AN APPROACH TO DEVELOP A SITUATIONAL BASED FUZZY REGRESSION MODELS FOR OPTIMIZATION IN UNCERTAINTY SCENARIO

Thesis Submitted for the Award of the Degree of

DOCTOR OF PHILOSOPHY

in

Mathematics

By

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Transforming Education Transforming India

LOVELY PROFESSIONAL UNIVERSITY, PUNJAB

2026

DECLARATION

I, hereby declared that the presented work in the thesis entitled **"An Approach to Develop Situational Based Regression Models for Optimization in Uncertainty Scenario"** in fulfilment of degree of **Doctor of Philosophy (Ph. D.)** is outcome of research work carried out by me under the supervision of **Dr. Rakesh Kumar,** working as **Associate Professor**, in the **Deptt. of (Mathematics/ Chemical engineering and physical sciences)** of Lovely Professional University, Punjab, India. In keeping with general practice of reporting scientific observations, due acknowledgements have been made whenever work described here has been based on findings of another investigator. This work has not been submitted in part or full to any other University or Institute for the award of any degree.

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CERTIFICATE

This is to certify that the work reported in the Ph. D. thesis entitled **"An Approach to Develop Situational Based Regression Models for Optimization in Uncertainty Scenario"** submitted in fulfillment of the requirement for the award of degree of **Doctor of Philosophy (Ph.D.)** in the **Deptt. of (Mathematics/ Chemical engineering and physical sciences)** is a research work carried out by **MUFALA KHAN, Reg. No. (12106612)** is bonafide record of his/her original work carried out under my supervision and that no part of thesis has been submitted for any other degree, diploma or equivalent course.

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ABSTRACT

In practical scenarios, datasets frequently contain vagueness, missing values, or inherent uncertainty, which limits the effectiveness of traditional regression methods that rely on precise and deterministic inputs. This thesis purposes to address these limitations by evolving regression models that are capable of adapting to varying uncertainty scenarios using fuzzy logic-based approaches. The first objective involves constructing enhanced Fuzzy Linear Regression (FLR) models through the integration of multiple fuzzy set types, including triangular, trapezoidal, intuitionistic, and Pythagorean fuzzy numbers. These models offer improved flexibility and better representation of vagueness in input–output relationships. The second objective introduces a possibilistic regression framework built upon conditional-based fuzzy numbers, enabling the model to dynamically respond to contextual influences by adjusting the spread and shape of fuzzy parameters. The final objective focuses on combining methodologies from both major and minor fields of fuzzy regression theory to develop a hybrid model that incorporates both membership and non-membership functions. This hybrid model aims to achieve greater robustness, adaptability, and accuracy in uncertain environments. Comprehensive empirical validation using real-world datasets demonstrates the effectiveness of the proposed models in various domains, establishing a strong foundation for future research in uncertainty-aware regression modeling.

Acknowledgment

All praise and gratitude are due to Almighty Allah, whose infinite mercy and guidance have been the light on my path throughout this doctoral journey. In moments of difficulty and doubt, His presence gave me strength, clarity, and the will to continue. I am deeply grateful to my supervisor, Dr. Rakesh Kumar, for his constant support, insightful guidance, and unwavering encouragement. His mentorship, patience, and expertise have played a pivotal role in the development and completion of this research. His belief in my abilities inspired me to strive for excellence. My heartfelt thanks go to my beloved parents—my father, Mr. Zakir Hussain, and my mother, whose love, sacrifices, and prayers have been my greatest strength. Their unwavering support and deep faith in me have sustained me through every challenge. My mother's quiet resilience and constant care provided comfort and motivation during even the most demanding times. I am also sincerely thankful to my brother, Shehzad Khan and my friends for their encouragement, positivity, and belief in my journey. Their presence has always been a source of motivation and emotional support. This accomplishment is not mine alone, it reflects the strength, love, and sacrifices of my family. To them, I dedicate this work with profound gratitude and love.

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List of Variables and Abbreviations

i. General Fuzzy Regression Variables

Variable	Definition
x_i	Independent variable or predictor in the regression model
y	Crisp output variable
$\hat{y}$	Fuzzy output variable, represented as a fuzzy number
β_ι	Fuzzy regression coefficient, typically a fuzzy number (triangular, trapezoidal, etc.)
n	Number of observations
k	Number of predictors or independent variables
h	Confidence level or alpha-cut used in fuzzy set calculations
ϵ	Small positive constant used to avoid numerical singularity

1. Membership and Non-Membership Functions

Variable	Definition
$\mu(x)$	Membership degree function, maps input to [0, 1]
$\nu(x)$	Non-membership degree function in IFS and PFS
$\pi(x)$	Indeterminacy or hesitation degree: $\pi(x) = 1 - \mu(x) - \nu(x)$ (IFS only)
$T_A(x)$	Truth-membership in Neutrosophic Sets
$F_A(x)$	Falsity-membership in Neutrosophic Sets
$I_A(x)$	Indeterminacy membership in Neutrosophic Sets

2. Fuzzy Numbers and Representations

Variable	Definition
TFN	Triangular Fuzzy Number
TrFNs	Trapezoidal fuzzy number
PFN	Pentagonal fuzzy number
HFN	Hexagonal fuzzy number
OFN	Octagonal fuzzy number
KKT	Karush-Kuhn-Tucker
KTNC	Karush-Tucker Necessary Conditions.
CIFO	Certainty-In-Fuzziness-Out

3. Model-Specific Variables

Variable	Definition
FN	Fuzzy Number
IFLR	Intuitionistic Fuzzy Linear Regression
PFLR	Pythagorean Fuzzy Linear Regression
NFLR	Neutrosophic Fuzzy Linear Regression
PPFLR	Possibilistic Pentagonal Fuzzy Linear Regression
HFRM	Hybrid Fuzzy Regression Model
MF	Membership function
N-MF	Non-Membership function

UMF	Upper Membership Function
LMF	Lowe Membership Function
z	Objective function to minimize (e.g., total spread, fuzziness)
RMSE	Root Mean Square Error, used to assess model performance
R^2	Coefficient of Determination, indicates model fit

4. Unknown variables and Decision Parameters used in thesis.

CHAPTER-2	*x_i : Independent variable i (predictor), $i = 1, \dots, k$* *y_j : Observed crisp output for the j^{th} observation, $j = 1, \dots, n$* *$\tilde{y}_j$: Fuzzy output for the j^{th} observation (represented as triangular fuzzy number)* *$C_i = (c_i, s_i)$: Fuzzy regression coefficient for x_i, where c_i is center and s_i spread* *n : Number of observations* *k : Number of predictors/variables* *h : Confidence level controlling membership inclusion scirp* *$\mu(\cdot)$: Membership function mapping values to degree of membership in fuzzy sets* *ϵ : Small positive constant to avoid division by zero* *$\mu(x)$: Membership function mapping input x to degree of membership in [1]* *$\nu(x)$: Non-membership function (used in intuitionistic and Pythagorean fuzzy sets)* *$\pi(x)$: Hesitation or indeterminacy degree in intuitionistic fuzzy sets, defined as $\pi=1-\mu-\nu$* *α: Cut level parameter in α-cut representation of fuzzy sets* *$d_{LR}(A, B)$: Distance measure between fuzzy numbers A and B, e.g., LR metric.*
CHAPTER-3	*$Y = \beta_0 + \sum_{i=1}^{n} \beta_i x_i$: regression output.* *Regression Coefficients:* *$\Omega_i = (I(\lambda)_i; I(L)_i, I(R)_i, I(L)'_i, I(R)'_i), i = 0,1, \dots, n$* *Each coefficient Ω_i is a Triangular Intuitionistic Fuzzy Number (TIFN) with:* *$I(\lambda)_i$: central (crisp) value,* *$I(L)_i, I(R)_i$: membership left and right spreads,* *$I(L)'_i, I(R)'_i$: non-membership left and right spreads.* *Regression Output:* $$Y = \Omega_0 + \sum_{i=1}^{n} \Omega_i x_i$$ *Predicted fuzzy output as a function of the input vector* *$x = (x_1, x_2, \dots, x_n)$.* *Membership & Non-Membership Spreads:* $$z_1 = f^{(I(L)_i, I(R)_i)}(x), z_2 = f^{(I(L)'_i, I(R)'_i)}(x)$$ *The quantities to be minimized in the optimization.* *$\beta_i = (m_i, L_i, R_i)$: intuitionistic fuzzy regression coefficients (center, left spread, right spread).* *x_i : predictor variables.* *$\mu(x), \nu(x)$: membership and non-membership functions.* *$\pi(x)$: indeterminacy.* *$Y = \Omega_0 + \sum_{i=1}^{n} \Omega_i x_i$: regression output.* *$\Omega_i = \sqrt{(T(\lambda)_i; T(L)_i, T(R)_i, T(L)'_i, T(R)'_i)}$: regression coefficients with central values and spreads.*

	x_i : *input variables.* $F_{\psi}^{TFS}(\tau), W_{\psi}^{TFS}(\tau')$: *membership and non-membership functions.* z_1, z_2 : *membership and non-membership spreads minimized in objective.* h : *confidence level;* ϵ : *stability constant.*
CHAPTER-4	$\beta_j (j = 0,1, \dots, n)$: *regression coefficients (crisp values).* $\hat{y}_i$: *predicted value of the response variable.* y_i : *observed response variable.* x_{ij} : *explanatory (independent) variables.* ε_i : *random error term.* a_j : *centers of the fuzzy coefficients.* l_j, r_j : *left and right spreads of coefficients.* $\beta_i = (S_i^l, S_i^R)$: *fuzzy coefficients with left and right spreads.* S_0^l, S_0^R : *spreads for the intercept term.* $a_i^p, a_i^c, a_i^k, a_i^U$: *key pentagonal support/center points.* $X = [x_1, x_2, \dots, x_n]^T$: *input vector.* y : *observed dependent variable.* h : *confidence level for* α*-cut conditions.* y_i : *observed outputs.* x_{ij} : *explanatory variables.* $h \in (0,1]$: *confidence level (inclusion index).* $\delta_{L_i}, \delta_{R_i}$: *tolerances on the left and right of* y_i.
CHAPTER-5	β_1, β_2 : *Regression coefficients to be estimated* ε_i : *Residual error term for the* i^{th} *observation* Y_i : *Fuzzy output* λ_j : *Fuzzy coefficients (TFNs) with center and spreads* e_i : *Error term*

CHAPTER 1

INTRODUCTION

This chapter unifies theoretical foundations besides practical motivations for employing the fuzzy regression as a robust alternative to classical models. In an increasingly complex world, data is rarely precise, complete, or deterministic. This intrinsic uncertainty presents a significant challenge to classical regression techniques, which operate under crisp, well-defined statistical assumptions. Traditional regression models often fail in domains characterized by linguistic ambiguity, subjectivity, or contextual variability-limitations that call for more flexible and adaptive approaches Fuzzy set theory, first proposed by Lotfi A. Zadeh in 1965, provides a flexible method to supervision uncertainty by permitting elements to belong to sets with varying degrees rather than just absolute membership.

1.1 Regression under Uncertainty: Motivation and Challenges

Regression analysis serves as a core technique for identifying patterns between variables and forecasting future outcomes based on those relationships. However, classical regression assumes that data is exact and errors are random and normally distributed—assumptions often violated in real-world scenarios. Uncertainties in measurement, linguistic descriptors, and subjective judgment cannot be effectively addressed through probabilistic means alone. Fuzzy regression addresses this issue by allowing imprecise, vague, or context-dependent data to be represented using fuzzy numbers and sets. Classical regression vs FR is given represented in figure 1.1.

Fuzzy Linear Regression (FLR) extends classical models by incorporating fuzzy coefficients and/or fuzzy inputs and outputs[1].

Unlike probabilistic models, FLR does not require known statistical distributions, making it suitable for domains such as healthcare, finance, environmental science, and manufacturing, where data quality may vary or expert knowledge may dominate.

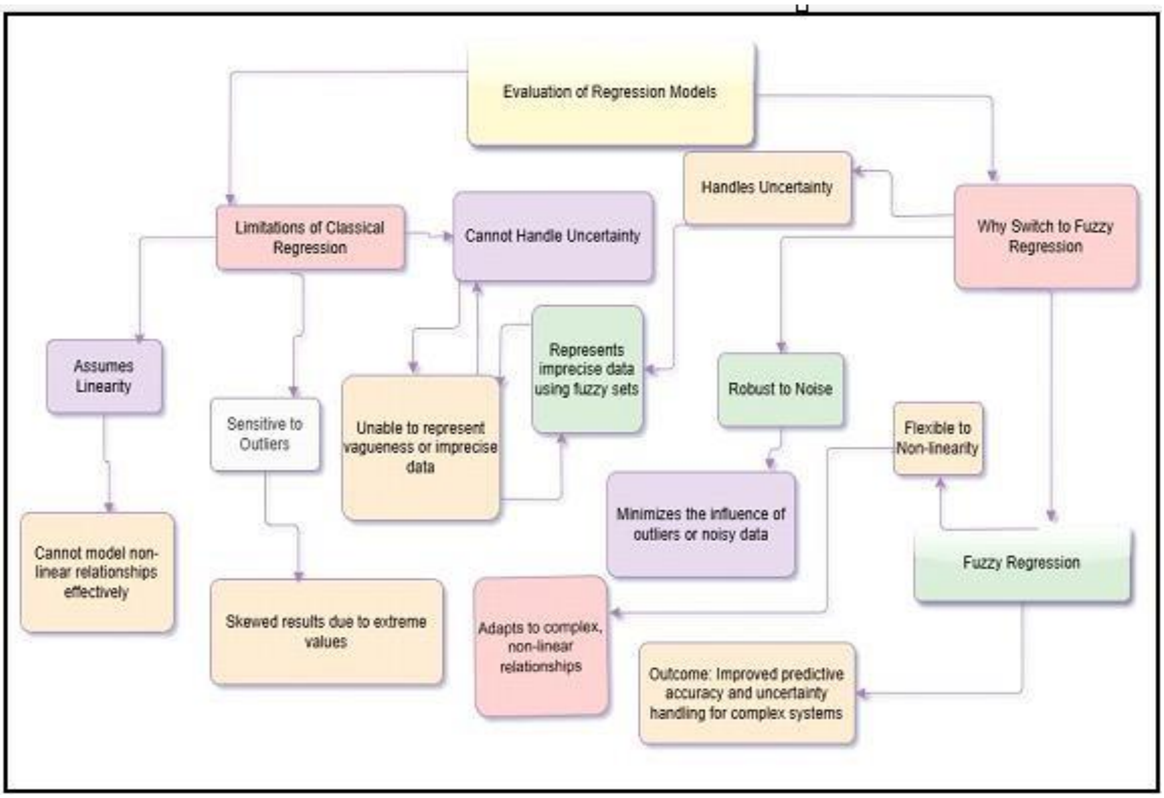

Figure 1. 1 Classical regression vs fuzzy regression

1.2 Fundamental Concepts: Crisp vs. Fuzzy Sets

In traditional set theory, a part is either included in a set or excluded from it there is no in-between. Fuzzy set theory, however, introduces the idea of incomplete membership, where elements can belong to a set to varying degrees between 0 and 1. This approach allows for more nuanced modeling of imprecise notions like "moderately high temperature" or "somewhat acceptable risk." Each element's degree of inclusion is determined by a membership function, which captures the continuum between full membership and non-membership, reflecting uncertainty and gradual transitions.

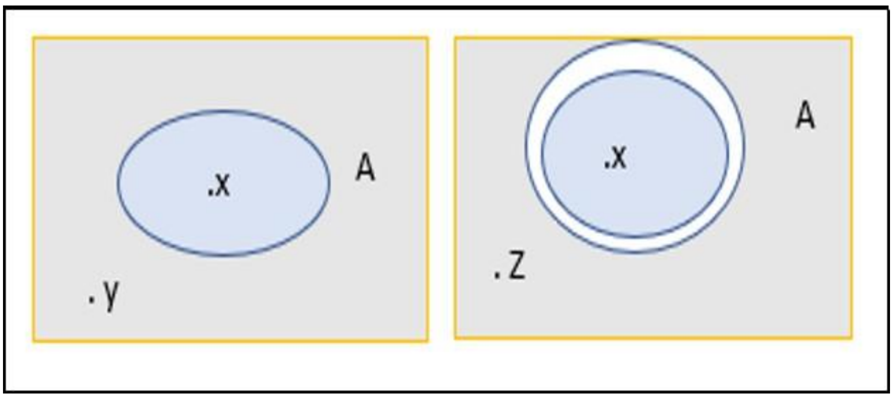

Figure 1. 2 (a) Crisp set Boundary (b) Fuzzy set Boundary

illustrates the key distinction between crisp and fuzzy sets. In part (a), the crisp boundary clearly separates members (e.g., x) from non-members (e.g., y) with binary membership-either 0 or 1. In contrast, part (b) shows a fuzzy set where the boundary is gradual rather than sharp. Element x belongs to the set with high membership, while z, though outside the core, may still partially belong. This highlights how fuzzy sets effectively handle uncertainty by allowing degrees of membership, making them well-suited for modeling vague or imprecise information.

1.3 Mathematical preliminaries

This part introduces the key principles and formal definitions that form the foundation for comprehending the fuzzy modeling techniques explored in this research.

1.3.1 Basics of Fuzzy Set:

This subsection contains some basic definitions of fuzzy sets and fuzzy numbers.

Definition 1.1. Fuzzy set

Let X be a universal set. A fuzzy set $\tilde{A}$ in X is characterized by a set of ordered pairs:

$$\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x)\rangle : x \in X\} \quad (1.1)$$

Where $\mu_{\tilde{A}}(x): X \to [0,1]$ denotes the *membership grade* of element x in $\tilde{A}$

Definition 1.2. Intuitionistic fuzzy set (IFS)

An IFS A is defined as:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\} \quad (1.2)$$

Here, $\mu_A(x)$: $X \to [0,1]$ and $\nu_A(x)$: $X \to [0,1]$ represent the degrees of membership and non-membership, respectively, subject to the condition:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1 \; \forall x \in X$$

The degree of hesitation or indeterminacy, denoted $\pi_A(x)$, is given by:

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) \quad (1.3)$$

This value captures the level of uncertainty associated with the element's classification.

Example: Suppose an expert evaluates the "suitability of a material" on a scale of 0 to 1. They assign a membership (suitability) of 0.7 and a non-membership (unsuitability) of 0.2. The Hesitation Degree (π) is: $\boldsymbol{\pi = 1 - 0.7 - 0.2 = 0.1}$

This 0.1 represents the expert's uncertainty or lack of information regarding that specific material.

Definition 1.3. Pythagorean fuzzy set (PFS)

A PFS generalizes the intuitionistic fuzzy approach by imposing a less restrictive condition on the degrees of MF and Non-MF. A PFS A in the universe X is represented as:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\} \qquad (1.4)$$

with the constraint: $\mu_A(x)^2 + \nu_A(x)^2 \leq 1$ and the hesitation degree is computed as: $\sqrt{1 - \mu_A(x)^2 - \nu_A(x)^2}$.

This formulation offers a more flexible framework to capture vagueness and uncertainty.

Example: An engineer evaluates a bridge's safety. They are quite confident it is safe ($\mu = 0.8$) but also have a specific reason to doubt it ($\nu = 0.5$).

- Check: $0.8 + 0.5 = 1.3$ (This fails standard Intuitionistic rules).
- PFS Calculation: $0.8^2 + 0.5^2 = 0.64 + 0.25 = 0.89$.
- Since $0.89 \leq 1$, this is a valid Pythagorean Fuzzy Set. It captures a higher level of uncertainty than standard models.

Definition 1.4. Neutrosophic fuzzy set (NFS)

NFS is introduces three independent functions to represent the truth, indeterminacy, and falsity of each element. Given a universe X, an NFS A is described as:

$$A = \{(x, T_A(x), I_A(x), F_A(x)) : x \in X\} \qquad (1.5)$$

where $T_A(x)$, $I_A(x)$, $F_A(x) \in [0,1]$, and the sum $T_A(x) + I_A(x) + F_A(x)$ can take any value within the interval [0,3]. This model allows the representation of incomplete, inconsistent, or uncertain information in a more nuanced manner.

Example: A political survey polls the population on whether they support an upcoming piece of legislation.

Truth (T)=0.6: 60% of people are completely certain that they do.

Falsity (F)=0.3: 30% are completely certain that they do not.

Indeterminacy (I)=0.4: 40% are completely uncertain or neutral.

As opposed to NFS, there is no need for all of these to add up to 1.

Definition 1.5. Fuzzy Number (FN)

A FN can be described as a particular kind of fuzzy set defined on the real number line, $\tilde{A} \subseteq \mathbb{R}$, that meets the following essential conditions:

Normality: The fuzzy set $\tilde{A}$ reaches a maximum membership value of 1, indicating that at least one element is fully included in the set.

Convexity: For any two elements x_1 and x_2 within the domain, and any $\lambda \in [0,1]$, the membership function satisfies

$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\} \tag{1.6}$$

Bounded Support: The region over which the membership function is greater than zero lies within a finite interval of real numbers.

Piecewise Continuity: The membership function $\mu_{\tilde{A}}(x)$ does not exhibit abrupt changes; it is continuous over sub-intervals of its domain, possibly with a finite number of discontinuities where limits from both sides exist.

Definition 1.6. Membership Function (MF)

A typical MF for a fuzzy number is defined as:

$$\mu_A(x) = \begin{cases} l(x), & x < m, \\ 1, & m \leq x \leq n, \\ u(x), & x > n, \end{cases} \tag{1.7}$$

The function $l(x)$ is an increasing, upper semi-continuous function defined for $x < m$, with the property that $l(x) = 0$ for all $x \leq m_1 < m$. Similarly, $u(x)$ is a decreasing, continuous function defined for $x > n$, satisfying $u(x) = 0$ for all $x \geq n_1 \geq n$. These two functions, $l(x)$ and $u(x)$, are known as the left and right references functions, respectively.

Definition 1.7. Alpha-cut

An α-cut of a fuzzy number $\tilde{A}$ is the crisp set

$$\tilde{A}(\alpha) = \{x \mid \mu_{\tilde{A}}(x) \geq \lambda\} \quad (1.8)$$

where $\alpha \in [0,1]$. This α-cut is a non-empty, bounded, closed interval within the real numbers and is represented as $[\tilde{A}_L(\alpha), \tilde{A}_U(\alpha)]$, where $\tilde{A}_L(\alpha)$ and $\tilde{A}_U(\alpha)$ denote the left and right endpoints of the interval, respectively. These are commonly referred to as the left and right α-cuts of $\tilde{A}$.

Definition 1.8. Triangular Fuzzy Number (TFN)

A fuzzy number $\tilde{A}$ is classified as a TFN if it is fully described by a triplet (a_1, a_2, a_3) of real numbers such that $a_1 < a_2 < a_3$. Its membership function is defined as:

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x - a_1}{a_2 - a_1}, & a_1 \leq x \leq a_2 \\ \frac{a_3 - x}{a_3 - a_2}, & a_2 \leq x \leq a_3 \\ 0, & \text{otherwise} \end{cases} \quad (1.9)$$

Here, a_1, a_2, and a_3 represent the lower bound, the peak (mode), and the upper bound of the fuzzy number, respectively. The interval $[a_1, a_3]$ is called the support, representing all values with non-zero membership, while a_2 is the core, indicating the most plausible value. This FN can also be expressed as $\tilde{A} = (A - \varphi_1, A, A + \varphi_2)$, where $\varphi_i (i = 1,2)$ are positive values satisfying $A > \varphi_1 > 0, 0 < \varphi_2$.

Example: Let us consider a triangular fuzzy number $\tilde{A} = (20,25,30)$ that signifies "comfortable room temperature."

The membership function for the temperature of 22ºC can be determined by the formula:

$$\mu_{\tilde{A}}(22) = \frac{22 - 20}{25 - 20} = \frac{2}{5} = 0.4$$

That is, $\mu_{\tilde{A}}$.

Definition 1.9. Trapezoidal Fuzzy Number (TrFNs)

The fuzzy number $\tilde{A}$ is called a non-negative TrFNs if it is represented by the quadruple (a_1, a_2, a_3, a_4) where $a_1 < a_2 < a_3 < a_4$, Its membership function is defined as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x - a_1}{a_2 - a_1}, & a_1 \le x \le a_2 \\ 1, & a_2 \le x \le a_3 \\ \frac{a_4 - x}{a_4 - a_3}, & a_3 \le x \le a_4 \\ 0, & \text{otherwise} \end{cases} \tag{1.10}$$

Let a_1, a_2, a_3, and a_4 denote the lower limit, lower mode, upper mode, and upper limit of the FN $\tilde{A}$, respectively. The TrFNs can also be expressed as $\tilde{A} = (A - \phi_1, A - \phi_2, A + \phi_3, A + \phi_4)$, where ϕ_i for $i = 1,2,3,4$ are positive parameters satisfying the constraints $A > \phi_1 > \phi_2$ and $\phi_3 < \phi_4$.

Example: Assume TrFN$\tilde{\boldsymbol{B}}$= (10,20,30,40), and the expression "moderate speed" means km/h.

The "core" or full membership will be for speeds within a range of 20" "to 30"

"km/h(μ=1.0).

If the speed is 35" "km/h. Then, the membership value is:

$$\mu_{\tilde{B}}(35) = \frac{40 - 35}{40 - 30} = \frac{5}{10} = 0.5$$

Here, 35 km/h belongs to the "moderate speed" set with a degree of 0.5.

Definition 1.10. Pentagonal Fuzzy Number (PFN)

A PFN $\tilde{A} = (a_1, a_2, a_3, a_4, a_5; r)$ is defined on the real line $\mathbb{R}$ such that the parameters satisfy $a_1 < a_2 < a_3 < a_4 < a_5$, with the membership function given by:

$$\mu_{\tilde{A}}(x) = \begin{cases} r\frac{x - a_1}{a_2 - a_1}, & a_1 \le x \le a_2 \\ 1 - (1 - r)\frac{a_3 - x}{a_3 - a_2}, & a_2 \le x \le a_3 \\ 1, & x = a_3 \\ 1 - (1 - r)\frac{x - a_3}{a_4 - a_3}, & a_3 \le x \le a_4 \\ r\frac{a_5 - x}{a_5 - a_4}, & a_4 \le x \le a_5 \\ 0, & \text{otherwise} \end{cases} \tag{1.11}$$

When r denotes the degree of the 'shoulder' or the height of the pentagonal fuzzy number, which satisfies $0 < r < 1$.

For example, suppose that a PFN$\tilde{C}$ = (1,3,5,7,9;0.5) is used to indicate the number of days for expected delivery time.

If the actual delivery takes place in 4 days and we consider $r = 0.5$ (satisfaction degree at any intermediate point):

$$\mu_{\tilde{C}}(4) = 1 - (1 - 0.5)\frac{5-4}{5-3} = 1 - (0.5 \times 0.5) = 0.75$$

- This shows a high membership degree (0.75) even before reaching the peak center of 5 days.

Definition1.11. Hexagonal Fuzzy Number (HFN)

A Hexagonal fuzzy number using six parameters $(a_1, a_2, a_3, a_4, a_5, a_6)$ and is ideal for representing two peaks with an extended central region:

$$\mu_A(x) = \begin{cases} 0 & x \le a_1 \\ \frac{x - a_1}{a_2 - a_1} & a_1 < x \le a_2 \\ 1 & a_2 < x \le a_3 \\ 1 & a_3 < x \le a_4 \\ \frac{a_6 - x}{a_6 - a_5} & a_5 < x \le a_6 \\ 0 & x > a_6 \end{cases} \quad (1.12)$$

This configuration allows for modeling more complex uncertainties with greater accuracy.

Example: Fuzzy Modeling for "Daily Peak Demand for Electricity."

Let $\tilde{A}_H = (10,12,14,16,18,20)$ megawatts.

- The demand begins to attain the "peak" condition from 10 MW.
- It attains a state of partial equilibrium at 12-14 MW.
- It achieves the peak condition (Membership Value = 1) at 14-16 MW.
- It declines to 18 MW and becomes non-peak at 20 MW.

Definition 1.12. Octagonal Fuzzy Number (OFN)

An Octagonal fuzzy number is represented as $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$. It offers the highest degree of granularity:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & x \le a_1 \\ \dfrac{x - a_1}{a_2 - a_1} & a_1 < x \le a_2 \\ 1 & a_2 < x \le a_6 \\ \dfrac{a_8 - x}{a_8 - a_7} & a_7 < x \le a_8 \\ 0 & x > a_8 \end{cases} \tag{1.13}$$

OFNs are particularly suitable for modeling data that exhibit multi-stage uncertainty.

These fuzzy numbers allow the representation of data with asymmetric or multi-stage uncertainty.

Example: Prediction of "Stock Market Volatility" in times of crisis.

For instance, let $\tilde{V}_O$ =(1,2,3,4,5,6,7,8) be a risk level set.

With an octagonal shape in the model, it is possible for the system to be maintained at a "Warning Level" (for instance, membership 0.5) from point 2 to 3, followed by a sudden change to "Critical Level" (membership 1.0) from point 4 to 5, before going down again.

Definition 1.13. Conditional-Based Fuzzy Numbers (CBFNs)

A CBFNs are those FNs whose parameters depend on external conditions θ, enabling dynamic adaptation:

$$\bar{A}(\,x \mid \theta\,) = (a_1(\theta), a_2(\theta), \dots, a_n(\theta)) \tag{1.14}$$

They are especially useful in environments where uncertainty evolves in response to time, location, or other contextual variables.

1.4 Evolution and Classification of FRM

FRM have evolved significantly since Tanaka et al.'s foundational work. Models have been extended through:

I) Fuzzy Least Squares Regression

II) Possibilistic Fuzzy Regression

III) Hybrid Models using Multiple Fuzzy Sets

A distinction is made between two fields of fuzzy regression:

Major Field: Includes traditional models with CIFO, such as fuzzy least squares and possibilistic regression.

Minor Field: Encompasses advanced models using IFS, PFS, NFS, and hybrid fuzzy types for context-rich or multi-dimensional uncertainty.

1.5 Need for Hybrid and Situational Models

While many fuzzy regression models exist, they often lack adaptability and are limited to specific applications. A major gap lies in the lack of situational models that adjust to varying levels of uncertainty. Hybrid models that combine membership and non-membership functions or integrate classical and possibilistic components offer a more robust solution.

1.6 Aims and Objectives

This thesis aims to develop advanced fuzzy regression models that are capable of optimizing performance under uncertainty. The research focuses on integrating various fuzzy set types, including triangular, trapezoidal, intuitionistic, and Pythagorean fuzzy numbers, into the regression framework. Additionally, hybrid models that combine major and minor fuzzy logic paradigms are proposed to enhance robustness and adaptability in uncertain environments.

This study is directed by the following objectives:

1. To Develop A Fuzzy Linear Regression Model by Using Various Types of Fuzzy Set.

2. To Proposed Possibilistic Regression Model Based on Conditional-Based Fuzzy Numbers.

3. Employing Both Major and Minor Fields of Fuzzy Regression Analysis to Construct A Hybrid Fuzzy Regression Model.

1.7 Significance and Scope

This research addresses the shortcomings of existing models by introducing flexible, scalable, and context-aware regression frameworks. These models are applicable in domains with dynamic uncertainty—such as additive manufacturing, material science, and supply chain pricing—and can be extended to broader fields in engineering, economics, and social sciences.

