

IMPLICATION OF GENERALIZED UNCERTAINTY PRINCIPLE ON THE FABRIC OF SPACETIME

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I, **Gaurav Bhandari**, bearing Registration No. **12318372**, hereby declare that the research work embodied in the Ph.D. thesis titled "**Implication of Generalized Uncertainty Principle on the Fabric of Spacetime**" submitted to Lovely Professional University, Punjab is the original and bonafide work carried out by me under the supervision of **Dr. Harjit Singh Ghotra, Professor, School of Chemical Engineering and Physical Sciences, Lovely Professional University, Punjab**.

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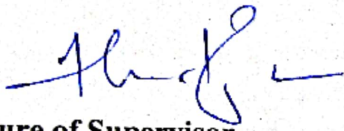
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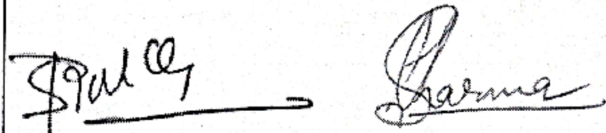


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Abstract

The search for a consistent theory of quantum gravity remains one of the most profound challenges in modern theoretical physics. The conceptual tension between General Relativity, which governs the structure of spacetime at macroscopic scales, and Quantum Mechanics, which dictates the behavior of matter and fields at microscopic scales, strongly suggests the existence of a deeper underlying framework. A recurring prediction across several quantum gravity approaches is the existence of a minimal measurable length, typically of the order of the Planck scale. The Generalized Uncertainty Principle (GUP), a deformation of the Heisenberg uncertainty relation, provides an effective and phenomenologically accessible way to encode such minimal-length effects.

This thesis, titled *“Implication of the Generalized Uncertainty Principle on the Fabric of Spacetime”*, presents a comprehensive exploration of how GUP and its relativistic extension (RGUP) modify the foundations of physics across cosmology, black hole thermodynamics, and quantum mechanical systems. The work begins by motivating the need for quantum gravity, discussing the limitations of canonical and covariant quantization, and tracing the historical development of the minimal-length concept through *Gedanken* experiments, black hole horizon arguments, and insights from string theory and loop quantum gravity.

The thesis develops a consistent formulation of GUP and RGUP within relativistic frameworks and examines their mathematical and physical implications. In cosmology, the modified Poisson brackets and deformed commutation relations are incorporated into the Friedmann, Raychaudhuri, and Klein–Gordon equations to analyze scalar-field dynamics in both phantom and quintessence scenarios. A detailed dynamical-systems analysis reveals how minimal-length corrections influence inflationary behavior, slow-roll evolution, and late-time acceleration.

In black hole physics, GUP-induced corrections to Hawking radiation are derived using both the Parikh–Wilczek tunneling method and the Hamilton–Jacobi approach. The analysis shows that GUP provides a natural ultraviolet cutoff, leading to finite remnant masses and offering potential resolutions to the information loss paradox.

The final part of the thesis investigates quantum mechanical systems under GUP and RGUP deformations. Modifications to the angular momentum algebra, ladder operators are obtained, and their consequences for the hydrogen atom spectrum are analyzed. The framework is further applied to the Zeeman and Stark effects, where subtle yet systematic RGUP-induced shifts in energy levels and degeneracy structures arise as possible low-energy signatures of Planck-scale physics.

Overall, this thesis provides a unified and phenomenological study of GUP and RGUP across cosmological models, black hole physics, angular momentum algebra, and atomic systems. The results demonstrate how minimal-length effects can leave observable imprints in diverse physical settings, thereby offering promising pathways for probing quantum gravitational phenomena at both high-energy and low-energy scales.

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*“I think Nature’s imagination is so much greater than man’s,
she’s never gonna let us relax!”*

— Richard P. Feynman

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According to me, nature's reality is coded in an infinite sequence of steps- each unveiling a deeper layer of truth. This is the very nature of nature itself.-

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Chapter 1

WHY QUANTUM GRAVITY?

From the dawn of human civilization, the fundamental questions about space and time have fascinated human beings, from those who painted on the walls of ancient caves to those who ponder over a cup of tea today. What are space and time? Did space have a beginning, and does it extend forever? What was the first moment, and what came before it? These are among the hardest questions we can ask, yet with every scientific revolution, new insights emerge. We now find ourselves in the midst of such a revolution. Equally captivating are the questions about the building blocks of nature. If we continue to probe to smaller scales, what is the smallest possible entity? Are electrons and quarks the ultimate constituents of matter, or are they themselves built from even more fundamental objects, such as the vibrating strings proposed by modern theories?

