RICCI SOLITONS ON CONTACT METRIC MANIFOLDS

Thesis Submitted for the Award of the Degree of

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In

Mathematics

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Transforming Education Transforming India

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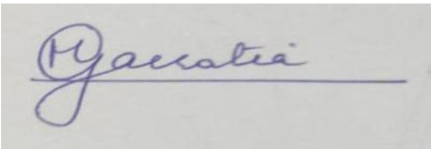
DECLARATION OF AUTHORSHIP

I, hereby declared that the presented work in the thesis entitled “**Study of Ricci Solitons on contact metric manifolds**” in fulfilment of degree of **Doctor of Philosophy (Ph.D)** is outcome of research work carried out by me under the supervision of **Dr. Pankaj Pandey**, working as Associate Professor, in the School of Chemical Engineering and Physical Sciences of Lovely Professional University, Punjab, India. In keeping with general practice of reporting scientific observations, due acknowledgements have been made whenever work described here has been based on findings of other investigator. This work has not been submitted in part or full to any other University or Institute for the award of any degree.



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CERTIFICATE FROM THE SUPERVISOR

This is to certify that the work reported in the Ph. D. thesis entitled "**Ricci Solitons on Contact metric Manifolds**" submitted in fulfillment of the requirement for the award of degree of **Doctor of Philosophy (Ph.D.)** in the School of Chemical Engineering and Physical Sciences, is a research work carried out by **Mona Jasrotia**, **41800186**, is bonafide record of his/her original work carried out under my supervision and that no part of thesis has been submitted for any other degree, diploma or equivalent course.

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Name of supervisor: Dr. Pankaj Pandey

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Abstract

Differential geometry is the study of spaces with certain geometrical features, such as metric, distance, geodesic, connection, curvature and isometries. Differential geometry during the eighteenth and nineteenth centuries was based on the geometry of surfaces in three dimensional plane curves, Euclidean spaces, and space curves. Differential geometry is that field of mathematics which is concerned with geometric point of view of differentiable manifolds, was founded in the early twentieth century. The emphasis switched to examine the regional and worldwide characteristics of smooth manifolds with fixed signature metric tensor on tangent spaces of that encodes the geometric qualities in manifolds. The distinction in a metric tensor's signature is of vital importance when analyzing geometrical structures on differentiable manifolds. Subjects of our research are the Ricci solitons on Contact metric Manifolds. Sections, subsections, theorems, lemmas, corollaries and definitions are all numbers using bold decimal notations in this thesis. Following is the chapter by chapter summary of work demonstrated in this thesis.

Chapter 1: The 1st chapter of this thesis offers brief introduction and summary of various important methods of Differential geometry of manifolds that will use in our thesis whenever these are required. Introduction of manifolds in differential geometry, scope of thesis and some standard tools on contact metric manifolds, Kenmotsu manifolds, para Kenmotsu manifolds, LP-Sasakian manifolds, f-Kenmotsu manifolds, Cotton tensor, Concircular curvature tensor. Data obtained from many reputed research journals, books, review articles is reflected in this chapter that helped a lot to develop the terminology for the chaperts given below.

Chapter 2: We study various properties of Yamabe Ricci soliton in f-Kenmotsu manifold. In this chapter we derived conditions for Yamabe Ricci soliton on f-Kenmotsu manifold to be steady, shrinking, as well as expanding. It is also proved that Yamabe Ricci soliton in equipped with a torse forming vector field has generalized geodesic vector field in f-Kenmotsu manifold. We established a relationship between η-Einstein manifold along with Yamabe Ricci soliton in f-Kenmotsu manifold. The content of this chapter is published in **Chhattisgarh Journal of Science and Technology, ISSN: 0973-7219, Volume 17, Issue 4, 2020, pp-33-36.**

Chapter 3: Some properties of generalized form of Ricci soliton, particularly η-Ricci soliton of dimension n with concircular structure(briefly(LCS) manifold) whose metric is Ricci soliton are discussed in chapter 3rd of the thesis. In addition to this, condition of η-Ricci flat(LCS) manifold derived for expanding, shrinking, as well as steady and parallel condition is

also derived. We also throw some light on η-Ricci soliton related to concircular vector field. Curvature of any plane section is also discussed in this chapter. We also proved some results of η-Ricci semisymmetric manifolds to be Einstein manifold.

This whole chapter is published in **European Chemical Bulletin Journal**, **2022, 11(Issue 10), 177-180**.

Chapter 4: The main goal of this chapter is introducing Cotton tensor on a para Kenmotsu manifold of dimension three admitting η-Ricci soliton. Concept of Cotton pseudo-symmetric manifold is also introduced in this chapter. Derivation of some important results on Cotton flat para Kenmotsu 3-manifold is also discussed. Besides this, study of Codazzi type of Cotton tensor admitting η-Ricci soliton is also conducted. Some applications of Cotton tensor in fluid space and its significances are briefly discussed in this chapter.

The whole content of this chapter is published in **IAENG International Journal of Applied Mathematics (SCOPUS Indexed).**

Acknowledgment

I would be pleased to thank God for always guiding me in the straight forward approach and offering me a chance to continue higher study. Thank you, God! for always helping me out and giving me the patience to deal with everything. My faith in you keeps me strong and resilient. Studying can sometimes feel overwhelming, but having a little motivation can make all the difference. A few encouraging words can help us remember our goals.

I am specifically grateful for my father Sh. Lal Singh and my mother Smt. Tara Devi who have always supported my decisions to continue learning and improving. It was the dream of my revered dad that I should persue my Doctoral degree. I convey deep thanks to my spouse Mr. Sandeep Singh because of his academic, emotional and monetary assistance. Special thanks to my brothers (Dr. Ajay Singh and Mr. Vicky) for their emotional and technical support, my daughters (Hezal and Smaira), whose understanding and encouragement have been pivotal. Without their unwavering support, completing my study would have been an unconquerable challenge.

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SYMBOLS AND NOTATIONS

$\mathbb{N}$	Represents a collection of natural numbers
$\mathbb{R}$	Represents a collection of all real numbers
$\mathbb{C}$	Represents a set of all complex numbers
$\mathbb{Z}$	Stands for integers
∇	Connection / Operator of Covariant differentiation
$\chi(\mathbb{M})$	Set of all vector fields
$\exists$	there exists
$\forall$	for all
$\in$	belongs to
$\neq$	not equal to
$\mathbb{M}$	Manifold
ς	Ricci tensor
R	Riemannian curvature tensor
$T_P\mathbb{M}$	Tangent space at point P
$\mho$	Riemannian metric
$\mathfrak{C}$	Cotton tensor
λ	Soliton constant
ξ	Vector field
Q	Ricci operator
r	Scalar curvature tensor
ϕ	(1, 1) type tensor

η 1-form

$\mathcal{Z}$ Concircular curvature tensor

$L_V\mho$ The Riemannian metric's Lie derivative $\mho$

A, B On manifold $\mathbb{M}$, these are arbitrary vector fields

[,] Lie bracket

V Vector field

LCS Lorentzian concircular structure

$\cup$ Union

$\cap$ Intersection

Table of content

CHAPTER 1

Introduction

In this chapter, an overview to Riemannian/ semi-Riemannian/ semi-Euclidean spaces have been provided in brief. To help the readers comprehend the study that was conducted, basic definitions, key concepts, current debates and terminologies that are employed in the following chapters are provided as well. Along with a brief history, the thesis' primary theme is offered.
By reviewing the literature a research gap is found and as a result, the goals of the current work are suggested. This core objective of this chapter is to set up prior knowledge of fundamental subjects that will be necessary for the presentation of the thesis' subsequent chapters. The result of this thesis will have potential applications in computer science, physics and engineering. Thesis results also contribute to in depth knowledge of some geometric and a handful of topological properties of Riemannian manifolds and will have potential applications encompassing computer science, engineering, and physics, among other disciplines.

1.1 Historical sketch of differential geometry

A study of geometric objects, such as curves, surfaces and manifolds using differential calculus and linear algebra is called Differential geometry. Differential geometry has its roots in ancient Greece and evolved over centuries. Euclid's "Elements" (300 BCE) laid foundations for geometry. Archimedes (250 BCE) introduced the concept of tangents, normals and curvature in differential geometry. Differential geometry is a vast and fascinating field. Here is an overview of its subfields:

- Riemannian geometry: Studies manifolds with Riemannian metrics (lengths and angles).
- Differential topology: Explores topological properties of manifolds.
- Symplectic geomertry: Concerned with symplectic manifolds (phase spaces).
- Complex geometry: Studies complex manifolds and their properties.
- Algebraic geometry: Intersects differential geometry with algebraic techniques.
- Foliations and Lie group actions.

Other subfields: Discrete geometry, Computational Geometry, Differential geometry,etc.

Several significant discoveries in mathematics from the 18th and 19th centuries served as forerunners to the contemporary notion of a manifold. The non-Euclidean geometry refers to the geometric systems that differ from fifth postulate of Euclid, which declares that, “Through a point not on a line, there is exactly one line parallel to the original line”. It was further refined 100 years later by Lobachevsky, Bolyai and Riemann.

The two types of spaces they discovered are known as elliptic geometry and hyperbolic geometry, and they both have geometric structures that are distinct from those of traditional Euclidean space. These concepts relate to constant, ative, and positive curvature manifolds in the theory of manifolds that is now in use.

In general, the study of topology deals with the problem of continuous maps connecting things. In this context, topologists distinguish between a sphere and an ellipsoid as one and the same object (by a continuous deformation in a specific sense), but they disagree when the ellipsoid is replaced by a torus. To be more specific, the study of topology seeks to categorize items according to similar topological types. One can have two infinite sequences of topological types, orientable and non-orientable, in the situation of closed surfaces. A topological invariant known as the Euler characteristic (or hole) is used to categorize orientable surfaces. However, this type of classification is not feasible in higher dimensions due to various technological problems. Similar attempts are made in the field of differential geometry.

These spaces feature an inner product that smoothly fluctuates from one point to another at any location on the tangent space. In other words, whereas topological information, which refers to neighborhood or connectivity information, must be understood in order to solve many geometric problems, spatial information is not necessary. Riemann’s technique provides an augmentation of the differential geometry of facets to multi dimensions.

The basic unit is the Riemann curvature tensor. Topological manifolds are the name given to these types of spaces. To continue the analysis in differential geometry, differential structure is required. In order to be more accurate, one needs smooth manifolds, which are topological manifolds that also have a smooth structure. Differential geometry has many applications

In Physics

- General relativity: Describes gravity, Spacetime curvature.
- String theory: Studies vibrating strings, extra dimensions.
- Quantum field theory: Describes particle interactions
- Cosmology: studies universe evolution, expansion.

In Engineering

- Computer vision: Image processing, object recognition.
- Robotics: Motion planning, control system.
- Computer-aided designs: Geomertic modeling.
- Medical imaging: MRI, CT scans.

In Biology

- Shape analysis: Studying biological shapes, morphology.
- Medical imaging: Analyzing anatomical structures.
- Protein folding: Understanding protein geometry
- Population Dynamics: Modeling biological system.

In Economics

- Geometric models: Economics system equilibrium.
- Game theory: Strategic decision making.
- Financial modeling: Risk analysis, portfolio optimization.
- Econophysics: Applying physics method to economics.

In other fields

- Navigation: GPS, geodesy.
- Material science: Studying material property.

- Architecture: Designing curved surfaces.
- Neuroscience: Studying brain geometry.

The differential geometry of manifolds is an important area of study for the Riemannian and semi-Riemannian geometries, which have applications not just in mathematics but in every branch of science. Semi-Riemannian geometry and its sub-manifolds have yielded a wealth of information regarding the differences between them. In real world, GPS relies on differential geometry for location tracking. Medical imaging uses differential geometry for tumor detection. Self- driving cars use differential for motion planning.

1.2 Concise history of Manifold

The foundation for differential geometry was laid in the last quarter of the 18th century when the differential calculus was first applied to clarify geometry difficulties. The word "Mannigfaltigkeit" that Riemann used to characterize the manifold was eventually translated as "Manifoldness" by William Kingdon Clifford. In the domains of group theory, relativity theory, topology, and linear algebra, manifolds are a generalisation of curves and surfaces. Several important mathematical fields are combined in the study of manifolds, extending concepts from topology and linear algebra to include ideas about surfaces and curves. Additionally, some particular classes of manifolds exhibit additional algebraic structure; for example, they might exhibit group behavior. Then they are known as Lie Groups. Alternatively, they can be defined in terms of algebraic varieties, which are expressed in terms of polynomial equations. If these groups also have a group structure, they are referred to as algebraic groups. Abstract spaces were first thought of as distinct mathematical objects apparently by Carl Friedrich Gauss. His theorema egregium provides a way for calculating a surface's curvature without taking the surrounding space into account.

The theorem showed that the surface's curvature is an intrinsic property. The extrinsic qualities of the surrounding space are generally ignored in manifold theory, which now focuses solely on these intrinsic properties (or invariants).
In 1854, Bernhard Riemann delivered a habilitation lecture in G¨ottingen in which he discussed various approaches of endowing any space with a metric.

Development of Charts and Coordinate are also discussed by him. Initially the notion of a Riemannian surface had been proposed by Bernhard Riemann in 1857. He also made the concept of a surface with higher dimensions famous. Riemannian geometry is a natural and advantageous enhancement of differential surface geometry in R. Euclidean and non-Euclidean geometry are the specific categories of Riemannian geometry. The theory of sub-manifolds is as fundamental to differential geometry as the subject matter itself. Formerly, the only element of Riemannian geometry, sub-manifold geometry is now one of many distinct. Sub-manifold geometry is currently one of several separate components of multi-dimensional expansions of the classical theory of surfaces, whereas it was originally the sole component of Riemannian geometry, while Riemannian geometry is a logical and effective progression of Gauss's concept of intrinsic geometry. Hermann Weyl was the first to provide a fundamental description of a differentiable manifold and later on Whitney demonstrated that a manifold that is naturally can be embedded in a Euclidean space also defined by charts. The Euler characteristic is yet another example of an inherent quality of a manifold that is more topological.

By extending the idea to more than three dimensions, Riemann (1854) expanded the abstract surface, or manifold, which resembles a particular dimension of Euclidean space known as the dimension of the manifold. Due to their ability to construct increasingly complex structures in terms of Euclidean spaces, manifolds are regarded as central objects in the subject of differential geometry. Many areas of physics, geometry and mathematics are based on the idea of a manifold due to arising the solution sets in equation system and graphs easily. The analysis of surfaces in Euclidean space was expanded to Riemannian geometry, which retains many of the intuitive properties of Euclidean geometry. There is an additional algebraic structure in some particular special classes of manifolds and they might act as groups.

It generalizes the notions and concepts of topology and linear algebra and higher dimension surface. The study of distinctive special structures on geometric manifolds is an important problem in geometry. A contact structure on a space assigns a plane to each point in the space, allowing the planes to change smoothly as one goes through various locations in the space and to always twist in one direction in order to satisfy the non-integrability criterion.

In other words, a contact structure is a hyperplanes on an odd-dimensional manifold that accomplishes a particular twisting requirement or non integrability condition.

Boothby and Wang conducted the first study on spaces with the hypothesis of structure of contact (almost) in 1958. The methods of contact geometry are crucial in current mathematics. In order to examine a set of differential equations, Lie used contact geometry as a geometric tool in 1896. Mathematical physics applications such as geometric quantization, classical mechanics, thermodynamics, and geometric optics have all made extensive use of the notion of contact manifolds.

1.3 Motivation of the research work

Earlier it was considered that the higher-dimensional space time manifold that contains the cosmos is a 4-dimensional manifold. Later on many mathematicians analyzed that semi-Riemannian manifold in addition to the sub manifold of Riemannian manifolds, which is driven by relativity, string theory and the Kaluza-Kleins theory.