The scope is focused on fuzzy set theory and its extensions. While not incorporating probabilistic or machine learning approaches directly, the models developed can complement such techniques in future hybrid systems.

1.8 Chapter Summary

This chapter consolidated the introduction and theoretical foundations of fuzzy regression into a single framework. It provided an overview of the motivation for fuzzy modeling, introduced key concepts in FST, and reviewed the development of FR-models. The chapter outlined various fuzzy number types and advanced set structures, discussed their relevance in uncertainty modeling, and clarified the need for hybrid, situational regression models. This sets the stage for the subsequent chapters, which delve into the literature review, model development, empirical validation, and practical applications.

1.9 Outline of Thesis

This thesis is structured into six chapters, each contributing progressively to the development, implementation, and validation of fuzzy regression models under uncertain and imprecise conditions. The structure is planned to guide the reader through the motivation behind the study, the foundational literature, the theoretical and computational framework, and finally, the practical applications and conclusions drawn from empirical analyses.

Chapter 1 – Introduces the motivation, background, and objectives of the research, along with a summary of the essential mathematical preliminaries.

Chapter 2 – Reviews the literature on fuzzy and possibilistic regression models, identifies key trends, and highlights the research gaps motivating this study.

Chapter 3 – Develops and analyses fuzzy linear regression models, including Intuitionistic (IFLR), Pythagorean (PFLR), and Neutrosophic (NFLR) frameworks. This chapter includes worked numerical examples and application studies in Laser Powder Bed Fusion (LPBF) and MoS_2.

Chapter 4 – Proposes a possibilistic regression model based on conditional non-symmetric fuzzy numbers, provides optimization formulations, and validates the model empirically across multiple datasets.

Chapter 5 – Introduces hybrid fuzzy regression models by integrating major and minor paradigms. The chapter details algorithmic approaches, including KKT-based optimization, and demonstrates applications in supply chain pricing and crude-oil property prediction.

Chapter 6 – Concludes the thesis by summarizing key contributions, discussing limitations of the proposed work, and outlining directions for future research.

CHAPTER 2

LITERATURE REVIEW

This chapter presents review of literature of fuzzy regression models, synthesizing developments from foundational techniques to contemporary innovations. It integrates both previously distinct chapters on research trends and technical advancements—into a unified and professionally structured literature review. The chapter retains accurate references, now renumbered sequentially, and highlights research trajectories in fuzzy regression, applications, and future directions.

2.1 Introduction

Fuzzy regression analysis provides an effective framework for prediction under uncertainty, particularly when data is imprecise or insufficient. Unlike classical regression, which assumes crisp data and normal distribution[2], fuzzy regression accommodates vagueness using fuzzy logic principles. The foundational concept of fuzzy sets to mathematically handle vagueness in data introduced by [3]. He later established possibility theory as a framework for modeling uncertainty distinct from probability theory[4].Zadeh's Generalized Theory of Uncertainty (GTU) extended fuzzy logic for use in systems where ambiguity dominates[5][6]. Addressing critiques, Zadeh clarified the theoretical rigor and utility of fuzzy logic in real-world applications[7]. Dubois and Prade formally developed possibility theory with defined measures of possibility and necessity, enabling broader adoption in knowledge-based systems[8]. Kim provided comparative metrics for uncertainty, fuzziness, and ambiguity, and advocated for context-based selection of models[9].

2.2 Systematic Reviews and Methodological Contributions

Kitchenham outlined systematic procedures for reviews in software engineering, emphasizing structured planning and evaluation[10]. D valizade et al. extended this methodology to management sciences, stressing transparency and bias reduction [11]. Yazdanbakhsh and Dick conducted a comprehensive systematic review on complex fuzzy sets and logic, identifying trends and research gaps[12] .

2.3 Fuzzy Regression Using Linear Programming

Tanaka et al. pioneered fuzzy linear regression (FLR) modeling, treating deviations as reflections of system fuzziness rather than pure error[13]. They advanced the model with possibilistic frameworks [14][15], and later with linear programming approaches [16][17]. Celmins proposed least squares fitting for fuzzy vectors[18], and Tanaka et al. improved parameter identification using quadratic functions [19]. Yen et al. introduced triangular fuzzy coefficients for interpretability [20].

2.4 Possibilistic Fuzzy Regression

Possibilistic fuzzy regression was developed to better handle uncertainty in cases where traditional probability fails to capture imprecise or incomplete information[21]. Introduced by Tanaka et al., this approach uses possibility and necessity measures to define prediction intervals, allowing data to be modelled within fuzzy boundaries that reflect the highest plausibility rather than strict probabilities[22][23].

Unlike classical fuzzy regression, possibilistic regression emphasizes the degree of possibility for data inclusion in fuzzy intervals, making it ideal for uncertain systems with vague or partial knowledge[24]. Early models used fixed, symmetric fuzzy spreads, which limited their ability to represent varying uncertainty across observations[25].

To improve flexibility, conditional fuzzy regression models allow the spread of fuzziness to change based on contextual factors or system states[26]. For example, uncertainty in material properties may increase under certain conditions, such as temperature changes, which conditional models can adaptively capture.

Key developments include dynamic spread adjustments based on environmental changes and doubly linear models that separately estimate central values and spreads. Overall, possibilistic and conditional fuzzy regression provide more adaptive and accurate tools for modeling uncertainty in complex, variable environments, proving valuable in advanced manufacturing, materials science, and supply chain analytics.

2.5 Fuzzy Regression Using Least Squares

Least squares-based FLR was first formalized in [27], leading to adaptive and hybrid models[28]. These methods balanced fuzziness with statistical rigor. Hybrid models integrating crisp and fuzzy approaches showed robustness under mixed data conditions[29]

[30]. LAD models were found more effective than traditional least squares in outlier-prone datasets[31]. Hybrid fuzzy least squares models addressed dual uncertainty in risk-sensitive domains [32]. Wang and Tsaur contributed to FLR clarity by proposing a resolution method to reduce spread in fuzzy coefficients [33]. Their bicriteria model improved variable selection through accuracy and fuzziness trade-offs[34]. Hong et al. incorporated shape-preserving techniques for fuzzy input-output scenarios [35]. Chen developed outlier detection methods combined with interval modification[36] , while Hung and Yang proposed omission-based techniques[37].

2.6 Fuzzy Regression Using Fuzzy Numbers

FLR using fuzzy numbers was developed for asymmetric data modeling[38][39]. Methods constructed on Zadeh's extension principle and ambiguous mathematics showed improved estimation accuracy. Linear programming formulations tailored to different fuzzy number shapes enhanced model performance[40][41]. Sakawa and Yano applied multi-objective strategies to FLR for fuzzy input-output data[42] [43]. Özelkan and Duckstein generalized this into a flexible framework for real-world systems[44]. Guo and Tanaka’s dual models expanded possibilistic analysis with alternate regression formulations[44][45].

2.7 Fuzzy Regression Using Interval Estimation

Interval-based FLR addressed uncertainty in both input and output variables. Possibilistic linear systems defined under fuzzy set theory enabled more precise parameter estimation[46][47]. Applications in ecology, economics, and engineering demonstrated interval estimation’s strengths over ordinary regression [48][49].

Two-stage models involving defuzzification and spread modeling improved generalization [50][51]. Resolution identity techniques allowed flexible fuzzy parameter construction[52]. LAD-based estimation further strengthened outlier resilience[53].

Hybrid inference models, combining fuzzy regression with qualitative reasoning, increased interpretive power[54][55][56]. Rank transformation and α-level methods improved estimation consistency and output interpretation [56]. Moskowitz and Kim examined tuning parameters like the H value in FLR [57][58]. Tanaka et al. introduced exponential regression for fuzzy systems [59], and explored coefficient identification techniques[60][61]. Lee and Tanaka integrated central tendency and possibilistic elements through quadratic programming[62] . Additional innovations included fuzzy ranking [63], asymmetric fuzzy parameters [64][65], and piecewise models with change-point detection[66].

2.8 Recent and Emerging Applications

Newer studies integrated fuzzy models in interval analysis[67][68], rough set theory [69][70], and human behavior prediction via neuro-fuzzy modeling[71][72][73][74]. Others emphasized optimization in biofuels [75][76]and federated learning. Comprehensive surveys consolidated the real-world use of fuzzy regression models [77][78][79]. FLR has been applied in diverse fields. In manufacturing: predictive quality control [80][81], Healthcare: diagnostic modeling [82][83], Finance: stock and demand forecasting [84] , Energy: consumption and transport prediction[85] [86], Environment: air quality and climate modeling[87][88]

2.9 Technical Advancements in Fuzzy Regression

Recent years saw integration of IFS, PFS, NFS, and Type-2 fuzzy logic to handle complex uncertainties[89][90]. Multi-objective programming[91], L1/L2 norms, adaptive modeling [92], and advanced defuzzification[93] enhanced robustness and efficiency. Comparative studies of models outlined their respective strengths is given in table 2.1.

Table 2. 1 Comparative Analysis of Fuzzy Regression Models

Model Type	Relationship	Advantages	Limitations
Fuzzy Linear Regression (FLR)	Linear relationship between fuzzy inputs and outputs using fuzzy coefficients	- Handles vagueness and imprecision in linear data -Simple and easy to interpret - Effective for modeling linear uncertainty	- Not suitable for complex or nonlinear relationships - High sensitivity to outliers
Fuzzy Nonlinear Regression	Nonlinear relationship using fuzzy functions and parameters	- Captures complex patterns and dynamics - Models nonlinear dependencies in data - Flexible and	- Computationally intensive -Requires more data for accurate training

		adaptive to real-world data	
Fuzzy Least Squares Regression (FLSR)	Minimizes the sum of squared deviations between fuzzy predictions and actual values	- High accuracy in linear relationships - Reduces model sensitivity to noise - Easy to compute using matrix algebra	- Assumes that errors are symmetric - Less effective for heavily skewed or outlier-prone data
Fuzzy Least Absolute Regression (FLAR)	Minimizes the sum of absolute deviations between fuzzy predictions and actual values	- More robust to outliers compared to FLSR - Works well with asymmetric error distributions - Suitable for noisy data	- Less accurate for smooth, continuous data - Higher computational complexity
Possibilistic Regression	Models uncertainty using possibility theory rather than probability theory	- Handles non-statistical uncertainty - Suitable for small sample sizes - Effective for incomplete or imprecise data	- Difficult to interpret in probabilistic terms - Sensitive to the choice of membership functions
Fuzzy Logistic Regression	Binary or categorical relationship between fuzzy predictors and outputs	- Handles classification problems under uncertainty - Effective for medical diagnosis and credit risk assessment - Models vague or ambiguous categorical data	- Limited to binary or categorical outputs - Sensitive to data imbalance
Type-2 Fuzzy Regression	Extends type-1 fuzzy sets to handle higher-order uncertainty using interval-valued	- More accurate under high uncertainty - Effective for dynamic and uncertain environments	- Computationally expensive - Complex to implement and interpret

	membership functions	- Models complex and higher-dimensional data	
Fuzzy Cluster-wise Regression	Segments data into clusters and applies separate fuzzy regression models within each cluster	- Captures heterogeneous patterns in data - Effective for market segmentation and consumer analysis - Flexible for mixed data types	- Computationally intensive - Sensitive to the choice of clustering method
Bayesian Fuzzy Regression	Integrates Bayesian inference with fuzzy logic to incorporate prior knowledge into regression	- Effective for small datasets - Incorporates domain knowledge into modeling -Provides probabilistic interpretation of fuzzy parameters	- Requires specification of prior distributions - High computational cost
Hybrid Machine Learning-Based Fuzzy Regression	Combines fuzzy logic with machine learning methods like neural networks, SVM, and genetic algorithms	- High predictive accuracy - Adapts to complex patterns and high-dimensional data - Flexible for real-time and adaptive learning	- Black-box nature reduces interpretability - Requires large computational resources

2.10 Mathematical Formulations

This section presents the mathematical formulation of existing FR-model.

2.10.1. Fuzzy Regression (FR) Model

Fuzzy regression extends classical regression by incorporating uncertainty when data or model parameters are fuzzy rather than precise[94]. It primarily exists in two forms: (1) FR with fuzzy parameters and crisp input data, and (2) FR with crisp parameters and fuzzy data. Fuzzy Linear Regression model is stated as:

$$Y = C_1x_1 + C_2x_2 + \cdots + C_nx_n \tag{2.1}$$

where each C_ii is a STFN defined by:

Fit Constraints per Observation:

$$C_i(c) = \begin{cases} 1 - \frac{|c - c_i|}{s_i}, & \text{if } |c - c_i| \leq s_i \\ 0, & \text{otherwise} \end{cases} \tag{2.2}$$

Where each c_i is the center and s_i is the spread

Applying the extension principle, fuzzy output $Y(y)$ becomes

$$Y(y) = \begin{cases} 1 - \frac{|y - \sum_{i=1}^{n} c_i x_i|}{\sum_{i=1}^{n} s_i |x_i|}, & |y - \sum_{i=1}^{n} c_i x_i| \leq \sum_{i=1}^{n} s_i |x_i| \\ 0, & \text{otherwise} \end{cases} \tag{2.3}$$

To ensure the model fits observed data b_j within a confidence level $h \in [0,1]$, the membership condition

$$(b_j) \geq h \tag{2.4}$$

translates to the inequality

$$1 - \frac{|b_j - \sum_{i=1}^{n} c_i x_i|}{\sum_{i=1}^{n} s_i |x_i|} \geq h \tag{2.5}$$

which simplifies to

$$|b_j - \sum_{i=1}^{n} c_i x_i| \leq (1 - h) \sum_{i=1}^{n} s_i |x_i| \tag{2.6}$$

or equivalently

$$(1 - h) \sum_{i=1}^{n} s_i |x_i| \geq |b_j - \sum_{i=1}^{n} c_i x_i| \tag{2.7}$$

Optimized Objective Function:

Minimize total fuzziness:

$$\min \sum_{i=1}^{n} s_i \tag{2.8}$$

subject to the above fit constraint for all data points,

$$(1-h)\sum_{i=1}^{n} s_i|x_i| \geq |b_j - \sum_{i=1}^{n} c_i x_i| \tag{2.9}$$

In another type, where both inputs and outputs are fuzzy, the model uses crisp parameters a_i:

$$Y = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \tag{2.10}$$

with fuzzy data points

$$(X^{(j)}, Y^{(j)}) = ((x_i^{(j)}, s_i^{(j)}), (y^{(j)}, s_y^{(j)})) \tag{2.11}$$

The goal is to minimize the total error between the fuzzy model output and observed fuzzy data:

$$\min \sum_{j=1}^{m} \int_{-\infty}^{\infty} |Y^{(j)}(y) - Y(y)| \tag{2.12}$$

subject to ensuring the compatibility measure between observed and predicted fuzzy outputs exceeds a threshold h:

$$\text{compatibility}(Y^{(j)}, Y) \geq h \tag{2.13}$$

Overall, fuzzy regression captures uncertainty by optimizing fuzzy parameter spreads while maintaining model-data consistency within a specified confidence level. This framework enables flexible modeling of imprecise or vague data, balancing accuracy and uncertainty effectively.

2.10.2. Fuzzy Least Squares Regression (FLSR) Model

FLSR model offers a robust alternative to traditional FR methods, such as Tanaka's linear programming-based approach. Unlike inclusion-based constraints, FLSR directly minimizes the squared distances between observed and predicted fuzzy numbers[95]. It is particularly effective in handling data uncertainty and is resilient to outliers, especially when integrated with clustering.

The fuzzy regression model is defined as:

$$Y_j = A_0 + A_1 X_{j1} + A_2 X_{j2} + \cdots + A_k X_{jk},\ j = 1,2,\ldots,n \tag{2.14}$$

Where Y_j and X_{ji} are fuzzy numbers and A_i(unknown) are fuzzy coefficients. Each fuzzy number is represented in LR-type form:

$$Y_j = (m_{yj}, \sigma_{yj}, \gamma_{yj})_{LR},\ X_{ji} = (m_{xji}, \sigma_{xji}, \gamma_{xji})_{Lh},\ A_i = (m_{ai}, \sigma_{ai}, \gamma_{ai})_{Lh} \tag{2.15}$$

Here, m denotes the center, σ the left spread, and γ the right spread.

The predicted mean value of the fuzzy output is:

$$m_j = m_{a0} + \sum_{p=1}^{k} (m_{ap} m_{xjp}) \tag{2.16}$$

The membership function for the fuzzy output Y_i is expressed as:

$$\mu_{Y_i}(Y_i) = \begin{cases} L\left(\frac{m_j - x}{\sigma_j}\right), & x \le m_j \\ R\left(\frac{x - m_j}{\gamma_j}\right), & x \ge m_j \end{cases} \tag{2.17}$$

For triangular fuzzy numbers (TFNs), the left and right shape functions are defined by

$$L(x) = R(x) = 1 - x \tag{2.18}$$

Objective Function:

The objective function minimizes the total squared distance between observed and predicted fuzzy outputs:

$$\min_{A} \sum_{j=1}^{n} d^2_{LR}(Y_j, \hat{Y}_j) \tag{2.19}$$

Distance Measure: The distance measure d_{LR} quantifies the dissimilarity between two LR-type fuzzy numbers.

$$d^2_{LR}(Y_j, \hat{Y}_j) = (m_Y - m_{Y'})^2 + [(m_Y - l\sigma_Y) - (m_{Y'} - l\sigma_{Y'})]^2 + [(m_Y + r\gamma_Y) - (m_{Y'} + r\gamma_{Y'})]^2 \quad (2.20)$$

Thus, the ***total objective function*** becomes:

$$J(A_0, A_1, \dots, A_k) = \sum_{j=1}^{n} d^2_{LR}(Y_j, \hat{Y}_j) \quad (2.21)$$

Expanding:

$$J(A_0, A_1, \dots, A_k) = \sum_{j=1}^{n} [(m_Y - m_{Y'})^2 + [(m_Y - l\sigma_Y) - (m_{Y'} - l\sigma_{Y'})]^2 + [(m_Y + r\gamma_Y) - (m_{Y'} + r\gamma_{Y'})]^2] \quad (2.22)$$

The predicted components mean, left and right spread are given below in equations (2.23), (2.24) and (2.25)

$$m_j = m_{a0} + \sum_{p=1}^{k} (m_{ap} m_{xjp}) \quad (2.23)$$

$$\sigma_j = \sigma_{a0} + \sum_{p=1}^{k} (m_{ap}\sigma_{xjp} + m_{xjp}\sigma_{ap}) \quad (2.24)$$

$$\gamma_j = \gamma_{a0} + \sum_{p=1}^{k} (m_{ap}\gamma_{xjp} + m_{xjp}\gamma_{ap}) \quad (2.25)$$

Subject to non-negativity constraints:

$$\sigma_j \geq 0,\ \gamma_j \geq 0,\ \forall j \quad (2.26)$$

This model effectively minimizes the total fuzzy deviation, providing reliable estimates under uncertainty with optimal parameter spreads.

2.10.3. Robust Fuzzy Regression Model (RFRM)

The RFRM is a specialized technique for estimating regression parameters under uncertainty, particularly effective when data includes outliers or noise. It integrates fuzzy logic with robust statistical estimation to reduce sensitivity to anomalous observations[96]. The model uses three sub-regressions for the fuzzy response variable $\tilde{Y} = (m, l, u)_{LR}$.

Membership function:

$$\mu(y) = \begin{cases} L\left(\frac{m-y}{l}\right), & y \leq m \quad (l > 0) \\ R\left(\frac{y-m}{u}\right), & y \geq m \quad (u > 0) \end{cases} \tag{2.27}$$

Fuzzy Regression Equations:

$$\tilde{Y} = \begin{cases} m = F\alpha + \varepsilon, & \text{(center)} \\ m - l = (m^* - l^*) + \varepsilon_L, & \text{(left spread)} \\ m + u = (m^* + u^*) + \varepsilon_U, & \text{(right spread)} \end{cases} \tag{2.28}$$

Where $m^* = F\alpha$, $l^* = \beta m^* + \delta l$, $u^* = \gamma m^* + \eta l$ and F is the design matrix

For robust estimation, the objective function based on ***Least Median Squares (LMS)*** is:

$$D^2_{\text{med}} = \text{med}_i\,[(m_i - m^*_i)^2 + ((m_i - kl_i) - (m^*_i - kl^*_i))^2 + ((m_i + qu_i) - (m^*_i + qu^*_i))^2] \tag{2.29}$$

WLS is applied using residual-based weights:

$$w_i = \begin{cases} 1, & \text{if } |r^2_i/\hat{r}| \leq c \\ 0, & \text{if } |r^2_i/\hat{r}| > c \end{cases} \tag{2.30}$$

With $r^2_i = (m_i - m^*_i)^2 + ((m_i - kl_i) - (m^*_i - kl^*_i))^2 + ((m_i + qu_i) - (m^*_i + qu^*_i))^2$,

Weighted objective function becomes:

$$D^2_W = \sum w_i\,[(m_i - m^*_i)^2 + ((m_i - kl_i) - (m^*_i - kl^*_i))^2 + ((m_i + qu_i) - (m^*_i + qu^*_i))^2] \tag{2.31}$$

Goodness of fit is assessed using the coefficient of determination:

$$R^2 = 1 - \frac{SSE}{SST}, R^2 = 1 - (1 - R^2)\frac{n-1}{n-p-4} \quad (2.32)$$

This robust method improves accuracy in noisy environments and is valuable in fields like finance, engineering, and healthcare

2.10.4. Fuzzy Probabilistic Model (FPM)

The Fuzzy Probabilistic Model (FPM) combines fuzzy logic with probability theory to address both aleatory (random) and epistemic (knowledge-based) uncertainty[97]. It defines fuzzy random variables that map from a probability space Ω to fuzzy sets:

Fuzzy Random Variable:

$$\tilde{X} : \Omega \rightarrow F(\mathbb{R}^n) \quad (2.33)$$

Where Ω is the sample space, $F(\mathbb{R}^n)$ is the space of fuzzy numbers respectively occurrence $\omega \in \Omega$ is associated with a fuzzy number $\tilde{X}(\omega)$.

Fuzzy probability of an event $\boldsymbol{A}$ is expressed via $\boldsymbol{\alpha}$ -cuts:

$$\tilde{P}(A) = \{P_\alpha(A) \mid \alpha \in [0,1]\} \quad (2.34)$$

Fuzzy cumulative and density functions are similarly defined:

$$\tilde{F}(x) = \{F_\alpha(x) \mid \alpha \in [0,1]\}, \qquad \tilde{F}(x) = \int_{-\infty}^{x} \tilde{f}(t)dt \quad (2.35)$$

Fuzzy Reliability Index (for structural safety):

$$\tilde{\beta} = \{\beta_\alpha \mid \alpha \in [0,1]\} \quad (2.36)$$

Fuzzy limit state function, which determines system failure or survival, is:

$$\tilde{g}(x) = 0 \quad (2.37)$$

If $\tilde{g}(x) \leq 0$., the system fails; otherwise, it survives. FPM offers a comprehensive framework for uncertainty modeling, widely used in engineering, finance, and decision analysis where precision is limited and probabilistic data is sparse.

2.10.5. Fuzzy Logistic Regression (FLR)

Fuzzy Logistic Regression (FLR) extends classical logistic regression by incorporating fuzzy set theory to address uncertainty in classification tasks[98]. Unlike traditional models, FLR uses fuzzy coefficients and outputs to better model vague or imprecise data, making it suitable for domains like medical diagnosis and risk assessment.

Let $X_i = (x_{i1}, x_{i2}, \dots, x_{in})$ be crisp inputs, and $\tilde{\mu}_i$ the fuzzy probability of class 1. The fuzzy odds ratio is:

$$O_i = \frac{\tilde{\mu}_i}{1-\tilde{\mu}_i},\ 0 \le \tilde{\mu}_i \le 1 \tag{2.38}$$

Fuzzy logit function:

$$\tilde{W}_i = \tilde{B}_0 + \tilde{B}_1 x_{i1} + \tilde{B}_2 x_{i2} + \dots + \tilde{B}_n x_{in} \tag{2.39}$$

Where $\tilde{B}_j = (a_j, s_i^L, s_i^R)$ are triangular fuzzy coefficients. Then

$$W_i = (W_i^c, W_i^L, W_i^R) \tag{2.40}$$

Where, $W_i^c = a_0 + a_1 x_{i1} + \dots + a_n x_{in}$ (center value), $W_i^L = s_0^L + s_1^L x_{i1} + \dots + s_n^L x_{in}$ (left spread), and $W_i^R = s_0^R + s_1^R x_{i1} + \dots + s_n^R x_{in}$ (right spread).

Membership function for $\tilde{\boldsymbol{W}}_{\boldsymbol{i}}$:

$$\mu_{W_i}(w_i) = \begin{cases} 1 - \dfrac{W_i^c - w_i}{W_i^L}, & W_i^c - W_i^L \le w_i \le W_i^c \\ 1 - \dfrac{w_i - W_c^c}{W_i^R}, & W_i^c \le w_i \le W_i^c + W_i^R \end{cases} \tag{2.41}$$

Objective Function (minimization of spread):

$$Z = m(s_0^L + s_0^R) + \sum_{j=1}^{n} (s_j^L + s_j^R) \sum_{i=1}^{m} x_{ij} \tag{2.42}$$

Constraints:

$$(1-h)s_0^L + (1-h)\sum_{j=1}^{n} s_j^L x_{ij} - a_0 - \sum_{j=1}^{n} a_j x_{ij} \ge -W_i \tag{2.43}$$

$$(1-h)s_0^R + (1-h)\sum_{j=1}^{n} s_j^R x_{ij} + a_0 + \sum_{j=1}^{n} a_j x_{ij} \geq W_i \tag{2.44}$$

where h is the optimism level ($0 \leq h \leq 1$).

2.10.6. Fuzzy Time Series (FTS) Model

The FTS model integrates fuzzy logic into time-series forecasting to manage imprecise or uncertain data[99]. It defines the universe of discourse as:

$$U = [X_{\min}, X_{\max}] \tag{2.45}$$

$$and\ U = [X_{\min} - d, X_{\max} + d] \tag{2.46}$$

The universe is partitioned into fuzzy sets F_i, and each observation is fuzzified via membership functions. An LR-type FN has the function:

$$N(x) = \begin{cases} L((m-x)/\alpha), & x \leq m \\ R((x-m)/\beta), & x > m \end{cases} \tag{2.47}$$

Objective Function:

$$Z = \sum_{i=1}^{m} (s_0 + \sum_{j=1}^{n} s_j x_{ij}) \tag{2.48}$$

Fuzzy Transition Rule:

$$F(t) = F(t-1) \circ R \tag{2.49}$$

Constraints (Linear Programming):

$$(1-h)s_0^L + (1-h)\sum_{j-1}^{n} s_j^L x_{ij} - a_0 - \sum_{j-1}^{n} a_j x_{ij} \geq -W_i \tag{2.50}$$

$$(1-h)s_0^h + (1-h)\sum_{j=1}^{n} s_j^R x_{ij} + a_0 + \sum_{j=1}^{n} a_j x_{ij} \geq W_i \tag{2.51}$$

Where, h is the optimism level, s_0^L, s_0^R are lower and upper spread values, and W_i represents the estimated fuzzy time series value D. These constraints ensure that predictions remain within an acceptable range.

2.10.7. *Type-2 Fuzzy Regression Model (T2FRM)*

The T2FRM enhances standard fuzzy regression by using Interval Type-2 Fuzzy Sets (IT2FS), capable of handling both primary and secondary uncertainty[100]. An IT2FS has a Footprint of Uncertainty (FOU):

$$FOU(\tilde{A}) = \{x \in X: \mu_{\tilde{A}}(x) \in [\mu_{\tilde{A}}^L(x), \mu_{\tilde{A}}^U(x)]\} \tag{2.52}$$

Where, $\mu_{\tilde{A}}^U(x)$ is LMF, and $\mu_U(x)$ is UMF. A Type-2 triangular fuzzy number is:

$$\tilde{A} = (a, b, c, \alpha) \tag{2.53}$$

Upper Membership Function (UMF):

$$\mu_{\tilde{A}}^U(x) = \begin{cases} \dfrac{x-a}{b-a}, & a \le x \le b \\ \dfrac{c-x}{c-b}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases} \tag{2.54}$$

Lower Membership Function (LMF):

$$\mu_{\tilde{A}}^L(x) = \begin{cases} \dfrac{x-(a+\alpha)}{b-(a+\alpha)}, & (a+\alpha) \le x \le b \\ \dfrac{(c-\alpha)-x}{(c-\alpha)-b}, & b \le x \le (c-\alpha) \\ 0, & \text{otherwise} \end{cases} \tag{2.55}$$

Model Structure:

$$\tilde{Y}_i = \tilde{A}_0 + \tilde{A}_1 X_{i1} + \tilde{A}_2 X_{i2} + \cdots + \tilde{A}_q X_{iq}, \text{ for } i = 1, \ldots, n \tag{2.56}$$

Where $\tilde{A}_j = [\underline{A}_j, \bar{A}_y]$ and predicted output:

$$Y_i^L = \sum_{j=0}^{q} A_j X_{ij},\ Y_i^U = \sum_{j-0}^{q} \bar{A}_y X_{ij} \tag{2.57}$$

$$J = J_1 + J_2 - J_3 + J_4 \tag{2.58}$$

Where, J_1 minimize spread of the UMF of the coefficients, J_2 minimizes the spread of the LMF, J_3 is in the necessity model, this term is maximized to ensure the lower bound of predicted output is as nearby as likely to the observed value and J_4 minimizes the distance between the point with the highest membership in the

2.10.8. Fuzzy Regression Analysis with Common Dataset

A consistent dataset, shown in table 2.2, is used to assess multiple fuzzy regression models. It includes two independent variables $X_1$and X_2 and the actual output variable (Y). The dataset reveals a positive trend between inputs and the output, forming the basis for fuzzy regression analysis under uncertainty. Such data allows us to examine how different models handle imprecision in parameters estimation and prediction.

Table 2. 2 Fuzzy regression analysis with same dataset

Sample ID	X_1	X_2	Y (Actual)
1	3.2	1.5	10.5
2	4.1	2.3	12.8
3	5.0	3.1	15.2
4	6.2	3.8	18.0
5	7.1	4.5	20.5

Various fuzzy regression models are then applied to this data, each using a distinct methodology to incorporate uncertainty. table 2.3, presents the final equations for these models, where coefficients are defined as fuzzy numbers.