The journey to these questions has spanned centuries from the speculations of ancient philosophers such as Aristarchus, Democritus, and Aristotle to the breakthroughs of Newtonian physics. In the 17th century, Newton's framework, which treated space and time as absolute and independent, achieved great success in describing the motion of planets, falling bodies, and the mechanics of everyday life. However, this picture carried deep limitations. According to Newtonian relativity, the speed of light should depend on the motion of the inertial observer, yet experiments such as the Michelson-Morley experiment showed otherwise. This limitation was overcome in the early 20th century by Einstein's revolutionary view of space and time, which reshaped our understanding

of the universe. At the same time, Newtonian physics and Maxwell's theory of electromagnetism faced another serious challenge at small scale physics as they could not explain the stability of the atom. According to classical physics, an electron orbiting a nucleus should radiate energy continuously, spiraling inward and collapsing into the nucleus within a finite time. Yet atoms are stable. This puzzle, and many others, triggered the quantum revolution, led by pioneers such as Niels Bohr, Werner Heisenberg, Erwin Schrödinger, and many others, unveiling the strange and beautiful laws of the microscopic world.

Today, quantum mechanics (QM), special relativity (SR), and general relativity (GR) resolve these old shortcomings of Newtonian physics and Maxwell theory. QM forbids continuous radiation by allowing only discrete energy spectrum of atoms, thereby ensuring their stability whereas SR overturned the Newtonian notion of absolute space and time, replacing it with a unified spacetime continuum governed by causal relationships between events. GR went further by generalizing the flat Minkowskian metric of SR into a dynamical theory of the metric field, reshaping our understanding of gravity not as a force but as a manifestation of spacetime geometry. The foundations of GR rest on two guiding principles. The *equivalence principle* asserts that the effects of gravity are independent of the composition or nature of matter and *Mach principle*, summarized by John Archibald Wheeler's "*Spacetime tells matter how to move; matter tells spacetime how to curve*" highlights the deep interplay between matter and geometry. At each point in spacetime, the presence of matter determines curvature, and this curvature in turn dictates the straightest possible paths (geodesics) along which matter moves.

Despite their great successes, QM and GR remain incomplete and mutually incompatible. QM excels in describing the microscopic world of atoms, molecules, and radiation, while GR provides a profound account of spacetime and gravity on cosmic scales. Yet attempts to extend QM to spacetime itself, or to apply GR at atomic scales, encounter paradoxes and breakdowns. Each theory is reliable within its domain, but neither can claim universal validity. A truly fundamental physical theory called *the quantum theory of gravity* must do more than catalog of particles and forces that must offer a seamless description that works consistently across all regimes, from the smallest quantum fluctuations to the vast structure of the universe. There are several compelling reasons for the need for a quantum theory of gravity:

- ***Inconsistency between Theories:*** The Mach principle is expressed through the Einstein field equations, which are classical in nature represented [1–3] as

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu}. \quad (1.1)$$

Here, the left-hand side represents the spacetime geometry in terms of the Ricci curvature tensor $R_{\mu\nu}$, while the right-hand side corresponds to the matter content described by the energy-momentum tensor $T_{\mu\nu}$. A fundamental inconsistency arises because the geometry is treated classically, taking definite values, whereas the matter fields are governed by quantum theory. To reconcile this, one commonly replaces the classical energy-momentum tensor with its quantum expectation value in a given state [2], that is,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa \langle \hat{T}_{\mu\nu} \rangle. \quad (1.2)$$

However, if we consider a quantum state of matter is localized in two disjoint regions P and Q , each with probability $1/2$. According to the above semiclassical Eq.(1.2), the gravitational field would respond as though half of the matter were in P and half in Q . And, when one makes the measurement, the matter is found entirely in either P or Q . This implies that, if the equation hold, then the gravitational field would change discontinuously and in an acausal manner depending on the measurement outcome. This leads to serious difficulty if we treat gravity classically. A consistent resolution requires treating the spacetime geometry itself in a probabilistic fashion by quantizing the gravitational field. In such an approach, one postulates a quantum version of the Einstein equations,

$$\hat{R}_{\mu\nu} - \frac{1}{2}\hat{R}\hat{g}_{\mu\nu} = \kappa\hat{T}_{\mu\nu}, \quad (1.3)$$

where both matter and geometry are promoted to operators defined within a Hilbert space.

- ***Unification of forces:*** It is widely believed that a true fundamental theory of nature will arise through the unification of all four fundamental interactions, describing them as different aspects of a single underlying entity. Classical theories of electricity and magnetism were unified by Maxwell more than a century ago.

The unification of weak and electromagnetic interactions was accomplished in the 20th century through the Weinberg-Salam theory [4, 5]. Grand unified models suggest that strong and electroweak interactions may also be brought together under a common framework. Extending this logic, the unification of gravity with the strong-electroweak interactions would be the next essential step in this program.

- ***Singularities***: The classical description of spacetime as a smooth manifold becomes inadequate in the presence of singularities, regions where the energy density and curvature became infinite [6, 7]. The most prominent examples are the black hole and big bang singularities, as demonstrated in the singularity theorems of Hawking and Penrose. Singularities arise because general relativity is pushed beyond its domain of validity, near such regions, one cannot expect a homogeneous and isotropic model to remain sufficient for describing the universe. When curvature reaches magnitudes of order 10^{66} or higher, quantum effects necessarily become important. It is therefore expected that a quantum theory of gravity will resolve these singularities in much the same way that quantum mechanics explains the stability of atoms, which could not be accounted for within classical physics alone.
- ***Problem of Time***: In quantum mechanics, time acts as an external variable, or say it is not described by an operator but rather serves as the parameter governing the evolution of other observables. In special relativistic quantum field theory, time remains external, embedded in the fixed Minkowskian spacetime background. In contrast, general relativity treats spacetime as a dynamical entity, where space and time are placed on an equal footing and evolve according to Einstein equations. Thus, any consistent unification of quantum theory and gravitation will inevitably require a reformulation of the notion of time [8, 9].