In addition to being a key tool in numerous scientific and commercial applications, Ricci solitons are significant because they were utilized to resolve the Poincare hypothesis, a question that had remained open for a century. There are several uses of Ricci solitons in physics, biology, economics and chemistry. Furthermore, the implication of Ricci flow and Ricci soliton has been demonstrated in brain surface medical imaging. We started our research motivated by a number of researches working on some interesting generalizations of Ricci solitons in various types of contact metric manifolds.

We study an interesting generalization of Ricci soliton satisfying properties of f-Kenmotsu manifold described as Ricci Yamabe soliton.

In addition, it is noted that f-Kenmotsu manifold also satisfies the property of Einstein manifold. These results will help many research scholars studying contact metric manifold to find further Ricci solitons generalizations.

An important concept of η-Ricci soliton is also outlined by us on Lorentzian concircular structure manifold and its different properties like concircular flat η-Ricci soliton, sectional curvature in a plane section, etc. are also studied. This could serve as inspiration for future studies on η-Ricci solitons in Lorentzian concircular structure manifolds: Natural extensions of Ricci solitons, that have been researched widely within the context of geometry of Riemannian, are η-Ricci solitons.

There are several factors that motivated us for study of η-Ricci solitons in Lorentzian manifolds:

- **Geometric significance:** Natural extensions of Ricci solitons, researched broadly within context of Riemannian, are η-Ricci solitons. The introduction of the η-function allows for a more nuanced understanding of the geometric and topological properties of these solitons.
- **Physical applications:** Cosmology and general relativity both use Lorentzian concircular structure manifolds, where they can be used to model spacetime. The investigation of η-Ricci solitons on these manifolds provides insight into the properties of gravitational fields and the structure of the universe.
- **Mathematical challenges:** The investigation of Ricci solitons on different manifolds poses significant mathematical challenges. The development of new techniques and methods for studying these solitons can lead to advances in our understanding of geometric analysis and partial differential equations.
- **Interdisciplinary connections:** Ricci soliton research on contact metric manifolds is related to differential geometry, among other areas of mathematics and topology, physics, and theoretical physics. Exploration of these connections can lead to new insights and perspectives on the behavior of complex systems.

Next, we examine the Cotton tensor that admits the η-Ricci soliton in the para Kenmotsu manifolds of dimension 3 as well as also discover some of its characteristics. In another manifold i.e para Kenmotsu manifold we study Cotton tensor which if of Codazzi type in para Kenmotsu manifold. Surveys are undertaken on generalized form of Ricci soliton because there is ambiguity about a circumstance that either has or has not occurred. In the meantime, the main purpose of writing and publishing is to promulgate research outcomes and to pass on new knowledge with other investigators in relevant fields. The highlighted motivation for the research is that the results of research are benefiting the community.

Some other motives of research are as under:

- To get curiosity satisfaction of doing some innovative work.
- Desiring to take on challenging tasks in order to resolve a number of unresolved issues That is, research is sparked by practical issues. .

- A desire for a research degree and the significant advantages that come with it.
- Hoping to be a part of community development.
- Wish to get honor and reputation.

A lot more, like government employment regulations, curiosity about new things, the drive to comprehend causal linkages, social thinking, and awakening.

Some professional motivations such as career advancement collaborate with peers and experts, contribute to societal or economic development.

1.4 Some important definitions

In this section, some important terms within the domain of differential geometry are defined. A synopsis of manifold and its different types are discussed. This section provides some background information on Ricci soliton in contact metric manifolds, including its significances and relevance. This section also includes necessary historical or theoretical background of Ricci solitons and its various generalizations. Outline of the research work and discussion of applicable mathematical and physical notions are investigated in this section.

Manifold or Smooth Manifold [50]

A topological space that is resembling with Euclidean space close to every point is called a manifold in differential geometry, but may have a complicated global structure.

A Manifold $\mathbb{M}$, which is a topological space that has dimension n, satisfies the following results:

(i) Locally Euclidean: Neighborhood U of each point p on $\mathbb{M}$ is homeomorphic to a subset which is open in R^n.

(ii) Hausdorff Space: For any pair of unique points p, q in $\mathbb{M}$, $\exists$ neighborhoods represented by U, V of points p, q resp. such as $U \cap V = \phi$.

(iii) Second Countable: $\mathbb{M}$ has a basis that is countable in topology.

Types of manifolds

(i) Topological manifold

(ii) Differentiable manifold

(iii) Riemannian manifold

(iv) Complex manifold

Examples of manifold

(i) Euclidean space R^n

(ii) Sphere S^n

(iii) Torus T^n

(iv) Circle S^1

(v) Cylinder RXS^1

(vi) Calabi-Yau manifolds (complex geometries)

Manifolds in Physics plays a crucial role in general relativity, string theory, in engineering (Computer vision, robotics), Biology (morphogenesis, population dynamics), Mathematics(Differential geometry, topology), etc.

Riemannian metric and Riemannian manifold [43]

For all $V_1,\ V_2 \in \chi(\mathbb{M})$, a tensor field $\mho$ which is (0, 2) type, called a Riemannian metric iff following important conditions are fulfilled:

(i) $\mho$ is symmetric iff $\mho(V_1, V_2) = \mho(V_2, V_1)$ for every $V_1,\ V_2 \in \chi(\mathbb{M})$.

(ii) $\mho$ satisfies the condition of positive definiteness, i.e., $\mho(V_1, V_1) \geq 0$ for $V_1 \in \chi(\mathbb{M})$, iff $V_1 = 0$.

In a manifold $\mathbb{M}$ which is manifold of n dimension, a Riemannian metric $\mho$ exists. A Riemannian manifold($\mathbb{M}$, $\mho$) has been smooth manifold as well as associated with this metric $\mho$.

Differentiable Manifold [13]

Assume that the Hausdorff separation axiom is satisfied by topological space $\mathbb{M}$. Group of open charts (U_i, ϕ_i), $i \in \Delta$ on $\mathbb{M}$ is open subset of R^n and also differentiable structure in $\mathbb{M}$ such as $\mathbb{M} = \cup\, U_i$, $i \in \Delta$. For i, $j \in \Delta$ the map $\phi_j \phi_i^{-1}$ is a differentiable mapping from $(U_i \cap U_j)$ onto $(U_i \cap U_j)$. To put it simply, a Hausdorff space with a differentiable structure having n dimensions represents a differentiable manifold. An analytic structure having n dimension is defined similarly as differentiable structure except in second condition we are just replacing differentiable structure by a structure which is analytic, we call this structure as analytic structure. Let $\mathbb{M}$ and N be any 2 manifolds, then function $f: \mathbb{M} \longrightarrow$ N is referred to as differentiable, if for each chart (U_i, ϕ_i) in $\mathbb{M}$ and each chart (V_j, ψ_j) $(Vj,\ \psi j)$ of N such that (U_i) $\subset Vj$. A' being a vector field on the manifold constitutes an assignment of vector ${A'}_p$ to every point p within the manifold $\mathbb{M}$. On a manifold, a tangent space is a set of the tangent vectors at point generally represented by $T_p\mathbb{M}$.

Define the brackets $[A', B']$ as a map on $\mathbb{M}$ from the ring of functions into itself if A and B are the vector fields by

$[A', B']\, f' = A'\ (B'f') - B'(\, A'f')$.

Above equation shows that $[A', B']$ is representing a vector field. A topological manifold which has n dimension is Hausdorff space which is second countable and also an n-dimensional locally Euclidean space. A mapping f of $\mathbb{M}$ to N which is an into mapping is an injective at each point P in $\mathbb{M}$, which is immersion for injective F∗P. Thus $\mathbb{M}$ is defined as the immersed manifold of N and f' is called imbedded submanifold of N if it is one-one map.

Contact manifold [63]

The presence of a global differentiable 1-form denoted as η s.t $\eta\wedge(d\eta)^n$ is non zero indicates that a differentiable C^∞ (2n+1) dimensional manifold denoted as $\mathbb{M}^{2n+1}$ have contact structure or is a contact manifold at every point of manifold.

Lie Bracket [63]

For a pair of vector fields A, B, a Lie bracket denoted by [,] and is stated as

$[A, B]F' = A(BF') - B(AF')$, for $F' \in C^\infty(\mathbb{M})$.

Generally speaking, the Lie bracket can be written as

$[A, B] = AB - BA$, representing tangent vector field in manifold $\mathbb{M}$ that's smooth also.

Semi-Riemannian metric [84]

A (0,2) type tensor field Ʊ called to be a semi-Riemannian pseudo-Riemannian metric if it fulfills the subsequent conditions:

(i) Ʊ exhibits the symmetric condition i.e., $\mho(B, A) = \mho(A, B)$.

(ii) $\mho(A, B) = 0\ \forall\ B \Leftrightarrow A = 0$, for $p \in \mathbb{M}$ A, B are vectors in $T_p\mathbb{M}$.

Semi Riemannian Manifold [84]

"In differential geometry, a pseudo-Riemannian manifold also called a semi-Riemannian manifold, is a differentiable manifold with a semi-Riemannian metric tensor $\mho$ that is everywhere non degenerate. This is a generalization of a Riemannian manifold in which the requirement of positive definiteness is relaxed. Lorentzian manifold is a semi-Riemannian manifold with a metric that has signature (-, +, +, +…)".

Manifold $\mathbb{M}$ which is differentiable and has smooth, symmetric, metric tensor $\mho$ which is not degenerate everywhere, is known as pseudo-Riemannian manifold ($\mathbb{M}$, $\mho$). We call such a metric as pseudo-Riemannian metric. An application to a vector field can result in a scalar field value that is zero, positive, or negative at any point on the manifold. For any two non-negative integers p and q, a pseudo-Riemannian metric has the signature (p, q).

Riemannian Connection An Affine connection in $(\mathbb{M}, \mho)$ denoted by ∇, which is n-dimensional Riemannian manifold associated with Riemannian metric $\mho$, is Levi-Civita connection or Riemannian connection if:

(i) ∇ is symmetric or torsion free connection.

(ii) ∇ is metric compatible connection.

Ricci and Scalar Curvature Tensor[17]

For vector fields $A, B, C \in T_p\mathbb{M}$, the Scalar curvature symbolically represented by R and as defined by

$R(A, B)C = \nabla_A \nabla_B Z - \nabla_B \nabla_A Z - \nabla[A, B]C.$

Tensor field which is of type (0,2) is Ricci tensor ς and defined as [64]

$Ric(A, B) = \varsigma(A, B) = Trace\{Z \to R(A, B)C\}$, for all A, B, C lying in manifold $\mathbb{M}$.

For some function λ, if the Ricci tensor ς satisfies $\varsigma = \lambda\, \mho$ in a Riemannian manifold, then that manifold is described as Einstein manifold and if its curvature tensor is curvature constant, it is called locally symmetric space $i.e$, ∇R is zero.

A scalar function with a scalar curvature r is defined as

$r = TrQ,$

where, Q is given by

$\varsigma(A, B) = \mho(QA, B).$

Almost Contact Manifold [8], [43]

Suppose that structure (ϕ,ξ,η) be (1,1) type tensor field, ξ being vector field, 1-form η respectively in manifold $\mathbb{M}$ of dimension ($2n$ + 1) satisfying the equations given below:

$\eta(\xi) = 1$

$\phi^2(A) = -A + \eta(A)\xi,$

for each vector field A in $\mathbb{M}$,

then $\mathbb{M}$ with structure (ϕ,ξ,η) referred to as almost contact manifold. Each almost contact manifold must possess a non-singular vector field in $\mathbb{M}$.

Sasakian Manifold [2], [53], [64]

Manifold $\mathbb{M}$ with dimension ($2n$ + 1) associated with a normal metric structure(ϕ,ξ,η,$\mho$) that is contact structure, referred as Sasakian manifold additionally has a Sasakian structure.

A structure (ϕ,ξ,η,$\mho$) with almost contact metric on $\mathbb{M}$ can only be considered a Sasakian structure if

$(\nabla_A\phi)(B) = \mho(A, B)\xi - \eta(B)A.$

A Sasakian manifold ($\mathbb{M}^{2n+1}$, $\mho$) satisfying the relation

$R(A, B).R = 0$ is said to be semi symmetric.

Additionally it is Weyl semi symmetric Sasakian manifold if its conformal curvature tensor C satisfies relation:

$R(A, B).C = 0$, where $R(A, B)$ is curvature operator.

If $\mathbb{M}$ satisfies the properties of above stated Sasakian manifold, we've the relation given below:

$R(A, B)\xi = \eta(B)A - \eta(A)B,$

where Riemannian curvature tensor in $\mathbb{M}$ is represented by the symbol R.

Along with this Sasakian manifold satisfies the properties given below:

$\mho(\xi, \xi) = 1,$

$\eta(\xi) = 1,$

$\phi(\xi) = 0,$

$\mho(\phi A, \phi B) = \mho(A, B) - \eta(A)\eta(B),$

$d\eta(A, B) = \mho(\phi A, B).$

Examples of Sasakian manifold:

a. Sphere

b. Torus

c. Heisenberg

Para Sasakian Manifold [71]

Considering $(\mathbb{M}, \mho)$ as the n-dimensional manifold accepting η as a 1-form which satisfies the conditions

$(\nabla_A \eta)(B) - (\nabla_B \eta)(A) = 0,$

$(\nabla_A \nabla_B \eta)(C) = 2\eta(C)\eta(B)\eta(A) - \eta(B)\mho(A, C) - \eta(C)\mho(A, B).$

Here the covariant differentiation operator indicated by the symbol $\nabla$with regard to the metric tensor $\mho$.

Additionally, if tensor field of (1,1) type, ξ as vector field and are admitted in $(\mathbb{M}, \mho)$ such that

$\eta(A) = \mho(A, \xi)$,

$\eta(\xi) = 1,$

$\nabla_A \xi = \phi A,$

Thus para Sasakian manifold is the term used for describing such manifold.

Cosymplectic Manifold [89]

A manifold $\mathbb{M}$ being almost contact manifold associated with (ϕ, ξ, η) almost contact metric structure where ξ represents vector field, η representing 1-form, ϕ a tensor as well as $\mho$ stands for Riemannian metric fulfilling the conditions:

$\mho(\phi A, \phi B) = \mho(A, B) - \eta(A)\eta(B)$, for all $A, B \in \chi(\mathbb{M})$.

If both the fundamental forms η and ω satisfy $d\eta$ = $d\omega$ = 0,
where $d(A, B) = \mho(\phi A, B)$,
then cosymplectic almost contact metric structure is metric structure in this case, and a manifold that possesses this metric structure is known as a Cosymplectic manifold.
One of the example of Cosymplectic manifold is Kahler manifold.

Paracontact manifold[54]

Any odd-dimensional manifold $\mathbb{M}^{2n+1}$ that has a η as a global 1-form everywhere has been referred as contact manifold. Given such a shape, there's a special vector field known as Reeb vector field of or characteristic vector field, which satisfies:

$\eta(\xi) = 1$
and in the case of any vector field A on $\mathbb{M}^{2n+1}$, $d(A, \xi) = 0$.
If there's a tensor field of type (1,1) such that a semi-Riemannian metric is associated metric, then

$\eta(A) = \mho(A, \xi),$

$d(A, B) = \mho(A, \phi B)$ and

$\phi^2(A) = A - \eta(A)\xi$, for all vector field A, B in odd dimensional manifold $\mathbb{M}^{2n+1}$.

Consequently, a manifold $\mathbb{M}^{2n+1}$ that has $(\phi, \xi, \eta, \mho)$ structure on it has been defined as paracontact metric manifold, also this structure itself is representing a metric structure which is paracontact.