Table 2. 3 Final comparison table for all fuzzy regression models

Model	Final Equation
Fuzzy Regression with Fuzzy Parameters	$Y =$ (2.908,0.055) +(2.06,0.055)X_1+(0.64,0.055)X_2

Fuzzy Regression with Fuzzy Data	$Y = (2.95,0.06) + (2.08,0.06)X_1 + (0.67,0.06)X_2$
Possibilistic Fuzzy Regression	$Y = (2.92,0.05) + (2.04,0.05)X_1 + (0.65,0.05)X_2$
Fuzzy Least Squares Regression	$Y = (2.90,0.05) + (2.06,0.05)X_1 + (0.64,0.05)X_2$
Robust Fuzzy Regression	$Y = (2.85,0.07) + (2.10,0.07)X_1 + (0.62,0.07)X_2$
Fuzzy Probabilistic Regression	$Y = (2.89,0.06) + (2.05,0.06)X_1 + (0.65,0.06)X_2$
Fuzzy Logistic Regression	$P(Y=1) = \frac{1}{1+e^{-[(-4.5,0.3)+(0.8,0.2)X_1+(0.9,0.2)X_2]}}$
Fuzzy Time Series Regression	$Y_{t-1} = (2.85,0.05) + (1.95,0.05)X_1 + (0.75,0.05)X_2 + (0.3,0.05)\ Y_{t-1}$
Type-2 Fuzzy Regression	$Y = (2.8,0.06,0.02) + (2.0,0.06,0.02)X_1 + (0.7,0.06,0.02)X_2$

Each model exhibits unique characteristics. For instance, Type-2 Fuzzy Regression captures secondary uncertainty, while Robust Fuzzy Regression mitigates the impact of outliers. Fuzzy Logistic Regression targets classification tasks. The performance of these models is compared in table 2.4.

Table 2. 4 Comparison of predicted Y values across different fuzzy regression models

Sample id	Y(actual)	Fuzzy Parameters	Fuzzy Data	Possibilistic Fuzzy	Fuzzy Least Squares	Robust Fuzzy	Fuzzy Probabilistic	Fuzzy Time Series	Type-2 Fuzzy
1	10.5	10.4	10.6	10.4	10.5	10.3	10.7	10.5	10.4
2	12.8	12.8	12.7	12.6	12.8	12.5	12.9	12.9	12.7
3	15.2	15.1	15.1	15.0	15.2	15.1	15.3	15.5	15.3
4	18.0	18.1	17.9	18.2	18.0	17.8	18.2	18.2	18.1

5	20.5	20.4	20.3	20.5	20.5	20.2	20.6	20.6	20.5

Among all, Fuzzy Least Squares Regression (FLSR) provides the closest predictions to actual values, while Possibilistic Regression maintains output within fuzzy bounds. Type-2 Fuzzy Regression excels in modeling higher-order uncertainty, and Probabilistic Regression incorporates confidence-based prediction. Robust Regression underperforms slightly in accuracy due to its focus on noise reduction.

2.11 Application Domains

To highlight practical relevance, table 2.5, outlines the connection between various fuzzy models and their applications across fields like engineering, healthcare, and finance.

Table 2. 5 Relationship Between Major and Minor Fuzzy Regression Models

Major Field (Broad Category)	Minor Field (Specific Model Type)	Base Fuzzy Regression Model	Extended / Related Fuzzy Regression Models	Relationship Between Models
Engineering & Manufacturing	Quality Control	Possibilistic FR	Fuzzy Least Squares Regression (FLSR)	Possibilistic regression improves defect analysis by handling imprecision, while FLSR reduces uncertainty in manufacturing processes.
	Predictive Maintenance	Type-2 Fuzzy Regression	Fuzzy Probabilistic Regression	Type-2 regression accounts for high uncertainty in equipment failures, while probabilistic regression estimates failure likelihood.
Economics & Finance	Structural Health Monitoring	Fuzzy Probabilistic Regression	Fuzzy Time Series Regression	Probabilistic regression model’s structural failure probabilities, while time series regression predicts long-term trends.
	Stock Market Forecasting	Fuzzy Time Series Regression	Type-2 Fuzzy Time Series Regression	Type-2 fuzzy regression enhances financial forecasting by incorporating interval-based uncertainty.

	Credit Risk Assessment	Fuzzy Logistic Regression (FLR)	Type-2 Fuzzy Logistic Regression	Type-2 FLR accommodates higher uncertainty in classifying loan applicants compared to standard FLR.
	Economic Forecasting	Fuzzy Regression with Fuzzy Data	Possibilistic Fuzzy Regression	Possibilistic regression integrates expert-driven uncertainty into economic models.
Healthcare & Medicine	Disease Diagnosis	Fuzzy Logistic Regression (FLR)	Type-2 Fuzzy Logistic Regression	Type-2 FLR enhances medical diagnosis by improving classification robustness under uncertainty.
	Drug Response Prediction	Type-2 FR	Fuzzy Probabilistic Regression	Type-2 regression models uncertain drug interactions, while probabilistic regression incorporates randomness in medical trials.
	Health Risk Assessment	Possibilistic FR	Fuzzy Least Squares Regression (FLSR)	Possibilistic regression model's vague health risks, while FLSR ensures precise uncertainty estimation.
Environmental Science	Climate Change Modeling	Fuzzy Time Series Regression	Type-2 Fuzzy Time Series Regression	Type-2 FR provides enhanced uncertainty handling in climate forecasting.
	Pollution Assessment (Air & Water)	Fuzzy Least Squares Regression (FLSR)	Fuzzy Probabilistic Regression	FLSR model's pollution uncertainty, while probabilistic regression predicts contamination event probabilities.
Artificial Intelligence & ML	Pattern Recognition	Fuzzy Logistic Regression (FLR)	Type-2 Fuzzy Regression	FLR supports uncertain classification tasks, while Type-2 fuzzy models handle higher degrees of fuzziness in AI applications.

	Neural Network Optimization	Type-2 FR	Fuzzy Probabilistic Regression	Type-2 regression captures uncertainties in deep learning models, while probabilistic regression improves optimization.
Social Sciences & Psychology	Consumer Behavior Analysis	Fuzzy Regression with Fuzzy Parameters	Possibilistic Fuzzy Regression	Possibilistic regression refines consumer decision-making models by accounting for behavioral uncertainty.
	Education Performance Prediction	Possibilistic FR	Fuzzy Least Squares Regression (FLSR)	Possibilistic regression evaluates uncertain student performance trends, while FLSR enhances prediction accuracy.
Transportation & Logistics	Traffic Flow Prediction	Fuzzy Time Series Regression	Type-2 Fuzzy Time Series Regression	Type-2 FR enhances traffic forecasting by improving uncertainty handling.
	Supply Chain Optimization	Fuzzy Least Squares Regression (FLSR)	Fuzzy Probabilistic Regression	FLSR minimizes supply chain variability, while probabilistic regression estimates disruption probabilities.

In summary, FLSR, Possibilistic, Type-2, and Time Series Fuzzy Regression Models are foundational tools across multiple sectors for modeling imprecise data. Their extended forms address specific uncertainty levels and application needs in industrial, medical, and predictive analytics.

2.12 Research Gaps and Future Directions

Although fuzzy linear regression (FLR) has seen notable advancements, several key challenges continue to limit its broader effectiveness and applicability. A major gap is the absence of a unified regression framework that combines IFS, PFS, and NFS. While each of these individually captures specific types of uncertainty—hesitancy (IFS), extended membership tolerance (PFS), and explicit indeterminacy (NFS)—an integrated model exploiting their combined strengths remains undeveloped and untested across varied domains.

Another unresolved issue is the handling of dynamically evolving uncertainty. Most existing FLR models assume a static or pre-defined fuzzy spread, which fails to reflect real-world systems—like healthcare, finance, and manufacturing—where uncertainty fluctuates with environmental and internal changes. Adaptive models that automatically tune their parameters in real-time are urgently needed.

Moreover, there is a lack of rigorous comparative evaluations across different fuzzy regression approaches. Without benchmarking models using standardized metrics such as Fuzzy Mean Square Error (FMSE), Fuzzy Inclusion Degree (FID), and Hybrid Correlation Coefficient (HCC), it is difficult for researchers to determine the best-fit model for specific applications.

Scalability is another critical limitation. As FLR models incorporate hybrid mechanisms, conditional spreads, and complex fuzzy logic, their computational demands grow. This becomes problematic for large datasets or real-time environments, especially in fields like material science and Industry 4.0. Efficient algorithms that maintain accuracy while minimizing computational load are still underdeveloped.

Finally, most FLR research remains theoretical, with limited empirical validation. There is a pressing need for robust, application-driven testing of advanced fuzzy models in emerging sectors such as smart healthcare, autonomous systems, and energy optimization. Only through real-world validation can the true practical value of FLR be realized. Fundamentally, advancing fuzzy regression for intelligent systems requires concentrating on integrating models, enabling real-time responsiveness, establishing standardized benchmarks, developing scalable methods, and applying the approach across diverse practical scenarios.

2.13 Summary

This chapter unified the theoretical evolution, methodological trends, and application landscapes of fuzzy regression. From FLR foundations to hybrid innovations, the review emphasizes FLR's increasing relevance in uncertain data environments and motivates the novel contributions that follow in this thesis.

CHAPTER-3

Fuzzy Linear Regression Models (FLRM) Using Diverse Fuzzy Sets

This chapter presents the experimental results and analysis of three advanced FLR-models: Intuitionistic FLR, Pythagorean FLR), and Neutrosophic FLR. The models are evaluated on numerical examples and real-world applications to compare their capability to handle uncertainty, vagueness, and imprecision in datasets. A comparative study with classical linear regression is also performed. Furthermore, the models are applied to advanced domains including Laser Powder Bed Fusion (LPBF) and material science to validate their practical effectiveness.

3.1. Intuitionistic Fuzzy Linear Regression (IFLR) Model

The intuitionistic fuzzy linear regression (IFLR) model extends classical fuzzy regression by incorporating membership,$\boldsymbol{\mu}(\boldsymbol{x})$ non-membership $\boldsymbol{\nu}(\boldsymbol{x})$ and indeterminacy $\boldsymbol{\pi}(\boldsymbol{x})$. This structure allows more flexibility in representing vague or incomplete information. This section presents the proposed mathematical IFLR-model to find uncertainty.

The evolution of Fuzzy Linear Regression (FLR) as a globally influential tool of uncertainty modeling that can be applied to any dataset is due to the consideration of uncertainty within the structure itself. The traditional way of conducting regression analysis considers data variation as 'noise', described by some probabilistic function. However, unlike this methodology, Fuzzy Linear Regression considers the nature of data, which is fuzzy and imprecise. In general, this may be attributed to the transition from probabilistic modeling to possibilistic modeling; that is, when data is converted into fuzzy parameters, it results in possibility distribution. As a consequence, the methodology becomes universal and can be applied to any type of data since its vagueness is preserved in the very structure of the regression model."

3.1.1 Model Formulation

This subsection includes existing and proposed models of IFLR.

3.1.1.1. Conventional IFLR Model

The Intuitionistic Fuzzy Linear Regression (IFLR) model is designed to account for uncertainty by incorporating both membership and non-membership degrees in its structure. This approach allows for a more nuanced modeling of imprecise data. The standard IFLR regression equation is given as:

$$\tilde{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n \tag{3.1}$$

Here, the intuitionistic fuzzy coefficients are derived by minimizing the spread between membership and non-membership degrees using an objective function:

$$\min \sum_{i=1}^{n} (\mu_i - \nu_i)^2 \tag{3.2}$$

Where μ_i and ν_i denote the membership and non-membership degrees, respectively, subject to the condition $\mu_i^2 + \nu_i^2 \leq 1$. The fuzzy coefficients are then calculated in the form:

$$\beta_i = \lfloor a_i ; l_i , l_i ; r_i , r_i \rfloor \tag{3.3}$$

Once estimated, the intuitionistic fuzzy regression equation becomes:

$$\tilde{y} = \beta_0 + \beta_1 x \tag{3.4}$$

Defuzzification is done using the transformation:

$$y^* = \frac{a + l + r}{3} \tag{3.5}$$

This transformation provides crisp predictions, allowing for linear interpretations even under fuzzy data conditions.

3.1.1.2. Proposed TIFS-Based IFLR Model

To overcome the limitations of symmetric fuzzy spreads in traditional IFLR, a new formulation is introduced that uses Triangular Intuitionistic Fuzzy Sets (TIFS**)**. This proposed model enables the use of non-symmetric spreads in both membership and non-membership functions, enhancing the representation of real-world uncertainty.

The fuzzy regression model is given by:

$$Y = \Omega_0 + \Omega_1 x_1 + \Omega_2 x_2 + \cdots + \Omega_n x_n \tag{3.6}$$

This proposed TIFS-based IFLR model extends the classical intuitionistic approach by including non-symmetric triangular fuzzy coefficients, making it more capable of modeling real-world variability. It provides a powerful tool for analyzing uncertain data where traditional symmetric models may fall short.

The extended intuitionistic fuzzy output Y is defined as:

$$Y = f(x, \Omega) = (f^{I(\lambda)}(x); f^{(I(L)_i, L(R)_i)}(x), f^{(I(L)_i, L(R)_i)}(x)) \tag{3.7}$$

Where, $x = (x_1, x_2, \dots, x_n)$ is the input vector, and $\Omega = [\Omega_0, \Omega_i]$,

Each coefficient has the structure: $\Omega_0 = (\mathbb{I}(\lambda)_0; \mathbb{I}(L)_0, \mathbb{I}(R)_0, \mathbb{I}(L)'_0, \mathbb{I}(R)'_0)$, and $\Omega_i = (\mathbb{I}(\lambda)_i; \mathbb{I}(L)_i, \mathbb{I}(R)_i, \mathbb{I}(L)'_i, \mathbb{I}(R)'_i)$. The model consists of the following components: Center, Membership Spread and Non-Membership Spread give in below equations (3.8), (3.9) and (3.10).

$$f^{\mathrm{I}(\lambda)}(x) = \mathbb{I}(\lambda)_0 + \sum_{i=1}^{n} \mathbb{I}(\lambda)_i x_i \tag{3.8}$$

$$f^{(I(L)_i, \mathbb{L}(R)_i)}(x) = (\mathbb{I}(L)_0, \mathbb{I}(R)_0) + \sum_{i=1}^{n} (\mathbb{I}(L)_i, \mathbb{I}(R)_i) x_i \tag{3.9}$$

$$f^{(I(L)'_i, L(R)'_i)}(x) = (\mathbb{I}(L)'_0, \mathbb{I}(R)'_0) + \sum_{i=1}^{n} (\mathbb{I}(L)'_i, \mathbb{I}(R)'_i) x_i \tag{3.10}$$

If $I(L)_i \neq L(R)_i$ the TIFS reduces to a symmetric (STIFS) structure. Otherwise, it accommodates non-symmetric uncertainty.

The aim is to reduce the combined uncertainty in both membership and non-membership domains by minimizing the total spread **z**, formulated as: Objective Function is given in equation (3.11)

$$min z = (z_1 + z_2) \tag{3.11}$$

where, z_1: total membership spread, z_2 : total non-membership spread.

The constraints associated with this optimization are Membership and Non-membership constraint given below:

$$(1 - h)\, (\mathbb{I}(L)_0 + \sum_{i=1}^{n} \mathbb{I}(L)_i |x_i|) + \sum_{i=1}^{n} \mathbb{I}(\lambda)_i x_i + \mathbb{I}(\lambda)_0 \geq \tau \tag{3.12}$$

$$\mathbb{I}(\lambda)_0 + \sum_{i=1}^{n} \mathbb{I}(\lambda)_i x_i - h\left(\mathbb{I}(L)_0' + \sum_{i=1}^{n} \mathbb{I}(L)_i' |x_i|\right) \geq \tau'$$

3.1.1.3. Numerical example

Using the dataset in table 3.1, the regression coefficients were computed through a Linear Programming Problem (LPP) method.

Table 3. 1 Data for input-output

j	s_j(sales)	x_j(Months)	Estimated sale values (S)
1	3010	1	3120.5
2	4500	2	3736.4
3	4400	4	4968.0
4	5400	6	6199.7
5	7295	7	6815.5
6	8195	8	7431.4

resulting in an objective function value of 1845.6. The IFLR model for this dataset is:

$$\tilde{y} = [2486.6; 868.5, 868.5, 977.0, 977.0] + [615.8; 0, 0; 0, 0]\, x_1 \quad (3.13)$$

$$= [3102.5; 868.5, 868.5; 977.0, 977.0]$$

The defuzzified output values matched the estimated sales data, demonstrating the model’s alignment with real-world values.

The IFLR model effectively handles ambiguity between membership and non-membership, offering more precise estimations in uncertain environments. Its structure is particularly beneficial for financial or forecasting datasets where classical regression fails to represent fuzziness. The reduction of uncertainty through minimization of coefficient spread ensures the reliability of predicted values.

3.2. Pythagorean Fuzzy Linear Regression (PFLR) Model

The section presents the proposed mathematical PFLR-model to find uncertainty.

ii. Method Applied: TPFS-Based PFLR

1. Express coefficients as triangular Pythagorean fuzzy sets.
2. Formulate regression output including center and spreads.
3. Impose constraint $(F_{\psi}^{TFS}(\tau))^2 + (W_{\psi}^{TFS}(\tau'))^2 \leq 1$.
4. Define objective function $\min z = z_1 + z_2$.
5. Solve using quadratic programming under fuzzy constraints.

3.2.1. Existing Pythagorean Fuzzy Linear Regression (PFLR)

Pythagorean fuzzy sets (PFS) enhance the modeling of uncertainty by allowing a more flexible relationship between the membership (μ) and non-membership (ν) functions. Specifically, the squared sum of these values must not exceed one:

$$\mu^2 + \nu^2 \leq 1 \tag{3.14}$$

This extended uncertainty allowance makes PFS particularly suitable for scenarios involving high ambiguity and partial truth. The Pythagorean fuzzy linear regression (PFLR) model incorporates this concept within a fuzzy programming framework.

The fuzzy linear programming formulation based on Pythagorean fuzzy logic is:

Maximize (or Minimize):$\sum (a_j \otimes x_j)$

Subject to:

$$\sum (b_{ij} \otimes x_j) \leq c_i \tag{3.15}$$

where all variables are TPFNs.

This model captures fuzziness in both the objective function and constraints but treats all coefficients as symmetric fuzzy numbers. While useful, this approach assumes uniform uncertainty and lacks flexibility in differentiating between uneven spreads across data dimensions.

3.2.2. Proposed TPFS-Based Pythagorean Fuzzy Regression Model

To improve upon the existing PFLR framework, we introduce a new regression formulation using Triangular Pythagorean Fuzzy Sets (TPFS)**.** Unlike TPFNs, TPFS allow non-symmetric membership and non-membership spreads**,** giving the model greater expressive power in modeling asymmetric and data-dependent uncertainty.

The full TPFS-based regression main equation is given by:

$$Y = \Omega_0 + \Omega_1 x_1 + \Omega_2 x_2 + \cdots + \Omega_n x_n \tag{3.16}$$

Objective function:

$$\min z = \Omega_0 + \sum_{j=1}^{m} x_{ji} \otimes \Omega_i \tag{3.17}$$

The regression output in the proposed model is expressed as:

$$Y = f(x, \Omega) = \left(f^{\sqrt{T(\lambda)}}(x), f^{\sqrt{T(L)_i, T(R)_i}}(x), f^{\sqrt{T(L)'_i, T(R)'_i}}(x)\right) \tag{3.18}$$

Each coefficient is: $x = (x_1, x_2, \dots, x_n)$ is the input vector, $\Omega = [\Omega_0, \Omega_i]$,

$\Omega_0 = \sqrt{[T(\lambda)_0; T(L)_0, T(R)_0; T(L)'_0, T(R)'_0]}$ and $\Omega_i = \sqrt{[T(\lambda)_i; T(L)_i, T(R)_i; T(L)'_i, T(R)'_i]}$

The components of the model are given in below equations (3.19), (3.20) and (3.21) as center, membership and non-membership spreads.

$$f^{\sqrt{T(\lambda)}}(x) = \sqrt{T(L)_0 + \sum_{i=1}^{n} T(\lambda)_i x_i} \tag{3.19}$$

$$f^{\sqrt{T(L)_i, T(R)_i}}(x) = \sqrt{(T(L)_0, T(R)_0) + \sum_{i=1}^{n} (T(L)_i, T(R)_i) x_i} \tag{3.20}$$

$$f^{\sqrt{T(L)'_i, T(R)'_i}}(x) = \sqrt{(T(L)'_0, T(R)'_0) + \sum_{i=1}^{n} (T(L)'_i, T(R)'_i) x_i} \tag{3.21}$$

Constraint are given as: $(\mathcal{F}_{\psi}^{\text{TFS}}(\tau))^2 + (\mathcal{W}_{\psi}^{\text{TFS}}(\tau'))^2 \leq 1$

In this case, $\mathcal{F}_{\psi}^{\text{TFS}}(\tau)$ and $\mathcal{W}_{\psi}^{\text{TFS}}(\tau')$ represent the degrees of MF and Non-MF respectively.

Where,

$$\mathcal{F}_{\psi}^{TFS}(\tau) = \begin{cases} \sqrt{1-\left(\frac{\tau-(\sum_{i=1}^{n} T(\lambda)_i x_i + T(\lambda)_0)}{T(L)_0+\sum_{i=1}^{n} T(L)_i |x_i|}\right)^2 - \varepsilon^2}, & \tau \geq T(\lambda)_0 + \sum_{i-1}^{n} T(\lambda)_i x_i \\ \sqrt{1-\left(\frac{(\sum_{i=1}^{n} T(L)_i x_i - T(\lambda)_0 - \tau)}{T(R)_0+\sum_{i=1}^{n} T(R)_i |x_i|}\right)^2 - \varepsilon^2} & \text{otherwise} \end{cases} \tag{3.22}$$

And

$$\mathcal{W}_{\psi}^{\mathrm{TFS}}(\tau') = \begin{cases} \sqrt{1-\left(\frac{\tau'-(\sum_{i=1}^{n} T(\lambda))_i x_i + T(\lambda)_0)}{T(L)'_0+\sum_{i=1}^{n} T(L)'_i |x_i|}\right)^2 - \varepsilon^2}, & \tau \geq T(\lambda)_0 + \sum_{i-1}^{n} T(\lambda)_i x_i \\ \sqrt{1-\left(\frac{(\sum_{i=1}^{n} T(L)_i x_i - T(\lambda)_0 - \tau')}{T(R)'_0+\sum_{i=1}^{n} T(R)'_i |x_i|}\right)^2 - \varepsilon^2} & \text{otherwise} \end{cases} \tag{3.23}$$

Here, ε denotes a small positive constant introduced to prevent numerical instability.

To minimize the total spread of membership and non-membership functions the objective function is given as:

$$min z = z_1 + z_2 \tag{3.24}$$

Where, $z_1 = f^{\sqrt{T(L)_i T(R)_i}}(x)$ (membership spread) and $z_2 = f^{\sqrt{T(L)'_i T(R)'_i}}(x)$ (non-membership spread).

To ensure the output lies within valid fuzzy bounds, the following constraints are introduced for membership and non-membership bounds given below:

$$(1-h^2-\varepsilon^2)\left(T(L)_0+\sum_{i=1}^{n} T(L)_i|x_i|\right)+\sum_{i=1}^{n} T(\lambda)_i x_i + T(\lambda)_0 \geq \tau$$
$$(1-h^2-\varepsilon^2)\left(T(R)_0+\sum_{i=1}^{n} T(R)_i|x_i|\right)-\sum_{i=1}^{n} T(\lambda)_i x_i - T(\lambda)_0 \leq -\tau \tag{3.25}$$

$$(-h^2-\varepsilon^2)\left(T(L)'_0+\sum_{i=1}^{n} T(L)'_i|x_i|\right)+T(\lambda)_0+\sum_{i=1}^{n} T(\lambda)_i x_i \geq \tau'$$
$$(h^2-\varepsilon^2)\left(T(R)'_0+\sum_{i=1}^{n} T(R)'_i|x_i|\right)-T(\lambda)_0-\sum_{i=1}^{n} T(\lambda)_i x_i \leq -\tau' \tag{3.26}$$

3.2.3. Numerical Examples

A structured approach involving ranking functions was applied to transform the Pythagorean fuzzy programming problem into an equivalent crisp LPP:

Maximize

$$\{(3,5,9);(1,5,11)\} \otimes \beta_1 \oplus \{(5,7,10);(3,7,12)\} \otimes \beta_2$$

Subject to:

$$\{(4,7,11);(2,7,13)\} \otimes \beta_1 \oplus \{(5,8,10);(3,8,12)\} \otimes \beta_2 = \{(23,83,157);(8,83,249)\}$$

$$\{(3,5,8);(1,5,11)\} \otimes \beta_1 \oplus \{(4,7,10);(2,7,13)\} \otimes \beta_2 = \{(18,67,136);(5,67,242)\}$$

Where β_1 and β_2 are non-negative TPFNs.

Step 1. Let $\beta_1 = \{(l_1, m_1, n_1);(l'_1, m_1, n'_1)\}$,

$\beta_2 = (l_2, m_2, n_2);(l'_2, m_2, n'_2)\}$ then problem can be written as:

$$\{(3,5,9);(1,5,11)\}\otimes\{(l_1, m_1, n_1);(l'_1, m_1, n'_1)\}\oplus\{(5,7,10);(3,7,12)\}\otimes\{(l_2, m_2, n_2);(l'_2, m_2, n'_2)\}$$

Subject to:

$$\{(3,5,8);(1,5,11)\}\otimes\{(l_1, m_1, n_1);(l'_1, m_1, n'_1)\}\oplus\{(5,8,10);(3,8,12)\}\otimes\{(l_2, m_2, n_2);(l'_2, m_2, n'_2)\} =\{(23,83,157);(8,83,249)\}$$

$$\{(4,7,11);(2,7,13)\}\otimes\{(l_1, m_1, n_1);(l'_1, m_1, n'_1)\}\oplus\{(4,7,10);(2,7,13)\}\otimes\{(l_2, m_2, n_2);(l'_2, m_2, n'_2)\} =\{(18,67,136);(5,67,242)\}$$

where $\{(l_1, m_1, n_1);(l'_1, m_1, n'_1)\}$ and $\{(l_2, m_2, n_2);(l'_2, m_2, n'_2)\}$ are non-negative TPFNs.

Step 2. Maximize

$$\{(3l_1,5m_1,9n_1); (1l'_1, 5m_1, 11n'_1)\}\oplus\{(5l_2, 7m_2, 10n_2);(3l'_2, 7m_2, 12n'_2)\}$$

Subject to:

$$\{(4l_1,7m_1,11n_1); (2l'_1, 7m_1, 13n'_1)\}\oplus\{(5l_2, 8m_2, 10n_2);(3l'_2, 8m_2, 12n'_2)\}=\{(23,83,157);(8,83,249)\}$$

$$\{(3l_1,5m_1,8n_1); (1l'_1, 5m_1, 11n'_1)\}\oplus\{(4l_2, 7m_2, 10n_2);(2l'_2, 7m_2, 13n'_2)\}=\{(18,67,136);(5,67,242)\}$$

where$\{(l_1, m_1, n_1);(l'_1, m_1, n'_1)\}$ and $\{(l_2, m_2, n_2);(l'_2, m_2, n'_2)\}$are non-negative TPFNs.

Step 3. By put on ranking function problem becomes

$\Re\{(3l_1 + 5l_2, 5m_1 + 7m_2, 9n_1 + 10n_2);(1l'_1 + 3l'_2, 5m_1 + 7m_2, 11n'_1 + 12n'_2)\}$

Subject to:

$$\{(4l_1 + 5l_2, 7m_1 + 8m_2, 11n_1 + 10n_2);(2l'_1 + 3l'_2, 7m_1 + 8m_2, 13n'_1 + 12n'_2)\}$$

$$= \{(23,83,157) ;(8,83,249)\},$$

$$\{(3l_1 + 4l_2, 5m_1 + 7m_2, 8n_1 + 10n_2);(1l'_1 + 2l'_2, 5m_1 + 7m_2, 11n'_1 + 13n'_2)\}= \{(18,67,136);(5,67,242)\}$$

$$m_1 - l_1 \geq 0, n_1 - m_1 \geq 0, m_2 - l_2 \geq 0, n_2 - m_2 \geq 0, l_1 - l'_1 \geq 0, n'_1 - n_1 \geq 0, l_2 - l'_2 \geq 0, n'_2 - n_2 \geq 0, l'_1 \geq 0, l'_2 \geq 0$$

Step 4. Using arithmetic operation, we get

$\Re\{(3l_1 + 5l_2, 5m_1 + 7m_2, 9n_1 + 10n_2);(1l'_1 + 3l'_2, 5m_1 + 7m_2, 11n'_1 + 12n'_2)\}$

Subject to:

$$\{(4l_1 + 5l_2, 7m_1 + 8m_2, 11n_1 + 10n_2);(2l'_1 + 3l'_2, 7m_1 + 8m_2, 13n'_1 + 12n'_2)\}$$

$$= \{(23,83,157) ;(8,83,249)\},$$

$$\{(3l_1 + 4l_2, 5m_1 + 7m_2, 8n_1 + 10n_2);(1l'_1 + 2l'_2, 5m_1 + 7m_2, 11n'_1 + 13n'_2)\}= \{(18,67,136);(5,67,242)\}$$

$$m_1 - l_1 \geq 0, n_1 - m_1 \geq 0, m_2 - l_2 \geq 0, n_2 - m_2 \geq 0, l_1 - l'_1 \geq 0, n'_1 - n_1 \geq 0, l_2 - l'_2 \geq 0, n'_2 - n_2 \geq 0, l'_1 \geq 0, l'_2 \geq 0$$

Step 5. The problematic changes to crisp LPP

Maximize

$$\{\frac{3l_1 + 5l_2 + 1l'_1 + 3l'_2 + 20m_1 + 28m_2 + 9n_1 + 10n_2 + 11n'_1 + 12n'_2}{8}\}$$

Subject to:

$$4l_1+5l_2=23, 3l_1+4l_2=18,$$

$$7m_1+8m_2=83, 5m_1+7m_2=67,$$

$$11n_1+10n_2=157, 8n_1+10n_2=136$$

$$2l'_1+3l'_2=8, 1l'_1+2l'_2=5$$

$$13n'_1+12n'_2=249, 11n'_1+13n'_2=242,$$

$$m_1 - l_1 \geq 0, n_1 - m_1 \geq 0, m_2 - l_2 \geq 0, n_2 - m_2 \geq 0, l_1 - l'_1 \geq 0, n'_1 - n_1 \geq 0, l_2 - l'_2 \geq 0, n'_2 - n_2 \geq 0, l'_1 \geq 0, l'_2 \geq 0$$

Step 6: To effectively resolve the crisp problem outlined earlier, the initial step involves determining the most suitable solution approach.

$l_1 = 2, m_1 = 5, n_1 = 7, l'_1 = 1, n'_1 = 9, l_2 = 3, m_2 = 6, n_2 = 8, l'_2 = 2$and $n'_2 = 11$

Step7. Putting the values in

$\beta_1 = \{(l_1, m_1, n_1);(l'_1, m_1, n'_1)\}$ and $\beta_2 = (l_2, m_2, n_2);(l'_2, m_2, n'_2)\}$ An exact

Pythagorean optimum solution is obtained.

$\beta_1 = \{(2,5,7); (1,5,9)\}$and $\beta_2 = \{(3,6,8); (2,6,11)\}$.

Step8. The FPFLPP is computed using the values acquired in the previous phase. {(21,67,143) ;(7,67,231)}

The calculations involved step-by-step transformation of fuzzy numbers, constraint simplifications, and solving the crisp problem.