The fundamental inconsistencies in the description of spacetime are reminiscent of the inconsistencies that existed prior to the development of General Relativity (GR) and Quantum Mechanics (QM). Resolving these inconsistencies is expected to revolutionize our fundamental understanding of nature. The quest to formulate a consistent theory of quantum gravity remains an active area of investigation, with different research communities pursuing diverse approaches. It is possible that many of the currently proposed frameworks may ultimately fail, or alternatively, that seemingly distinct theories may

turn out to be complementary descriptions of the same underlying reality. In light of this, we provide a brief review of some of the leading approaches to quantum gravity, including String Theory, Loop Quantum Gravity, Doubly Special Relativity, and Non-Commutative Geometry. In the following chapter, we further emphasize the importance of quantum gravity phenomenology and discuss the motivation for pursuing concrete predictions and experimental signatures.

1.1 Relevant scales for quantum gravity

Despite their differences, most approaches to quantum gravity agree on one central idea that there exists a fundamental physical scale at which the classical description of spacetime ceases to be valid. Just as classical mechanics breaks down at atomic scales, necessitating quantum mechanics, the classical description of spacetime provided by general relativity is expected to break down near the Planck scale. This breakdown is determined by fundamental constants of nature, and in those regimes, the effects of quantum gravity can't be ignored. Max Planck, in 1899 [10], demonstrated that a natural system of units can be constructed from the quantum of action ($\hbar$), the speed of light (c), and Newton's gravitational constant (G). These yield the characteristic Planck mass (m_P), Planck time (t_P), and Planck length (l_P) expressed as

$$m_P = \frac{\hbar}{l_P c} = \sqrt{\frac{\hbar c}{G}} \approx 1.22 \times 10^{19} \text{ GeV}, \quad (1.4)$$

$$t_P = \frac{l_P}{c} = \sqrt{\frac{\hbar G}{c^5}} \approx 5.40 \times 10^{-44} \text{ s}, \quad (1.5)$$

$$l_P = \sqrt{\frac{\hbar G}{c^3}} \approx 1.62 \times 10^{-33} \text{ cm}. \quad (1.6)$$

To directly probe quantum gravitational effects, energies of the order of the Planck mass would need to be localized within a region of size comparable to the Planck length. Current experiments, however, are far from reaching this regime. The Standard Model has been tested down to scales of about 10^{-22} m, which is still 13 orders of magnitude below this, as of yet, is the highest ever achieved accuracy. When considering an elementary particle, the Planck length corresponds to the scale at which Compton wavelength of a

particle equals its Schwarzschild radius [1, 11],

$$\frac{\hbar}{m_P c} \approx R_s \equiv \frac{2Gm_P}{c^2}, \quad (1.7)$$

implies that gravitational effects on such small scales cannot be neglected.

It is also well known that gravity is significantly weaker than the other three fundamental forces. A simple everyday demonstration is lifting pieces of paper using a charged comb, thereby overcoming the gravitational pull of the entire Earth. More quantitatively, the weakness of gravity can be seen by comparing the acceleration due to the gravitational field with that due to the electrostatic field between a proton and an electron, known as *Dirac's large number hypothesis*

$$\frac{a_g}{a_{\text{elec}}} \approx 10^{-39}. \quad (1.8)$$

This extreme smallness can alternatively be expressed through the *gravitational fine-structure constant*, defined as the ratio of the proton mass to the Planck mass written as

$$\alpha_{\text{grav}} = \frac{Gm_{pr}^2}{\hbar c} = \left(\frac{m_{pr}}{m_P} \right)^2 \approx 5.91 \times 10^{-39}, \quad (1.9)$$

where m_{pr} denotes the proton mass. These considerations highlight both the weakness of gravity and the enormous gap between experimentally accessible scales and the Planck scale, where quantum gravitational phenomena are expected to manifest [12]. A related argument, concerning experimental limitations, was put forward by Freeman Dyson, who coherently reasoned that any detector capable of observing gravitons—the mediators of gravity in the weak-field limit—would necessarily have to be so dense that it would collapse into a black hole [13, 14]. Likewise, in accordance with Penrose's cosmic censorship conjecture [15, 16], it is widely believed that regions of extremely strong curvature, such as those near black holes, which constitute the natural arena for quantum gravity, are typically hidden from external observation by horizons.