It is evident that following relations hold in a metric manifold that is paracontact:

$\phi(\xi) = 0, \quad \eta o \phi = 0$ and

$\mho(\phi A, \phi B) = -\mho(A, B) + \eta(A)\eta(B),$

for any vector fields on $\mathbb{M}^{2n+1}$ A, B.

Kenmotsu Manifold [21], [57], [93]

Suppose that $\mathbb{M}$ is a manifold of ($2n$+1)-dimensions which is almost contact metric manifold that has an almost contact metric structure $(\phi, \xi, \eta, \mho)$ made up of vector field ξ, a 1-form η , a (1,1) tensor field ϕ along with suitable Riemannian metric $\mho$ providing:

$\phi^2(A) = -1 + \eta \otimes \xi,$

$\eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta o \phi = 0,$

$\mho(B', A') = \mho(\phi B', \phi A') - \eta(B')\eta(A'),$

$\mho(A, \phi B) = -\mho(\phi A, B),$

$\mho(A, \xi) = \eta(A).$

Such a manifold that satisfying the following condition and almost contact metric manifold is representing Kenmotsu manifold.

$(\nabla_A \phi)B = \mho(\phi A, B) - \eta(B)\phi A,$

for every vector field A, B and Riemannian metric's Levi civita connection ∇.

The equation above indicates that

$\nabla_A \xi = A - \eta(A)\xi,$

$(\nabla_A \eta)B = \mho(A,B) - \eta(A)\eta(B)$.

Furthermore, Ricci operator Q, Ricci tensor ς, and curvature tensor R all satisfy
$R(A,B)\xi = \eta(A)B - \eta(B)A$,
$\mho(A,\xi) = \eta(A)$ and $Q\xi = -2n\xi$.

α-Kenmotsu Manifold

Real C^∞ manifold in n dimensions, when tensor field of (1,1) type is admitted by $\mathbb{M}$, vector field that is contravariant vector field, and η as a 1-form that satisfy the following, this metric structure is regarded as almost contact (,,,).

$\eta(\xi) = 1$,

$\phi^2(A) = -A + \eta(A)\xi$,

$\phi\xi = 0,\ \eta(\phi A) = 0$,

$\mho(\phi A, \phi B) = \mho(A,B) - \eta(A)\eta(B)$,

$\mho(A,\xi) = \eta(A)$,

for vector fields A, B on $\chi(\mathbb{M})$, where Lie algebra of vector fields represented by $\chi(\mathbb{M})$ on $\mathbb{M}$.
A real C^∞ manifold $\mathbb{M}$ that has almost contact metric structure $(\phi, \xi, \eta, \mho)$and is of n-dimension, is referred to as an almost contact metric manifold.
An almost contact metric manifold $\mathbb{M}$ referred to be an α-Kenmotsu manifold if $d\eta = 0$ and $d\phi = 2\alpha\eta\wedge\phi$, here ϕ= a fundamental 2-form defined as
$\phi(A,B) = \mho(\phi A, B)$ and being a real constant that is not zero.

β-Kenmotsu manifold

If the following criteria is met, the dimensional nearly contact metric manifold ($2n$ +1)-dimensional $\mathbb{M}^{2n+1}(\phi, \xi, \eta, \mho)$ is considered as a Kenmotsu manifold:

$\nabla_A \xi = \beta(A - \eta(A))$ and

$(\nabla_A \phi)(B) = \beta(\mho(\phi A, B)\xi - \eta(B)(\phi A))$,

where, ∇ is used for the Riemannian connection of $\mho$.
If $\beta = 1$, When β= constant, Kenmotsu manifold is called a Homothetic Kenmotsu manifold; otherwise, it is called a β-Kenmotsu manifold.

Para Kenmotsu manifold [6]

When manifold of dimension n that has a Riemannian metric $\mho$, it admits a tensor field ξ, a 1-form η and tensor field ϕ of type (1,1) that satisfy:

$\eta(\xi) = 1$, $\mho(A, \xi) = \eta(A)$, along with

$(\nabla_A \eta)(B) - (\nabla_B \eta)A = 0$.

$(\nabla_A \nabla_B \eta)C = [-\mho(A, C) + \eta(A)\eta(C)\eta(B)] + [-\mho(A, B) + \eta(A)\eta(B)\eta(C)]$ and

$\nabla_A \xi = \phi^2(A) = A - \eta(A)\xi$.

Lorentzian manifold [59]

It is smooth manifold $\mathbb{M}$ with signature (-,+,+…) or (+,-,-,...) which is a symmetric bilinear form associated with Lorentzian metric $\mho$.
In Lorentzian manifold $(\mathbb{M}, \mho)$, we can define P as vector field as

$\mho(A', P) = \mathrm{F}(A')$,
for any $A' \in T\mathbb{M}$,
A concircular vector field may be described as portions of a tangent vector field that is smooth on $\mathbb{M}$ if

$(\nabla_X \mathrm{F})(B) = \alpha\{\mho(A, B) + \omega(A)\mathrm{F}(B)\}$,
Here in above equation, covariant differentiation operator about Lorentzian metric
$\mho$ represented by ∇, ω is a closed 1-form, and α is used for scalar function that is non zero.

Lorentzian Concircular Structure Manifold [1], [32], [33], [59], [97]

A Lorentzian concircular structure manifold that has a concircular structure is a Lorentzian manifold $(\mathbb{M}, \mho)$, which is a geometric configuration that expands on the geometric concept of conformal flatness. In a concircular structure $\exists$ a 1-form η such that $\nabla_A \eta = 0$, for all $A \in T\mathbb{M}$, the portions of all tangent vector fields which are smooth on $\mathbb{M}$ is referred to as a concircular vector field. Some of its properties are:

$\mho(\xi, \xi) = -1, \quad \mho(K, \xi) = \eta(K),$

here ξ used here represents unit concircular vector field.

Also $\nabla_K \xi = \alpha\{K + \eta(K)\xi\}$.

Ricci Solitons [2], [3], [37], [53], [57]

"A Ricci soliton is a natural generalization of Einstein metric. A Ricci soliton $(\mho, V, \lambda)$ is defined on a pseudo-Riemannian manifold $(\mathbb{M}, \mho)$ by

$$(L_V \mho)(A, B) + 2\varsigma(A, B) + 2\lambda\mho(A, B) = 0,$$

where, $L_V \mho$ denotes the Lie derivative of Riemannian metric $\mho$ along a vector field V, λ is a constant, and A, B are arbitrary vector fields on $\mathbb{M}$. A Ricci soliton is said to be shrinking, steady, and expanding pursuant to λ is negative, zero, and positive respectively. Theoretical physicists have also been exhibiting curiosity in the equation of Ricci soliton concerning with the string theory and is a particular instance of Einstein field equations. The Ricci soliton in Riemannian geometry was brought forth as a symmetrical solution of the Ricci flow".

Examples of Ricci solitons:

- Einstein manifolds: Any Einstein manifold is a Ricci soliton with $\lambda = 0$.
- Guassian solitons : The Guassian soliton a Ricci soliton on the Euclidean space R^n.
- Hamilton's cigar soliton: The cigar soliton is a Ricci soliton on the two dimensional sphere S^2.

$\boldsymbol{\eta}$-Ricci soliton [21], [32], [71]

A more comprehensive idea of Ricci flow is an η-Ricci soliton.

Makoto Kimura and J.J. Cho suggested this concept and provided its equation as

$(L_\xi \mho)(A,B) + 2\varsigma(A,B) + 2\lambda\mho(A,B) + 2\mu\eta(A)\eta(B) = 0,$

for $A, B \in \chi(\mathbb{M})$

where A and B are the vector fields, μ and λ are constants.

Ricci Yamabe Soliton [15], [28], [51], [69], [82]

Yamabe-Ricci soliton is another generalization of Ricci soliton, which is either Riemannian or Pseudo Riemannian manifold $\mathbb{M}$ $(\mho, V, \lambda, \alpha, \beta)$ if it fulfills:

$$(L_V \mho)(A,B) + (2\lambda - \beta r)\mho(A,B) + 2\alpha\varsigma(A,B) = 0,$$

where, ς is denoting Ricci tensor, r being scalar curvature and $\lambda, \alpha, \beta \in \mathbb{R}$.

If λ is greater than zero, then there is expanding Yamabe Ricci soliton,

if λ is equal to zero, then there is steady Yamabe Ricci soliton and

if λ is less than zero, then there is sgrinking Yamabe Ricci soliton.

A generalization of both Yamabe and Ricci soliton is Yamabe Ricci soliton. The Yamabe Ricci soliton is another well-known Einstein soliton of the (1,-1) kind.

Additionally, special Yamabe flow solutions that fulfill the following equation are known as Ricci Yamabe solitons:

$\frac{\partial \mho}{\partial t} = -2\varsigma\alpha + \beta r \mho.$

Cotton tensor [24], [52]

Since, Weyl conformal curvature tensor vanishes for a pseudo-Riemannian manifold in three dimensions, a new kind of conformal invariant known as cotton tensor in differential geometry can be created. The type (1,2) tensor known as cotton tensor $\mathfrak{C}$ is described by

$\mathfrak{C}(A,B) = (\nabla_A Q)B - (\nabla_B Q)A - \frac{1}{4}\{(Ar)B - (Br)A\},$

for smooth vector fields A and B.

Eisenhart shared the idea that if metric is conformally flat, then Cotton tensor becomes zero in a three dimensional pseudo-Riemannian manifold.

A Cotton tensor in three dimensions has been identified by numerous authors in recent years as a n extremely intriguing subject that connects the energy momentum tensor and Ricci tensor of matter in Einstein's equations.

The Weyl tensor, a measurement of a manifold's conformal curvature, strongly associated with Cotton tensor. Additionally, Ricci tensor, a measurement of trace of curvature tensor, is associated with it.

CHAPTER 2

Some results on Ricci Yamabe soliton equipped with f-Kenmotsu manifold

We examine the Yamabe Ricci soliton in f-Kenmotsu manifold, which is an important generalized form of Ricci soliton. We determine certain circumstances under which Ricci-Yamabe soliton on f-Kenmotsu manifold is shrinking, steady or expanding. We also demonstrated that the generalized geodesic vector field is present in Ricci-Yamabe soliton on f-Kenmotsu manifold with torse- forming vector field.

Investigating and comprehending these geometric formations' characteristics and behavior is the goal of Ricci−Yamabe soliton study.Our main goal is to categorize Ricci−Yamabe soliton accor ding to their topology, symmetry, and curvature characteristics.
The Ricci−Yamabe soliton clarified the topological and geometric characteristics of manifolds.
Non−linear partial differential equations can be better understood by studying Ricci−Yamabe soliton.

2.1 Introduction

The Ricci flow concept was proven to exist in 1982 by Hamilton [29].
Additionally, Hamilton [30] categorized all compact manifolds in dimension four that have a positive curvature.
An Einstein metric's natural extension is the Ricci soliton. The definition of a Ricci soliton in Riemannian manifold ($\mathbb{M}$, $\mho$) can be described in following way [32], [37]:

$$(L_V\mho)(A,B) + 2\varsigma(A,B) + 2\lambda\mho(A,B) = 0, \qquad (2.1.1)$$

here A and B are arbitrary vector fields in $\mathbb{M}$, $\boldsymbol{\lambda}$ being a constant, and $L_V\mho$ represents a Lie derivative of Riemannian metric $\mho$ across vector field V.

A shrinking, steady, and expanding Ricci soliton is indicated by λ as negative, zero, as well as positive, correspondingly.

Hamilton presented idea of Yamabe flow, the metric on the Riemannian manifold in this case is warped through advancing in accordance with

$$\frac{\partial \mho(t)}{\partial t} = -R(t)\mho(t),$$

The metric $\mho(t)$ has scalar curvature denoted by (t). .

S. Deshmukh and B.Y. Chen and [15] studied Yamabe soliton in 2018. Yamabe soliton correlates to homogeneous solution of Yamabe flow. Yamabe flow can be compared to Ricci flow in 2-dimensions. Recently, C Xia, Gomes, Q.Wang [27] continued the idea behind Ricci soliton and examined Ricci soliton on complete Riemannian manifold with h-almost soliton. If Riemannian manifolds satisfy

$$\frac{h}{2} L_\xi \mho + \lambda \mho + \varsigma = 0, \tag{2.1.2}$$

where the smooth function $h: \mathbb{M} \to \mathbb{R}$ is known as the h-almost Ricci soliton.

Now we investigate Yamabe Ricci soliton in f-Kenmotsu manifold of Ricci soliton in its generalized version. Ricci Yamabe soliton associated with semi-Riemannian manifold [15], [28], [51], is

defined as

$$(L_V \mho)(A, B) + (2\lambda - \beta r)\mho(A, B) + 2\alpha\varsigma(A, B) = 0, \tag{2.1.3}$$

where, Lie derivative is represented by L_V, Ricci tensor by ς, scalar curvature by r and λ, α, $\beta \in \mathbb{R}$.

Some special solutions to Ricci Yamabe flow is presented by Crasmareanu and Guler [28], is the Ricci Yamabe soliton.

$$\frac{\partial \mho}{\partial t} = -2\alpha\varsigma + \beta r \mho, \tag{2.1.4}$$

In particular, for α = 1 and β = 0, the equation (2.1.4) reduces to the Ricci soliton.

When a vector field ξ is smooth in ($\mathbb{M}$,$\mho$), is known as geodesic vector field if it satisfies:

$$\nabla_\xi \xi = 0. \tag{2.1.5}$$

and is considered to be generalized geodesic vector field if it satisfies:

$\nabla_\xi \xi = f\xi,$ (2.1.6)

where, A potential function on $\mathbb{M}$ is a smooth function denoted as f.

This chapter is organized as under:

A brief overview of Ricci-Yamabe soliton equipped with f-Kenmotsu manifold is presented in 2.1 section of this chapter. In 2.2 section, certain properties of Yamabe Ricci soliton in f-Kenmotsu manifold are studied and proved certain theorems which are reflecting certain conditions under which f-Kenmotsu manifold is shrinking, steady and expanding. We discuss Ricci Yamabe soliton on f-Kenmotsu manifold admitting torse forming in 2.3 section. We in this section also investigated that a soliton ξ which is a vector field, is a generalized geodesic vector field. Ultimately, in our concluding section 2.4, we find a relationship between η-Einstein manifold and Ricci Yamabe soliton in f-Kenmotsu manifold.

Definition 2.1.1 Kenmotsu manifold [21], [25], [57], [93]:

Define $\mathbb{M}^{2n+1}$ as a Riemannian manifold that is nearly contact, which includes a (1,1) tensor field ϕ, a 1-form η as well as Riemannian metric $\mho$. $\mathbb{M}^{2n+1}$ is recognized as a Kenmotsu manifold provided it adheres to the following specifications :

$\phi\xi = 0, \qquad \eta(\phi A) = 0, \qquad \eta(\xi) = 1, \quad \eta(A) = \mho(A, \xi),$

$\phi^2(A) = -A + \eta(A)\xi, \qquad \mho(\phi A, \phi B) = \mho(A, B) - \eta(A)\eta(B),$ (2.1.7)

$\nabla_X \xi = A - \eta(A)\xi, \qquad (\nabla_A \phi)B - \eta(B)\phi(A) - \mho(A, \phi B)\xi,$ (2.1.8)

Characterization of Kenmotsu manifold

- Harmonic curvature is a defining feature of Kenmotsu manifolds, which are a type of contact metric manifold.
- A Sasakian manifold characterized by constant scalar curvature is called a Kenmotsu manifold.

Definition 2.1.2.Torse forming vector field [8]**:**

A vector field in a manifold is considered as a torse forming, if vector field ξ, compiles with:

$\nabla_A \xi = \phi(A) - \eta(A)\xi,$

where, we are using ∇ for Levi-civita connection, ϕ for (1,1) tensor field, η for a 1-form and A for a vector field.

