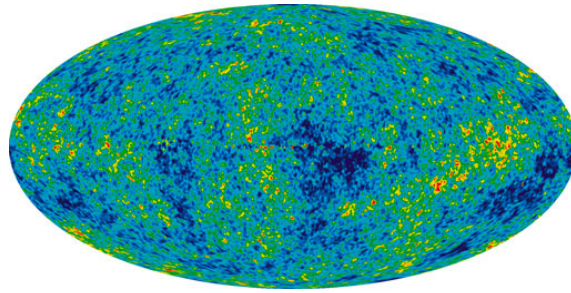


FIGURE 1.1: The CMB temperature from WMAP 7 year data showing anisotropies at the level approximately 10^{-5} . (Courtesy NASA/ WMAP Science Team.)

that the early universe's expansion was highly anisotropic, which isotropized later. The cosmic microwave background radiation observations and results reveal that the level of anisotropy left out by the era of decoupling is of the order of 10^{-5} . This motivates

several authors to consider models of the universe with an anisotropic background. The simple anisotropic and homogeneous space-times are the Bianchi (I - IX) space-times of Bianchi- III , Bianchi- V , and Bianchi- IX space-times are the simplest generalizations of Friedmann-Laimetre-Robertson-Walker (FLRW) space-time metric of zero, negative and positive curvatures respectively. Therefore the models of the Bianchi type are very relevant to representing the early universe with the anisotropic background. Again, in the present scenario, many observations are conducted for obtaining the homogeneity and isotropic properties of the universe. It is accepted as a fact that at the end of the inflationary era, the universe's geometry was homogeneous and isotropic [30], where the FLRW models played an important role in representing the universe both spatially homogeneous and invariant concerning direction, respectively. After the announcement of Plank probe results [31], it is believed that the early universe possibly not has been exactly uniform. The homogeneous and isotropic cosmological model describes the present large-scale universe structure with Robertson-Walker space-time geometry. Theoretical models and experimental data support the existence of an anisotropic phase that tends to become an isotropic phase at late times [32]. Observations like CMBR [33, 34] and Wilkinson Microwave Anisotropy Probe (WMAP) [35, 36] provides the evidence of anisotropy during cosmic expansion Fig. 1.1.

1.7 Bianchi Universes

The experimental and theoretical observations need a Bianchi type formation that resolves the departure from global isotropy problems [37]. The Bianchi universes are the series of homogenous cosmological models not compulsorily isotropic on spatial slices, formulated by famous cosmologist Luigi Bianchi. Bianchi universes are the top exciting topic for researchers in observational cosmology. The study of cosmological models containing anisotropic fluids is not allowed by the FRW universe as the fluids are co-moving. Therefore, the spatially anisotropic space-times are considerable for studying the anisotropic energy-momentum sources like magnetic fields, bulk viscous fluid, cosmic strings, and the hypothetical fluids observed in the view of DE. The theoretical point of view shows that anisotropic universes have greater generality than isotropic models. We are presenting a detailed description of Bianchi's space times. The space-times admitting an automorphism of the three-parameter group are essential

for discussing the cosmological models. In 1898, Bianchi concluded that there are only nine diverse sets of structural constants for groups of such type so that it will be easy to classify homogeneous space-times [37]. Thus, a three-parameter group of motions is admitted by Bianchi type space-times and hence has only a feasible number of degrees of freedom.

The observable universe is best expressed with the FLRW (Friedmann-Lemaitre-Robertson-Walker) space-time, which is homogeneous and isotropic. It is fascinating to understand the early-stage evolution of the universe. In the present scenario, the universe is described by homogeneous and isotropic cosmological models. However, at the same time, we cannot expect that they are equally described in the early stages of the universe. Thus Bianchi spaces are crucial for constructing cosmological models in GR as well as alternative gravity theories. Bianchi's models are the foundation of the modern cosmological models as the zeroth approximation. In cosmology, Bianchi's models are known as minisuperspace models. Firstly, Bianchi type models strongly restrict space topology. There is nonexistence of a spatially compact model with local symmetry of the Bianchi type IV or VI_h . Secondly, there is an appearance of new dynamical degrees of freedom known as the moduli, which describes the degrees of freedom during the deformation of the global space structure. During the last three decades, there has been exciting research work on anisotropic universes. Taub, in 1951 concluded nine types of Bianchi models that are spatially homogeneous [38]. In 1973, Collins and Hawking [39] concluded that out of these nine types of Bianchi models, only I , V , VH_0 , and VII_h , can tend towards isotropy at random significant times. Hence, these models permit the formation of galaxies. The type I and VII_0 represents the generalized flat FRW models, while the generalized open FRW models are represented by the types V and VII_h . Furthermore, the most general non-flat models are represented by the types II , VI , $VIII$, and IX . The considerable study between the open and flat models, which is physically more relevant due to the 'hot spots', occurs only in these models. In 1983 according to Barrow et al. [40] the positive detection of a 'hot spot' would unequivocally demonstrate that the universe is open.

Bianchi type models or space-times are characterized by the kinematic quantities of the similar spatial hypersurface: bulk expansion rate H , trace free shear tensor $\sigma_{\alpha\beta}$ an auxiliary three-vector Ω_α that measure the rotation of the frame concerning fermi-propagated one and a symmetric three-tensor $\eta_{\alpha\beta}$ which determines the internal geometry on the spatial hypersurface.

1.7.1 Classifications of Bianchi Universes

The Bianchi universes are classified as a spatially homogeneous space-times with three-parameteric isometry groups which acts on spatial slices are introduced by Luigi Bianchi [37]. Furthermore, the Bianchi cosmologies are classified into two classes: class A and class B by Ellis and MacCallum [41] and Krasinski et al. [42]. They were further divided into subgroups using the matrix's eigenvalue. The observable universe is best expressed with the FLRW (Friedmann-Lemaitre-Robertson-Walker) space-time, which is homogeneous and isotropic. It is exciting to understand the early-stage evolution of the universe. In the present scenario, the universe is described by homogeneous and isotropic cosmological models. However, at the same time, we cannot expect that they are equally described in the early stages of the universe. Thus Bianchi spaces are crucial for constructing cosmological models in GR and alternative theories of gravitation, which are homogeneous and anisotropic. The description of the early stage universe is well expressed through the homogeneous and anisotropic cosmological models. Thus, investigation of Bianchi types anisotropic cosmological models is essential in GR and other alternative and modified theories of gravitation. Sharif and Zubair [43] have analyzed the qualitative behavior of massless scalar field and energy conditions for the Bianchi type-*I* universe in view of $f(R, T)$ gravity. In some particular cases, the type-*I* of Bianchi models include *Kasner metric*, which helps to regulate the dynamics near the singularity point. The dynamics of the Bianchi type-*I* cosmological model has been discussed in the condition of a viscous fluid and observed that the viscosity could be a reason for the qualitative behavior of solutions near the singularity points without removal of the total initial big bang singularity [44].

Cosmologists show more interest in discussing and investigating Bianchi type-*III* models in alternative theories of gravity. The Bianchi type-*III* cosmological models in scalar-tensor theory were discussed by Singh and Agrawal [45]. Shamir studied the vacuum solutions of type *I*, *III* of Bianchi models and Kantowski-Sachs space-times in the metric in $f(R)$ gravity [31]. Reddy et al. have obtained a Bianchi type *III* cosmological model by considering the matter as a perfect fluid within the cases of $f(R, T)$ gravity with negative deceleration parameter [46]. In our another chapter we have studied LRS Bianchi type *V* models in $f(R, T)$ theory in presence of perfect fluid [47].

Ahmed and Pradhan in 2014 have investigated Bianchi type-*V* cosmological model in $f(R, T)$ gravity for the choice of functions as $f(R, T) = f_1(R) + f_2(T)$ [48]. In 2015

a big rip singularity in LRS Bianchi type-*I* universe by concerning the linear varying deceleration parameter in $f(R, T)$ gravity was discussed by Sahoo and Sivakumar [49]. In this thesis, we also restrain ourselves to Bianchi type-*III* and LRS Bianchi type-*V* metric with perfect fluid, Bulk viscous fluid, Zeldovich fluid, and Cosmic fluid with one-dimensional string solutions in $f(R, T)$ gravity and Lyra geometry. The above said space-time has a primal role in representing details of the early stages of evolution of the universe, and it is the most straightforward spatially homogeneous and anisotropic metric. It is also known as the generalization of the FLRW flat metric with various scale factors in each spatial direction.

There are nine types of Bianchi models introduced by Taub [38], which are necessarily spatially homogeneous. Now we are presenting Bianchi models in the respective metric form of Bianchi type-*I* metric:

$$ds^2 = -dt^2 + X_1^2 dx^2 + X_2^2 dy^2 + X_3^2 dz^2, \quad (1.19)$$

where X_1, X_3 and X_4 are functions of t only. The LRS Bianchi type-*I* metric:

$$ds^2 = -dt^2 + X_1^2 dx^2 + X_2^2 (dy^2 + dz^2), \quad (1.20)$$

Bianchi type-*II* metric:

$$ds^2 = -dt^2 + X_1^2 (dx - z dy)^2 + X_2^2 dy^2 + X_3^2 dz^2, \quad (1.21)$$

Bianchi type-*III* metric:

$$ds^2 = -dt^2 + X_1^2 dx^2 + X_2^2 e^{-2mx} dy^2 + X_3^2 dz^2, \quad (1.22)$$

where m is a constant.

Bianchi type-*IV* metric:

$$ds^2 = -dt^2 + e^{-2z} [X_1 dx^2 + (X_1 z^2 X_2) dy^2 + 2X_1 z dx dy] + X_3^2 dz^2, \quad (1.23)$$

Bianchi type-*V* metric:

$$ds^2 = -dt^2 + X_1^2 dx^2 + e^{2mx} (X_2^2 dy^2 + X_3^2 dz^2), \quad (1.24)$$

The LRS Bianchi type V metric of the form

$$ds^2 = dt^2 - X_1^2 dx^2 - X_2^2 e^{2x} [dy^2 + dz^2], \quad (1.25)$$

Bianchi type- VI_0 metric:

$$ds^2 = -dt^2 + X_1^2 dx^2 + X_2^2 e^{-2mx} dy^2 + X_3^2 e^{2mx} dz^2, \quad (1.26)$$

Bianchi type- VII_0 metric:

$$ds^2 = -dt^2 + (X_1 \cos^2 z + X_2 \sin^2 z) dx^2 + (X_1 \sin^2 z + X_2 \cos^2 z) dy^2 \\ + (X_1 + X_2) \sin 2z dx dy + X_3 dz^2, \quad (1.27)$$

Bianchi type- $VIII$ metric:

$$ds^2 = -dt^2 + X_1^2 dx^2 + X_2^2 dy^2 + (X_3^2 \sinh^2 y + X_4^2 \cosh^2 y) dz^2 \\ - 2X_2^2 \cosh y dx dz, \quad (1.28)$$

Bianchi type- IX metric:

$$ds^2 = -dt^2 + X_1^2 dx^2 + X_2^2 dy^2 + (X_2^2 \sin^2 y + X_1^2 \cos^2 y) dz^2 - 2X_1^2 \cos y dx dz \quad (1.29)$$

1.8 The observational status of the Universe

1.8.1 Baryonic Matter

Cosmologists referred the term "Baryonic Matter" for all the objects made up of atomic matter. Ordinary matter is made up of protons and neutrons is called *Baryonic Matter*. The concept of Baryonic matter is one of the most crucial discoveries, which possesses less than 5% of the universe's mass. Baryons and anti-baryons are both a part of the matter in the universe and do not affect the macroscopic structure, which is governed by gravity. However, the excess of baryons over anti-baryons is explained by the mechanism of particle physics. As per the energy conditions and the results from the CMB and Big-Bang Nucleosynthesis (BBN), 5% is the Baryonic matter content of the universe. Due to this mismatch between baryonic mass with the total matter content, one should

suggest that dark matter is Non-baryonic. The Non-baryonic dark matter is relatively non-relativistic, concluding that heavy and slow-moving particles form the backbones of galaxies and galaxy clusters.

1.8.2 Dark Matter

We know from the various astronomical data that the visible universe is only 4% of the total mass content, and the rest, 96% is the missing matter which we cannot explain. This missing content of the universe is invisible so, this 96% of the total mass content of the universe is in the form of non-baryonic dark matter, which is known as *Dark matter* (DM). The DM is referred to as "dark" because it does not seem to emit, reflect or absorb any electromagnetic radiation. There are various ways to estimate the amount of DM, in which the most direct way depends on a detailed analysis of the fluctuations in the Cosmic Microwave Background (CMB). The DM is five times more abundant than baryonic matter [50]. The DM was controversial initially, but now it is widely accepted due to the experimental facts, observations of the anisotropies in the CMB, galaxy cluster velocity dispersion, large scale structure distribution, x-ray measurements from galaxy clusters. The growth of the structure is an essential argument in favor of the existence of DM. Observational data revealed the collapsing of the minor structure, firstly followed by galaxies and then galaxy structure. The DM hypothesis agrees with the statistical surveys and observations of the visible structure. Simulation of DM particles confirms that the structure formation DM model is consistent with the structural observations through galaxy surveys like Sloan Digital Sky Survey, observations of the Lyman-alpha forest, and 2dF Galaxy Radiation Survey [51, 52]. The DM will be split into two possible components, i.e., hot dark matter and cold dark matter. The hot dark matter comprises very light particles moving close to the speed of light, and the cold dark matter contains relatively massive particles moving more slowly. Many cosmologists worked in the direction of the DM and the Dark energy (DE) to understand their importance in the universe. Smith [53] settled up the terrestrial experiments to directly detect galactic dark matter. Finkbeiner [54] proposed a comprehensive theory of DM that explains the recent profile of unexpected observations in high-energy astrophysics. Bertone [55] discussed the history of the DM very briefly. Bergstrom [56] proposed the

evidence of the presence of DM in the universe. Peebles [57] explains the importance of DM in a classical scalar field.

1.8.3 Dark Energy

The late-time accelerated universe expansion is the most significant discovery in cosmology. However, so far, there is no clear evidence or proof of why this is happening. So the responsible thing for this cosmic acceleration must be mysterious. This Mysterious and unknown form of energy is termed *Dark Energy* (DE). DE is the most accepted hypothesis to explain the observations since late 1990, which indicate that the universe is expanding. In the past research history, cosmologists explained the existence of DE and Universe expansion theoretically based on equations, concepts, and theories. The most significant feature of DE is its negative pressure which behaves like a force opposite to gravity. The SN Ia observational data indicate that DE contributes 73% of total energy in the present-day observable universe, and ordinary matter contributes about 27% of total energy. DE appears to be homogeneous, not very dense. Copeland [58] reviewed the models that have been formulated to try and explain remarkable observations of our accelerating universe. Sahani [59] restudied the models to explain the expansion history of the universe and to study the exact nature of DE. Buchert [60] proposed a detailed report of the DE from its structure. Cai [61] constructed the models of DE in which the age of the universe is introduced as the length measure. Cosmologists explained the concept of DE in various theories of gravitation like $f(R)$ gravity [62], $f(R, T)$ gravity [63], Lyra geometry theory [64], Rastall gravity theory [65], Brans-Dicke theory [66].

1.8.3.1 Cosmological Constant Λ

Einstein introduced the concept of the cosmological constant in the gravitation field equation with the motive for a finite, closed, and static universe where the matter-energy density determines the geometry. It is also called Einstein's cosmological constant. But according to Einstein "*The cosmological constant Λ is the biggest blunder of my life,*". After several decades in the 1980s, Peebles et al. [67] used the cosmological constant Λ to research the universe's accelerating expansion. Further, they concluded that the existence of dark energy could be related to the cosmological constant Λ . Ratra and

Peebles [68] discussed various aspects of dark energy and the cosmological constant Λ . The cosmological constant Λ is the simplest candidate for the study of dark energy. It is assumed that the cosmological constant Λ is the simplest form of dark energy. Its existence is also predicted by quantum physics, where it exists in the form of vacuum energy. Cosmologists have considered cosmological constant Λ for building models to study and explore the dynamics of dark energy. The homogeneous active mass represented by the cosmological constant Λ changes the relative motion equation of freely moving test particle in a non-relativistic limit to,

$$\frac{d^2 \vec{r}}{dt^2} = \vec{g} + \Omega_{\Lambda 0} H_0^2 \vec{r} \quad (1.30)$$

where $\vec{g}$ is the gravitational acceleration produced by the distribution of ordinary matter. The ratio of acceleration g_{Λ} produced by the cosmological constant Λ to the total gravitational acceleration $g = \frac{v_c^2}{r}$ is

$$\frac{g_{\Lambda}}{g} = \frac{\Omega_{\Lambda 0} H_0^2 r^2}{v_c^2} \sim 10^{-5} \quad (1.31)$$

The equation of motion in GTR is given as

$$\frac{\ddot{\mathcal{R}}}{\mathcal{R}} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \quad (1.32)$$

Where G is the constant of gravitation, a is a scale factor, p is pressure, ρ is the energy density. We have observed that if in the equation (1.32) $\rho + 3p = 0$ and Λ is positive, then the cosmological constant Λ alone is responsible for cosmic acceleration. Furthermore, if $\Lambda > -4\pi G(\rho + 3p)$, and Λ is positive, then also universe will be accelerating and the cosmological constant Λ will also contribute in some amount in the universe's acceleration.

Carroll [69] presented a pedagogical overview of cosmological models in the existence of cosmological constant. According to recent observations and studies, Padmanabham [70] discussed the appearance of a cosmological constant with the magnitude of $\Lambda(\frac{Gh}{v_c^3}) \approx 10^{-123}$ where G is gravitational constant and h is Planks constant.

1.8.3.2 Quintessence

For solving the fine-tuning problem of the cosmological constant in the recent period, the quintessence scalar field model has been introduced by Caldwell [71], Ratra and Peebles [72]. The quintessence is a time-varying inhomogeneous field with an EoS parameter $-1 < \omega < 0$ [73]. In the quintessence model, dark energy dominates the cosmic acceleration of the universe for future evolution. Various cosmological models based on quintessence dark energy models are developed in the research history [74–76].

1.8.3.3 Phantom

The phantom model is another hypothetical form of dark energy with EoS parameter $\omega < -1$. The difference between the dark energy with $\omega > -1$ and $\omega < -1$ becomes apparent if we consider the universe's expansion. Phantom energy is not sufficient to generate Big Rip because ω does not appear to be constant during the universe's evolution [77]. Scherrer [78] studied the phantom dark energy model in the existence of negative kinetic term fluid. Ludwick [79] constructed a model to examine the theoretical consistency and viability of phantom dark energy. Nojiri [80] demonstrated the properties of future singularities by constructing a phantom dark energy model.

1.8.3.4 Quintom

Quintom model is a unified dark energy model with EoS parameter getting across the cosmological constant boundary $\omega = -1$ from either side. The quintom model forecast some exciting facts linked with the universe's evolution. In the quintom model, the universe would neglect the singularities like Big Bang [81] and Big Rip [82]. The dynamical quintom DE models are constructed by Elizade et al. [83] and Anisimov et al. [84].

1.9 Other Theories of Gravity

1.9.1 Lyra's Geometry

As we know, Einstein's GTR can describe the gravitation geometrically in the most successful way. Thus many researchers were inspired by Einstein's work and tried to geometrize the other field of physics. In that period, one such theory was known as *the theory of gravity in Lyra's geometry*. We know Einstein's GTR is based upon Riemannian geometry with metric tensor $g_{\mu\nu}$. In 1918 Weyl [85] proposed a modification of the Riemannian manifold to geometrize both gravitation and electromagnetism. A Weyl manifold is characterized by two independent quantities i) metric tensor $g_{\mu\nu}$ and ii) vector field ϕ_μ , in which under parallel transfer, a vector not only changes direction but it may also change the length and change of length is depending on ϕ_μ . Einstein criticized the change of length of a vector under parallel transfer. It states that the frequency of spectral lines emitted by atoms wouldn't remain constant but, depend on their past histories, which contradicts the uniformity of their properties. Sen and Dunn [86] proposed a scalar-tensor theory of gravitation, which is based on Lyra's manifold rather than Riemannian manifold. Halford [87] shows that this new theory predicts the same effect as Einstein's theory within the observational limits. Lyra's geometry is based on the modification of Riemannian geometry in a direction similar to Weyl's geometry, in which he introduces a gauge function into the structureless manifold.

1.9.1.1 Field equation of Lyra's geometry

In Lyra's geometry, the displacement vector PP' between both arbitrary neighbouring points $P(x^\mu)$ and $P'(x^\mu + dx^\mu)$ is resolved by the component βdx^μ , where $\beta = \beta(x^\gamma)$ is a gauge function. The coordinate system x^μ and the gauge function β together form a reference frame (β, x^μ) . The transformation to a new reference frame (β', x'^μ) is given by

$$\begin{aligned}\beta' &= \beta(\beta, x^\gamma), \\ x'^\mu &= x^\mu(x^\gamma)\end{aligned}\tag{1.33}$$

where $\frac{\partial \beta'}{\partial \beta} \neq 0$ and jacobian $|\frac{\partial x'^{\mu}}{\partial x^{\gamma}}| \neq 0$. The Levi-civita connection coefficients in Lyra geometry is defined as

$$*\Gamma_{\nu\gamma}^{\mu} = \frac{1}{\beta}\Gamma_{\nu\gamma}^{\mu} - \frac{1}{2}(\delta_{\nu}^{\mu}\phi_{\gamma} + \delta_{\gamma}^{\mu}\phi_{\nu} - g_{\mu\gamma}\phi^{\mu}) \quad (1.34)$$

where $\Gamma_{\nu\gamma}^{\mu}$ is defined in terms of metric tensor $g_{\mu\gamma}$ just like as in Riemannian geometry and ϕ_{γ} is the displacement vector field. The line element on the Lyra's geometry is determined by

$$ds^2 = \beta^2 g_{\mu\gamma} \beta dx^{\mu} dx^{\gamma} \quad (1.35)$$

which is invariant concerning to the coordinate and gauge transformations. As a result the parallel transport of a vector ξ^{μ} is given by

$$d\xi^{\mu} = -\tilde{\Gamma}_{\nu\gamma}^{\mu} \xi^{\nu} dx^{\gamma} = \Gamma_{\nu\gamma}^{\mu} \xi^{\nu} dx^{\gamma} - \frac{1}{2}\delta_{\nu}^{\mu}\phi_{\gamma} \xi^{\nu} dx^{\gamma} \quad (1.36)$$

where

$$\tilde{\Gamma}_{\nu\gamma}^{\mu} = *\Gamma_{\nu\gamma}^{\mu} - \frac{1}{2}\delta_{\nu}^{\mu}\phi_{\gamma} \quad (1.37)$$

Thus we see that $\tilde{\Gamma}_{\nu\gamma}^{\mu}$ is not symmetric with respect to ν and γ . The curvature tensor in Lyra geometry is given by

$$*R_{\nu\gamma l}^{\mu} = \frac{1}{\beta^2} \left[-(\beta \tilde{\Gamma}_{\nu\gamma}^{\mu}), l + (\beta \tilde{\Gamma}_{\nu l}^{\mu}), \gamma - \beta^2 (\tilde{\Gamma}_{\nu\gamma}^m \tilde{\Gamma}_{ml}^{\mu} - \tilde{\Gamma}_{\nu l}^m \tilde{\Gamma}_{\gamma m}^{\mu}) \right] \quad (1.38)$$

where $\tilde{\Gamma}_{\nu\gamma}^{\mu}$ is determined from (1.37). The curvature scalar is given as

$$*R = \frac{1}{\beta^2} R + 3 \frac{1}{\beta} \phi_{;\mu}^{\mu} + \frac{3}{2} \phi^{\mu} \phi_{\mu} + 2 \frac{1}{\beta} \frac{\partial(\ln \beta^2)}{\partial x^{\mu}} \phi^{\mu} \quad (1.39)$$

where R is the Riemannian curvature scalar. The semicolon denotes the covariant differentiation with respect to Christoffel symbol of second kind. The action integral in four dimension is as follows

$$I = \int L \beta^4 \sqrt{-g} d^4x \quad (1.40)$$

which is invariant under the gauge function and coordinate transformations. Also, d^4x and L denotes the volume element and the scalar function respectively. In normal gauge,

the curvature scalar function in the equation (1.39) takes the form

$$*R = R + 3\phi_{;\mu}^{\mu} + \frac{3}{2}\phi^{\mu}\phi_{\mu} \quad (1.41)$$

Again in normal gauge and $L = *R$ the action in (1.40) is given by

$$I = \int (R + 3\phi_{;\mu}^{\mu} + \frac{3}{2}\phi^{\mu}\phi_{\mu})\sqrt{-g}d^4x \quad (1.42)$$

$$J = \int L_m\sqrt{-g}d^4x \quad (1.43)$$

be the action for the matter of the universe. The total action for the scalar tensor theory based on Lyra's geometry is as follows

$$S = I + J = \int (\beta^4 L + L_m)\sqrt{-g}d^4x \quad (1.44)$$

In the normal gauge and with the equation (1.42), the action in the equation (1.44) takes the form

$$S = \int (R + 3\phi_{;\mu}^{\mu} + \frac{3}{2}\phi^{\mu}\phi_{\mu} + L_m)\sqrt{-g}d^4x \quad (1.45)$$

The field equations of scalar tensor theory based on Lyra's geometry in normal gauge is obtained from the variational principle as

$$\delta S = \delta \left(\int (R + 3\phi_{;\mu}^{\mu} + \frac{3}{2}\phi^{\mu}\phi_{\mu} + L_m)\sqrt{-g}d^4x \right) \quad (1.46)$$

Which leads to the field equation of the form

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \frac{3}{2}\phi_{\mu}\phi_{\nu} - \frac{3}{4}g_{\mu\nu}\phi^m\phi_m = -\chi T_{\mu\nu}. \quad (1.47)$$

In which $\chi = \frac{8\pi G}{c^2}\phi_{\mu}$ is the displacement vector defined through the gauge function β . The displacement vector field is considered as

$$\phi_{\mu} = (0, 0, 0, \beta(t)). \quad (1.48)$$

1.9.2 $f(R)$ Gravity

It has been accepted that the universe is undergoing a phase of expansion as tested by the Hubble diagram of Type Ia supernovae and several large-scale structure observations. $f(R)$ **modified theories of gravity** consist of infrared modifications of GTR that becomes crucial only at low curvatures, late in the matter era. The Einstein-Hilbert action is given by.

$$S_{EH} = \frac{1}{2k} \int d^4x \sqrt{-g} R + s^{matter}$$

which is modified to

$$S = \frac{1}{2k} \int d^4x \sqrt{-g} f(R) + s^{matter} \quad (1.49)$$

where, $f(R)$ is a non-linear function of its arguments and R is Ricci scalar. The field equation resulting from equation (1.49) in $f(R)$ gravity theory is given as

$$F(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu \nabla_\nu F(R) + g_{\mu\nu} \square F(R) = T_{\mu\nu} \quad (1.50)$$

where $F(R) \equiv \frac{df(R)}{dR}$ and $\square \equiv \nabla^\mu \nabla_\mu$, ∇_μ is the covariant derivative. Furthermore, by contracting the field equation (1.50) it takes the form as

$$F(R)R - 2f(R) + 3\square F(R) = T \quad (1.51)$$

by using (1.50) and (1.51) the field equation of $f(R)$ theory takes the form

$$F(R)R_{\mu\nu} - \nabla^\mu \nabla_\nu F(R) - T_{\mu\nu} = g_{\mu\nu} \left(\frac{F(R)R - \square F(R) - T}{4} \right) \quad (1.52)$$

In the past time $f(R)$ gravity received the serious attention of many researchers. One could vary the action concerning the metric and connections. The metric $f(R)$ gravity is the variation with respect to the metric by a variational principle. Similarly, variation with respect to both the metric and the connection results in the palatini $f(R)$ gravity, where the connection is independent of the metric and vice versa. The most general way is the metric-affine $f(R)$ gravity, where we use the palatini formalism, abandoning the assumption that the matter action is independent of connections. Sotiriou [88] constructed a model by considering palatini formalism in which he investigated the relationship between scalar-tensor theory and $f(R)$ theories of gravity. Hwang et al. [89]

investigated the model in which they considered the Large-scale cosmic structure formation due to inflation based on a pure $f(R)$ gravity theory. Brax et al. [90] analyzed the modification of Einstein's gravity by $f(R)$ gravity as a dark energy model by introducing their connections with chameleon theories. The $f(R)$ gravity theory provides us with the existence of a strong coupling of the scalar field to matter. Dunsby et al. [91] constructed Λ CDM models to explore the reconstruction of $f(R)$ modified theories of gravity by investigating FLRW expansion history. Azadi et al. [92] investigated the static cylindrically symmetric solutions in the presence of Weyl's coordinates in the framework of $f(R)$ modified theory of gravity. Li et al. [93] explored the cosmologies that were raised in a subclass of $f(R)$ gravity, also constructed the model in which they derived the cosmological perturbation equations in the covariant form within the framework of $f(R)$ gravity

1.9.3 $f(R, T)$ Gravity

The recent observational data based on the universe's late-time acceleration and the existence of DM have proposed a theoretical challenge to gravity theories. There is a possibility of explaining the observations based on the assumption that at a large scale, Einstein's gravity model of GTR breaks down, and more general action expresses the gravitational field. Harko [95] in 2008 made an extension of this model to the case of arbitrary coupling in both geometry and matter. The $f(R, T)$ **modified theory of gravity** will be considered as another extension of GTR. The gravitational Lagrangian is denoted by an arbitrary function of the Ricci scalar R and the stress-energy tensor is denoted by T . The $f(R, T)$ gravity model depends on a source term, which represents the variation of the matter stress-energy tensor concerning the metric. The gravitational field equation of $f(R, T)$ gravity is given in the following section.

1.9.3.1 Field equations of $f(R, T)$ gravity

The action for $f(R, T)$ gravity is proposed as

$$S = \int \sqrt{-g} \left(\frac{1}{16\pi G} f(R, T) + L_m \right) d^4x \quad (1.53)$$

where the gravitational Lagrangian consists of an arbitrary function of Ricci scalar R and T is the trace of the energy momentum tensor $T_{\mu\nu}$ of the matter source. L_m is the matter lagrangian density and G is gravitational constant. For the matter source, the stress energy tensor $T_{\mu\nu}$ is given as follows

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}L_m)}{\delta g^{\mu\nu}} \quad (1.54)$$

and its trace is $T = g^{\mu\nu}T_{\mu\nu}$. Here, we assumed that the matter lagrangian L_m depends only on the metric tensor component $g_{\mu\nu}$ rather than its derivatives. Therefore, we get

$$T_{\mu\nu} = g_{\mu\nu}L_m - \frac{\partial L_m}{\partial g^{\mu\nu}} \quad (1.55)$$

Now, by varying the action S in (1.53) with respect to $g_{\mu\nu}$, the $f(R, T)$ gravity field equation are obtained as

$$f_R(R, T)R_{\mu\nu} - \frac{1}{2}f(R, T)g_{\mu\nu} + (g_{\mu\nu}\square - \nabla_\mu \nabla_\nu)f_R(R, T) = 8\pi T_{\mu\nu} - f_T(R, T)T_{\mu\nu} - f_T(R, T)\Theta_{\mu\nu} \quad (1.56)$$

where

$$\Theta_{\mu\nu} = -2T_{\mu\nu} + g_{\mu\nu}L_m - 2g^{\ln} \frac{\partial^2 L_m}{\partial g^{\mu\nu} \partial g^{\ln}} \quad (1.57)$$

Here, $f_R(R, T) = \frac{\partial f(R, T)}{\partial R}$, $f_T(R, T) = \frac{\partial f(R, T)}{\partial T}$, $\square \equiv \nabla^\mu \nabla_\mu$ where ∇_μ is the covariant derivative. From the (1.56) we get

$$f_R(R, T)R + 3\square f_R(R, T) - 2f(R, T) = (8\pi - f_T(R, T))T - f_T(R, T)\Theta \quad (1.58)$$

where $\Theta = g^{\mu\nu}\Theta_{\mu\nu}$. Therefore from equation (1.56) and (1.58), The $f(R, T)$ gravity field equation takes the form

$$f_R(R, T)(R_{\mu\nu} - \frac{1}{3}Rg_{\mu\nu}) + \frac{1}{6}f_R(R, T)g_{\mu\nu} = 8\pi(T_{\mu\nu} - \frac{1}{3}Tg_{\mu\nu}) - f_T(R, T)(T_{\mu\nu} - \frac{1}{3}Tg_{\mu\nu}) - f_T(R, T)(\Theta_{\mu\nu} - \frac{1}{3}\Theta g_{\mu\nu} + \nabla_\mu \nabla_\nu f_R(R, T)) \quad (1.59)$$

Since different cosmological model of $f(R, T)$ gravity are possible depending on the nature of the matter source.

Harko et al. [96] formulated three types of frames of $f(R, T)$ gravity and their corresponding field equations are as follows

1.9.3.2 $f(R, T) = R + 2f(T)$

For an arbitrary choice of function $f(T)$, the field equation from general equation (1.56) takes the form

$$G_{\mu\nu} = 8\pi T_{\mu\nu} - 2f'(T)T_{\mu\nu} - 2f'(T)\Theta_{\mu\nu} + f(T)g_{\mu\nu} \quad (1.60)$$

For a perfect fluid matter source $\Theta_{\mu\nu} = -2T_{\mu\nu} - pg_{\mu\nu}$. The field equation will be written as

$$G_{\mu\nu} = 8\pi T_{\mu\nu} + 2f'(T)T_{\mu\nu} + (2pf'(T) + f(T))g_{\mu\nu} \quad (1.61)$$

where the prime stand for the derivative with respect to argument.

1.9.3.3 $f(R, T) = f_1(R) + f_2(T)$

With the choice of both arbitrary functions of R , $f_1(R)$, and function of T , $f_2(T)$, the equation (1.56) will be written as

$$\begin{aligned} f_1'(R)R_{\mu\nu} - \frac{1}{2}f_1(R)g_{\mu\nu} + (g_{\mu\nu}\square - \nabla_\mu\nabla_\nu)f_1(R) &= 8\pi T_{\mu\nu} - f_2'(T)T_{\mu\nu} \\ &- f_2'(T)\Theta_{\mu\nu} + \frac{1}{2}f_2(T)g_{\mu\nu} \end{aligned} \quad (1.62)$$

Similarly, for perfect fluid source,

$$\begin{aligned} f_1'(R)R_{\mu\nu} - \frac{1}{2}f_1(R)g_{\mu\nu} + (g_{\mu\nu}\square - \nabla_\mu\nabla_\nu)f_1(R) &= 8\pi T_{\mu\nu} + f_2'(T)T_{\mu\nu} \\ &+ \left(f_2'(T)p + \frac{1}{2}f_2(T)\right)g_{\mu\nu} \end{aligned} \quad (1.63)$$

1.9.3.4 $f(R, T) = f_1(R) + f_2(R)f_3(T)$

The field equation for the third generalized $f(R, T)$ gravity matter source is

$$\begin{aligned} & (f'_1(R) + f'_2(R)f_3(T))R_{\mu\nu} - \frac{1}{2}f_1(R)g_{\mu\nu} + (g_{\mu\nu}\square - \nabla_\mu\nabla_\nu)(f'_1(R) + f'_2(R)f_3(T)) \\ &= 8\pi T_{\mu\nu} + f_2(R)f'_3(T)T_{\mu\nu} - f_2(R)f'_3(T)\Theta_{\mu\nu} + \frac{1}{2}f_2(R)f_3(T)g_{\mu\nu} \end{aligned} \quad (1.64)$$

Similarly, for perfect fluid source,

$$\begin{aligned} & (f'_1(R) + f'_2(R)f_3(T))R_{\mu\nu} - \frac{1}{2}f_1(R)g_{\mu\nu} + (g_{\mu\nu}\square - \nabla_\mu\nabla_\nu)(f'_1(R) + f'_2(R)f_3(T)) \\ &= 8\pi T_{\mu\nu} + f_2(R)f'_3(T)T_{\mu\nu} + f_2(R)\left(f'_3(T)p + \frac{1}{2}f_3(T)\right)g_{\mu\nu} \end{aligned} \quad (1.65)$$

1.9.4 Rastall Gravity

Peter Rastall developed a theory of gravity that includes the Super-Newtonian theory, and the gravitational field is introduced in terms of two fundamental functions, which is known as the ***Rastall gravity theory***. Rastall's theory of gravity is based on the non-conservation of the energy-momentum tensor. At a late time, the background of the Hubble parameter in Rastall gravity theory shows similar results compared to the Λ CDM models. Visser [97] proposed that the Rastall gravity is equivalent to usual general relativity. Mello et al. [98] investigated Abelian-High strings that take quantum effects in curved space-time in the framework of Rastall gravity. Cruz et.al. [99] discussed some aspects of the thermodynamical consistency of the model in the framework of Rastall gravity. Babma et al. [100] discussed the thermodynamical aspects at the apparent horizon in the framework of Rastall gravity theory. Lobo et.al. [101] discussed the thermodynamical effects of black holes in the framework of Rastall gravity theory. At the cosmological level, the Rastall gravity theory gives fascinating results. Bronnikov et al. [102] constructed a static, spherically symmetric cosmological model with a scalar field in the context of Rastall Gravity theory. In Rastall's gravity theory, geometry and matter fields are coupled in a very nonminimal way. Abbas et al. [103] developed a cosmological model to investigate a solution of anisotropic compact stars in the context of Rastall gravity theory. Maurya et.al. [104] explored the stellar interior for the static self-gravitating system through gravitational decoupling in the background

of Rastall gravity. The wormhole solutions with the phantom regime through conformal symmetry were discovered by Javed et al. [105] in the framework of Rastall gravity theory. Oliveira et al. [106] investigated the static and spherically symmetric solution to describe neutron stars in the framework of Rastall gravity theory. Ali et.al. [107] considered the model to describe the tunneling radiation and quantum gravity effect on tunneling radiation due to the regular black hole in the context of Rastall gravity. Spallucci et al. [108] investigated the solution for gaussian matter distribution in the framework of Rastall gravity theory. The expressions in the Rastall gravity are characterised as follows

$$R_{\mu\nu} - \frac{\lambda}{2}g_{\mu\nu}R = KT_{\mu\nu} \quad (1.66)$$

$$T^{\mu\nu};_{\mu} = \frac{1-\lambda}{2k}R^{;\nu} = \frac{1-\lambda}{2(1-2\lambda)}T^{;\nu} \quad (1.67)$$

where λ is a free parameter. Here we can notice that if we put $\lambda = 1$ we get the field equation Einstein's general relativity. Cosmologists developed the field equation of Rastall gravity theory in their own way as Mello et al. [98] proposed the field equation of Rastall gravity theory as

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{\gamma-1}{2}g_{\mu\nu}T \right) \quad (1.68)$$

The field equation in Rastall gravity theory with variable cosmological constant was formulated by Abbas et al. [103] as

$$G_{\mu\nu} + \beta g_{\mu\nu}R + \Lambda_r g_{\mu\nu} = 8\pi T_{\mu\nu} \left(\frac{4\beta-1}{6\beta-1} \right) \quad (1.69)$$

Also, Kumar et al. [109] represents the field equation of Rastall gravity as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa(T_{\mu\nu} - \lambda g_{\mu\nu}R) \quad (1.70)$$

it can be rewritten as

$$R_{\mu\nu} + \left(\psi - \frac{1}{2}\right)g_{\mu\nu}R = \kappa T_{\mu\nu} \quad (1.71)$$

where κ is the constant of gravitational coupling in Rastall theory of gravity and $\psi = \kappa\lambda$ is a Rastall coupling parameter.