1.2 Approaches to quantum theory of gravity

As we have already discussed, experimental evidence for quantum gravity remains far beyond the reach of current accelerator technology. It is generally believed that a true fundamental theory should possess a rigid structure such that all low-energy phenomena

can be predicted as limiting cases. Since no direct experimental input is yet available, many efforts focus on constructing mathematically and conceptually consistent frameworks. In this pursuit, various approaches have emerged, largely because there is no universally accepted methodological starting point. Isham distinguishes between two broad classes of such approaches, namely “*primary theories*” and “*secondary theories*”. In the primary approach, one begins with classical general relativity and attempts to quantize it heuristically, in a manner similar to the quantization procedures used in quantum electrodynamics (QED). For instance, one might also consider starting from the Brans-Dicke theory. Within this category, one usually distinguishes between the *covariant approach*, which maintains spacetime diffeomorphism invariance, and the *canonical approach*, which reformulates general relativity in Hamiltonian form and applies canonical quantization rules.

The secondary approach, on the other hand, begins with a more fundamental quantum framework intended to describe all interactions, from which quantum general relativity is expected to emerge as a limiting case or effective approximation. The classic example of this category is string/M-theory, which has the advantage of naturally incorporating unification of forces. By contrast, in the primary approach, unification is not immediate, since one begins with general relativity. However, secondary approaches face the drawback that their starting point is highly speculative, with no direct empirical motivation.

For the purpose of this thesis, it is useful to present a concise overview of some of the most popular approaches to quantum gravity. The following discussion is intentionally brief and does not attempt a rigorous treatment, since the detailed technicalities are beyond the scope of thesis.

1.2.1 Covariant Approach

The covariant approach represents the first and most straightforward attempt to quantize gravity, relying on perturbation theory. In one of the earliest works, Leon Rosenfeld investigated the quantization of the gravitational field produced by the electromagnetic field. However, the calculation of the gravitational self-energy led to divergences, with the resulting quantities turning out to be infinite [17].

The starting point of this approach is the decomposition of the spacetime metric into a background part and a perturbation,

$$g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu}, \quad (1.10)$$

where $\eta_{\mu\nu}$ is the flat Minkowski metric, and the manifold is assumed to be $M = \mathbb{R}^4$ so that $(M, \eta_{\mu\nu})$ represents Minkowski spacetime. More generally, one may replace $(\mathbb{R}^4, \eta_{\mu\nu})$ with any background solution $(M, \bar{g}_{\mu\nu})$ of Einstein's field equations. In this formulation, the classical Einstein equations can be interpreted as the equations of motion for a free massless spin-2 field propagating on the background spacetime [18, 19]. If we view the perturbation not to be an approximation (if $\gamma_{\mu\nu}$ is not small), then the full Einstein equation can be expressed as a free spin-2 field plus non-linear (interaction) terms, which describe the self-interaction of gravitons as well as their coupling to the background metric.

The main difficulty with this approach is that the perturbation theory for $\gamma_{\mu\nu}$ turns out to be *non-renormalizable*. The Dimensional argument suggests that the expansion parameter in the perturbative series is the squared Planck length, l_P^2 . Consequently, each successive term in the series involves higher powers of l_P^2 , which correspond to interactions with increasingly high dimensions in momentum space. This leads to divergences with a growing superficial degree of divergence at higher loop orders [19, 20]. At each order in perturbation theory, the causality condition is satisfied with respect to the background metric $\eta_{\mu\nu}$ rather than the true metric $g_{\mu\nu}$, which is an awkward way of representing the role of the spacetime metric in causal structure. The difficulty is the *violation of background independence*, in which the true metric splits into a classical background $\eta_{\mu\nu}$ and a dynamical field $\gamma_{\mu\nu}$ which is to be quantized, that is unnatural from the viewpoint of classical General Relativity. In the covariant approach, the perspective of *gravitons and geometry* treats gravitational interaction as resulting from graviton exchange on a fixed background, which contradicts Einstein's original idea that gravity is geometry, not a force in the conventional sense. The Classical GR focuses on the evolution of the metric in time and its interplay with the matter present. In situations with singularities or regions of high curvature, such as the Big Bang, the background metric can be far from Minkowskian, and the perturbative term $\gamma_{\mu\nu}$ may no longer be small, which makes the perturbative approach dynamically unstable. On the other hand, the advantage of the covariant approach is that in the weak-field regime

it provides a good approximation and yields concrete expressions for physical quantities within perturbation theory, without requiring the development of an entirely new conceptual framework.

1.2.2 Canonical Approach

“In the canonical framework, the goal is to develop a mathematically consistent, background-independent, and non-perturbative quantum field theory describing four-dimensional Lorentzian general relativity coupled to matter fields.”

Thus, the given figures serve as a conceptual prelude to the canonical quantization program: while Fig. 1.1 represents the standard QFT picture with fixed geometry, Fig. 1.2 reflects the background-independent setting in which canonical quantum gravity is formulated.