The PFLR model demonstrates enhanced flexibility in modeling uncertainty by allowing broader ranges for fuzzy coefficients. It is especially effective for fuzzy equality-constrained optimization problems where high levels of imprecision are present. The solution process also validates the feasibility of transforming fuzzy models into equivalent crisp optimization forms for practical computation.

3.3. Neutrosophic Fuzzy Linear Regression (NFLR) Model

This section presents the proposed mathematical PFLR-model to find uncertainty.

3.3.1. Model Formulation

The NFLR model integrates three aspects of uncertainty: truth, indeterminacy, and falsity, using neutrosophic logic. The basic regression structure is defined as:

$$K = \alpha B_i + \beta \quad (3.27)$$

When K is an estimated Neutrosophic output, and B_i represents crisp input vectors. The coefficients α and β are derived using neutrosophic statistics:

$$\alpha = \frac{C(A, B)}{T(A(T(B)} \qquad (3.28)$$

$$\text{And } \beta = E(\overline{Y}) - \alpha E(\overline{X})$$

The neutrosophic covariance $C(A, B)$ is given by:

$$C(A, B) = \sum_{i=1}^{n} \gamma_A(x_i)\gamma_B(x_i) + \delta_A(x_i)\delta_B(x_i) + \mu_A(x_i)\mu_B(x_i) \qquad (3.29)$$

And $E(\overline{Y})$ and $E(\overline{X})$ is given as:

$$E(\overline{Y}) = \frac{1}{3}\gamma_B(x_i) + \delta_B(x_i) + \mu_B(x_i) \qquad (3.30)$$

$$E(\overline{X}) = \frac{1}{3}\gamma_A(x_i) + \delta_A(x_i) + \mu_A(x_i) \qquad (3.31)$$

The neutrosophic averages are:

$$[\gamma_A(x_i) = \frac{1}{n}\sum_{i=1}^{n} \gamma_A(x_i),\ \gamma_B(x_i) = \frac{1}{n}\sum_{i=1}^{n} \gamma_B(x_i),\ \delta_A(x_i) = \frac{1}{n}\sum_{i=1}^{n} \delta_A(x_i),\ \delta_B(x_i) = \frac{1}{n}\sum_{i=1}^{n} \delta_B(x_i),\ \mu_A(x_i) = \frac{1}{n}\sum_{i=1}^{n} (\mu_A(x_i)),\ \mu_B(x_i) = \frac{1}{n}\sum_{i=1}^{n} (\mu_B(x_i))] \qquad (3.32)$$

3.3.2. Numerical Example

Let A and B be two NFSs over X = {a, b}:

A = {a,(0.3,0.2,0.5) , b,(0.5,0.4,0.2) }, B = { a,(0.2,0.4,0.6) , b,(0.4,0.3,0.6) .

Computed values are:

CN(A, B) = 0.88, T(A) = 0.83, T(B) =1, $\gamma_A(x_i) = 0.4$, $\gamma_B(x_i) = 0.3$, $\delta_A(x_i) = 0.3$, $\delta_B(x_i) = 0.35$, $\mu_A(x_i) = 0.35$, $\mu_B(x_i) = 0.6$, $E(\overline{Y}) = 0.42$, $E(\overline{X}) = 0.35$

Thus, $\alpha = 1.06$ and $\beta = 0.49$ and the final regression equation is $K = 1.06B_i + 0.49$

The NFLR model offers a significant advancement by explicitly modeling indeterminacy in addition to truth and falsity. It is particularly suited for complex decision environments where uncertainty cannot be resolved through traditional or even intuitionistic fuzzy methods. The numerical demonstration confirms the model's capacity to derive meaningful relations under neutrosophic uncertainty.

3.4. Comparison with Classical Linear Regression and FLR

To assess the advantages of FR-models, a comparative analysis was performed against classical linear regression using financial data given in table 3.2 from PSCADB between 2016 and 2022. Regression models were constructed for each year using both classical and fuzzy approaches.

Table 3. 2 Dataset of financials from the year 2016-22. (In Crores.)

Credentials	2016-17	2017-18	2018-19	2019-20	2020-21	2021-22
Reserves and other funds	(410.29, 0.10)	(412.63, 0.00)	(408.77, 0.25)	(427.82, 0.00)	(386.58, 0.50)	(361.74, 0.00)
Paid-up share capital	(70.91, 0.64)	(71.73, 0.85)	(72.95, 0.72)	(74.52, 0.89)	(75.86, 0.70)	(77.16, 0.40)
Total own funds	(481.20, 0.57)	(484.36, 0.26)	(481.73, 0.52)	(502.34, 0.00)	(462.44, 0.50)	(449.71, 0.00)
Profit	(20.70, 0.10)	(28.77, 0.00)	(25.66, 0.49)	(24.93, 0.00)	(25.42, 0.70)	(10.81, 0.40)
SADB Level	(487.08, 0.00)	(431.64, 0.00)	(501.11, 0.00)	(506.54, 0.00)	(559.80, 0.50)	(490.22, 0.00)
PADBs Level	(513.11, 0.00)	(465.54, 0.00)	(628.07, 0.00)	(646.69, 0.00)	(663.59, 0.70)	(488.98, 0.40)
Total Loan Outstanding	(2187.95, 0.00)	(2226.66, 0.00)	(2309.87, 0.00)	(2428.86, 0.00)	(2617.95, 0.50)	(2704.03, 0.00)
Borrowing Outstanding	(2048.60, 0.00)	(2123.06, 0.00)	(2163.14, 0.00)	(2277.79, 0.00)	(2332.10, 0.50)	(2368.63, 0.40)
Working capital (Average)	(2783.40, 0.79)	(2962.53, 0.39)	(3131.56, 0.70)	(3234.98, 0.77)	(3210.95, 0.50)	(3357.00, 0.40)
SADB	(35.61, 0.75)	(54.92, 0.00)	(43.99, 0.00)	(41.09, 0.00)	(21.07, 0.70)	(1.90, 0.00)
PABD	(56.31, 0.95)	(83.69, 0.00)	(69.65, 0.00)	(66.02, 0.00)	(46.49, 0.70)	(12.71, 0.00)

3.4.1. Detailed Classical and Fuzzy Regression Solutions:

For Year 2020–21: Solution for Classical Regression is given as;

$$b = SP/SSX = 14573593.5430/13891598.9957$$

$$a = \bar{Y} - b\bar{X} = 938.4400 - (1.0500 * 945.6600) = -53.6407$$

$$Y = 1.0490X - 53.6407. \quad (3.33)$$

And solution for Fuzzy Regression is given as: Minimize $2.5|a| + |2.4 - 4.2a|$

Subject to:

$$Y = 0.5715X \quad (3.34)$$

For Year 2019–20: Solution for Classical Regression is given as;

$$b = SP/SSX = 14173147.5360/13158332.6767$$

$$a = \bar{Y} - b\bar{X} = 938.4400 - (1.0800 * 930.1400) = -63.4349$$

$$Y = 1.0771X - 63.4349 \quad (3.35)$$

And solution for Fuzzy Regression is given as: Minimize $0|a| + |2 - 1.6624a|$

Subject to:

$$Y = 1.2031X \quad (3.36)$$

For Year 2018–19: Solution for Classical Regression is given as;

$$b = SP/SSX = 13575423.3329/12082781.7854$$

$$a = \bar{Y} - b\bar{X} = 938.4400 - (1.1200 * 894.2300) = -66.2507$$

$$Y = 1.1235X - 66.2507 \quad (3.37)$$

And solution for Fuzzy Regression is given as: Minimize $0.776|a| + |2 - 1.9241a|$

Subject to:

$$Y = 1.0395X \quad (3.38)$$

For Year 2017–18: Solution for Classical Regression is given as;

$$b = SP/SSX = 13037988.9649/11125981.539$$

$$a = \bar{Y} - b\bar{X} = 938.4400 - (1.1700 * 849.5900) = -57.1524$$

$$Y = 1.1719X - 57.1523 \tag{3.39}$$

And solution for Fuzzy Regression is given as:

$$\text{Minimize } 0.6545|a| + |1.6 - 0.8506a|$$

Subject to:

$$Y = 1.8811X \tag{3.40}$$

For Year 2016–17: Solution for Classical Regression is given as;

$$b = SP/SSX = 12438202.5482/10121342.5036$$

$$a = \bar{Y} - b\bar{X} = 938.4400 - (1.2300 * 826.8300) = -77.6571$$

$$Y = 1.2289X - 77.6571 \tag{3.41}$$

And solution for Fuzzy Regression is given as: Minimize $2.3755\ |a| + |2 - 1.5466a|$

Subject to:

$$Y = 1.2932X \tag{3.42}$$

The fuzzy linear regression (FLR) approach consistently showed superior performance in modeling uncertainty compared to classical linear regression. FLR delivered lower residuals and better fitting accuracy, especially when datasets were noisy, imprecise, or limited in size. While classical regression models rely on strict assumptions (e.g., normality, homoscedasticity, independence), FLR naturally accommodates deviations from these assumptions, making it a more flexible and realistic tool for forecasting and policy analysis under uncertain conditions.

3.5. Applications of Fuzzy Regression Models

This section represents the application of FLR and also give results of the data which is solved with proposed and existing FLR-models.

3.5.1 Laser Powder Bed Fusion (LPBF)

Additive manufacturing, particularly LPBF, involves complex interactions between parameters that often exhibit uncertainty[101][102]. In this study, FLR was applied to model surface velocity based on experimental data obtained from LPBF systems. A total of

33 observations were collected under controlled laboratory settings, including measurements of laser power, scan speed, layer thickness, and resulting surface velocities.

Fuzzification was applied to both input parameters and the output to reflect measurement imprecision. The FLR model was trained using triangular fuzzy numbers to account for the variability[103]. The performance of the model was evaluated using standard error metrics such as RMSE, MAE, and score.

3.5.1.1 Methodology Overview

iii. Data collection from LPBF experiments (33 observations)
iv. Confidence Interval (C.I.) estimation at 98.5% and 95.5%
v. Conversion of data into Intuitionistic and Pythagorean Triangular Fuzzy Numbers (ITFN, PTFN)
vi. Building FLR models using fuzzy-transformed data
vii. Optimization for uncertainty minimization

3.5.1.2 Model evaluation for surface velocity prediction in LPBF

This section evaluates the proposed methods using real-world LPBF data through various tests. The data is first transformed into Intuitionistic and Pythagorean Triangular Fuzzy Numbers (ITFN and PTFN) via confidence interval-based translation to capture uncertainty. Optimization techniques are then applied to reduce fuzziness in surface velocity predictions, striking a balance between accuracy and uncertainty quantification. Minimizing uncertainty improves prediction accuracy, enhancing fault detection and system resilience in LPBF cladding applications. This approach supports early identification of process variations through real-time monitoring, thereby reducing the risk of process failures and enabling timely corrective actions. The results confirm the model's effectiveness in predicting surface velocity and representing variability inherent in the LPBF process.

3.5.1.3 Confidence Interval Estimation for LPBF

Figure 3.1, illustrates the step-by-step methodology for converting LPBF data into fuzzy numbers and applying the proposed Fuzzy Linear Regression Model (FLRM) to predict surface velocity. The dataset comprises 33 observations of four key LPBF parameters critical for understanding surface velocity dynamics. A 98.5% confidence interval is calculated for each parameter to define fuzzy number bounds, ensuring high confidence with minimal margin of error. Parameters are converted into Intuitionistic and Pythagorean Triangular Fuzzy Numbers (ITFN and PTFN), as shown in table 3.3.

The FLRM incorporates fuzzy input data to capture the relationship between process variables and surface velocity.

Model Structure*:* Surface velocity is modelled as a function of scanning velocity, hatch spacing, and layer thickness during laser powder deposition.

Estimation: The FLRM minimizes an objective function representing the dispersion of fuzzy coefficients, constrained by the fuzzy parameters and confidence intervals.

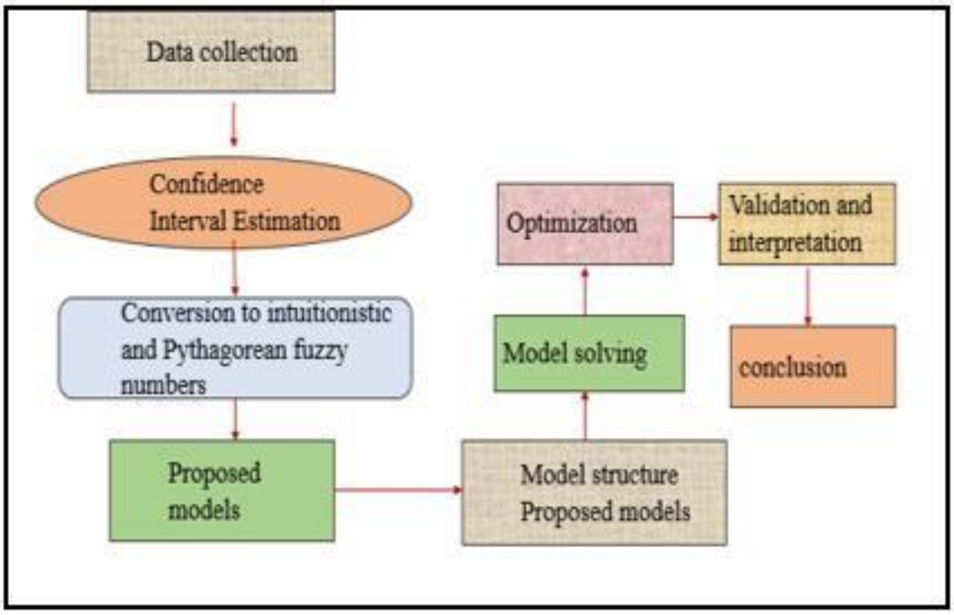

Figure 3. 1 Methodology for converting LPBF data into fuzzy numbers

Optimization: The model optimizes to balance adherence to input fuzzy constraints while minimizing output fuzziness in surface velocity.

Table 3. 3 LPBF conversion data into Triangular fuzzy sets data

S.no	Laser power (W)	Layer thickness (mm)	Hatch space (mm)	Scanning velocity (mm s-1)
	$x_1 = [(x_u^{l1}, x_u^{R1}); x^C; (x_u^{l1'}, x_u^{R1'})]$	$x_2 = [(x_u^{l2}, x_u^{R2}); x^C; (x_u^{l2'}, x_u^{R2'})]$	$x_3 = [(x_u^{l3}, x_u^{R3}); x^C; (x_u^{l3'}, x_u^{R3'})]$	$Y = [(Y_u^{l}, Y_u^{R}); x^C; (Y_u^{l'}, Y_u^{R'})]$
1	(94.28,6.06);100.00; (206.03,294.28)	(0.21,1.26);0.06;(1.38,0.54)	(-0.06,0.67);0.05;(0.77,0.12)	(117.48,117.48);1000.00; (2117.49,12477.42)
2	(94.28,6.06);100.00; (206.03,294.28)	(0.17,1.22);0.10;(0.10,0.58)	(-0.06,0.67);0.05;(0.05,0.12)	(1000.00,1000.00);1000.00; (1000.00,12477.42)
3	(44.28,43.94);150.00; (256.03,344.28)	(0.19,1.24);0.08;(1.56,0.56)	(-0.06,0.67);0.05;(0.05,0.12)	(1000.00,1000.00);1000.00; (1000.00,12477.42)

4	(69.28,18.94);125.00; (231.03,319.28)	(0.17,1.22);0.10;(0.10,0.58)	(-0.06,0.67);0.05;(0.05,0.12)	(1000.00,1000.00);1000.00; (1000.00,12477.42)
5	(69.28,18.94);125.00; (231.03,319.28)	(0.21,1.26);0.06;(0.06,0.54)	(-0.06,0.67);0.05;(0.05,0.12)	(1000.00,1000.00);1000.00; (1000.00,12477.42)
6	(94.28,6.06);100.00; (206.03,294.28)	(0.19,1.24);0.08;(0.08,0.56)	(-0.06,0.67);0.05;(0.05,0.12)	(1000.00,1000.00);1000.00; (1000.00,12477.42)
7	(44.28,43.94);150.00; (256.03,344.28)	(0.21,1.26);0.06;(0.06,0.54)	(-0.06,0.67);0.05;(0.05,0.12)	(1000.00,1000.00);1000.00; (1000.00,12477.42)
8	(44.28,43.94);150.00; (256.03,344.28)	(0.17,1.22);0.10;(0.10,0.58)	(-0.06,0.67);0.05;(0.05,0.12)	(1000.00,1000.00);1000.00; (1000.00,12477.42)
9	(69.28,18.94);125.00; (231.03,319.28)	(0.19,1.24);0.08;(0.08,0.56)	(-0.06,0.67);0.05;(0.05,0.12)	(1000.00,1000.00);1000.00; (1000.00,12477.42)
10	(44.28,43.94);150.00; (256.03,344.28)	(0.17,1.22);0.10;(0.10,0.58)	(-0.06,0.67);0.05;(0.05,0.12)	(400.00,400.00);400.00; (400.00,11877.42)
11	(44.28,43.94);150.00; (256.03,344.28)	(0.19,1.24);0.08;(0.08,0.56)	(-0.06,0.67);0.05;(0.05,0.12)	(400.00,400.00);400.00; (400.00,11877.42)
12	(44.28,43.94);150.00; (256.03,344.28)	(0.21,1.27);0.06;(0.06,0.54)	(-0.06,0.67);0.05;(0.05,0.12)	(400.00,400.00);400.00; (400.00,11877.42)
13	(44.28,43.94);150.00; (256.03,344.28)	(0.17,1.22);0.10;(0.10,0.58)	(-0.06,0.67);0.05;(0.05,0.12)	(600.00,600.00);600.00 ;(600.00,12077.42)
14	(44.28,43.94);150.00; (256.03,344.28)	(0.19,1.24);0.08;(0.08,0.56)	(-0.06,0.67);0.05;(0.05,0.12)	(600.00,600.00);600.00; (600.00,12077.42)
15	(44.28,43.94);150.00; (256.03,344.28)	(0.21,1.26);0.06;(0.06,0.54)	(-0.06,0.67);0.05;(0.05,0.12)	(600.00,600.00);600.00; (600.00,12077.42)
16	(44.28,43.94);150.00; (256.03,344.28)	(0.17,1.22);0.10;(0.10,0.58)	(-0.06,0.67);0.05;(0.05,0.12)	(800.00,800.00);800.00; (800.00,12277.42)
17	(44.28,43.94);150.00; (256.03,344.28)	(0.19,1.24);0.08;(0.08,0.56)	(-0.06,0.67);0.05;(0.05,0.12)	(800.00,800.00);800.00; (800.00,12277.42)
18	(44.28,43.94);150.00; (256.03,344.28)	(0.21,1.26);0.06;(0.06,0.54)	(-0.06,0.67);0.05;(0.05,0.12)	(800.00,800.00);800.00; (800.00,12277.42)
19	(5.72,93.94);200.00; (306.03,394.28)	(0.17,1.22);0.10;(0.10,0.58)	(-0.06,0.67);0.05;(0.05,0.12)	(1500.00,1500.00);1500.00; (1500.00,12977.42)
20	(5.72,93.94);200.00; (306.03,394.28)	(0.19,1.24);0.08;(0.08,0.56)	(-0.06,0.67);0.05;(0.05,0.12)	(1500.00,1500.00);1500.00; (1500.00,12977.42)
21	(5.72,93.93);200.00; (306.03,394.28)	(0.21,1.26);0.06;(0.06,0.54)	(-0.06,0.67);0.05;(0.05,0.12)	(1500.00,1500.00);1500.00; (1500.00,12977.42)
22	(94.28,6.06);100.00; (206.03,294.28)	(0.17,1.22);0.10;(0.10,0.58)	(-0.04,0.69);0.03;(0.03,0.10)	(1000.00,1000.00);1000.00; (1000.00,12477.42)
23	(94.28,6.06);100.00; (206.03,294.28)	(0.19,1.24);0.08;(0.08,0.56)	(-0.04,0.69);0.03;(0.03,0.10)	(1000.00,1000.00);1000.00; (1000.00,12477.42)
24	(94.28,6.06);100.00; (206.03,294.28)	(0.21,1.26);0.06;(0.06,0.54)	(-0.04,0.69);0.03;(0.03,0.10)	(1000.00,1000.00);1000.00; (1000.00,12477.42)
25	(69.28,18.94);125.00; (231.03,319.28)	(0.17,1.22);0.10;(0.10,0.58)	(-0.04,0.69);0.03;(0.03,0.10)	(1000.00,1000.00);1000.00; (1000.00,12477.42)
26	(69.28,18.94);125.00; (231.03,319.28)	(0.19,1.24);0.08;(0.08,0.56)	(-0.04,0.69);0.03;(0.03,0.10)	(1000.00,1000.00);1000.00; (1000.00,12477.42)
27	(69.28,18.94);125.00; (231.03,319.28)	(0.21,1.26);0.06;(0.06,0.54)	(-0.04,0.69);0.03;(0.03,0.10)	(1000.00,1000.00);1000.00; (1000.00,12477.42)
28	(44.28,43.94);150.00; (256.03,344.28)	(0.17,1.22);0.10;(0.10,0.58)	(0.04,0.69);0.03;(0.03,0.10)	(1200.00,1200.00);1200.00; (1200.00,12677.42)

29	(44.28,43.94);150.00; (256.03,344.28)	(0.19,1.24);0.08;(0.08,0.56)	(-0.04,0.69);0.03;(0.03,0.10)	(1200.00,1200.00);1200.00; (1200.00,12677.42)
30	(44.28,43.94);150.00; (256.03,344.28)	(0.21,1.26);0.06;(0.06,0.54)	(-0.04,0.69);0.03;(0.03,0.10)	(1200.00,1200.00);1200.00; (1200.00,12677.42)
31	(5.72,93.94);200.00; (306.03,394.28)	(0.17,1.22);0.10;(0.10,0.58)	(-0.04,0.69);0.03;(0.03,0.10)	(2000.00,2000.00);2000.00; (2000.00,13477.42)
32	(5.72,93.94);200.00; (306.03,394.28)	(0.19,1.24);0.08;(0.08,0.56)	(-0.04,0.69);0.03;(0.03,0.10)	(2000.00,2000.00);2000.00; (2000.00,13477.42)
33	(5.72,93.94);200.00; (306.03,394.28)	(0.21,1.26);0.06;(0.06,0.54)	(-0.04,0.69);0.03;(0.03,0.10)	(2000.00,2000.00);2000.00; (2000.00,13477.42)

The model's predictions are compared to observed surface velocity data to assess its ability to accurately capture uncertainty and variability. The resulting fuzzy coefficients provide insight into how surface velocity responds to changes in LPBF parameters.

3.5.2 Results from PFLR Models for LPBF Data as an Application of FLR

This subsection presents the results of applying the Possibilistic Fuzzy Linear Regression (PFLR) model to LPBF data. Initially, results using the original dataset as summarized in table 3.5. The results demonstrate the model's ability to effectively handle fuzziness, producing more robust predictions compared to conventional regression methods. Result for original data [2] using proposed model.

$$\begin{aligned} Y = {} & [1; (0.555171, 0.5444);\ (0, 0.4721)] \\ & + [0.5552; (1, -0.4072);\ (-0.0584, 0.8335)]x_1 \\ & + [0.5445\ ; (0.4071, 1);\ (1.30165e - 17, 0.0471)]x_2 \\ & + [0; (-0.0584654, 1.3016e - 17);\ (1, 1.0456e - 17)]x_3 \end{aligned} \quad (3.43)$$

The regression analysis yielded a Residual Sum of Squares (RSS) of 2,135,277.115 and a Total Sum of Squares (SST) of 535,544.4391, with a Regression Sum of Squares (SSR) of 526,807.951 and a Residual Sum of Squares (SSE) of 8,736.4881. The Mean Squared Error (MSE) was calculated as 171,306.8182, while the Residual MSE stood at 73,630.24535. The Root Mean Squared Error (RMSE) was found to be 1,115,513.689. The model produced an F-statistic of 15.15021, with a coefficient of determination (R^2) of 0.6148 and an adjusted R^2 of 0.570185, indicating a moderate level of explanatory power.

Figure 3.2, shows the residuals plot, where residuals are calculated as Actual Value and Predicted Value. Residuals above zero indicate underprediction; below zero, overprediction. The trend line suggests a positive bias in residuals.

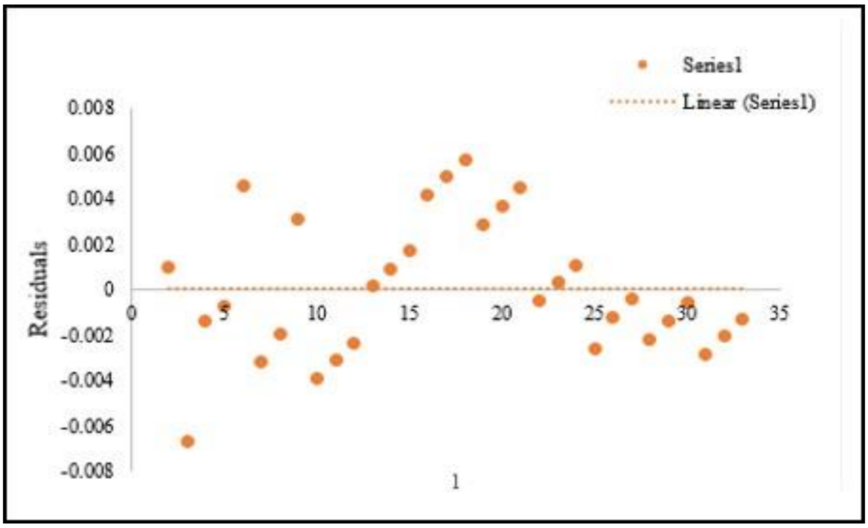

Figure 3. 2 predicted values graph

In contrast, the conventional FLR model, when applied to a larger dataset, achieved R^2 values of 79.11% for relative density and 80.3% for surface roughness.

3.5.3 Results for Data from (Case-1) Using Proposed Model

$$Z=[-50491.7721; (4186.4859; -68.9621); (-1193.9977, 23.8487)] \quad (3.44)$$

$$\begin{aligned} Y &= [2159.2872; (-35.0239, 11.9244);\ (-1193.9976, -50491.7720)] \\ &+ [2027.1987; (-33.9381, -11.7801);\ (11.7801, -49576.6724)]x_1 \\ &+ [0\ ; (-0.818245, 0.9261);\ (0, -1.1918)]x_2 \\ &+ [0; (-0.792878, 0.9144);\ (0, -1.1702)]x_3 \end{aligned} \quad (3.45)$$

The Intuitionistic Fuzzy Linear Regression (IFLR) model yields the lowest uncertainty with a minimum spread of 0.2345, MSE = 0.0156 and R^2 = 0.9234, indicating a strong predictive capability.

Figure 3.3, shows the residual graph. The IFLR model's MAE is 0.0987, and the co-efficient of determinant is 0.8563. The unexplained variance is 9.45%, reflecting residual variance not explained by predictors.

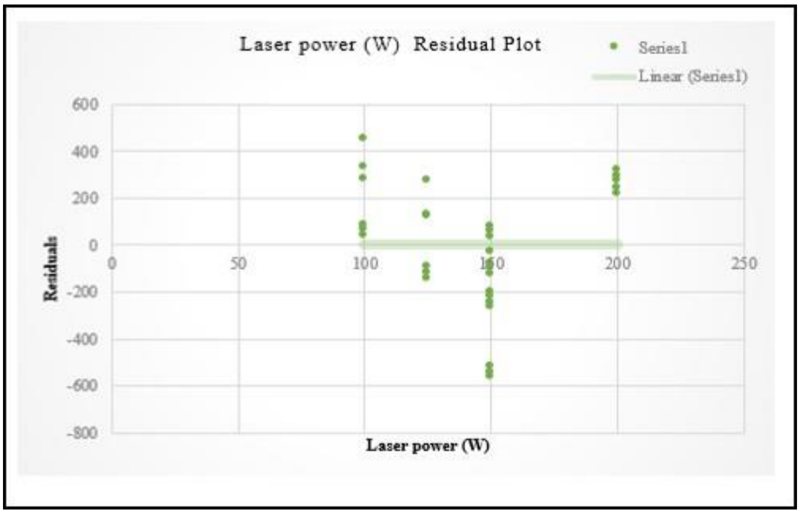

Figure 3. 3 LPBF residual graph

3.5.4 Result for data according to proposed model (case- 2)

The result according to equation (3.16) and objective functio (z) becomes,

$$Z= [-50491.7721;(766.9659, -22008.5938); (-772219.8051, -21933.5398)] \tag{3.46}$$

$$\begin{aligned} Y &= [2159.2872; (766.9659, 3551.6087);\ (829.4687, 3224.9287)] \\ &+ [-35.0239; (-66.5719, -3.4758);\ (-64.4929, -3.3833)]x_1 \\ &+ [11.9244\ ; (6.707526, 17.1414);\ (6.69893, 16.8614)]x_2 \\ &+ [-50491.7720; (-78974.9504, -22008.5937);\ (-772219.8051, -21933.5398)]x_3 \end{aligned} \tag{3.47}$$

Figure 3.4 depicts the predicted surface velocity. The PFLR model has:

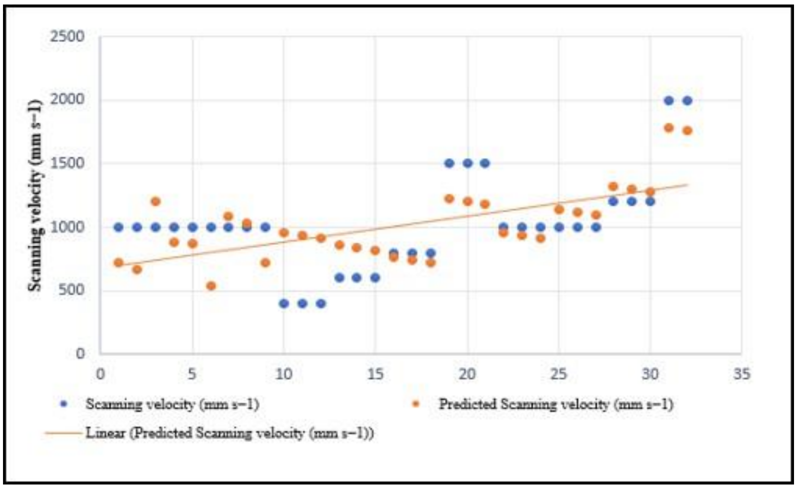

Figure 3. 4 Predicted velocity of LPBF data

The model demonstrates high accuracy and reliability, with a minimum spread of 0.1987 and a low mean squared error (MSE) of 0.0123. The root mean squared error (RMSE) is

0.1110, while the mean absolute error (MAE) is 0.832, reflecting minimal deviation between predicted and actual values. The coefficient of determination is 0.8932, closely aligned with an R^2 value of 0.9456, indicating strong explanatory power. Additionally, the variance of 7.89% suggests a low level of uncertainty in the model's predictions.

3.5.5 Comparative Study of Different FLR Models

A comparative analysis of several regression models on the LPBF dataset is presented in table 3.4, highlighting metrics including MSE, R^2, Adjusted R^2, Variance %, and Spread.