Properties of torse-forming vector field:

- One type of characteristic vector fields is the torse forming vector field.
- They have a relation with contact structures and nearly contact metric structures.
- They satisfy Kenmotsu manifold conditions.
- On manifolds, they are encountered in the study of geometric structures.

Definition 2.1.3 $\boldsymbol{\eta}$-Einstein manifold [32]:

An η-Einstein manifold $(\mathbb{M}, \mho)$ is a Riemannian manifold with a given vector field ξ and a 1-form η that meets following requirements.

$$Ric + \eta(A) \times \eta(A) = \lambda \mho, \tag{2.1.9}$$

where, Ric is Ricci tensor, λ is η-Einstein constant, 1-form dual to ξ is η and $\mho$ is Riemannian metric.

Definition 2.1.4. Geodesic vector field

Manifold $\mathbb{M}$ with vector field A' fulfills following equation to be a geodesic vector field:

$$\nabla_{A'} A' = 0,$$

here, ∇ has been utilized for Levi Civita connection.

A few examples of geodesic vector fields includes:

Vector fields in Euclidean spaces, killing vector field in Riemannian manifolds, etc.

2.2 Ricci Yamabe soliton equipped with $\boldsymbol{f}$-Kenmotsu manifold [51]

A smooth manifold $(\mathbb{M}, \mho)$ of dimension n with a metric structure $(\phi, \xi, \eta, \mho)$ is almost paracontact, $\mho$ is pseudo Riemannian metric, ϕ is a tensor field of type (1,1), ξ is used for vector field and η for 1-form .

$$\phi^2(A') = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1,$$

$$\eta(A') = \mho(A', \xi), \qquad \phi\xi = 0, \qquad \eta o \phi = 0 \tag{2.2.1}$$

$$\mho(\phi A', \phi B') = \mho(A', B') - \eta(A')\eta(B'), \tag{2.2.2}$$

for all $A', B' \in \chi(\mathbb{M})$.

2.2.1. f-Kenmotsu manifold [9], [101] and Cosymplectic manifold [89]

A manifold $\mathbb{M}^{2n+1}(\phi,\ \xi,\ \eta,\ \mho)$ of dimension 2n+1 is referred to as f-Kenmotsu manifold [13] in the event that covariant differentiation of ϕ meets the following:

$$(\nabla_A \phi)B = f\{\mho(\phi A, B)\xi - \eta(B)\phi A\}, \tag{2.2.3}$$

where $f \in C^\infty(\mathbb{M})$ when $df \wedge n = 0$.

$f = \beta$, a non-zero constant indicates that manifold is β- Kenmotsu manifold.

When $f = 0$, manifold is referred to as a cosymplectic manifold.

If $f^2 + f' \neq 0$, then a f-Kenmotsu manifold is considered a regular manifold, where $f' = f\xi$.

2.2.2 Ricci soliton equipped with f-Kenmotsu manifold.

The Riemannian curvature tensor in a three−dimensional Riemannian manifold can be found using [43].

$$R(A,B)C = \mho(B,C)QA - \mho(A,C)QB + \varsigma(B,C)A - \varsigma(A,C)B - \frac{r^2}{2}\{\mho(B,C)A - \mho(A,C)B\}.$$

In an f-Kenmotsu manifold which is of dimension three, Ricci tensor as well as Riemannian curvature tensor become

$$R(A,B)C = \left(\frac{r}{2} + 2f^2 + 2f'\right)(A \wedge B)C - \left(\frac{r}{2} + 3f^2 + 3f'\right)\{\eta(A)(\xi \wedge B)C + \eta(B)(A \wedge \xi)C\}. \tag{2.2.4}$$

$$\varsigma(A,B) = (r^2 + f^2 + f')\mho(A,B) - (r^2 + 3f^2 + f')\eta(A)\eta(B). \tag{2.2.5}$$

$$QA = \left(\frac{r}{2} + f^2 + f'\right)A - \left(\frac{r}{2} + 3f^2 + f'\right)\eta(A)\xi. \tag{2.2.6}$$

here, r=scalar curvature in $\mathbb{M}$.

$$R(B,A)\xi = (f^2 + f')[\eta(A)B - \eta(B)A].$$

Put $Y = \xi$ in (2.2.5), we get

$$\varsigma(A,\xi) = -2(f^2 + f')\eta(A). \tag{2.2.7}$$

$$Q\xi = -2(f^2 + f')\xi.$$

From (2.1.3), we get

$$2\alpha\varsigma(A,B) = -L_\xi \mho(A,B) - (2\lambda - \beta r)\mho(A,B). \tag{2.2.8}$$

The above equation can be written as

$$2\alpha\varsigma(A,B) = -\mho(\nabla_A \xi, B) - \mho(A, \nabla_B \xi) - (2\lambda - \beta r)\mho(A,B). \tag{2.2.9}$$

Using (2.2.4) in (2.2.9), we get

$$2\alpha\varsigma(A,B) = -\mho(f(A - \eta(A)\xi), B) - (2\lambda - \beta r)\mho(A,B) - \mho(A, f(B - \eta(B)\xi). \tag{2.2.10}$$

Simplifying (2.2.10), we have

$$\varsigma(A,B) = \frac{1}{\alpha}\left[\left(\frac{\beta r}{2} - f - \lambda\right)\mho(A,B) + f\eta(A)\eta(B)\right]. \tag{2.2.11}$$

Putting $B = \xi$, the equation (2.2.11) yields

$$\varsigma(A,\xi) = \frac{1}{\alpha}\left[\left(\frac{\beta r}{2} - f - \lambda\right)\mho(A,\xi) + f\eta(A)\eta(\xi)\right]. \tag{2.2.12}$$

Using the value $\eta(\xi) = 1$, (2.2.12) reduces to

$$\varsigma(A,\xi) = \frac{1}{\alpha}\left(\frac{\beta r}{2} - \lambda\right)\eta(A). \tag{2.2.13}$$

We know that a paracontact manifold with dimension (2n+1) satisfy the following properties:

$$\varsigma(A,\xi) = -\dim(\mathbb{M} - 1)\,\eta(A).$$

$$\varsigma(A,\xi) = -2n\eta(A). \tag{2.2.14}$$

Comparing (2.2.12) and (2.2.14), we get

$$\lambda = \frac{\beta r+4n\alpha}{2}. \tag{2.2.15}$$

Thus, the following can be said:

Theorem 2.2.1: In an f-kenmotsu manifold, Yamabe-Ricci soliton is

• expanding if , $\lambda = \frac{\beta r+4n\alpha}{2} > 0.$

• steady if $\lambda = \frac{\beta r+4n\alpha}{2} = 0.$ and

• shrinking $\lambda = \frac{\beta r+4n\alpha}{2} < 0.$

Also, (2.2.7) and (2.2.13) yield

$$\frac{1}{\alpha}\left(\frac{\beta r}{2} - \lambda\right) = -2(f^2 + f'). \tag{2.2.16}$$

which implies

$$f^2 = \frac{2\lambda-\beta r-4\alpha f\prime}{4\alpha}. \tag{2.2.17}$$

Hence, we have $f = \frac{1}{2}\sqrt{\frac{2\lambda-\beta r-4\alpha f\prime}{\alpha}}$. (2.2.18)

From above, we can state:

Theorem 2.2.2: In a Ricci Yamabe soliton equipped with f- Kenmotsu manifold, f satisfies the relation $f = \frac{1}{2}\sqrt{\frac{2\lambda-\beta r-4\alpha f\prime}{\alpha}}$.

2.2.3. Ricci Yamabe soliton on $\boldsymbol{f}$-Kenmotsu manifold admitting torse forming vector field

From equation (2.2.4), we get

$$\mho(\nabla_A \xi, \xi) = \mho(fA + \eta(A)\xi, \xi), \tag{2.2.19}$$

(2.2.19) also gives us

$$\mho(\nabla_A \xi, \xi) = (f+1)\eta(A). \tag{2.2.20}$$

When we swap out A for ξ in (2.2.20), we obtain

$$\mho\left(\nabla_\xi \xi, \xi\right) = (f+1)\eta(\xi). \tag{2.2.21}$$

which implies $\nabla_\xi \xi = (f+1)\xi$.

Hence, we have

Theorem 2.2.4 : In a Ricci Yamabe soliton equipped with f- Kenmotsu manifold vector field which is forse forming, soliton vector field ξ is generalized geodesic vector field.

2.3 $\boldsymbol{\eta}$-Einstein Ricci Yamabe soliton in $\boldsymbol{f}$-kenmotsu manifold

In equation (2.2.11) arranging $A = B = \xi$, we have

$$2\alpha\varsigma(\xi,\xi) = (-2\lambda - 2f + \beta r)\mho(\xi,\xi) + 2f\eta(\xi)\eta(\xi). \tag{2.3.1}$$

Using (2.2.14) in (2.2.22), we get

$$4n\alpha = 2\lambda - \beta r. \tag{2.3.2}$$

Using the (2.2.23), (2.2.11) becomes

$$\varsigma(A,B) = \frac{1}{\alpha}[\lambda - 2n\alpha - f - \lambda)\mho(A,B) + \mho(A,B) + f\eta(A)\eta(B)]. \tag{2.3.3}$$

Simplifying (2.2.24), we have

$$\varsigma(A,B) = -\frac{1}{\alpha}[(2n\alpha - f)\mho(A,B) + f\eta(A)\eta(B)], \tag{2.3.4}$$

This is a η-Einstein manifold's form.

Therefore, we have the statement:

Theorem 2.3.1: A Ricci Yamabe soliton stating f-Kenmotsu manifold satisfies condition of η-Einstein manifold.

2.4 Applications of Ricci Yamabe soliton in f-Kenmotsu manifolds:

(i) Physical applications:

- **Gravitational Physics:** Ricci-Yamabe solitons in f-Kenmotsu manifolds can be used to model gravitational fields in general relativity.
- **Cosmology:** These solitons can be used to study the evolution of the universe, especially when it comes to Kenmotsu manifolds.
- **String theory:** f-Kenmotsu manifolds admitting Ricci Yamabe solitons can be used to study the behavior of strings in curved spacetime.

(ii) Mathematical applications:

- **Geometric analysis:** Ricci-Yamabe solitons in f-Kenmotsu manifolds can be used to study some geometric and topological properties of manifolds.
- **Partial differential equation:** The behaviour of partial differential equation solutions in f-Kenmotsu manifolds can be investigated using these solitons.
- **Differential geometry:** Ricci-Yamabe soliton in f-Kenmotsu manifolds can be used to study the properties of curvature and torsion on these manifolds.

(iii) Other applications:

- **Image processing:** Ricci Yamabe solitons in f-Kenmotsu manifolds can be used to develop new algorithms for image processing and computer vision.
- **Machine learning:** These solitons can be used to develop new models for machine learning and artificial intelligence.
- **Network analysis:** Ricci-Yamabe solitons in f-Kenmotsu manifolds can be used to study the properties of complex networks.

These are just a few examples of the potential applications of Ricci-Yamabe solitons in f-Kenmotsu manifolds. As the subject develops, new applications are probably going to appear, and the study of these solitons is an active area of research.

2.5 Significances

Here are some potential significances of Yamabe-Ricci solitons with manifold, f-Kenmotsu:

(i) Geometric significances

- **Generalization of Ricci solitons:** A more extensive application of thought of Yamabe Ricci soliton was provided by Yamabe-Ricci soliton in f-Kenmotsu manifold.
- **New insights into curvature:** New information about the behaviour of curvature on f-Kenmotsu manifold can be gained by studying Ricci-Yamabe solitons in these manifolds.
- **Relationship with other geometric structures:** Ricci-Yamabe solitons in f-Kenmotsu manifolds are related to other geometric structures such as Kenmotsu structures and f-structures.

(ii) Physical significances:

- **Gravitational Physics:** Ricci-Yamabe solitons in f-Kenmotsu manifolds can be used to model gravitational fields in general relativity.
- **Cosmology:** These solitons can be used to study the evolution of the universe particularly with reference to Kenmotsu manifolds.
- **String theory:** Ricci-Yamabe solitons in f-Kenmotsu manifolds can be used to study the behavior of strings in curved spacetime.

(iii) Mathematical significances:

- **New techniques and methods:** New approaches and procedures that can be used in other branches of physics and mathematics must be developed in order to explore Ricci-Yamabe solitons in f-Kenmotsu manifold.
- **Advances in geometric analysis:** The investigation of Yamabe-Ricci soliton with f-Kenmotsu manifold promotes geometric evaluation, especially in the areas of geometric flows and curvature.
- **Connections of other areas of mathematics:** There are links between investigation of Yamabe-Ricci soliton in f-Kenmotsu manifold with some other branches of mathematics, including differential geometry and partial differential equations.

2.6 Conclusion

We have examined η-Ricci solitons in f-Kenmotsu manifold in chapter 2nd. This category of manifolds is a generalization of the concept of Kenmotsu manifold. We have demonstrated η-Ricci solitons in f-Kenmotsu manifolds possess several interesting properties, including the fact that they are steady, shrinking or expanding solitons.

Our results provide new insights into the geometry and topology of -Kenmotsu manifold and demonstrate the significance of η-Ricci solitons in understanding the behavior of these manifolds. Additionally, our study offers a fresh viewpoint on the function of the η-function in geometric analysis and emphasizes the significance of that η-Ricci soliton with context of Ricci soliton.

Studying the η-Ricci solitons in different kinds of manifolds could be one of the future research avenues. Additionally, it would be interesting to investigate the relationship between that η-Ricci soliton and other geometric structures such as killing spinors of twistor spinors.

Overall our study demonstrate the richness and complexity of the geometry and topology of that f-Kenmotsu manifolds and highlights the importance of that η-Ricci solitons in understanding these manifolds.

CHAPTER 3

η-Ricci soliton on Lorentzian Concircular structure manifold

This chapter focuses on the investigation of η-Ricci solitons with Lorentzian concircular structure manifolds. In this chapter, we will explore a few definitions, properties, and behavior of these solitons as well as the relationship with other geometric structures. Additionally we will discuss recent research developments and open problems in this field. The circumstances behind -Ricci as well as η-Ricci flat$(LCS)_n$manifolds has acquired to be shrinking, expanding, and steady. Additionally the parallel conditions have been developed. This chapter gives a thorough description of these conditions.

We have discussed the concircular flat η-Ricci soliton as well as the requirements for a concircular vector field related to η-Ricci soliton being a steady soliton. Any plane segment in a manifold has a constant sectional curvature, as has been demonstrated. Additionally, the Einstein manifold shown to be η-Ricci semisymmetric manifold.

3.1 Introduction

An η-Ricci soliton is the crucial concept in studying the geometric structures in manifolds. A.K. Mandal [53] in 2014, investigated Ricci solitons in LP-Sasakian as well as gradient Ricci soliton in manifold. The generalised Ricci soliton, or η-Ricci soliton in Lorentzian para-Sasakian manifold along with para-Sasakian manifolds, was introduced by A.M. Blaga in 2015 [6], [7]. Hui and Chakraborty examined a few manifold features [32, 33]. Extended generalised phi-recurrent manifolds were studied by D.L. Suthar,S.K. Yadav & M. Hailu [95], [96], and [97].

S. Deshmukh [15], [16] and B.Y. Chen, working on the compact shrinking Ricci soliton geometry, came up with a few requirements to be shrinking in 2014. Bejan and Crasmareanu [3] investigated quasi-constant curvature Ricci solitons.

Ingalahalli and Bagewadi [2] produced idea of Ricci soliton on α-Sasakian manifolds in 2013 during their working on geometry of Ricci solitons in manifolds.