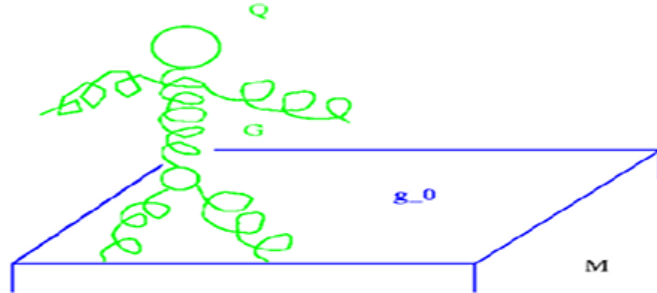


FIGURE 1.1: Illustration of QFT on a background spacetime (M, g_0) , where the **Matter fields** constitute the actors and the fixed **Geometry** on the manifold M serves as the stage [1].

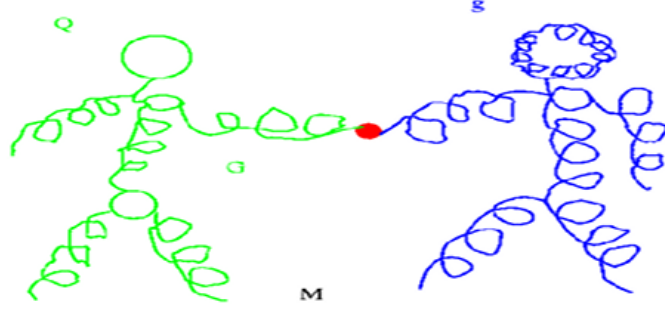


FIGURE 1.2: Quantum gravity on a differential manifold M : Actors: **matter** and **geometry**; Stage: the manifold M [1].

This framework is based on the Hamiltonian formulation of general relativity, which requires a decomposition of spacetime variables into temporal and spatial components. In other words, classical general relativity is reformulated through a $(3 + 1)$ splitting of spacetime, thereby introducing a distinguished notion of time and revealing the geometric structure of the theory. Let us first denote a space-like hypersurface Σ corresponding to a fixed time, representing all events simultaneous with respect to this slicing. This denote [21] as

$$X^\alpha = g^\alpha(x^i), \quad (1.11)$$

where x^i ($i = 1, 2, 3$) are intrinsic coordinates on Σ . The tangent vectors to the hypersurface are then given by

$$\zeta_i \equiv \partial_i g^\alpha \partial_\alpha, \quad (1.12)$$

from which the induced spatial metric is constructed as

$$h_{ij} = g_{\alpha\beta} \zeta_i^\alpha \zeta_j^\beta, \quad (1.13)$$

and the unit normal vector n^α to the hypersurface is defined through the orthogonality condition

$$g_{\alpha\beta} n^\alpha \zeta_i^\beta = 0, \quad (1.14)$$

together with the normalization

$$g_{\alpha\beta} n^\alpha n^\beta = -1. \quad (1.15)$$

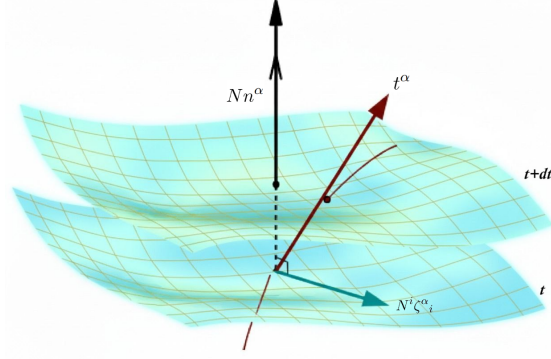


FIGURE 1.3: ADM 3+1 decomposition showing the lapse Nn^α , shift $N^i \zeta_i^\alpha$, and the time evolution vector $t^\alpha = Nn^\alpha + N^i \zeta_i^\alpha$ between hypersurfaces Σ_t and Σ_{t+dt} .

Now, if we take a set of such spacelike hypersurfaces that together cover the entire spacetime, a foliation of spacetime into a one-parameter family (with time as the parameter), then each slice is a disjoint spacelike hypersurface as

$$\Sigma_t = X^\alpha = g^\alpha(x^i, t), \quad (1.16)$$

where t labels the hypersurfaces.

The Arnowitt-Deser-Misner (ADM) formalism characterizes the embedding through two functions namely the *lapse function* N and the *shift vector* N^i . From the Fig.(1.3) one can defining

$$t^\alpha \equiv \frac{\partial X^\alpha}{\partial t}, \quad (1.17)$$

one decomposes this time-evolution vector as

$$t^\alpha = Nn^\alpha + N^i \zeta_i^\alpha, \quad (1.18)$$

where N measures the rate of advance of proper time along the normal direction (the lapse), while N^i describes how the coordinates shift tangentially along the hypersurface (the shift). Using this decomposition, the spacetime line element can be written in terms of dt and dx^i as