1.10 Cosmological Parameters

The results of cosmological observational data have been studied by the researchers Riess et al. 1998 [110], Perlmutter et al. 1999 [111], Gold et al. 2009 and established theory about universe's accelerated expansion [112]. This late time accelerated expansion has been confirmed by high redshift type SN Ia supernovae by Riess et al. 1998 [110], Perlmutter et al. 1999 [111], Bennett et al. 2003[113]. Along with these experiments, observations such as CMB (cosmic microwave background) radiations by Spergel et al. 2003 and 2007 [36, 114] and large scale structure by Tegmark et al. 2004 [115] provided the evidence about the accelerated expansion of the universe. Cosmological parameters are very important to study because they are helpful for describing the different phenomenon of the cosmological models of the universe. We have presented some cosmological parameters, which are studied in this thesis. The standard procedure is to expand the scale factor $a(t)$ can be expanded in a Taylor series with respect to time t around the present time t_0 as [116, 117]

$$\begin{aligned} \mathcal{R}(t) = \mathcal{R}_0 & \left[1 + \frac{d\mathcal{R}}{dt} \Big|_{t_0} (t - t_0) + \frac{1}{2!} \frac{d^2\mathcal{R}}{dt^2} \Big|_{t_0} (t - t_0)^2 + \frac{1}{3!} \frac{d^3\mathcal{R}}{dt^3} \Big|_{t_0} (t - t_0)^3 \right. \\ & \left. + \frac{1}{4!} \frac{d^4\mathcal{R}}{dt^4} \Big|_{t_0} (t - t_0)^4 + \frac{1}{5!} \frac{d^5\mathcal{R}}{dt^5} \Big|_{t_0} (t - t_0)^5 + \frac{1}{6!} \frac{d^6\mathcal{R}}{dt^6} \Big|_{t_0} (t - t_0)^6 + \mathbf{o}((t - t_0)^7) \right] \end{aligned} \quad (1.72)$$

where we truncated the series at the sixth order in small change of time $\Delta t = t - t_0$. Here, we assume that $t - t_0 > 0$. As we know that the present scale factor $\mathcal{R}_0 = 1$ and putting this value in above eq.(1.72) can be written as

$$\begin{aligned} a(t) = 1 - H_0 \Delta t - \frac{q_0}{2} H_0^2 \Delta t^2 - \frac{j_0}{6} H_0^3 \Delta t^3 + \frac{s_0}{24} H_0^4 \Delta t^4 \\ - \frac{l_0}{20} H_0^5 \Delta t^5 + \frac{m_0}{720} H_0^6 \Delta t^6 + \mathbf{o}(\Delta t^7). \end{aligned} \quad (1.73)$$

The coefficients in the expansion are called cosmographic coefficients. Proper understanding of these coefficients provides a better geometrical description of the model

proposed. A set of geometrical parameters can be defined from the expansion coefficients that involve derivatives of the scale factor at any time t :

$$H = \frac{1}{\mathcal{R}} \frac{d\mathcal{R}}{dt} = \frac{\dot{\mathcal{R}}}{\mathcal{R}}, \quad (1.74)$$

$$q = -\frac{1}{\mathcal{R}H^2} \frac{d^2\mathcal{R}}{dt^2} = -\frac{\mathcal{R}\ddot{\mathcal{R}}}{\dot{\mathcal{R}}^2}, \quad (1.75)$$

$$j = \frac{1}{\mathcal{R}H^3} \frac{d^3\mathcal{R}}{dt^3} = \frac{\ddot{\mathcal{R}}}{\mathcal{R}H^3}, \quad (1.76)$$

$$s = \frac{1}{\mathcal{R}H^4} \frac{d^4\mathcal{R}}{dt^4} = \frac{\mathcal{R}^{(4)}}{\mathcal{R}H^4}, \quad (1.77)$$

$$l = \frac{1}{\mathcal{R}H^5} \frac{d^5\mathcal{R}}{dt^5} = \frac{\mathcal{R}^{(5)}}{\mathcal{R}H^5}, \quad (1.78)$$

$$m = \frac{1}{\mathcal{R}H^6} \frac{d^6\mathcal{R}}{dt^6} = \frac{\mathcal{R}^{(6)}}{\mathcal{R}H^6}, \quad (1.79)$$

where $\mathcal{R}^{(4)}, \mathcal{R}^{(5)}, \mathcal{R}^{(6)}$ are the fourth, fifth, sixth-order derivative of the scale factor with respect to cosmic time. H and q are the Hubble parameter and the deceleration parameter; j and s are called the jerk parameter and snap parameter; l and m are the lerk parameters and Om diagnostic parameter respectively, and form the state finder pairs that involve higher-order derivatives of the scale factor. These parameters pair can be a better tool to describe a model.

1.10.1 Hubble Parameter

Let us consider a galaxy far away from an observer, which is located at $r = 0$. If the radial coordinate distance of the galaxy is D , the proper distance will be given by

$$d_p = \mathcal{R}(t)D, \quad (1.80)$$

where $\mathcal{R}(t)$ is the scale factor of the universe. The velocity v of the galaxy relative to the observer will be

$$v = \frac{d}{dt}(d_p) = \frac{\dot{\mathcal{R}}}{\mathcal{R}}(\mathcal{R}D) = Hd_p, \quad (1.81)$$

where H is called the Hubble parameter from Eq.(1.74) is $H = \frac{\dot{\mathcal{R}}}{\mathcal{R}}$. The overhead dot denotes the differentiation with respect to time(t). It's present value is called the

H_0 determination [$km s^{-1} Mpc^{-1}$]	Number of clusters	Reference
$76.9^{+3.9+10.0}_{-3.4-8.0}$	38	[118]
66.9^{+11+9}_{-10-8}	5	[119]
67^{+30+15}_{-18-6}	7	[120]
69 ± 8	3	[121]
66^{+14+15}_{-11-15}	7	[122]
60^{+4+13}_{-4-18}	18	[123]

TABLE 1.1: Some measurements of H_0

Hubble constant and generally denoted as H_0 . Hubble constant is a measure of the current expansion rate of the universe. The SI unit of H_0 is s^{-1} but frequently used one is $km s^{-1} Mpc^{-1}$. The inverse of Hubble constant is called the Hubble time, an estimate of the age of the universe i.e.

$$t_H(\text{Hubble time}) = \frac{1}{H_0}. \quad (1.82)$$

The speed of light (c) multiplied by the Hubble time (t_H) provides the Hubble length or Hubble distance. i.e.

$$\text{Hubble length or Hubble distance} = ct_H = \frac{c}{H_0}. \quad (1.83)$$

Hubble length is a unit of distance in cosmology, which measure the size of observable universe. On the value of H , the following type of expansion is exhibited by our universe..

1. $H = 0$: static universe.
2. $H = \text{constant}$:exponential scale factor(de sitter universe).
3. $H > 0$:expanding universe.
- 4 $H \in (-h, h)$ through 0: bouncing Universe. Here h is a positive number.

The basic principle of the Sunyaev-Zeldovich (S-Z) method,some measurements of H_0 using the S-Z effect [124]. Some pioneer authors suggest the different measurements of H_0 by observe the number of cluster samples. The details are given in the table (1.1).

It is worth to mention here that, despite the success of Λ CDM model, it suffers from the well known H_0 tension arises out of the variation between the local distance ladder measurement of the Hubble parameter by Reiss et al. yielding $H_0 = 74.03 \pm 1.42 \text{ km s}^{-1} \text{ Mpc}^{-1}$ [125, 126] and an derived measurement depends on Λ CDM and the CMB temperature from Planck collaboration providing a value of $H_0 = 67.36 \pm 0.54 \text{ km s}^{-1} \text{ Mpc}^{-1}$ [127]. While recent determination from SH0ES provides $H_0 = 73.5 \pm 1.4 \text{ km s}^{-1} \text{ Mpc}^{-1}$ [128], H0LiCOW collaboration estimates $H_0 = 73.3_{-1.8}^{+1.7} \text{ km s}^{-1} \text{ Mpc}^{-1}$ [129]. The statistical significance of the H_0 tension is $\geq 4\sigma$. Till date there is no satisfactory explanation for the tension arising on the systematic errors. In the recent time, a bulk viscous anisotropic universe have been investigated by Tripathy et al. and compiled its parametric model with recent $H(z)$ and Pantheon compilation data by applying cosmic chronometric technique and evaluate the present value of Hubble's constant as $H_0 = 69.39 \pm 1.54 \text{ km s}^{-1} \text{ Mpc}^{-1}$, $70.016 \pm 1.65 \text{ km s}^{-1} \text{ Mpc}^{-1}$ and $69.36 \pm 1.42 \text{ km s}^{-1} \text{ Mpc}^{-1}$ [130]. In the year 2009 Xu et al. discussed the three different ranges observational values for Hubble parameter: $H_0 = 0.716 \pm 0.053$, $0.709_{-0.053}^{+0.053}$ and $0.716_{-0.053}^{+0.053}$ [131].

1.10.2 Deceleration Parameter

The deceleration parameter is generally denoted by q , which indicates the rate at which the expansion of universe is slowing down. It is defined from Eq.(1.75) as

$$q = -\frac{1}{\mathcal{R}H^2} \frac{d^2\mathcal{R}}{dt^2} = -\frac{\mathcal{R}\ddot{\mathcal{R}}}{\dot{\mathcal{R}}^2}.$$

It is useful to classify the cosmological models in the sense that, accelerating model or decelerating model. For $q > 0$ the cosmological model is a decelerating one where as for $q < 0$, the cosmological model is accelerating. The Eq.(1.74) indicates that, any expanding Universe should have a decreasing Hubble parameter. Depending on the value of q , we observed the following types of universe expansion [132, 133].

1. $q < -1$: (Super exponential expansion).
2. $-1 \leq q < 0$: (exponential expansion).
3. $q = 0$: (expansion with constant rate).
4. $-1 < q < 1$: (accelerating power expansion).

5. $q > 0$: (decelerating expansion).

6. $q = -1$: (de-Sitter expansion).

Recent observations have declared that the universe's rate of expansion is currently accelerating due to the effect of DE. In 2009 Xu et al. discussed the three different ranges observational values for deceleration parameter: $q_0 = -0.715 \pm 0.045$, $-0.658^{+0.061}_{-0.057}$ and $+0.461^{+0.031}_{-0.033}$ [131]. In the year 2009 Gruber et al. discussed the six different ranges observational values for deceleration parameter : $q_0 = -0.528^{+0.092}_{-0.088}$, $-0.683^{+0.084}_{-0.105}$, $-0.658^{+0.098}_{-0.098}$, $-0.069^{+0.051}_{-0.055}$, -0.513 and $-0.561^{+0.055}_{-0.042}$ [134]. Capozziello and Izzo used 27 GRBs to derive the values of the cosmographic parameters and found $q_0 = -0.78 \pm 0.20$ [135]. Wang et al. used the 192 supernovae (SNe Ia) and 69 high redshift gamma-ray bursts (GRBs), find the best fit parameter is $q_0 = -0.85 \pm 0.19$ and result is consistent with the flat Λ CDM model [136].

1.10.3 Jerk Parameter

Galileo theoretically argued that the speed of a falling body increases in proportion to distance, determined by the law given by empirical, kinematic measurement [137]. The cosmic jerk is a phenomenon where the universe transitions from deceleration to acceleration. Modern cosmology began with the universe's expansion discovery, quantified by the Hubble parameter can be written in the form of scale factor from Eq.(1.74) as $H(t) = \frac{\dot{\mathcal{R}}}{\mathcal{R}}$. The pursuits of observational cosmology was dominated by the measurement of the deceleration parameter q_0 , where deceleration parameter can be written in the form of scale factor from Eq.(1.75) as $q(t) = \frac{-\ddot{\mathcal{R}}\mathcal{R}}{\dot{\mathcal{R}}^2}$. In terms of the scale factor and Hubble parameter from Eq.(1.75) as $q = -\frac{1}{\mathcal{R}H^2} \frac{d^2\mathcal{R}}{dt^2}$. As the universe was decelerating and now it is accelerating, it is convenient to consider the third derivative of $\mathcal{R}$. The dimensionless jerk is given as [138]

$$j = \frac{\ddot{\mathcal{R}}\mathcal{R}^2}{\dot{\mathcal{R}}^3}, \quad (1.84)$$

While the current value is denoted by j_0 . Although some authors eschew the dynamics, j can be explained in fluid terms through the easily derived relation $j = 1 - \frac{4\pi\dot{P}}{H^3}$, where P is the pressure, ρ the total energy density, and H is the Hubble parameter while they ignore the spatial curvature

In the year 2009 Xu et al. discussed the three different ranges observational values for jerk parameter: $j_0 = -2.196^{+0.254}_{-0.241}$, $-1.382^{+0.219}_{-0.225}$ and $-0.147^{+0.110}_{-0.094}$ [131]. Rapetti et al. find the observational value of jerk parameter $j = 2.16^{+0.81}_{-0.75}$ [139]. In the year 2009 Gruber et al. discussed the seven different ranges observational values for jerk parameter : $j_0 = 0.506^{+0.489}_{-0.428}$, $2.044^{+1.002}_{-0.705}$, $2.412^{+1.065}_{-0.978}$, $-0.955^{+0.228}_{-0.175}$, $-0.785^{+0.220}_{-0.208}$, $2.227^{+0.245}_{-0.237}$ and $0.999^{+0.346}_{-0.468}$ [134]. Recently Suresh kumar observed the constraints on the jerk parameter of power-law cosmology from $H(z)$ (measurement of the universe's expansion rate as a redshift's function, i.e., $H(z)$), SNe Ia and primordial big bang nucleosynthesis (BBN) observations and he discussed the $H(z)$ data constrain the jerk parameter as $j = -0.09^{+0.04}_{-0.03}$ while the SNe Ia data constraints on jerk parameter as $j = -0.09^{+0.03}_{-0.02}$ and both data($H(z)$ +SNe Ia) constraints on jerk parameter as: $j = -0.11^{+0.02}_{-0.01}$ but the primordial nucleosynthesis(BBN) restricts $j \geq 1.76$ [140]. Capozziello and Izzo used 27 GRBs to derive the values of the cosmographic parameters and found $j_0 = 0.62 \pm 0.86$ [135]. Wang et al. used the 192 supernovae (SNe Ia) and 69 high redshift gamma-ray bursts (GRBs),find the best fit parameter is $j_0 = 1.50 \pm 0.80$ and results is consistent with the flat Λ CDM model[136]. Tripathy et al. investigated some bouncing models in the framework of an extended gravity theory by using an exponential and a power law scale factors to derive the values of the cosmographic parameters and found three different ranges observational values for jerk parameter: $j = -3.7, -3, -2.39$ [117].

1.10.4 Snap parameter

The universe's expansion rate can be described in terms of the Hubble parameter, $H = \frac{\dot{\mathcal{R}}}{\mathcal{R}}$. As we know that the deceleration parameter q and the jerk parameter j are related to the second derivative and third derivative of the scale factor, and s is the so-called snap parameter, given by the relation of the fourth derivative of the scale factor. These quantities are defined by Eq.(1.75)-(1.78) and from Eq.(1.78) we can write the snap parameter as:

$$s = \frac{1}{\mathcal{R}H^4} \frac{d^4\mathcal{R}}{dt^4} = \frac{\mathcal{R}^{(4)}}{\mathcal{R}H^3} = \frac{1}{H^4} \frac{\ddot{\mathcal{R}}}{\mathcal{R}}. \quad (1.85)$$

Capozziello and Izzo used 27 GRBs to derive the values of the cosmographic parameters, found $s_0 = 8.32 \pm 12.16$ [135]. Wang et al. used the 192 supernovae (SNe Ia) and 69 high redshift gamma-ray bursts (GRBs), find the best fit parameter is $s_0 = 3.49 \pm 4.50$ and result is consistent with the flat Λ CDM model [136]. Tripathy et al. investigated some bouncing models in the framework of an extended gravity theory by using an exponential and a power law scale factors to derive the values of the cosmographic parameters and found three different ranges observational values for snap parameter: $s = 0.9, 0.89, 0.87$ [117]. Recently Suresh kumar observed the constraints on the snap parameter of power-law cosmology from $H(z)$ (measurement of the universe's expansion rate as a redshift's function, i.e., $H(z)$), SNe Ia and BBN observational data, and he discussed the $H(z)$ data compel the snap parameter as $s = 0.58^{+0.04}_{-0.12}$ while the SNe Ia data constraints on snap parameter as $s = 0.41^{+0.03}_{-0.03}$ and both data ($H(z)$ +SNe Ia) constraints on snap parameter as: $s = 0.44^{+0.03}_{-0.03}$ but the primordial nucleosynthesis (BBN) restricts $s \geq 1.15$ [140]. The Monte Carlo Markov (MCM) Chain analysis is used by Aviles et al. to compare the compilation data of supernova Ia Union 2, aligned with the Hubble Space Telescope (HST) calculations of the Hubble constant, and the parametric data sets of Hubble parameter by considering cosmographic series (CS merged in three sets (A,B,C) with distinct maximum parameter's order:

$$\begin{aligned} A &= [H_0, q_0, j_0, s_0], \\ B &= [H_0, q_0, j_0, s_0, l_0], \\ C &= [H_0, q_0, j_0, s_0, l_0, m_0], \end{aligned}$$

along with each parameter set has two sets of observations, i.e., observations of set 1 is Union 2 + HST and observations of set 2 is Union 2 + HST + $H(z)$ and suggests different fit values of s are $2.59^{+60.61}_{-10.45}, 0.16^{+1.45}_{-1.03}, -5.59^{+33.74}_{-34.55}, -1.47^{+4.20}_{-10.72}, -25.52^{+65.60}_{-10.90}, -7.71^{+14.77}_{-7.83}, 1.21^{+61.24}_{-2.24}, -88.91^{+57.62}_{-11.08}, -180.95^{+331.75}_{-18.93}, -112.63^{+156.60}_{-82.53}, -30.97^{+90.96}_{-43.47}, -16.95^{+73.68}_{-38.79}, 9.85^{+74.69}_{-26.69}, 3.151^{+3.920}_{-1.771}, -37.67^{+80.51}_{-60.10}, 5.278^{+13.076}_{-14.732}, -13.48^{+71.65}_{-31.28}, 2.179^{+42.126}_{-35.919}$ [116].

1.10.5 Om diagnostic

In this section, we have used another concept for differentiation between the standard Λ CDM model and the dark energy models [141]. This approach has been developed to examine the dynamical scenario of the dark energy models by linking the redshift z , and the geometric parameter H , and it is called *Om diagnostic* [141–143]. The Om diagnostic can distinguish among various dark energy models without concern for the exact present value of the matter density parameter and without involving the EoS parameter. It is quite simple to formulate the Om diagnostic as it is a single parameter evaluation technique. The Om diagnostic is defined as follows [142, 143]

$$Om(z) = \frac{\left(\frac{H(z)}{H_0}\right)^2 - 1}{(1+z)^3 - 1},$$

$$\Rightarrow Om(z) = \frac{\left(\frac{H(z)}{H_0}\right)^2 - 1}{z(z^2 + 3z + 3)}. \quad (1.86)$$

The Om diagnostic can be expressed as the scale factor's sixth-order derivative concerning cosmic time as from Eq.(1.79) we can write

$$Om(z) = \frac{1}{\mathcal{R}H^6} \frac{d^6 \mathcal{R}}{dt^6}. \quad (1.87)$$

The present values of Hubble parameter (H_0) and energy density parameter of matter Ω_{0m} for a constant EoS parameter is expressed as

$$H^2(z) = H_0^2 [\Omega_{0m}(1+z)^3 + (1 - \Omega_{0m})(1+z)^{3(1+\omega)}]. \quad (1.88)$$

The expression for $Om(z)$ corresponding to Eq.(1.88) is written as

$$Om(z) = \Omega_{0m} + (1 - \Omega_{0m}) \left(\frac{(1+z)^{3(1+\omega)} - 1}{z(z^2 + 3z + 3)} \right). \quad (1.89)$$

From Eq.(1.89), one can notice that, $Om(z) = \Omega_{0m}$ for $\omega = -1$ (Λ CDM), whereas $Om(z) > \Omega_{0m}$ for $\omega > -1$ (quintessence model) and $Om(z) < \Omega_{0m}$ for $\omega < -1$ (phantom model).

The distinct behavior of the dark energy models from Λ CDM is based on the slope of the $Om(z)$ diagnostic. Agarwal et al. discussed the three different evolution of $Om(z)$ diagnostic in comparison with redshift z for various DE models such as $\omega = -1$ (Λ CDM),

-1.4 (phantom model), -0.6 (quintessence model) [144]. The Monte Carlo Markov (MCM) Chain analysis is used by Aviles et al. to compare the compilation data of supernova Ia Union 2, aligned with the Hubble Space Telescope (HST) calculations of the Hubble constant, and the parametric data sets of Hubble parameter by considering cosmographic series(CS merged in three sets(A,B,C) with distinct maximum parameter's order:

$$\begin{aligned} A &= [H_0, q_0, j_0, s_0], \\ B &= [H_0, q_0, j_0, s_0, l_0], \\ C &= [H_0, q_0, j_0, s_0, l_0, m_0], \end{aligned}$$

along with each parameter set has two sets of observations, i.e., observations of set 1 is Union 2 + HST and observations of set 2 is Union 2 + HST + $H(z)$ and suggests different fit values of $Om(z)$ are $71.93^{+382.17}_{-315.76}$, $70.9^{+2497.8}_{-2254.5}$, $-8.827^{+2.263}_{-2.941}$ [116].

1.10.6 Equation of State

The equation of state (EoS) of a perfect fluid is characterized by a dimensionless quantity ω . The EoS is given by the ratio of its pressure p to its energy density ρ

$$\omega = \frac{p}{\rho}. \quad (1.90)$$

The EoS of DE indicates cosmic inflation and the universe's accelerated expansion. The EoS of the cosmological constant is $\omega = -1$, and for this case, the above scale factor's expression is not valid, and

$$\mathcal{R} \propto e^{Ht}, \quad (1.91)$$

where H is the Hubble parameter. In general, the universe's accelerated expansion for any EoS $\omega < -\frac{1}{3}$. The hypothetical phantom energy would have an EoS $\omega < -1$, and would cause a Big Rip. According to existing observational data, it is complicated to distinguish between phantom $\omega < -1$ and non-phantom $\omega \geq -1$. The origin of the flatness problem curvature has $\omega = -\frac{1}{3}$ and monopole problems of the Big Bang

curvature have $\omega = 0$. Cosmic inflation solved these types of problems where $\omega \approx -1$. The EoS calculations of DE is one of the greatest endeavour of observational cosmology. The following type of expansion is exhibited, which depends on the value of ω are given as follows:

1. $\omega < -1$: phantom behaviour.
2. $\omega \geq -1$: non-phantom behaviour.
3. $\omega = -\frac{1}{3}$: radiation dominating models.
4. $\omega > -\frac{1}{3}$: accelerating behaviour /decelerating behaviour.
5. $\omega < -\frac{1}{3}$: accelerating behaviour.
6. $\omega = 0$: dust models.
7. $\omega = -1$: dark energy models.
8. $-1 \leq \omega \leq -\frac{1}{3}$: quintessence behaviour.

In the year 2009 Gruber et al. discussed the six different ranges observational values for equation of state (ω) parameter [134]: $\omega_0 = -0.685^{+0.061}_{-0.059}$, $-0.789^{+0.056}_{-0.07}$, $-0.772^{+0.065}_{-0.065}$, $-0.380^{+0.034}_{-0.036}$, -0.675 and $-0.707^{+0.037}_{-0.028}$. The Monte Carlo Markov (MCM) Chain analysis is used by Aviles et al. to compare the compilation data of supernova Ia Union 2, aligned with the Hubble Space Telescope (HST) calculations of the Hubble constant, and the parametric data sets of Hubble parameter by considering cosmographic series(CS) combined together in three redshifts, using set 2 of observations (Union 2 + HST + $H(z)$) and suggests different fit values of ω are $-0.7174^{+0.0922}_{-0.0964}$, $-0.7439^{+0.3085}_{-0.3222}$, $-0.7315^{+0.1193}_{-0.1373}$ [116].

1.10.7 Expansion, Shear and Vorticity

Let us start with a family of observers with non-intersecting world lines. The time-like 4-velocity vector which is tangent to these world lines is $u^\mu = \frac{dx^\mu}{d\tau}$ and satisfies $u^\mu u_\mu = 1$. Here τ denotes the proper time measured along the world lines. The space-time metric can be expressed the longitudinal ($-u^\mu u_\nu$) and transverse parts ($h_{\mu\nu}$) as

$$h_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu, \quad (1.92)$$

which projects the space-time orthogonal to the 4-velocity into the observers instantaneous rest space at each event. $u_\mu = (1; 0; 0; 0)$ be the velocity of a co-moving particle.

It follows that

$$h_\mu^\gamma h_\gamma^\nu = h_\mu^\nu, h_\mu^\nu u_\mu = 0, h_\mu^\mu = 3.$$

The vector field u_μ and its associated tensor counterpart $h_{\mu\nu}$ allows for a unique decomposition of every space-time quantity into its irreducible time-like and space-like parts [145]. The world line's evolution would be characterized by the covariant derivative of the velocity field. Further, it will be split into its fundamental parts, which are defined by their symmetric properties:

$$u_{\nu;\mu} = \sigma_{\mu\nu} + \omega_{\mu\nu} + \frac{1}{3}\Theta h_{\mu\nu} - u_\mu \dot{u}_\nu. \quad (1.93)$$

Here $\sigma_{\mu\nu}$ is trace-free symmetric, $\omega_{\mu\nu}$ is skew-symmetric and $u^\mu \sigma_{\mu\nu} = 0 = u^\mu \omega_{\mu\nu}$. It follows that

$$\begin{aligned} \Theta &= u^\mu_{;\mu}, \\ \dot{u}_\mu &= u^\mu_{;\nu} u^\nu, \\ \sigma_{\mu\nu} &= u_{(\mu;\nu)} - \frac{1}{3}\Theta h_{\mu\nu} - \dot{u}_{(\mu} u_{\nu)} = \frac{1}{2}(u_{\nu;\mu} + u_{\mu;\nu}), \\ \omega_{\mu\nu} &= u_{[\mu;\nu]} - \dot{u}_{[\mu} u_{\nu]} = \frac{1}{2}(u_{\mu;\nu} - u_{\nu;\mu}). \end{aligned} \quad (1.94)$$

The scalar Θ is the rate of expansion scalar and $\dot{u}_\mu = u^\mu_{;\nu} u^\nu$ is the 4 acceleration vector. The shear tensor $\sigma_{\mu\nu}$ with magnitude σ is given by

$$\sigma^2 = \frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu},$$

which describes the distortion rate of the matter flow and $\omega_{\mu\nu}$ is the vorticity (rotation) tensor with magnitude ω is given as

$$\omega^2 = \frac{1}{2}\omega_{\mu\nu}\omega^{\mu\nu},$$

which describes the matter rotation relative to a non-rotational frame like a Fermi-propagated frame [146]. The governing equations of Θ , $\sigma^{\mu\nu}$ and $\omega_{\mu\nu}$ are the propagation equations which give the overall evolution of these quantities [147, 148]. Some times, the rate of expansion scalar $\Theta_{\mu\nu}$ is defined as the combination of rate of shear tensor and rate of expansion scalar because in cosmological context we usually replace $\Theta = 3H$.

Thus

$$\Theta_{\mu\nu} = \sigma_{\mu\nu} + \frac{1}{3}\Theta h_{\mu\nu}, \quad (1.95)$$

which determines the rate-of-change of distance of neighbouring particles (in our case, cluster of galaxies) in the fluid.

1.11 Scope of future work

In the future, more work within the modified theories of gravitation is required to fully understand how these solutions should be interpreted and what their physical effects could be. There are several directions in which future research on these subjects can proceed.

- The thesis examines the cosmological models of the universe using Lyra geometry, $f(R, T)$ gravity, and Rastall theory of gravitation.
- These models can also be studied in future using other theories, such as $f(R)$ gravity, $f(G)$ gravity, or $f(T, G)$ gravity.
- The thesis is based on cosmological models of first class $f(R, T)$ gravity, which may be extended to other classes.
- To study the dynamics of the universe in various aspects, the same models can also be regenerated with different ansatz equations or constraint equations.
- Furthermore, these models can be used to constrain parameters of our model in light of observational data, such as data from type Ia supernovae, HZ data and BAO data.

Chapter 2

Impact of Particle creation in Lyra geometry under different deceleration parameter

2.1 Introduction

Theoretical arguments for late-time cosmic acceleration are a major issue amongst the cosmologists of the twenty-first century. Most of the researchers in the field of cosmology have a curiosity about the behavior of the universe. In relativistic cosmological models, the study of particle creation has been done by many authors. There are two approaches to understand the physical concept of particle creation i) the technique of adiabatic vacuum state, and ii) the technique of instantaneous Hamiltonian diagonalization. Trunin [149] has discussed a toy model of particle production in de Sitter space with the Casimir effect. Garg and Pradhan [150] worked out the problem of gravitationally influenced particle creation in a $(d + 2)$ -dimensional homogeneous, isotropic, and flat FLRW model in the presence of a cosmological constant. They have obtained cosmological solutions under a periodic varying deceleration parameter with and without particle creation. In view of Lyra's geometry, Desikan and Das [151] obtained a new class of cosmological models in the presence of particle creation.

Singh and Beesham [152] have investigated a few properties of the Friedmann-Lemaitre-Robertson-Walker (FLRW) line element during the early universe with particle creation in general relativity. A gamma-law equation of state was used to get the solution of the field equations, and they obtained some exact expressions for the distance parameters. A higher-dimensional FLRW cosmology with variable gravitational parameter G and Λ was studied by Singh and Beesham [153] in the framework of open thermo-dynamical systems of particle creation. They found that in the process of the evolution of the universe, a decaying Λ and constant G always produce particle creation, but $\Lambda = 0$ and $G = \text{constant}$, or decaying Λ and varying G produce no particle creation.

A flat FLRW universe in the presence of particle creation was analyzed by Setare and Houndjo [154] in $f(T)$ gravity. The solution is derived by considering that the matter is minimally coupled. In general, a comparison in $f(R)$ and $f(T)$ gravity was presented in view of particle production. Pereira and Holanda [155] examined the cosmological particle creation process to super massive scalar particles in the context of $f(R) = R - \frac{\beta}{R^n}$ gravity. As compared to the standard model, modified gravity slightly increases the total particle number density as noticed by Pereira and Holanda [155]. Lemaitre-Tolman-Bondi (LTB) cosmology with back reaction to particle creation was examined by Zecca [156], and they used the separation of variable technique to obtain an analytical solution. A flat FLRW cosmological model was presented in the presence of particle creation by Singh [157]. He has derived the solution through the simple parameterization of the Hubble parameter in the form $H(t) = hn \tanh(m - nt) + c$, where h, n, m, c are constants. Singh et al. [158] have explored particle creation for a flat FLRW cosmology in a higher derivative theory and examined the cosmological models by taking different laws of variation of the scale factor, such as a hybrid expansion law (HEL), emergent law (EL) and intermediate expansion law (IEL). Further, Singh et al. [159] extended their study to particle creation under different forms of the scale factor and a quadratic equation of state.

Recently, Mandal and Sahoo [160] have examined an FLRW cosmological model in $f(T)$ gravity with particle creation. They presented three different cosmological models, depending on the different forms of $f(T)$, and pointed out that particle creation produces negative pressure, which may be responsible for the universe's accelerated expansion. Harko et al. [161] explored the concept of particle production through the

mechanism of non-minimal torsion–matter coupling. Also, they pointed out that particle production occurs in an expanding universe and there may exist a strong relationship between cosmological dynamics and irreversible thermodynamics, so that large levels of particle production trigger the transition from deceleration to acceleration.

In the following years, various surprising results have been obtained by several groups of astronomers to estimate the expansion rate of the universe. Akarsu and Dereli [162] introduced a linearly varying deceleration parameter (LVDP), which generalizes many exact solutions obtained through a constant deceleration parameter. Berman's law related to a constant deceleration parameter is also a particular case of the LVDP. Singh and Kumar [163] made an attempt to formulate a law for the Hubble parameter, which came up with a constant deceleration parameter for the Bianchi type-II space-time in GTR. They have obtained a new class of solutions for Bianchi type-II space-time and investigated four different physical models.

The most important problems related to standard cosmological models based on Riemannian geometry are the existence of a big-bang singularity and the accelerating phase of the universe. This is the major setback for the theory of general relativity. On the other hand, Lyra's geometry is capable of constructing cosmological models, which are not only free of the big-bang singularity but also exhibit acceleration during late-time evolution. Therefore, we have analysed the importance of particle creation in Lyra's geometry by taking the following quadratic deceleration parameter $q = (8n^2 - 1) - 12nt + 3t^2$ [164] and time-varying deceleration parameter $q = -\frac{\alpha_1}{t^2} + \alpha_2 - 1$ [165], and $q = \frac{\alpha(1-t)}{1+t}$ [166].