They also derived some results of Ricci solitons associated with α-Sasakian manifolds [37]. Later, Ricci soliton in contact with Kenmotsu manifolds was investigated by Nagaraja and Premalatha [57]. Additionally, M.M. Tripathi [86] contributed to the research of the Ricci soliton along with contact metric manifolds. In Riemannian manifold, Ricci flow was defined by Hamilton [29], [30] as

$$\frac{\partial \mho}{\partial t} = -2Ric(\mho), \tag{3.1.1}$$

Ricci soliton basically triplet$(\mho, \lambda, \xi)$ in association with Riemannian manifold $(\mathbb{M}, \mho)$ satisfying:

$$(L_\xi \mho)(B, A) + 2\varsigma(B, A) + 2\lambda\mho(B, A) = 0, \tag{3.1.2}$$

where, a Ricci tensor is denoted by ς, L_ξ along a vector field ξ is used for Lie derivative and for a Riemannian metric $\mho$, $\lambda \in \mathbb{R}$ on $\mathbb{M}$ for a pair of vector fields along with A & B.

An η-Ricci soliton has been generalization of Ricci soliton firstly presented by Kimura and Cho [17] also defined by

$$(L_\xi \mho)(A, B) + 2\varsigma(A, B) + 2\lambda(A, B) + 2\mu\eta(A)\eta(B) = 0, \tag{3.1.3}$$

For $\mu = 0$, the reduction of η-Ricci soliton to Ricci soliton occurs.

Structure of this chapter is as follows: Definition and basic properties of $(LCS)_n$ manifold is discussed in introduction in section 1. Section 2 explores the geometric and topological implications of Lorentzian concircular structure manifold in association with Ricci soliton. Section 3 discusses some open problems and applications of η-Ricci soliton in $(LCS)_n$ manifold. Finally, section 4 presents recent research developments in the field.

Definition 3.1.1 If a $(LCS)_n$ manifold's Ricci tensor ς has a non-vanishing constant of form (2.1), it is considered an Einstein manifold and satisfies

$\varsigma(A, B) = a\mho(A, B)$,

and it turns into Ricci flat manifold when $a = 0$. .

Definition 3.1.2 If a manifold's Ricci tensor ς has the following form, it is considered a η-Einstein manifold:

$$\varsigma(A,B) = a\mho(A,B) + b\eta(A)\eta(B), \tag{3.1.4}$$

where a and b are non-vanishing constants.

Definition 3.1.3 If a vector field fulfills relation given below, is referred to as concircular vector field:

$$(\nabla_A\eta)B = \alpha\{\mho(A,B) + \eta(A)\eta(B)\}, \tag{3.1.5}$$

with η being a 1-form and α being a scalar function that is non zero.

If we consider ξ to be vector field with concircular features, having

$\nabla_A\xi = \alpha\{A + \eta(A)\xi\}$, for pair of vector fields A and B. (3.1.6)

$$\nabla_A\alpha = d\alpha = A\alpha = \rho\eta(A), \tag{3.1.7}$$

ρ is a scalar function given by $\rho = -\xi\alpha$.

The following relationships hold in Lorentzian concircular structure manifold [32], [33], [59], [97] briefly manifold $(\mathbb{M}^n, \mho)$ (n>2):

$$\eta(\xi) = -1, \quad \phi\xi = 0 \text{ and } \quad \mho(\phi A', \phi B') = \mho(A', B') - \eta(A')\eta(B') \tag{3.1.8}$$

$$\phi^2 A' = A' + \eta(A')\xi, \tag{3.1.9}$$

$$\varsigma(A',\xi) = (n-1)(\alpha^2 - \rho)\eta(A'), \tag{3.1.10}$$

$$R(A',B')\xi = (\alpha^2 - \rho)(\eta(B')A' - \eta(A')B', \tag{3.1.11}$$

$$R(\xi,B')W' = (\alpha^2 - \rho)[-\eta(W')B' + \mho(B',W')\xi], \tag{3.1.12}$$

$$(\nabla_{A'}\phi)B' = \{R(A',B') + 2\eta(A')\eta(B')\xi + \eta(B')A'\}, \tag{3.1.13}$$

$$R(A',B')W' = \phi R(A',B')W' + (\alpha^2 - \rho)\{\mho(B',W')\eta(A') - \mho(A',W')\eta(B')\xi. \tag{3.1.14}$$

$$\varsigma(\phi A', \phi B') = \varsigma(A',B') + (n-1)(\alpha^2 - \rho)\eta(A')\eta(B'), \tag{3.1.15}$$

where each vector field A' B', W' has a corresponding Ricci and curvature tensors denoted by R and ς . The following equation is satisfied by Lie derivative.

$$(L_\xi \mho)(A,B) = \mho(\nabla_A \xi, B) + \mho(A, \nabla_B \xi). \qquad (3.1.16)$$

3.2 η-Ricci soliton associated with $(LCS)_n$ manifold

When we use (3.1.6) in (3.1.16), we obtain

$$\frac{1}{2}(L_\xi \mho)(A,B) = \alpha[\mho(A,B) + \eta(A)\eta(B)]. \qquad (3.2.1)$$

Using (3.2.1) in (3.1.3), we get

$$\varsigma(A,B) = -(\alpha+\lambda)\mho(A,B) - (\alpha+\mu)\eta(A)(B), \qquad (3.2.2)$$

From(3.2.2), we may say that:

Theorem 3.2.1η-Ricci soliton has been also η-Einstein manifold in an $(LCS)_n$ manifold.

Proof: When in (3.2.2) we enter $B = \xi$, we are obtaining

$$\varsigma(A,\xi) = (\mu - \lambda)\eta(A) \qquad (3.2.3)$$

When we contract (3.2.2), we obtain

$$r = -(\alpha+\lambda)n + (\alpha+\mu), \qquad (3.2.4)$$

Corollary 3.2.2 This connection (3.2.3) and (3.2.4) are satisfied by a η-Ricci soliton equipped with $(LCS)_n$ manifold.

Theorem 3.2.3 The Ricci soliton is steady when the concircular vector field is geodesic equipped with manifold $(LCS)_n$.

Proof: In (3.1.3), using (3.2.2), we obtain

$$\mho(\nabla_A \xi, B) + \mho(A, \nabla_B \xi) - 2(\alpha-\lambda)\mho(A,B) - 2(\alpha+\mu-\mu)\eta(A)\eta(B) = 0, \qquad (3.2.5)$$

When in (3.2.5) we choose $A = B = \xi$, we obtained

$$\mho(\nabla_\xi \xi, \xi) + \mho(\xi, \nabla_\xi \xi) - 2\alpha\eta(\xi)\eta(\xi) - 2(\alpha - \lambda)\mho(\xi, \xi) = 0, \tag{3.2.6}$$

$$2\mho(\nabla_\xi \xi, \xi) - 2\alpha + 2(\alpha - \lambda)\lambda = 0, \tag{3.2.7}$$

which suggests

$$(\nabla_\xi \xi, \xi) = \lambda, \tag{3.2.8}$$

Given that $\nabla_\xi \xi = 0$ It suggests that λ=0 consequently, the outcome is demonstrated.

3.3 Concircular curvature tensor in $\boldsymbol{\eta}$-Ricci soliton [59]

An $(LCS)_n$ manifold's concircular curvature tensor for vector fields A', B' and W' is defined by

$$\mathcal{L}(A'B')W = R(A', B')\,W - \frac{r}{n(n-1)}[\mho(B', W')A' - \mho\,(A',\ W')B']. \tag{3.3.1}$$

Theorem 3.3.1: For flat concircular η-Ricci soliton, $(LCS)_n$ manifold is

- shrinking iff $\mu < \frac{r}{n}$,
- steady iff $\mu = \frac{r}{n}$,
- expanding iff $\mu > \frac{r}{n}$.

Proof: Taking into account equation (3.3.1), we arrive at

$$\mathcal{L}(A', B', W', D') = R(A', B', W', D') - \frac{r}{n(n-1)}[\mho(B', W')\varpi(A', D') - \mho(A', W')\mho(B', D')], \tag{3.3.2}$$

where, $\mathcal{L}(A', B', W', D') = \mho(\mathcal{L}(A', B')W', D')$ and $A', B', W', D' \in \chi(\mathbb{M})$.

For vanishing concircular curvature tensor, equation(3.3.2)yields

$$R(A', B', W', D') = \frac{r}{n(n-1)}[\mho(B', W')\mho(A', D') - \mho(A', W')\mho(B', D')], \tag{3.3.3}$$

Contracting (3.3.3) by taking $A' = D' = e_i$, we get

$$\varsigma(A',\ B') = \frac{r}{n\,(n-1)}[\mho(B', W')n - \mho(e_i, W')\ \ \mho(B', e_i)], \tag{3.3.4}$$

$$\varsigma(A', B') = \frac{r}{n(n-1)}[\mho(B', W')n - \mho(B', W')], \tag{3.3.5}$$

Comparing (3.2.2) and (3.3.5), we get

$$\left[-\alpha - \lambda - \frac{r}{n}\right]\mho(B', W') = [\alpha + \mu]\eta(W')\eta(B'). \tag{3.3.6}$$

By choosing $B' = W' = \xi$ in (3.3.6), we are obtaining following value

$$\lambda = \mu - \frac{r}{n}. \tag{3.3.7}$$

Consequently, η- Ricci soliton with $(LCS)_n$ manifold for the concircular curvature tensor is

- Expanding iff $\mu > \frac{r}{n}$,
- Steady iff $\mu = \frac{r}{n}$,
- Shrinking iff $\mu < \frac{r}{n}$.

Theorem 3.3.2: When $\mu \neq 0$, the soliton in a η-Ricci $(LCS)_n$ flat manifold is not steady.

Proof: When in equation (3.1.3), we use definition (3.1.4) to obtain

$$\mho(\nabla_A \xi, B) + \mho(A, \nabla_B \xi) + 2\mho(A, B) + 2\mu\eta(A)\eta(B) = 0, \tag{3.3.8}$$

$$(\alpha + \lambda)\mho(A, B) + (\alpha + \mu)\eta(A)\eta(B) = 0. \tag{3.3.9}$$

Taking $A = B = \xi$ in (3.3.9), we get

$$\lambda = \mu. \tag{3.3.10}$$

Therefore, if $\mu \neq 0$, η-Ricci $(LCS)_n$ flat manifold is non-steady .

Theorem 3.3.3: The Ricci tensor ς in a η-Ricci flat Einstein manifold has the following form:

$$\varsigma(A', B') = \frac{3r - (n-1)(\alpha^2 - \rho) - 4\alpha n}{n}\mho(A', B') + (2r - (n-1)(\alpha^2 - \rho) - 4\alpha\, n)\eta(A')\eta(B').$$

Proof: In $(LCS)_n$ manifold,

$\varsigma(A', B') = [(n-1)\ (\alpha^{2} - \rho) - r\ +\ 2\alpha n]\mho(A', B') + (\ r\ -\ 2\alpha n)\eta(\ A')\eta(B')$.
(3.3.11)

On contracting (3.2.4), we get

$$r = na - b. \tag{3.3.12}$$

Taking $A' = B' = \xi$ in (3.1.5) and comparing the result with (3.3.11), we get

$$b - a = 2r - 4\alpha n - (n-1)(\alpha^2 - \rho). \tag{3.3.13}$$

Solving (3.3.12) and (3.3.13) for a and b, we get

$a = \frac{3r-(n-1)(\alpha^2-\rho)-4\alpha n}{n}$ and

$b = 2r - 4\alpha n - (n-1)(\alpha^2 - \rho)$.

Thus, the outcome is demonstrated.

Theorem 3.3.4: Any plane section's sectional curvature in the $(LCS)_n$ manifold exhibiting metric structure $(\phi,\ \xi,\ \eta,\ \mho)$ equals $-\alpha^2$.

Proof: Using Riemannian curvature tensor, we have

$$R(\xi, A')\xi \ \ = \ \ \nabla_\xi \nabla_{A'}\xi - \ \nabla_{A'}\nabla_\xi \xi - \nabla_{[\ \xi, A']}\xi.$$

$$R(\xi, A')\xi \ \ = \ \ \alpha\nabla_{\ \xi}(\phi A') - \ \alpha\phi\left(\nabla_\xi A'\right) - \alpha\phi\nabla_{A'}\xi),$$

which implies

$$R(\xi, A')\xi = -\alpha^2\phi^2 A'. \tag{3.3.14}$$

Using (3.1.9) in (3.3.14), we get

$R(\xi, A')\xi = -\alpha^2\left(A' + \mho(A', \xi)\right)$,

If A' is orthogonal to ξ, then $R(\xi, A')\xi = -\alpha^2 A'$ and

$\varpi(R(\xi, A')\xi, A') = -\alpha^2$.

Hence the result is proved.

3.4 Applications

Here are some potential applications of η-Ricci solitons in Lorentzian concircular structure manifolds:

(i) Information theory:

- Data compression: η-Ricci solitons can be used to develop new algorithms for data compression.
- Error-correcting codes: These solitons can be used to develop new error correcting codes.

(ii) Computer science

- Computer vision: η-Ricci solitons can be used to develop new algorithms for computer vision.
- Robotics: These solitons can be used to develop new algorithms for robotics.

(iii) Engineering:

- Signal processing: η-Ricci solitons can be used to develop new algorithm for medical imaging.
- Protein folding: These solitons can be used to study protein folding and misfolding.

(iv) Finance and Economics:

- Risk management: η-Ricci solitons can be used to develop new algorithms for risk management.
- Portfolio optimization: These solitons can be used to develop new algorithms for portfolio optimization.

(v) Materials science:

- Materials modeling: η-Ricci solitons can be used to model the behavior of materials under different conditions.
- Non structure designs: These solitons can be used to design and optimize nanostructures.

(vi) Astronomy and Astrophysics:

- Cosmological modeling: η-Ricci solitons is capable of modelling the universe's evolution.
- Black hole Physics: These solitons can be used to study the behavior of ecosystems.

(vii) Environmental science:

- Climate modeling: η-Ricci solitons can be used to model the behavior of the climate system.
- Ecosystem modeling: These solitons can be used to model the behavior of ecosystem.

(viii) Neuroscience:

- Brain modeling: η-Ricci solitons can be used to model the behavior of brain.
- Neural network analysis: These solitons can be used to analyze and optimize neural networks.

3.5 Significances

(i) η-Ricci solitons in Lorentzian concircular structure manifolds expand on the idea of Ricci Soliton in a broader context.

(ii) The study of η-Ricci solitons in $(LCS)_n$ manifold provides new insights into the behavior of curvature on these manifolds.

(iii) η-Ricci solitons in $(LCS)_n$ manifold can be used to study differential geometry including the properties of curvature and torsion.

(iv) These solitons can be used to study partial differential equations including the Ricci flow equation.

(v) η-Ricci solitons in $(LCS)_n$ manifolds provide a framework for unifying geometry and physics.

(vi) These solitons offer insights on traditional physics and geometry issues, like universe evolution as well as black holes behaviour.

3.6 Conclusion

We have examined η-Ricci solitons in $(LCS)_n$ manifold in this chapter. We have shown that these solitons possess several interesting properties including the fact that they are shrinking, steady or expanding solitons. Our results provide new insights into the geometry and physics of $(LCS)_n$ manifold and demonstrate the significance of η-Ricci solitons to study the behavior of $(LCS)_n$ manifold.

Potential avenues for further research could involve examining an η-Ricci solitons in many kinds of manifolds, like contact metric manifolds or Sasakian manifolds. Our research emphasises how crucial η-Ricci solitons are to comprehending the geometric and physical characteristics of the $(LCS)_n$ manifold. Our research further emphasises the significance of the η-function in relation to the Ricci flatness of Einstein manifolds.