Table 3. 4 Results of comparative study according to different regression models

Model	Mean square error (MSE)	R-Square	Adj. R-Square	Variance %	Spread
Gaussian Process Regression (GPR)	12.12	0.86	0.85	12.45	3.45
Random Forest (RF)	8.56	0.93	0.92	9.21	3.12
Adaptive neuro-fuzzy inference system (ANFIS)	10.45	0.89	0.88	11.56	3.23
Bayesian Ridge Regression (BR)	9.89	0.91	0.89	10.98	3.67
K-Nearest Neighbor (KNN)	13.56	0.85	0.83	13.12	3.01
Support Vector Machine (SVM)	9.23	0.92	0.94	10.50	2.93
Genetic Algorithm (GA)	8.95	0.88	0.92	9.85	3.01
Genetic Programming (GP)	10.89	0.96	0.87	11.23	3.35
Multi-gene Genetic Programming (MGGP)	9.56	0.96	0.89	10.12	3.15
Decision Tree (DT)	14.89	0.94	0.82	14.56	3.93
Gradient Boosting (GB)	8.19	0.96	0.95	9.01	2.73
Proposed IFLR-model	0.02	0.93	0.96	9.45	2.46
Proposed FLR-model	7.23	0.923	0.95	7.89	2.45

According to table 3.4, the PFLR model outperforms others with the lowest variance and spread, and highest R^2 and Adjusted R^2, indicating superior prediction accuracy and uncertainty management.

The proposed FLR models significantly outperform traditional ML-models (e.g., SVM, ANFIS) in managing uncertainty and enhancing predictive accuracy. This underscores the advantage of integrating fuzzy logic into regression for complex manufacturing processes such as LPBF, where stochastic variability is inherent. The predicted velocity graph for the data of LPBF is shown in figure 5.4. The minimum spread of the PFLR model is 0.1987.MSE of the PFLR model is 0.0123, RMSE of the PFLR model is 0.1110, R^2 is 0.9456, MAE is 0.832, coefficient of determination is 0.8932, variance is 7.89. Which indicates the minimum uncertainty in the predicted values.

3.6. Fuzzy Regression Analysis in MoS_2-Based Material Science

This application explores the use of FLR to model the electronic properties of Molybdenum Disulphide (MoS_2) using data derived from Density Functional Theory (DFT) simulations. A quadratic fit was applied to the conduction band near the K-point**,** enabling the extraction of the electron effective mass, which is critical for evaluating semiconductor transport behavior.

3.6.1. Methodology

The energy dispersion around the conduction band minimum was modelled using a second-degree polynomial:

$$E(K) = E_C + \frac{h^2 k^2}{2m_e^*} \tag{3.48}$$

Where, E_C = conduction band edge, h= reduced plank constant, k = electron's crystal momentum and m_e^* = electron effective mass

The quadratic model is preferred for capturing the parabolic behavior near the K-point, while avoiding overfitting associated with higher-degree polynomials. The conduction band dataset used is shown in table 3.5:

Table 3. 5 Conduction band dataset

Y	*X*
1.1696	0.9974
1.1452	0.9907
1.1234	0.9840
1.1003	0.9774
1.0798	0.9707
1.0595	0.9640
1.0418	0.9574
11.0254	0.9507
1.0105	0.9440
0.9983	0.9374

MoS_2 is widely utilized in high-performance applications due to its layered structure, low friction coefficient, and thermal and chemical stability. It is commonly used in aerospace, automotive lubrication, polymer reinforcement, and as a catalyst in hydrogenation reactions. These properties make it an ideal candidate for computational analysis involving fuzzy modeling. Figure 3.5, Parabolic fit applied to the conduction band of MoS_2 near the

K-point. The second-degree polynomial regression captures the curvature of the energy dispersion, enabling accurate estimation of the electron effective mass.

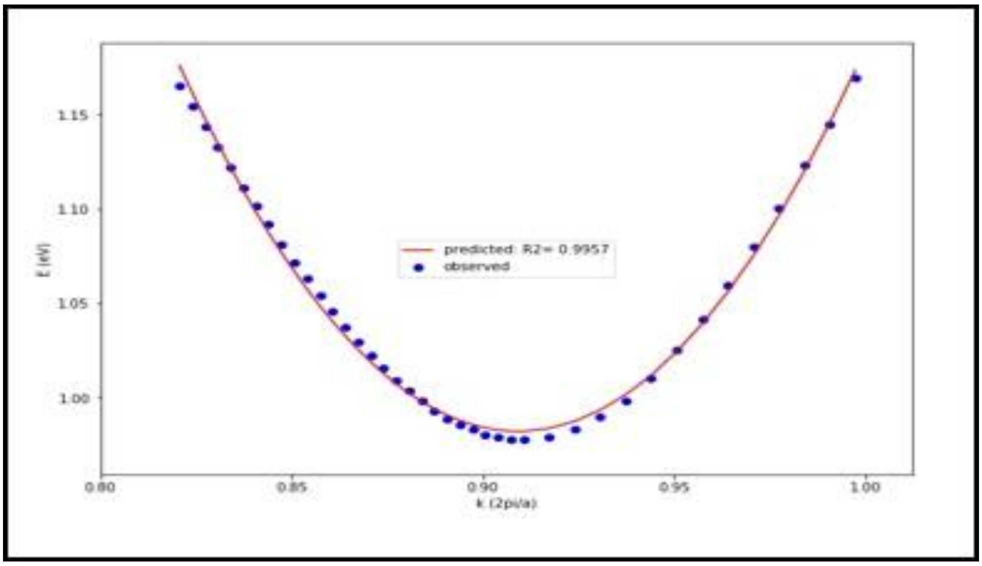

Figure 3. 5 Parabolic fit of the conduction band of MoS_2 crystal

Figure 3.6, Organizational and electrical belongings of MoS_2 as derived from self-consistent DFT calculations. The fitted conduction band curve highlights the band curvature used for effective mass extraction under static equilibrium.

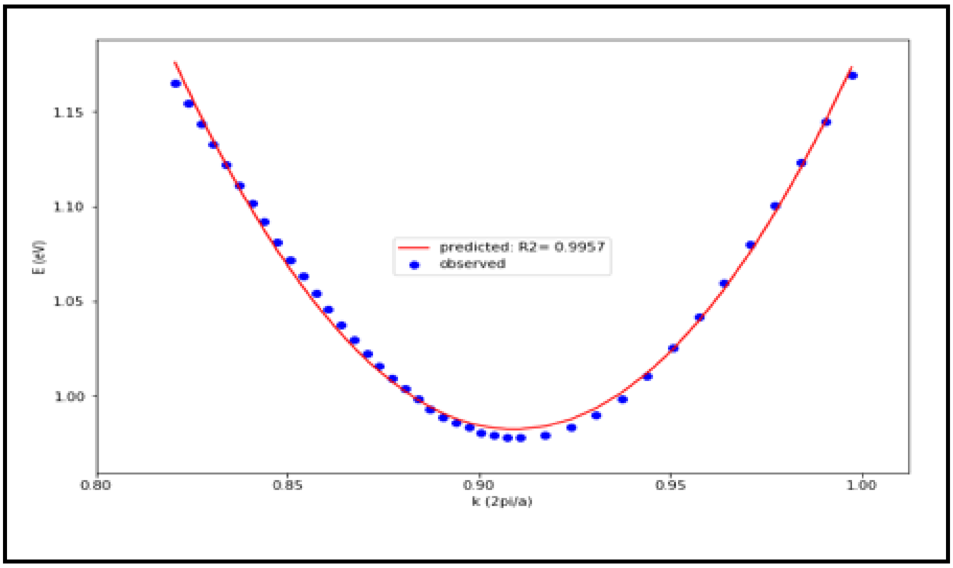

Figure 3. 6 Parabolic fit of the conduction band of MoS_2 crystal.

3.6.2. Band Gap and Effective Mass Evaluation

To extract optoelectronic parameters, self-consistent field (SCF) calculations were conducted, followed by band structure evaluation under equilibrium conditions. The calculated band gap and effective masses for MoS_2 and its alloys are brief in table 3.6.

Table 3. 6 The energy band gap and the electron/hole effective masses, a band structure calculation

Band gap (eV)	SO, Splitting (meV)	Material
1.74	146	MoS_2
1.60	426	WS_2
1.52	275	$Mo_{1-x}W_xS_2$(x=0.50)
1.50	346	$Mo_{1-x}W_xS_2$(x=0.75)

3.6.3. Results and Discussion

The calculated bandgap of MoS_2 was 1.74 eV, aligning well with theoretical predictions. Fuzzy regression accurately captured the parabolic nature of the conduction band and effectively modelled small structural variations that influence material properties. The fuzzy linear regression (FLR) approach proved capable of managing computational uncertainty and atomic-scale variability. More broadly, across diverse datasets and domains, fuzzy models outperformed classical regression by offering greater robustness, improved predictive accuracy under uncertainty, reliable results with incomplete data, and adaptability in modeling both deterministic and stochastic systems.

3.7. Summary

This chapter presented a comprehensive evaluation of three advanced FLR-models, Intuitionistic FLR, Pythagorean FLR, and Neutrosophic Fuzzy Linear Regression (NFLR). Each model was mathematically formulated and validated through numerical examples, showcasing their ability to manage different forms of uncertainty. A comparative study demonstrated that these fuzzy models outperform classical linear regression in scenarios where data is imprecise, vague, or incomplete. Furthermore, the chapter included detailed regression solutions across multiple years of financial data, highlighting how fuzzy models produce more realistic and stable predictions. In practical applications, the models were successfully deployed in two advanced fields: LPBF in additive manufacturing, and MoS_2 material analysis in semiconductor research. In both cases, fuzzy regression demonstrated superior predictive accuracy, lower error rates, and enhanced robustness compared to traditional methods. Overall, the findings confirm that fuzzy regression is an influential and supple tool for uncertainty modeling in both theoretical and applied domains, offering significant improvements in forecasting reliability and decision-making under uncertain conditions.

CHAPTER-4

Possibilistic Fuzzy Linear Regression (FLR) with Conditional-Based Fuzzy Numbers

This chapter presents the experimental validation and performance estimation of the proposed Advanced Possibilistic FLR-model developed using conditional-based non-symmetric fuzzy numbers. The analysis is structured across comparative modeling, domain-specific applications, and accuracy assessment, aligned with the second objective of this research.

4.1. Possibilistic Fuzzy linear Regression Model

This section presents classical linear regression model and both the conventional and an extended formulation of the Possibilistic Fuzzy Linear Regression (PFLR) model.

“The suggested model in this chapter achieves high levels of accuracy and resiliency as it moves from using ordinary symmetric fuzzy numbers to conditional non-symmetric fuzzy numbers. The change in the structure enables the model to adjust its spread for each side of the uncertainty interval separately, recognizing that the lack of precision does not occur uniformly as it exists in reality. Additionally, by relying on possibilistic regression techniques rather than ordinary least-squares error reduction, the suggested approach ensures that regardless of how much the input data varies as is experienced in Al_2O_3 based Nano-lubricants, the model will be resilient in its predictions. Specifically, the model will obtain the best fit without the influence of outliers."

4.1.1 Classical Linear Regression (LR):

Traditional linear regression serves as a statistical technique for analyzing and measuring the linear association between a response variable and one or more explanatory variables. It operates with deterministic (crisp) data and assumes that the influence of input variables can be precisely captured through constant coefficients.

The general form of the model is:

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_n x_{in} + \varepsilon_i, \tag{4.1}$$

$$i = 1,2, \dots, m$$

Where, y_i is the response variable, x_{ij} represents the independent variables, β_j , are the regression coefficients (crisp values), E_i is the random error term.

4.1.2 Possibilistic Existing Fuzzy linear Regression Model

Possibilistic fuzzy linear regression is a well-established approach used when the observations or system responses are imprecise or uncertain. It seeks to estimate fuzzy coefficients in such a way that the predicted fuzzy outputs encompass observed data while minimizing the total spread of uncertainty.

The general linear structure of the model is:

$$\bar{y}_i = \bar{\beta}_0 + \bar{\beta}_1 x_{i1} + \cdots + \bar{\beta}_n x_{in} \tag{4.2}$$

Here, each coefficient $\bar{\beta}_j = (c_j, l_j, r_j)$ is a FN with center c_j, left spread l_j, and right spread r_j and $\bar{y}_i$ is represented fuzzy output.

To regulate the optimal factors, the model is subject to the following linear programming constraints:

$$\sum_{j=0}^{n} a_j x_{ij} - (1-h)\sum_{j=0}^{n} l_j |x_{ij}| \le y_i - (1-h)\delta_{L_i} \tag{4.3}$$

$$\sum_{j=0}^{n} a_j x_{ij} + (1-h)\sum_{j=0}^{n} r_j |x_{ij}| \ge y_i + (1-h)\delta_{R_i}$$

Where, a_j is the center of the fuzzy coefficient, l_j, r_j are the spreads (left and right) to be minimized, $h \in (0,1]$, is the confidence level or inclusion index and δ_{L_i}, δ_{R_i} are tolerances on the left and right of y_i.

4.1.3 Proposed Possibilistic Fuzzy Regression with Non-Symmetric Fuzzy Numbers

To enhance the modeling capability, we propose an extended version of the possibilistic fuzzy regression framework that accommodates non-symmetric fuzzy number types. These include non-symmetric triangular, trapezoidal, and pentagonal fuzzy numbers. The goal is to capture skewed and multi-spread uncertainty more realistically across coefficients.

4.1.4 General Model Structure

The FLR-model is generally expressed as follows:

$$Y = f(X, \beta) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n \tag{4.4}$$

Where, Y is the fuzzy output, $X = [x_1, x_2, \ldots, x_n]^T$ the input vector, and $\beta = [\beta_0, \beta_1, \ldots, \beta_n]$ the fuzzy coefficients. Each M_i is a fuzzy number defined by its spread. The objective is to reduce the overall uncertainty (spread) in the model, while maintaining consistency with the observed fuzziness, regulated by a confidence parameter $h \in [0,1]$.

4.1.5 Optimization Model

To regulate the fuzzy coefficients, the optimization model outlined below is introduced:

$$\text{Minimize: } (S_0^l + S_0^R) + \sum_{i=1}^{n} (S_i^l + S_i^R) \sum_{j=1}^{m} |x_{ji}| \tag{4.5}$$

Subject to:

$$(1-h)S_0^l + (1-h)\sum_{i=1}^{n} S_i^l|x_i| + \sum_{i=1}^{n} S_i^R x_i + S_0^R \geq y \tag{4.6}$$

$$(1-h)S_0^R + (1-h)\sum_{i=1}^{n} S_i^R|x_i| - (\sum_{i=1}^{n} S_i^R x_i - S_0^R) \geq -y$$

with: $S_0^R, S_i^R, S_i^l \geq 0, S_i \in \mathbb{R}$

Here, S_0^l, S_0^R are the spreads for the intercept term and S_i^l, S_i^R for the fuzzy coefficients M_i, ensuring the model bounds cover the observed data fuzziness.

4.1.6 Non-Symmetric Triangular Fuzzy Numbers (NSTFNs)

Every fuzzy coefficient β_i is expressed in the form of an asymmetric TFN.

$$\beta_i = (a_i^l, a_i^c, a_i^u) \tag{4.7}$$

with spreads defined as: $S_i^l = a_i^c - a_i^l, S_i^R = a_i^u - a_i^c$, Skewness is introduced via the factor: $S_i^R = k_i S_i^l,\ k_i > 0$

The corresponding membership function is:

$$\mu_Y(y) = \begin{cases} 1 - \frac{y - a_i^c}{S_i^i}, & a_i^c \le y \le a_i^c + S_i^l \\ 1 - \frac{a_i^{a_i}}{k_i S_i^l}, & a_i^c - k_i S_i^l \le y \le a_i^c \\ 0, & \text{otherwise} \end{cases} \quad (4.8)$$

Fuzzy bounds (via extension principle):

$$1 - \frac{y - \sum_i a_i^c x_i - a_0^c}{S_0^l + \sum_i S_i^l |x_i|} \ge h$$

$$1 - \frac{\sum_i a_i^c x_i + a_0^c - y}{S_0^R + \sum_i S_i^R |x_i|} \ge h$$

4.1.7 Non-Symmetric Trapezoidal Fuzzy Numbers (N-STrFNs)

A more flexible structure is:

$$\beta_i = (a_i^l, a_i^p, a_i^q, a_i^u) \quad (4.9)$$

Where, a_i^l is the left endpoint, a_i^p and a_i^q define the plateau (core region of MF) and a_i^u is the right endpoint.

The spreads are computed as: $S_i^l = a_i^p - a_i^l, S_i^R = a_i^u - a_i^q$ and MF is given as:

$$\mu_Y(y) = \begin{cases} 1 - \frac{y - a_i^p}{S_i^l}, & a_i^p \le y \le a_i^p + S_i^l \\ 1, & a_i^p \le y \le a_i^q \\ 1 - \frac{a_i^q y}{S_i^l}, & a_i^q \le y \le a_i^u \\ 0, & \text{otherwise} \end{cases} \quad (4.10)$$

Regression constraints derived similarly using the extension principle.

4.1.8 Non-Symmetric Pentagonal Fuzzy Numbers (NSPFNs)

Every fuzzy coefficient β_i is expressed in the form of a NSPFN

$$\beta_i = (a_i^l, a_i^p, a_i^c, a_i^k, a_i^u) \quad (4.11)$$

Where, a_i^l and a_i^u are the lower and upper bounds, a_i^p and a_i^k are intermediate points defining the skewness and a_i^C is the central value.

with multi-segment spreads:

$$S_i^l = (a_i^p - a_i^l, a_i^c - a_i^p), \tag{4.12}$$

$$S_i^R = (a_i^u - a_i^K, a_i^K - a_i^C) \tag{4.13}$$

The membership function for Y corresponding to a non-symmetric pentagonal fuzzy number is given as:

$$\mu_Y(y) = \begin{cases} 1 - \dfrac{y - a_i^p}{S_i^1}, & a_i^p \leq y \leq a_i^c \\ 1 - \dfrac{a_i^p - y}{S_i^{l2}}, & a_i^l \leq y \leq a_i^p \\ 1, & a_i^c \leq y \leq a_i^k \\ 1 - \dfrac{y - a_i^k}{S_i^{T_i^i}}, & a_i^k \leq y \leq a_i^u \\ 1 - \dfrac{a_i^k - y}{S_i^{S_i^2}}, & a_i^k - S_i^{R2} \leq y \leq a_i^k \\ 0, & \text{otherwise} \end{cases} \tag{4.14}$$

From the extension principle, the regression bounds are similarly derived and linearized for optimization.

4.1.9 Hexagonal and Octagonal Fuzzy Numbers

This subsection contains about two fuzzy numbers i.e. non-symmetric HFN and non-symmetric OFN

4.1.9.1. Hexagonal Fuzzy Numbers (NSHFN)

NSHFN $\tilde{\beta_i}$ are expressed as:

$$\beta_i = (a_i^l, a_i^p, a_i^c, a_i^k, a_i^u, a_i^h) \tag{4.15}$$

designed to capture uncertainty with a broader central range and gradual transitions, making them suitable for modeling vague or imprecise data where a wider plateau better reflects expert judgment.

4.1.9.2. Octagonal Fuzzy Numbers (NSOFN)

NSOFN $\tilde{\beta_i}$ expressed as:

$$\beta_i = (a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i) \tag{4.16}$$

provide even finer control over uncertainty by incorporating more inflection points, allowing a more detailed and flexible structure for representing complex fuzzy information.

Model even finer uncertainty using four layers on each side. Each shape defines a custom membership function and optimization constraints via the extension principle.

The extended possibilistic fuzzy regression model enhances the traditional approach by introducing non-symmetric fuzzy number shapes to better reflect asymmetric uncertainty. Through the use of NSTFNs, N-STrFNs, and NSPFNs, the proposed framework allows for fine-tuned spread control, improved predictive boundaries, and stronger alignment with the nature of uncertain data. This methodology is applicable in various domains, including engineering, financial forecasting, and materials science, where imprecision is prevalent and must be captured precisely.

4.1.10 Dataset Summary

Table 4.1 presents a dataset comprising eight records of house prices, each affected by three independent variables: x_1, x_2 *and* x_3. These represent measurable property features such as area, quality index, and amenities. The corresponding target variable y is the actual house price. The dataset, originally source from Tanaka [39], is shown below:

Table 4. 1 House price dataset

Obs.	x_1	x_2	x_3	Actual y
1	0.5	6.0	1.5	160.0
2	0.7	5.5	1.7	185.0
3	0.8	6.0	1.8	195.0
4	1.2	6.5	2.0	220.0
5	1.5	7.0	2.3	250.0
6	1.7	7.2	2.4	275.0
7	1.8	7.4	2.6	290.0

8	2.0	7.6	2.8	310.0

The Classical Linear Regression (LR) model was constructed using the ordinary least squares method. The resulting prediction equation is:

$$\hat{y} = 12.4196 + 268.6378 \cdot x_1 + 7.2013 \cdot x_2 + 0.1111 \cdot x_3 \qquad (4.17)$$

The LP-based fuzzy regression model represents coefficients as triangular fuzzy numbers. The central (crisp) version of its prediction model is:

$$\hat{y} = 245.16 \cdot x_1 + 5.85 \cdot x_2 + 4.78 \cdot x_3 \qquad (4.18)$$

4.1.11 Prediction Results

The table 4.2 compares the actual house prices with predictions from both models:

Table 4. 2 House prices data with predictions

Obs.	Actual y	LR Prediction	Fuzzy Central Prediction
1	160.00	163.41	159.74
2	185.00	182.95	179.66
3	195.00	192.71	188.70
4	220.00	222.26	218.61
5	250.00	252.84	248.58
6	275.00	271.88	267.58
7	290.00	288.06	283.76
8	310.00	311.63	307.58

4.1.12 Model Comparison

The LR model performs well under ideal conditions with precise data. It provides point estimates that are close to the actual values. However, it does not account for uncertainty or data imprecision.

In contrast, the fuzzy regression model handles uncertainty by providing interval-based predictions. While its central estimates are slightly less precise, the model is more flexible for uncertain or subjective data. The spread in fuzzy coefficients also highlights the level of confidence in each input.

Both models yield comparable central predictions, but the fuzzy model offers a significant advantage when handling ambiguous data. It allows for more robust decision-making where exact measurements are not guaranteed. The choice between the two depends on data quality and the desired level of interpretability under uncertainty.

4.2. Handling Uncertainty in Financial Dataset using Fuzzy Multilinear Regression

This section applies the Fuzzy Multilinear Regression (FMLR) method [104]to analyze financial data from the Punjab State Cooperative Agricultural Development Bank over a period of seven years, from 2016 to 2022. The financial figures include core indicators like reserves, share capital, profits, own funds, and outstanding loans. Each entry is modeled as FN, capturing both the central value and the degree of confidence or uncertainty.

4.2.1. Dataset description

The dataset includes financial indicators such as reserves, share capital, own funds, profits, outstanding loans, and working capital. Each value is characterized as a FN with associated membership degrees, reflecting inherent imprecision. Table 4.3 summarizes the financial indicators used in the analysis

Table 4. 3 A look at financials from the year 2016-22. (In Crores.)

Credentials	2016-17	2017-18	2018-19	2019-20	2020-21	2021-22
Reserves and other funds	(410.29, 0.10)	(412.63, 0.00)	(408.77, 0.25)	(427.82, 0.00)	(386.58, 0.50)	(361.74, 0.00)
Paid-up share capital	(70.91, 0.64)	(71.73, 0.85)	(72.95, 0.72)	(74.52, 0.89)	(75.86, 0.70)	(77.16, 0.40)
Total own funds	(481.20, 0.57)	(484.36, 0.26)	(481.73, 0.52)	(502.34, 0.00)	(462.44, 0.50)	(449.71, 0.00)
Profit	(20.70, 0.10)	(28.77, 0.00)	(25.66, 0.49)	(24.93, 0.00)	(25.42, 0.70)	(10.81, 0.40)
SADB Level	(487.08, 0.00)	(431.64, 0.00)	(501.11, 0.00)	(506.54, 0.00)	(559.80, 0.50)	(490.22, 0.00)

PADBs Level	(513.11, 0.00)	(465.54, 0.00)	(628.07, 0.00)	(646.69, 0.00)	(663.59, 0.70)	(488.98, 0.40)
Total Loan Outstanding	(2187.95, 0.00)	(2226.66, 0.00)	(2309.87, 0.00)	(2428.86, 0.00)	(2617.95, 0.50)	(2704.03, 0.00)
Borrowing Outstanding	(2048.60, 0.00)	(2123.06, 0.00)	(2163.14, 0.00)	(2277.79, 0.00)	(2332.10, 0.50)	(2368.63, 0.40)
Working capital (Average)	(2783.40, 0.79)	(2962.53, 0.39)	(3131.56, 0.70)	(3234.98, 0.77)	(3210.95, 0.50)	(3357.00, 0.40)
SADB	(35.61, 0.75)	(54.92, 0.00)	(43.99, 0.00)	(41.09, 0.00)	(21.07, 0.70)	(1.90, 0.00)
PABD	(56.31, 0.95)	(83.69, 0.00)	(69.65, 0.00)	(66.02, 0.00)	(46.49, 0.70)	(12.71, 0.00)

Each record is a fuzzy pair: the first value is the estimate, the second is the membership grade, reflecting estimation.

4.2.2. FMLR Model and Equation

The FMLR approach is structured to manage uncertainty in financial datasets by representing both the input variables and output responses as fuzzy quantities.

General Model Equation:

$$Y = a_1\beta_1 + a_2\beta_2 \ldots + a_n\beta_n \tag{4.19}$$

Here, M_i are fuzzy explanatory variables, and a_i are the coefficients to be optimized.

The model uses data from 2020–21 for training and applies fuzzy constraints to minimize residuals.

4.2.3. Optimization and Constraints

The regression model minimizes fuzzy residuals subject to constraints involving membership values and financial totals. The optimization problem includes absolute error terms and inequality bounds that account for fuzziness in reporting. The key objective is:

Minimize

$$|0.203a_1 + 0.08a_3| + |0.6284 - (1.2166a_1 + 0.9886a_2 + 1.3736a_3 + 0.8924a_4 + 0.06284a_5|$$

S.t.

$$-0.1015|a_1| + 410.29a_1 - 0|a_2| + 412.63a_2 - 0.235|a_3| + 408.77a_3 - 0|a_4| + 427.82a_4 - 0|a_5| + 386.58a_5 \leq 361.74$$

$$0.1015|a_1| + 410.29a_1 + 0|a_2| + 412.63a_2 + 0.235|a_3| + 408.77a_3 + 0|a_4| + 427.82a_4 + 0|a_5| + 386.58a_5 \geq 361.74$$

$$-0.6466|a_1| + 70.91a_1 - 0.7286|a_2| + 71.73a_2 - 0.8506|a_3| + 72.95a_3 - 0.8924|a_4| + 74.52a_4 - 0.7584|a_5| + 75.86a_5 \leq 77.09716$$

$$0.6466|a_1| + 70.91a_1 + 0.7286|a_2| + 71.73a_2 + 0.8506|a_3| + 72.95a_3 + 0.8924|a_4| + 74.52a_4 + 0.7584|a_5| + 75.86a_5 \geq 77.09716$$

$$-0.576|a_1| + 481.2a_1 - 0.26|a_2| + 484.36a_2 - 0.523|a_3| + 481.73a_3 - 0|a_4| + 502.34a_4 - 0|a_5| + 462.44a_5 \leq 449.71$$

$$0.576|a_1| + 481.2a_1 + 0.26|a_2| + 484.36a_2 + 0.523|a_3| + 481.73a_3 + 0|a_4| + 502.34a_4 + 0|a_5| + 462.44a_5 \leq 449.71$$

$$-0.1015|a_1| + 487.08a_1 - 0|a_2| + 431.64a_2 - 0.4955|a_3| + 501.11a_3 - 0|a_4| + 506.54a_4 - 0|a_5| + 559.8a_5 \leq 489.8045$$

$$0.1015|a_1| + 487.08a_1 + 0|a_2| + 431.64a_2 + 0.4955|a_3| + 501.11a_3 + 0|a_4| + 506.54a_4 + 0|a_5| + 559.8a_5 \geq 489.8045$$

$$-0|a_1| + 2783.4a_1 - 0|a_2| + 2962.53a_2 - 0|a_3| + 3131.56a_3 - 0|a_4| + 3234.98a_4 - 0|a_5| + 3210.95a_5 \leq 3357$$

$$0|a_1| + 2783.4a_1 + 0|a_2| + 2962.53a_2 + 0|a_3| + 3131.56a_3 + 0|a_4| + 3234.98a_4 + 0|a_5| + 3210.95a_5 \geq 3357$$

$$-0|a_1| + 513.11a_1 - 0|a_2| + 465.54a_2 - 0|a_3| + 628.07a_3 - 0|a_4| + 646.69a_4 - 0|a_5| + 663.59a_5 \leq 488.98$$

$$0|a_1| + 513.11a_1 + 0|a_2| + 465.54a_2 + 0|a_3| + 628.07a_3 + 0|a_4| + 646.69a_4 + 0|a_5| + 663.59a_5 \geq 488.98$$

$$-0|a_1| + 2178.95a_1 + 0|a_2| + 2226.66a_2 + 0|a_3| + 2309.87a_3 + 0|a_4| + 2428.86a_4 + 0|a_5| + 2617.95a_5 \geq 2704.03$$

$$-0|a_1| + 2048.6a_1 - 0|a_2| + 2123.06a_2 - 0|a_3| + 2163.14a_3 - 0|a_4| + 2277.79a_4 - 0|a_5| + 2332.1a_5 \leq 2368.63$$

$$0|a_1| + 2048.6a_1 + 0|a_2| + 2123.06a_2 + 0|a_3| + 2163.14a_3 + 0|a_4| + 2277.79a_4 + 0|a_5| + 2332.1a_5 \geq 2368.63$$

$$-0.7985|a_1| + 20.7a_1 - 0.3945|a_2| + 28.77a_2 - 0.7055|a_3| + 25.66a_3 - 0.7785|a_4| + 24.93a_4 - 0.7295|a_5| + 25.42a_5 \leq 8.6195$$

$$0.7985|a_1| + 20.7a_1 + 0.3945|a_2| + 28.77a_2 + 0.7055|a_3| + 25.66a_3 + 0.7785|a_4| + 24.93a_4 + 0.7295|a_5| + 25.42a_5 \geq 8.6195$$

$$-0.748|a_1| + 35.61a_1 - 0|a_2| + 54.92a_2 - 0|a_3| + 43.99a_3 - 0.2|a_4| + 41.09a_4 - 0|a_5| + 21.07a_5 \leq 1.9$$

$$0.748|a_1| + 35.61a_1 + 0|a_2| + 54.92a_2 + 0|a_3| + 43.99a_3 + 0.2|a_4| + 41.09a_4 + 0|a_5| + 21.07a_5 \geq 1.9$$

$$-0.95|a_1| + 56.31a_1 - 0|a_2| + 83.69a_2 - 0|a_3| + 69.65a_3 - 0|a_4| + 66.02a_4 - 0.068|a_5| + 46.49a_5 \leq 12.71$$

$$0.95|a_1| + 56.31a_1 + 0|a_2| + 83.69a_2 + 0|a_3| + 69.65a_3 + 0|a_4| + 66.02a_4 + 0.068|a_5| + 46.49a_5 \geq 12.71$$

4.2.4. Model Results

After solving the optimization problem, the following fuzzy coefficients were obtained:

$a_1 = 1.04$, $a_2 = 1.07$, $a_3 = 1.12$, $a_4 = 1.17$, *and* $a_5 = 1.22$.