2.2 Field equations

Let us start with the FLRW metric

$$ds^2 = -dt^2 + \mathcal{R}^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right] \quad (2.1)$$

where $k = 1, -1$ or 0 . In the presence of creation of matter, the energy momentum tensor is given by

$$T_{\mu\nu} = (\rho + p + p_c)u_\mu u_\nu + (p + p_c)g_{\mu\nu} \quad (2.2)$$

where ρ and p are the energy density and pressure, respectively, p_c is the creation pressure, u_μ the fluid-four velocity vector and $g_{\mu\nu}$ is the metric tensor. The pressure p satisfies the equation of the state of the form

$$p = \gamma\rho, \quad -1 \leq \gamma \leq 1, \quad (2.3)$$

known as a barotropic equation of state. The field equations (1.47) with the help of equations (2.1) to (2.3) take the form

$$3H^2 + \frac{3k}{R^2} - \frac{3\beta^2}{4} = +\chi\rho \quad (2.4)$$

$$2\dot{H} + 3H^2 + \frac{k}{R^2} + \frac{3\beta^2}{4} = -\chi(p + p_c), \quad (2.5)$$

Here $H = \frac{\dot{R}}{R}$ is the Hubble parameter and $\chi = 8\pi G$. The above system leads to the continuity equation of the form

$$\chi\dot{\rho} + \frac{3}{2}\beta\dot{\beta} + 3\left[\chi(\rho + p + p_c) + \frac{3}{2}\beta^2\right]H = 0 \quad (2.6)$$

where the particle creation pressure is given by .

$$p_c = -(1 + \gamma)\frac{\rho}{N}\frac{dN}{dt}\frac{1}{3H} \quad (2.7)$$

Here $\frac{\dot{N}}{N}$ indicates the rate of change of particle number in a co-moving volume V . Secondly particular form of the source function is

$$\frac{\dot{N}}{N} = F(t) = aH, \quad (2.8)$$

where $a \geq 0$ is a constant. Now, by eliminating ρ from (2.4) and (2.5) we have

$$\chi p_c = -\left[2\dot{H} + 3(1 + \gamma)H^2 + (1 + 3\gamma)\frac{k}{R^2} + (1 - \gamma)\frac{3\beta^2}{4}\right] \quad (2.9)$$

Further, with the help of equations (2.4),(2.7),(2.8) and (2.9) we have

$$\beta^2 = -\frac{4}{a(1 + \gamma) + 3(1 - \gamma)}\left[2\dot{H} + (1 + \gamma)(3 - a)H^2 + (1 + 3\gamma - a(1 + \gamma))\frac{k}{R^2}\right] \quad (2.10)$$

Inserting the expression of β^2 in equations (2.4) and (2.9), the energy density and particle creation takes the form

$$\chi\rho = \frac{6}{(a-3)\gamma + (a+3)} \left[\dot{H} + 3H^2 + \frac{2k}{\mathcal{R}^2} \right] \quad (2.11)$$

$$\chi p_c = -\frac{2a(1+\gamma)}{(a-3)\gamma + (a+3)} \left[\dot{H} + 3H^2 + \frac{2k}{\mathcal{R}^2} \right] \quad (2.12)$$

The pressure of the derived model is expressed as

$$\chi p = \frac{6\gamma}{(a-3)\gamma + (a+3)} \left[\dot{H} + 3H^2 + \frac{2k}{\mathcal{R}^2} \right] \quad (2.13)$$

2.3 Exact solution of the field equation under varying deceleration parameters

2.3.1 Model-I

The system has two equations and four unknowns namely ρ , p_c , β and R . Thus, in order to obtain the unique solution we need two physically plausible relation among the variables. Hence, first we assume quadratic deceleration parameter of the form [164]

$$q = (8n^2 - 1) - 12nt + 3t^2 \quad (2.14)$$

The important information regarding the considered form of deceleration parameter is that, the universe begins and end with $q = 8n^2 - 1$. The acceleration phase ($q < 0$) and super exponential phase ($q < -1$) occurs in $t \in (2n - \frac{1}{3}\sqrt{12n^2 + 3}, 2n + \frac{1}{3}\sqrt{12n^2 + 3})$ and $t \in (2n - \sqrt{\frac{4n^2}{3}}, 2n + \sqrt{\frac{4n^2}{3}})$ respectively. It is interesting to note that, the presented models are non-singular for $q < -1$ and otherwise models are singular. One can observe that, the expression in equations (2.10) to (2.13) depends on the scale factor R . In order to obtain R and explicit form of the solution we have used the equation (2.14). The well know relation among Hubble parameter and deceleration parameter $q = \frac{d}{dt} \left(\frac{1}{H} \right) - 1$ is applied to equation (2.14) for the expression of Hubble parameter. The Hubble parameter and its derivative is

expressed as

$$H = \frac{\dot{\mathcal{R}}}{\mathcal{R}} = \frac{1}{t(t-2n)(t-4n)} \quad (2.15)$$

For sake of simplicity constant of integration is taken as unity.

$$\dot{H} = -\frac{3t^2 - 12tn + 8n^2}{t^2(t-2n)^2(t-4n)^2} \quad (2.16)$$

Integration of equation (2.15) leads to

$$\mathcal{R} = R_0 \frac{[t(t-4n)]^{\frac{1}{8n^2}}}{(t-2n)^{\frac{1}{4n^2}}}, \quad R_0 \text{ is the constant of integration.} \quad (2.17)$$

Incorporation of equations (2.15) to (2.17), the expression in (2.10) to (2.13) for the physical quantities take the form

$$\begin{aligned} \beta^2 = & -\frac{4}{a(1+\gamma) + 3(1-\gamma)} \left[\frac{-6t^2 + 24tn - 16n^2 - (1+\gamma)(-3+a)}{t^2(t-2n)^2(t-4n)^2} \right. \\ & \left. + (1+3\gamma - a(1+\gamma))k \frac{(t-2n)^{\frac{1}{2n^2}}}{R_0^2[t(t-4n)]^{\frac{1}{4n^2}}} \right] \end{aligned} \quad (2.18)$$

$$\begin{aligned} \rho = & \frac{6}{\chi[(a-3)\gamma + (a+3)]} \left[\frac{-3t^2 + 12tn - 8n^2 + 3}{t^2(t-2n)^2(t-4n)^2} + 2k \frac{(t-2n)^{\frac{1}{2n^2}}}{R_0^2[t(t-4n)]^{\frac{1}{4n^2}}} \right] \quad (2.19) \\ p_c = & -\frac{2a(1+\gamma)}{\chi[(a-3)\gamma + (a+3)]} \left[\frac{-3t^2 + 12tn - 8n^2 + 3}{t^2(t-2n)^2(t-4n)^2} + 2k \frac{(t-2n)^{\frac{1}{2n^2}}}{R_0^2[t(t-4n)]^{\frac{1}{4n^2}}} \right] \quad (2.20) \end{aligned}$$

The pressure of the derived model is expressed as

$$p = \frac{6\gamma}{\chi[(a-3)\gamma + (a+3)]} \left[\frac{-3t^2 + 12tn - 8n^2 + 3}{t^2(t-2n)^2(t-4n)^2} + 2k \frac{(t-2n)^{\frac{1}{2n^2}}}{R_0^2[t(t-4n)]^{\frac{1}{4n^2}}} \right] \quad (2.21)$$

Now as per the different values of γ we have different cosmological fluids. These are discussed in the subcases as follows

2.3.1.1 Dust filled Model

The solution of the dust filled universe is obtained by setting $\gamma = 0$ in equations (2.18) to (2.21). The physical quantities are as follows

$$\beta^2 = \begin{cases} -\frac{4}{a+3} \left[\frac{-6t^2+24tn-16n^2+3-a}{t^2(t-2n)^2(t-4n)^2} + (1-a) \frac{(t-2n)^{\frac{1}{2n^2}}}{R_0^2[t(t-4n)]^{\frac{1}{4n^2}}} \right], & k = 1 \\ -\frac{4}{a+3} \left[\frac{-6t^2+24tn-16n^2+3-a}{t^2(t-2n)^2(t-4n)^2} \right], & k = 0 \\ -\frac{4}{a+3} \left[\frac{-6t^2+24tn-16n^2+3-a}{t^2(t-2n)^2(t-4n)^2} - (1-a) \frac{(t-2n)^{\frac{1}{2n^2}}}{R_0^2[t(t-4n)]^{\frac{1}{4n^2}}} \right], & k = -1 \end{cases} \quad (2.22)$$

$$\rho = \begin{cases} \frac{6}{\chi(a+3)} \left[\frac{-3t^2+12tn-8n^2+3}{t^2(t-2n)^2(t-4n)^2} + 2 \frac{(t-2n)^{\frac{1}{2n^2}}}{R_0^2[t(t-4n)]^{\frac{1}{4n^2}}} \right], & k = 1 \\ \frac{6}{\chi(a+3)} \left[\frac{-3t^2+12tn-8n^2+3}{t^2(t-2n)^2(t-4n)^2} \right], & k = 0 \\ \frac{6}{\chi(a+3)} \left[\frac{-3t^2+12tn-8n^2+3}{t^2(t-2n)^2(t-4n)^2} - 2 \frac{(t-2n)^{\frac{1}{2n^2}}}{R_0^2[t(t-4n)]^{\frac{1}{4n^2}}} \right], & k = -1 \end{cases} \quad (2.23)$$

$$p_c = \begin{cases} -\frac{2a}{\chi(a+3)} \left[\frac{-3t^2+12tn-8n^2+3}{t^2(t-2n)^2(t-4n)^2} + 2 \frac{(t-2n)^{\frac{1}{2n^2}}}{R_0^2[t(t-4n)]^{\frac{1}{4n^2}}} \right], & k = 1 \\ -\frac{2a}{\chi(a+3)} \left[\frac{-3t^2+12tn-8n^2+3}{t^2(t-2n)^2(t-4n)^2} \right], & k = 0 \\ -\frac{2a}{\chi(a+3)} \left[\frac{-3t^2+12tn-8n^2+3}{t^2(t-2n)^2(t-4n)^2} - 2 \frac{(t-2n)^{\frac{1}{2n^2}}}{R_0^2[t(t-4n)]^{\frac{1}{4n^2}}} \right], & k = -1 \end{cases} \quad (2.24)$$

$$p = 0 \quad (2.25)$$

In this model the gauge function β^2 has the same expression for closed, flat and open

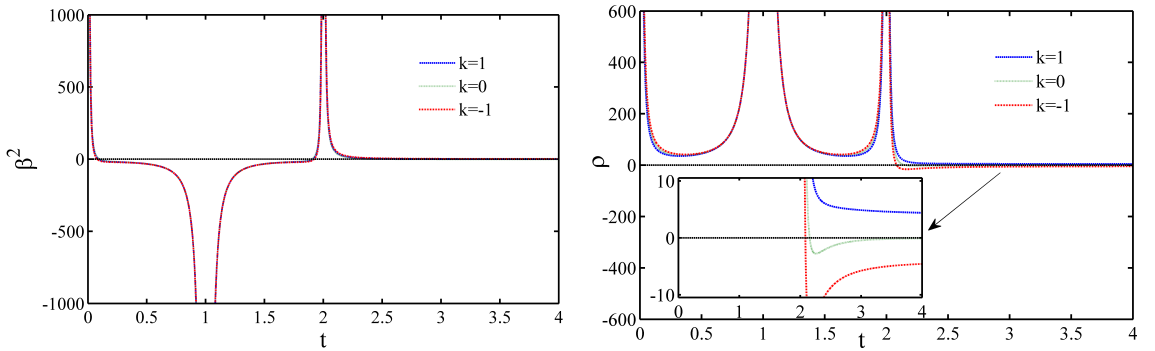


FIGURE 2.1: Profile for gauge function and energy density against cosmic time for flat, open and closed universe.

universe at $a = 1$. From the Figure 2.1 it is noticed that β^2 evolve with the singularities at $t = 0, 2n, 4n$. The singular point depends upon the choice of the parameter n . For this study we consider $n = 0.5$, thus the singularity occurs at $t = 0, 1, 2$. Further it is noticed that, for $t \in (0, 2)$, β^2 evolves with positive-negative-positive value and for $t > 2$, $\beta^2 > 0$ is a decreasing function of cosmic time and approaches to zero with the evolution of time in flat, open and closed universe.

Figure 2.1 (right) represents the profile of energy density against cosmic time. The energy density has singularity at $t = 0, 1, 2$ for $n = 0.5$. For $t \in (0, 2)$ the energy density is positive valued in flat, open and closed universe. For $t > 2$, positivity condition of energy density is satisfied only in closed universe where as it violates in flat and open universe. Figure 2.2 indicates the profile of the creation pressure against cosmic time.

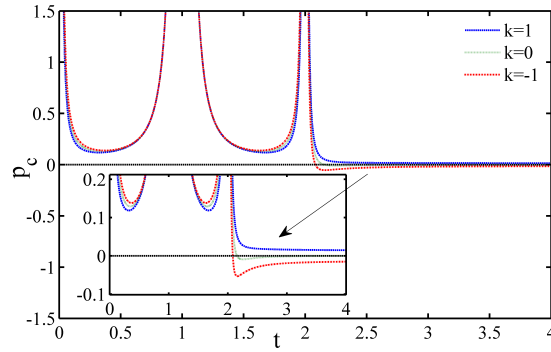


FIGURE 2.2: Profile for particle creation pressure against cosmic time for flat, open and closed universe.

The creation pressure p_c has also singularity at $t = 0, 1, 2$ for $n = 0.5$. It is positive valued for $t \in (0, 2)$ in flat, open and closed universe. Again for $t > 2$, $p_c > 0$ in closed universe where as $p_c < 0$ in flat and open universe.

2.3.1.2 Stiff Matter Model

The solution of the Stiff matter model is obtained by setting $\gamma = 1$ in equations (2.18) to (2.21). The physical quantities are obtained as follows

$$\beta^2 = \begin{cases} -\frac{4}{a} \left[\frac{-3t^2+12tn-8n^2+3-a}{t^2(t-2n)^2(t-4n)^2} + (2-a) \frac{(t-2n)^{\frac{1}{2n^2}}}{R_0^2[t(t-4n)]^{\frac{1}{4n^2}}} \right], & k = 1 \\ -\frac{4}{a} \left[\frac{-3t^2+12tn-8n^2+3-a}{t^2(t-2n)^2(t-4n)^2} \right], & k = 0 \\ -\frac{4}{a} \left[\frac{-3t^2+12tn-8n^2+3-a}{t^2(t-2n)^2(t-4n)^2} - (2-a) \frac{(t-2n)^{\frac{1}{2n^2}}}{R_0^2[t(t-4n)]^{\frac{1}{4n^2}}} \right], & k = -1 \end{cases} \quad (2.26)$$

$$p = \rho = \begin{cases} \frac{3}{\chi^a} \left[\frac{-3t^2+12tn-8n^2+3}{t^2(t-2n)^2(t-4n)^2} + 2 \frac{(t-2n)^{\frac{1}{2n^2}}}{R_0^2[t(t-4n)]^{\frac{1}{4n^2}}} \right], & k = 1 \\ \frac{3}{\chi^a} \left[\frac{-3t^2+12tn-8n^2+3}{t^2(t-2n)^2(t-4n)^2} \right], & k = 0 \\ \frac{3}{\chi^a} \left[\frac{-3t^2+12tn-8n^2+3}{t^2(t-2n)^2(t-4n)^2} - 2 \frac{(t-2n)^{\frac{1}{2n^2}}}{R_0^2[t(t-4n)]^{\frac{1}{4n^2}}} \right], & k = -1 \end{cases} \quad (2.27)$$

$$p_c = \begin{cases} -\frac{2}{\chi} \left[\frac{-3t^2+12tn-8n^2+3}{t^2(t-2n)^2(t-4n)^2} + 2 \frac{(t-2n)^{\frac{1}{2n^2}}}{R_0^2[t(t-4n)]^{\frac{1}{4n^2}}} \right], & k = 1 \\ -\frac{2}{\chi} \left[\frac{-3t^2+12tn-8n^2+3}{t^2(t-2n)^2(t-4n)^2} \right], & k = 0 \\ -\frac{2}{\chi} \left[\frac{-3t^2+12tn-8n^2+3}{t^2(t-2n)^2(t-4n)^2} - 2 \frac{(t-2n)^{\frac{1}{2n^2}}}{R_0^2[t(t-4n)]^{\frac{1}{4n^2}}} \right], & k = -1 \end{cases} \quad (2.28)$$

In this model it is observed that, the physical parameters like gauge function β^2 , energy density ρ , pressure p and particle creation pressure p_c have the singularity at $t = 0, 1, 2$ for $n = 0.5$. The gauge function β^2 has the same expression in flat, open and closed universe at $a = 2$. Figure 2.3 (left & right) indicates the profile of β^2 , ρ and p against cosmic time. One can notice from the Figure 2.3 (left) that $\beta^2 < 0$ for $t \in (0, 2)$ in flat, open and closed universe and for $t > 2$, $\beta^2 > 0$ in open and flat universe whereas $\beta^2 < 0$ in open universe.

For $t \in (0, 2)$ the energy density and pressure is positive valued in flat, open and closed universe. Again for $t > 2$, $\rho, p > 0$ in closed universe whereas $\rho, p < 0$ in open and flat universe (see Figure 2.3 (right)). Figure 2.4 indicates that, $p_c < 0$ for $t \in (0, 2)$ in flat, open and closed universe. Particle creation pressure is positive in both flat and open universe whereas $p_c < 0$ in closed universe for $t > 2$.

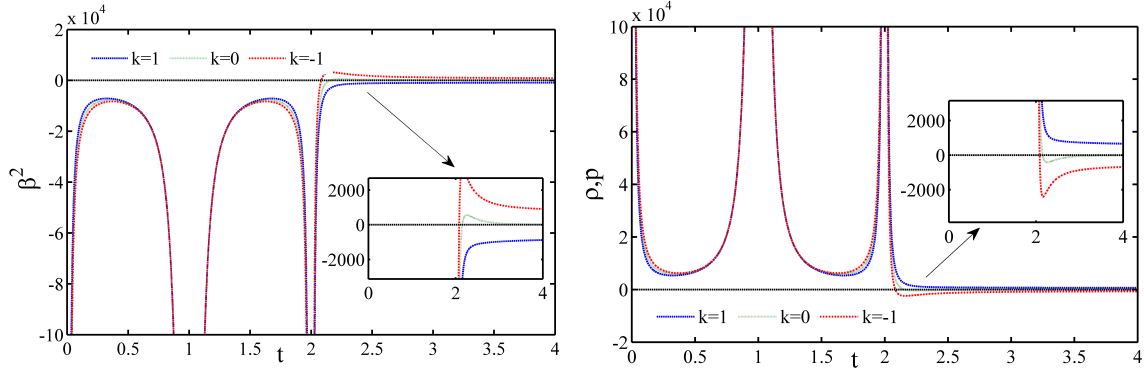


FIGURE 2.3: Profile for gauge function and energy density against cosmic time for flat, open and closed universe.

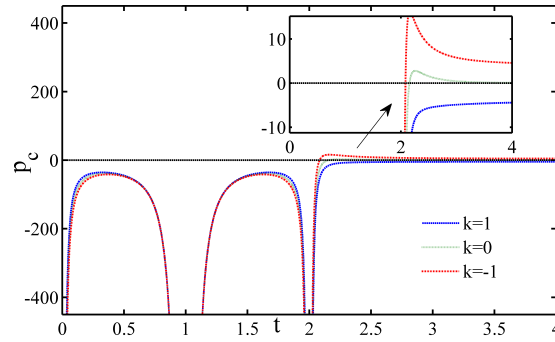


FIGURE 2.4: Profile for particle creation pressure against cosmic time for flat, open and closed universe.

2.3.1.3 Vacuum energy Model

The solution of the vacuum energy model is obtained by setting $\gamma = -1$ in equations (2.18) to (2.21). The physical quantities are as follows

$$\beta^2 = \begin{cases} -\frac{2}{3} \left[\frac{-6t^2 + 24tn - 16n^2}{t^2(t-2n)^2(t-4n)^2} \right] + \frac{4}{3} \frac{(t-2n)^{\frac{1}{2n^2}}}{R_0^2[t(t-4n)]^{\frac{1}{4n^2}}}, & k = 1 \\ -\frac{2}{3} \left[\frac{-6t^2 + 24tn - 16n^2}{t^2(t-2n)^2(t-4n)^2} \right], & k = 0 \\ -\frac{2}{3} \left[\frac{-6t^2 + 24tn - 16n^2}{t^2(t-2n)^2(t-4n)^2} \right] - \frac{4}{3} \frac{(t-2n)^{\frac{1}{2n^2}}}{R_0^2[t(t-4n)]^{\frac{1}{4n^2}}}, & k = -1 \end{cases} \quad (2.29)$$

$$\rho = -p = \begin{cases} \frac{1}{\chi} \left[\frac{-3t^2 + 12tn - 8n^2 + 3}{t^2(t-2n)^2(t-4n)^2} + \frac{2(t-2n)^{\frac{1}{2n^2}}}{R_0^2[t(t-4n)]^{\frac{1}{4n^2}}} \right], & k = 1 \\ \frac{1}{\chi} \left[\frac{-3t^2 + 12tn - 8n^2 + 3}{t^2(t-2n)^2(t-4n)^2} \right], & k = 0 \\ \frac{1}{\chi} \left[\frac{-3t^2 + 12tn - 8n^2 + 3}{t^2(t-2n)^2(t-4n)^2} - \frac{2(t-2n)^{\frac{1}{2n^2}}}{R_0^2[t(t-4n)]^{\frac{1}{4n^2}}} \right], & k = -1 \end{cases} \quad (2.30)$$

$$p_c = 0 \quad (2.31)$$

Here the physical quantities like gauge function, energy density and pressure is independent of parameter a . Thus the sign of the a does not affect the qualitative behavior of these physical quantities. The particle creation pressure is zero, which means that in this model no particle production takes place. Like the other models discussed above here also β^2, ρ and p have the singularity at $t = 0, 1, 2$ for $n = 0.5$.

The gauge function takes values positive-negative-positive for $t \in (0, 2)$ in flat, open and closed universe. In flat and closed universe $\beta^2 > 0$ where as it takes positive to negative valued in open universe for $t > 2$ (See Figure 2.5(left)). In Figure 2.5(right), profile of energy density and pressure is depicted above and below the zero line. The energy density and pressure is positive and negative valued respectively for $t \in (0, 2)$. Further it is noted that, $\rho > 0$ and it takes positive to negative valued in closed and open & flat universe respectively for $t > 2$. The pressure takes negative to positive values in flat and open universe whereas it is negative in closed universe for $t > 2$.

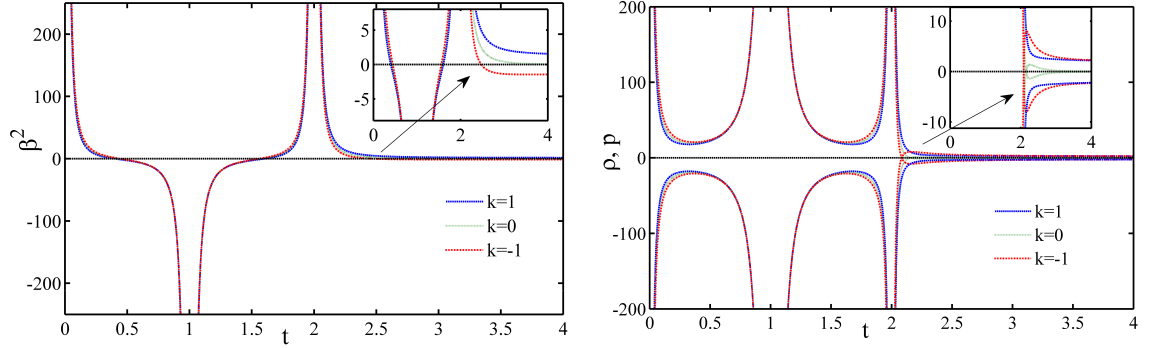


FIGURE 2.5: Profile for gauge function and energy density against cosmic time for flat, open and closed universe.

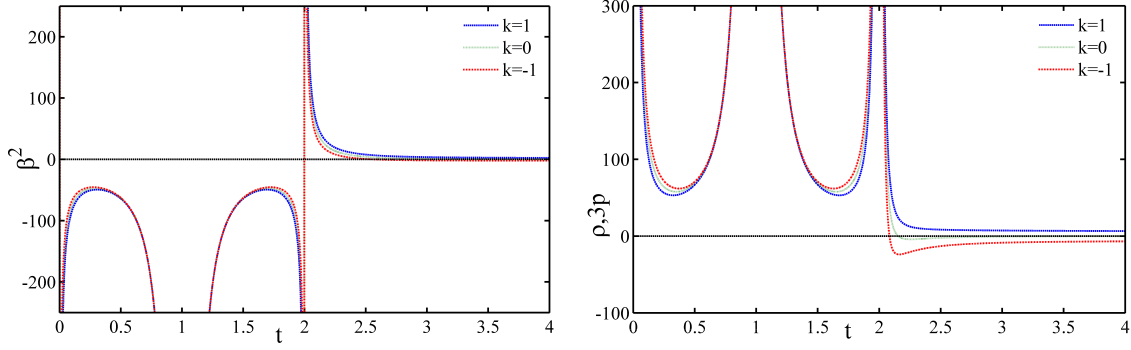


FIGURE 2.6: Profile for gauge function and energy density against cosmic time for flat, open and closed universe.

2.3.1.4 Radiation dominated Model

The solution of the radiation dominated model is obtained by setting $\gamma = \frac{1}{3}$ in equations (2.18) to (2.21). The physical quantities are as follows

$$\beta^2 = \begin{cases} -\frac{4}{1.33a+2} \left[\frac{-6t^2+24tn-16n^2-1.33a+4}{t^2(t-2n)^2(t-4n)^2} \right] + \frac{(2-1.33a)(t-2n)^{\frac{1}{2n^2}}}{R_0^2[t(t-4n)]^{\frac{1}{4n^2}}}, & k = 1 \\ -\frac{4}{1.33a+2} \left[\frac{-6t^2+24tn-16n^2-1.33a+4}{t^2(t-2n)^2(t-4n)^2} \right], & k = 0 \\ -\frac{4}{1.33a+2} \left[\frac{-6t^2+24tn-16n^2-1.33a+4}{t^2(t-2n)^2(t-4n)^2} \right] - \frac{(2-1.33a)(t-2n)^{\frac{1}{2n^2}}}{R_0^2[t(t-4n)]^{\frac{1}{4n^2}}}, & k = -1 \end{cases} \quad (2.32)$$

$$\rho = 3p = \begin{cases} \frac{6}{\chi(2+1.33a)} \left[\frac{-3t^2+12tn-8n^2+3}{t^2(t-2n)^2(t-4n)^2} + \frac{2(t-2n)^{\frac{1}{2n^2}}}{R_0^2[t(t-4n)]^{\frac{1}{4n^2}}} \right], & k = 1 \\ \frac{6}{\chi(2+1.33a)} \left[\frac{-3t^2+12tn-8n^2+3}{t^2(t-2n)^2(t-4n)^2} \right], & k = 0 \\ \frac{6}{\chi(2+1.33a)} \left[\frac{-3t^2+12tn-8n^2+3}{t^2(t-2n)^2(t-4n)^2} - \frac{2(t-2n)^{\frac{1}{2n^2}}}{R_0^2[t(t-4n)]^{\frac{1}{4n^2}}} \right], & k = -1 \end{cases} \quad (2.33)$$

$$p_c = \begin{cases} -\frac{8a}{3\chi(2+1.33a)} \left[\frac{-3t^2+12tn-8n^2+3}{t^2(t-2n)^2(t-4n)^2} + \frac{2(t-2n)^{\frac{1}{2n^2}}}{R_0^2[t(t-4n)]^{\frac{1}{4n^2}}} \right], & k = 1 \\ -\frac{8a}{3\chi(2+1.33a)} \left[\frac{-3t^2+12tn-8n^2+3}{t^2(t-2n)^2(t-4n)^2} \right], & k = 0 \\ -\frac{8a}{3\chi(2+1.33a)} \left[\frac{-3t^2+12tn-8n^2+3}{t^2(t-2n)^2(t-4n)^2} - \frac{2(t-2n)^{\frac{1}{2n^2}}}{R_0^2[t(t-4n)]^{\frac{1}{4n^2}}} \right], & k = -1 \end{cases} \quad (2.34)$$

In this scenario the gauge function $\beta^2 < 0$ for $t \in (0, 2)$ and $\beta^2 > 0$ for $t > 2$ in the open, flat and closed universe (See Figure 2.6(left)). The energy density ρ and pressure $3p$ are positive valued $t \in (0, 2)$ in the open, flat and closed universe. But $\rho, 3p > 0$ and

tending towards zero with the evolution of cosmic time in the closed universe only. In case of flat and open universe ρ and $3p$ takes positive to negative values and approaches to zero with the evolution of cosmic time (See Figure 2.6(right)). The particle creation pressure $p_c < 0$ for closed universe whereas $p_c > 0$ and it takes negative to positive values for $t > 2$ in open and flat universe (See Figure 2.7).

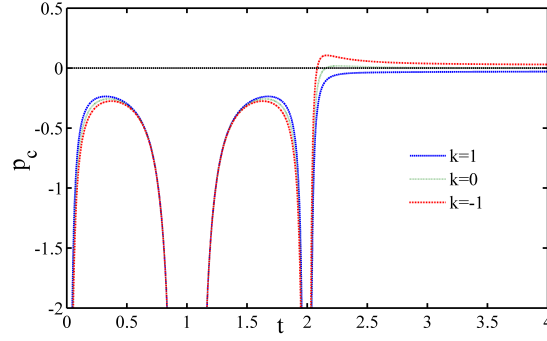


FIGURE 2.7: Profile for particle creation pressure against cosmic time for flat, open and closed universe.

2.3.2 Model-II

We know that, two observable parameter H and q are related as $q = \frac{d}{dt} \left(\frac{1}{H} \right) - 1$. This can be integrated to get the scale factor of the form

$$\mathcal{R}(t) = e^{\delta} \exp \left\{ \int \frac{dt}{\int (1+q)dt + \alpha_0} \right\} \quad (2.35)$$

Here α_0 and δ are integration constants. Here we are free to choose the form of q . The choice of q in the form

$$q = -\frac{\alpha_1}{t^2} + \alpha_2 - 1 \quad (2.36)$$

leads to an explicit form of the scale factor. The details about such consideration of deceleration parameter is discussed by Abdussattar and Prajapati [165]. Here $\alpha_1 > 0$ is a parameter having the dimension of square of time and $\alpha_2 > 1$ is a dimensionless constant. From equation (2.35), with the help of equation (2.36) we have

$$\mathcal{R}(t) = e^{\delta} \exp \left\{ \frac{1}{\alpha_2} \int \frac{t}{t^2 + \frac{\alpha_0}{\alpha_2}t + \frac{\alpha_1}{\alpha_2}} dt \right\} \quad (2.37)$$

Set $\alpha_0 = 0$ to obtain the simplest form of scale factor as

$$\mathcal{R} = e^\delta \left(t^2 + \frac{\alpha_1}{\alpha_2} \right)^{\frac{1}{2\alpha_2}} \quad (2.38)$$

In view of $\mathcal{R} = R_0$ when $t = 0$ we have $e^\delta = R_0 \left(\frac{\alpha_2}{\alpha_1} \right)^{\frac{1}{2\alpha_2}}$. Thus, Equation (2.38) takes the form

$$\mathcal{R} = R_0 \left(\frac{\alpha_2}{\alpha_1} \right)^{\frac{1}{2\alpha_2}} \left(t^2 + \frac{\alpha_1}{\alpha_2} \right)^{\frac{1}{2\alpha_2}}, \quad (2.39)$$

Equation (2.39), leads to the Hubble parameter in the form

$$H(t) = \frac{t}{\alpha_2 t^2 + \alpha_1} \quad (2.40)$$

Here we observe that, $H > 0$ for different choices of α_1, α_2 . As per the classification of Bolotin et al. [167], this choice of q leads to (i) expanding and decelerating models for sufficiently large value of t as $q \rightarrow \alpha_2 - 1$ when $t \rightarrow \infty$. (ii) expanding and accelerating models for $\alpha_2 = 1$. (iii) expanding with constant expansion/zero deceleration for $t^2 = \frac{\alpha_1}{\alpha_2 - 1}$. The system has two equations and four unknowns namely ρ, p_c, β and R . Thus, in order to obtain the unique solution we need two physically plausible relation among the variables. Hence, first we assume quadratic deceleration parameter of the form (2.36).

Incorporation of equations (2.39) to (2.40), the expression in (2.10) to (2.13) for the physical quantities take the form

$$\begin{aligned} \beta^2 = & -\frac{4}{a(1+\gamma) + 3(1-\gamma)} \left[\frac{2\alpha_1 - t^2(-3 + a + 2\alpha_2 - 3\gamma + a\gamma)}{(\alpha_2 t^2 + \alpha_1)^2} \right. \\ & \left. + (1 + 3\gamma) - a(1 + \gamma) \frac{k}{R_0^2 \left(\frac{\alpha_2}{\alpha_1} \right)^{\frac{1}{\alpha_2}} \left(t^2 + \frac{\alpha_1}{\alpha_2} \right)^{\frac{1}{\alpha_2}}} \right] \end{aligned} \quad (2.41)$$

$$\rho = \frac{6}{\chi[(a-3)\gamma + (a+3)]} \left[\frac{\alpha_1 - (\alpha_2 - 3)t^2}{(\alpha_2 t^2 + \alpha_1)^2} + \frac{2k}{R_0^2 \left(\frac{\alpha_2}{\alpha_1} \right)^{\frac{1}{\alpha_2}} \left(t^2 + \frac{\alpha_1}{\alpha_2} \right)^{\frac{1}{\alpha_2}}} \right] \quad (2.42)$$

$$p_c = -\frac{2a(1+\gamma)}{\chi[(a-3)\gamma + (a+3)]} \left[\frac{\alpha_1 - (\alpha_2 - 3)t^2}{(\alpha_2 t^2 + \alpha_1)^2} + \frac{2k}{R_0^2 \left(\frac{\alpha_2}{\alpha_1} \right)^{\frac{1}{\alpha_2}} \left(t^2 + \frac{\alpha_1}{\alpha_2} \right)^{\frac{1}{\alpha_2}}} \right] \quad (2.43)$$

The pressure of the derived model is expressed as

$$p = \frac{6\gamma}{\chi[(a-3)\gamma + (a+3)]} \left[\frac{\alpha_1 - (\alpha_2 - 3)t^2}{(\alpha_2 t^2 + \alpha_1)^2} + \frac{2k}{R_0^2 \left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{1}{\alpha_2}} \left(t^2 + \frac{\alpha_1}{\alpha_2}\right)^{\frac{1}{\alpha_2}}} \right] \quad (2.44)$$

2.3.3 Model-III

In this model, we consider the deceleration parameter proposed by Mishra and Chand [166] as

$$q = \frac{\alpha(1-t)}{1+t}, \quad (2.45)$$

here α is a non-negative constant. One can observe that, the expression in equations (2.10) to (2.13) depends on the scale factor R . In order to obtain R and explicit form of the solution we have used the equation (2.45). The well know relation among Hubble parameter and deceleration parameter $q = \frac{d}{dt} \left(\frac{1}{H} \right) - 1$ is applied to equation (2.45) for the expression of Hubble parameter. The Hubble parameter and its derivative is expressed as

$$H = \frac{1}{(1-\alpha)t + 2\alpha \log(1+t)} \quad (2.46)$$

For sake of simplicity constant of integration is taken as unity.

$$\dot{H} = -\frac{1-\alpha + 2\alpha \frac{1}{(1+t)}}{(t(1-\alpha) + 2\alpha(1+t))^2} \quad (2.47)$$

Integration of equation (2.46) leads to

$$\mathcal{R} = a_0 t^{\frac{1}{1+\alpha}} e^{G(t)}, \quad a_0 \text{ is the constant of integration.} \quad (2.48)$$

where

$$\begin{aligned} G(t) = & \frac{\alpha}{(1+\alpha)^2} t + \frac{-2\alpha + \alpha^2}{6(1+\alpha)^3} t^2 + \frac{3\alpha - 2\alpha^2 + \alpha^3}{18(1+\alpha)^4} t^3 \\ & + \frac{-18\alpha + 11\alpha^2 - 14\alpha^3 + 2\alpha^4}{180(1+\alpha)^5} t^4 + O(t^5) \end{aligned} \quad (2.49)$$

Incorporation of equations (2.46) to (2.48), the expression in (2.10) to (2.13) for the physical quantities take the form

$$\beta^2 = -\frac{4}{a(1+\gamma) + 3(1-\gamma)} \left[\frac{1 + 2\alpha + 3\gamma - a(1+\gamma) - \frac{4\alpha}{1+t}}{(t(-1+\alpha) + 2\alpha \log(1+t))^2} + ka_0 t^{-2-2\alpha} ((1+3\gamma) - a(1+\gamma)) e^{-2G(t)} \right] \quad (2.50)$$

$$\rho = \frac{6}{\chi[(a-3)\gamma + (a+3)]} \left[\frac{2 + \alpha - \frac{2\alpha}{1+t}}{(t(-1+\alpha) + 2\alpha \log(1+t))^2} + \frac{2k}{a_0 t^{\frac{1}{1+\alpha}} e^{G(t)}} \right] \quad (2.51)$$

$$p_c = -\frac{2a(1+\gamma)}{\chi[(a-3)\gamma + (a+3)]} \left[\frac{2 + \alpha - \frac{2\alpha}{1+t}}{(t(-1+\alpha) + 2\alpha \log(1+t))^2} + \frac{2k}{a_0 t^{\frac{1}{1+\alpha}} e^{G(t)}} \right] \quad (2.52)$$

The pressure of the derived model is expressed as

$$p = \frac{6\gamma}{\chi[(a-3)\gamma + (a+3)]} \left[\frac{2 + \alpha - \frac{2\alpha}{1+t}}{(t(-1+\alpha) + 2\alpha \log(1+t))^2} + \frac{2k}{a_0 t^{\frac{1}{1+\alpha}} e^{G(t)}} \right] \quad (2.53)$$

Now as per the different values of γ we have different cosmological fluids. These are illustrated in the Table 2.1 to Table 2.4. Figure 2.8 shows the profile of the energy density with respect to cosmic time in Vacuum Energy Model (VEM), Dust Filled Model (DFM), Radiation Dominated Model (RDM) and Stiff Matter Model (SMM) for open, flat and closed universes. Here we observed that, the energy density is positive valued through out the cosmic evolution of time in all the universes and all the models for $\alpha_1 \in [0, 0.5]$. In the quantitative level, the energy density is maximum in SMM and minimum in VEM for all the universes. The energy density is positive and goes on decreasing with the evolution of cosmic time. In the late time, ρ approaching to zero. For $\alpha_1 \in (0.5, 0.1)$, ρ takes negative valued to positive valued. Thus we neglect the interval of α_1 . Further $\alpha_1 \geq 1$, it is noticed that energy density is positive valued in all models for flat and close universe but negative valued in open universe for all the models.

Figure 2.9 represents the profile of pressure with respect to cosmic time. For $\alpha_1 = 0.25$ in all the universes (open, flat and closed) similar qualitative behaviour is noticed. In case of VEM the pressure is negative whereas pressure is positive for SMM and RDM in the universes. Further for $\alpha_1 = 1.5$, similar qualitative behaviour is noticed as that of $\alpha_1 = 0.25$ in flat and closed universe. At the late time, pressure is negligible gradually and vanishes with the evolution of the cosmic time.

The profile of the particle creation pressure with respect to cosmic time is presented in the Figure 2.10. For $\alpha_1 = 0.25$, particle creation pressure is negative in DFM,RDM and SMM whereas zero in VEM for all the universes. Further for $\alpha_1 = 1.5$ same qualitative behaviour is noticed for flat and closed universe. In case of open universe, p_c is positive and zero in DFM,RDM,SMM and VEM respectively.

The profile of the gauge function β^2 is plotted in the Figure 2.11 against cosmic time. In all the universes (open, flat and closed), for $\alpha_1 = 0.25$ the gauge function β^2 is negative in DFM, RDM and SMM but it takes negative to positive valued in VEM. For $\alpha_1 = 1.5$, in case of flat and closed universe $\beta^2 < 0$ in DFM, RDM and SMM. In VEM β^2 purely positive valued whereas negative to positive valued in closed and flat universe respectively. In case of open universe for $\alpha_1 = 1.5$, $\beta^2 > 0$ in SMM whereas $\beta^2 < 0$ in VEM, DFM and RDM.

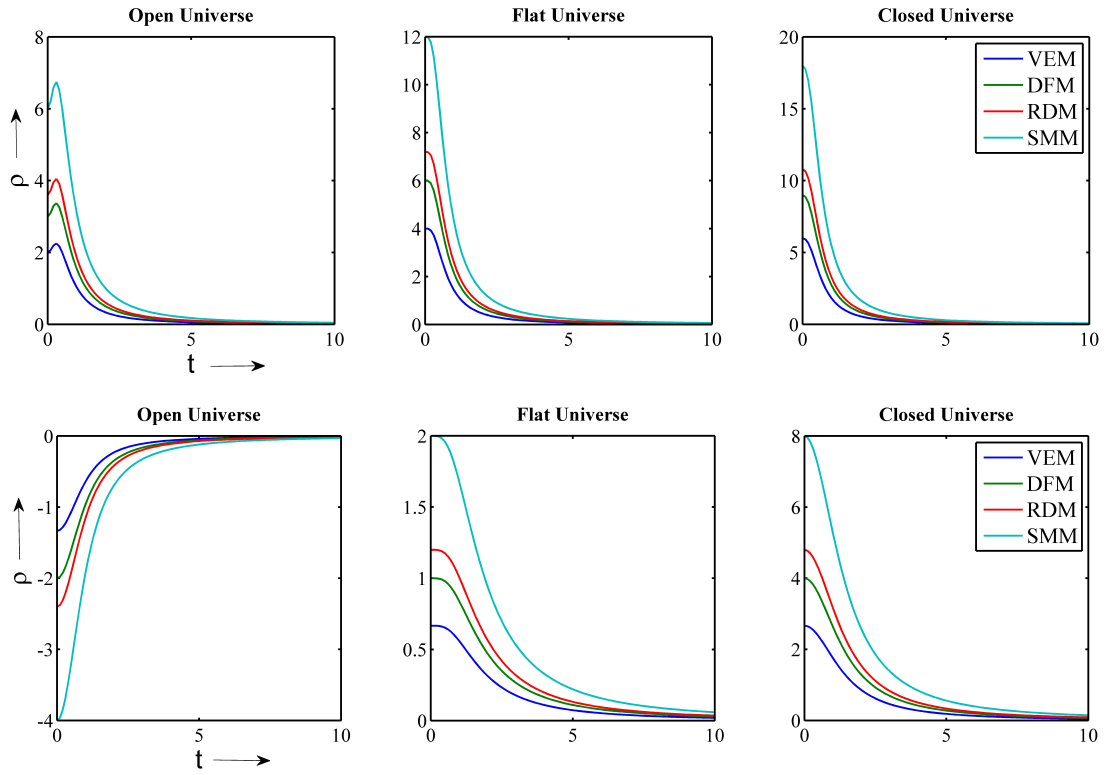


FIGURE 2.8: Profile of energy density in vacuum energy model (VEM), dust filled model (DFM), Radiation dominated model (RDM), stiff matter model (SMM) w.r.t cosmic time t for open, flat and closed universe. The upper panel and lower panel indicates for $\alpha_1 = 0.25$ and $\alpha_1 = 1.5$ respectively for model-II.