CHAPTER 4

Cotton tensor on para Kenmotsu three dimensional manifold admitting η-Ricci solitons

In this chapter, the aim is to examine certain characteristics of the Cotton tensor within para-Kenmotsu manifold which is three dimensional that supports η-Ricci solitons. Furthermore, this chapter also presents the concept of Cotton pseudo-symmetric manifolds. Results regarding Cotton flat para-Kenmotsu manifolds that support η-Ricci solitons are also presented. Expanding on this groundwork, this chapter explores the absence of specific geometric characteristics of the Cotton tensor associated with para-Kenmotsu 3-dimensional manifold. This chapter provides an extensive analysis of the present status of Codazzi type Cotton tensor, along with recent findings and insights that advance the progress of this research domain. Additionally, the paper explores various applications of the Cotton tensor within the context of fluid space time.

4.1 Introduction

First introduced by Sato in 1976, the idea of para-contact manifold. Subsequently, Takahashi defined Almost contact manifolds that are associated with a pseudo-Riemannian metric. In 1985, consept of almost para-contact structure that has been demonstrated by Kaneyuki and Williams on pseudo-Riemannian manifolds of dimension $2n + 1$. Concept of para-Kenmotsu manifolds first presented by Welyczko. Subsequent research on para-Kenmotsu manifolds and their special variants has been conducted by Blaga. Additionally, the study of $(LCS)^{2n+1}$ manifolds was undertaken by S.K. Yadav, D.L. Suthar, and others. In 1982 year, the Ricci flow was introduced by Hamilton, which has proven as valuable technique for simplifying the structure of manifolds.

$$\frac{\partial \mho}{\partial t} = -2\varsigma, \tag{4.1.1}$$

In this context, $\mho$ represents a Riemannian metric, Ricci curvature tensor is represented by ς, time is signified by t. A Ricci soliton serves as one of the natural extension of an Einstein metric [2], [3], [37].

A structure ($\mho$,V, λ) of Ricci soliton is characterized in manifold($\mathbb{M}$, $\mho$), which is pseudo-Riemannian, by

$$(\mathcal{L}_V\mho)(A,B) + 2\varsigma(A,B) + 2\lambda\mho(A,B) = 0, \tag{4.1.2}$$

where, ς represents the Ricci tensor associated with $\mho$, a Riemannian metric, the Lie derivative of $\mho$ is indicated by $(\mathcal{L}_V\mho)$ along with the V vector field, a constant λ, and A and B are some vector fields in the manifold $\mathbb{M}$. Ricci soliton has been classified as steady, shrinking, or expanding based on whether λ = zero, negative, or positive, correspondingly. Its links to string theory have recently attracted the attention of theoretical physicists. A Ricci soliton is recognized as the self-similar soliton within the context of the Ricci flow. In various contexts on Kahler manifolds, α-Sasakian manifolds, in contact and Lorentzian manifolds, in Sasakian manifolds, in LP-Sasakian manifolds, k-contact manifolds, in f-Kenmotsu manifolds, in Kenmotsu manifolds, in Lorentzian concircular structure manifolds, in para-Kenmotsu manifolds, etc. Ricci solitons were widely evaluated in framework of pseudo-Riemannian geometry and have been investigated.

U. C. De obtained findings related to ϕ-recurrent Kenmotsu manifolds [20]. C.S. Bagewadi, S. R. Ashoka, and G. Ingalahalli contributed to the derivation of specific results [1] concerning Ricci solitons in contact with α-Sasakian manifolds. Additionally, M. Crasmareanu & C. L. Bejan [3] established a connection between Ricci solitons and quasi-constant curvature.

A broader concept of Ricci soliton i.e an η-Ricci soliton, indicated by Kimura and Cho [17], is defined as a structure $(\mho, V, \lambda, \mu)$

$$(\mathcal{L}_V\mho)(A,B) + 2\varsigma(A,B) + 2\lambda\mho(A,B) + 2\mu\eta(A)\eta(B) = 0, \tag{4.1.3}$$

where, V on manifold on $\mathbb{M}$, is used for a vector field, constants are λ and μ and Riemannian (or pseudo-Riemannian) metric is denoted by symbol $\mho$.

Numerous generalizations of Ricci solitons exist, including almost Yamabe Ricci soliton, h-almost Yamabe Ricci soliton, Yamabe Ricci soliton, ⁎-Ricci solitons, among others. A novel type of soliton, referred to as the h-almost Yamabe Ricci soliton, expands upon concept of almost Yamabe Ricci soliton, was presented by A. Sardar and U. C. De. Additionally, and C. R. Premalatha and H. G. Nagaraja have presented findings on Ricci solitons

Related to para Kenmotsu manifolds, which pave the way for research scholars to explore new topics and derive further results in various other manifolds.

Idea of almost Ricci soliton is recently extended to an h-almost Ricci soliton in complete Riemannian manifold by Gomes et al. [27]. Findings on Ricci Solitons are recently derived in Contact Geometry metric manifolds by Pankaj Pandey and Kamakshi Sharma [63], [64] and [65] in LP Sasakian manifold and some important results in generalized form of Z-Ricci

soliton, on weakly cyclic Z-symmetric generalized manifolds and on Lorentzian Complex Space Form are also presented by Pankaj Pandey, BB Chaturvedi [61], [62] that form the basis for deriving new generalizations of Ricci soliton in different types of manifolds in Differential geometry.

A unique form of conformal invariant, known as the cotton tensor in differential geometry, can be characterized by the condition that a Weyl's conformal curvature tensor is zero in pseudo-Riemannian manifold of three-dimensions. A Cotton tensor$\mathfrak{C}$, that's tensor of type(1,2) described by [24], [52]

$$\mathfrak{C}(A, B) = (\nabla_A Q)B - (\nabla_B Q)A - \frac{1}{4}\{(Ar)B - (Br)A\}, \tag{4.1.4}$$

for smooth vector fields A and B.

Eisenhart proposed that in pseudo-Riemannian manifold of three-dimensions, Cotton tensor becomes zero conformally flat metric. In recent years, numerous authors have found the Cotton tensor in three-dimensional spaces to be a compelling subject, particularly because of its connection with energy-momentum tensor of matter and Ricci tensor within Einstein's equations. Pradip Majhi and Debabrata Kar [52] applied some elementary results to find some outcomes derived from Cotton tensors that accommodate η-Ricci solitons within Sasakian 3-manifolds.

Definition 4.1.1 Consider $(\mathbb{M}^n, \mho)$ as a smooth manifold of dimension three that possesses almost para-contact metric structure characterized by components $(\xi, \phi, \mho, \eta)$. Here, ϕ represents type (1,1) tensor , a vector field is denoted as ξOne-form is represented by η and symbol $\mho$ is a pseudo-Riemannian metric that satisfies:

$$\phi^2 A = A - \eta(A)\xi, \qquad \eta(\xi) = 1, \qquad \phi\xi = 0, \qquad \eta(\phi A) = 0, \tag{4.1.5}$$

$$\mho(\phi A, \phi B) = -\mho(A, B) + \eta(A)\eta(B), \tag{4.1.6}$$

$$\mho(A,\xi) = \eta(A), \tag{4.1.7}$$

$$\mho(\phi A, B) = -\mho(A, \phi B) \tag{4.1.8}$$

For a pair of vector fields $A, B \in TM^n$. Then this manifold denoted by $(\mathbb{M}^n, \mho)$ described as Para-contact metric manifold.

When following equation is satisfied by almost para-contact metric manifold

$$(\nabla_A \phi)B \;=\; \mho(A,B)\xi - \eta(B)\phi A,$$
(4.1.9)

for every vector field $A, B \in T\mathbb{M}^n$, then the structure $(\mathbb{M}^n, \mho)$ is referred to as almost para-Kenmotsu manifold. Standard nearly para-Kenmotsu manifold, known as para-Kenmotsu manifold. A framework of para-Kenmotsu with dimensions three normal structures has been presented by Welyczko [94]. Results of preceding equation helps us to obtain

$$\nabla_A \xi = A - \eta(A)\xi \;\text{ and}$$
$$(\nabla_A \eta)B = \mho(A,B) - \eta(A)\eta(B) \tag{4.1.10}$$

In para-Kenmotsu manifold [6] following relations holds:

$$R(A', B', C', W') = \mho(A', C')\mho(B', W') - \mho(B', C')\mho(A', W'),$$

$$R(\xi, A')B' = \{-\mho(A', B')\xi + \eta(B')A'\},$$

$$R(\xi, A')\xi = \{-\eta(B')\xi + B'\},$$

$$\varsigma(A', B') = -(n-1)\mho(A', B'),$$

$$\varsigma(\xi, \xi) = 1 - n,$$

$QA' = -(n-1)A'$, for each vector field A', B'.

Definition 4.1.2. Defining curvature tensor R of Riemannian manifold in 3-dimensions by

$$R(A,B)C = [\varsigma(B,C)A - \varsigma(A,C)B + \mho(B,C)QA - \mho(A,C)QB] - \frac{r}{2}[\mho(B,C)A - \mho(A,C)B],$$

where symbol Q is used for Ricci operator and is defined by

A para-Kenmotsu three dimensional manifold's Ricci tensor is provided by

$$\varsigma(A, B) = \frac{1}{2}[(2+r)\mho(A, B) - (6+r)\eta(A)\eta(B). \qquad (4.1.11)$$

Definition 4.1.3 Within the context of Riemannian n-dimensional manifold, a concircular curvature tensor, categorized as type(1,3), is defined as follows:

$$\mathcal{Z}(A', B')C' = R(A', B')C' - \frac{r}{n(n-1)}[\mho(B', C')A' - \mho(A', C')B']. \qquad (4.1.12)$$

In the realm of three-dimensional Riemannian manifolds, the formulation of the concircular curvature tensor is represented by:

$$\mathcal{Z}(A', B')C' = R(A', B')C' - \frac{r}{6}[\mho(B', C')A' - \mho(A', C')B']. \qquad (4.1.13)$$

Definition4.1.4 A Riemannian curvature tensor is labeled as concircularly flat when the concircular curvature tensor is null. Considering $(\mathbb{M}, \mho)$, a Riemannian manifold along with its associated connection ∇ (Levi civita connection). If equation $\nabla R = 0$ is satisfied, then Riemannian manifold is identified as locally where R is denoting Riemannian curvature tensor in manifold$(\mathbb{M}, \mho)$. Moreover, on semi-Riemannian curvature, Riemannian metric manifold$(\mathbb{M}^n, \mho)$, where n is greater than or equal to 3, termed semisymmetric if it meets the criterion $R.R = 0$. Metric of semi-Riemannian curvature manifold$(\mathbb{M}^n, \mho), n \geq 3$ called semi symmetric if $R.R = 0$.

Semi-Riemannian n-dimensional manifold$(\mathbb{M}^n, \mho), n \geq 3$ termed as Ricci semi-symmetric, when it fulfills condition $R.\varsigma = 0$ on the manifold $\mathbb{M}$.

Definition 4.1.5 Another classification of vector field V as conformal killing vector field, when relationship given below is satisfied:

$$\mathcal{L}_V(A, B) = 2\varpi\mho(A, B)$$

Here, ϖ in the coordinates denotes a function, known as a conformal scalar. A conformal killing vector field V referred to as proper if ϖ isn't constant. In contrast, if the value of ϖ have been constant, V recognized as a homothetic vector field. When non-zero value is taken by the constant ϖ, V is then called a proper homothetic vector field. Furthermore, when ϖ is equal to zero in the equation provided, V is classified as a killing vector field.

4.2 Properties of cotton tensor

This section focuses on a skew-symmetric (1, 2) type tensor within the context of a para-Kenmotsu 3-manifold, known as Cotton tensor $\mathfrak{C}$, might be represented as

$$\mathfrak{C}(A,B) = (\nabla_A Q)B - (\nabla_B Q)A - \frac{1}{4}\{(Ar)B - (Br)A\}, \tag{4.2.1}$$

$$\mathfrak{C}(A,B) = (\mu - 1)[\mho(\phi A,\phi B)\xi + A\eta(B) - \eta(A)\eta(B)\xi] - \frac{1}{4}[(Ar)B - (Br)A]. \tag{4.2.2}$$

$$\mathfrak{C}(A,B) = (\mu - 1)\{-\mho(A,B) + A\eta(B)\} - \frac{1}{4}\{(Ar)B - (Br)A\}. \tag{4.2.3}$$

Furthermore, as seen below, Cotton tensor may additionally be written as tensor of (0, 3)type.

$$\mathfrak{C}(A,B,C) = \mho(\mathfrak{C}(A,B),C). \tag{4.2.4}$$

By virtue of (4.1.4) and (4.2.4), it follows that

$$\mathfrak{C}(A,B,C) = (\mu - 1)\{\mho(\phi A,\phi B)\eta(C) + \eta(B)\mho(A,C) - \eta(A)\eta(B)\mho(\xi,C)\}. \tag{4.2.5}$$

As a consequence of (4.2.4), we get

$$\mathfrak{C}(A,\xi) = (\mu - 1)(A - \eta(A)\xi - \frac{1}{4}(Ar)\xi. \tag{4.2.6}$$

$$\eta\big(\mathfrak{C}(A,B)\big) = (\mu - 1)[\mho(\phi A,\phi B) - \frac{1}{4}\{(Ar)\eta(B) - (Br)\eta(A)\}, \tag{4.2.7}$$

Taking $B = \xi$ in (4.2.7), we get the following equation

$$\eta\big(\mathfrak{C}(A,\xi)\big) = -\frac{1}{4}\ (Ar). \tag{4.2.8}$$

equation (4.2.2) also implies

$$\mathfrak{C}(\phi A,B) = (\mu - 1)\{\mho(A,\phi B)\xi + \phi A\eta(B)\} - \frac{1}{4}\ \big\{\big((\phi A)r\big)B - (Br)\phi A\big\}. \tag{4.2.9}$$

$$\eta(\mathfrak{C}(\phi A,B)) = (\mu - 1)\{\mho(A,\phi B)\eta(\xi) + \eta(\phi A)\eta(B)\} - \frac{1}{4}\ \big\{\big((\phi A)r\big)\eta(B) - (Br)\eta(\phi A)\big\}.$$

$$\eta\big(\mathfrak{C}(\phi A,B)\big) = (\mu - 1)\{\mho(A,\phi B)\} - \frac{1}{4}\ \{\big((\phi A)r\big)\eta(\phi B). \tag{4.2.10}$$

$$\eta\big(\mathfrak{C}(\phi A,\phi B)\big) = (\mu - 1)\mho(A,\phi B).$$

$$\eta\big(\mathfrak{C}(\phi A, B)\big) = (\mu - 1)\eta(\phi A)\eta(\xi) - \tfrac{1}{4}\,\{\big((\phi A)r\big)\eta(\xi) - (\xi r)\eta(\phi A).$$

$$\eta\big(\mathfrak{C}(\phi A, \xi)\big) = -\tfrac{1}{4}(\phi A)r. \tag{4.2.11}$$

4.3 Main Results

This section focuses on the analysis of cotton flat para-Kenmotsu manifolds that support η-Ricci solitons. As a result for vector fields A', B' and C', we have

$$\mathfrak{C}(A', B', C') = 0, \tag{4.3.1}$$

Equation (4.3.1) and (4.2.5) implies that

$$(\mu - 1)\{\mho(\phi A', \phi B')\eta(C') + \eta(B')\mho(A', C') - \eta(A')\eta(B')\eta(C')\}\tfrac{1}{4}\{(A'r)\mho(B', C') - (B'r)\mho(A', C')\} = 0. \tag{4.3.2}$$

Replacing C' by ξ in above equation, we get

$$(\mu - 1)\mho(\phi A', \phi\xi) = \tfrac{1}{4}\,\{(A'r)\eta(\xi) - (\xi r)\eta(A')\}. \tag{4.3.3}$$

$$0 = \tfrac{1}{4}(A'r),$$ which implies

$A'r = 0$. Hence, r becomes constant.