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \quad (1.19)$$

$$= g_{\alpha\beta} (t^\alpha dt + \zeta_i^\alpha dx^i) (t^\beta dt + \zeta_j^\beta dx^j) \quad (1.20)$$

$$= (-N^2 + N_i N^i) dt^2 + 2N_i dt dx^i + h_{ij} dx^i dx^j, \quad (1.21)$$

where $N_i \equiv h_{ij}N^j$. This representation first describes the intrinsic geometry of a single spacelike slice and then extends the discussion to the full spacetime foliation via the lapse and shift, which is the natural order of exposition in the canonical approach. The full spacetime metric can be decomposed [21] as

$$g_{\mu\nu} = h^{ij}\zeta_{i\mu}\zeta_{j\nu} + n_\mu n_\nu, \quad (1.22)$$

now the extrinsic curvature of the hypersurface is defined as

$$K_{ij} = -\zeta_i^\alpha \nabla_\sigma n_\alpha \zeta_j^\sigma \quad (1.23)$$

In terms of ADM decomposition, the Einstein-Hilbert Lagrangian reads [2, 21] as

$$\mathcal{L} = \frac{1}{2\kappa} [N\sqrt{h}(R[h] + K_{ij}K^{ij} - K^2) - \partial_\alpha V^\alpha] \quad (1.24)$$

where $R[h]$ is the intrinsic curvature of the spatial hypersurface and $V^\alpha = 2\sqrt{-g}(n^\sigma \nabla_\sigma n^\alpha - n^\alpha \nabla_\sigma n^\sigma)$. Now, on defining the canonical momenta, the primary constraint and momenta conjugate for the spatial part appear respectively, as

$$p^\mu \equiv \frac{\partial \mathcal{L}}{\partial \dot{N}_\mu} \sim 0, \quad \pi^{ij} \equiv \frac{\partial \mathcal{L}}{\partial \dot{h}_{ij}} = -\sqrt{h}(K^{ij} - Kh^{ij}), \quad (1.25)$$

which gives the total Hamiltonian will be

$$H \equiv \int d^3x (\pi_\mu \dot{N}^\mu + \pi^{ij} \dot{h}_{ij} - \mathcal{L}) = \int d^3x (N\mathcal{H} + N^i \mathcal{H}_i) \quad (1.26)$$

where

$$\mathcal{H} = \sqrt{h} \left[h^{-1} \left(\pi_{ij} \pi^{ij} - \frac{1}{2} \pi^2 \right) - R[h] \right], \quad (1.27)$$

$$\mathcal{H}_i = -2h_{ij} \partial_k \pi^{jk} - (2\partial_k h_{ji} - \partial_i h_{jk}) \pi^{jk}. \quad (1.28)$$

Now, introducing the primary and secondary constraints, the central equations take the form

$$\mathcal{H} = 0, \quad (1.29)$$

$$\mathcal{H}_i = 0, \quad (1.30)$$

and upon quantization, the physical states in the Hilbert space must satisfy the operator constraints [19]

$$\hat{\mathcal{H}} |\psi\rangle = 0, \quad (1.31)$$

$$\hat{\mathcal{H}}_i |\psi\rangle = 0, \quad (1.32)$$

where Eq. (1.31) is known as the *Wheeler–DeWitt equation*, while Eq. (1.32) represents the *diffeomorphism constraints*. The canonical approach is thus built on the evolution of variables defined through the foliation of spacetime into spacelike three-dimensional hypersurfaces, with the intrinsic three-metric and its canonical conjugate momenta being related to the extrinsic curvature. A major development in this direction was achieved by Ashtekar in 1986 [22, 23], who identified the fundamental canonically conjugate variables of *spinorial gravity*, starting from the Einstein–Hilbert action expressed in the tetrad formalism [1, 24] as

$$\mathcal{S} = \int E^a \wedge E^b \wedge R^{cd} \epsilon_{abcd}, \quad (1.33)$$

where E^a denotes the tetrad one-forms, $E^a \equiv E^a_\mu dx^\mu$, R^{cd} the curvature two-forms and the $\wedge$ product is the antisymmetric tensor product of differential form

$$E^a \wedge E^b = E^a \otimes E^b - E^b \otimes E^a.$$

The tetrad also have the following property

$$h_{ab} = \delta_{jk} E^j_a E^k_b \quad (1.34)$$

One half of the Ashtekar variables are given by the *densitized triads*, defined as

$$\tilde{E}^a_i = \sqrt{\det h} E^a_i, \quad (1.35)$$

which serve as the canonical momenta in the Hamiltonian formulation. The other half are the connection variables, which behave like an $SU(2)$ Yang–Mills connection [25, 26],

$$A_a^i = \Gamma_a^i + \gamma K_a^i, \quad (1.36)$$

where $\Gamma_a^i \equiv \Gamma_{ajk} \epsilon^{jki}$ is the spin connection compatible with the triads, K_a^i is the extrinsic curvature, and γ is the *Barbero–Immirzi parameter*. The parameter γ is dimensionless and of order unity, with its value fixed by requiring consistency with the Bekenstein–Hawking entropy formula for black holes.