Figure 2.12 indicates the profile of the energy density w.r.t cosmic time for model-II. In figure we have plotted the energy density in various models (VEM,DFM, RDM and

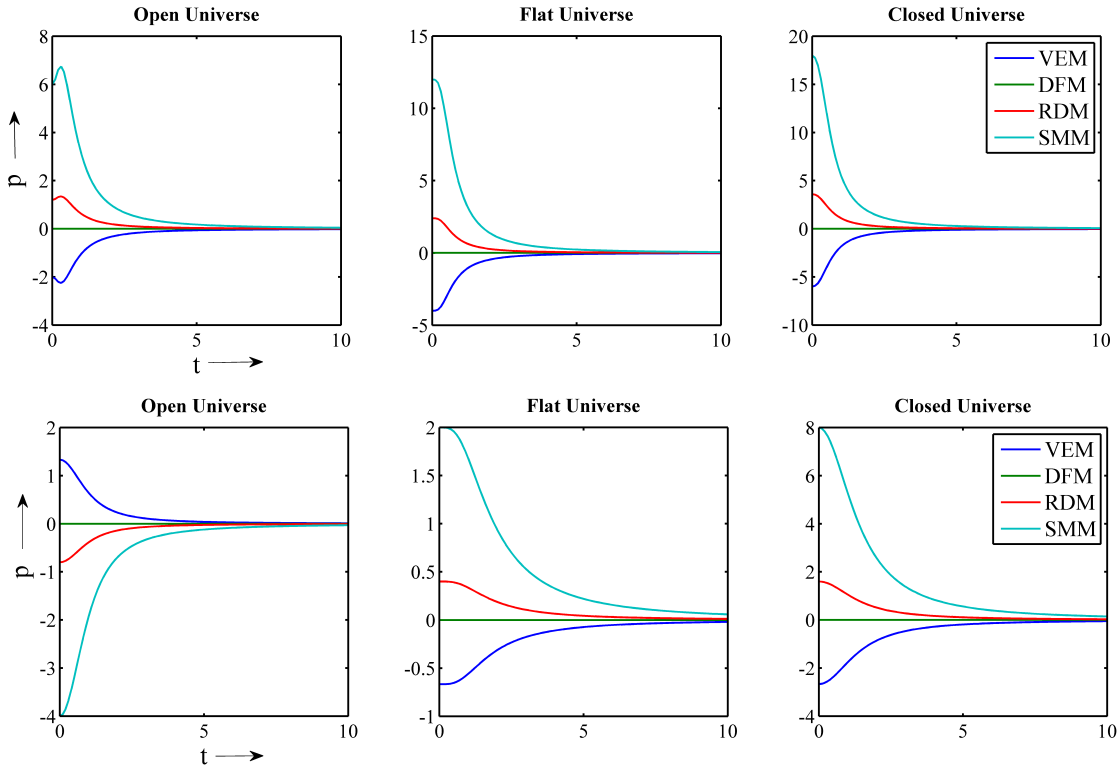


FIGURE 2.9: Profile of pressure in vacuum energy model (VEM), dust filled model (DFM), Radiation dominated model (RDM), stiff matter model (SMM) w.r.t cosmic time t for open, flat and closed universe. The upper panel and lower panel indicates for $\alpha_1 = 0.25$ and $\alpha_1 = 1.5$ respectively for model-II.

SMM) for flat, open and closed universe. It is noticed that, in case of flat universe energy density is a decreasing function of cosmic time and approaching to zero with evolution of cosmic time in all the models. For the case of closed universe, energy density goes on increasing with cosmic time for all the models. Further it takes positive to negative valued in case of open universe. In view of the above observations, the physically realistic one is the open universe for all the models.

The profile of pressure for model-II is depicted in the Figure 2.13 w.r.t the cosmic time. In case of open universe, the pressure is negative to positive valued for VEM whereas positive to negative valued for RDM and SMM. Further it is observed that, in SMM and RDM pressure is positive valued and negative valued in VEM for both flat and closed universe respectively.

The profile of particle creation pressure is presented in Figure 2.14. Figure reveals that, the particle creation pressure is zero in VEM for all the universes. In case of open universe, it is negative to positive valued and purely negative in flat & closed universe

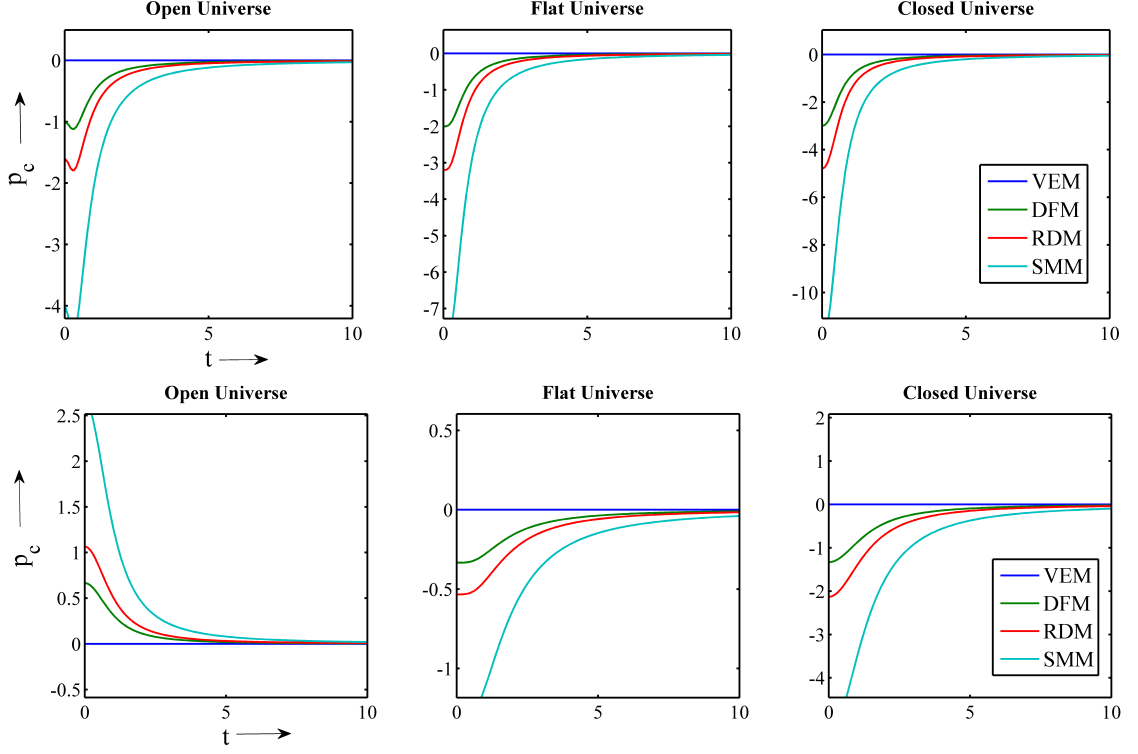


FIGURE 2.10: Profile of particle creation pressure in vacuum energy model (VEM), dust filled model (DFM), Radiation dominated model (RDM), stiff matter model (SMM) w.r.t cosmic time t for open, flat and closed universe. The upper panel and lower panel indicates for $\alpha_1 = 0.25$ and $\alpha_1 = 1.5$ respectively for model-II.

respectively in DFM, RDM and SMM. The qualitative behaviour is different for p_c also, negative valued in both flat and closed universe.

The gauge function β^2 is plotted in the Figure 2.15 with respect to cosmic time. In case of flat universe, β^2 is positive valued and approaching towards zero with the evolution of cosmic time in VEM and positive to negative valued in DFM, RDM and SMM. Again in closed universe, β^2 is positive valued in VEM and DFM whereas positive to negative valued in RDM and SMM.

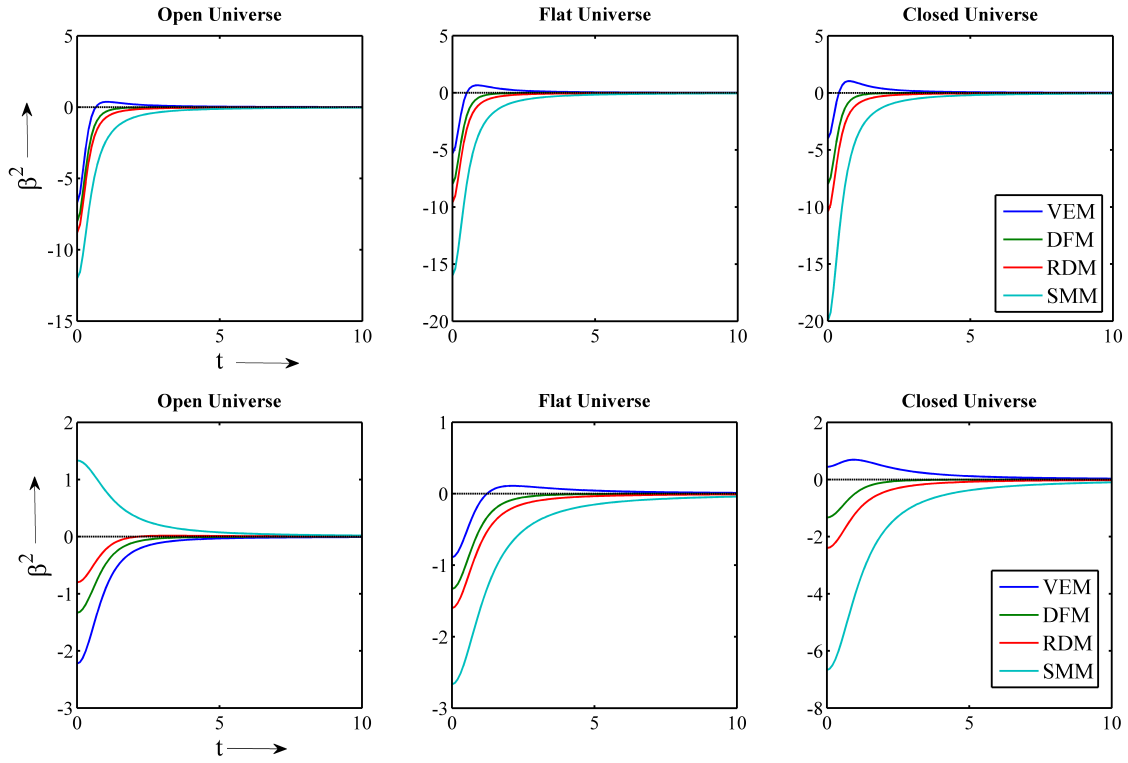


FIGURE 2.11: Profile of gauge function in vacuum energy model (VEM), dust filled model (DFM), Radiation dominated model (RDM), stiff matter model (SMM) w.r.t cosmic time t for open, flat and closed universe. The upper panel and lower panel indicates for $\alpha_1 = 0.25$ and $\alpha_1 = 1.5$ respectively for model-II.

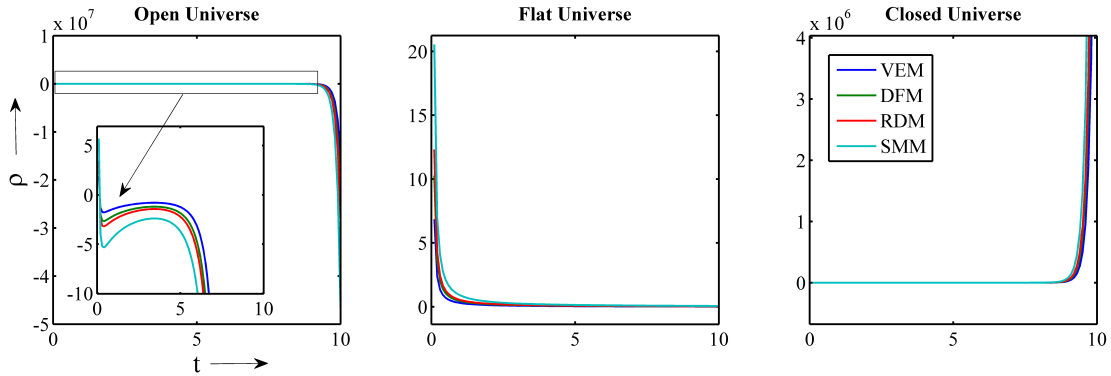


FIGURE 2.12: Profile of density w.r.t cosmic time t for model-III.

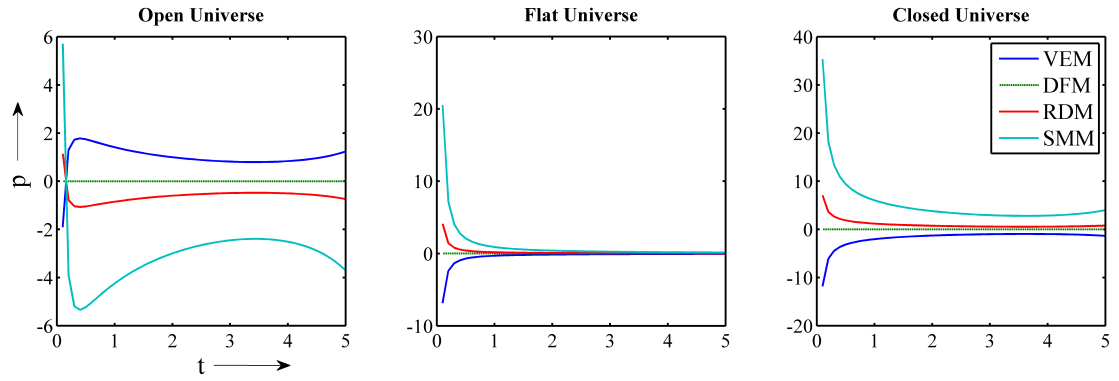


FIGURE 2.13: Profile of pressure with respect to cosmic time t for model-III.

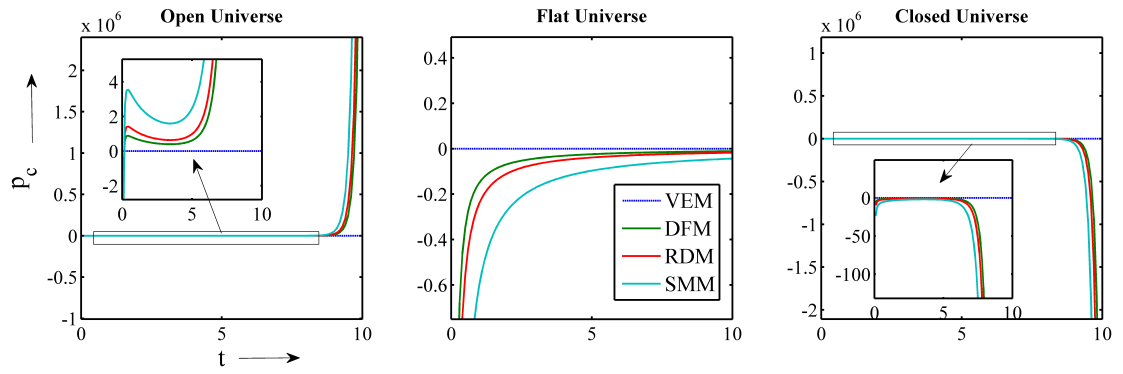


FIGURE 2.14: Profile of particle creation pressure w.r.t cosmic time t for model-III.

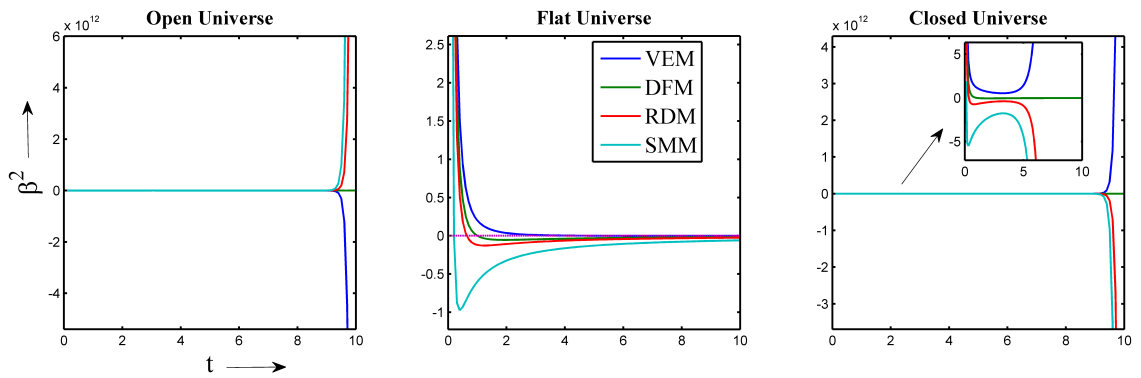


FIGURE 2.15: Profile of gauge function w.r.t cosmic time t for model-III.

Physical parameters ↓	Closed universe($k = 1$)	Flat universe($k = 0$)	Open universe($k = -1$)
Model-II			
β^2	$-\frac{4}{a+3} \left[\frac{2\alpha_1 - t^2(-3+a+2\alpha_2)}{(\alpha_2 t^2 + \alpha_1)^2} + \frac{(1-a)}{\left(R_0^2 \left(\frac{\alpha_2}{\alpha_1} \right)^{\frac{1}{\alpha_2}} \left(t^2 + \frac{\alpha_1}{\alpha_2} \right)^{\frac{1}{\alpha_2}} \right)} \right]$	$-\frac{4}{a+3} \left[\frac{2\alpha_1 - t^2(-3+a+2\alpha_2)}{(\alpha_2 t^2 + \alpha_1)^2} \right]$	$-\frac{4}{a+3} \left[\frac{2\alpha_1 - t^2(-3+a+2\alpha_2)}{(\alpha_2 t^2 + \alpha_1)^2} - \frac{(1-a)}{\left(R_0^2 \left(\frac{\alpha_2}{\alpha_1} \right)^{\frac{1}{\alpha_2}} \left(t^2 + \frac{\alpha_1}{\alpha_2} \right)^{\frac{1}{\alpha_2}} \right)} \right]$
ρ	$\frac{6}{\chi(a+3)} \left[\frac{\alpha_1 - (-3+\alpha_2)t^2}{(\alpha_2 t^2 + \alpha_1)^2} + \frac{2}{R_0^2 \left(\frac{\alpha_2}{\alpha_1} \right)^{\frac{1}{\alpha_2}} \left(t^2 + \frac{\alpha_1}{\alpha_2} \right)^{\frac{1}{\alpha_2}}} \right]$	$\frac{6}{\chi(a+3)} \left[\frac{\alpha_1 - (-3+\alpha_2)t^2}{(\alpha_2 t^2 + \alpha_1)^2} \right]$	$\frac{6}{\chi(a+3)} \left[\frac{\alpha_1 - (-3+\alpha_2)t^2}{(\alpha_2 t^2 + \alpha_1)^2} - \frac{2}{R_0^2 \left(\frac{\alpha_2}{\alpha_1} \right)^{\frac{1}{\alpha_2}} \left(t^2 + \frac{\alpha_1}{\alpha_2} \right)^{\frac{1}{\alpha_2}}} \right]$
p_c	$-\frac{2a}{\chi(a+3)} \left[\frac{\alpha_1 - (-3+\alpha_2)t^2}{(\alpha_2 t^2 + \alpha_1)^2} + \frac{2}{R_0^2 \left(\frac{\alpha_2}{\alpha_1} \right)^{\frac{1}{\alpha_2}} \left(t^2 + \frac{\alpha_1}{\alpha_2} \right)^{\frac{1}{\alpha_2}}} \right]$	$-\frac{2a}{\chi(a+3)} \left[\frac{\alpha_1 - (-3+\alpha_2)t^2}{(\alpha_2 t^2 + \alpha_1)^2} \right]$	$-\frac{2a}{\chi(a+3)} \left[\frac{\alpha_1 - (-3+\alpha_2)t^2}{(\alpha_2 t^2 + \alpha_1)^2} - \frac{2}{R_0^2 \left(\frac{\alpha_2}{\alpha_1} \right)^{\frac{1}{\alpha_2}} \left(t^2 + \frac{\alpha_1}{\alpha_2} \right)^{\frac{1}{\alpha_2}}} \right]$
p	0	0	0
Model-III			
β^2	$-\frac{4}{a+3} \left[\frac{1+2\alpha-a-\frac{4\alpha}{1+t}}{(t\alpha-t)+2\alpha \log(1+t))^2} + a_0 t^{-2-2\alpha} (1-a) e^{-2G(t)} \right]$	$-\frac{4}{a+3} \left[\frac{1+2\alpha-a-\frac{4\alpha}{1+t}}{(t\alpha-t)+2\alpha \log(1+t))^2} \right],$	$-\frac{4}{a+3} \left[\frac{1+2\alpha-a-\frac{4\alpha}{1+t}}{(t\alpha-t)+2\alpha \log(1+t))^2} - a_0 t^{-2-2\alpha} (1-a) e^{-2G(t)} \right]$
ρ	$\frac{6}{\chi(a+3)} \left[\frac{2+\alpha-\frac{2\alpha}{1+t}}{(t(-1+\alpha)+2\alpha \log(1+t))^2} + \frac{2}{a_0 t^{\frac{1}{1+\alpha}} e^{G(t)}} \right]$	$\frac{6}{\chi(a+3)} \left[\frac{2+\alpha-\frac{2\alpha}{1+t}}{(t(-1+\alpha)+2\alpha \log(1+t))^2} \right]$	$\frac{6}{\chi(a+3)} \left[\frac{2+\alpha-\frac{2\alpha}{1+t}}{(t(-1+\alpha)+2\alpha \log(1+t))^2} - \frac{2}{a_0 t^{\frac{1}{1+\alpha}} e^{G(t)}} \right]$
p_c	$-\frac{2a}{\chi(a+3)} \left[\frac{2+\alpha-\frac{2\alpha}{1+t}}{(t(-1+\alpha)+2\alpha \log(1+t))^2} + \frac{2}{a_0 t^{\frac{1}{1+\alpha}} e^{G(t)}} \right]$	$-\frac{2a}{\chi(a+3)} \left[\frac{2+\alpha-\frac{2\alpha}{1+t}}{(t(-1+\alpha)+2\alpha \log(1+t))^2} \right]$	$-\frac{2a}{\chi(a+3)} \left[\frac{2+\alpha-\frac{2\alpha}{1+t}}{(t(-1+\alpha)+2\alpha \log(1+t))^2} - \frac{2}{a_0 t^{\frac{1}{1+\alpha}} e^{G(t)}} \right]$
p	0	0	0

TABLE 2.1: Dust filled Model

Physical parameters ↓	Closed universe($k = 1$)	Flat universe($k = 0$)	Open universe($k = -1$)
Model-II			
β^2	$-\frac{4}{a} \left[\frac{\alpha_1 - t^2(a + \alpha_2 - 3)}{(\alpha_2 t^2 + \alpha_1)^2} + \frac{(2-a)}{R_0^2 \left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{1}{\alpha_2}} \left(t^2 + \frac{\alpha_1}{\alpha_2}\right)^{\frac{1}{\alpha_2}}} \right]$	$-\frac{4}{a} \left[\frac{\alpha_1 - t^2(a + \alpha_2 - 3)}{(\alpha_2 t^2 + \alpha_1)^2} \right]$	$-\frac{4}{a} \left[\frac{\alpha_1 - t^2(a + \alpha_2 - 3)}{(\alpha_2 t^2 + \alpha_1)^2} - \frac{(2-a)}{\left(R_0^2 \left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{1}{\alpha_2}} \left(t^2 + \frac{\alpha_1}{\alpha_2}\right)^{\frac{1}{\alpha_2}}\right)} \right]$
$p = \rho$	$\frac{3}{\chi a} \left[\frac{\alpha_1 - (\alpha_2 - 3)t^2}{(\alpha_2 t^2 + \alpha_1)^2} + \frac{2}{R_0^2 \left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{1}{\alpha_2}} \left(t^2 + \frac{\alpha_1}{\alpha_2}\right)^{\frac{1}{\alpha_2}}} \right]$	$\frac{3}{\chi a} \left[\frac{\alpha_1 - (\alpha_2 - 3)t^2}{(\alpha_2 t^2 + \alpha_1)^2} \right]$	$\frac{3}{\chi a} \left[\frac{\alpha_1 - (\alpha_2 - 3)t^2}{(\alpha_2 t^2 + \alpha_1)^2} - \frac{2}{R_0^2 \left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{1}{\alpha_2}} \left(t^2 + \frac{\alpha_1}{\alpha_2}\right)^{\frac{1}{\alpha_2}}} \right]$
p_c	$-\frac{2}{\chi} \left[\frac{\alpha_1 - (\alpha_2 - 3)t^2}{(\alpha_2 t^2 + \alpha_1)^2} + \frac{2}{R_0^2 \left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{1}{\alpha_2}} \left(t^2 + \frac{\alpha_1}{\alpha_2}\right)^{\frac{1}{\alpha_2}}} \right]$	$-\frac{2}{\chi} \left[\frac{\alpha_1 - (\alpha_2 - 3)t^2}{(\alpha_2 t^2 + \alpha_1)^2} \right]$	$-\frac{2}{\chi} \left[\frac{\alpha_1 - (\alpha_2 - 3)t^2}{(\alpha_2 t^2 + \alpha_1)^2} - \frac{2}{R_0^2 \left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{1}{\alpha_2}} \left(t^2 + \frac{\alpha_1}{\alpha_2}\right)^{\frac{1}{\alpha_2}}} \right]$
Model-III			
β^2	$-\frac{4}{a} \left[\frac{2 + \alpha - a - \frac{2\alpha}{1+t}}{(t\alpha - t + 2\alpha \log(1+t))^2} + (2-a)a_0 t^{-2-2\alpha} e^{-2G(t)} \right]$	$-\frac{4}{a} \left[\frac{2 + \alpha - a - \frac{2\alpha}{1+t}}{(t\alpha - t + 2\alpha \log(1+t))^2} \right]$	$-\frac{4}{a} \left[\frac{2 + \alpha - a - \frac{2\alpha}{1+t}}{(t\alpha - t + 2\alpha \log(1+t))^2} - (2-a)a_0 t^{-2-2\alpha} e^{-2G(t)} \right]$
$p = \rho$	$\frac{3}{\chi a} \left[\frac{2 + \alpha - \frac{2\alpha}{1+t}}{(t(-1+\alpha) + 2\alpha \log(1+t))^2} + \frac{2}{a_0 t^{\frac{1}{1+\alpha}} e^{G(t)}} \right]$	$\frac{3}{\chi a} \left[\frac{2 + \alpha - \frac{2\alpha}{1+t}}{(t(-1+\alpha) + 2\alpha \log(1+t))^2} \right]$	$\frac{3}{\chi a} \left[\frac{2 + \alpha - \frac{2\alpha}{1+t}}{(t(-1+\alpha) + 2\alpha \log(1+t))^2} - \frac{2}{a_0 t^{\frac{1}{1+\alpha}} e^{G(t)}} \right]$
p_c	$-\frac{2}{\chi} \left[\frac{2 + \alpha - \frac{2\alpha}{1+t}}{(t(-1+\alpha) + 2\alpha \log(1+t))^2} + \frac{2}{a_0 t^{\frac{1}{1+\alpha}} e^{G(t)}} \right]$	$-\frac{2}{\chi} \left[\frac{2 + \alpha - \frac{2\alpha}{1+t}}{(t(-1+\alpha) + 2\alpha \log(1+t))^2} \right]$	$-\frac{2}{\chi} \left[\frac{2 + \alpha - \frac{2\alpha}{1+t}}{(t(-1+\alpha) + 2\alpha \log(1+t))^2} - \frac{2}{a_0 t^{\frac{1}{1+\alpha}} e^{G(t)}} \right]$

TABLE 2.2: Stiff Matter Model

Physical parameters ↓	Closed universe($k = 1$)	Flat universe($k = 0$)	Open universe($k = -1$)
Model-II			
β^2	$-\frac{4}{3} \left[\frac{\alpha_1 - \alpha_2 t^2}{(\alpha_2 t^2 + \alpha_1)^2} \right] + \frac{4}{3 \left(R_0^2 \left(\frac{\alpha_2}{\alpha_1} \right)^{\frac{1}{\alpha_2}} \left(t^2 + \frac{\alpha_1}{\alpha_2} \right)^{\frac{1}{\alpha_2}} \right)}$	$-\frac{4}{3} \left[\frac{\alpha_1 - \alpha_2 t^2}{(\alpha_2 t^2 + \alpha_1)^2} \right]$	$-\frac{4}{3} \left[\frac{\alpha_1 - \alpha_2 t^2}{(\alpha_2 t^2 + \alpha_1)^2} \right] - \frac{4}{3 \left(R_0^2 \left(\frac{\alpha_2}{\alpha_1} \right)^{\frac{1}{\alpha_2}} \left(t^2 + \frac{\alpha_1}{\alpha_2} \right)^{\frac{1}{\alpha_2}} \right)}$
$\rho = -p$	$\frac{1}{\chi} \left[\frac{\alpha_1 - (\alpha_2 - 3)t^2}{(\alpha_2 t^2 + \alpha_1)^2} + \frac{2}{R_0^2 \left(\frac{\alpha_2}{\alpha_1} \right)^{\frac{1}{\alpha_2}} \left(t^2 + \frac{\alpha_1}{\alpha_2} \right)^{\frac{1}{\alpha_2}}} \right]$	$\frac{1}{\chi} \left[\frac{\alpha_1 - (\alpha_2 - 3)t^2}{(\alpha_2 t^2 + \alpha_1)^2} \right]$	$\frac{1}{\chi} \left[\frac{\alpha_1 - (\alpha_2 - 3)t^2}{(\alpha_2 t^2 + \alpha_1)^2} - \frac{2}{R_0^2 \left(\frac{\alpha_2}{\alpha_1} \right)^{\frac{1}{\alpha_2}} \left(t^2 + \frac{\alpha_1}{\alpha_2} \right)^{\frac{1}{\alpha_2}}} \right]$
p_c	0	0	0
Model-III			
β^2	$-\frac{2}{3} \left[\frac{\alpha - 1 - \frac{2\alpha}{1+t}}{(t\alpha - t + 2\alpha \log(1+t))^2} \right] + \frac{4}{3} a_0 t^{-2-2\alpha} e^{-2G(t)}$	$-\frac{2}{3} \left[\frac{\alpha - 1 - \frac{2\alpha}{1+t}}{(t\alpha - t + 2\alpha \log(1+t))^2} \right]$	$-\frac{2}{3} \left[\frac{\alpha - 1 - \frac{2\alpha}{1+t}}{(t\alpha - t + 2\alpha \log(1+t))^2} \right] - \frac{4}{3} a_0 t^{-2-2\alpha} e^{-2G(t)}$
$\rho = -p$	$\frac{1}{\chi} \left[\frac{2 + \alpha - \frac{2\alpha}{1+t}}{(t(-1+\alpha) + 2\alpha \log(1+t))^2} + \frac{2}{a_0 t^{\frac{1}{1+\alpha}} e^{G(t)}} \right]$	$\frac{1}{\chi} \left[\frac{2 + \alpha - \frac{2\alpha}{1+t}}{(t(-1+\alpha) + 2\alpha \log(1+t))^2} \right]$	$\frac{1}{\chi} \left[\frac{2 + \alpha - \frac{2\alpha}{1+t}}{(t(-1+\alpha) + 2\alpha \log(1+t))^2} - \frac{2}{a_0 t^{\frac{1}{1+\alpha}} e^{G(t)}} \right]$
p_c	0	0	0

TABLE 2.3: Vacuum energy Model

Physical parameters ↓	Closed universe($k = 1$)	Flat universe($k = 0$)	Open universe($k = -1$)
Model-II			
β^2	$-\frac{4}{1.33a+2} \left[\frac{2\alpha_1-(1.33a+2\alpha_2-4)t^2}{(\alpha_2 t^2 + \alpha_1)^2} \right] + \frac{(2-1.33a)}{R_0^2 \left(\frac{\alpha_2}{\alpha_1} \right)^{\frac{1}{\alpha_2}} \left(t^2 + \frac{\alpha_1}{\alpha_2} \right)^{\frac{1}{\alpha_2}}}$	$-\frac{4}{1.33a+2} \left[\frac{2\alpha_1-(1.33a+2\alpha_2-4)t^2}{(\alpha_2 t^2 + \alpha_1)^2} \right],$	$-\frac{4}{1.33a+2} \left[\frac{2\alpha_1-(1.33a+2\alpha_2-4)t^2}{(\alpha_2 t^2 + \alpha_1)^2} \right] - \frac{(2-1.33a)}{R_0^2 \left(\frac{\alpha_2}{\alpha_1} \right)^{\frac{1}{\alpha_2}} \left(t^2 + \frac{\alpha_1}{\alpha_2} \right)^{\frac{1}{\alpha_2}}}$
$\rho = 3p$	$\frac{6}{\chi(2+1.33a)} \left[\frac{\alpha_1-(\alpha_2-3)t^2}{(\alpha_2 t^2 + \alpha_1)^2} + \frac{2}{R_0^2 \left(\frac{\alpha_2}{\alpha_1} \right)^{\frac{1}{\alpha_2}} \left(t^2 + \frac{\alpha_1}{\alpha_2} \right)^{\frac{1}{\alpha_2}}} \right]$	$\frac{6}{\chi(2+1.33a)} \left[\frac{\alpha_1-(\alpha_2-3)t^2}{(\alpha_2 t^2 + \alpha_1)^2} \right],$	$\frac{6}{\chi(2+1.33a)} \left[\frac{\alpha_1-(\alpha_2-3)t^2}{(\alpha_2 t^2 + \alpha_1)^2} - \frac{2}{R_0^2 \left(\frac{\alpha_2}{\alpha_1} \right)^{\frac{1}{\alpha_2}} \left(t^2 + \frac{\alpha_1}{\alpha_2} \right)^{\frac{1}{\alpha_2}}} \right]$
p_c	$-\frac{8a}{3\chi(2+1.33a)} \left[\frac{\alpha_1-(\alpha_2-3)t^2}{(\alpha_2 t^2 + \alpha_1)^2} + \frac{2}{R_0^2 \left(\frac{\alpha_2}{\alpha_1} \right)^{\frac{1}{\alpha_2}} \left(t^2 + \frac{\alpha_1}{\alpha_2} \right)^{\frac{1}{\alpha_2}}} \right]$	$-\frac{8a}{3\chi(2+1.33a)} \left[\frac{\alpha_1-(\alpha_2-3)t^2}{(\alpha_2 t^2 + \alpha_1)^2} \right]$	$-\frac{8a}{3\chi(2+1.33a)} \left[\frac{\alpha_1-(\alpha_2-3)t^2}{(\alpha_2 t^2 + \alpha_1)^2} - \frac{2}{R_0^2 \left(\frac{\alpha_2}{\alpha_1} \right)^{\frac{1}{\alpha_2}} \left(t^2 + \frac{\alpha_1}{\alpha_2} \right)^{\frac{1}{\alpha_2}}} \right]$
Model-III			
β^2	$-\frac{4}{1.33a+2} \left[\frac{2+2\alpha-1.33a-\frac{4\alpha}{1+t}}{(t\alpha-t+2\alpha \log(1+t))^2} \right] + a_0 t^{-2-2\alpha} (2 - 1.33a) e^{-2G(t)}$	$-\frac{4}{1.33a+2} \left[\frac{2+2\alpha-1.33a-\frac{4\alpha}{1+t}}{(t\alpha-t+2\alpha \log(1+t))^2} \right]$	$-\frac{4}{1.33a+2} \left[\frac{2+2\alpha-1.33a-\frac{4\alpha}{1+t}}{(t\alpha-t+2\alpha \log(1+t))^2} \right] - a_0 t^{-2-2\alpha} (2 - 1.33a) e^{-2G(t)}$
$\rho = 3p$	$\frac{6}{\chi(2+1.33a)} \left[\frac{2+\alpha-\frac{2\alpha}{1+t}}{(t(-1+\alpha)+2\alpha \log(1+t))^2} + \frac{2}{a_0 t^{\frac{1}{1+\alpha}} e^{G(t)}} \right]$	$\times \left[\frac{\frac{6}{\chi(2+1.33a)} \left[\frac{2+\alpha-\frac{2\alpha}{1+t}}{(t(-1+\alpha)+2\alpha \log(1+t))^2} \right]}{\left[\frac{2+\alpha-\frac{2\alpha}{1+t}}{(t(-1+\alpha)+2\alpha \log(1+t))^2} \right]^2} \right]$	$\frac{6}{\chi(2+1.33a)} \left[\frac{2+\alpha-\frac{2\alpha}{1+t}}{(t(-1+\alpha)+2\alpha \log(1+t))^2} - \frac{2}{a_0 t^{\frac{1}{1+\alpha}} e^{G(t)}} \right]$
p_c	$-\frac{8a}{3\chi(2+1.33a)} \left[\frac{2+\alpha-\frac{2\alpha}{1+t}}{(t(-1+\alpha)+2\alpha \log(1+t))^2} + \frac{2}{a_0 t^{\frac{1}{1+\alpha}} e^{G(t)}} \right]$	$-\frac{8a}{3\chi(2+1.33a)} \left[\frac{2+\alpha-\frac{2\alpha}{1+t}}{(t(-1+\alpha)+2\alpha \log(1+t))^2} \right]$	$-\frac{8a}{3\chi(2+1.33a)} \left[\frac{2+\alpha-\frac{2\alpha}{1+t}}{(t(-1+\alpha)+2\alpha \log(1+t))^2} - \frac{2}{a_0 t^{\frac{1}{1+\alpha}} e^{G(t)}} \right]$

TABLE 2.4: Radiation dominated Model

2.4 Concluding Remarks

In this chapter, we have presented FLRW cosmological models in the framework of Lyra's geometry together with particle creation. The modified field equations are investigated in the presence of particle creation and we also analyzed the qualitative impact of the particle creation pressure on the derived models. Specifically, we have obtained exact solutions of the field equations under quadratic deceleration parameter and two different forms of a time varying deceleration parameter. The observations for the discussed models are presented below. The qualitative behaviour of gauge function β , energy density ρ , pressure p and particle creation pressure p_c are examined.

For Model-I The discussed models of quadratic deceleration parameter have singularity at $t = 0, 1, 2$ for $n = 0.5$. The choice of the parameter n is important as it determined the singularity. All the models are non-singular for $q < -1$. In view of the positivity of the gauge function β^2 , the physically viable models derived under quadratic deceleration parameter are dust filled models and vacuum energy model within the cosmic time interval of $t \in (0, 2)$. The positivity condition of energy density ρ within the cosmic time interval of $t \in (0, 2)$. The particle creation pressure $p_c < 0$ in stiff matter model and radiation dominated model. In case of dust filled model $p_c > 0$ and $p_c = 0$ in the vacuum energy model within the cosmic time interval of $t \in (0, 2)$. The pressure $p > 0$ in the stiff matter model & radiation dominated while $p < 0$ in vacuum energy model for $t \in (0, 2)$. It is obvious that, $p = 0$ in the dust filled universe.