From (4.3.3) we get

$$(\mu - 1)\mho(\phi A', \phi B') = 0,$$

which implies $\mu = 1$.

Lemma 4.3.1 The structure of Ricci tensor in para-Kenmotsu manifold of three dimension that possesses an η-Ricci soliton is given as follows:

$$\varsigma(A, B) = -(1 + \lambda)\mho(A, B) - (\mu - 1)\eta(A)\eta(B). \tag{4.3.4}$$

As a consequence, above lemma implies

$\varsigma(A,\xi) = \varsigma(\xi,A) = -(\mu+\lambda)\eta(A)$.

Reference [21] addresses a para-contact manifold situated in a $(2n+1)$-dimensional framework. $\varsigma(A,\xi) = -(\dim(\mathbb{M})-1)\eta(A) = -2n\eta(A)$,

for 3-dimensional para-Kenmotsu manifold, we get

$$\lambda + \mu = 6\,. \tag{4.3.5}$$

using $\mu = 1$ in (4.3.5) we get

$$\lambda = 5. \tag{4.3.6}$$

Hence, following theorem can be stated:

Theorem 4.3.2: An η-Ricci soliton associated with cotton flat three dimensional para-Kenmotsu manifold is classified as expanding soliton.

Proof: From lemma (4.3.1), we obtain

$$\varsigma(e_i\,, e_i) = -(\mu-1)\eta(e_i)\eta(e_i) - (\lambda+1)\mho(e_i, e_i). \tag{4.3.7}$$

$$r = -3\lambda - 2 - \mu. \tag{4.3.8}$$

Using $\mu = 1$ in (4.3.6) and (4.3.7) we get

$$r = -18.$$

As a result, following theorem can be declared:

Theorem 4.3.3: Three dimensional para-Kenmotsu manifold which was Cotton flat also and supports an η-Ricci soliton, exhibits a scalar curvature of -18.

Theorem 4.3.4: A three dimensional cotton flat para-Kenmotsu manifold that accommodates

η-Ricci soliton is recognized as η-Einstein manifold.

Proof: In the context of a three dimensional para-Kenmotsu manifold, formulation of Ricci tensor has been widely acknowledged to be:

$$\varsigma(A,B) = \frac{1}{2}\{(r+2)\mho(A,B) - (r+6)\eta(A)\eta(B)\}. \tag{4.3.9}$$

As a consequence of above equation and by using $r = -18$, we get

$$\varsigma(A,B) = \frac{1}{2}\{-16\mho(A,B) + 12\eta(A)\eta(B)\},$$

which implies

$$\varsigma(A,B) = -8\mho(A,B) + 6\eta(A)\eta(B). \tag{4.3.10}$$

Hence the result is proved.

4.4 Codazzi type

Here, we analyze η-Ricci solitons within Cotton para-Kenmotsu manifolds which feature a Ricci tensor of Codazzi-type. Idea of Ricci tensor which has been cyclic parallel was proposed by Gray, along with Ricci tensor Codazzi-type.

Theorem 4.4.1 Cotton tensor in Codazzi type of η-Ricci soliton in para-Kenmotsu manifold satisfies following relation in three dimensions.

$$\mathfrak{C}(A,B) = -\frac{1}{4}\{(Ar)B - (Br)A\}.$$

Proof: A para-Kenmotsu manifold can defined as having a Codazzi kind of Ricci tensor, if its Ricci tensor ς, which is of type(0,2), is not equal to 0 that adheres to various stipulation

$$(\nabla_C\varsigma)(A,B) = (\nabla_A\varsigma)(B,C), \text{ for vector fields } A,B,C \text{ in } \mathbb{M}. \tag{4.4.1}$$

Computing covariant derivative of (4.3.4), we get

$$(\nabla_C\varsigma)(A,B) = (1-\mu)[(\nabla_C\eta)A\eta(B) - \eta(A)(\nabla_C\eta)B]. \tag{4.4.2}$$

using (4.1.10) in (4.4.2), we get

$$(\nabla_C\varsigma)(A,B) = (1-\mu)\{(\mho(A,C) - \eta(A)\eta(C))\eta(B) - \eta(A)\big(\mho(B,C) - \eta(B)\eta(C)\big)\}. \tag{4.4.3}$$

which on simplifying implies

$$(\nabla_C \varsigma)(A,B) = (1-\mu)\{(\mho(A,C)\eta(B) - \eta(A)\mho(B,C)\}. \qquad (4.4.4)$$

In view of (4.4.1), (4.4.4) reduces to

$$(\nabla_C \varsigma)(A,B) - (\nabla_A \varsigma)(B,C) = (1-\mu)\{2\mho(A,C)\eta(B) - \eta(A)\mho(B,C) - \eta(C)\mho(A,B)\}. \qquad (4.4.5)$$

Taking $C = \xi$ in (4.4.5) and using (4.4.1), we get

$$(1-\mu)\{2\mho(A,\xi)\eta(B) - \mho(B,\xi)\eta(A) - \mho(A,B)\eta(\xi)\} = 0. \qquad (4.4.6)$$

$$(1-\mu)\{-\mho(A,B) + \eta(A)\eta(B)\} = 0.$$

using (4.1.5) in (4.4.6), we get

$(1-\mu)\mho(\phi A, \phi B) = 0$, which implies $\mu = 1$. (4.4.7)

In view of (4.4.6) and (4.2.3), Cotton tensor reduces to

$$\mathfrak{C}(A,B) = -\frac{1}{4}\{(Ar)B - (Br)A\}. \qquad (4.4.8)$$

Corollary 4.4.1*:* Codazzi Tensor in Cotton flat η-Ricci soliton in para-Kenmotsu manifold if provided that it meets the following criteria

$$(Ar)B = (Br)A.$$

equation (4.3.3) and (4.4.7) also implies the following preposition:

Proposition 4.4.1: A para-Kenmotsu manifold admitting a η-Ricci soliton has a cotton tensor that vanishes iff it is of the Codazzi type.

Definition 4.4.1 A Para-Kenmotsu manifold is considered to exhibit a Ricci tensor if its Ricci tensor ς which is cyclic parallel, categorized as (0,2) type, is not zero and adheres to the below stipulation:

$$(\nabla_A \varsigma)(B,C) + (\nabla_B \varsigma)(C,A) + (\nabla_C \varsigma)(A,B) = 0,$$

for all vector fields A, B, C in $\mathbb{M}$.

Theorem 4.4.2 The condition of cyclic parallel Ricci tensor is satisfied by Cotton tensor in η-Ricci soliton admitting para Kenmotsu manifold.

Proof: Assume $(\mho, \mu, \xi, \lambda)$ is an η -Ricci soliton structure situated in n-dimensional para-Kenmotsu manifold with cyclic parallel Ricci tensor; consequently, definition (4.4.1) is satisfied. Taking the covariant derivative of (4.3.4) and employing (4.1.5) leads us to the following conclusion.

Taking covariant derivative of (4.3.4) and using (4.1.5), we get

$$(\nabla_C \varsigma)(A, B) = (1 - \mu)\{\eta(B)\mho(A, C) - \eta(A)\mho(B, C)\}. \tag{4.4.9}$$

Similarly, by calculating covariant derivative of(4.3.4) for A & B and using (4.1.5), we get

$$(\nabla_A \varsigma)(A, B) = (1 - \mu)\{\eta(C)\mho(A, B) - \eta(B)\mho(A, C)\} \tag{4.4.10}$$

$$(\nabla_B \varsigma)(A, B) = (1 - \mu)\{\eta(A)\mho(B, C) - \eta(C)\varpi(A, B)\}. \tag{4.4.11}$$

adding (4.4.9), (4.4.10) and (4.4.11), we get

$$(\nabla_A \varsigma)(B, C) + (\nabla_B \varsigma)(C, A) + (\nabla_C \varsigma)(A, B) = 0.$$

Thus, a para-Kenmotsu manifold admitting Cotton tensor in η-Ricci soliton is cyclic parallel Ricci tensor.

4.5 Applications

The Cotton tensor is used in the study of perfect fluid space-times in several ways, including:

(i) Cotton tensor is used in Cotton gravity to specify the energy- momentum tensor. The Bertotti-Robinson, Nariai and Stephani space-times are the examples of space-times that involve cotton gravity.

(ii) In three dimensions, the Cotton tensor is utilized to formulate a perfect fluid conformally flat solution of Einstein's field equations.

(iii) Cotton tensor serves as a tool to explore the connection between energy momentum and the Cotton tensor within the framework of Einstein's theory.

(iv) Cotton tensor finds its uses to modify Einstein’s equation resulting in the equation of Cotton gravity.

4.6 Significance of the work

The significance of our research work can be illustrated in following points:

(i) Securing the condition of a Cotton flat para-Kenmotsu manifold which permits η-Ricci soliton, where Ricci tensor remains constant.

(ii) Analyzing specific properties of the Cotton tensor on para-Kenmotsu 3-manifolds that accommodate an η-Ricci soliton in section (iii).

(iii) In this study, a relationship is derived between Cotton flat 3-D para-Kenmotsu manifolds that possess η-Ricci solitons and an η-Einstein manifolds.

(iv) A key component of Hamiltonian formalism of general relativity, the Cotton tensor is crucial in establishing the relationship between energy-momentum tensor along with Ricci tensor of matter as expressed in Einstein's equations.

4.7 Conclusion

The geometric relevance of cotton tensors is notably distinctive. They are also extensively applicable in the recently proposed Harada’s Cotton gravity. Moreover, we determine that cotton tensor is a third-order tensor in differential geometry found in relation to the Bianchi identities and can be employed to investigate fluid spacetime. We investigated that vanishing a cotton tensor is an essential and adequate requirement for a manifold to be locally conformal flat in three dimensions. In the context of higher dimensions, while the Cotton tensor must vanish, this condition alone is not enough to draw conclusions.

Future Scope

Here are some potential future research directions for Ricci solitons:

- Examine as well as categorize Ricci solitons in light of diverse contact metric manifolds, specifically focusing in Sasakian, k-contact and Kenmotsu manifolds.
- Explore the properties of Ricci solitons within contact metric manifolds that are equipped with additional structures, such as symplectic, complex, or CR structures.
- Examine the peculiarities of Ricci solitons in relation to manifolds of contact metric. and their relationships with a range of manifold types.
- Numerical methods can be developed to simulate Ricci solitons associated with contact metric manifolds.
- Utilize computational algebraic geometry to analyze the algebraic features of Ricci solitons alongwith contact metric manifolds. Furthermore, explore the potential applications of machine learning approaches in the examination of Ricci solitons in this context.
- Extend concept of Ricci soliton in relation to the contact metric manifolds to include manifolds with torsion.
- Examine the relationship between contact metric manifolds and Ricci solitons that possess non-constant curvature, particularly those with varying sectional curvature.

★★★★★★★★

Paper Publications

Paper published from the thesis

1. Mona Jasrotia and Pankaj Pandey, Some results on Ricci Yamabe soliton equipped with f-Kenmotsu manifold, Chhattisgarh Journal of Science and Technology, ISSN: 0973-7219, Volume 17, Issue 4, 2020, pp-33-36.

2. Mona Jasrotia and Pankaj Pandey, $\boldsymbol{\eta}$-Ricci soliton on Lorentzian Concircular structure manifold, published by European Chemical Bulletin Journal, 2022, 11(Issue 10), 177-180.

3. Mona Jasrotia and Pankaj Pandey, Cotton tensor on a 3-dimensional para Kenmotsu manifold admitting η-Ricci soliton in IAENG International Journal of Applied Mathematics (SCOPUS Indexed), vol.55, no.8, pp2390-2397,2025.

Paper Presentations

1. Mona Jasrotia and Pankaj Pandey, $\boldsymbol{\eta}$-Ricci soliton on Lorentzian Concircular structure manifold in 2nd International Conference on Computational Applied Sciences and its Applications(ICCASA-2023), held by University of Engineering and Management(UEM), Jaipur.

Conferences and Workshops attended

1. Attended the "Two Days International Colloquium on Recent Trends in Mathematical Sciences (ICRTMS-2021)" organized by the Department of Mathematics and the IQAC at Mata Sundri College for Women, University of Delhi, held on November 15-16, 2021.

2. Participated in "4th International Conference on Recent Advances in Fundamental and Applied Sciences" (RAFAS 2023), on March 24 to 25, 2023. The conference was organized by the School of Chemical Engineering and Physical Sciences at the Lovely Faculty of Technology and Sciences, Lovely Professional University, Punjab.

Bibliography

[1] S.R. Ashoka, C.S. Bagewadi and G. Ingalahalli, "A geometry on Ricci solitons in Lorentzian Concircular structure manifold", Diff. Geom.-Dynamical Systems, 16 (2014), 50–62.

[2] S.R. Ashoka, C.S. Bagewadi, and G. Ingalahalli, "Certain results on Ricci solitons in α-Sasakian manifolds", Hindawi Publ. Corporation, Geometry, Vol. 2013, Article ID 573925, 4 pages.

[3] C.L. Bejan and M. Crasmareanu, "Ricci solitons in manifolds with quasi constant curvature", Publ. Math. Debrecen, 78 (2011), 235-243.

[4] J. Bergem, S.B. Edgar and M. Herberthson, "The bach tensor and other divergence free tensors", Int. J. Geom. Methods Mod.Phys.2(01),13-21(2005).

[5] A.L. Besse, "Einstein manifolds", Springer Science and Business Media, New York(2007).

[6] A.M. Blaga, "η-Ricci solitons on Para-Kenmotsu manifolds", Balkan J. Geom. Appl., 20, (2015), 1-13.

[7] A.M. Blaga, "η-Ricci solitons on Lorentzian para-Sasakian manifolds", Filomat, 30, (2016), 489-496.

[8] A.M. Blaga, "Torse-forming η -Ricci solitons in almost paracontact Einstein geometry", Filomat, 31(2), (2017), 499-504.

[9] A.M. Blaga, S.Y. Perktas, B.Y. Acet and F.E.Erdogan, "η-Ricci solitons in (ϙ)-almost paracontact metric manifolds", Glasnik Math., 53(2018), 205-22.

[10] G. Catino, "Generalized quasi-Einstein manifolds with harmonic Weyl tensor" Mathematische Zeitschrift, 271, (2013): 751-756.

[11] G. Catino, P. Mastrolia, D.D. Monticelli and M. Rigoli, "Conformal Ricci solitons and related integrability condition. Advances in Geometry 16.3 (2016): 301-328.

[12] G. Catino, P. Mastrolia and D.D. Monticellia, "Gradient Ricci solitons with vanishing conditions on Weyl. J. Math. Pures Appl. 108 (2017): 1-13.

[13] G. Catino, P. Mastrolia, D.D. Monticellia and M. Rigoli, "On the geometry of gradient Einstein-typemanifolds. Pacific j. math. 286 (2017): 39-67.

[14] M.C. Chaki, M.L. Gosh, "On quasi Einstein manifolds". Int. J. Math. **42**,211-220(2000).

[15] B.Y. Chen, and S. Deshmukh,“A note on Yamabe solitons”, Balkan Journal of Geometry and its Applications, 23 (1), 2018, 37-43.

[16] B.Y. Chen and S. Deshmukh, “Geometry of compact shrinking Ricci solitons”, Balkan J. Geom. Appl.

[17] J.T. Cho and M. Kimura, “Ricci soliton and real hypersurfaes in a complex space form”, Tohoku Math. J.,61(2), 205-212, (2009).

[18] B. Chow, P. Lu and L. Ni, “Hamilton's Ricci flow”, Graduate Studies in Mathematics,77, AMS, Providence, RI, USA, 2006.