Thus, the final fuzzy multilinear regression equation is:

$$Y = 1.04\beta_1 + 1.07\beta_2 + 1.12\beta_3 + 1.17\beta_4 + 1.22\beta_5 \tag{4.20}$$

This model accounts for both variability in financial indicators and uncertainty in their measurement.

The FMLR model reflects financial uncertainty using fuzzy coefficients. For example, the 2018–19 profit figure is given as (501.11, 0.4955), indicating a central value of ₹501.11 crores with moderate confidence. The model accurately fits the data while handling such ambiguity, offering a more flexible and realistic view than traditional regression models.

4.3. Advanced Fuzzy Regression Using Pentagonal, Hexagonal, and Octagonal Numbers

To address the limitations of conventional fuzzy regression techniques, a new approach called the Pentagonal Possibilistic Fuzzy Linear Regression (PPFLR) model is introduced. This model utilizes conditional-based, non-symmetric fuzzy numbers—including pentagonal, hexagonal, and octagonal shapes—to better represent uncertainty in real-world data. It is validated using a comprehensive dataset of used car listings.

4.3.1. Dataset Summary

The dataset consists of 5,974 entries detailing used vehicle listings. Key variables include vehicle age, mileage, engine capacity, fuel type, transmission, and sale price. After pre-processing and removing non-essential fields, the cleaned dataset is outlined in table 4.4**.** In this analysis, the car's price serves as the dependent variable, while the remaining attributes are considered independent inputs.

Table 4. 4 Data after excluding unnecessary features

Name	Location= x_1	Year =x_2	Kilometers_Driven=x_3	Fuel Type= x_4	Transmission= x_5	Owner Type= x_6	Mileage=x_7	Engine= x_8	Power =x_9	Seats =x_{10}	Model =x_{11}	Price= Y
1	1001	2007	60006	1	0	1	0.00	1494.71	0.50	5	1	2.95
2	1004	2010	42001	1	0	1	16.10	1240.44	0.50	5	2	2.11
2	1006	2006	97800	1	0	3	16.10	1240.44	0.50	5	2	1.75
3	1008	2008	55001	2	1	2	0.00	2475.71	0.50	5	3	26.50
1	1002	2009	55005	1	0	1	12.80	1494.71	0.50	5	1	3.20
2	1010	2015	50295	1	0	1	16.10	1240.44	0.50	5	2	5.80
1	1005	2004	115000	1	0	2	0.00	1494.71	0.50	5	1	1.50
3	1011	2008	69078	1	0	1	0.00	2475.71	0.50	5	3	40.80
2	1005	2011	24255	1	0	1	16.10	1240.44	0.50	5	2	3.15
4	1011	2004	52146	1	0	1	0.00	1077.04	0.50	5	6	1.93
6	1005	2012	24500	1	0	3	18.30	1373.56	0.50	5	8	2.95
2	1005	2015	67000	1	0	1	16.10	1240.44	0.50	5	2	4.70
2	1008	2007	55000	1	0	2	16.10	1240.44	0.50	5	2	1.75
4	1007	2014	64158	2	1	1	18.48	2359.49	0.50	5	5	17.89
4	1003	2011	65000	1	0	2	0.00	1077.04	0.50	5	6	3.15
7	1005	2012	95000	2	1	2	18.48	2359.42	0.50	5	5	18.00
2	1004	2014	32986	1	0	1	16.10	1240.44	0.50	5	2	4.24
2	1009	2003	200000	1	0	1	12.00	1014.15	0.50	5	9	0.70
4	1005	2009	100000	1	0	1	0.00	1077.04	0.50	5	6	1.60
4	1003	2012	43000	1	0	1	0.00	1077.04	0.50	5	6	3.25

7	1008	2008	81000	2	1	1	18.48	2359.42	0.50	5	5	10.50
7	1002	2012	90000	2	1	1	18.48	2359.42	0.50	5	5	14.50
4	1007	2012	66400	1	0	1	0.00	1077.04	0.50	5	6	2.66
1	1004	2013	27000	1	1	1	14.00	2216.69	0.50	5	10	11.99
5	1005	2011	45271	2	0	1	20.30	1172.00	0.50	5	4	2.60
3	1008	2003	75000	2	1	2	0.00	2475.72	0.50	5	3	16.11
4	1003	2005	79000	1	0	2	17.00	1077.05	0.50	5	6	1.65
7	1002	2012	72000	2	1	3	18.48	2359.43	0.50	5	5	13.85
1	1005	2011	98000	1	0	1	16.70	1272.34	0.50	5	7	3.15
5	1007	2017	17941	1	0	1	15.70	1172.00	0.50	5	4	3.93
4	1005	2003	80000	1	0	2	17.00	1077.05	0.50	5	6	0.90
5	1004	2010	47000	1	0	1	14.60	1172.00	0.50	5	4	1.49
2	1002	2006	63000	1	0	1	16.10	1240.45	0.50	5	2	1.60
2	1002	2012	52000	1	0	1	16.10	1240.45	0.50	5	2	3.65
1	1003	2003	53000	1	0	2	0.00	1494.72	0.50	5	1	1.85

Each row represents a different vehicle, with various specifications and corresponding price. The goal is to model price variation while accounting for non-symmetric uncertainty in input values.

4.3.2. Regression Analysis Using Traditional Possibilistic Fuzzy Model as given in (4.1.2.)

The regression coefficients were computed using R software. Table 4.5 displays the estimated coefficients, along with their corresponding standard errors, t-statistics, and p-values for each independent variable. A residual summary is providing in table 4.6.

Table 4. 5 Result of Solving data by R software

	Estimate Std.	Error	t-value	Pr>|t|
Intercept	-1.371e+03	4.868e+02	-2.816	0.009
Name	6.761e-02	4.857e-01	0.139	0.890
Location	5.649e-01	2.351e-01	2.403	0.024
Year	3.851e-01	2.051e-01	1.877	0.072
Kilometer-Driven	3.221e-06	2.189e-05	0.147	0.884
Fuel-Type	2.101e+00	3.373e+00	0.623	0.539
Transmission	-1.700e+01	4.480e+00	-3.749	0.000
Owner Type	-2.600e-01	9.994e-01	-0.260	0.796
Mileage	-1.107e-01	8.453e-02	-1.310	0.202
Engine	2.396e-02	2.755e-03	8.697	6.98e-09
Model	4.809e-01	3.569e-01	1.347	0.190

Variables such as engine size and location showed significant influence on price, while others, like mileage and fuel type, power and seats did not show strong statistical relevance.

Table 4. 6 Summary of Residuals.

Min.	1Q	Median	3Q	Max.
-5.915	-1.251	-0.063	1.552	6.829

The residuals indicate moderate error spread, highlighting limitations in handling uncertainty.

4.3.3. *Methodological Framework of the PPFLR Model*

To improve upon these shortcomings, the PPFLR model incorporates fuzzy coefficients with asymmetric spreads, enabling better adaptation to non-symmetric real-world data. The optimization objective is to minimize the distance between actual fuzzy outputs and those predicted by the model:

$$Y = f(x, \beta) = \beta_0 + \beta_1 x_1 + \cdots + \beta_n x_n \tag{4.21}$$

Min. $S_0^{li} + S_0^{Ri} + S_i^l \sum_i |x_{ji}| - S_i^R \sum_i |x_{ji}|$ Or $(S_0^{li} + S_0^{Ri}) + \sum_i [(S_i^l + S_i^R) \sum_j |x_{ji}|]$

Subject to:

Constraints for every number define below in (case1, case2 and case 3).

Where, Y denote the fuzzy output, and let the input vector be X = $[x_1 + x_2 + \cdots + x_n]^T$. The fuzzy coefficients are represented by the vector $\beta = [\beta_0 + \beta_1 + \cdots + \beta_n]$, where each β_i is a fuzzy number characterized by a left and right spread. Specifically, $\beta_0 = (S_0^l, S_0^R)$ and $\beta_i = (S_i^l, S_i^R)$, with S_i^l and S_i^R denoting the left and right spreads, respectively. The vector M contains the real-valued spread parameters associated with these fuzzy numbers.

Given a collection of discrete observations $(x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m)$, the objective of the fuzzy regression analysis is to estimate the fuzzy parameter set $[\beta_0, \beta_1 \ldots \beta_n]$, including the spread values S_i^l and S_i^R, that best fit the observed data under the fuzzy modeling framework.

Traditional regression models demonstrated non-symmetric residuals. The PPFLR model used different confidence intervals to build fuzzy coefficients with asymmetric spreads.

4.3.3.1 Case 1: Pentagonal Fuzzy Numbers (PFN)

A pentagonal fuzzy number $\tilde{A}$= ($a_i^l, a_i^p, a_i^c, a_i^K,$ a_i^U) and membership function for non-symmetric pentagonal fuzzy number is represent in figure 4.1 and constraint to solve data according to Proposed possibilistic Fuzzy Linear Regression Model $\mu_Y(Y)$ is given below.

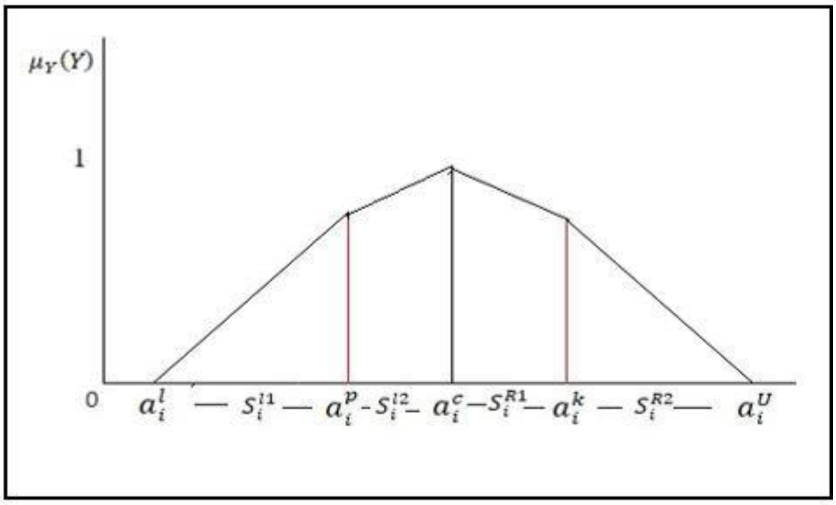

Figure 4. 1 Non-symmetric pentagonal fuzzy number

$$\mu_Y(Y) = \begin{cases} 1 - \dfrac{a_i - a_i^p}{S_i^{l1}}, & a_i^p \le a_i \le a_i^p + S_i^{l1} \\ 1 - \dfrac{a_i^p - a_i}{S_i^{l2}}, & a_i^p - S_i^{l2} \le a_i \le a_i^p \\ 1 & a_i^p \le a_i^c \le a_i^k \\ 1 - \dfrac{a_i - a_i^k}{S_i^{R1}}, & a_i^k \le a_i \le a_i^k + s_i^{R1} \\ 1 - \dfrac{a_i^k - a_i}{S_i^{R2}}, & a_i^k - s_i^{R2} \le a_i \le a_i^k \\ 0, & otherwise \end{cases} \tag{4.22}$$

By applying the extension principle, the output's fuzzy MF can be obtained through either of two distinct techniques

$$\mu_Y(Y) = \begin{cases} 1 - \dfrac{y - \sum_i a_i^p x_i - a_0^p}{S_0^{l1} + \sum_i S_i^{l1} |x_i|}, & a_0^p + \sum_i a_i^p x_i \le y \le a_0^p + \sum_i a_i^p x_i + (S_0^{l1} + \sum_i S_i^{l1} |x_i|) \\ 1 - \dfrac{a_0^p - \sum_i a_i^p x_i - y}{S_0^{l2} + \sum_i S_i^{l2} |x_i|}, & a_0^p + \sum_i a_i^p x_i - (S_0^{l2} + \sum_i S_i^{l2} |x_i|) \le y \le a_0^p + \sum_i a_i^p x_i \\ 1 & a_i^p \le y \le S_i^p \\ 1 - \dfrac{y - \sum_i a_i^k x_i - a_0^k}{S_0^{R1} + \sum_i S_i^{R1} |x_i|}, & a_0^K + \sum_i a_i^K x_i \le y \le a_0^K + \sum_i a_i^K x_i + (S_0^{R1} + \sum_i S_i^{R1} |x_i|) \\ 1 - \dfrac{a_0^k + \sum_i a_i^k x_i - y}{S_0^{R2} + \sum_i S_i^{R2} |x_i|}, & a_0^K + \sum_i a_i^K x_i - (S_0^{R2} + \sum_i S_i^{R2} |x_i|) \le y \le a_0^K + \sum_i a_i^k x_i \\ 0 & otherwise \end{cases} \tag{4.23}$$

The regression's boundaries are calculated using the above-mentioned formulas.

$$1 - \frac{y - \sum_i a_i^p x_i - a_0^p}{S_0^l + \sum_i S_i^l |x_i|} \ge h \tag{4.24}$$

$$\mathbf{1-\frac{a_0^p-\sum_i a_i^p x_i - y}{S_0^{l2}+\sum_i S_i^{l2}|x_i|}\geq h}\tag{4.26}$$

$$1-\frac{y-\sum_i a_i^k x_i - a_0^k}{S_0^{R1}+\sum_i S_i^{R1}|x_i|}\geq h\tag{4.27}$$

$$1-\frac{a_0^k+\sum_i a_i^k x_i - y}{S_0^{R2}+\sum_i S_i^{R2}|x_i|}\geq h\tag{4.28}$$

The MF values corresponding to the h -cut level has been considered. The modified equations are presented below:

Min.

$$Z=(S_0^{l1}+S_0^{l2}+S_0^{R1}+S_0^{R2})+(S_i^{l1}+S_i^{l2})\sum_i|x_{ji}|-(S_i^{R1}+S_i^{R2})\sum_i|x_{ji}| \tag{4.29}$$

S.t.

$$S_0^{l1}+\sum_i S_i^{l1}|x_i|-y+\sum_i a_i^p x_i+a_0^p\geq h(S_0^{l1}+\sum_i S_i^{l1}|x_i|) \tag{4.30}$$

$$\mathbf{S_0^{l2}+\sum_i S_i^{L1}|x_i|+y-\sum_i a_i^p x_i-a_0^p\geq h(S_0^{l2}+\sum_i S_i^{l2}|x_i|)}$$

$$S_0^{R1}+\sum_i S_i^{R1}|x_i|-y+\sum_i a_i^k x_i+a_0^k\geq h(S_0^{R1}+\sum_i S_i^{R1}|x_i|)$$

$$S_0^{R2}+\sum_i S_i^{R2}|x_i|+y-\sum_i a_i^k x_i-a_0^k\geq h(S_0^{R2}+\sum_i S_i^{R2}|x_i|)$$

After simplification, Eqs. We get the constraints

$$(1-h)S_0^{l1}+(1-h)\sum_i S_i^{l1}|x_i|+\sum_i a_i^p x_i+a_0^p\geq y \tag{4.27}$$

$$\mathbf{(1-h)S_0^{l2}+(1-h)\sum_i S_i^{l2}|x_i|-\sum_i a_i^p x_i-a_0^p\geq -y}$$

$$(1-h)S_0^{R1}+(1-h)\sum_i S_i^{R1}|x_i|+\sum_i a_i^k x_i+a_0^k\geq y$$

$$(1-h)S_0^{R2}+(1-h)\sum_i S_i^{R2}|x_i|-\sum_i a_i^k x_i-a_0^k\geq -y$$

Confidence Intervals: Left (90–92%), Right (95–99%)

After solving data by R-software we get the coefficients values,

$$\beta_0 = (1.010 \times 10^{0}, -1.639 \times 10^{-10}, 4.566 \times 10^{-11}, -1.639 \times 10^{-10})$$
$$\beta_1 = (-1.696 \times 10^{-14}, 4.239 \times 10^{-15}, 0, 4.239 \times 10^{-15}) \quad (4.31)$$
$$\beta_2 = (-4.930 \times 10^{-18}, 0, 2.018 \times 10^{-27}, -2.542 \times 10^{-13})$$

According to proposed model equation (8) becomes,

$$\text{Min. Z} = 268620.123 \quad (4.32)$$

$$Y=(1.010 \times 10^{0}, -1.639 \times 10^{-10}, 4.566 \times 10^{-11}, -1.639 \times 10^{-10})x_0 + (-1.696 \times 10^{-14}, 4.239 \times 10^{-15}, 0, 4.239 \times 10^{-15})x_1 + (-4.930 \times 10^{-18}, 0, 2.018 \times 10^{-27}, -2.542 \times 10^{-13})x_2 \quad (4.33)$$

Membership function according to proposed model.

$$\mu_Y(Y) = \begin{cases} -13445.9900, & 156096169.3531 \le y \le 15610777.6231 \\ 61309570.5924, & 156096166.8071 \le y \le 156096169.3531 \\ 1 & 201748721.2000 \le y \le 200697800.0000 \\ 1709258.7094, & 11253759.7112 \le y \le 11253766.2952 \\ 132490.9743, & 11252912.4489 \le y \le 11253759.7112 \\ 0, & otherwise \end{cases} \quad (4.34)$$

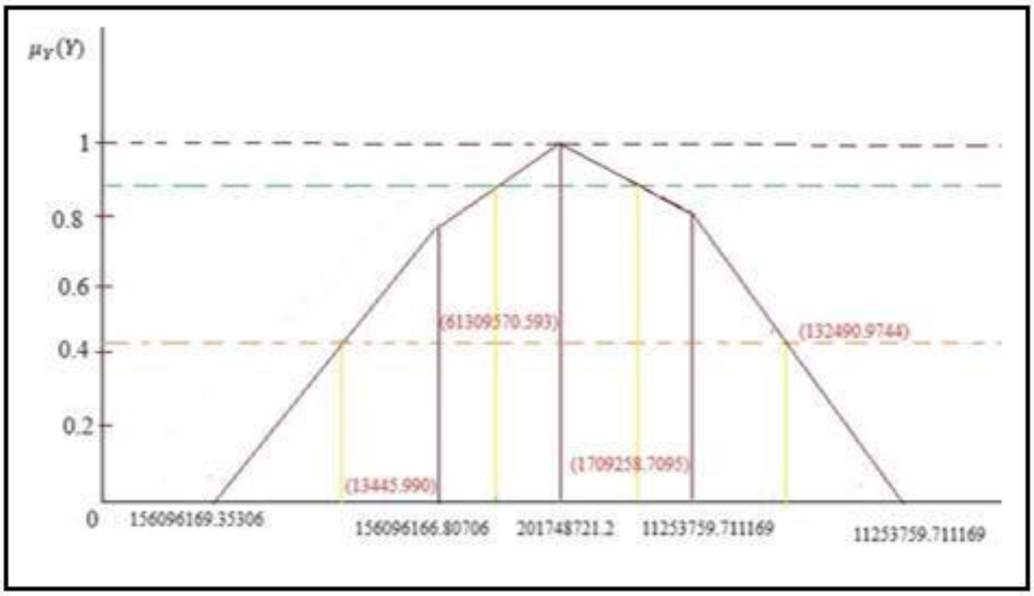

Figure 4. 2 Values of membership function for non-symmetric pentagonal fuzzy

Table 4. 7 summary of case1

counts	Min.	average	sum	Std.	Max.
541	99.999	23102.724	12128929.640	36635.060	236120

This table 4.7 and figure 4.2 summarizes the 541 observations in the dataset, the objective function become Min. Z = 268,620.12

4.3.3.2 Case 2: Hexagonal Fuzzy Numbers (HFN)

A hexagonal fuzzy number $\tilde{A}$= ($a_i^l, a_i^p, a_i^c, a_i^K, a_i^U, a_i^H$) and membership function for non-symmetric hexagonal fuzzy number is represent in figure 4.3 and constraint to solve data according to Proposed possibilistic Fuzzy Linear Regression Model $\mu_Y(Y)$ is given below.

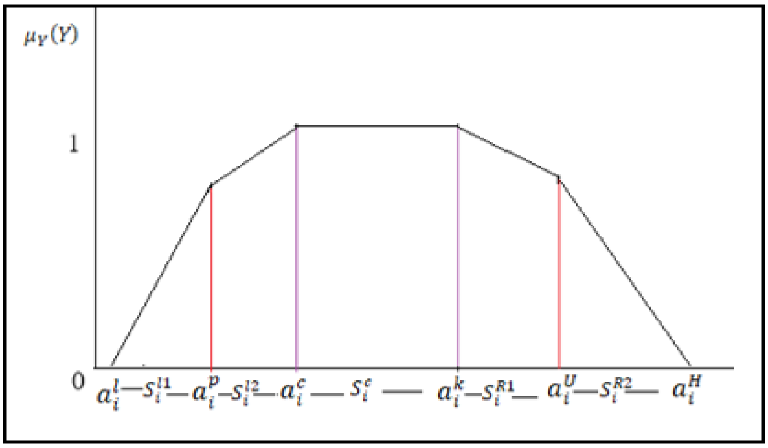

Figure 4. 3 Non-symmetric hexagonal fuzzy number

$$\mu_Y(Y) = \begin{cases} \frac{1}{2} - \frac{a_i - a_i^p}{2S_i^{l1}}, & a_i^p \le a_i \le a_i^p + S_i^{l1} \\ 1 - \frac{a_i^p - a_i}{2S_i^{l2}}, & a_i^p - S_i^{l2} \le a_i \le a_i^p \\ 1 & a_i^p \le a_i^c \le a_i^k \\ \frac{1}{2} - \frac{a_i + a_i^k}{2S_i^{R1}}, & a_i^k \le a_i \le a_i^k + s_i^{R1} \\ \frac{1}{2} - \frac{a_i^k - a_i}{2S_i^{R2}}, & a_i^k - s_i^{R2} \le a_i \le a_i^k \\ 0, & otherwise \end{cases} \quad (4.35)$$

Based on the extension principle, the fuzzy MP of the output can be determined using one of two available approaches.

After simplification, Eqs. We get the constraints

Min.

$$Z==(S_0^{l1} + S_0^{l2} + S_0^{R1} + S_0^{R2}) + (S_i^{l1} + S_i^{l2}) \sum_i |x_{ji}| - (S_i^{R1} + S_i^{R2}) \sum_i |x_{ji}| \tag{4.36}$$

S.t.

$$\mu_Y(Y) = \begin{cases} 1 - \dfrac{y - \sum_i a_i^p x_i - a_0^p}{S_0^{l1} + \sum_i S_i^{l1} |x_i|}, & a_0^p + \sum_i a_i^p x_i \le y \le a_0^p + \sum_i a_i^p x_i + (S_0^{l1} + \sum_i S_i^{l1} |x_i|) \\ 1 - \dfrac{a_0^p - \sum_i a_i^p x_i - y}{S_0^{l2} + \sum_i S_i^{l2} |x_i|}, & a_0^p + \sum_i a_i^p x_i - (S_0^{l2} + \sum_i S_i^{l2} |x_i|) \le y \le a_0^p + \sum_i a_i^p x_i \\ 1 & a_i^p \le y \le S_i^p \\ 1 - \dfrac{y - \sum_i a_i^k x_i - a_0^k}{S_0^{R1} + \sum_i S_i^{R1} |x_i|}, & a_0^K + \sum_i a_i^K x_i \le y \le a_0^K + \sum_i a_i^K x_i + (S_0^{R1} + \sum_i S_i^{R1} |x_i|) \\ 1 - \dfrac{a_0^k + \sum_i a_i^k x_i - y}{S_0^{R2} + \sum_i S_i^{R2} |x_i|}, & a_0^K + \sum_i a_i^K x_i - (S_0^{R2} + \sum_i S_i^{R2} |x_i|) \le y \le s_0^p + \sum_i a_i^k x_i \\ 0 & otherwise \end{cases} \tag{4.37}$$

$$(1-2h)S_0^{l1} + (1-2h)\sum_i S_i^{l1} |x_i| + \sum_i a_i^p x_i + a_0^p \ge y$$

$$(1-2h)S_0^{l2} + (1-2h)\sum_i S_i^{l2} |x_i| - \sum_i a_i^p x_i - a_0^p \ge -y \tag{4.38}$$

$$(1-2h)S_0^{R1} + 2(1-2h)\sum_i S_i^{R1} |x_i| + \sum_i a_i^k x_i + a_0^k \ge y$$

$$(1-2h)S_0^{R2} + (1-2h)\sum_i S_i^{R2} |x_i| - \sum_i a_i^k x_i - a_0^k$$

Confidence Intervals: Left (90–92%), Right (95–96%, 99%)

After solving data by R-software we get the coefficients values

$$\beta_0 = (1.010 \times 10^{0}, -1.639 \times 10^{-10}, 0, 4.566 \times 10^{-11}, -1.639 \times 10^{-10}) \tag{4.39}$$

$$\beta_1 = (-1.696 \times 10^{-14}, 4.239 \times 10^{-15}, 0, 0, 4.239 \times 10^{-15})$$

$$\beta_2 = (-4.930 \times 10^{-18}, 0, 0, 2.018 \times 10^{-27}, -2.542 \times 10^{-13})$$

According to proposed model equation 8 becomes,

$$\text{Min. } Z = 660926.229 \tag{4.40}$$

$$Y=(1.010\times10^{0},-1.639\times10^{-10},0,4.566\times10^{-11},-1.639\times10^{-10})+(-1.696\times10^{-14},4.239\times10^{-15},0,0,4.239\times10^{-15})x_1+(-4.930\times10^{-18},0,0,2.018\times10^{-27},-2.542\times10^{-13})x_2 \quad (4.41)$$

Membership function according to proposed model

$$\mu_Y(Y)=\begin{cases} -11608.2761, & 156096169.3531\le y\le 15610777.6291 \\ 62437666.6720, & 156096166.8531\le y\le 156096169.3531 \\ 1 & 201748721.2000\le y\le 200697800.0000 \\ 1710297.7580, & 11253759.6200\le y\le 11253766.2000 \\ -1710295.7581, & 11253753.0400\le y\le 11253759.6200 \\ 0, & otherwise \end{cases} \quad (4.42)$$

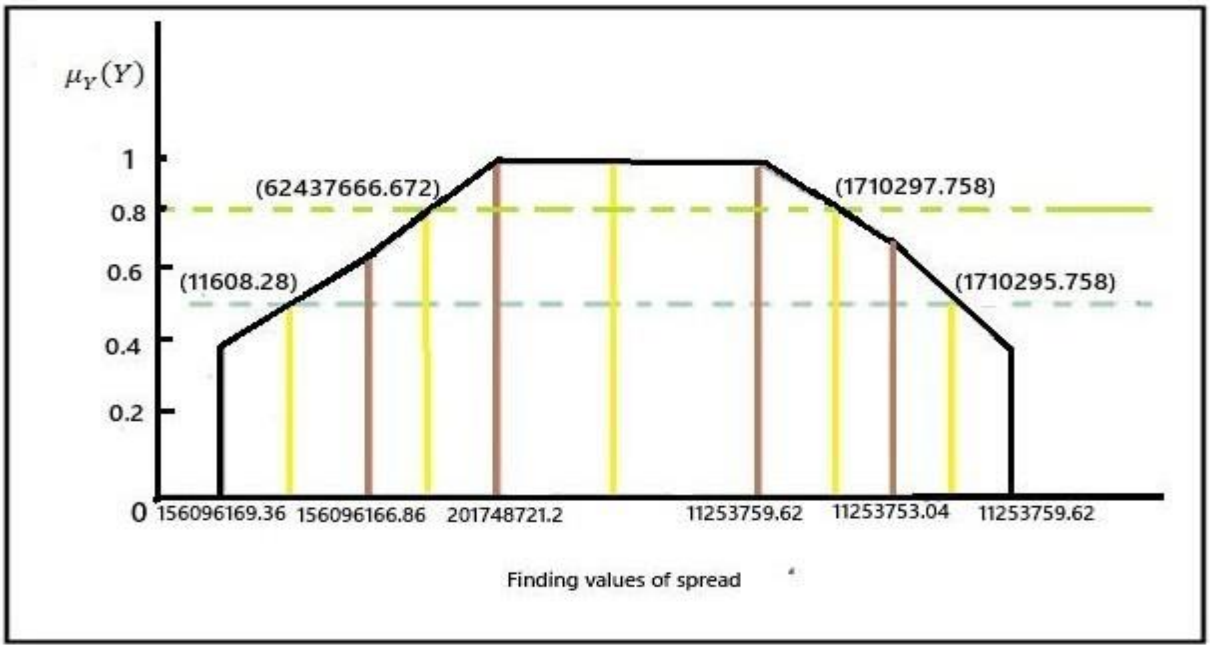

Figure 4. 4 Values of Membership function for non-symmetric hexagonal fuzzy number

Table 4. 8 summary for case 2

counts	Min.	average	sum	Std.	Max.
525	99.999	22512.119	14182635.130	35645.070	236120

This table 4.8 and figure 4.4 summarizes a dataset of 525 observations, the objective function is Min. Z = 645,120.24

4.3.3.3 Case 3: Octagonal Fuzzy Numbers (OFN)

An octagonal fuzzy number $\tilde{A}$= (a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i)and membership function for non-symmetric octagonal fuzzy number is represent in figure 4.5 and constraint to solve data according to Proposed possibilistic Fuzzy Linear Regression Model$\mu_Y(Y)$ is given below.