The considered forms of the deceleration parameter in model-II reduce to a constant for $\alpha_1 = 0$, which is the famous singular model of Berman [168]. Further under a suitable choice of the parameters involved in q , we obtain the models of Abdel-Rahman [169] and Ozer and Taha [170]. The next form of the deceleration parameter used is $q = \frac{\alpha(1-t)}{1+t}$. This is usually treated as a bilinear varying deceleration parameter in the literature, and widely implemented in the construction of cosmological models in different contexts [166, 170, 171]. One notices that, in suitable units, $q < 0$ for $t > 1$, $q = 0$ for $t = 1$ and $q > 0$ for $t \in (0, 1)$. This suggests a decelerating model at early times ($t < 1$) and an accelerating model at late times ($t > 1$).

During the early universe, the physical parameters (gauge function, pressure, energy density, particle creation pressure) start with a constant value as $\beta^2 \rightarrow$

$$\frac{4}{a(1+\gamma)+3(1-\gamma)} \left[(a(1+\gamma) - (1+3\gamma)) \frac{k}{R_0^2} - \frac{2}{\alpha_1} \right], \rho \rightarrow \frac{6}{\chi[(a-3)\gamma+(a+3)]} \left[\frac{2k}{R_0^2} + \frac{1}{\alpha_1} \right], p_c \rightarrow$$

$$-\frac{2a(1+\gamma)}{\chi[(a-3)\gamma+(a+3)]} \left[\frac{2k}{R_0^2} + \frac{1}{\alpha_1} \right], p \rightarrow \frac{6\gamma}{\chi[(a-3)\gamma+(a+3)]} \left[\frac{2k}{R_0^2} + \frac{1}{\alpha_1} \right]$$
 and all are approaching zero at late times for model-II. In other words, the impact of all the parameters are negligible at late times. For model-III with the flat universe, similar qualitative behavior is observed at late times for all the physical parameters. In the early universe, all the parameters evolve with huge values, or the model has an initial singularity.

In model-II, physically acceptable qualitative behaviour of the energy density is obtained for all the universes (flat, open and closed) and all the models (VEM, DFM, RDM and SMM) with $\alpha_1 \in [0, 0.5]$. With this specified range of α_1 , negative pressure is observed in only the VEM for all the universes. Particle creation pressure is negative valued in the DFM, RDM and SMM in all the universes. Furthermore, a physically viable gauge function is obtained in the VEM only for all the universes. In the case of $\alpha_1 \geq 1$, similar qualitative behaviour is noted for all the models in the flat and closed universes.

In model-III, the only physically viable universe is the flat one in terms of physically acceptable behaviour of the physical parameters. In both the models, the particle creation pressure is zero in case of the VEM for all the universes. Previously, Singh and Beesham [152] and Singh et al. [173] have discussed such types of results. One can conclude that the deceleration parameter used in model-II widely encompasses all the universes, whereas model-III encompasses only the flat universe in terms of acceptability of physical quantities involved in the models. However, the most accepted form of the universe is a flat universe due to the generally accepted inflationary scenario. Thus both the forms of the deceleration parameter are promising for constructing physically viable cosmological models.

Chapter 3

Dark energy cosmological models in Lyra geometry for Bianchi-I space time

3.1 Introduction

Physicists and cosmologists have spent more time researching the future evolution of the universe as well as understanding the past and present state of the universe. Thus, it is important to conduct more and more research to understand the unknown phenomenon and the diverse objectives of the universe. Theory has been drawn to the anisotropic cosmological model, partly due to the realization that standard models, which agree well with our present-day universe, fail to account for the early phases of the universe's evolution. The exotic type of matter with large negative pressures may provide the necessary mechanism to overcome the gravitational pull.

Back to Einstein, the cosmological constant has a repulsive character, which makes it an ideal candidate for the dark energy. The cosmological constant is subject to the "fine-tuning" problem and to the "cosmic coincidences". Thus, dynamical cosmological constants with negative pressure appear to be the most promising choice. Also we have found different candidates for dark energy such as Quintessence [174], phantom [175], k-essence [176], tachyons [177] and Chaplygin gas [178] in the literature. Modern cosmology and modern astrophysics grapple with the issue of cosmic acceleration, despite the existence of these candidates of dark energy.

We can characterize dark energy models by the EoS parameter $\omega = \frac{p}{\rho}$, where p is the pressure of the fluid and ρ is the energy density. It is important to note that ω can either be a constant or a variable. Generally, one may say that ω depends on time t , redshift z , or scale factor a . With respect to variable ω , we have $\omega > -1$, $\omega = -1$, and $\omega < -1$, denoting respectively the quintessence, Λ CDM, and phantom dark energy models [179–182]. Additionally, the equation of state parameter has limits $-1.67 < \omega < -0.62$, $-1.33 < \omega < -0.79$ and $-1.44 < \omega < -0.92$ [183–186]. More details relating to dark matter, dark energy and alternate models can be found in Arun et al.[187]. The constant ω usually has the value $-1, 0, \frac{1}{3}, 1$ for vacuum, dust, radiation, and stiff-dominated universes, respectively. In Lyra geometry, several models with a constant displacement vector field [87, 188] and a time-dependent displacement vector field [189–193] have been analyzed. In these models, the universe appears to be decelerating, which is incompatible with current observations. However, in references [194–200] it is reported that the universe has been accelerating in late time in Lyra geometry. A dark energy cosmological model in lyra geometry was studied by Hova [201], which involved matter energy and a constant equation-of-state parameter interacting with the geometric displacement field. According to the study, the late-time universe’s accelerated expansion for $\omega_m \neq 1$ can be explained by nonzero effective time-dependent gravitational constants. By taking into account the time-varying deceleration parameter, Ram et al. [202] have presented the Kantowski–Sachs model of the universe filled with anisotropic dark energy. A five dimensional spherically symmetric cosmological model is analyzed by Singh and Singh [203]. The authors pointed out that, in addition to vacuum energy, the displacement vector also acts as time-dependant dark energy. The universe undergoes super-exponential expansion throughout the entire process of evolution. Ram et al. [204] investigate the Bianchi type-VI0 cosmological model through the lens of the Lyra manifold when an anisotropic dark energy is present along with a massless scalar field. Exact solutions of the field equations are obtained by utilizing the Hubble parameter, which leads to a time-varying deceleration parameter. In the framework of general relativity and Lyra’s geometry, Gusu and Shanthi [205] studied anisotropic and homogeneous Bianchi type V space-time in the presence of dark matter and holographic dark energy model components. Additionally, they demonstrated that Lyra geometry converges to general relativity in late time. Moreover, the models are similar to the Λ CDM model in late time.

In this chapter, we have presented the exponential-law expansion model and power-law expansion model in Bianchi type-I universe in view of Lyra geometry. Also, we have developed cosmological models with a variable deceleration parameter of the form $q = -\frac{\alpha_1}{t^2} + \alpha_2 - 1$.

3.2 Field equations for Bianchi type-I line element

let us consider the Bianchi type-I line element of the form

$$ds^2 = -dt^2 + A^2 dx^2 + B^2 dy^2 + C^2 dz^2. \quad (3.1)$$

Here A , B and C are function of t only. The energy momentum tensor T_ν^μ is considered as

$$T_\nu^\mu = (\rho + p)v_\mu v^\nu + pg_\mu^\nu \quad (3.2)$$

Here ρ and p represents the proper energy density and pressure respectively. The quantity θ is the scalar of expansion which is given by,

$$\theta = v^\mu_{|\mu} \quad (3.3)$$

and v^μ satisfies the relation

$$g_{\mu\nu}v^\mu v^\nu = -1. \quad (3.4)$$

we assume that coordinates to be co-moving, so that

$$v^1 = v^2 = v^3 = 0, v^4 = 1. \quad (3.5)$$

Equation (3.2) yield

$$\begin{aligned} T_1^1 &= p_x = \omega_x \rho = \omega \rho, \\ T_2^2 &= p_y = \omega_y \rho = (\omega + \gamma) \rho, \\ T_3^3 &= p_z = \omega_z \rho = (\omega + \delta) \rho, \\ T_4^4 &= p_x = -\rho. \end{aligned} \quad (3.6)$$

where p_x , p_y , p_z and ω_x , ω_y , ω_z indicates the directional pressure and equation of state (EoS) parameters along x , y and z axes respectively. The EoS parameter of the fluid, which is deviation free is denoted by ω . We have parameterized the deviation from isotropy by setting $\omega_x = \omega$ and then introducing skewness parameter γ and δ that are the deviation from ω along y and z axes, which are $\omega_y = (\omega + \gamma)$ and $\omega_z = (\omega + \delta)$ respectively. The field equations (1.47) in Lyra's geometry, for the metric (3.1), using equation (1.48) and (3.2) takes the form

$$\frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{B_4 C_4}{BC} + \frac{3}{4}\beta^2 = -\omega\rho \quad (3.7)$$

$$\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4 B_4}{AB} + \frac{3}{4}\beta^2 = -(\omega + \gamma)\rho \quad (3.8)$$

$$\frac{A_{44}}{A} + \frac{C_{44}}{C} + \frac{A_4 C_4}{AC} + \frac{3}{4}\beta^2 = -(\omega + \delta)\rho \quad (3.9)$$

$$\frac{A_4 B_4}{AB} + \frac{B_4 C_4}{BC} + \frac{A_4 C_4}{AC} - \frac{3}{4}\beta^2 = \rho \quad (3.10)$$

where $A_4 = \frac{dA}{dt}$, $A_{44} = \frac{d^2 A}{dt^2}$. The energy conservation $T_{i,j}^j = 0$ leads to

$$\frac{3}{2}\beta\beta_4 + \frac{3}{2}\beta^2 \left(\frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C} \right) = 0 \quad (3.11)$$

The spatial volume V for model and the average scale factor are defined as

$$V = \mathcal{R}^3 = ABC \quad (3.12)$$

In terms of scale factor or metric potentials, the Hubble parameter may be define as

$$H = \frac{\mathcal{R}_4}{\mathcal{R}} = \frac{1}{3} \left(\frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C} \right)$$

Or

$$H = \frac{1}{3} (H_x + H_y + H_z), \quad (3.13)$$

here $H_x = \frac{A_4}{A}$, $H_y = \frac{B_4}{B}$ and $H_z = \frac{C_4}{C}$ are the directional Hubble's parameter in directions of x , y and z axis respectively. The deceleration parameter, q is expressed as

$$q = -\frac{\mathcal{R}\mathcal{R}_{44}}{\mathcal{R}_4^2} \quad (3.14)$$

The following physical parameters are defined as

$$\text{Scalar expansion} = \theta = \frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C} \quad (3.15)$$

$$\text{Shear scalar} = \sigma^2 = \frac{1}{2} \sum_{i=1}^3 \left(H_i^2 - \frac{1}{3} \theta^2 \right) \quad (3.16)$$

$$\text{Average anisotropy parameter} = A_m = \frac{1}{3} \sum_{i=1}^3 \left(\frac{\Delta H_i}{H} \right)^2 \quad (3.17)$$

where $\Delta H_i = H_i - H$, $i = 1, 2, 3$

Here, We assume the second condition as $\sigma_1^1 \propto \theta$, which leads

$$A = (BC)^m \quad (3.18)$$

where $m > 0$. As a third assumption, we take $\gamma = \delta$ that means the deviation from ω along y and z axes are equal. From the differential equations (3.8) and (3.9), we deduced

$$\frac{B}{C} = \alpha_3 \exp \left[\alpha_4 \int (ABC)^{-1} dt \right] \quad (3.19)$$

where α_3 and α_4 are integration constants. From (3.19) the metric potentials can be explicitly written as

$$A = \mathcal{R}^{\frac{3m}{m+1}} \quad (3.20)$$

$$B = \sqrt{\alpha_3} \mathcal{R}^{\frac{3}{2(m+1)}} \exp \left[\frac{\alpha_4}{2} \int (ABC)^{-1} dt \right] \quad (3.21)$$

$$C = \frac{1}{\sqrt{\alpha_3}} \mathcal{R}^{\frac{3}{2(1+m)}} \exp \left[\frac{-\alpha_4}{2} \int (ABC)^{-1} dt \right] \quad (3.22)$$

3.3 Solutions of field equation under constant deceleration parameter

The field equations (3.7) to (3.10) are the four differential equations with eight unknown parameters A , B , C , β , ρ , ω , δ and γ . To make the system complete, we need the additional four conditions. Special law of variation for generalized Hubble's parameter is used as the first condition, which provides a constant value of deceleration parameter. The mean Hubble parameter H and the average scale factor are related through the

expression as

$$H = l\mathcal{R}^{-n} \quad (3.23)$$

From which we write (using equation (3.12))

$$\mathcal{R}_4 = l\mathcal{R}^{-n+1} \quad (3.24)$$

$$\mathcal{R}_{44} = -l^2(n-1)\mathcal{R}^{-2n+1} \quad (3.25)$$

Using equations (3.24) and (3.25) in the equation (3.14) we have

$$q = n - 1 \quad (3.26)$$

The inflation of a model is determined by the sign of q . q has positive sign for $n > 1$ corresponding to "standard" decelerating model whereas it has negative sign for $0 \leq n < 1$ indicating accelerating model [206]. Integration of equation (3.24), leads to

$$\mathcal{R} = \begin{cases} (nlt + \alpha_1)^{\frac{1}{n}}, & \text{for } n \neq 0 \\ \alpha_2 e^{lt}, & \text{for } n = 0 \end{cases} \quad (3.27)$$

where α_1 and α_2 are constants and which come out from the integration. Thus, the law (3.23) generate volumetric expansion in the form of power-law (PL) and exponential-law (EL)(One can see from equation (3.27)).

3.3.1 Model-I (Exponential-Law expansion)

In this case, lengthy but straight forward calculation leads to the metric potentials (3.20)-(3.22) takes the following forms:

$$\begin{aligned} A(t) &= \alpha_2^{\frac{3m}{m+1}} \exp\left(\frac{3ml}{m+1}t\right) \\ B(t) &= \sqrt{\alpha_3} \alpha_2^{\frac{3}{2(m+1)}} \exp\left[\frac{3lt}{2(m+1)} - \frac{\alpha_4}{6l\alpha_2^3} e^{-3lt}\right] \\ C(t) &= \frac{\alpha_2^{\frac{3}{2(m+1)}}}{\sqrt{\alpha_3}} \exp\left[\frac{3lt}{2(m+1)} + \frac{\alpha_4}{6l\alpha_2^3} e^{-3lt}\right] \end{aligned} \quad (3.28)$$

As a result, the required metric (3.1) is

$$ds^2 = -dt^2 + \alpha_2^{\frac{6m}{m+1}} \exp\left(\frac{6mlt}{m+1}\right) dx^2 + \alpha_3 \alpha_2^{\frac{3}{m+1}} \exp\left[\frac{3lt}{m+1} - \frac{\alpha_4}{3l\alpha_2^3} e^{-3lt}\right] dy^2 + \frac{\alpha_3^{\frac{3}{m+1}}}{\alpha_3} \exp\left(\frac{3lt}{m+1} + \frac{\alpha_4}{3l\alpha_2^3} e^{-3lt}\right) dz^2 \quad (3.29)$$

At early time, the scale factor is constant but it increases with time and reaches infinity at the late epoch of time. It shows that at first, the universe starts with a constant volume and expands exponentially to infinity. Let us calculate all physical quantities $\beta(t)$, H_1 , H_2 , H_3 , θ , σ , A_m , ρ , ω , γ and δ for exponential law. The gauge function $\beta(t)$ from equation (3.11), using equation (3.28) will takes the form

$$\beta(t) = \frac{N}{\alpha_2^3 e^{3lt}}, \quad (3.30)$$

where N is constant of integration. At initial stage of time, gauge function $\beta(t)$ return constant value and it is decreasing exponentially with increasing in time at later time. So that gauge function $\beta(t)$ of the model is goes over to the model of general relativity (Anirudh Pradhan et. al.[207]). The directional Hubble parameter H along x , y and z axes are given by

$$H_1 = \frac{A_4}{A} = \frac{3ml}{m+1} \quad (3.31)$$

$$H_2 = \frac{B_4}{B} = \frac{3l}{2(m+1)} + \frac{\alpha_4}{2\alpha_2^3} e^{-3lt} = \frac{3l}{2(m+1)} + \frac{\alpha_4}{2} \left(\frac{\beta(t)}{N} \right) \quad (3.32)$$

$$H_3 = \frac{C_4}{C} = \frac{3l}{2(m+1)} - \frac{\alpha_4}{2\alpha_2^3} e^{-3lt} = \frac{3l}{2(m+1)} - \frac{\alpha_4}{2} \left(\frac{\beta(t)}{N} \right) \quad (3.33)$$

It is noticed from the above equations that, H_i ($i = 2, 3$) depends upon gauge function $\beta(t)$. When $\beta(t) = 0$, this shows the nature of directional Hubble parameter and they are constant at early stage as well as lateral stage of time t . The following physical parameters for the model (3.29) are deduced as

$$\theta = 3H = 3l \quad (3.34)$$

$$\sigma^2 = \frac{3l^2(2m-1)^2}{4(m+1)^2} + \frac{\alpha_4^2}{4\alpha_2^6} e^{-6lt} = \frac{3l^2(2m-1)^2}{4(m+1)^2} + \frac{\alpha_4^2}{4N^2} \beta(t)^2 \quad (3.35)$$

$$A_m = \frac{(2m-1)^2}{2(m+1)^2} + \frac{\alpha_4^2}{6l^2\alpha_2^6} e^{-6lt} = \frac{(2m-1)^2}{2(m+1)^2} + \frac{\alpha_4^2}{6l^2N^2} \beta(t)^2 \quad (3.36)$$

The expansion scalar θ exhibit constant value for whole range of time t . This show uniform exponential expansion of the model. The shear of the model is depending upon gauge function $\beta(t)$. Initially it has constant value and goes on decreasing with increasing time t . When $t \rightarrow \infty$, $\sigma^2 \rightarrow \frac{(2m-1)^2}{2(m+1)^2}$ and $A_m \rightarrow \frac{(2m-1)^2}{2(m+1)^2}$. Further, it is noticed that at late time shear and average anisotropy parameter vanishes for $m = \frac{1}{2}$. Using the equation (3.19) and (3.30), in differential equation (3.10), we obtained ρ as

$$\begin{aligned}\rho &= \frac{9l^2(4m+1)}{4(m+1)^2} - \frac{3N^2}{4\alpha_2^6}e^{-6lt} - \frac{\alpha_4^2}{4\alpha_2^6}e^{-6lt} \\ &= \frac{9l^2(4m+1)}{4(m+1)^2} - \frac{3}{4}\beta(t)^2 - \frac{\alpha_4^2}{4N^2}\beta(t)^2\end{aligned}\quad (3.37)$$

The energy density depends upon the gauge function $\beta(t)$. At initial time, the energy density has a constant value $9(4m+1)\alpha_2^6l^2 - (m+1)^2(\alpha_4^2 + 3N^2)$. The positivity of energy density leads to the constraint on m such that $\frac{4m+1}{(m+1)^2} > \frac{(\alpha_4^2 + 3N^2)}{9\alpha_2^6l^2}$. When $t \rightarrow \infty$, the energy density $\rho \rightarrow \frac{9l^2(4m+1)}{4(m+1)^2}$. With the help of equations (3.7), (3.10), and (3.30) the EoS parameter ω has been calculated as

$$\begin{aligned}\omega &= \frac{27l^2\alpha_2^6e^{6lt} - (m+1)^2\alpha_4^2 - 3N^2(m+1)^2}{-9l^2(4m+1)\alpha_2^6e^{6lt} - (m+1)^2 - 3N^2(m+1)^2} \\ &= \frac{27N^2l^2 + 3N^3(m+1)^2\beta(t)^2 + \alpha_4^2(m+1)^2\beta(t)^2}{-9l^2(4m+1)N^2 - \alpha_4^2(m+1)^2\beta(t)^2 - 3N^2(m+1)^2\beta(t)^2}\end{aligned}\quad (3.38)$$

In this exponential-law of expansion, the EoS parameter ω depends upon gauge function $\beta(t)$. In early time of the universe, the ω takes a constant value equal to $\frac{27\alpha_2^6l^2 + (m+1)^2(3N^2 + \alpha_4^2)}{(m+1)^2(3N^2 + \alpha_4^2) - 9(4m+1)\alpha_2^6l^2}$. In late time, it approaches $-\frac{3}{4m+1}$. We write the values of the skewness parameter γ and δ , using the equation (3.7), (3.8), (3.30) and (3.38) as follows :

$$\begin{aligned}\gamma = \delta &= \frac{18l^2e^{6lt}\alpha_2^6(m+1)(2m-1)}{3\beta(t)^2e^{6lt}\alpha_2^6(m+1)^2 - \alpha_4^2(m+1)^2 - 9l^2(4m+1)e^{6lt}\alpha_2^6} \\ &= \frac{18l^2N^2(2m-1)(m+1)}{3\beta(t)^2N^2(m+1)^2 - \alpha_4^2\beta(t)^2(m+1)^2 - 9l^2N^2(4m+1)^2}\end{aligned}\quad (3.39)$$

For the value of time $t = \tau_0$ given by

$$\tau_0 = \frac{1}{6l} \ln \left(\frac{(m+1)^2(\alpha_4^2 + 3N^2)}{27l^2\alpha_2^6} \right) \quad (3.40)$$

which leads to equation of state parameter $\omega = 0$. So that dusty universe at time $t = \tau_0$ given by equation (3.40). Again, it is to be noted that when time t lies in open interval $\tau_1 < t < \tau_2$ with

$$\tau_1 = \frac{1}{6l} \ln \left(\frac{(m+1)^2(\alpha_4^2 + 3N^2)}{3.37l^2\alpha_2^6(6.68m - 1.33)} \right) \quad (3.41)$$

$$\tau_2 = \frac{1}{6l} \ln \left(\frac{(m+1)^2(\alpha_4^2 + 3N^2)}{5.6l^2\alpha_2^6(2.48m - 2.38)} \right) \quad (3.42)$$

the equation of state parameter ω lies in the range $-1.67 < \omega < -0.62$, which consistent and in good agreement with the limiting observational results [184].

Further for the value of time $t = \tau$, given by

$$\tau = \frac{1}{6l} \ln \left(\frac{(m+1)^2(\alpha_4^2 + 3N^2)}{9l^2\alpha_2^6(2m - 1)} \right) \quad (3.43)$$

The values of ω comes out to be -1 . For flat model, the matter energy density Ω_M and dark energy Ω_Λ , satisfies

$$\Omega_M + \Omega_\Lambda = 1 \quad (3.44)$$

where

$$\Omega_M = \frac{\rho}{3H^2} \quad \text{and} \quad \Omega_\Lambda = \frac{\Lambda}{3H^2} \quad (3.45)$$

Thus, equations (3.44) and (3.45) gives

$$\rho + \Lambda = 3H^2 \quad (3.46)$$

Then using equation (3.23) and (3.37) in (3.46), we write the expressions of Λ as

$$\begin{aligned} \Lambda &= \frac{3l^2(2m-1)^2}{4(m+1)^2} + \left(\frac{\alpha_4^2 + 3N^2}{4} \right) \frac{e^{-6lt}}{\alpha_2^6} \\ &= \frac{3l^2(2m-1)^2}{4(m+1)^2} + \left(\frac{\alpha_4^2}{4} \right) \frac{\beta(t)^2}{N^2} + \frac{3}{4}\beta(t)^2 \end{aligned} \quad (3.47)$$

From equation (3.47), one can noticed that cosmological constant Λ takes a positive constant value $\frac{3l^2(2m-1)^2\alpha_2^6 + (m+1)^2(3N^2 + \alpha_4^2)}{4\alpha_2^6(m+1)^2}$ at early stage of the universe and approaches

to a positive value $\frac{3l^2(2m-1)^2}{4(m+1)^2}$ at late time for all values of m and l . From equations (3.23), (3.37) and (3.47), we have

$$\Omega_M = \frac{3(4m+1)}{4(m+1)^2} - \frac{\alpha_4^2 e^{-6lt}}{12l^2 \alpha_2^6} - \frac{\beta(t)^2}{4l^2} \quad (3.48)$$

$$\Omega_\Lambda = \frac{(4m^2 - 4m + 1)}{4(m+1)^2} + \frac{\alpha_4^2 e^{-6lt}}{12l^2 \alpha_2^6} + \frac{\beta(t)^2}{4l^2} \quad (3.49)$$

From equation (3.48), we have observed that the matter energy density Ω_m takes the value $\frac{3(4m+1)}{4(m+1)^2} - \frac{\alpha_4^2}{12l^2 \alpha_2^6}$ at initial time. At late time, Ω_m is positive valued with the value $\frac{3(4m+1)}{4(m+1)^2}$. Further equation (3.49) indicates that, the dark energy Ω_Λ is positive valued at initial and late time for all the values of parameters involved in the expression. Ω_Λ is a decreasing function of cosmic time and approaches to $\frac{(4m^2-4m+1)}{4(m+1)^2}$ when $t \rightarrow \infty$.

3.3.2 Model-II (Power-Law expansion)

We solved the differential equation (3.7), (3.8), by using the power-law expansion ($n \neq 0$) (3.27), leads to the metric potentials (3.20)-(3.22) that takes the following forms:

$$\begin{aligned} A(t) &= (nlt + \alpha_1)^{\frac{3m}{n(m+1)}} \\ B(t) &= \sqrt{\alpha_3} (nlt + \alpha_1)^{\frac{3}{2n(m+1)}} \exp \left[\frac{\alpha_4}{2l(n-3)} (nlt + \alpha_1)^{\frac{n-3}{n}} \right] \\ C(t) &= \frac{1}{\sqrt{\alpha_3}} (nlt + \alpha_1)^{\frac{3}{2n(m+1)}} \exp \left[-\frac{\alpha_4}{2l(m+1)} (nlt + \alpha_1)^{\frac{n-3}{n}} \right] \end{aligned} \quad (3.50)$$

Thus the metric (3.1) reduced to (for $n \neq 0$), for power law expansion

$$\begin{aligned} ds^2 &= -dt^2 + (nlt + \alpha_1)^{\frac{6m}{n(m+1)}} dx^2 + \alpha_3 (nlt + \alpha_1)^{\frac{3}{n(m+1)}} \exp \left[\frac{\alpha_4}{2l(n-3)} (nlt + \alpha_1)^{\frac{n-3}{n}} \right] dy^2 \\ &+ \frac{1}{\alpha_3} (nlt + \alpha_1)^{\frac{3}{n(m+1)}} \exp \left[-\frac{\alpha_4}{2l(n-3)} (nlt + \alpha_1)^{\frac{n-3}{n}} \right] dz^2 \end{aligned} \quad (3.51)$$

At early time, the scale factor is constant but it increases with time and reaches infinity at the late epoch of time. It shows that at first, the universe starts with a constant volume and expands rapidly to infinity. Let us calculate all physical quantities $\beta(t)$, H_1 , H_2 , H_3 , θ , σ , A_m , ρ , ω , γ and δ for power law. The gauge function $\beta(t)$ from equation (3.11),

using equation (3.50) will takes the form

$$\beta(t) = N(nlt + \alpha_1)^{-\frac{3}{n}} \quad (3.52)$$

where N is constant of integration. At initial stage of time, gauge function $\beta(t)$ return constant value and it is decreasing with evolution of time. The directional Hubble parameters H_i along x , y and z axes are given by

$$H_1 = \frac{A_4}{A} = \frac{3ml}{m+1} \frac{1}{(nlt + \alpha_1)} \quad (3.53)$$

$$H_2 = \frac{B_4}{B} = \frac{1}{2} \left[\frac{3l}{(1+m)(\alpha_1 + nlt)} + \alpha_4(\alpha_1 + nlt)^{-\frac{3}{n}} \right] \quad (3.54)$$

$$H_3 = \frac{C_4}{C} = \frac{1}{2} \left[\frac{3l}{(1+m)(\alpha_1 + nlt)} - \alpha_4(\alpha_1 + nlt)^{-\frac{3}{n}} \right] \quad (3.55)$$

The physical parameters for the model (3.51) are deduced as

$$\theta = 3H = 3l(nlt + \alpha_1)^{-1} \quad (3.56)$$

$$\begin{aligned} \sigma^2 &= \frac{1}{4} \left[\frac{3l^2(1-2m)^2}{(1+m)^2(\alpha_1 + nlt)^2} + \alpha_4^2 \beta(t)^2 N^{-2} \right] \\ &= \frac{1}{4} \left[\frac{3l^2(1-2m)^2}{(1+m)^2(\alpha_1 + nlt)^2} + \alpha_4^2 (\alpha_1 + nlt)^{-\frac{6}{n}} \right] \end{aligned} \quad (3.57)$$

$$\begin{aligned} A_M &= \frac{1}{6} \left[\frac{3(1-2m)^2}{(1+m)^2} + \frac{\alpha_4^2 \beta(t)^2 N^{-2} (\alpha_1 + nlt)^2}{l^2} \right] \\ &= \frac{1}{6} \left[\frac{3(1-2m)^2}{(1+m)^2} + \frac{\alpha_4^2 (\alpha_1 + nlt)^{\frac{2n-6}{n}}}{l^2} \right] \end{aligned} \quad (3.58)$$

The expansion scalar θ is a decreasing function of cosmic time t . It exhibit a constant value initially and approaching to zero with the evolution of cosmic time. The shear and average anisotropy parameters of the model is depending upon the gauge function $\beta(t)$. Both the parameters are decreasing function of cosmic time t . Initially σ^2 has constant value and approaching towards zero at late time. The anisotropy parameter A_M has constant value initially and $A_M \rightarrow \infty$ and $A_M \rightarrow \frac{(1-2m)^2}{2(1+m)^2}$ for $n > 3$ and $0 < n < 3$ respectively. Using the equation (3.50) and (3.52), in differential equation (3.10), we

obtained ρ as

$$\begin{aligned}\rho &= \frac{1}{4} \left[\frac{9l^2(1+4m)}{(1+m)^2(\alpha_1 + nlt)^2} - \alpha_4^2 \beta(t)^2 N^{-2} - 3\beta(t)^2 \right] \\ &= \frac{1}{4} \left[\frac{9l^2(1+4m)}{(1+m)^2(\alpha_1 + nlt)^2} - (\alpha_1 + nlt)^{-\frac{6}{n}} (\alpha_4^2 + 3N^2) \right]\end{aligned}\quad (3.59)$$

The energy density depends upon the gauge function $\beta(t)$. At initial time, the energy density has a constant value $\frac{1}{4} \left[\frac{9l^2(1+4m)}{\alpha_1^2(1+m)^2} - \alpha_1^{-\frac{6}{n}} (\alpha_4^2 + 3N^2) \right]$. The positivity of energy density leads to the constraint on m such that $\frac{4m+1}{(m+1)^2} > \frac{4\alpha_1^{\frac{2n-6}{n}} (\alpha_4^2 + 3N^2)}{9l^2}$. When $t \rightarrow \infty$, the energy density $\rho \rightarrow \frac{9l^2(4m+1)}{\alpha_1^2(m+1)^2}$. Using the equation (3.50) and (3.59) in equation (3.7), the EoS parameter ω have been calculated as

$$\begin{aligned}\omega &= \frac{\frac{3l^2(4(1+m)n-9)}{(1+m)^2(\alpha_1+nlt)^2} - \alpha_4^2 \beta(t)^2 N^{-2} - 3\beta(t)^2}{\frac{9l^2(1+4m)}{(1+m)^2(\alpha_1+nlt)^2} - \alpha_4^2 \beta(t)^2 N^{-2} - 3\beta(t)^2} \\ &= \frac{\frac{3l^2(4(1+m)n-9)}{(1+m)^2(\alpha_1+nlt)^2} - (\alpha_1 + nlt)^{-\frac{6}{n}} (\alpha_4^2 + 3N^2)}{\frac{9l^2(1+4m)}{(1+m)^2(\alpha_1+nlt)^2} - (\alpha_1 + nlt)^{-\frac{6}{n}} (\alpha_4^2 + 3N^2)}\end{aligned}\quad (3.60)$$

In this power-law of expansion, the EoS parameter ω depends upon gauge function $\beta(t)$ from (3.52). In early time of the universe, the ω takes a constant value equal to $\frac{\frac{3l^2(-9+4(1+m)n)}{\alpha_1^2(1+m)^2} - \alpha_1^{-\frac{6}{n}} (\alpha_4^2 + 3N^2)}{\frac{9l^2(1+4m)}{\alpha_1^2(1+m)^2} - \alpha_1^{-\frac{6}{n}} (\alpha_4^2 + 3N^2)}$. In late time, it approaches $\frac{4n(1+m)-3}{1+4m}$ for $0 < n < 2$. We write the values of the skewness parameter γ and δ , using the equation (3.52), (3.59) and (3.60) in (3.8) as follows

$$\begin{aligned}\gamma = \delta &= \frac{\frac{3l^2}{n(m+1)^2} (3-n) (4mn(m+\frac{1}{2}) - 2n) (nlt + \alpha_1)^{-2}}{\alpha_4^2 \frac{\beta(t)^2}{N^2} - \frac{9l^2(4m+1)}{(m+1)^2} (nlt + \alpha_1)^{-2} + 3\beta(t)^2} \\ &= \frac{\frac{3l^2}{n(m+1)^2} (3-n) (4mn(m+\frac{1}{2}) - 2n) (nlt + \alpha_1)^{-2}}{(\alpha_4^2 + 3N^2) (nlt + \alpha_1)^{-\frac{6}{n}} - \frac{9l^2(4m+1)}{(m+1)^2} (nlt + \alpha_1)^{-2}}\end{aligned}\quad (3.61)$$

For the value of time $t = \tau_0$ given by

$$\tau_0 = \frac{1}{nl} \left[\frac{l}{m+1} \sqrt{\frac{3[4n(m+1) - 9]}{\alpha_4^2 + 3N^2}} \right]^{\frac{n}{n-3}} - \frac{\alpha_1}{nl}\quad (3.62)$$

the equation of state parameter $\omega = 0$. So that we have dusty universe at time $t = \tau_0$ given by equation (3.62). It is to be noted that when time t lies in open interval $\tau_1 <$

$t < \tau_2$, where

$$\tau_1 = \frac{1}{nl} \left[\frac{1.06l}{m+1} \sqrt{\frac{[4n(m+1) + 4(5m-1)]}{\alpha_4^2 + 3N^2}} \right]^{\frac{n}{n-3}} - \frac{\alpha_1}{nl} \quad (3.63)$$

$$\tau_2 = \frac{1}{nl} \left[\frac{1.36l}{m+1} \sqrt{\frac{[4n(m+1) + 7.44m - 7.1]}{\alpha_4^2 + 3N^2}} \right]^{\frac{n}{n-3}} - \frac{\alpha_1}{nl} \quad (3.64)$$

then the equation of state parameter ω lies in the range $-1.67 < \omega < -0.62$, which consistent and in good agreement with the limiting observational results [184]. When the domain of time t is $\tau_3 < t < \tau_4$, in which

$$\tau_3 = \frac{1}{nl} \left[\frac{1.13l}{m+1} \sqrt{\frac{[4n(m+1) + 15.96m - 5.01]}{\alpha_4^2 + 3N^2}} \right]^{\frac{n}{n-3}} - \frac{\alpha_1}{nl} \quad (3.65)$$

$$\tau_4 = \frac{1}{nl} \left[\frac{1.29l}{m+1} \sqrt{\frac{[4n(m+1)9.48m - 6.63]}{\alpha_4^2 + 3N^2}} \right]^{\frac{n}{n-3}} - \frac{\alpha_1}{nl} \quad (3.66)$$

then we observed the values of ω as $-1.33 < \omega < -0.79$ which coincides with the values obtained from observational results [209]. Also it is noticed that when t lies in the open interval $\tau_5 < t < \tau_6$ where

$$\tau_5 = \frac{1}{nl} \left[\frac{1.10l}{m+1} \sqrt{\frac{[4n(m+1) + 17.28m - 4.68]}{\alpha_4^2 + 3N^2}} \right]^{\frac{n}{n-3}} - \frac{\alpha_1}{nl} \quad (3.67)$$

$$\tau_6 = \frac{1}{nl} \left[\frac{1.25l}{m+1} \sqrt{\frac{[4n(m+1) + 11.04m - 6.24]}{\alpha_4^2 + 3N^2}} \right]^{\frac{n}{n-3}} - \frac{\alpha_1}{nl} \quad (3.68)$$

then the values of ω are found to be $-1.44 < \omega < -0.92$ which are very much consistent with the values of ω of latest observational results in 2009 [210, 211]. Further for the value of time $t = \tau$, given by

$$\tau = \frac{1}{nl} \left[\frac{1.22l}{m+1} \sqrt{\frac{[4n(m+1) + 12m - 6]}{\alpha_4^2 + 3N^2}} \right]^{\frac{n}{n-3}} - \frac{\alpha_1}{nl}, \quad (3.69)$$

the values of ω comes out to be -1 . The geometrical and physical behaviour of the dark energy model is to be discussed on the values of EoS parameter and the gauge function $\beta(t)$. The gauge function $\beta(t)$ is appeared in all physical quantity like

$H, \sigma, A_M, \rho, \omega, \delta, \gamma$. If the constant of integration $N = 0$ then $\beta(t) = 0$ from equation (3.52). Thus for $N = 0$, the physical quantity ρ, ω and time t in our model goes over to the result of Pradhan et al.[207]. If $N \neq 0$, then the gauge function $\beta(t)$ play a role in our models. Thus in this note, an attempt has been made to generalize the model of Pradhan et al.[207], in Lyra's geometry. For flat model (in the absence of curvature), the matter energy density Ω_M and dark energy Ω_Λ , satisfies the relation

$$\Omega_M + \Omega_\Lambda = 1 \quad (3.70)$$

where

$$\begin{aligned} \Omega_M &= \frac{\rho}{3H^2} \\ \Omega_\Lambda &= \frac{\Lambda}{3H^2}. \end{aligned} \quad (3.71)$$

Thus, equation (3.70) and (3.71) gives

$$\rho + \Lambda = 3H^2 \quad (3.72)$$

Then using equation (3.23) and (3.59) in (3.72), we write the values of Λ as

$$\begin{aligned} \Lambda &= \frac{1}{4} \left[\frac{3l^2(3+4m+4m^2)}{(1+m)^2(\alpha_1+nlt)^2} + \alpha_4^2\beta(t)^2N^{-2} + 3\beta(t)^2 \right] \\ &= \frac{1}{4} \left[\frac{3l^2(3+4m+4m^2)}{(1+m)^2(\alpha_1+nlt)^2} + (\alpha_1+nlt)^{-\frac{6}{n}}(\alpha_4^2+3N^2) \right] \end{aligned} \quad (3.73)$$

From equation (3.73), one can noticed that cosmological constant Λ takes a positive constant value $\frac{1}{4} \left[\frac{3l^2(3+4m+4m^2)}{(1+m)^2\alpha_1^2} + \alpha_1^{-\frac{6}{n}}(\alpha_4^2+3N^2) \right]$ at early stage of the universe and approaches to zero at late time for all the parameters involved in the expression (3.73). The cosmological term Λ is a decreasing function of cosmic time t . From equation (3.23), (3.59) and (3.73), we write the values of the matter energy density $\Omega_M = \frac{\rho}{3H^2}$ and dark energy $\Omega_\Lambda = \frac{\Lambda}{3H^2}$ as,

$$\Omega_M = \frac{3(4m+1)}{4(m+1)^2} - \frac{\alpha_4^2(\alpha_1+nlt)^{2-\frac{6}{n}}}{12l^2} - \frac{\beta(t)^2}{4l^2(nlt+\alpha_1)^{-2}} \quad (3.74)$$

$$\Omega_\Lambda = \frac{(4m^2-4m+1)}{4(m+1)^2} + \frac{\alpha_4^2(\alpha_1+nlt)^{2-\frac{6}{n}}}{12l^2} + \frac{\beta(t)^2}{4l^2(nlt+\alpha_1)^{-2}} \quad (3.75)$$

Here we noticed that both the matter energy density Ω_M and dark energy Ω_Λ start evolve with a constant value. Further, $\Omega_M \rightarrow \frac{3}{4} \frac{(4m+1)}{(m+1)^2}$ and $\Omega_\Lambda \rightarrow \frac{(4m^2-4m+1)}{4(m+1)^2}$ when $t \rightarrow \infty$ for $0 < n < 2$. The Ω_Λ is a decreasing function of cosmic time t .