Recasting the ADM Hamiltonian in terms of the Ashtekar variables makes the quantization of gravity *background independent*, a key property that distinguishes this approach from perturbative methods. The quantization procedure is inspired by techniques from non-Abelian gauge theory, particularly those developed in the study of Quantum Chromodynamics (QCD). Following the pioneering work of K. G. Wilson (1974), one introduces *Wilson loops* as fundamental objects, providing a non-perturbative representation of the quantum theory. In the context of gravity, these loop variables are defined [27] as

$$W(\gamma, A) \equiv \text{Tr} \left[P \exp \left\{ \oint_{\gamma} A \right\} \right], \quad (1.37)$$

where the trace is taken in the fundamental representation of $SU(2)$, P denotes path-ordering, and the integration is performed along a closed loop γ . These Wilson loops form the basis of the *loop representation of quantum gravity*, which ultimately gave rise to the framework of Loop Quantum Gravity (LQG).

In this context, Jacobson and Smolin discovered that Wilson loops of the Ashtekar connection constitute solutions of the Wheeler–DeWitt equation, the formal canonical equation of quantum gravity. This remarkable result led Rovelli and Smolin, between 1988 and 1990, to formulate the “loop representation of quantum general relativity,” which laid the foundations of Loop Quantum Gravity (LQG). Early developments of the subject were summarized in the review of Ref. [28–33]. In the 1990s, Brügmann, Gambini, and Pullin demonstrated that intersections of loops are essential for the consistency of the theory, and that quantum states of geometry should be described in terms of intersecting loops, i.e. graphs with links and nodes. Inspired by Penrose earlier ideas on the combinatorial structure of space, Rovelli and Smolin introduced the notion of spin

networks, later formalized mathematically by Baez, which provide an explicit orthonormal basis of states for quantum geometry. Spin networks are graphs whose links and nodes are labeled by representations of $SU(2)$, and their emergence marked a decisive step in the realization of background-independent quantum geometry. The discrete spectra of area and volume operators, first obtained in 1995, confirmed the long-suspected quantization of geometry at the Planck scale. Subsequent rigorous studies established spin networks (and their diffeomorphism-invariant classes, or s-knots) as the kinematical backbone of LQG, representing quantized 3-dimensional geometries without reference to any fixed background spacetime.

A particularly fruitful line of development has been the application of LQG to cosmology. In the symmetry-reduced setting of homogeneous spacetimes, this program, known as Loop Quantum Cosmology (LQC), was pioneered by Bojowald and has since matured into a well-established field. One of its most striking predictions is the resolution of the classical big bang singularity via a quantum bounce, suggesting that our expanding universe was preceded by a contracting phase.

Finally, LQG also admits a covariant, path-integral-like formulation known as spin foam theory. Introduced in the early 1990s and developed systematically in the 2000s, spin foams describe the evolution of spin networks and provide a discrete, combinatorial notion of quantum spacetime. The interplay between the canonical loop representation and the covariant spin foam approach has become a central theme in recent years, leading to the framework now often referred to as covariant loop quantum gravity. This synthesis continues to guide ongoing efforts toward a consistent and complete background-independent theory of quantum gravity.

1.3 String Theory

String Theory emerged in the late 1960s as an attempt to explain the observed properties of strongly interacting particles, known as hadrons, including protons, neutrons, and mesons. Experimental data, notably by G. Chew and S. Frautschi, revealed that mesons exhibited a striking linear relationship between their squared masses M^2 and their spin

J , forming patterns called *Regge trajectories*. Empirically [34],

$$J = \alpha' M^2 + \alpha_0 \quad (1.38)$$

where α' is the Regge slope, or inverse of the string tension. Such a relationship could not be easily explained within the standard point-particle framework of quantum field theory. A significant breakthrough came from the pioneering work of Y. Nambu [35–37], who independently proposed that the observed mass-spin relationship could be understood by modelling hadrons as one-dimensional relativistic strings under tension. In this picture, various vibrational modes of the string give rise to different particle states [38], while the rotational energy of the string accounts for the increase in mass with spin. This insight provided a profound new perspective [39]: rather than being fundamental point particles, hadrons might emerge as excitations of underlying string-like objects.

Around the same time, another remarkable property of hadronic interactions was discovered by R. Dolen, D. Horn, and C. Schmid [40]. Their analysis of hadronic scattering experiments showed that the s-channel (direct resonance production) and the t-channel (particle exchange) contributions appeared to describe the same total scattering amplitude. This empirical observation, known as *duality*, hinted at a deep and previously unexplored structural principle governing particle interactions. Building on these insights, Gabriele Veneziano constructed the first explicit scattering amplitude (the now-famous *Veneziano amplitude*) [41] which manifestly exhibited the property of duality. This achievement is now understood as an early glimpse of what later became known as *worldsheet duality* in string theory.

Further progress was made when C. Lovelace calculated a loop amplitude in the theory and found that to avoid inconsistency one required the spacetime dimension to be exactly 26 [42]. This was a remarkable and unexpected result, implying that for the theory to be consistent at the quantum level, the target spacetime must have a specific number of dimensions. Initially, string theory described only bosonic states, which meant it was incapable of accounting for half-integer spin particles (fermions) present in nature. This limitation was overcome in 1971 when Pierre Ramond [43] introduced fermionic degrees of freedom by formulating a two-dimensional supersymmetry on the string worldsheet. His work laid the foundation for *superstring theory*, allowing both bosons and fermions to emerge from string vibrations. Shortly thereafter, John Schwarz and André Neveu

extended the model [44], which resulted in the reduction of the required spacetime dimensions to 10.