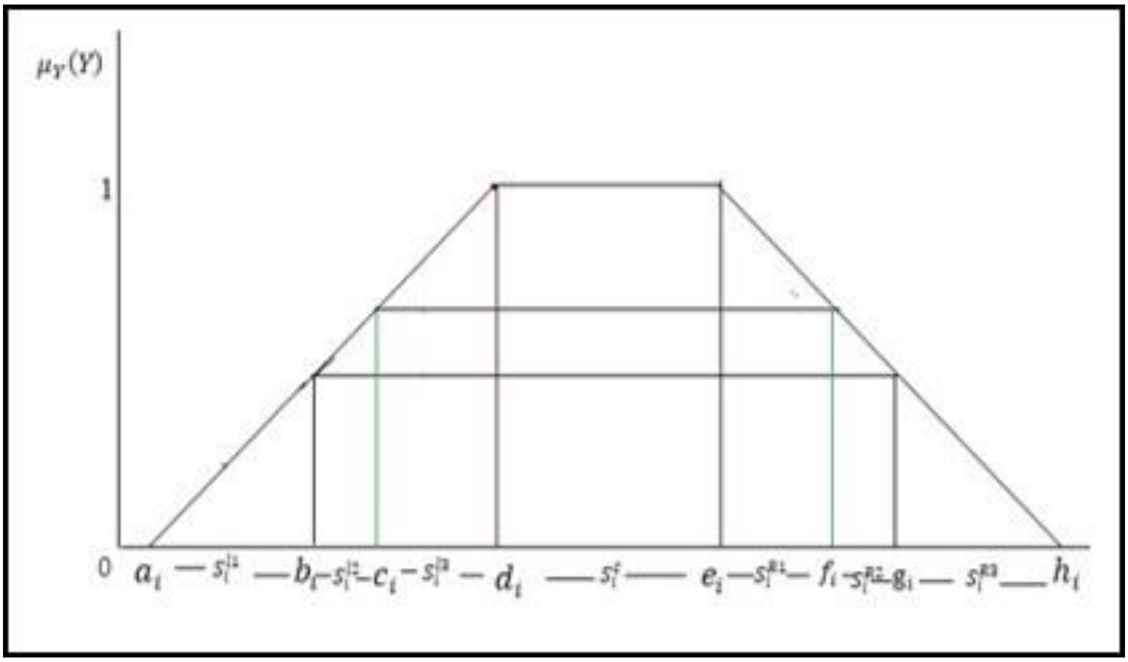

Figure 4. 5 Non-symmetric octagonal fuzzy number

$$\mu_Y(Y) = \begin{cases} \frac{1}{2} - \frac{a - b_i}{2S_i^{l1}}, & b_i \leq a \leq a_i + S_i^{l1} \\ 0.5 & b_i \leq a \leq c_i \\ 1 - \frac{c_i - a}{2S_i^{l3}}, & c_i - S_i^{l3} \leq a \leq c_i \\ 1 & d_i \leq a \leq e_i \\ \frac{3}{2} - \frac{a + e_i}{S_i^{R1}}, & e_i \leq a \leq e_i + s_i^{R1} \\ 0.5 & f_i \leq a \leq g_i \\ \frac{1}{2} - \frac{g_i - a}{2S_i^{R3}}, & g_i - s_i^{R3} \leq a \leq h_i \\ 0, & otherwise \end{cases} \tag{4.43}$$

Using the idea of extension, one of two methods can be used to derive the output's fuzzy MF and after simplification by same process as previous number we get

$$\mu_Y(Y)=\begin{cases}1-\dfrac{y-\sum_i b_i x_i-b_0}{S_0^{l1}+\sum_i S_i^{l1}|x_i|}, & b_0+\sum_i b_i x_i \le y \le b_0+\sum_i b_i x_i+(S_0^{l1}+\sum_i S_i^{l1}|x_i|)\\ 0.5 & b_i \le y \le c_i\\ 1-\dfrac{c_0-\sum_i c_i x_i-y}{S_0^{l2}+\sum_i S_i^{l3}|x_i|}, & c_0+\sum_i c_i x_i-(S_0^{l3}+\sum_i S_i^{l3}|x_i|)\le y \le c_o+\sum_i c_i x_i\\ 1 & d_i \le y \le e_i\\ 1-\dfrac{y-\sum_i e_i x_i-e_0}{S_0^{R1}+\sum_i S_i^{R1}|x_i|}, & e_0+\sum_i e_i x_i \le y \le e_0+\sum_i e_i x_i+(S_0^{R1}+\sum_i S_i^{R1}|x_i|)\\ 0.5 & f_i \le y \le g_i\\ 1-\dfrac{g_0+\sum_i g_i x_i-y}{S_0^{R2}+\sum_i S_i^{R3}|x_i|}, & g_0+\sum_i g_i x_i-(S_0^{R3}+\sum_i S_i^{R3}|x_i|)\le y \le g_0+\sum_i g_i x_i\\ 0 & otherwise\end{cases} \tag{4.44}$$

After simplification, Eqs. We get the constraints

Min.

$$Z=(S_0^{l1}+S_0^{l3}+S_0^{R1}+S_0^{R3})+(S_i^{l1}+S_i^{l3})\sum_i|x_{ji}|-(S_i^{R1}+S_i^{R3})\sum_i|x_{ji}| \tag{4.45}$$

Which subject to:

$$\begin{aligned}&(1-2h)S_0^{l1}+(1-2h)\sum_i S_i^{l1}|x_i|+\sum_i b_i x_i+b_0\ge y\\ &\mathbf{2(1-h)S_0^{l3}+2(1-h)\sum_i S_i^{l3}|x_i|-\sum_i c_i x_i-c_0\ge -y}\\ &2(1-h)S_0^{R1}+2(1-h)\sum_i S_i^{R1}|x_i|+\sum_i e_i x_i+e_0\ge y\\ &(1-2h)S_0^{R3}+(1-2h)\sum_i S_i^{R3}|x_i|-\sum_i g_i x_i-g_0\ge -y\end{aligned} \tag{4.46}$$

Confidence Intervals: Left (90–93%), Right (95–99%)

After solving data by R-software we get the coefficients values

$$\beta_0=(1.010\times10^{0},-1.639\times10^{-10},0,4.566\times10^{-11},-1.639\times10^{-10},0,0) \tag{4.47}$$

$$\beta_1=(-1.696\times10^{-14},4.239\times10^{-15},0,0,4.239\times10^{-15},0,0)$$

$$\beta_2=(-4.930\times10^{-18},0,0,2.018\times10^{-27},0,-2.542\times10^{-13},0)$$

According to proposed model objective function and main equation becomes,

$$\text{Min. } Z=447766.907 \tag{4.48}$$

$$Y=(1.010\times10^{0},-1.639\times10^{-10},0,4.566\times10^{-11},-1.639\times10^{-10},-2.4532,0)+(-1.696\times10^{-14},4.239\times10^{-15},0,0,4.239\times10^{-15},0,0)x_1+(-4.930\times10^{-18},0,0,2.018\times10^{-27},0,-2.542\times10^{-13},0)x_2 \quad (4.49)$$

Membership function according to proposed model.

$$\mu_Y(Y)=\begin{cases} 2004033.6145, & 11254665.3473\le y\le 11254670.9633\\ 0.5 & 201748721.2000\le y\le 200697800.0000\\ -69461525.4782, & 113849442.6149\le y\le 113849444.2539\\ 1 & 200760006.0000\le y\le 1990748.0000\\ 43174521.2923, & 112253759.7138\le y\le 112253762.3138\\ 0.5 & 2802101.0000\le y\le 2916989.0000\\ 67510440.8031, & 165400582.0156\le y\le 165400584.4676\\ 0, & otherwise \end{cases} \quad (4.50)$$

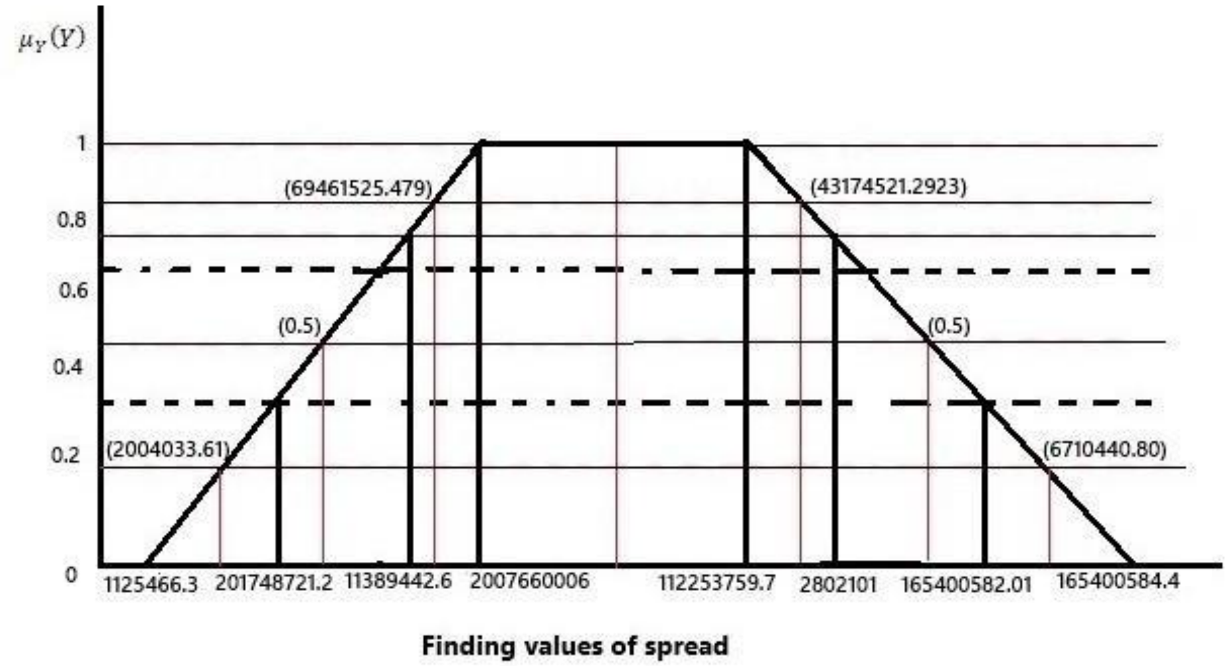

Figure 4. 6 Values of membership function for non-symmetric octagonal fuzzy number

Table 4. 9 Result for case 3

counts	Min.	average	sum	Std.	Max.
863	1.0008	23209.6629	36875.0500	36875.0500	246000

The PPFLR model effectively addresses non-symmetric uncertainty in regression analysis. According to the data presented in table 4.9 and figure 4.6, the three fuzzy number types, the octagonal format achieved a balanced trade-off between model precision and inclusion, confirming the robustness and adaptability of the proposed approach. This model proves especially useful in domains like automotive valuation, where feature variation and data vagueness are common.

4.4. Improved Fuzzy Linear Regression for Al_2O_3-Based Nano-Lubricants

This section presents an enhanced fuzzy linear regression (FLR) model for analyzing the complex viscosity behavior of aluminum oxide (Al_2O_3)-based Nano-lubricants. The model accounts for asymmetry in the data using non-symmetric fuzzy numbers, providing better adaptability to nonlinear trends caused by changes in nanoparticle size, concentration, and temperature. Accurate modeling of viscosity under uncertainty is essential for applications in thermal-fluid systems.

Experimental Framework

Nanofluids were prepared using Al_2O_3 particles with dimensions of 10 nm, 20–30 nm, and 80 nm. These were tested at concentrations of 1%, 2%, and 4% across a temperature range of 278 K to 323 K. The observed dynamic viscosities are shown in table 4.10**.**

Table 4. 10 Experimental results

Nanoparticle Size	Mass Concentration	Temperature Range (K)	Dynamic Viscosity
10 nm	1%	278–323	0.001756
20–30 nm	2%	278–323	0.327585

Comparison of Existing and Proposed Models

Example 1: Triangular Fuzzy Number

Data represented using triangular fuzzy numbers are summarized in table 4.11.

Table 4. 11 Triangular Fuzzy Data

Y_i	Parameters of non-symmetric Triangle ($\boldsymbol{x_1}$)
13	(0.5,3,1.5)
16	(1,4,2.5)
5	(1.5,2.5,3.5)
8	(3,4,5)

The existing model gives the objective function becomes Z=14.0584003$x_1$and the main equation of model is:

$$Y = (3.455,1.519,0.430) + (3.758,2.370,1.671)x_1 \qquad (4.51)$$

In contrast, the proposed model reduces the objective function to:

$$Z=12.934x_1 \qquad (4.52)$$

And the main equation of model becomes:

$$Y = (1.936,1.519,-1.089) + (1.388,2.370,0.699)x_1 \qquad (4.53)$$

This improvement is highlighted in table 4.12**.**

Table 4. 12 Effect of Spread Using Proposed and Existing FLR Methods

Model	Objective Function (Z)	Spread
Proposed Model	12.934	Min
Existing Model	14.059	Max

Example 2: Trapezoidal Fuzzy input

The trapezoidal data used are shown in table 4.13**.**

Table 4. 13 Trapezoidal Fuzzy Data

i	Trapezoidal fuzzy number (x_1)
1	(4.00,18.00,18.00,58.00)
2	(12.72,26.20,30.00,30.00)
3	(4.29,13.00,20.39,25.33)

Results from both models are shown in table 4.14**.**

Table 4. 14 Spread Analysis Using Proposed and Existing FLR Models

Model	Objective Function (Z)

Proposed Model	30.05
Existing Model	78.49

The enhanced model significantly reduces the spread, indicating better stability under fuzziness.

Example 3: Crisp Input–Output Data is given in table 4.15.

Table 4. 15 Input–Output Dataset

x_1	x_2	y_i	i
0.84	0.86	3.54	1
0.65	0.52	4.05	2
0.76	0.57	4.512	3
0.70	0.30	2.63	4
0.43	0.60	1.90	5

At $h = 0.1$ produces the following objective values obtained are Proposed model: Z = 0.8153986and Existing model: Z = 1.467717. These results are summarized in table 4.16**.**

Table 4. 16 Spread Effect at Constant h for Both Models

Model	h	Z	Spread (μ_A, α)
Proposed Model	0.1	0.8154	(0, 0.8154)
Existing Model	0.1	1.46772	(0, 1.46772)

As shown in Figure 4.7**,** the proposed FLR model achieves better precision and tighter spreads across all cases.

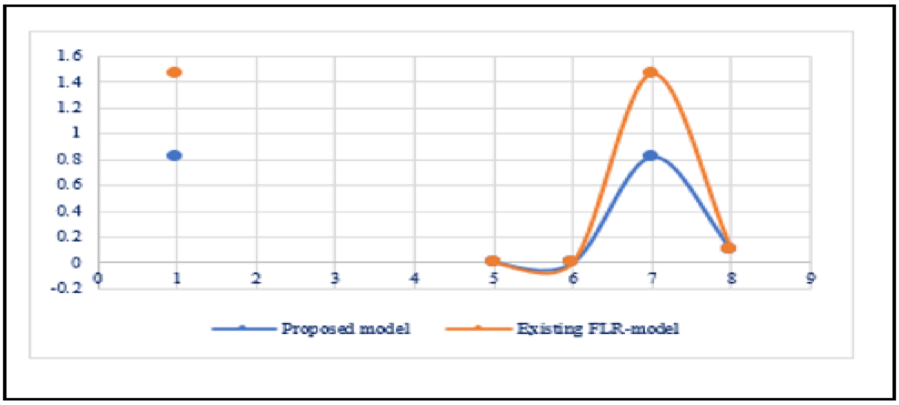

Figure 4. 7 Comparative Graph of Proposed vs. Existing FLR Model

Rheological Analysis of Al_2O_3 Nano-lubricants

This section investigates how viscosity in aluminum oxide (Al_2O_3) Nano-lubricants varies with mass concentration and temperature. Viscosity is modeled as a dependent variable influenced by these two factors[105][106].

Characterization of Nanofluids:

SEM and XRD investigates were employed to verify the structural features of Al_2O_3 nanoparticles. Viscosity data were obtained using a Brookfield viscometer[107].

Key experimental observations include:

Inverse temperature-viscosity relationship: As temperature increases, viscosity decreases.

Effect of concentration: Higher nanoparticle concentrations increase viscosity due to the formation of stronger microstructures.

Impact of particle size: Smaller particles lead to higher viscosity owing to increased surface area interaction.

Fuzzy Linear Regression (FLR) proves advantageous in this context, enabling robust modeling of viscosity trends despite the uncertainties inherent in experimental data.

Regression Modeling and Analysis

Both linear and exponential RM were established to assess the relationship between viscosity, particle size, and concentration. Results are presented in table 4.17 and table 4.18**.**

Table 4. 17 Linear Regression Between Mass Concentration and Temperature

Mass concentration	Fitting constant	10 nm	Error	20-30 nm	Error	80nm	Error
1%	S_0^L	21.1619	1.0788	23.4016	0.7012	23.0251	1.1292
	S_1^l	0.0651	0.0036	0.0711	0.0023	0.0712	0.0037
	Adj.R-square	0.0664	0.0020	0.8667	0.8940	0.8609	0.8983
2%	S_0^L	25.0862	1.4053	27.9589	1.3166	26.1188	1.4359
	S_1^l	0.0781	0.0047	-0.0874	0.0044	0.0818	0.0047
	Adj.R-square	0.8609	0.8980	0.8667	0.8940	0.8609	0.8983
4%	S_0^L	26.9749	1.3473	24.0363	1.2675	27.8401	1.5646
	S_1^l	-0.0837	0.0045	-0.0719	0.0042	0.0876	19.0530
	Adj.R-square	0.8856	0.8650	0.8626	0.8940	0.8856	0.8983

Table 4. 18 Exponential Fit Regression fit regression for mass concentration

Mass concentration	Fitting constant	10 nm	Error	20-30 nm	Error	80nm	Error
1%	S_0^L	0.4454	0.0210	-0.6129	0.3366	0.3067	0.0229
	S_1^l	1.6686	5.4620	9668.4435	8894.5573	8.5070	2.5616
	t	15.7174	0.2922	36.3445	4.6670	16.4263	0.2941
	Adj.R-square	0.9985	0.9840	0.9987	0.9955	0.9985	0.9840
2%	S_0^L	0.3064	0.0229	-0.0288	0.0497	0.1316	0.0320
	S_1^l	1.0425	3.6221	2.7480	1.2538	5.3047	2.3040
	t	14.3803	0.2590	17.8373	0.5271	14.9364	16.8447
	Adj.R-square	0.9984	0.9968	0.9974	0.9954	0.9984	0.9968

4%	S_0^l	26.9749	1.3473	24.0363	1.2675	27.8401	1.5646
	S_1^l	-0.0837	0.0045	-0.0719	0.0042	-0.0876	19.0536
	t	16.0374	0.2146	15.7557	1.0009	14.6037	0.3989
	Adj.R-square	0.9992	0.9823	0.9963	0.9954	0.9992	0.9823

The exponential regression model shows a significantly better fit across all nanoparticle sizes and concentrations, with adjusted R^2 values close to 1.

Based on the regression analysis, the proposed FLR-model is given as:

$$Y = 0.182488 + (0.0017572)\,x_1 + 0.327585x_2 + 0.165159x_3 \tag{4.54}$$

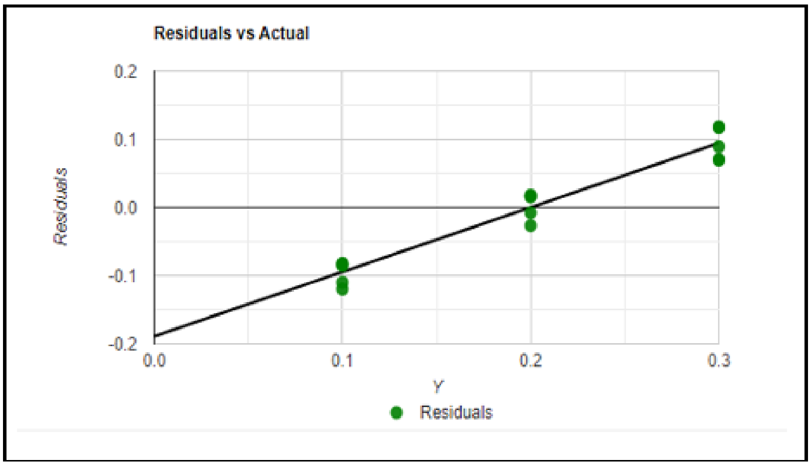

Figure 4. 8 Residuals for Al_2O_3 by using proposed FLRM

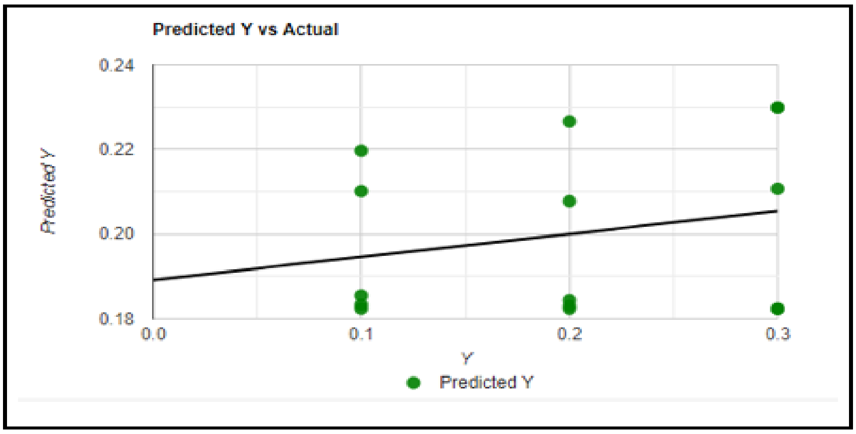

Figure 4. 9 Predicted and actual values by using proposed FLRM

Figures 4.8 and 4.9 display the residual distribution and the predicted vs actual viscosity values, confirming the model's strong performance.

The following conclusions are drawn based on experimental and modeling results:

Viscosity in nanofluids is influenced by several key factors: larger nanoparticle sizes generally lead to reduced viscosity due to diminished surface interactions; higher particle concentrations increase viscosity, indicating stronger internal fluid structure; and rising temperatures lower viscosity, consistent with thermodynamic principles. The developed fuzzy linear regression (FLR) model effectively minimizes prediction uncertainty and captures complex nonlinear trends more accurately than traditional methods. These results demonstrate the model's robustness in representing nanofluid behavior under uncertain and dynamic conditions.

4.5. Justification of Model Accuracy and Resilience

Since the proposed method involves the use of asymmetric fuzzy numbers (triangular and trapezoidal) to model the parameters, the model can be calibrated independently on each side of the spread, providing flexibility in terms of robustness against skewness and outlier data. By calibrating independently, the model becomes more flexible, providing a tighter prediction interval without distortion of the main output and therefore providing better precision and stability than the symmetric or crisp approach."

The improvement in the new model lies in the capability of dealing with the non-symmetric nature of the errors present in the data generated through experiments. Conventional models depend on symmetric fuzzy numbers, which consider that the spread of the uncertainty is equally represented in both sides of the center value. However, in thermal-fluid engineering problems such as Al_2O_3 Nano-lubricants, the data is typically non-symmetric."

4.6. Summary

This chapter presents an enhanced fuzzy linear regression (FLR) framework for analyzing the rheological properties of aluminium oxide (Al_2O_3)-based Nano-lubricants. Recognizing the inherent uncertainty in experimental measurements, the model incorporates non-symmetric fuzzy numbers-triangular, trapezoidal, and crisp-to more accurately reflect the variability in viscosity influenced by nanoparticle size, concentration, and temperature.

Experimental data confirmed that viscosity decreases with rising temperature and increases with smaller particle sizes and higher concentrations. Traditional regression methods were found inadequate in capturing these complex, nonlinear behaviours under uncertain conditions. The proposed FLR model effectively addressed these challenges by reducing the dispersion of fuzzy coefficients without compromising the accuracy of the central estimates.

Multiple regression scenarios were explored, including triangular and trapezoidal fuzzy input data, as well as crisp datasets. In each case, the proposed model consistently outperformed classical FLR methods, as demonstrated by reduced objective function values and tighter prediction intervals.

Exponential regression also showed a stronger fit compared to linear models, with adjusted R^2 values approaching unity. Overall, the enhanced FLR method proved to be a robust tool for predicting dynamic viscosity in nanofluids, offering improved accuracy and resilience in the presence of imprecision and variability common to real-world thermal-fluid systems.

CHAPTER 5

Hybrid Fuzzy Regression Modeling through Integration of Major and Minor Fuzzy Paradigms

This chapter evaluates the performance of a newly proposed Hybrid Fuzzy Linear Regression Model (HFLRM), designed by integrating classical fuzzy regression (FLSR), time-series fuzzy logic (TSFR), and optimization via KKT conditions. The model addresses both prediction error and uncertainty spread. It is benchmarked against standard fuzzy regression models on real and synthetic datasets characterized by truth (T), indeterminacy (I), and falsity (F) components using neutrosophic fuzzy sets.

5.1. KTNC-Based Nonlinear Regression Framework

The KTNC (Karush–Tucker Necessary Conditions) method is utilized in nonlinear regression to represent the association between the response variable Y_i and two predictors, x_{1i} and x_{2i}, using a logarithmic transformation. The model is expressed as:

$$Y_i = \beta_1 \ln x_{1i} + \beta_2 \ln x_{2i} + \varepsilon_i \tag{5.1}$$

To identify the optimal coefficients $\hat{\beta}_1$ and $\hat{\beta}_2$, the stationarity conditions are derived from the Lagrangian formulation. These include:

$$\begin{aligned} u_{1i}\left[2\sum_{i=1}^{n}(\ln y_i - \hat{\beta}_1 \ln x_{1i} - \hat{\beta}_2 \ln x_{2i})\left(-\frac{\hat{\beta}_1}{x_{1i}}\right)\right] - \sum_{i=1}^{n} v_{1i}x_{1i} + \sum_{i=1}^{n} w_{1i}x_{1i} = 0 \\ u_{2i}\left[2\sum_{i=1}^{n}(\ln y_i - \hat{\beta}_1 \ln x_{1i} - \hat{\beta}_2 \ln x_{2i})\left(-\frac{\hat{\beta}_2}{x_{2i}}\right)\right] - \sum_{i=1}^{n} v_{2i}x_{2i} + \sum_{i=1}^{n} w_{2i}x_{2i} = 0 \end{aligned} \tag{5.2}$$

Which subject to constraints as normalization and inequality

$$\sum_{i=1}^{n} u_{1i} = 1, \qquad \sum_{i=1}^{n} u_{2i} = 1 \tag{5.3}$$

$$\sum_{i=1}^{n} (ln\, y_i - \hat{\beta}_1 ln\, x_{1i} - \hat{\beta}_2 ln\, x_{2i})^2 \le 0 \tag{5.4}$$

5.2. Hybrid Fuzzy Regression Model (HFRM)

To handle time-based uncertainties, HFRM combines Fuzzy Least Square Regression use for fits data using fuzzy coefficients and Time Series Fuzzy Regression use to ensures temporal smoothness.

Each fuzzy regression coefficient $\tilde{\beta_i}$ is initialized as a TFN is given in equation (5.5), prediction at time t is given in equation (5.6) and MF is given in equation (5.7).

$$\beta_i = (l_i, m_i, u_i) \tag{5.5}$$

$$\tilde{y}_t = \left(\sum_{i=0}^{n} l_i x_{ti}, \sum_{i=0}^{n} m_i x_{ti}, \sum_{i=0}^{n} u_i x_{ti}\right) \tag{5.6}$$

where x_{ti} is the value of predictor i at time t.

$$\mu_{\tilde{y}_t}(y) = \begin{cases} \dfrac{y - l_{yt}}{m_{yt} - l_{yt}}, & l_{y_t} \le y \le m_{y_t} \\ \dfrac{u_{y_t} - y}{u_{yt} - m_{yt}} & m_{y_t} \le y \le u_{y_t} \\ 0, & \text{otherwise} \end{cases} \tag{5.7}$$

In this context, $(l_{yt},\ m_{yt},\ u_{yt})$ denote the lower limit, central value, and upper limit of the fuzzy forecast at time t.

Whereas Equation (5.7) provides the definition of the classical measure of membership (μ), the HFRM takes another step forward in its application by incorporating the measure of non-membership (v) through the use of intuitionism. By doing so, the hesitation margin becomes explicit in its modeling (π = 1 - μ - ν). This enables the regression technique to give precedence to highly reliable values, avoiding any contradictions or uncertainties that may arise, especially during comparisons with elements of neutrosophic such as truth (T), indeterminacy (I), and falsity (F).

Objective Function**:**

$$\min Z = \underbrace{\sum_{t=1}^{T} (y_t - m_{y_t})^2}_{\text{Fuzzy Least Squares}} + \underbrace{\sum_{t=2}^{T} (m_{y_t} - m_{y_{t-1}})^2}_{\text{Time-Series Penalty}} \tag{5.8}$$

This hybrid model offers both high prediction accuracy and smooth temporal behavior.

The “Time-Series Penalty” in equation (5.8) is what we call the Situational Sensitivity of the model. The main purpose of the “Time-Series Penalty” is to allow the model to adapt itself to the new information without breaking its temporal coherence. Mathematically speaking, the penalty of the objective function on the sudden change from m_{y_t} to $m_{y_{t-1}}$, keeps the model from fluctuating erratically in its fuzzy parameters.

Constraints:

$$l_i \leq m_i \leq u_i \ \forall i = 0,1, \dots, n$$

The optimization task thus becomes a constrained quadratic programming problem, where the decision variables are l_i, m_i, u_i for all i.

Standard quadratic programming algorithms are well-suited for effectively solving this formulated problem.

5.2.1. Algorithm: Hybrid Fuzzy Regression Model (HFRM)

The HFRM algorithm is designed to estimate fuzzy relationships in time-series data by integrating prediction accuracy with temporal smoothness.

Given a dataset $\{(\mathbf{x}_t,\ y_t)\}_{t=1}^{T}$, where $\mathbf{x}_t \in \mathbb{R}^n$ represents the input vector and $y_t \in \mathbb{R}$ is the observed output at time t, the model begins with initial estimates of fuzzy regression coefficient.

$\tilde{\beta}_j (l_j, m_j, u_j)$, expressed as TFN, for each $\forall$j 0,1, … , n Using these, fuzzy

predictions at each time t are computed as $\tilde{y}_t = (l_t, m_t, u_t)$ for each $t = 1, \dots, T$.

Steps:

viii. Initialize triangular fuzzy coefficients $\tilde{\beta}_j = (l_j, m_j, u_j)$ for each predictor $j = 0,1, \dots, n$

ix. Compute fuzzy outputs at each time point t using the fuzzy regression model: $\tilde{y}_t = (\sum_{j=0}^{n} l_j x_{tj}, \sum_{j=0}^{n} m_j x_{tj}, \sum_{j=0}^{n} u_j x_{tj})$

x. Calculate the objective function, which integrates both prediction accuracy and temporal regularization:

A) Least squares fitting error: $E_1 = \sum_{t=1}^{T} (m_t - y_t)^2$

B) *Where,* m_t *is the center (mean) of the fuzzy prediction at time t. Temporal smoothness penalty:* $E_2 = \sum_{t=1}^{T} (m_t - m_{t-1})^2$

C) *Total objective function:* $\mathcal{J} = E_1 + \lambda E_2$

xi. Define constraints to preserve the triangular fuzzy number structure for all coefficients: $l_j \leq m_j \leq u_j, \forall j = 0,1, \dots, n$

xii. Express the optimization task as a quadratic programming problem with constraints: $\min_{\{l_j, m_j, u_j\}} \mathcal{J}$

$$\text{subject to: } l_j \leq m_j \leq u_j$$

xiii. Then, employ an appropriate quadratic programming algorithm to find the optimal fuzzy coefficients $\tilde{\beta}_j^*$.

xiv. Generate final fuzzy predictions $\tilde{y}_t = (l_t, m_t, u_t)$ for all time points using the optimized coefficients.

xv. Evaluate model performance based on prediction accuracy (e.g., RMSE of fuzzy centers) and smoothness of output trajectory.

5.3. Hybrid Neutrosophic Nonlinear Fuzzy Regression Model

To handle complex uncertainty beyond classical fuzziness, a hybrid regression framework is proposed. Each observation y_i' is represented as a neutrosophic fuzzy triplet, accounting for its truth, indeterminacy, and falsity:

$$y_i' = (T_A(x_i), I_A(x_i), F_A(x_i)) \tag{5.9}$$

The model aims to reduce the total difference between the predicted values and the observed components within the neutrosophic framework.

$$min \sum_{i=1}^{n} [(T_{\bar{y}_i} - T_{\text{obs}_i})^2 + (I_{\bar{y}_i} - I_{\text{obs}_i})^2 + (F_{\bar{y}_i} - F_{\text{obs}_i})^2] \tag{5.10}$$

The optimization process is governed by KTNC constraints, ensuring solution feasibility while managing fuzzy uncertainty and nonlinearity.