3.4 Solutions of field equation under varying deceleration parameter :

The field equations (3.7) to (3.10) are the four differential equations with eight unknown parameters $A, B, C, \beta, \rho, \omega, \delta$ and γ . We know that, two observable parameter H and q are related as $q = \frac{d}{dt} \left(\frac{1}{H} \right) - 1$. This can be integrated to get the scale factor of the form

$$\mathcal{R}(t) = e^\delta \exp \left\{ \int \frac{dt}{\int (1+q)dt + \alpha_0} \right\} \quad (3.76)$$

Here α_0 and δ are integration constants. Here we are free to choose the form of q . The choice of q in the form

$$q = -\frac{\alpha_1}{t^2} + \alpha_2 - 1 \quad (3.77)$$

leads to an explicit form of the scale factor. The details about such type of consideration of the deceleration parameter is discussed by Abdussattar and Prajapati [165]. From equation (3.76), with the help of equation (3.77) we have

$$\mathcal{R}(t) = e^\delta \exp \left\{ \frac{1}{\alpha_2} \int \frac{t}{t^2 + \frac{\alpha_0}{\alpha_2}t + \frac{\alpha_1}{\alpha_2}} dt \right\} \quad (3.78)$$

Set $\alpha_0 = 0$ to obtain the simplest form of scale factor as

$$\mathcal{R} = e^\delta \left(t^2 + \frac{\alpha_1}{\alpha_2} \right)^{\frac{1}{2\alpha_2}} \quad (3.79)$$

In view of $\mathcal{R} = R_0$ when $t = 0$ we have $e^\delta = R_0 \left(\frac{\alpha_2}{\alpha_1} \right)^{\frac{1}{2\alpha_2}}$. Thus, Equation (3.79) takes the form

$$\mathcal{R} = R_0 \left(\frac{\alpha_2}{\alpha_1} \right)^{\frac{1}{2\alpha_2}} \left(t^2 + \frac{\alpha_1}{\alpha_2} \right)^{\frac{1}{2\alpha_2}}, \quad (3.80)$$

Equation (3.80), leads to the Hubble parameter in the form

$$H(t) = \frac{t}{\alpha_2 t^2 + \alpha_1} \quad (3.81)$$

Here we observe that, $H > 0$ for different choices of α_1, α_2 . As per the classification of Bolotin et al. [167], this particular choice of q leads to (i) expanding and decelerating models for sufficiently large value of t as $q \rightarrow \alpha_2 - 1$ when $t \rightarrow \infty$. (ii) expanding and accelerating models for $\alpha_2 = 1$. (iii) expanding with constant expansion/zero deceleration for $t^2 = \frac{\alpha_1}{\alpha_2 - 1}$. In order to obtain a deterministic solution, first we assume quadratic deceleration parameter of the form (3.77). We assume the second condition as $\sigma_1^1 \propto \theta$, which leads

. Equation (3.77), which shows that $q \rightarrow \alpha_2 - 1$ when $t \rightarrow \infty$. This puts a limit on α_2 as $1 \leq \alpha_2 \leq 2$. Hence, we have developed two different cosmological models based on two different values for α_2 , such as $\alpha_2 = 1$ and $\alpha_2 = \frac{3}{2}$. On the other hand, for some other choices of α_2 , we encountered unrealistic expressions or were unable to find the integral expressions.

3.4.1 Model-III, For the case: $\alpha_2 = 1$

The metric potentials (3.20)-(3.22), in this case, take the form of

$$\begin{aligned} A(t) &= \alpha_5^{\frac{3m}{m+1}} (t^2 + \alpha_1)^{\frac{3m}{2(m+1)}} \\ B(t) &= \sqrt{\alpha_3} \alpha_5^{\frac{3}{2(m+1)}} (t^2 + \alpha_1)^{\frac{3}{4(m+1)}} \exp \left[\frac{\alpha_4 t}{2\alpha_1 \alpha_5^3 \sqrt{t^2 + \alpha_1}} \right] \\ C(t) &= \frac{1}{\sqrt{\alpha_3}} \alpha_5^{\frac{3}{2(m+1)}} (t^2 + \alpha_1)^{\frac{3}{4(m+1)}} \exp \left[-\frac{\alpha_4 t}{2\alpha_1 \alpha_5^3 \sqrt{t^2 + \alpha_1}} \right] \end{aligned} \quad (3.82)$$

As a result, the required metric (3.1) is

$$ds^2 = -dt^2 + \alpha_5^{\frac{6m}{m+1}} (t^2 + \alpha_1)^{\frac{3m}{(m+1)}} dx^2 + \alpha_3 \alpha_5^{\frac{3}{(m+1)}} (t^2 + \alpha_1)^{\frac{3}{2(m+1)}} \left[e^{t^*} dy^2 + \frac{e^{-t^*}}{\alpha_3^2} dz^2 \right] \quad (3.83)$$

where $\alpha_5 = \frac{R_0}{\sqrt{\alpha_1}}$ and $t^* = \frac{\alpha_4 t}{2\alpha_1 \alpha_5^3 \sqrt{t^2 + \alpha_1}}$. The gauge function $\beta(t)$ from equation (3.11), using equation (3.82) will takes the form

$$\beta(t) = \frac{N}{\alpha_5^3 (t^2 + \alpha_1)^{\frac{3}{2}}} \quad (3.84)$$

Where N is constant of integration. The directional Hubble parameter H_i along x , y and z axes are given by

$$H_1 = \frac{A_4}{A} = \frac{3mt}{(m+1)(t^2 + \alpha_1)} \quad (3.85)$$

$$H_2 = \frac{B_4}{B} = \frac{3\alpha_5^3 t \sqrt{t^2 + \alpha_1} + \alpha_4(m+1)}{2(t^2 + \alpha_1)^{\frac{3}{2}} \alpha_5^3 (m+1)} \quad (3.86)$$

$$H_3 = \frac{C_4}{C} = \frac{3\alpha_5^3 t \sqrt{t^2 + \alpha_1} - \alpha_4(m+1)}{2(t^2 + \alpha_1)^{\frac{3}{2}} \alpha_5^3 (m+1)} \quad (3.87)$$

The following physical parameters for the model (3.83) are deduced as

$$\theta = 3H = \frac{3t}{t^2 + \alpha_1} \quad (3.88)$$

$$\sigma^2 = \frac{12(m-0.5)^2 t^2 (t^2 + \alpha_1) \alpha_5^6 + \alpha_4^2 (m+1)^2}{4\alpha_5^6 (m+1)^2 (t^2 + \alpha_1)^3} \quad (3.89)$$

$$A_m = \frac{12(m-0.5)^2 t^2 (t^2 + \alpha_1) \alpha_5^6 + \alpha_4^2 (m+1)^2}{6\alpha_5^6 (m+1)^2 t^2 (t^2 + \alpha_1)} \quad (3.90)$$

The expansion scalar θ decreases with cosmic time. At a initial time t , the expansion scalar is zero, and it approaches zero when the cosmic time approaches infinity. The shear scalar σ^2 of the model depends on the gauge function β , and it has the same qualitative behavior as the expansion scalar θ . The shear scalar σ^2 evolves with constant value $\frac{\alpha_4^2}{4\alpha_1^3 \alpha_5^6}$ initially, and it approaches zero as cosmic time approaches infinity. Likewise, the average anisotropy parameter A_m decreases as cosmic time progresses. Initially, it takes a large value and continues to decrease. In late time, A_m has a constant value $\frac{2(m-0.5)}{(m+1)^2}$, and it shows that the parameter A_m vanishes for $m = 0.5$. Using the

equations (3.84) to (3.87), in differential equation (3.10), we obtained ρ as

$$\rho = \frac{9 t^2 (4 m + 1) (t^2 + \alpha_1) \alpha_5^6 - (m + 1)^2 (3 N^2 + \alpha_4^2)}{4 \alpha_5^6 (m + 1)^2 (t^2 + \alpha_1)^3} \quad (3.91)$$

The energy density ρ of the model depends on the gauge function β . It begins with a negative value $-\frac{3 N^2 + \alpha_4^2}{4 \alpha_5^6 \alpha_1^3}$, changes to a positive value, and then reaches zero when $t \rightarrow \infty$ [See Fig. 3.2]. Using the equations (3.84) and (3.91) in equation (3.7), the EoS parameter ω have been calculated as

$$\omega = \frac{3 \alpha_5^6 (4 \alpha_1 (m + 1) + (-4 m + 5) t^2) (t^2 + \alpha_1) + (m + 1)^2 (3 N^2 + \alpha_4^2)}{(m + 1)^2 (3 N^2 + \alpha_4^2) - 9 t^2 (4 m + 1) (t^2 + \alpha_1) \alpha_5^6} \quad (3.92)$$

In Fig 3.3 and Fig 3.4 , we see the profile of the EoS parameter ω for different N and m respectively, where ω starts with a positive value $\frac{(m+1)\alpha_4^2+(3m+3)N^2+12\alpha_1^2\alpha_5^6}{(3N^2+\alpha_4^2)(m+1)}$ and then gradually slides into a negative value as time passes. At late time $t \rightarrow \infty$, the EoS parameter ω approaches $\frac{4m-5}{3(4m+1)}$ and vanishes for $m = \frac{5}{4}$. In other words, as time progresses, the EoS parameter ω starts in the quintessence region and moves along into the phantom region and then in the quintessence region. We write the values of the skewness parameter γ and δ , using the equations (3.7), (3.8), (3.84) and (3.91) as follows :

$$\gamma = \delta = \frac{6 (2 t^2 + \alpha_1) (t^2 + \alpha_1) (m + 1) (2 m - 1) \alpha_5^6}{(m + 1)^2 (3 N^2 + \alpha_4^2) - 9 t^2 (4 m + 1) (t^2 + \alpha_1) \alpha_5^6} \quad (3.93)$$

Fig. 3.5 presents the skewness parameter's (γ) profile for different N and m . Initially, gamma evolves with $\frac{(12m-6)\alpha_5^6\alpha_1^2}{(3N^2+\alpha_4^2)(m+1)}$ as its value, but it ends at $-\frac{4(2m-1)(m+1)}{3(4m+1)}$. The skewness parameter vanishes for $m = 0.5$ throughout the evolution. For the value of time $t = (\tau_0^2)_\pm$ given by

$$(\tau_0^2)_\pm = \frac{27 \alpha_1 \alpha_5^3 \pm \sqrt{12 (m + 1)^2 (4 m - 5) (3 N^2 + \alpha_4^2) + 9 \alpha_1^2 \alpha_5^6 (8 m - 1)^2}}{6 (4 m - 5) \alpha_5^3} \quad (3.94)$$

which leads to equation of state parameter $\omega = 0$. So that dusty universe at time $t = (\tau_0^2)_\pm$ given by equation (3.94). It is to be noted that when time t lies in open interval

$(\tau_1^2)_\pm < t < (\tau_2^2)_\pm$ where

$$(\tau_1^2)_\pm = \frac{3(133 - 668m)\alpha_1\alpha_5^3 \pm \sqrt{356(m+1)^2(2404m+1)(3N^2 + \alpha_4^2) + \alpha_5^6\alpha_1^2(2804m+401)^2}}{2(2404m+1)\alpha_5^3} \quad (3.95)$$

$$(\tau_2^2)_\pm = \frac{3(119 - 124m)\alpha_1\alpha_5^3 \pm \sqrt{108(572m-157)(m+1)^2(3N^2 + \alpha_4^2) + \alpha_5^6\alpha_1^2(772m+43)^2}}{2(572m-157)\alpha_5^3} \quad (3.96)$$

the EoS parameter ω lies in the interval $-1.67 < \omega < -0.62$. This range of ω follows the observational limit of Knop et al. [184]. Further for the value of time $t = (\tau^2)_\pm$, given by

$$(\tau^2)_\pm = \frac{9(1-2m)\alpha_1\alpha_5^3 \pm \sqrt{12(m+1)^2(8m-1)(3N^2 + \alpha_4^2) + 9\alpha_1^2(10m+1)^2\alpha_5^6}}{6(8m-1)\alpha_5^3} \quad (3.97)$$

the values of ω comes out to be -1 . For flat model, the matter-energy density Ω_M and dark energy Ω_Λ , satisfies

$$\Omega_M + \Omega_\Lambda = 1 \quad (3.98)$$

where

$$\Omega_M = \frac{\rho}{3H^2} \quad \text{and} \quad \Omega_\Lambda = \frac{\Lambda}{3H^2} \quad (3.99)$$

Thus, equations (3.98) and (3.99) gives

$$\rho + \Lambda = 3H^2 \quad (3.100)$$

and then using equations (3.81) and (3.91) in (3.100), we write the values of Λ as

$$\Lambda = \frac{(m+1)^2(3N^2 + \alpha_4^2) + 3(2m-1)^2t^2(t^2 + \alpha_1)\alpha_5^6}{4\alpha_5^6(m+1)^2(t^2 + \alpha_1)^3} \quad (3.101)$$

The profile of cosmological constant Λ for different N and m is presented in Fig. 3.6.

It shows that cosmological constant Λ is positive and decreasing function of cosmic time t for different N whereas for certain values of m , it increases for certain period of time and then decreases. Further, a positive cosmological constant Λ indicates the negative effective mass density. The expansion of the universe should therefore tend to accelerate due to positive cosmological constants. The matter energy density Ω_M and dark energy Ω_Λ are expressed as

$$\Omega_M = \frac{9t^2(4m+1)(t^2+\alpha_1)\alpha_5^6 - (m+1)^2(3N^2+\alpha_4^2)}{12(m+1)^2\alpha_5^6(t^2+\alpha_1)t^2} \quad (3.102)$$

$$\Omega_\Lambda = \frac{(m+1)^2(3N^2+\alpha_4^2) + 3(2m-1)^2t^2(t^2+\alpha_1)\alpha_5^6}{12(m+1)^2\alpha_5^6(t^2+\alpha_1)t^2} \quad (3.103)$$

A plot of the matter energy density Ω_m and dark energy Ω_Λ is shown in Fig. 3.7 and Fig. 3.8, respectively. The dark energy Ω_Λ is a decreasing function of cosmic time t , taking positive values over time for different N and m , while the matter energy density Ω_m is an increasing function of cosmic time t , taking values from negative to positive for different N and m .

3.4.2 Model-IV, For the case: $\alpha_2 = \frac{3}{2}$

The metric potentials (3.20)-(3.22), in this case, take the form of

$$\begin{aligned} A(t) &= \left(\frac{\alpha_6}{3}\right)^{\frac{3m}{m+1}} (27t^2 + 18\alpha_1)^{\frac{m}{m+1}} \\ B(t) &= \sqrt{\alpha_3} \left(\frac{\alpha_6}{3}\right)^{\frac{3}{2(m+1)}} (27t^2 + 18\alpha_1)^{\frac{1}{2(m+1)}} \exp \left[\frac{\alpha_4\sqrt{6} \arctan(\frac{t\sqrt{6}}{2\sqrt{\alpha_1}})}{4\alpha_6^3\sqrt{\alpha_1}} \right] \\ C(t) &= \frac{1}{\sqrt{\alpha_3}} \left(\frac{\alpha_6}{3}\right)^{\frac{3}{2(m+1)}} (27t^2 + 18\alpha_1)^{\frac{1}{2(m+1)}} \exp \left[-\frac{\alpha_4\sqrt{6} \arctan(\frac{t\sqrt{6}}{2\sqrt{\alpha_1}})}{4\alpha_6^3\sqrt{\alpha_1}} \right] \end{aligned} \quad (3.104)$$

As a result, the required metric (3.1) is

$$\begin{aligned} ds^2 &= -dt^2 + \left(\frac{\alpha_6}{3}\right)^{\frac{6m}{m+1}} (27t^2 + 18\alpha_1)^{\frac{2m}{m+1}} dx^2 \\ &\quad + \left(\frac{\alpha_6}{3}\right)^{\frac{3}{m+1}} (27t^2 + 18\alpha_1)^{\frac{1}{m+1}} \left[\alpha_3 e^{t^*} dy^2 + \frac{1}{\alpha_3} e^{-t^*} dz^2 \right] \end{aligned} \quad (3.105)$$

where $\alpha_6 = R_0 \left(\frac{3}{2\alpha_1} \right)^{\frac{1}{3}}$ and $t_1^* = \frac{\alpha_4 \sqrt{6} \arctan(\frac{t\sqrt{6}}{2\sqrt{\alpha_1}})}{2\alpha_6^3 \sqrt{\alpha_1}}$. The displacement vector $\beta(t)$ from equation (3.11), using equation (3.104) will takes the form

$$\beta(t) = \frac{N}{\alpha_6^3(t^2 + \frac{2\alpha_1}{3})} \quad (3.106)$$

Where N is constant of integration. The directional Hubble parameter H_i along x , y and z axes are given by

$$H_1 = \frac{A_4}{A} = \frac{54mt}{(m+1)(27t^2 + 18\alpha_1)} \quad (3.107)$$

$$H_2 = \frac{B_4}{B} = \frac{3(2\alpha_6^3 t + \alpha_4 m + \alpha_4)}{2(m+1)(3t^2 + 2\alpha_1)\alpha_6^3} \quad (3.108)$$

$$H_3 = \frac{C_4}{C} = \frac{1}{3t^2 + 2\alpha_1} \left[\frac{3t}{m+1} - \frac{3\alpha_4}{2\alpha_6^3} \right] \quad (3.109)$$

The following physical parameters for the model (3.105) are deduced as

$$H = \frac{2t}{3t^2 + 2\alpha_1} \Rightarrow \theta = 3H = \frac{6t}{3t^2 + 2\alpha_1} \quad (3.110)$$

$$\sigma^2 = \frac{3(4(2m-1)^2\alpha_6^6 t^2 + 3(m+1)^2\alpha_4^2)}{4(m+1)^2(3t^2 + 2\alpha_1)^2\alpha_6^6} \quad (3.111)$$

$$A_m = \frac{4(2m-1)^2\alpha_6^6 t^2 + 3(m+1)^2\alpha_4^2}{8\alpha_6^6(m+1)^2 t^2} \quad (3.112)$$

The expansion scalar θ increases for a certain period of time, then decreases with cosmic time t until it approaches zero in late time. Initially, the shear scalar σ^2 of the model has a constant value $\frac{9\alpha_4^2}{16\alpha_1^2\alpha_6^6}$, then increases for a period of time, then decreases with cosmic time t until it approaches zero. The shear scalar σ^2 tends to zero as cosmic time t approaches infinity, indicating that our model is shear free at late time. The average anisotropy parameter A_m of the model decreases as cosmic time t increases. It approaches $\frac{2(m-0.5)^2}{(m+1)^2}$, when cosmic time t tends to infinity. Using the equation (3.106) to (3.109), in differential equation (3.10), we obtained ρ as

$$\rho = \frac{9[4t^2(4m+1)\alpha_6^6 - (m+1)^2(3N^2 + \alpha_4^2)]}{4(m+1)^2(3t^2 + 2\alpha_1)^2\alpha_6^6} \quad (3.113)$$

The profile of energy density can be seen in Fig. 3.9 for different N and m . Starting from a constant value $-\frac{9(3N^2+\alpha_4^2)}{16\alpha_6^6\alpha_1^2}$, it increases for a while and then decreases with cosmic time until it reaches zero when cosmic time reaches infinity. Using the equations (3.113) and (3.106) in equation (3.7) the EoS parameter ω have been calculated as

$$\omega = \frac{3(m+1)^2(3N^2+\alpha_4^2) + 16(m+1)\alpha_6^6\alpha_1 - 12t^2\alpha_6^6(2m-1)}{3(m+1)^2(3N^2+\alpha_4^2) - 12t^2(4m+1)\alpha_6^6} \quad (3.114)$$

In Fig 3.10 and Fig 3.11, we see the profile of the EoS parameter ω for different N and m , where ω starts with a positive value $\frac{(3m+3)\alpha_4^2+(9m+9)N^2+16\alpha_6^6\alpha_1}{3(m+1)(3N^2+\alpha_4^2)}$. and then gradually slides into a negative value as time passes. In terms of qualitative behavior, it is similar to Model-III. We write the values of the skewness parameter γ and δ , using the equation (3.7), (3.8),(3.106) and (3.113) as follows :

$$\gamma = \delta = \frac{4(m+1)(3t^2+2\alpha_1)(2m-1)\alpha_6^6}{3(m+1)^2(3N^2+\alpha_4^2) - 12t^2(4m+1)\alpha_6^6} \quad (3.115)$$

The skewness parameter γ at the early stage $t \rightarrow 0$ gives the constant value equals to $\frac{8\alpha_6^6(2m-1)\alpha_1}{3(m+1)(3N^2+\alpha_4^2)}$ and approaches $-\frac{(m+1)(2m-1)}{4m+1}$ when cosmic time approaches infinity [See Fig.3.12]. When $m = 0.5$, the skewness parameter γ vanishes over time. For the value of time $t = (\tau_0)_{\pm}$ given by

$$(\tau_0)_{\pm} = \pm \frac{\sqrt{3(2m-1)(m+1)(3(3N^2+\alpha_4^2)(m+1) + 16\alpha_6^6\alpha_1)}}{6(2m-1)\alpha_6^3} \quad (3.116)$$

which leads to equation of state parameter $\omega = 0$. So that dusty universe at time $t = \tau_0$ given by equation (3.116). It is to be noted that when time t lies in open interval $(\tau_1)_{\pm} < t < (\tau_2)_{\pm}$ where

$$(\tau_1)_{\pm} = \pm \frac{\sqrt{3(868m+67)(m+1)(801(3N^2+\alpha_4^2)(m+1) + 1600\alpha_6^6\alpha_1)}}{6(868m+67)\alpha_6^3} \quad (3.117)$$

$$(\tau_2)_{\pm} = \pm \frac{\sqrt{3(224m-19)(m+1)(243(3N^2+\alpha_4^2)(m+1) + 800\alpha_6^6\alpha_1)}}{6(224m-19)\alpha_6^3} \quad (3.118)$$

the EoS parameter ω lies in the interval $-1.67 < \omega < -0.62$. This range of ω follows the observational limit of Knop et al. [184]. Further for the value of time $t = (\tau)_{\pm}$,

given by

$$(\tau)_{\pm} = \frac{\sqrt{m(m+1)(3(3N^2 + \alpha_4^2)(m+1) + 8\alpha_6^6\alpha_1)}}{6m\alpha_6^3} \quad (3.119)$$

the values of ω comes out to be -1 . For flat model, the matter-energy density Ω_M and dark energy Ω_Λ , satisfies

$$\Omega_M + \Omega_\Lambda = 1 \quad (3.120)$$

where

$$\Omega_M = \frac{\rho}{3H^2} \text{ and } \Omega_\Lambda = \frac{\Lambda}{3H^2} \quad (3.121)$$

Thus, equations (3.120) and (3.121) gives

$$\frac{\rho}{3H^2} + \frac{\Lambda}{3H^2} = 1$$

and then using equation (3.81) and (3.113), we write the values of Λ as

$$\Lambda = \frac{3[3(m+1)^2(3N^2 + \alpha_4^2) + 4t^2(2m-1)^2\alpha_6^6]}{4(m+1)^2(3t^2 + 2\alpha_1)^2\alpha_6^6} \quad (3.122)$$

It can be seen from Fig. 3.13 that the cosmological constant Λ exhibits the same qualitative behavior as model-III. From equations (3.81), (3.113) and (3.122), we have

$$\Omega_M = \frac{3[4t^2(4m+1)\alpha_6^6 - (m+1)^2(3N^2 + \alpha_4^2)]}{16t^2(m+1)^2\alpha_6^6} \quad (3.123)$$

$$\Omega_\Lambda = \frac{3(m+1)^2(3N^2 + \alpha_4^2) + 4t^2(2m-1)^2\alpha_6^6}{16t^2(m+1)^2\alpha_6^6} \quad (3.124)$$

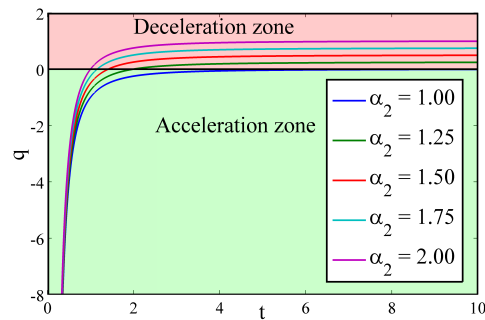


FIGURE 3.1: Profile of deceleration parameter q against time

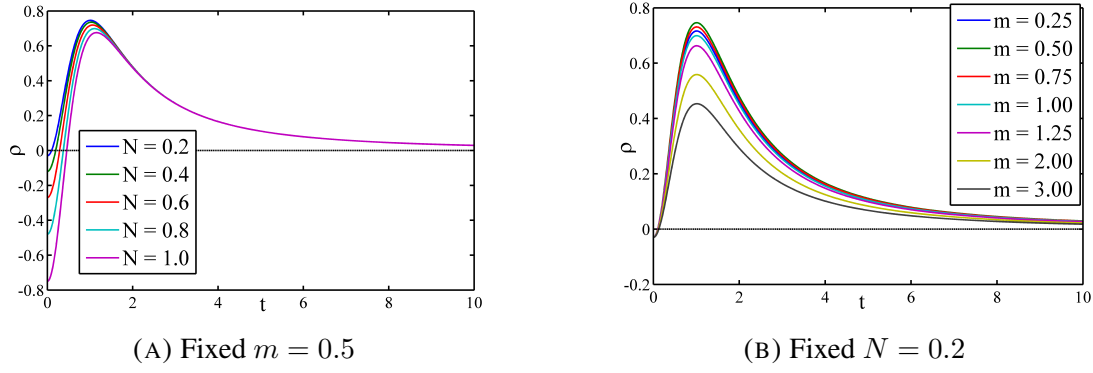


FIGURE 3.2: Profile of energy density for Model-III against time for different N and m

Mass energy density Ω_M and dark energy Ω_Λ exhibit the same qualitative characteristics as model-III [See Fig. 3.14 and Fig. 3.15].

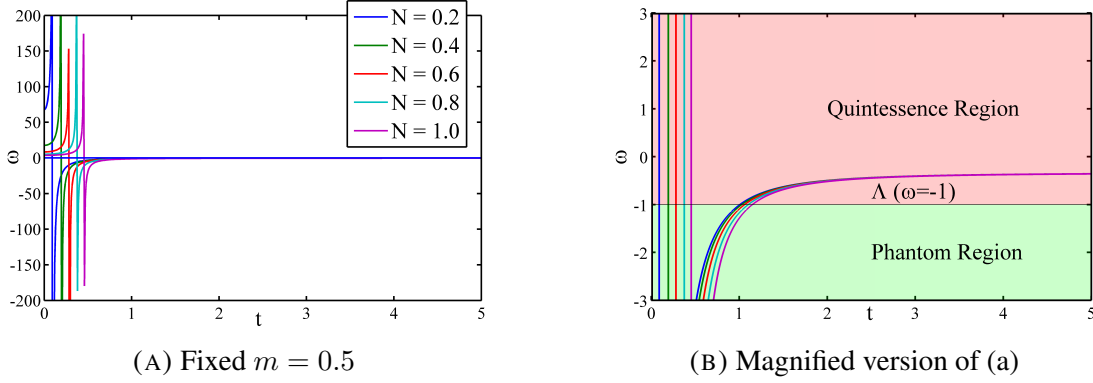


FIGURE 3.3: Profile of EoS parameter ω for Model-III against time for different N .

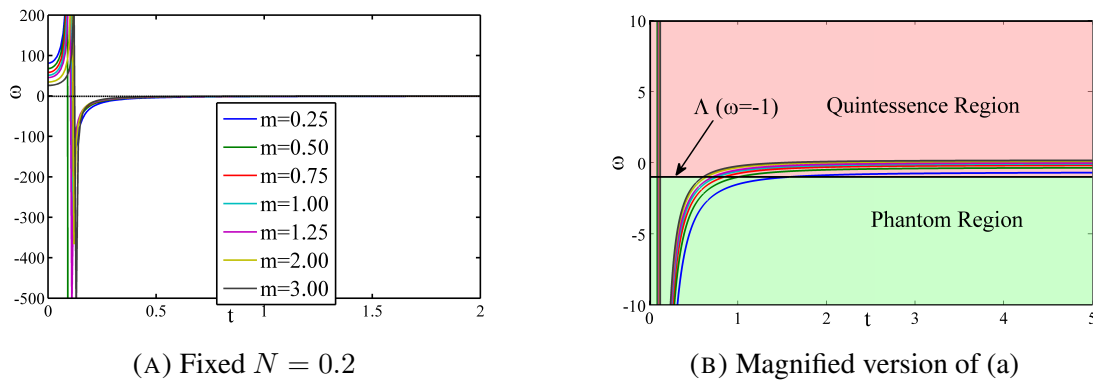


FIGURE 3.4: Profile of EoS parameter ω for Model-III against time for different m .

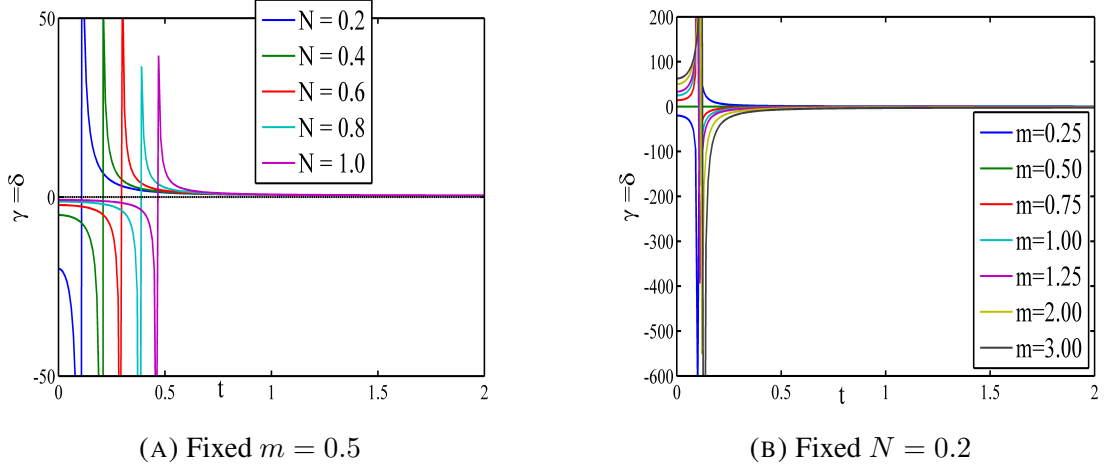


FIGURE 3.5: Profile of Skewness parameter $\gamma = \delta$ for Model-III against time for different N and m .

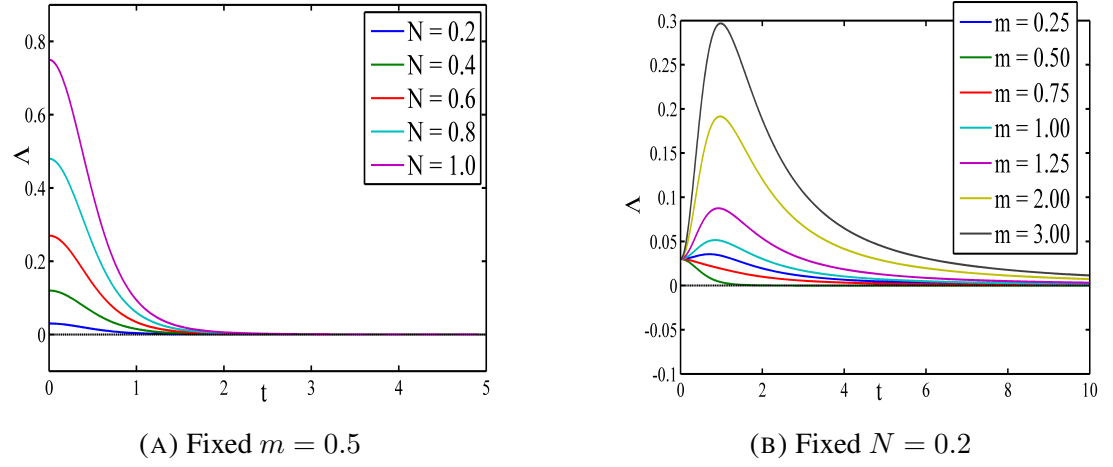


FIGURE 3.6: Profile of Cosmological constant Λ for Model-III against time for different N and m .

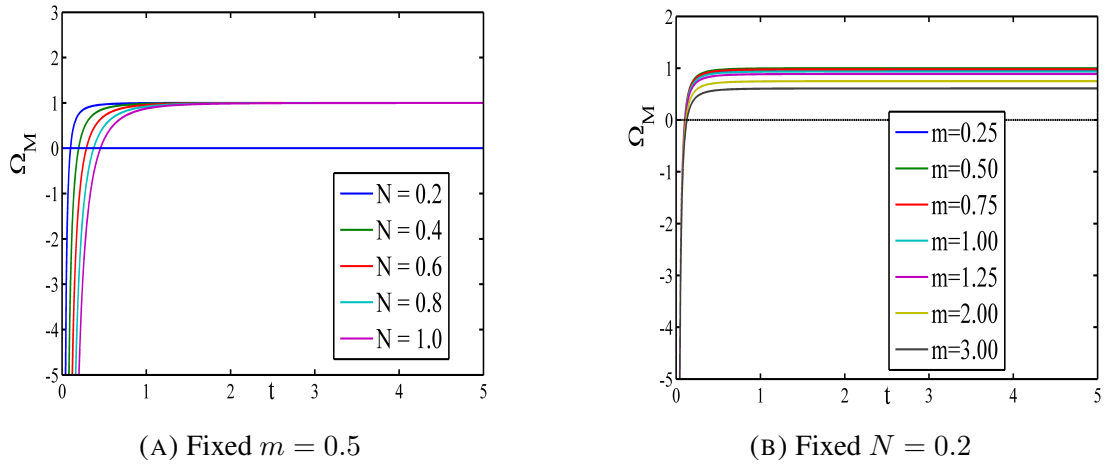


FIGURE 3.7: Profile of matter energy density Ω_M for Model-III against time for different N and m .

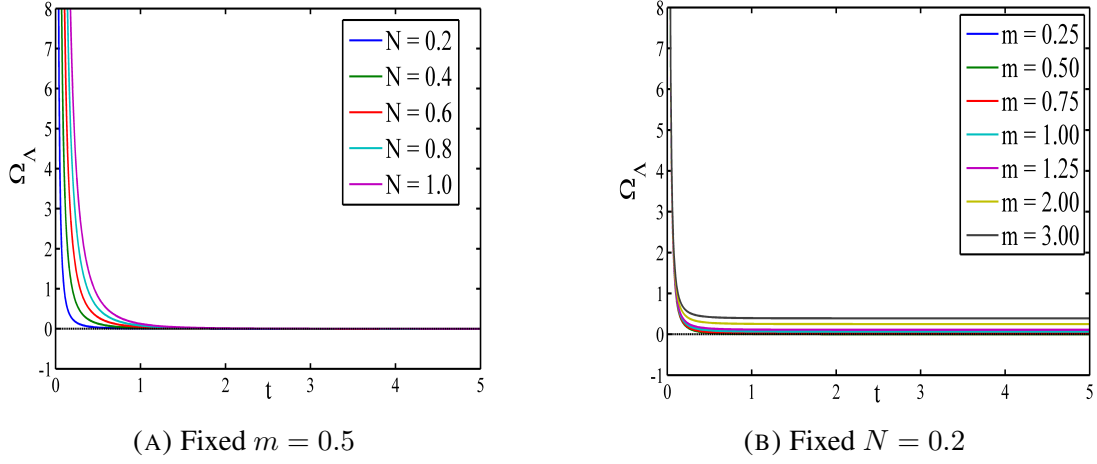


FIGURE 3.8: Profile of dark energy Ω_Λ for Model-III against time for different N and m .

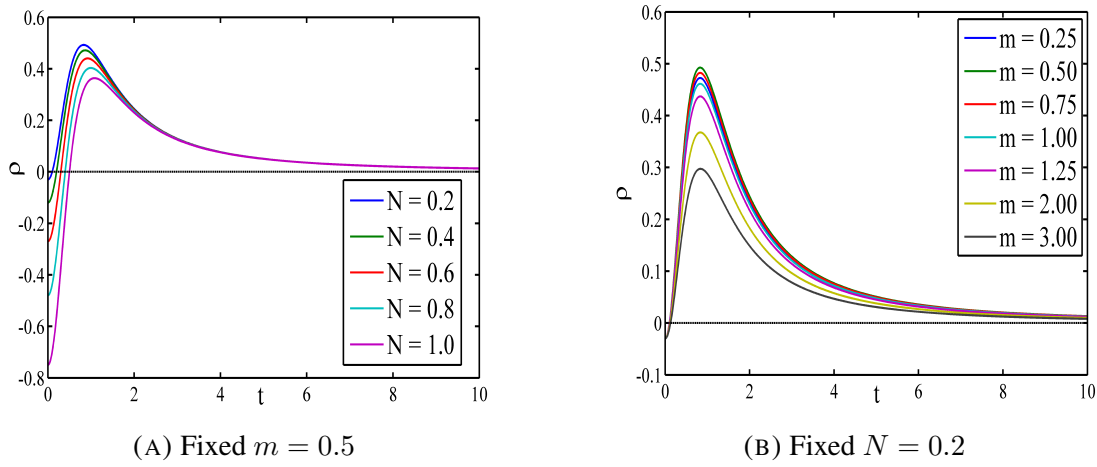


FIGURE 3.9: Profile of energy density for Model-IV against time for different N and m .