[19] P. Daskalopoulos and N. Sesum, “The classification of locally conformally flat Yamabe Solitons”, Adv. Math. **240** (2013) 346–369.

[20] U.C. De, “On ϕ-recurrent Kenmotsu manifolds”, Turk J. Math. 33 (2009), 17-35.

[21] K. De and U.C. De, “η-Ricci solitons on Kenmotsu 3-manifolds”, Analele Univ. de Vest, LVI, (2018), 51-63.

[22] S. Deshmukh and N. Turki, “A note on ϕ-analytic conformal vector fields”, Anal. Math. Phys.(2017), https://doi.org/10.1007/s13324-017-0190-8.

[23] F.T. Fong and O. Chodosh,“Rotational symmetry of conical Kahler-Ricci solitons”, Math. Ann., 364(2016), 777-792.

[24] A. Garcia, F.W. Hehl, C. Heinicke and A. Macias, “The Cotton tensor in Riemannian spacetimes”,Class.Quant.Grav.21(2004)1099-1118, https://doi.org/10.1088/0264-9381/21/4/024.

[25] A. Ghosh, “Yamabe soliton and quasi Yamabe soliton on Kenmotsu manifold”, Math.Slovaca 70 (2020), 151-160.

[26] A. Ghosh and R. Sharma, “Sasakian manifolds with purely transversal Bach tensor”, J. Math. Phys. 58(10), 103502(2017).

[27] J.N. Gomes, Q. Wang and C. Xia, “On the h-almost Ricci soliton”, J. Geom. Phys., 14, 216-222(2017).

[28] S. Guler, and M. Crasmareanu, “Ricci-Yamabe maps for Riemannian flow and their volume variation and volume entropy”, Turk. J. Math., 43, (2019), 2631-2641.

[29] R.S. Hamilton, “The Ricci flow on surfaces”, Math. Gen. Relativity. (Santa Cruz, CA, 1986), Contemp. Math.71 (1998) 237–262.

[30] R.S. Hamilton, “Three Manifold with positive Ricci curvature”, J. Differential Geom. 17(2), (1982), 255-306.

[31] A. Haseeb and S.K. Choubey, “Lorentzian para Sasakian manifolds and $*$ Ricci solitons, Kragujevac Journal of Mathematics, 48(2), (2024), 167–179.

[32] S.K. Hui and D. Chakaraborty, “η-Ricci soliton on η-Einstein $(LCS)_n$ manifold”, Acta Univ. Palacki, Olomuc, 52(2), 101-109, (2016).

[33] S. K. Hui and D. Chakraborty, “Some types of Ricci solitons on $(LCS)_n$ manifold”, J. Math. Sci. Advances and Applications, 37 (2016), 1-17.

[34] S. Hwang, J. Chang and G. Yun, “Non-existence of multiple black holes in static space-times and weakly harmonic curvature. Gen. Rel. Grav. 48.9(2016).

[35] S. Hwang and G. Yun, “Vacuum static spaces with vanishing of complete divergence of Bach tensor and Weyl tensor. J. Geom. Anal. (2020). https://doi.org/10.1007/s12220-020-003844.

[36] T. Imai, “Notes on semi-symmetric metric connection, Tensor, N.S., 24, (1972), 293–296.

[37] G. Ingalahalli and C.S. Bagewadi, “Ricci soliton on α-Sasakian manifolds”, ISRN Geom., (2012) Art. ID 421384.

[38] W. Israel, “Event horizons in static vacuum space-times. Phys. Review,vol. 164: 5,(1967): 1776-1779.

[39] M. Jiang, C. Gao, Y. Yang and A.K. Pogodaev, “A novel containment control design scheme for second-order multi-agent systems aith adjustable reference signals”, IAENG International journal of Applied Mathematics, vol.54, no.6 ,pp1048-1052, 2024.

[40] V.R. Kaigorodov, “The curvature structure of space-time”, prob. Geom., 14(1983), 177-204.

[41] S. Kaneyuki and F.L. Williams, “Almost paracontact and parahodge structure on manifolds”, Nagoya Math.J., 99(2), 173-187, (1985).

[42] J.N. Karatsobanis and P. J. Xenos, “ On the new class of contact metric 3-manifolds” ,Journal of Geometry, 2004; 80:136-153.

[43] K. Kenmotsu, “A class of almost contact Riemannian manifolds”, Tohoku Math. J. 24 (1972), 93-103.

[44] P. Khamrot, A. Iampan and T. Gaketem, “Bipolar complex fuzzy interior ideals in

semigroups", IAENG International journal of Applied Mathematics, vol.54, no.6 ,pp1102-1109, 2024.

[45] J. Kim, "On pseudo semiconformally symmetric manifolds" Bull. Korean Math. Soc. **54**, 177-186(2017).

[46] T. Koufogiorgos, "On a class of contact Riemannian 3-manifolds Results in Mathematics", 1995; 27:51-62.

[47] W. Kuhnel and H.B. Rademecher, "Conformal vector fields on pseudo-Riemannian spaces" Differ.Geom.Appl.7(3), 237-250(1997).

[48] P.D. Kusuma, "Iteration controlled mixture optimizer: A new Metaheuristic and its applications to solve economic load dispatch problem" , IAENG International journal of Applied Mathematics, vol.54, no.6 ,pp1073-1082, 2024.

[49] B. Leandro and N. Sorl´ozano, "Static perfect fluid spacetime with half conformally flat spatial factor. Manuscripta Math. 160 (2019), no. 1-2, 51–63.

[50] E.S. Lone and P. Pandey, "Study of lightlike submanifolds on semi Riemannian manifolds", Lovely Professional University(2022), http://hdl.handle.net/10603/436083.

[51] L. Ma and L. Cheng, "Properties of non-compact Yamabe solitons", Ann. Glob. Anal. Geom. **40** (2011) 379–387.

[52] P. Majhi and D. Kar, "Cotton tensor on Sasakian 3-manifolds admitting etta Ricci solitons", 70(3), 569-581(2021).

[53] A.K. Mandal, "Ricci soliton and gradiant Ricci soliton in an LP-Sasakian manifolds", Facta Univ. Ser. Math. Inform., 29 (2), (2014), 145-153.

[54] K. Matsumoto, "On Lorentzian paracontact manifolds", Bull. Yamagata Univ. Natur. Sci., 12(2)(1989),151-156.

[55] P. Miao and L.F. Tam, "Einstein and conformally flat critical metrics of the volume functional. Trans.AMS. 363.6 (2011): 2907-2937.

[56] F. Mofarreh and U.C. De, "A note on gradient solitons in three-dimensional Riemannian manifolds", International Electronic Journal of Geometry, 2021; 14: 85-90.

[57] H.G. Nagaraja and C. R. Premalatha, "Ricci soliton in Kenmotsu manifolds", J. Math. Anal., 3(2), 18-24(2012).

[58] D.M. Naik, V. Venkatesha, H.A. Kumara, " Ricci solitons and certain related metrics on almost co-Kaehler manifolds" Journal of Mathematical Physics Analysis and Geometry, 2020; 16: 402-417.

[59] D. Narain and S. Yadav, "On weak concircular symmetries of $(LCS)_{2n+1}$-manifolds", Global J. Sci. Frontier Research, 12 , 85-94(2012).

[60] B. O' Neill, "Semi-Riemannian Geometry with Applications to Relativity", Academic Press, New York (1983).

[61] P. Pandey, "On weakly cyclic generalized Z-symmetric manifolds", National Academy Science Letters, 43(4), 347-350(2020).

[62] P.Pandey and B.B.Chaturvedi, "On a Lorentzian Complex Space Form", General Mathematics Notes, 29(2), 61-66(2015).

[63] P. Pandey and K. Sharma, "Some results on Ricci solitons on Contact metric manifolds", Facta University, Series: Mathematics and Informatics,473-486.(2023).

[64] P. Pandey and K. Sharma, "Generalized Z-Ricci soliton on Sasakian 3-manifold with generalized z-tensor", AIP conference proceddings (2024).

[65] P. Pandey and K. Sharma, "Some novel results on (alpha, beta) Ricci Yamabe soliton and its spacetime", arxiv:2407.05940, 1-18(2024).

[66] P. Pandey and K. Sharma, B.B. Chaturvedi and M.N.I. Khan, "η-Ricci soliton and its application on ϕ-recurrent LP Sasakian manifold", International Journal of Analysis and Applications, vol 22, 200-200(2024).

[67] K. Pathak, A.S. Yadav and Priyanka Agarwal , "Enhancing Two-Warehouse Inventory models for perishable goods: Time-Price dependent demand, inflation and partial backlogging", IAENG International journal of Applied Mathematics, vol.54, no.6 ,pp1089-1101, 2024.

[68] D. Perrone and G. Calvaruso, "Geometry of H-paracontact metric manifolds", Publ. Math. Debrecen, 86(2015), 325-346.

[69] V. Pirhadi and A. Razavi, "On the almost Quasi-yamabe solitons", Int. J.Geom., Methods Mod.Phys.14 (2017), 1750161(9 pages).

[70] G.P Pokhariyal and S.K. Chaubey, "Study of P-Curvature tensor in the space-time of general relativity, Honam Mathematical J., 45(2) (2023), 316-324, , https://doi.org/10.5831/HMJ.2023.45.2.316.

[71] D. Prakasha, B. Hadimani, "η-Ricci solitons on para-Sasakian manifolds", J. Geom. **108**(2), 383–392(2020).

[72] J. Qing and W. Yuan, "A note on static spaces and related problems", J. Geom. Phys. 74 (2013): 18-27.

[73] M. Sati and K.C. Petwal , " Properties of Weyl tensor and Cotton tensor in a Riemannian manifold", Aryabhatta Journal of Mathematics and Informatics vol.12, no.2(2020).

[74] Sato, "On a structure similar to almost contact structure I", Tensor N.S., 30, 219-224, (1976).

[75] A. Sardar, U.C. De and C. Ozgur, "h-almost Ricci Yamabe soliton in Paracontact geometry", arXiv:2111.13577v1 [math.DG] (2021).

[76] T. Seko and S. Maeta, "Classifications of almost Yamabe solitons in Euclidean spaces", J. Geom. Phys. 136 (2019), 97-103.

[77] S. Sharma and V. Bhatia, "Global approaches to immunity and epidemiological studies of SARS-COV-2: Does Universal vaccination policy play a role?", ECB, 10(2),103-106(2021).

[78] G.S. Shivaprasanna, P.G. Angadi, G. Somashekhara and P.s. Kota Reddy, "Ricci Yamabe solitons on submanifolds of some indefinite almost contact manifolds", Advances in Mathematics: Scientific journal 9(2020), no. 11, 10067-10080, ISSN: 1857-8438(electronic), http://doi.org/10.37418/amsj.9.11.114.

[79] M.D. Siddiqi, M.A. Akyol, "η-Ricci–Yamabe solitons on Riemannian submersions from Riemannian manifolds", arXiv:2004.14124.

[80] M.D. Siddiqi, S.A. Siddiqui, "Conformal Ricci soliton and geometrical structure in a perfect fluid spacetime" Int. J. Geom. Methods Mod. Phys. **17**, 2050083 (2020).

[81] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, E. Herlt, "Exact Solutions of Einstein's Field Equations", 2nd edn.Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge (2003).

[82] Y.J. Suh and U.C. De, "Yamabe solitons and gradient Yamabe solitons on three dimensional $N(k)$-contact manifolds", International Journal of Geometric Methods in Modern Physics, 2020; 17.

[83] J.K. Sutariya, J.L. Gajera, J. Soni, H. Ram and P.K. Godhaviya, "Synthesis and characterization of Trifluoromethyl substituted Pyrazole derivatives", ECB, 10(2), 107-

113(2021).
[84] T. Takahashi, "Sasakian manifold with pseudo Reimannian metric", Tohoku Math. J.,21(2), 644-653, (1969).
[85] W.P. Thurston, H.E. Winkelnkemper, " On the existence of contact forms" , Proceedings of the American Mathematical Society, 1975; 52: 345-347.
[86] M.M. Tripathi, "Ricci soliton in contact metric manifolds", http;//arxiv.org/abs/0801.4222.
[87] V. Venkatesha, H.A. Kumara, "Ricci soliton and geometrical structure in a perfect fluid spacetime with torse-forming vector field", Afr. Mat. **30**, 725–736 (2019).
[88] Y. Wang, "Yamabe solitons on three dimensional Kenmotsu manifolds", Bulletin of the Belgian Mathematical Society-Simon Stevin, 2016; 23: 345-355.
[89] Y. Wang, "Ricci solitons on 3-dimensional cosymplectic manifolds" Mathematica Slovaca, 2017; 67: 979-984.
[90] Y. Wang, "Ricci solitons on almost co-kahler manifolds", Canadian Mathematical Bulletin 2019; 62: 912-922.
[91] Y. Wang, "Contact 3-manifolds and *-Ricci soliton", Kodai Mathematical Journal 43 (2020), 256–267.
[92] Y. Wang, "Almost Kenmotsu (k, μ) -manifolds with Yamabe solitons", Revista de la Real Academia de Ciencias, Exactas, Físicas y Naturales. Series A. Matemáticas 2021; 115.
[93] Y. Wang and X. Liu, "Ricci solitons on three-dimensional η -Einstein almost Kenmotsu manifolds", Taiwanese Journal of Mathematics, 19(1), (2015), 91-100.
[94] J. Welyczko, "Slant curves in 3-dimensional normal almost paracontact metric manifolds", Mediterr. J. Math., 11(3), 965-978, (2014).
[95] S.K. Yadav, D.L. Suthar and S.K. Chaubey, "Certain Geometric properties of η-Ricci soliton on η-Einstein para-kenmotsu manifolds", Pure & Appl. Math., 9(1), 237–244, (2020).
[96] S.K. Yadav, P.K. Dwivedi and D.L. Suthar, "On $(LCS)_{2n+1}$-manifolds satisfying certain conditions on the concircular curvature tensor". Thai J. Math., 9(3), 597-603, (2011).
[97] S.K. Yadav, D.L. Suthar and M. Hailu, "On Extended generalized phi-recurrent $(LCS)_{2n+1}$-manifolds", Bol. Soc. Parana. Mat.,37(2), 9-21, (2019).
[98] S.K. Yadav, D.L. Suthar and A.K. Srivastava, "Some results on $M(f_1, f_2, f_3)$2n+1-manifolds", Int. J. Pure & Appl. Math., 70(3), 415–423, (2011).

[99] S.K. Yadav, S.K. Hui, and A. Patra,“On Almost conformal Ricci soliton in f-Kenmotsu manifolds”, Khayyam J. Math. 5 (1), (2019), 89-104.

[100] K. Yano, “Integral Formulas in Riemannian Geometry” Marcel Dekker,1970.

[101] A. Yildiz, U.C. De and M. Turan,“On 3-dimensional f-Kenmotsu manifolds and Ricci solitons”, Ukrainian Math. J. 65 (5), (2013), 684-693.

[102] Z. Yuan and T. Du, “The multipoint-based Hermite-Hadamard inequalities for fractional integrals with exponential kernels”, IAENG International Journal of Applied Mathematics, vol.54, no.6, pp1013-1025, 2024.

[103] J. Zhang, X. Zhao and Z. Bai, “Study on Heat transfer of Capillary Heat Exchanger for the Heat Pump in coastal areas of Diary Operation”, IAENG Journal of Applied Mathematics, 54(7), 1259-1263(2024).

[104] W. Zheng, X. Wang and J. Wang, “Predictive Control Algorithm for Speed and Displacement Tracking of Urban Rail Trains”, IAENG Journal of Applied Mathematics, 54(7), 1362-1370(2024).