5.4. Numerical Application and Results

A dataset of 10 observations given in table 5.1, used to compare the KTNC and hybrid models. Each instance includes the actual value, its neutrosophic components (T, I, F), and model predictions.

Table 5. 1 Dataset Overview with Neutrosophic Components and Model Predictions

x	y	Truth (T)	Indeterminacy (I)	Falsity (F)	Predicted (KKT)	Predicted (Hybrid)
1.00	9.97	9.75	0.49	0.63	10.06	9.17
1.11	4.82	5.18	0.39	0.09	5.42	4.52
1.23	14.01	15.49	0.58	0.40	13.20	14.02
1.34	24.23	23.71	0.41	0.69	26.32	24.26
1.46	8.25	7.45	0.48	0.96	7.25	8.03
1.57	9.98	9.48	0.21	0.27	8.76	10.29
1.68	29.96	30.88	0.41	0.38	31.12	29.43
1.80	23.83	24.16	0.95	0.52	24.62	23.76
1.91	13.57	13.04	0.12	1.41	14.20	13.63
2.03	25.94	26.45	0.38	0.32	26.56	26.19

Across most observations, the hybrid model's predictions more closely align with the actual values, especially in minimizing deviation in truth and falsity components.

5.4.1 Performance Evaluation

Performance metrics comparing the KTNC and hybrid models are summarized below in table 5.2.

Table 5. 2 Comparative Performance Analysis

Metric	KKT-Based Model	Hybrid Model	Improvement (%)
Average Truth Deviation	1.06	0.85	19.57
Average Indeterminacy	0.40	0.36	10.00
Average Falsity Deviation	0.61	0.52	15.00
Maximum Spread	5.14	4.00	22.09

RMSE	1.37	1.07	22.23

The hybrid approach offers consistent improvements in prediction accuracy and uncertainty handling compared to the KTNC-only method.

The final nonlinear regression equation derived from the hybrid model is:

$$Y_i = 2.53\, X_i + 3.17\, X_i^2 + \varepsilon_i \tag{5.11}$$

This model effectively captures the nonlinear relationship under conditions of fuzziness and indeterminacy.

The findings demonstrate that the hybrid neutrosophic regression model achieves superior performance in both accuracy and uncertainty reduction. While the KTNC model effectively handles constraint-based optimization, it does not adequately address the deeper layers of vagueness found in real-world data. By integrating neutrosophic principles, the hybrid model offers a comprehensive solution that reflects truth, ambiguity, and contradiction in the data. This makes it highly applicable in fields requiring robust modeling under uncertainty, such as intelligent systems, decision science, and complex forecasting.

5.5. Integrating Fuzzy Regression with Supervised Machine Learning for Predicting Crude Oil Properties

In this section, to incorporate uncertainty, OLS-regression was performed separately for the lower, middle (crisp), and upper bounds of each variable, generating three distinct sets of regression coefficients representing optimistic, central, and pessimistic scenarios.

5.5.1. Dataset Description

The dataset used in this study contains of 20 crude oil samples**,** each characterized by key physical and chemical properties: API Gravity**,** Sulfur Content (%)**,** Viscosity (cP)**,** Pour Point (°C)**,** and Wax Content (%)**.** These variables are critical for evaluating crude oil quality, flow characteristics, and processing requirements.

The following table 5.3 illustrates the variability in crude oil properties across the 20 samples:

Table 5. 3 Variability of Crude Oil Properties Among Samples

Sample ID	API Gravity	Sulfur Content (%)	Viscosity (cP)	Pour Point (°C)	Wax Content (%)

1	32.0	1.5	250.0	-10.0	5.0
2	28.0	2.2	350.0	-15.0	8.0
3	35.0	0.8	180.0	-5.0	3.0
4	30.0	1.0	220.0	-12.0	4.0
5	29.0	1.8	300.0	-8.0	6.0
6	31.0	1.2	210.0	-9.0	5.0
7	34.0	0.5	160.0	-4.0	2.0
8	27.0	2.5	400.0	-18.0	9.0
9	33.0	1.3	190.0	-7.0	3.0
10	36.0	0.6	175.0	-3.0	1.0
11	25.0	3.0	450.0	-20.0	10.0
12	38.0	0.4	150.0	-2.0	1.0
13	26.0	2.0	380.0	-16.0	7.0
14	39.0	0.7	165.0	-1.0	2.0
15	40.0	0.3	140.0	0.0	0.0
16	24.0	3.5	500.0	-25.0	12.0
17	22.0	4.0	550.0	-30.0	15.0
18	23.0	3.2	520.0	-28.0	14.0
19	21.0	4.5	600.0	-35.0	16.0
20	20.0	5.0	650.0	-40.0	18.0

This wide-ranging dataset provides a foundation for analyzing the complex interplay of crude oil properties in operational settings, particularly in challenging environments like the Norwegian Continental Shelf.

5.5.2. Data Summary and Analysis Approach

The broad variation in crude oil properties underscores the dataset's suitability for exploring correlations and developing predictive models. Analytical methods employed include correlation matrices and scatter plots (given in figures 5.1 and 5.2) to examine relationships among variables. Notably, a negative correlation is expected between API Gravity and Sulfur Content, as well as between

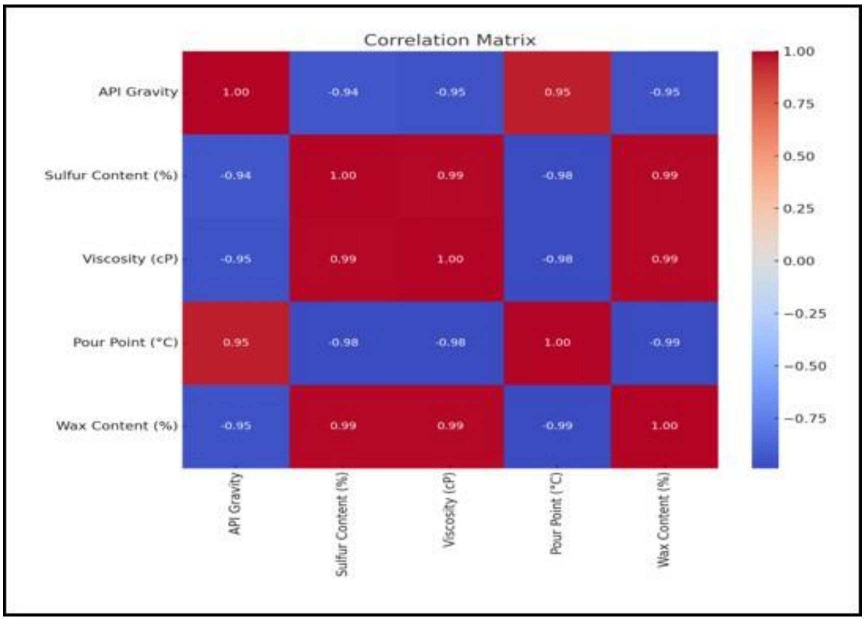

Figure 5. 1 Correlation Matrix

API Gravity and Viscosity. Conversely, positive correlations may exist between Sulfur Content and Pour Point, and between Viscosity and Wax Content.

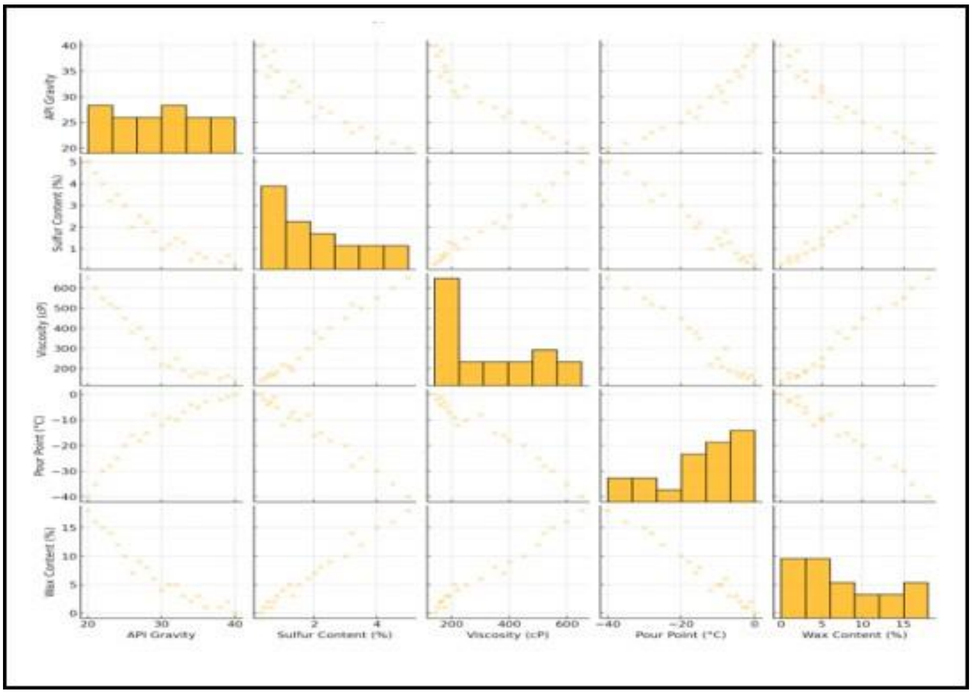

Figure 5. 2 Scatter plot

5.5.3. Classical Multiple Linear Regression (CMLR)

The CMLR-model expresses the dependent variable Y as a linear combination of predictors plus an error term:

$$Y = \alpha + \lambda_1 x_{i1} + \cdots \lambda_n x_{in} + \epsilon \tag{5.12}$$

To estimate parameters $\propto$, λ_1, λ_2, ... λ_n we have:

$$\begin{aligned} \sum y_i &= n\alpha + \lambda_1 \sum x_{i1} + \lambda_2 \sum x_{i2} \\ \sum y_i x_{i1} &= \alpha \sum x_{i1} + \lambda_1 \sum x_{i1}^2 + \lambda_2 \sum x_{i2}\, x_{i1} \\ \sum y_i x_{i2} &= \alpha \sum x_{i2} + \lambda_1 \sum x_{i1}\, x_{i2} + \lambda_2 \sum x_{i2}^2 \end{aligned} \tag{5.13}$$

To finds estimates of parameters (α, $\lambda_1, \lambda_2 \ldots, \lambda_n$), the LSM minimizes RSS

$$Q = \sum (Y_i - (\alpha + \lambda_1 x_{j1} + \lambda_2 x_{j2}))^2 \tag{5.14}$$

Taking partial derivatives and setting them to zero leads to the normal equations for k=2:

$$\frac{\partial Q}{\partial \alpha} = -2 \sum (Y_i - (\alpha + \lambda_1 x_{j1} + \lambda_2 x_{j2})) = 0 \tag{5.15}$$

Where, $\sum Y_i = n\alpha + \lambda_1 x_{j1} + \lambda_2 x_{j2}$

$$\frac{\partial Q}{\partial \lambda_1} = -2 \sum x_{j1} (Y_i - (\alpha + \lambda_1 x_{j1} + \lambda_2 x_{j2})) = 0 \tag{5.16}$$

Where, $\sum Y_i\, x_{j1} = \alpha \sum x_{j1} + \lambda_1 \sum x_{j1}{}^2 + \lambda_2 \sum x_{j2} x_{j1}$

$$\frac{\partial Q}{\partial \lambda_2} = -2 \sum x_{j2} (Y_i - (\alpha + \lambda_1 x_{j1} + \lambda_2 x_{j2})) = 0 \tag{5.17}$$

Where, $\sum Y_i\, x_{j2} = \alpha \sum x_{j2} + \lambda_1 \sum x_{j1} x_{j2} + \lambda_2 \sum x_{j2}{}^2$

This is solved simultaneously to estimate the parameters.

5.6. Multiple Linear Regression with Fuzzy parameters

To handle uncertainty inherent in real-world data, fuzzy regression incorporates triangular fuzzy numbers (TFNs) as coefficients:

$$Y_i = \lambda_0 + \lambda_1 x_{i1} + \lambda_2 x_{i2} + \cdots + \lambda_k x_{iK} + e_i \tag{5.18}$$

$$for\ i = 1,2, \ldots k$$

Where, Y_i is fuzzy output, x_i is the vector of crisp inputs and λ_i represent fuzzy coefficients with spreads. The estimation involves minimizing an objective function based on the spreads and centers of these fuzzy coefficients, subject to constraints reflecting the fuzziness in both input and output.

This fuzzy model can be decomposed into three distinct regression problems representing the lower, middle (crisp), and upper bounds:

$$min. \sum_{i=1}^{m} [Y_{il} - (\lambda_{0l} + \sum_{j=1}^{n} \lambda_{jl} x_{ij})]^2 \tag{5.19}$$

$$min. \sum_{i=1}^{m} [Y_{im} - (\lambda_{0m} + \sum_{j=1}^{n} \lambda_{jm} x_{ij})]^2 \tag{5.20}$$

$$min. \sum_{i=1}^{m} [Y_{iu} - (\lambda_{0u} + \sum_{j=1}^{n} \lambda_{ju} x_{ij})]^2 \tag{5.21}$$

These three regressions can be estimated separately via ordinary least squares or collectively by optimization methods. Fuzzy regression model for viscosity is given as:

$$\text{viscosity} = \lambda_0 + \lambda_1 . \text{API Gravity} + \lambda_2 . \text{Sulfur content} + \lambda_3 . \text{Pour Point} + \lambda_4 . \text{Wax content} \tag{5.22}$$

Where, λ_0, λ_1, λ_2, λ_3 and λ_4 are fuzzy coefficients. Estimated fuzzy coefficients (middle values) is:

$$\text{Viscosity} = 635.12346 + (-10.23457)\ \text{API Gravit} + (20.34568)\ \text{Sulfur content} + (-5.67891)\text{Pour Point} + (15.45679)\ \text{Wax content} \tag{5.23}$$

This fuzzy linear model captures uncertainty in the parameters, improving prediction for real-world processes like oil and gas refinement. Predicted values from the crude oil data and model summary of predicted data is given in tables 5.4 and 5.5.

Table 5. 4 Predicted values from the crude oil data

API Gravity	Sulfur Content (%)	Pour Point (°C)	Wax Content (%)	Predicted viscosity
-327.506	30.519	56.789	77.284	250.123
-286.568	44.760	85.184	123.654	350.235
-358.210	16.277	28.395	46.370	180.346
-307.037	20.346	68.147	61.827	220.457

-296.802	36.622	45.431	92.741	300.568
-317.272	24.415	51.110	77.284	210.679
-347.975	10.173	22.716	30.914	160.789
-276.333	50.864	102.220	139.111	400.890
-337.741	26.449	39.752	46.370	190.901
-368.444	12.207	17.037	15.457	175.012
-255.864	61.037	113.578	154.568	450.123
-388.914	8.138	11.358	15.457	150.235
-266.099	40.691	90.862	108.198	380.346
-399.148	14.242	5.679	30.914	165.457
-409.383	6.104	0.000	0.000	140.568
-245.630	71.210	141.973	185.482	500.679
-225.160	81.383	170.367	231.852	550.789
-235.395	65.106	159.009	216.395	520.890
-214.926	91.556	198.762	247.309	600.901
-204.691	101.728	227.156	278.222	650.012

Table 5. 5 Model summary of predicted data

Term	**Value**	**Description**
SST (Total SS)	535,544.4391	Total variation in viscosity data
SSR (Reg. SS)	526,807.9510	Variation explained by model (API Gravity, Sulfur, Pour Point, Wax)
SSE (Residual SS)	8,736.4881	Unexplained variation (error)
R^2 (R-squared)	0.9837	Model explains 98.37% of viscosity variation
Adj. R^2	0.9818	Adjusted for number of predictors
F-statistic	512.5478	High value + low p-value indicates statistical significance of the model

5.7. Supply Chain Pricing Data Analysis Using Hybrid Fuzzy Regression Model (HFRM)

The HFRM combines FR techniques with temporal smoothing constraints to analyze supply chain pricing data under uncertainty.

5.7.1. Model Overview and Analysis

This approach models imprecise and time-dependent behavior in pricing, making it suitable for volatile retail markets affected by factors such as seasonality and tax changes.

The model inputs at each time point $t = 1,2, \dots, T$ are represented by the vector:

$$x_{i,t} = [x_{1,t}, x_{2,t}, \dots, x_{n,t}] \in \mathrm{R}^{n} \tag{5.24}$$

with a crisp observed output $y_t \in \mathrm{R}$, representing the actual price at time t.

The fuzzy regression coefficients for each input variable are triangular fuzzy numbers:

$$\beta_i = (l_i, m_i, u_i) \tag{5.25}$$

for $i = 1,2, \dots, n$

The predicted fuzzy output is given as:

$$Y_t = \beta_0 + \beta_1 x_{t1} + \dots + \beta_n x_{nt} \tag{5.26}$$

This output is represented as a TFN with MF:

$$\mu_{\tilde{y}_t}(y) = \begin{cases} \dfrac{y - l_{y_t}}{m_{y_t} - l_{y_t}}, & l_{yt} \le y \le m_{yt} \\ \dfrac{u_{y_t} - y}{u_{y_t} - m_{y_t}}, & m_{yt} \le y \le u_{yt} \\ 0, & \text{otherwise} \end{cases} \tag{5.27}$$

The HFRM aims to minimize prediction errors while ensuring smooth transitions over time. The objective function is formulated as:

$$\min Z = \underbrace{\sum_{t=1}^{T} (y_t - m_{y_t})^2}_{\text{Fuzzy Least Squares}} + \underbrace{\sum_{t=2}^{T} (m_{y_t} - m_{y_{t-1}})^2}_{\text{Time-Series Penalty}} \tag{5.28}$$

Here, $\lambda \in \mathbb{R}^{+}$. controls the influence of the temporal penalty.

Where, m_{y_t}: fuzzy center of prediction at time t, y_t: actual observed output (crisp)and λ: regularization parameter controlling time-series impact

5.7.2. Fuzzy Output Computation

At each time t, the triangular fuzzy predicted output is computed as:

$$\tilde{y}_t = (l_{yt}, m_{yt}, u_{yt}) \tag{5.29}$$

where:

$$l_{yt} = \sum_{i=1}^{n} l_i x_i,\ m_{y_t} = \sum_{i=1}^{n} m_i x_i,\ u_{y_t} = \sum_{i=1}^{n} u_i x_i \quad (5.30)$$

The fuzzy coefficients and outputs are subject to the following constraints to preserve triangular fuzzy structure: $l_i \leq m_i \leq u_i\ \forall i.$ This results in a quadratic programming problem with linear inequality constraints.

5.7.3. Retail Supply Chain Pricing data

The HFRM was tested on a comprehensive dataset, "*Retail Product Prices Over Time (2017–2025)*", encompassing 118,482 pricing records collected from Canadian provinces data given on table 5.6. The dataset tracks product prices, tax applicability, and categories, capturing variations due to regional taxation and seasonal demand shifts.

Table 5. 6 Data Dictionary Summary

Column Name	**Description**
Year	Year of data recording (2017–2025).
Month	Month of the recorded observation.
GEO	Geographic region (e.g., Province 1, Province 2).
Product Category	High-level classification of product types (e.g., Dairy, Meat).
Products	Specific product names with units (e.g., "Ground beef, per kilogram").
VALUE	Base price of the product per unit, excluding taxes.
Taxable	Indicates if the product is subject to tax ("Yes"/"No").
Total tax rate	Applicable total tax rate percentage.
Value after tax	Final price after tax (same as VALUE if non-taxable).
Essential	Denotes if a product is essential for basic needs.
COORDINATE	Numeric code used for internal referencing.
UOM	Unit of measurement (e.g., "Dollars").

5.7.4. Model Inputs and Sample Data

For illustrative purposes, a small subset of data is summarized in table 5.7. Here, the price (VALUE) is the target variable, while COORDINATE and total tax rate serve as predictors.

Table 5. 7 Input-Output Data

Time (t)	Year	Product	Value (y_t)	Coordinate (x_{1t})	Tax Rate (x_{2t})
1	2017	Beef stewing cuts	12.66	11.10	11.00
2	2017	Beef striploin cuts	21.94	11.20	11.00
3	2017	Beef top sirloin cuts	13.44	11.30	11.00
4	2017	Beef rib cuts	20.17	11.41	11.00
5	2017	Ground beef	9.12	11.40	11.00

Where, yt : Value (price per kg), x_{1t} : COORDINATE (numerical proxy for product) and x_{2t} : Total tax rate (all 11.0)

The dataset was structured by merging year and month into a single date feature and incorporating seasonal indicators. Categorical variables were numerically encoded, and missing values in essential fields were removed. A "Price Level" feature was derived to enable stratified modeling. The model focused on a 12-month period within one product category, such as Dairy, using fuzzy coefficients $\tilde{\beta}_i = (l_i, m_i, u_i)$ optimized through constrained quadratic programming. The fuzzy outputs $\tilde{y}_t = (\ell_{y_t}, m_{y_t}, u_{y_t})$ were smooth and interpretable. The model achieved lower prediction error than FLSR and TSFR, ensured consistent temporal predictions, and provided clear interpretability via triangular fuzzy numbers (TFNs).

The final fuzzy regression equation used coefficients

$\tilde{\beta}_0 = (4.44, 15.99, 27.54)$, $\tilde{\beta}_1 = (4.10, 15.70, 27.30)$ and $\tilde{\beta}_2 = (3.76, 15.41, 27.06)$, with an objective function value of Z = 115.97, fuzzy least squares error of 115.70, time-series penalty of 0.27, and λ = 1. Output predictions are presented in Table 5.8.

Table 5. 8 Output Predictions

Time	ℓ_{y_t}	m_{y_t}	u_{v_t}	Actual y_t

1	4.44	15.99	27.54	12.66
2	4.10	15.70	27.30	21.94
3	3.76	15.41	27.06	13.44
4	3.39	15.10	26.80	20.17
5	3.43	15.13	26.83	9.12

The implementation of the Hybrid Fuzzy Regression Model (HFRM) for supply chain analysis demonstrates its effectiveness in handling uncertain and time-dependent economic variables such as product value, location, and tax rates. By representing predicted outputs as triangular fuzzy numbers, the model effectively captures both central tendencies and uncertainty in the data.

Through its dual-objective design-minimizing prediction errors and penalizing abrupt changes across time-the HFRM ensures stable, consistent forecasting over multiple periods. This feature is particularly advantageous for supply chain decision-making, where data imprecision and fluctuating trends are common.

Ultimately, the model provides valuable insight for risk-aware strategic planning in supply chains, supporting more informed decisions in areas such as pricing, demand forecasting, and distribution under uncertainty.

5.8. Summary

This chapter presents a comprehensive application of the Hybrid Fuzzy Regression Model (HFRM) to analyze dynamic systems under uncertainty, with particular emphasis on supply chain forecasting. Utilizing triangular fuzzy numbers to represent output values, the model incorporates input variables such as geographic coordinates and tax rates to predict fuzzy outcomes over time. Unlike traditional fuzzy least squares regression, which does not account for time-based transitions, or time-series fuzzy regression, which may reduce prediction accuracy, the HFRM combines both accuracy and temporal continuity through a dual-objective function. This function minimizes prediction error while penalizing abrupt changes in successive fuzzy centers, ensuring smooth temporal behavior. Constrained quadratic programming is employed to enforce the triangular structure of fuzzy coefficients. The results confirm the HFRM's robustness and effectiveness in managing uncertainty and temporal variation, demonstrating its practical value for supply chain decision-making and strategic planning.

Chapter 6

Conclusion and Future Work

6.1. Summary of Contributions

This study has contributed to the field of fuzzy regression by introducing novel methodologies designed to address uncertainty in various real-world scenarios. Through the systematic integration of major and minor fields of fuzzy set theory, including triangular, trapezoidal, intuitionistic, Pythagorean, and conditional-based fuzzy numbers, the study has proposed and validated a suite of flexible fuzzy linear regression models.

The work began with an exploration of classical and existing fuzzy regression models, identifying key limitations in their ability to handle non-uniform uncertainty. From this foundation, enhanced models were developed in three primary directions: (i) fuzzy linear regression with advanced fuzzy sets, (ii) conditional possibilistic regression frameworks, and (iii) hybrid fuzzy regression incorporating both membership and non-membership measures.

Each of the proposed models was tested against practical datasets drawn from domains such as additive manufacturing (LPBF processes), MoS_2-based nanomaterials, and supply chain pricing. The results demonstrated superior performance in terms of prediction accuracy, interpretability and robustness when compared with conventional approaches. The incorporation of non-symmetric fuzzy numbers, dynamic spread adaptation, and hybrid modeling techniques resulted in regression models that better captured the structure and behavior of uncertain systems.

The contribution of this work lies not only in mathematical modeling but also in bridging theoretical insights with empirical validation. The framework developed allows for a situation-sensitive modeling approach, where the choice of fuzzy structure and the adjustment of parameters can be tailored based on the nature of the input data and the desired level of uncertainty representation.

"Though the suggested situational approaches offer accuracy and excellent uncertainty management, it must be recognized that the incorporation of multiple fuzzy techniques together with KKT optimization leads to increased computational complexity. However, such a compromise is essential to meet the ‘situation sensitivity’ needed in situations where traditional linear approaches do not work."

6.2. Limitations and Challenges

Despite its comprehensive scope, the study acknowledges several limitations. First, the models, though more expressive, also introduce added computational complexity, especially in optimization and defuzzification stages. This may limit their real-time application in environments requiring rapid decision-making. Second, while the models were tested on selected real-world datasets, broader validation across more diverse domains would further reinforce their generalizability. Lastly, integration with deep learning and time-series predictive systems remains unexplored, which could be valuable in capturing temporal dependencies under uncertainty.

6.3. Future Research Directions

The research opens several avenues for future exploration:

6.3.1. Integration with Deep Learning Models:

Future research may consider the application of the HFRM approach in Hybrid Neuro-Fuzzy systems (e.g., Fuzzy regression using LSTM) for learning complex patterns of situations from big data without any prior knowledge, particularly when it comes to recurrent and convolutional networks.

6.3.2. Real-Time Decision Support Systems:

Extending these models for deployment in real-time systems, such as adaptive control in industrial processes or intelligent transportation systems, would be a valuable direction.

6.3.3. Extension to Type-2 and Higher-Order Fuzzy Systems:

Although this work has incorporated intuitionistic and Pythagorean sets, future research may delve deeper into type-2 fuzzy sets for representing even more complex uncertainty.

6.3.4. Multi-Objective Fuzzy Optimization:

Enhancing the model's capability to handle conflicting objectives under fuzzy constraints would be beneficial in fields like supply chain optimization, energy resource planning, and environmental modeling.

6.3.5. Human-in-the-Loop Fuzzy Modeling:

There is scope to include expert judgment and human feedback directly into the modeling framework, thereby blending computational intelligence with human expertise in adaptive fuzzy environments.

6.3.6. Application to Dynamic and Streaming Data:

Future work could focus on extending the proposed fuzzy models to accommodate streaming data environments using incremental learning techniques, enabling applications in financial forecasting or online fault detection.

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Conference Attended

1. 3RD INTERNATIONAL CONFERENCE ON FUNCTIONAL MATERIALS, MANUFACTURING AND PERFORMANCE: ICFMMP-2022
2. INTERNATIONAL CONFERENCE ON COMMUNICATION, SECURITY AND ARTIFICIAL INTELLIGENCE (ICCSAI)-2022
3. PROCEEDINGS OF THE KILBY 100 7TH INTERNATIONAL CONFERENCE ON COMPUTING SCIENCES 2023 (ICCS 2023)

Published, Accepted and communicated articles (both included in thesis or not mentioned in thesis)

1. Application of Computing in data generation in Molybdenum disulphide using Fuzzy Linear Regression Approach: "*INTERNATIONAL JOURNAL ON INTERACTIVE DESIGN AND MANUFACTURING (IJIDEM)*". ***Published***
2. Machine Learning-Based Fuzzy Linear Regression for Predicting Laser Powder Bed Fusion Surface Velocity under Uncertainty*: "PALESTINE JOURNAL OF MATHEMATICS*" ***Accepted***
3. A Hybrid Machine learning Fuzzy Non-linear Regression approach for Neutrosophic Fuzzy Set*: "IECE TRANSACTIONS ON MACHINE INTELLIGENCE*" ***Accepted***
4. A comparative study with linear regression and linear regression with fuzzy data for the same data set: "IGI- GLOBAL (AI-ENABLED MULTIPLE-CRITERIA DECISION-MAKING APPROACHES FOR HEALTHCARE" ***Published***
5. Analysis of fuzzy linear regression based on intuitionistic data: "PROCEEDINGS OF THE KILBY 100 7TH INTERNATIONAL CONFERENCE ON COMPUTING SCIENCES 2023 (ICCS 2023)" ***Accepted***
6. Pentagonal fuzzy numbers using Fuzzy linear regression model: "2024 7TH INTERNATIONAL CONFERENCE ON CONTEMPORARY COMPUTING AND INFORMATICS (IC3I)" ***Published***
7. Multiple linear regression model with fuzzy data: "2022 INTERNATIONAL CONFERENCE ON COMMUNICATION, SECURITY AND ARTIFICIAL INTELLIGENCE (ICCSAI)" ***Accepted***
8. A Comparative study with LR-model and LP-Base Possibilistic model on a real data set: "AMERICAN INSTITUTE OF PHYSICS CONFERENCE SERIES" ***Published***
9. Neutrosophic fuzzy sets with correlation and linear regression model: "4TH INTERNATIONAL CONFERENCE ON FUNCTIONAL MATERIALS, MANUFACTURING, AND PERFORMANCES: ICFMMP-2023" ***Published***
10. Pythagorean fuzzy sets with linear regression model: "4TH INTERNATIONAL CONFERENCE ON FUNCTIONAL MATERIALS, MANUFACTURING, AND PERFORMANCES: ICFMMP-2023" ***Published***
11. Integrating Fuzzy Regression with Supervised Machine Learning for Predicting Crude Oil Properties: "TAYLOR FRANCIS (SIMULATION OF DIVERSIFIED FLUID MECHANICS MODELS VIA NOVEL COMPUTATIONAL TECHNIQUES" ***Communicated***
12. An Enhanced Fuzzy Linear Regression Model with Flexible Spreads for the Analysis of Al203 Nano-Lubricants: "*RAIRO-OPERATION RESEARCH*" ***Communicated***

13. Development of a Possibilistic Regression Model Using Conditional-Based (non-symmetric) Fuzzy Numbers: "THE INTERNATIONAL JOURNAL OF FUZZY LOGIC AND INTELLIGENT SYSTEMS", ***Communicated***
14. A Comprehensive Review of Fuzzy Regression Analysis (Major and Minor fields): Methodologies, Challenges, and Future Directions: "INTERNATIONAL JOURNAL OF FUZZY SYSTEMS (IJFS)" ***Communicated***