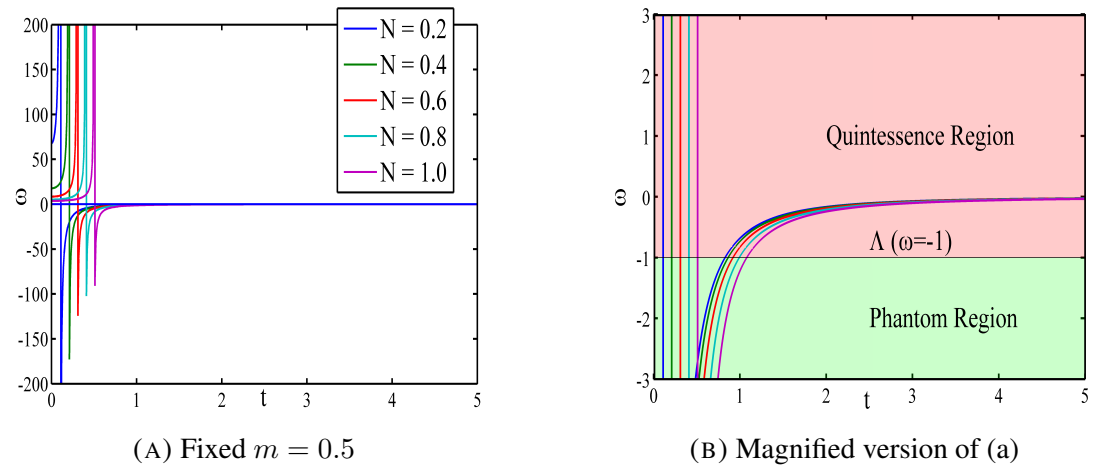


FIGURE 3.10: Profile of EoS parameter ω for Model-IV against time for different N .

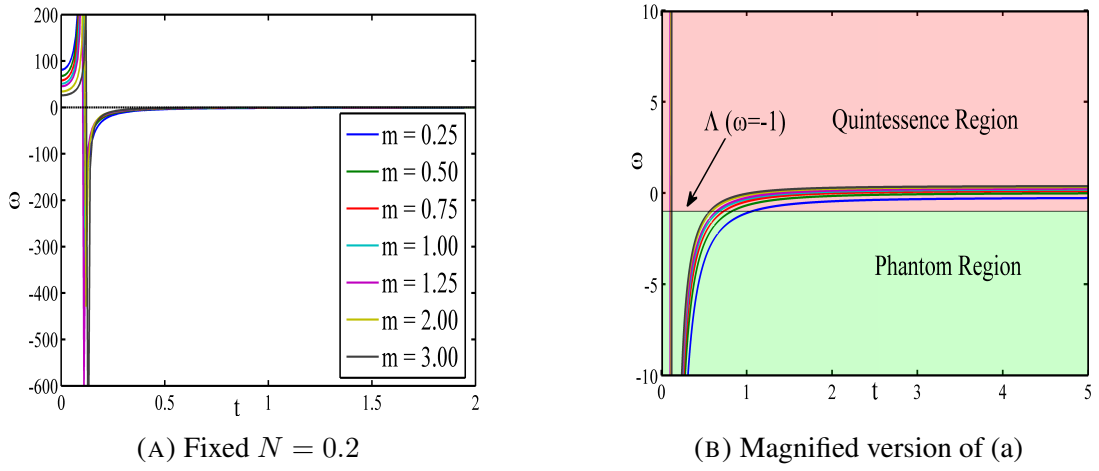


FIGURE 3.11: Profile of EoS parameter ω for Model-IV against time for different m .

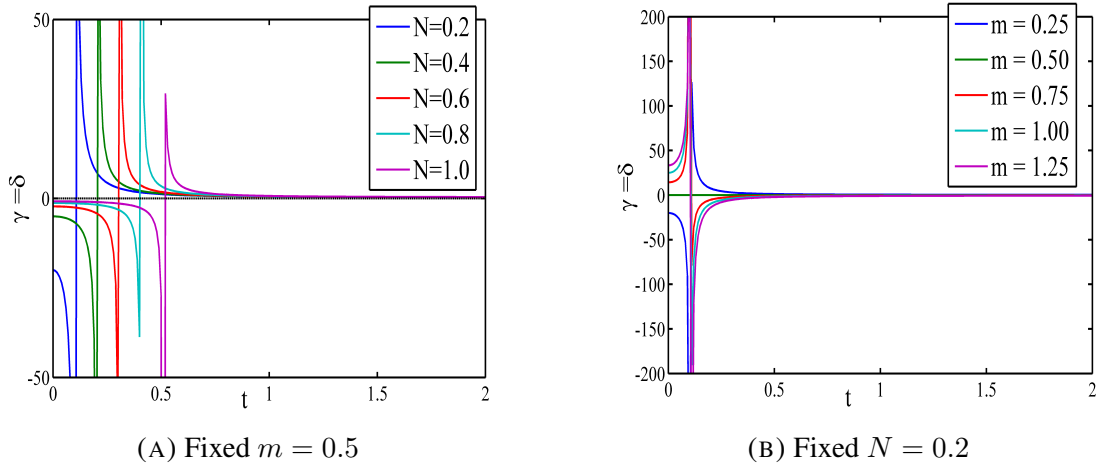


FIGURE 3.12: Profile of Skewness parameter $\gamma = \delta$ for Model-IV against time for different N and m .

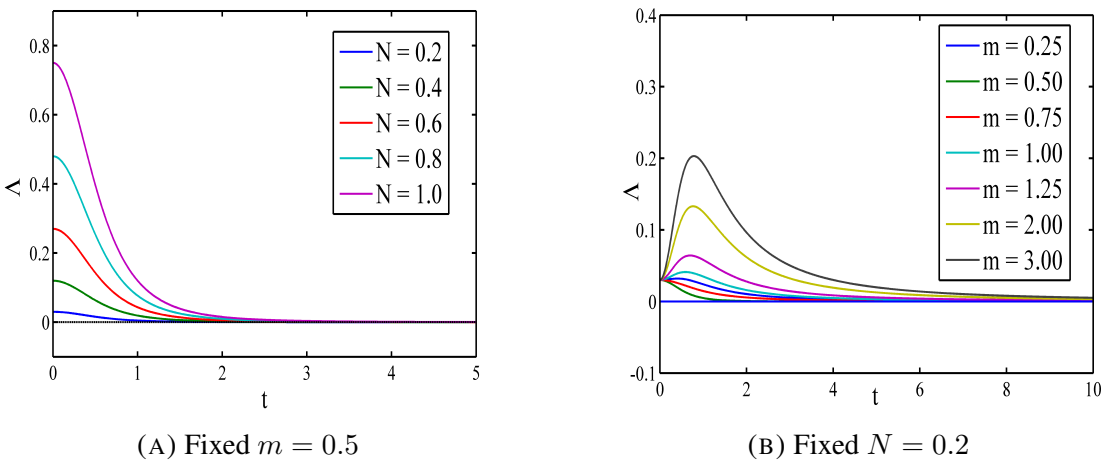


FIGURE 3.13: Profile of Cosmological constant Λ for Model-IV against time for different N and m .

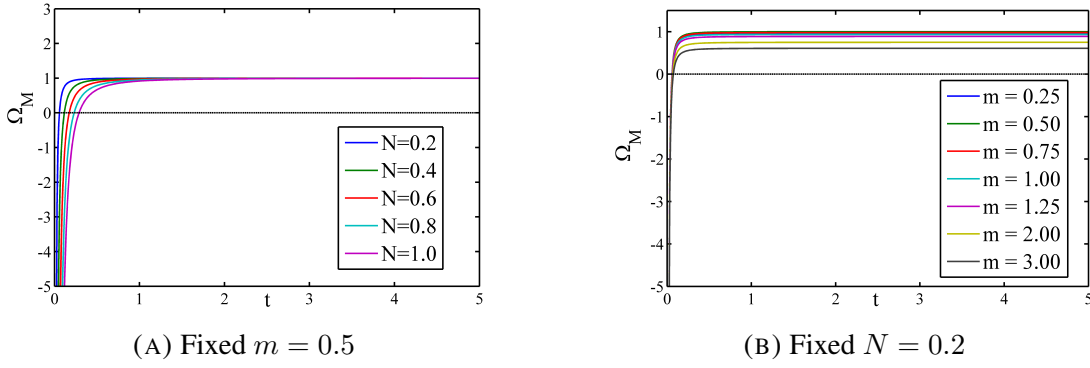


FIGURE 3.14: Profile of matter energy density Ω_M for Model-IV against time for different N and m .

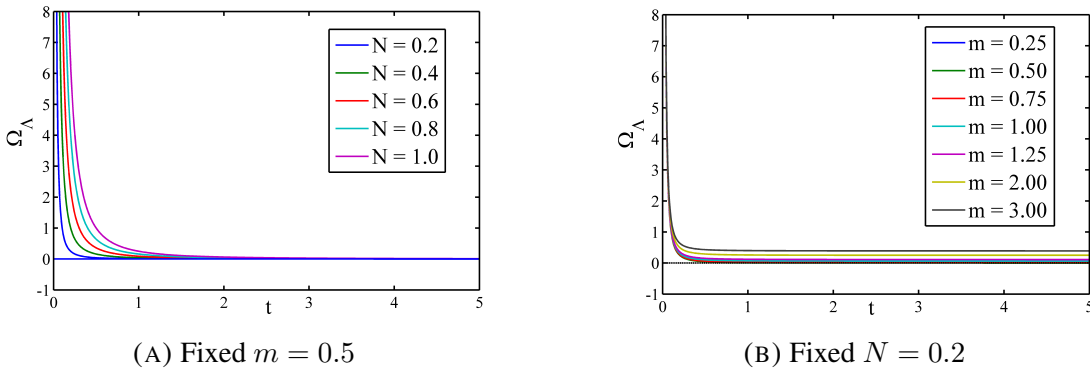


FIGURE 3.15: Profile of dark energy Ω_Λ for Model-IV against time for different N and m .

3.5 Concluding Remarks

The chapter offers a Bianchi type-I dark energy cosmological model with variable EoS parameter ω under variable deceleration parameter, $q = n - 1$ and $q = -\frac{\alpha_1}{t^2} + \alpha_2 - 1$ in the framework of Lyra geometry. The outcomes obtained from these models are presented below: In model-I and model-II, the energy density ρ starts with a constant value initially and approaches to same value $\frac{9l^2(4m+1)}{4(m+1)^2}$ at late time for $\alpha_1 = \pm 2$. For model I and II, the cosmological constant Λ is depending on the gauge function and in terms of cosmic time, the qualitative behaviour of Λ is decreasing in nature for both the discussed models. In case of model-I (exponential model) and model-II (power law model), cosmological constant Λ approaches to $\frac{3l^2(2m-1)^2}{4(m+1)^2}$ and zero at late time respectively. In both the model-I and II, the dark energy Ω_Λ is a decreasing function of cosmic time t . In model-II ($n = 0$) and in Model-I ($0 < n < 2$) $\Omega_\Lambda \rightarrow 1 - \frac{3(4m+1)}{4(m+1)^2}$ and $\Omega_\Lambda \rightarrow \frac{(2m-1)^2}{4(m+1)^2}$ respectively at late time.

Depending upon the parameter α_2 we have discussed two cosmological models. Model-III is dominated by phantom dark energy whereas Model-IV, follow quintessence-phantom-quintessence dark energy with respect to the evolution of time. According to the recent observations, EoS parameters ω presented in equations (3.92) and (3.114) for models III and IV are in line with observations [181, 184]. For model III and IV, the cosmological constant Λ is positive and decaying with respect to time, indicating a negative effective mass density. Therefore, the toy models of our universe are accelerating. For model III and IV,, the energy density ρ starts off as negative and increases over time, and then decreases until it approaches zero with the evolution of cosmic time. The EoS parameter ω is time-dependent here, such a type of $\omega(t)$ was previously discussed by Pradhan et al. [207]. It can also be determined by redshift z or scale factor R . The redshift dependence of ω can be linear or non-linear [212, 213]. If $N = 0 \Rightarrow \beta = 0$ and $\alpha_1 = 0$, then our solution resembles that for a constant deceleration parameter $q = \alpha_2 - 1$ as in general relativity, as studied by Pradhan et al. [207] and Kumar & singh [214].

Chapter 4

Gödel universe in various functional form of $f(R, T)$ gravity

4.1 Introduction

Observations like the high red shift supernova experiments (HRSSE)[215–217], cosmic microwave background (CMB) fluctuations [218, 219] and large scale structure (LSS)[220] directly or indirectly reveal that the present universe is expanding in an accelerated way. The most successful theory of gravitation is general relativity. But as per the present scenario of our universe, general relativity cannot explain the cause of this acceleration. Researchers are trying to find the cause of this acceleration, in which galaxies are moving away from each other with accelerating expansion. It appears that some mysterious energy exists, which is responsible for the cosmic acceleration. This mysterious form of energy is known as dark energy. Still today it is a matter of discussion amongst the scientific community whether dark energy really exists or not. To address such an issue, we need modified gravity theories, but keeping the basic concepts of general relativity introduced by Einstein. The important question is how to define gravity in modified theories. These theories approach the definition of gravity in a slightly different way from general relativity. The simplest theories are based on the scalar curvature R . But such theories based on the scalar curvature work well in explaining solar system observations, whereas they face a problem to explain dark matter and dark energy beyond the galaxy and inside the disk of a galaxy.

In the literature, various alternative theories of gravity are available. The development of alternative or modified theories is due mainly to two reasons, i.e., quantum and cosmological. In the quantum case, one has to construct a perturbatively consistent gravity theory, whereas in the cosmological case, one constructs the models to handle the problem of late time acceleration and the smallness of the cosmological constant at present. A number of cosmological models have been discussed, addressing the above-stated issues. Various modified theories of gravity have been proposed in the last few decades such as $f(T)$ gravity [221–223], $f(Q)$ gravity [224–226], $f(Q, T)$ gravity [227–229], R^2 gravity [230–232], $f(G)$ gravity [233–235], $f(R, G)$ gravity [236–238], $f(R)$ gravity [239–244] and $f(R, T)$ gravity [96]. In this manuscript, we deal with $f(R, T)$ gravity, which is a generalization of $f(R)$ gravity. $f(R)$ gravity deals with a general function of the Ricci scalar R in the action principle. This gravity theory is a family of theories depending upon the functional form of R . The simplest form $f(R) = R$ is the case of general relativity. Thus the general function $f(R)$ may give extra freedom to deal with problems like accelerated expansion of the universe and structure formation without any addition of dark energy and dark matter. Further, it is generalized to $f(R, T)$ gravity, in which the functional form is considered to be an arbitrary function of the Ricci scalar R and the trace T of the energy momentum tensor of matter. Like $f(R)$ gravity, $f(R, T)$ gravity is also a family of theories depending on the form of the functional term $f(R, T)$. Basically three forms of $f(R, T)$ gravity are usually considered in the literature. Some of the recent work based on $f(R, T)$ gravity is discussed below.

Recently, Singh and Beesham [245, 246] studied the plane symmetric and LRS Bianchi type-I cosmological models in the first class of $f(R, T)$ gravity. Sahoo et al. [247] investigated traversable wormholes in traceless $f(R, T)$ gravity. In that article, a spherically symmetric static traversable wormhole was investigated, which led to satisfactory compliance with the energy conditions. Godani and Samanta [248] devoted their study to charged wormholes, and evaluated the deflection angle using strong field limit coefficients. Also, they derived an exact solution of charged wormholes in $f(R, T)$ gravity. A traversable wormhole with a logarithmic shape function was investigated by Dixit et al. [249]. The presented logarithmic shape function in the linear case $f(R, T) = R + 2\lambda T$ represents the existence of exotic and non-exotic matter. A wormhole solution in exponential $f(R, T)$ gravity was investigated by Moraes and Sahoo [250] and they showed that it is possible to obtain wormhole solutions satisfying the energy conditions.

Known for its unique unusual global properties, the Gödel universe is an exact solution to Einstein's field equations with rotation. It is both stationary and spatially homogeneous [18]. In this space-time, the most intriguing feature is the existence of closed timelike curves which violate global causality and allow time travel to be theoretically possible. In 1921, H. Weyl proposed that time travel might be possible in general relativity [251] if there are "considerable fluctuations in the space-time metric", but he believed that this kind of fluctuation does not occur in the region where we live. There has been growing interest in quantum computation, and the presence of closed timelike curves in space-time has led to a model for quantum computation in compact regions. The NP-complete problems can be solved with a quantum computer based on closed timelike curves with only a polynomial number of quantum logic gates. String theories have also taken into account the insights gained from a detailed analysis of the Gödel space-time in order to explore the possibility of closed timelike paths (See [252] and references therein). In the years following the Gödel solution, a large amount of research has been done on rotating solutions for the Einstein field equations, and on rotational cosmological models. In spite of the fact that the Gödel space-time does not resemble our universe well, it is an important theoretical tool for investigating a variety of global features of space-time in various gravity theories.

Bonnor et al. [253] examined a spacetime with Einstein's equations when the cosmological constant was negative, describing an infinite cylinder within the Gödel universe surrounded by empty space. In their paper, they discussed the possibility that closed timelike curves (CTCs) exist to the matching problem on the boundary of the cylinder. In $f(R)$ gravity, Reboucas and Santos [254] studied the Gödel universe and showed that this theory provides causal and non-causal Gödel-type solutions. Furtado et al [255] devoted their study to the Gödel universe in Chern–Simons modified gravity in the presence of a cosmological constant, and have shown that the Gödel metric is compatible with this theory when the cosmological constant is a variable one. Furthermore, they continued their study in Horava-Lifshitz gravity and checked the consistency of the Gödel metric [256]. Santos [257] studied the Gödel universe in $f(R, T)$ gravity for the cases $f(R, T) = R + \lambda T$ and $f(R, T) = R + \lambda RT$. In his research, he pointed out the resulting smallness of the cosmological constant. Santos and Ulhoa [258] studied the Gödel universe in the context of Rastall gravity, and found a solution for the rotating universe. These researchers have verified that Rastal gravity is inconsistent with a

Gödel-type solution exhibiting expansion. In Brans-Dicke theory, Gödel and Gödel type solutions were analyzed by Agudelo et al [259], and they pointed out the existence of CTCs, causality violation and consistency of the equation of motion. Further, such solutions were investigated in k-essence theory [260], which allows causal and non-causal Gödel-type solutions. The study by Shamir and Zia [261] deals with Gödel's universe in the context of $f(R, G)$ gravity, analyzing the energy conditions for the two functional forms of $f(R, G)$. Their results suggest that for a certain range of radial coordinates, Gödel's universe supports expansion of the universe. In Palatini $f(R, T)$ gravity, Gödel and Gödel type space-times were analyzed and the causality problem investigated by Gonçalves and Santos [262]. A number of modified gravity theories have been studied in the literature based on Gödel and Gödel-type solutions for different forms of the Gödel space-time [See [263–268] and references therein].

In the last decade, $f(R, T)$ gravity has gained much attention in the study of cosmological models due to its generalization of general relativity and $f(R)$ gravity. Following the above stated research work based on the Gödel universe in view of different modified theories of gravitation, we have been motivated to investigate the Gödel universe in the first class of $f(R, T)$ gravity, i.e., $f(R, T) = R + 2h(T)$, by considering different forms of $h(T)$. The novelty of this paper is to address the problem of the smallness of the cosmological constant and compatibility of closed timelike curves in $f(R, T) = R + 2h(T)$ gravity with a different form of $h(T)$ for a Gödel metric in the form (4.12). As we know, the Gödel metric is the simplest one to allow closed timelike curves, which may possible for time travel theoretically.

4.2 Basic Formalism

In $f(R, T)$ gravity [96], the gravitational Lagrangian is given by an arbitrary function of the Ricci scalar and the trace of the stress energy tensor, and also it depends on a source term representing the variation of the matter stress energy tensor with respect to the metric. The source term expression is obtained as a function of the matter Lagrangian (L_m). As a result, for each choice of L_m , a specific set of field equations is generated. The field equations of $f(R, T)$ gravity are obtained from a Hilbert-Einstein

type variational principle. The action for this gravity is given by [96, 257]

$$S = \frac{1}{2} \int \sqrt{-g} [f(R, T) + 2L_m] d^4x \quad (4.1)$$

In the above expression (4.1) $f(R, T)$ is an arbitrary function of the R and T , which indicates the scalar curvature and trace of matter energy tensor respectively. The main objective of introducing T in to the functional form is to introduce the exotic fluid or quantum effects (conformal analogy). Here we used the system of units, where $8\pi G = 1$. Let us define the energy momentum tensor as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}L_m)}{\delta g^{\mu\nu}} \quad (4.2)$$

Further, assume that the Lagrangian density L_m depends on the metric tensor components $g_{\mu\nu}$ and not on its derivatives. Thus we have

$$T_{\mu\nu} = g^{\mu\nu} L_m - 2 \frac{\partial L_m}{\partial g_{\mu\nu}} \quad (4.3)$$

Now we will derive the equation of motion for the above said gravity. Varying the action S as in (4.1) with respect to the metric components, we have

$$\begin{aligned} \delta S = \frac{1}{2} \int & \left[f_R(R, T) \delta R + f_T(R, T) \frac{\delta T}{\delta g^{\mu\nu}} \delta g^{\mu\nu} - \frac{1}{2} g^{\mu\nu} f(R, T) \delta g^{\mu\nu} \right. \\ & \left. + \frac{2}{\sqrt{-g}} \frac{\delta \sqrt{-g} L_m}{\delta g^{\mu\nu}} \right] \sqrt{-g} d^4x \end{aligned} \quad (4.4)$$

In the above expression $f_R(R, T)$ and $f_T(R, T)$ stands for $\frac{\partial f}{\partial R}$ and $\frac{\partial f}{\partial T}$ respectively. Now we have to find the variation of both scalar curvature R and the trace T of energy momentum tensor. The variation of R is expressed as

$$\delta R = R_{\mu\nu} \delta g^{\mu\nu} + g_{\mu\nu} \square \delta g^{\mu\nu} - \nabla_\mu \nabla_\nu g^{\mu\nu} \quad (4.5)$$

The expression for the variation of T is expressed as

$$\frac{\delta T}{\delta g^{\mu\nu}} = \frac{\delta g^{\mu\nu} T_{\mu\nu}}{\delta g^{\mu\nu}} = \frac{\delta g^{\alpha\beta}}{\delta g^{\mu\nu}} T_{\alpha\beta} + g^{\alpha\beta} \frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}} = T_{\mu\nu} + \Theta_{\mu\nu} \quad (4.6)$$

where $\Theta_{\mu\nu} = g_{\alpha\beta} \frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}}$. The modified field equations for $f(R, T)$ gravity model is derived as

$$f_R(R, T)R_{\mu\nu} - \frac{1}{2}f(R, T)g_{\mu\nu} + (g_{\mu\nu}\square - \nabla_\mu \nabla_\nu) f_R(R, T) = T_{\mu\nu} - f_T(R, T) (T_{\mu\nu} + \Theta_{\mu\nu}) \quad (4.7)$$

Note that when $f(R, T) = f(R)$ and $f(R, T) = f(R) = R$, equation (4.7) reduces to the field equations of $f(R)$ gravity and general relativity respectively. With the help of equation (4.3) and $g_{\alpha\beta}g^{\alpha\beta} = \delta^\beta_\alpha$, the tensor $\Theta_{\mu\nu}$ takes the form

$$\Theta_{\mu\nu} = -2T_{\mu\nu} + g_{\mu\nu}L_m - 2g^{\alpha\beta} \frac{\partial^2 L_m}{\partial g^{\mu\nu} \partial g^{\alpha\beta}} \quad (4.8)$$

For the stress-energy tensor of the matter in the form and matter Lagrangian $L_m = -p$ we arrive at

$$\Theta_{\mu\nu} = -2T_{\mu\nu} - pg_{\mu\nu} \quad (4.9)$$

Therefore, the modified field equations (4.7) finally takes the form

$$\begin{aligned} f_R(R, T)R_{\mu\nu} - \frac{1}{2}f(R, T)g_{\mu\nu} + (g_{\mu\nu}\square - \nabla_\mu \nabla_\nu) f_R(R, T) &= (1 + f_T(R, T)) T_{\mu\nu} \\ &+ pf_T(R, T)g_{\mu\nu} \end{aligned} \quad (4.10)$$

Now we focus on equation (4.10) to analyze the Gödel universe.

4.3 Analysing the Gödel solution

In this section, we deal with the case $f(R, T) = R + 2h(T)$. For this case the modified field equation (4.10) for $f(R, T)$ gravity model takes the form

$$G_{\mu\nu} = [1 + 2h_T(T)]T_{\mu\nu} + [2ph_T(T) + h(T)]g_{\mu\nu} \quad (4.11)$$

Let us consider the Gödel metric in the form

$$ds^2 = a^2 \left[dt^2 - dx^2 + \frac{1}{2}e^{2x}dy^2 - dz^2 + 2e^x dt dy \right], \quad (4.12)$$

here a is a positive number. The non-zero components of the Ricci tensor for the Gödel metric (4.12) is expressed as

$$R_{22} = e^{2x}, R_{24} = R_{42} = e^x, R_{44} = 1$$

. The Ricci scalar or scalar curvature $R = g^{\mu\nu} R_{\mu\nu}$ is $R = \frac{1}{a^2}$. In the present study, we consider the energy momentum tensor in the following form

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + (p + \Lambda)g_{\mu\nu}, \quad (4.13)$$

here u is the unit vector along the t line with $u_\mu = (a, 0, ae^x, 0)$. ρ , p and Λ represents the energy density, pressure and cosmological constant respectively. The non-zero components of energy momentum tensor are

$$\begin{aligned} T_{11} &= -(p + \Lambda)a^2 \\ T_{22} &= \left(\rho + \frac{3p}{2} + \frac{\Lambda}{2}\right)a^2 e^{2x} \\ T_{33} &= -(p + \Lambda)a^2 \\ T_{44} &= (\rho + 2p + \Lambda)a^2 \\ T_{24} &= (\rho + 2p + \Lambda)a^2 e^x \end{aligned} \quad (4.14)$$

The trace of the energy momentum tensor is expressed as $T = \rho + 5p + 4\Lambda$.

4.3.1 Solution for the choice of $h(T) = \lambda T^2$

Santos [257] has already discussed the case $f(R, T) = f_1(R) + f_2(T)$ by choosing $f_1(R) = R$ and $f_2(T) = 2\lambda T$, which means $f(R, T) = R + 2\lambda T$. When we compare it to our form, $h(T) = \lambda T$. Therefore, it is interesting to consider the T -dependence term to be quadratic in nature. As a result, we consider $h(T) = \lambda T^2$. In this case the modified field equations (4.11) takes the form

$$G_{\mu\nu} = [1 + 4\lambda T]T_{\mu\nu} + \lambda T[4p + T]g_{\mu\nu} \quad (4.15)$$

The non-zero components of equation (4.15) for the Gödel metric (4.12) along with (4.14) are expressed as

$$-65\lambda p^2 + (-92\Lambda\lambda - 18\lambda\rho - 1)p - 32\Lambda^2\lambda - 12\Lambda\lambda\rho - \lambda\rho^2 - \Lambda = \frac{1}{2a^2} \quad (4.16)$$

$$105\lambda p^2 + (124\Lambda\lambda + 66\lambda\rho + 3)p + 32\Lambda^2\lambda + 44\Lambda\lambda\rho + 9\lambda\rho^2 + \Lambda + 2\rho = \frac{3}{2a^2} \quad (4.17)$$

$$85\lambda p^2 + (108\Lambda\lambda + 42\lambda\rho + 2)p + 32\Lambda^2\lambda + 28\Lambda\lambda\rho + 5\lambda\rho^2 + \Lambda + \rho = \frac{1}{2a^2} \quad (4.18)$$

The above system is non linear algebraic system with three variables ρ , p and Λ . Let us assume, universe is filled with perfect fluid and, for simplicity, we choose the case in which pressure is zero i.e. $p = 0$. Thus the system of equations (4.16)-(4.18) reduces to

$$-32\Lambda^2\lambda - 12\Lambda\lambda\rho - \lambda\rho^2 - \Lambda = \frac{1}{2a^2} \quad (4.19)$$

$$32\Lambda^2\lambda + 44\Lambda\lambda\rho + 9\lambda\rho^2 + \Lambda + 2\rho = \frac{3}{2a^2} \quad (4.20)$$

$$32\Lambda^2\lambda + 28\Lambda\lambda\rho + 5\lambda\rho^2 + \Lambda + \rho = \frac{1}{2a^2} \quad (4.21)$$

We now discuss the solution for two cases, namely, $\lambda = 0$ and $\lambda \neq 0$. For general relativity and $f(R, T) = R + 2\lambda T^2$ gravity, solutions are found when $\lambda = 0$ and $\lambda \neq 0$, respectively. For $\lambda = 0$, the system of equations (4.19)-(4.21) has solution in the form $\rho = \frac{1}{a^2}$ and $\Lambda = -\frac{1}{2a^2}$. This solution recovered the solution of Santos [257] and Gödel [18]. Let us discuss the case when $\lambda \neq 0$. With the removal of the Λ^2 term from the system of equations, we now have a single equation

$$4\lambda\rho^2 + (16\Lambda\lambda + 1)\rho = \frac{1}{a^2}. \quad (4.22)$$

This equation (4.22) has infinite solutions depending on ρ or Λ . In view of the solution obtained by Santos [257] for linear case $f(T) = \lambda T$, Let us assume that

$$\rho = \frac{1}{a^2(1 + 2\lambda)} \quad (4.23)$$

is a solution of the equation. Then the corresponding cosmological constant is given as

$$\Lambda = \frac{-2 + a^2(1 + 2\lambda)}{8(1 + 2\lambda)a^2} \quad (4.24)$$

The obtained matter energy density ρ and cosmological constant Λ satisfies the field equations under the constraint $\lambda = -\frac{a^2+2}{4a^2}$. Thus matter energy density ρ and cosmological constant Λ takes the form

$$\rho = \frac{2}{a^2 - 2} \quad (4.25)$$

and

$$\Lambda = \frac{a^2 - 6}{8(a^2 - 2)} \quad (4.26)$$

Given the physical acceptability of the matter energy density ρ and cosmological constant Λ , we can restrict a^2 to $2 < a^2 < 6$ such that the energy density is positive and the cosmological constant is negative. The relationship between matter energy density and cosmological constant for this model is expressed as

$$\Lambda = \left(\frac{a^2 - 6}{16} \right) \rho \quad (4.27)$$

It is interesting to note that, for $a^2 > 6$ both matter energy density ρ and cosmological constant Λ is positive.

4.3.2 Solution for the choice of $h(T) = \lambda \ln(T)$

The T -dependence function $h(T)$ is considered in this section as $h(T) = \lambda \ln(T)$. This choice of $h(T)$ automatically limits the trace of the energy-momentum tensor $T > 0$, otherwise $f(R, T) = R + 2h(T)$ is not defined. In this case the modified field equations (4.11) takes the form

$$G_{\mu\nu} = \left[\frac{T + 2\lambda}{T} \right] T_{\mu\nu} + \lambda \left[\frac{2p + T \ln(T)}{T} \right] g_{\mu\nu} \quad (4.28)$$

The non-zero components of equation (4.28) for the Gödel metric (4.12) along with (4.14) are expressed as

$$\begin{aligned} \frac{\rho + 5p + 4\Lambda}{2a^2} = & -(\rho + 5p + 4\Lambda + 2\lambda)(p + \Lambda) \\ & - \lambda [2p + (\rho + 5p + 4\Lambda) \ln(\rho + 5p + 4\Lambda)] \end{aligned} \quad (4.29)$$

$$\begin{aligned} \frac{3(\rho + 5p + 4\Lambda)}{4a^2} &= (\rho + 5p + 4\Lambda + 2\lambda) \left(\rho + \frac{3p}{2} + \frac{\Lambda}{2} \right) \\ &+ \frac{\lambda}{2} [2p + (\rho + 5p + 4\Lambda) \ln(\rho + 5p + 4\Lambda)] \end{aligned} \quad (4.30)$$

$$\begin{aligned} \frac{\rho + 5p + 4\Lambda}{2a^2} &= (\rho + 5p + 4\Lambda + 2\lambda) (\rho + 2p + \Lambda) \\ &+ \lambda [2p + (\rho + 5p + 4\Lambda) \ln(\rho + 5p + 4\Lambda)] \end{aligned} \quad (4.31)$$

The above system is non linear system with three variables ρ , p and Λ . Let us assume, universe is filled with perfect fluid and, for simplicity, we choose the case in which pressure is zero i.e. $p = 0$. Thus the system of equations (4.29)-(5.24) reduces to

$$\frac{\rho + 4\Lambda}{2a^2} = -(\rho + 4\Lambda + 2\lambda)\Lambda - \lambda(\rho + 4\Lambda) \ln(\rho + 4\Lambda) \quad (4.32)$$

$$\frac{3(\rho + 4\Lambda)}{4a^2} = (\rho + 4\Lambda + 2\lambda) \left(\rho + \frac{\Lambda}{2} \right) + \frac{\lambda}{2} (\rho + 4\Lambda) \ln(\rho + 4\Lambda) \quad (4.33)$$

$$\frac{\rho + 4\Lambda}{2a^2} = (\rho + 4\Lambda + 2\lambda)(\rho + \Lambda) + \lambda(\rho + 4\Lambda) \ln(\rho + 4\Lambda) \quad (4.34)$$

For general relativity and $f(R, T) = R + 2\lambda \ln(T)$ gravity, solutions are found when $\lambda = 0$ and $\lambda \neq 0$, respectively. For $\lambda = 0$, the system of equations (4.32)-(4.34) reduces to

$$\frac{\rho + 4\Lambda}{2a^2} = -4\Lambda^2 - \Lambda\rho \quad (4.35)$$

$$\frac{3(\rho + 4\Lambda)}{4a^2} = 2\Lambda^2 + \frac{9}{2}\Lambda\rho + \rho^2 \quad (4.36)$$

$$\frac{\rho + 4\Lambda}{2a^2} = 4\Lambda^2 + 5\Lambda\rho + \rho^2 \quad (4.37)$$

For the above system of equations (4.35)-(4.37) the solutions are $(\rho, \Lambda) = \left\{ \left(\frac{1}{a^2}, -\frac{1}{2a^2} \right), \left(\frac{2}{a^2}, -\frac{1}{2a^2} \right) \right\}$ and $\Lambda = -\frac{1}{4}\rho$. As per the requirement, the required solution is $(\rho, \Lambda) = \left(\frac{1}{a^2}, -\frac{1}{2a^2} \right)$, which recovered solution as that of Santos [257] and Gödel [18]. For $\lambda \neq 0$ and elimination of $\lambda(\rho + 4\Lambda) \ln(\rho + 4\Lambda)$ from system of equations (4.32)-(4.34) reduces to a single equation

$$\frac{\rho + 4\Lambda}{a^2} = 4\Lambda\rho + 2\lambda\rho + \rho^2 \quad (4.38)$$

This equation has infinite solution depending on ρ or Λ . As a result of using ρ in the form 4.23, we get $\rho < 0$ and $\Lambda > 0$, which are unrealistic behavior of the physical

parameters. Therefore, let us assume that

$$\rho = \frac{1}{a^2 (1 + k\lambda)} \quad (4.39)$$

is a solution of (4.38). Where $k \neq 2$ is a constant quantity, which needs to be determined. The corresponding cosmological constant has the form

$$\Lambda = \frac{2a^2(k\lambda + 1) - k}{4a^2(k\lambda + 1)k} \quad (4.40)$$

The obtained matter energy density ρ and cosmological constant Λ satisfy the field equations (4.32)-(4.34) under the constraint

$$\lambda = -\frac{2a^2 + k}{2a^2k \left(2 \ln\left(\frac{2}{k}\right) + 1\right)}. \quad (4.41)$$

Therefore, matter energy density ρ and cosmological constant Λ will take the form

$$\rho = \frac{4 \ln\left(\frac{2}{k}\right) + 2}{4a^2 \ln\left(\frac{2}{k}\right) - k} \quad (4.42)$$

$$\Lambda = \frac{(2a^2 - k) \ln\left(\frac{2}{k}\right) - k}{(4a^2 \ln\left(\frac{2}{k}\right) - k)k} \quad (4.43)$$

For this model, the relationship between matter energy density and cosmological constant is as follows:

$$\Lambda = \frac{(2a^2 - k) \ln\left(\frac{2}{k}\right) - k}{(4 \ln\left(\frac{2}{k}\right) + 2)k} \rho \quad (4.44)$$

We have the restriction on the parameters k and a^2 as $k \in (0, 2)$ and $\frac{k}{4 \ln\left(\frac{2}{k}\right)} < a^2 < \frac{k}{2} \left(\frac{1}{2 \ln\left(\frac{2}{k}\right)} + 1 \right)$ for matter energy density $\rho > 0$ and cosmological constant $\Lambda < 0$.

4.3.3 Solution for the choice of $h(T) = \lambda e^T$

As a result of exponential $f(R)$ gravity, it makes sense to consider the T dependence function $h(T)$ as $h(T) = \lambda e^T$. In this case the modified field equations (4.11) takes the form

$$G_{\mu\nu} = [1 + 2\lambda e^T] T_{\mu\nu} + \lambda e^T [2p + 1] g_{\mu\nu} \quad (4.45)$$

The non-zero components of equation (4.45) for the Gödel metric (4.12) along with (4.14) are expressed as

$$\frac{1}{2a^2} = - (1 + 2\lambda e^{\rho+5p+4\Lambda}) (p + \Lambda) - \lambda e^{\rho+5p+4\Lambda} (2p + 1) \quad (4.46)$$

$$\frac{3}{4a^2} = (1 + 2\lambda e^{\rho+5p+4\Lambda}) \left(\rho + \frac{3p}{2} + \frac{\Lambda}{2} \right) + \frac{\lambda}{2} e^{\rho+5p+4\Lambda} (2p + 1) \quad (4.47)$$

$$\frac{1}{2a^2} = (1 + 2\lambda e^{\rho+5p+4\Lambda}) (\rho + 2p + \Lambda) + \lambda e^{\rho+5p+4\Lambda} (2p + 1) \quad (4.48)$$

The above system is non linear system with three variables ρ , p and Λ . Let us assume, universe is filled with perfect fluid and, for simplicity, we choose the case in which pressure is zero i.e. $p = 0$. Thus the system reduces to

$$\frac{1}{2a^2} = - (1 + 2\lambda e^{\rho+4\Lambda}) \Lambda - \lambda e^{\rho+4\Lambda} \quad (4.49)$$

$$\frac{3}{4a^2} = (1 + 2\lambda e^{\rho+4\Lambda}) \left(\rho + \frac{\Lambda}{2} \right) + \frac{\lambda}{2} e^{\rho+4\Lambda} \quad (4.50)$$

$$\frac{1}{2a^2} = (1 + 2\lambda e^{\rho+4\Lambda}) (\rho + \Lambda) + \lambda e^{\rho+4\Lambda} \quad (4.51)$$

For general relativity and $f(R, T) = R + 2\lambda e^T$ gravity, solutions are found when $\lambda = 0$ and $\lambda \neq 0$, respectively. For $\lambda = 0$, the system of equations (4.49)-(4.51) has solution $(\rho, \Lambda) = \left(\frac{1}{a^2}, -\frac{1}{2a^2} \right)$, which recovered Solution as that of Santos [257] and Gödel [18]. For $\lambda \neq 0$ and elimination of $\lambda e^{\rho+4\Lambda}$ from the above system of equations (4.49)-(4.51) reduces to a single equation

$$\rho (1 + 2\lambda e^{\rho+4\Lambda}) = \frac{1}{a^2} \quad (4.52)$$

Following the above procedure, let us assume that

$$\rho = \frac{1}{a^2 (1 + 2\lambda)} \quad (4.53)$$

is a solution of the equation (4.52). Then the corresponding cosmological constant is given as

$$\Lambda = -\frac{1}{4a^2 (1 + 2\lambda)} \quad (4.54)$$

The obtained matter energy density ρ and cosmological constant Λ satisfy the field equations (4.49)-(4.51) under the constraint $\lambda = -\frac{1}{4a^2}$. Thus, matter energy density ρ and cosmological constant Λ will take the form

$$\rho = \frac{2}{2a^2 - 1} \quad (4.55)$$

and

$$\Lambda = -\frac{1}{2(2a^2 - 1)} \quad (4.56)$$

Based on this model, matter energy density ρ and cosmological constant Λ are related as

$$\Lambda = -\frac{1}{4}\rho \quad (4.57)$$

The positive density of matter causes $a^2 > \frac{1}{2}$, which leads to a negative cosmological constant.

4.4 Issue of energy-momentum conservation in $f(R, T)$ gravity

Since the Friedmann model satisfies the continuity equation, energy-momentum conservation can be expressed as

$$\dot{\rho} + 3H(\rho + p) = 0, \quad (4.58)$$

which leads to $d(\rho V) = -pdV$ and $V = a^3$. Taking into account the volume scale factor and the quantity ρV , we determine the total energy. The situation may be different with modified gravity theories. A violation of energy momentum conservation can be reconciled with the metric theory of gravity by taking the fundamental theory of space and time as uni-modular gravity, according to Josset et al. [269] In Friedmann's equation, this paradigm shift yields an effective cosmological constant that can be interpreted as a record of the evolution of non-conservation of energy-momentum during the history of the universe. In their study, Shabani and Ziaie [270] used a dynamical system approach to examine the consequences of energy conservation violation for the subclass

$f(R, T) = R + \beta\Lambda(-T)$. In the class $f(R, T) = R + 2f_1(T)$ with $f_1(T) = \lambda T$, Sahoo et al. [271] pointed out that energy-momentum conservation is violated.

It is generally acknowledged that the energy-momentum tensor is not conserved in $f(R, T)$ gravity for the general functional form $f(R, T)$. Several researchers [272–274] have addressed this issue. We will begin with the action of $f(R, T)$ gravity as $S = \int \left[\frac{f(R, T)}{16\pi} + L_m \right] \sqrt{-g} d^4x$ [257]. For general functional forms of $f(R, T)$, when varying such action w.r.t. $g_{\mu\nu}$, we have

$$f_R(R, T)R_{\mu\nu} - \frac{1}{2}f(R, T)g_{\mu\nu} + (g_{\mu\nu}\square - \nabla_\mu \nabla_\nu)f_R(R, T) = (8\pi - f_T(R, T))T_{\mu\nu} - f_T(R, T)\Theta_{\mu\nu} \quad (4.59)$$

where $\Theta_{\mu\nu} = g^{\alpha\beta} \frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}}$. By contracting and eliminating the term $\square f_R(R, T)$, the gravitational field equations (4.59) can be expressed as

$$f_R(R, T) \left(R_{\mu\nu} - \frac{1}{3}Rg_{\mu\nu} \right) + \frac{1}{6}f(R, T)g_{\mu\nu} = (8\pi - f_T(R, T)) \left(T_{\mu\nu} - \frac{1}{3}Tg_{\mu\nu} \right) - f_T(R, T) \left(\Theta_{\mu\nu} - \frac{1}{3}\Theta g_{\mu\nu} \right) + \nabla_\mu \nabla_\nu f_R(R, T) \quad (4.60)$$

The covariant divergence of (4.59) gives

$$\nabla^\mu T_{\mu\nu} = \frac{f_T(R, T)}{8\pi - f_T(R, T)} [(T_{\mu\nu} + \Theta_{\mu\nu})\nabla^\mu \ln f_T(R, T) + \nabla^\mu \Theta_{\mu\nu}] \quad (4.61)$$

This shows that energy-momentum is not conserved in $f(R, T)$ gravity.

Now let us discuss some particular classes of $f(R, T)$. For the functional form $f(R, T) = R + h(T)$, by varying the action w.r.t. $g_{\mu\nu}$, we have

$$G_{\mu\nu} = 8\pi T_{\mu\nu} + \frac{1}{2}h(T)g_{\mu\nu} + h_T(T)(T_{\mu\nu} - L_m g_{\mu\nu}) \quad (4.62)$$

The covariant derivative of $T_{\mu\nu}$ in (4.62) is expressed as

$$\nabla^\mu T_{\mu\nu} = \frac{h_T(T)}{8\pi + h_T(T)} \left[(L_m g_{\mu\nu} - T_{\mu\nu})\nabla^\mu \ln h_T(T) + \nabla^\mu (L_m - \frac{1}{2}T)g_{\mu\nu} \right] \quad (4.63)$$

For $h(T) = 2\lambda T$, equations (4.62) can be expressed as

$$G_{\mu\nu} = 8\pi T_{\mu\nu}^{eff}, \quad (4.64)$$

with $T_{\mu\nu}^{eff} = T_{\mu\nu} + \tilde{T}_{\mu\nu}$ and $\tilde{T}_{\mu\nu} \equiv \frac{\lambda}{8\pi} [2(T_{\mu\nu} - L_m g_{\mu\nu}) + T g_{\mu\nu}]$. Applying the Bianchi identities to (4.64) gives $\nabla^\mu [8\pi(T_{\mu\nu} + \tilde{T}_{\mu\nu})] = 0$, or simultaneously $\nabla^\mu T_{\mu\nu} = 0$ and $\nabla^\mu \tilde{T}_{\mu\nu} = 0$. The first of these two equations yields the usual conservation law of standard cosmology, while the second yields the EoS of stiff matter. Chakraborty [273] has shown that a part of the arbitrary function $f(R, T)$ can be determined if one imposes $\nabla_\mu (T^{\mu\nu}) = 0$.

However, for the case $f(R, T) = f_1(R) + f_2(T)$ in an FLRW background sourced by the energy–momentum tensor of a perfect fluid and $p = \omega\rho$, the $f(R, T)$ model gives rise to the conservation law as

$$\dot{\rho} + 3H\rho(1 + \omega) = -\frac{1}{8\pi + f_2'} \left[(1 + \omega)\rho\dot{f}_2' + \omega\dot{\rho}f_2' + 0.5\dot{f}_2 \right] \quad (4.65)$$

where the prime and dot denote derivatives with respect to the argument and to the cosmic time, respectively. This shows a non-conservation of the matter–energy content as a whole. This class is conservative if we impose $\left[(1 + \omega)\rho\dot{f}_2' + \omega\dot{\rho}f_2' + 0.5\dot{f}_2 \right] = 0$. This provides a solution of the form $f_2(T) = \sigma T^{\frac{3\omega+1}{2(\omega+1)}} + \sigma_0$ where σ and σ_0 are constants of integration. Thus we can conclude that under this choice of $f_2(T)$, the class $f(R, T) = f_1(R) + f_2(T)$ obeys the conservation law.

4.5 Conclusion

We discussed the Gödel universe in the first class of $f(R, T) = R + 2h(T)$ gravity in this chapter. It is noted that various modified theories have been developed by researchers after the accelerated expansion of the universe. Because of its generalization of the most discussed $f(R)$ gravity, $f(R, T)$ gravity has recently piqued the interest of cosmologists in analyzing cosmological models in various contexts. The modified field equations are developed for various trace dependence functions $h(T)$, including $h(T) = \lambda T^2$, $h(T) = \lambda \ln(T)$, and $h(T) = \lambda e^T$. For all the discussed forms of $h(T)$, the functional form $f(R, T) = R + 2h(T)$ produced the same results as Santos [257] and Gödel [18] for $\lambda = 0$. For $\lambda \neq 0$, the cosmological constant depends on matter and geometry for $h(T) = \lambda T^2$ and $h(T) = \lambda \ln(T)$, but only on matter for $h(T) = \lambda e^T$. The expressions in (4.26), (4.43) and (4.56) indicate the smallness of the cosmological constant Λ depending on a followed by the prescribed λ . Further, it is noted that the λ parameter is a small

negative quantity and depends on a . The parameter λ lies in the interval $-0.5 < \lambda < -0.33$ and $-\infty < \lambda < -0.5$ for $h(T) = \lambda T^2$ and $h(T) = \lambda e^T$ respectively. For the scenario $h(T) = \lambda \ln(T)$, the range of λ is determined by k . We'll use $k = 1$ and $a^2 \in (0.36068, 1.2214)$ as an example, so $-1 < \lambda < -0.59$. We have the matching a^2 depending on the choice of k , which leads to the smallness of the parameter λ .

Chapter 5

Impact of Particle creation in Rastall gravity

5.1 Introduction

Researchers have always been curious for understanding the universe from the past to the future scientifically. Nowadays, it is well accepted that the modifications of Einstein's gravity or extension of Einstein's GTR are necessary to discover and study the phenomenological dynamics of the universe that have been exposed by using current observations. The Λ CDM models seem to be sufficient to describe the current scenario of the universe. In the present chapter we are interested in Rastall's gravity theory. Rastall's gravity theory was developed in 1972. Rastall's theory of gravity is the generalization of Einstein's GTR which describes the conservation law of energy ($\nabla_\mu T^{\mu\nu} = 0$) and energy-momentum tensor $T^{\mu\nu}$. The expression for the Rastall's gravity is given by [275]

$$R_{\mu\nu} - \frac{\lambda}{2}g_{\mu\nu}R = KT_{\mu\nu} \quad (5.1)$$

$$T^{\mu\nu};_{\mu} = \frac{1-\lambda}{2k}R^{,\nu} = \frac{1-\lambda}{2(1-2\lambda)}T^{,\nu} \quad (5.2)$$

Rastall's theory of gravity is a nonconservative extension of Einstein's GTR. Rastall's gravity contains a rich structure that may be easily get connected with the fundamental aspects of gravity theory. Since the energy-momentum tensor is not conserved anymore, there will be no variational principle in Riemannian geometry. In Rastall's gravity,

the energy-momentum tensor can be formulated from the lagrangian of matter fields, such that $T_{\mu\nu} = -2 \frac{\delta L_m}{\delta g^{\mu\nu} + g_{\mu\nu} L_m}$. However, there is no corresponding equation of motion for the matter fields due to the absence of total action for Rastall's gravity. The Rastall's parameter λ , is selected conveniently by accepting the Anastaz philosophy. In empty space, i.e., for the vanishing energy-momentum tensor, Rastall's gravity theory agrees with the concept of GTR with direct implementation of Mach's principle, which states that the inertial mass distribution should depend on the mass and energy of the extremal space-time [284]. Static, spherically symmetric solutions with a canonical scalar field are concluded by considering the canonical energy-momentum tensor ($T_{\mu\nu} = \epsilon(\phi_{;\mu}\phi_{;\nu} - \frac{1}{2}g_{\mu\nu}\phi^{;\rho}\phi_{;\rho}) + g_{\mu\nu}V(\phi)$) in the framework of Rastall gravity [285]. Rastall's gravity is based on the considerable deviation from the general relativity in the high curvature regime. Nevertheless, it would be interesting to detect whether it substantially affects the horizon scale, where the curvature is comparatively tiny. Some structures are similar to Rastall's gravity theory are found in the framework of Weyl's geometry [286]. Various exact solutions of Rastall gravity has also been obtained [276–278].

In Rastall gravity, the exact solutions of the dynamical equations for interacting, non-interacting dark energy and dark matter are derived by considering holographic parameter $C^2(Z)$ where, $(C^2(Z) = \left(\frac{1-4\kappa\lambda}{\kappa(1-3\kappa\lambda(1+\omega))}\right) \left(\frac{1-\Omega_\kappa(Z)}{1+r(Z)}\right))$ as a slowly varying function of redshift for non-interacting models and $C^2(Z)$ as a constant function of redshift for the interacting model of dark energy. Many cosmologists developed the models to study the thermodynamical aspects of Rastall's gravity [279–281]. The Newtonian-Janis approach is used to compute the charged black string solution in the framework of Rastall gravity [282]. Also, they investigate the graphical representation of Hawking temperature through the event horizon to examine the stability conditions of charged black string within Rastall gravity. The Wormhole solutions are investigated with the phantom regime ($\omega < -1$) through conformal symmetry, and they explored the physical aspects of the epicyclic frequencies to explore the self-gravitating system [283].

Saleem et al. [287] discussed the phenomenon of warm inflation in the modified cosmological scenario by modifying standard FRW equations for Tsallis, i.e., non-additive entropy ($s_T = \alpha A^b$) and Barrow holographic dark energy entropies ($s_B = (\frac{A}{A_0})^{\Delta+1}$). Madsen et al. [288] discussed the universe's evolution for inflationary ($R \ll R^*$) and radiation-dominated phase ($R \gg R^*$) in a unified manner by using the gamma-law

equation of state, i.e., $(p = (\gamma - 1)\rho)$ as a scale factor's function in the background of FRW metric.

In this study, our primary focus is on the study of particle creation in the influence of Rastall gravity. Particle creation remains one of the most unsolved problems in cosmology. Several cosmologists have discussed the particle creation phenomenon and its effects on the universe's evolution. A detailed exploration of the thermodynamics of particle creation with the change of specific entropy has been discussed [289–291]. Hamil et al. [292] discussed the mechanism of particle creation in the absence of time-like singularity in the emergent universe scenario. The creation of matter, i.e., particle creation was investigated in the framework of various modified gravity theories like cubic gravity [293], $f(T)$ gravity [294], $f(R)$ gravity, [295] $f(R, T)$ gravity [296], $f(G)$ gravity [297], $f(G, T)$ gravity [298], and the alternative theories like Brans-Dicke theory [299] and Lyra's geometry [300]. Some of the recent works based on particle creation are discussed below. Lyth et al. [16] discussed the cosmological consequences of the particle creation during the inflation era of the universe. Particle creation arises due to a change of space-time metric at the end of an inflationary era at the time of the early universe [7]. The time dependence of particle creation of matter is because of a quantized, massless, minimally coupled scalar field in two-dimensional flat space-time with an accelerating mirror [15]. In the current time period Bishi et al. [302] studied the influence of deceleration parameter with the particle creation mechanism.

Following the above-stated research work based on Rastall gravity and particle creation mechanism, we have been motivated to investigate the impact of particle creation in Rastall gravity theory by considering different types of scale factors $a(t)$ and deceleration parameters q . This manuscript is organized as: In section 2, we obtained the cosmological equations of Rastall gravity with particle creation and obtained the cosmological solutions by assuming following types of scale factor $a(t)$ as $a(t) = \frac{-1}{t} + t^2$, $a(t) = \exp(mt^l)$. The general characteristics of the models with the observational characteristics of the universe are discussed in section 3. In section 4, we conclude the obtained results.

5.2 Field equations

The corresponding field equation of the Rastall theory given by Kumar et al. [109]

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = k(T_{\mu\nu} - \lambda g_{\mu\nu}R) \quad (5.3)$$

it can be rewritten as

$$R_{\mu\nu} + (\psi - \frac{1}{2})g_{\mu\nu}R = kT_{\mu\nu} \quad (5.4)$$

Let us consider the FRW metric

$$ds^2 = -dt^2 + R^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right] \quad (5.5)$$

where $k = 1, -1$ and 0 . In the presence of creation of matter the energy momentum tensor is given by

$$T_{\mu\nu} = (\rho + p + p_c)u_\mu u_\nu + (p + p_c)g_{\mu\nu} \quad (5.6)$$

where ρ and p are the energy density and pressure respectively, p_c is the creation pressure, u_μ the fluid-four velocity vector and $g_{\mu\nu}$ is the metric tensor. The trace of the energy momentum tensor is given as

$$T = \rho - 3(p + p_c) \quad (5.7)$$

The certain creation pressure in case of conserved specific entropy $\sigma = \frac{S}{N}$ is given by

$$p_c = -\frac{(\rho + p)\Gamma}{3nH} \quad (5.8)$$

here $\Gamma = 3\eta H n$ therefore we get

$$p_c = -(\rho + p)\eta \quad (5.9)$$

The modified gravitational field equation can be written as

$$\begin{aligned}\lambda_1 \dot{H} + \lambda_2 H^2 + \frac{\kappa}{a^2} &= -\lambda_3 \rho \\ \dot{H} + \lambda_4 H^2 + \frac{\lambda_5 \kappa}{a^2} &= \lambda_6 (p + p_c)\end{aligned}\tag{5.10}$$

where $\lambda_1 = \frac{2\psi}{2\psi-1}$, $\lambda_2 = \frac{4\psi-1}{2\psi-1}$, $\lambda_3 = \frac{k}{3(2\psi-1)}$, $\lambda_4 = \frac{3}{2}\lambda_2$, $\lambda_5 = \frac{6\psi-1}{6\psi-2}$, $\lambda_6 = \frac{k}{6\psi-2}$

using (5.9) in (5.10) the modified gravitational field equation takes the form

$$\begin{aligned}\lambda_1 \dot{H} + \lambda_2 H^2 + \frac{\kappa}{a^2} &= -\lambda_3 \rho \\ \dot{H} + \lambda_4 H^2 + \frac{\lambda_5 \kappa}{a^2} &= \lambda_6 [(1-\eta)p - \eta\rho]\end{aligned}\tag{5.11}$$

from (5.11) we get the general equation for energy density ρ , pressure p , and particle creation p_c

$$\rho = -\frac{(\lambda_2 H^2 + \lambda_1 \dot{H})a^2 + \kappa}{\lambda_3 a^2}\tag{5.12}$$

$$p = \frac{(\lambda_1 \lambda_6 \eta - \lambda_3)}{\lambda_3 \lambda_6 (\eta - 1)} \dot{H} + \frac{\lambda_2 \lambda_6 \eta - \lambda_3 \lambda_4}{\lambda_3 \lambda_6 (\eta - 1)} H^2 + \frac{\kappa(\lambda_6 \eta - \lambda_3 \lambda_5)}{a^2 (\lambda_3 \lambda_6 (\eta - 1))}\tag{5.13}$$

$$p_c = -\eta \left[\frac{\lambda_1 \lambda_6 - \lambda_3}{\lambda_3 \lambda_6 (\eta - 1)} \dot{H} + \frac{\lambda_2 \lambda_6 - \lambda_3 \lambda_4}{\lambda_3 \lambda_6 (\eta - 1)} H^2 + \frac{\kappa(\lambda_6 - \lambda_3 \lambda_5)}{a^2 \lambda_3 \lambda_6 (\eta - 1)} \right]\tag{5.14}$$

5.3 Solution of the modified field equations

5.3.1 Model I

let us consider the scale factor of the form

$$a = \frac{-1}{t} + t^2\tag{5.15}$$

The scale factor (5.15) gives a time-dependent deceleration parameter. We have to note that for $t > 1$, $\ddot{a} > 0$. Hence, we can observe the universe's inflationary scenario for $t > 1$. The above-stated scale factor (5.15) is investigated in the FLRW universe with the interacting and noninteracting two-fluids atmosphere for DE [303]. This scale factor

is also used in Bianchi type-I space-time with generalized chaplygin gas to explore the universe's evolution with cosmological const Λ and gravitational constant G . [304] the Hubble parameter is given as

$$H = \frac{2t^3 + 1}{t(t^3 - 1)} \quad (5.16)$$

From the equation (5.16) we observed that when $t \rightarrow \infty$ we get $H \rightarrow 0$

The deceleration parameter q for the scale factor (5.15) is given by

$$q = \frac{-\dot{H} + H^2}{H^2} = -2 \frac{(t^3 - 1)^2}{(1 + 2t^3)^2} \quad (5.17)$$

In this model the universe evolves with $-1 < q < 0$ and at the late time $q \rightarrow -0.5$. therefore we get the energy density ρ by using the equations (5.16) and (5.15) in (5.12)

$$\rho = - \left[\frac{\lambda_1 + \kappa t^4 - 2\lambda_1 t^3(4 + t^3) + \lambda_2(1 + 2t^3)^2}{\lambda_3 t^2(t^3 - 1)^2} \right] \quad (5.18)$$

With the help of equations (5.16) and (5.15) in (5.13) the pressure p can be calculated as

$$p = \frac{\lambda_6 \eta (\lambda_1 + \kappa t^4 - 2\lambda_1 t^3(4 + t^3) + \lambda_2(1 + 2t^3)^2) - \lambda_3 (1 + \lambda_4(1 + 2t^3)^2 + t^3(\kappa t \lambda_5 - 2t^3 - 8))}{\lambda_3 \lambda_6 (\eta - 1) t^2 (t^3 - 1)^2} \quad (5.19)$$

By incorporating equations (5.16) and (5.15) in (5.14) we get the particle creation P_c as

$$p_c = - \frac{\eta \left[(\lambda_1 + \lambda_2) \lambda_6 + \lambda_6 t^3 (\kappa t + 4\lambda_2(1 + t^3) - 2\lambda_1(4 + t^3)) - \lambda_3 (1 + \lambda_4(1 + 2t^3)^2 + t^3(\kappa t \lambda_5 - 2t^3 - 8)) \right]}{\lambda_3 \lambda_6 (\eta - 1) t^2 (t^3 - 1)} \quad (5.20)$$

As seen in Figure 5.1, the deceleration parameter q is negative-valued and $q \rightarrow -0.5$ when $t = \infty$. The Hubble parameter H decreases as cosmic time t increases, and $H \rightarrow 0$ when $t = \infty$.

We see in Figure 5.2 the profile of the energy density ρ against cosmic time t for different ψ values. We have observed that the energy density ρ takes positive and positive to negative values for $\phi = 0.2$ and $\phi = 0.4$. However, for other ψ values (0.6, 0.8, 1, 2), it is a negative function for $k = 1$. Furthermore, for the case of $k = -1$, it is a negative and negative to positive value for $\phi = 0.2$ and $\psi = 0.4$. For other values of

$\psi(= 0.6, 0.8, 1, 2)$, energy density ρ is decreasing positive valued function of cosmic time t and $\rho \rightarrow 0$ when $t = \infty$.

The Figures 5.3 and 5.4 show the profile of pressure p and particle creation p_c against cosmic time t for different ψ . In the case of $k = 1$ and $k = -1$, the pressure is positive and negative for different ψ , respectively. With respect to cosmic time, the particle creation p_c takes a negative value for $k = 1$ and a positive value for $k = -1$ for different ψ .

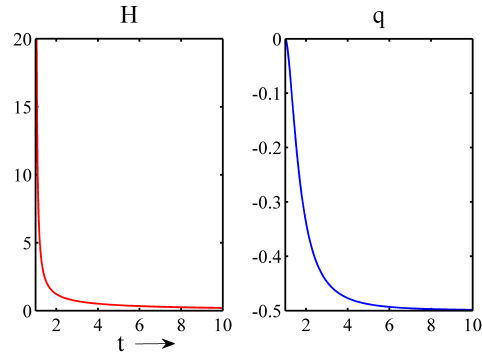


FIGURE 5.1: Profile of Hubble parameter H and deceleration parameter q against time

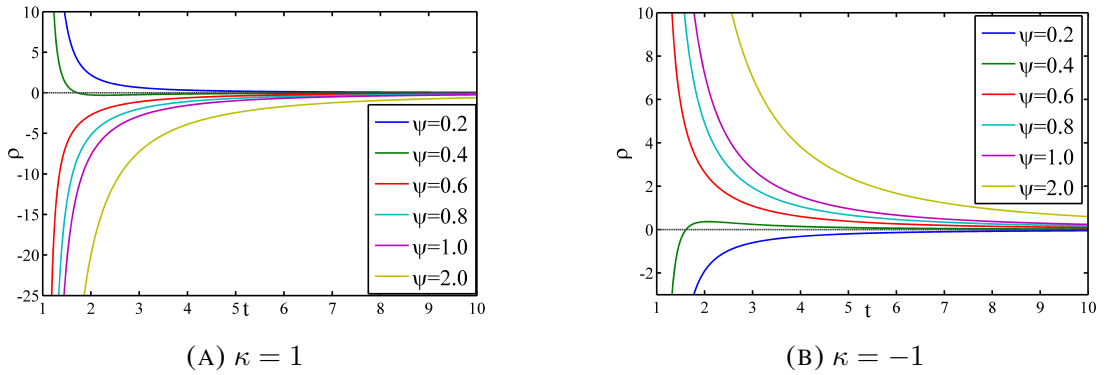


FIGURE 5.2: Profile of energy density for Model-I against time for different ψ

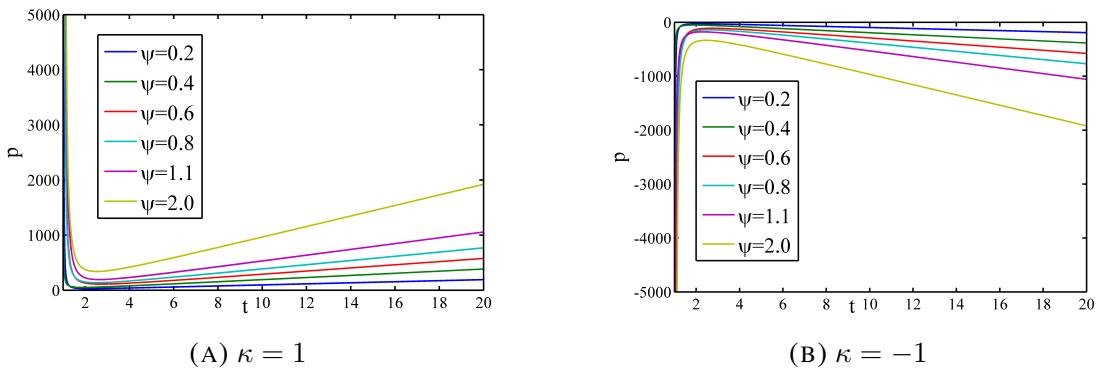


FIGURE 5.3: Profile of pressure for Model-I against time for different ψ

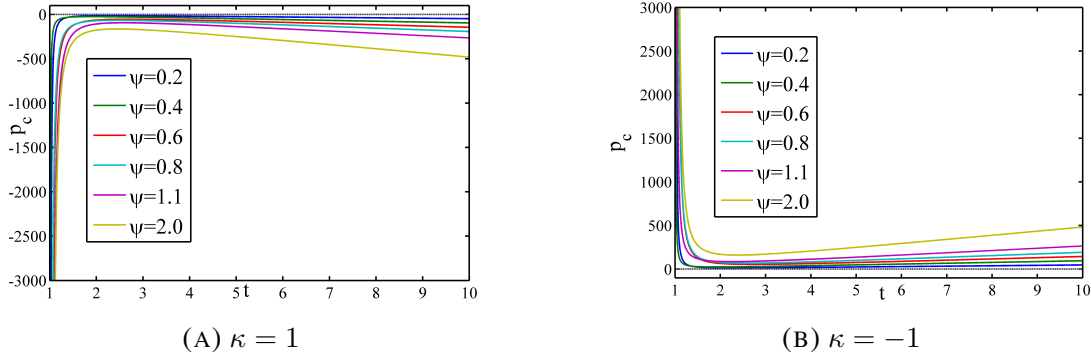


FIGURE 5.4: Profile of particle creation for Model-I against time for different ψ

5.3.2 Model II

Borrow [305] investigated the intermediate expansion law and developed the cosmological model. The intermediate form of scale factor yields an exponential function of the cosmic time as

$$a = \exp(mt^l) \quad (5.21)$$

where $m > 0$ and $0 < l < 1$ are constant. Number of research work based on intermediate scale factor in the isotropic and anisotropic metric background has been done in various gravity theories [306–308]. The Hubble parameter is given by

$$H = mlt^{l-1} \quad (5.22)$$

the deceleration parameter for the emergent universe takes the form

$$q = -1 - \frac{l-1}{mlt^l} \quad (5.23)$$

therefore we get the energy density ρ by using equations (5.21) and (5.22) in (5.12)

$$\rho = -\frac{1}{\lambda_3} \left[e^{-2mt^l} \kappa + lmt^{l-2} (\lambda_1(l-1) + \lambda_2 lmt^l) \right] \quad (5.24)$$

With the help of the equations (5.21) and (5.22) in (5.13) the pressure p can be calculated as

$$p = \frac{e^{-2mt^l} \left(\lambda_6 \eta (\kappa t^2 + e^{2mt^l} lmt^l (\lambda_1(l-1) + \lambda_2 lmt^l)) - \lambda_3 (e^{2mt^l} lmt^l (l-1 + \lambda_4 lmt^l) + \kappa t^2 \lambda_5) \right)}{\lambda_3 \lambda_6 (\eta - 1) t^2} \quad (5.25)$$

By incorporating equations (5.21) and (5.22) in (5.14) we get the particle creation P_c as

$$p_c = \frac{e^{-2mt^l} \eta \left(e^{2mt^l} (\lambda_3 - \lambda_1 \lambda_6) (l-1) l m t^l + e^{2mt^l} (\lambda_3 \lambda_4 - \lambda_2 \lambda_6) l^2 m^2 t^{2l} + \kappa t^2 (\lambda_3 \lambda_5 - \lambda_6) \right)}{\lambda_3 \lambda_6 (\eta - 1) t^2} \quad (5.26)$$

On Figure 5.5 and Figure 5.6 are plotted the deceleration parameter q and Hubble parameter H for different m against cosmic time t . The q changes from positive to negative values as t increases, which signifies that the universe is in a phase transition from decelerating to accelerating. The deceleration parameter $q \rightarrow -1$ for different m and l when $t \rightarrow \infty$. The Hubble parameter H decreases as cosmic time t increases, and $H \rightarrow 0$ when $t = \infty$. We consider $l = 0.5$ in all of the figures of model-II to be a representative case.

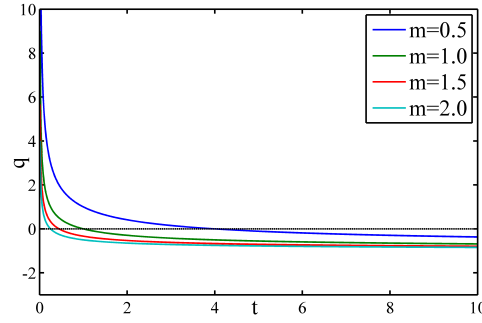


FIGURE 5.5: Profile of deceleration parameter q against time

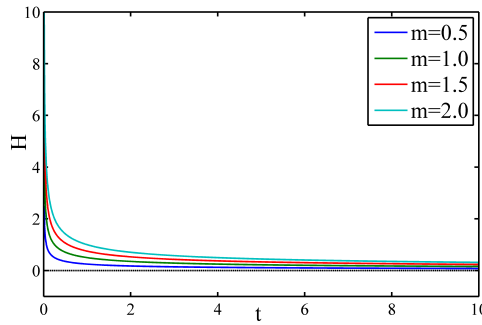


FIGURE 5.6: Profile of Hubble parameter H against time

Figure 5.7 shows the variation of energy density ρ against cosmic time t for different ψ . We observed that for $k = 1$, energy density ρ is a positive valued and decreasing function of cosmic time t for $\psi = 0.2$ and $\psi = 0.4$. For other ψ values ($= 0.6, 0.8, 1, 2$), it takes positive to negative values. Furthermore, for $k = -1$ and different ψ , energy

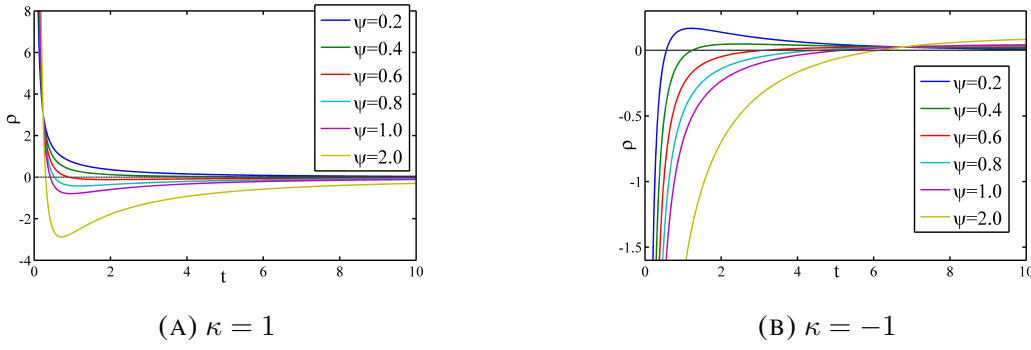


FIGURE 5.7: Profile of energy density for Model-II against time for different ψ

density ρ is negative valued during the initial phase and then becomes a positive valued function of cosmic time t .

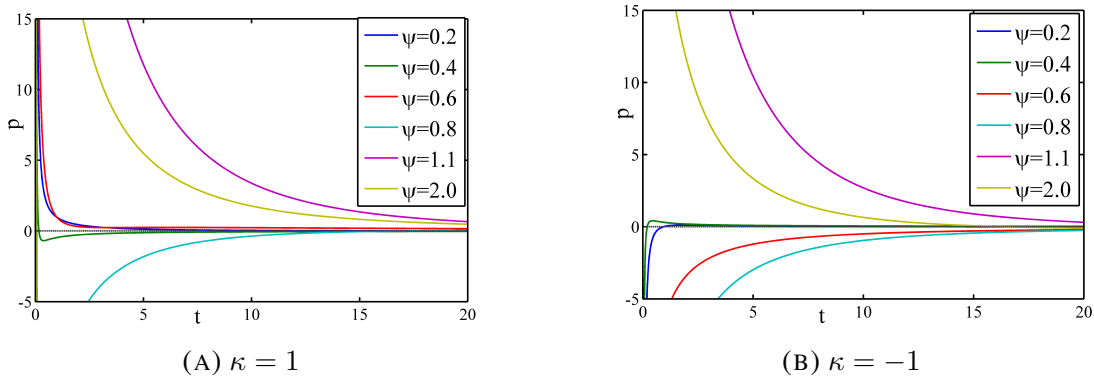


FIGURE 5.8: Profile of pressure for Model-II against time for different ψ

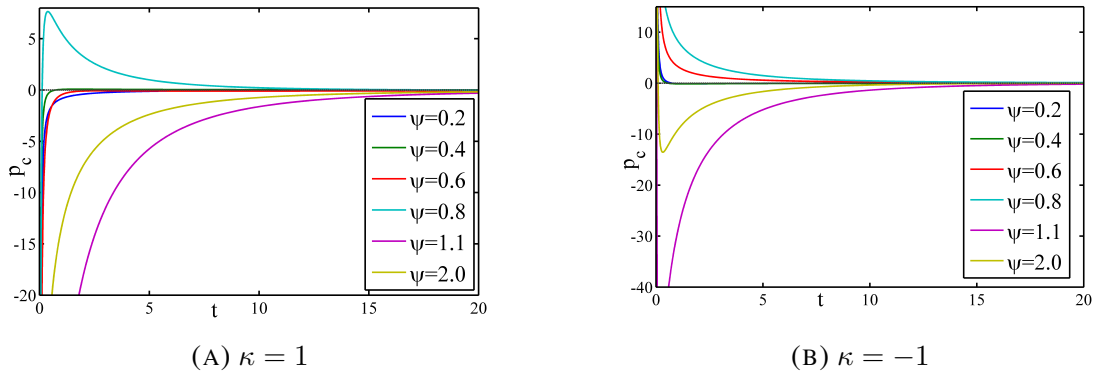


FIGURE 5.9: Profile of particle creation for Model-II against time for different ψ

Figure 5.8 indicates the pressure profile for different ψ against cosmic time t . For the case $k = 1$, the pressure is positive valued for $\psi = 0.2, 0.6, 1.1$, positive to negative valued for $\psi = 0.4, 0.8$ and negative to positive valued for $\psi = 2$. The pressure for $k = -1$ changes from negative to positive valued when $\psi = 0.2, 0.4$, negative valued

when $\psi = 0.6, 0.8$, positive valued when $\psi = 1.1$, and positive to negative valued when $\psi = 2$.

Figure 5.9 illustrate the profile of particle creation pressure p_c for different ψ against cosmic time. The particle creation pressure p_c for $k = 1$ changes from negative to positive valued for $\psi = 0.4, 0.8$ and negative valued for $\psi = 0.2, 0.6, 1.1, 2$. Furthermore, for $k = -1$, it changes from positive to negative valued for $\psi = 0.2, 0.4, 1.1, 2$ and positive valued for $\psi = 0.6, 0.8$.

5.4 Conclusion

We have examined particle creation in Rastall gravity in this chapter. Rastall gravity is a non-conservative theory and an extension of general relativity. The key element of this theory is that non-vacuum solutions are dependent on the Rastall coupling parameter and are significantly different from their corresponding solutions in general relativity. We derive and solve the Rastall gravity field equations under two different scale factors. Our results are as follows:

- In model-I, the energy density ρ shows physically acceptable behavior for Rastall coupling parameters $\psi = 0.2$ in $k = 1$ and $\psi \geq 0.4$ in $k = -1$
- Pressure p profiles for model-I show positive and negative values with $k = 1$ and $k = -1$, respectively.
- When $k = 1$ and $k = -1$, the particle creation pressure is negative or positive. The presence or absence of particle production is indicated by negative or zero particle creation pressure p_c . As a result, particle production occurs in model-I when $k = 1$.
- In model-II, the physically acceptable behavior of energy density ρ is illustrated by Rastall coupling parameter $\psi = 0.2$ and $\psi = 0.4$ for $k = 1$ and it takes negative values during the initial phase and positive values afterwards for $k = -1$.
- As shown in Figure 5.8, the pressure p profile in model-II is different for cases $k = 1$ and $k = -1$. Its behavior mainly depends on the Rastall coupling parameter ψ and k .

- There is also a difference in the particle creation pressure p_c under different ψ and k . Usually, we look for negative and zero values of particle creation pressure p_c . However, particle production is observed under $k = 1$ for $\psi = 0.6$ and 0.8 and under $k = -1$ for $\psi = 0.4$ and 0.8 .

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