To set the stage, we consider a fixed, pseudo-Riemannian spacetime $\mathcal{M}$ of dimension D , with coordinates X^μ ($\mu = 0, \dots, D-1$). The dynamics of relativistic strings in $\mathcal{M}$ are described on a two-dimensional worldsheet Σ with coordinates $\sigma^a = (\sigma^0, \sigma^1)$. The embedding of the worldsheet into spacetime is given by a smooth map:

$$X : \Sigma \rightarrow \mathcal{M}, \quad \sigma \mapsto X(\sigma). \quad (1.39)$$

The two-dimensional background metric induces a metric on a world-sheet as

$$G_{ab} = \frac{\partial X^\mu}{\partial \sigma^a} \frac{\partial X^\nu}{\partial \sigma^b} G_{\mu\nu}, \quad (1.40)$$

where $a, b = 0, 1$ are worldsheet indices. This induced metric G_{ab} is distinct from the intrinsic metric h_{ab} , which describes the intrinsic geometry of the worldsheet. In string theory, the intrinsic metric h_{ab} is treated as a dynamical object, acting as a background around which the string's vibrational modes oscillate.

The action describing a relativistic string is naturally proportional to the area of the worldsheet, leading to the *Nambu-Goto action* [35, 38, 45] written as

$$S_{\text{NG}} = \frac{1}{2\pi\alpha'} \int_{\Sigma} d^2\sigma \sqrt{\det G_{ab}}. \quad (1.41)$$

While this action has a direct geometric interpretation, the presence of the square root makes it technically cumbersome for quantization. A more convenient formulation is the *Polyakov action* [46] which is

$$S_P = \frac{1}{4\pi\alpha'} \int_{\Sigma} d^2\sigma \sqrt{h} h^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X), \quad (1.42)$$

where $h = |\det h_{ab}|$ and $\mu, \nu = 0, \dots, D-1$. This formulation describes the string as a two-dimensional field theory defined on the worldsheet. If one introduces a slowly varying background *dilaton field* $\Phi(X)$ and imposing conformal (Weyl) symmetry on the worldsheet, the action generalizes to

$$\mathcal{S} = \frac{1}{4\pi\alpha'} \int_{\Sigma} d^2\sigma \sqrt{h} \left(h^{ab} G_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu + \alpha' R \Phi(X) \right), \quad (1.43)$$

where R is the Ricci scalar associated with the worldsheet metric. From this action, one can derive local functionals for the coupling functions $G_{\mu\nu}$ and Φ . In particular, the functional $\beta_{\mu\nu}^g$ takes the form [47]:

$$\beta_{\mu\nu}^g + \frac{\alpha'}{8\pi^2} g_{\mu\nu} \beta^\Phi = -T_{\mu\nu}^{\text{matter}} + \alpha' \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) + \mathcal{O}(\alpha'^2). \quad (1.44)$$

Imposing the conformal invariance condition implies that these beta functionals vanish, yielding equations equivalent to the Einstein field equations in D -dimensional space-time. This demonstrates that gravity emerges naturally from the dynamics of strings. Compactification can then reduce the theory to the familiar four-dimensional spacetime.

After knowing that the theory of gravity naturally arises in string theory, we now move to prove that it's a quantum theory as well. The quantization of the string is performed by promoting the Fourier mode coefficients of the string coordinates to operators. For a closed string in the light-cone gauge, the general solution for the transverse coordinates can be decomposed into left- and right-moving modes or thought as a wave moving from right and one moving from left represented [48] as

$$X^i(\sigma, \tau) = x_{\text{CM}}^i + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \left(\frac{\alpha_n^i}{n} e^{in(\tau+\sigma)} + \frac{\tilde{\alpha}_n^i}{n} e^{in(\tau-\sigma)} \right), \quad (1.45)$$

where $i = 2, \dots, D-1$. The oscillators α_n^i and $\tilde{\alpha}_n^i$ satisfy the canonical commutation relations

$$[\alpha_n^i, \alpha_m^j] = n\delta_{n+m,0}\eta^{ij}. \quad (1.46)$$

The physical states are constrained by the *level-matching condition*:

$$L_0^\perp |\text{phys}\rangle = \tilde{L}_0^\perp |\text{phys}\rangle, \quad (1.47)$$

where $L_0^\perp$ and $\tilde{L}_0^\perp$ are the transverse Virasoro operators. The mass spectrum of string states is given by

$$M^2 = \frac{2}{\alpha'} (N^\perp + \tilde{N}^\perp - 2), \quad (1.48)$$

with $N^\perp$ and $\tilde{N}^\perp$ being the occupation number operators. Notably, the first excited states are massless and naturally include a symmetric, traceless rank-2 tensor, identified as the *graviton* and two massless vector fields, corresponding to gauge bosons.

Thus, string theory inherently incorporates both quantum gravity and gauge interactions within its framework.

The argument presented by Susskind [49, 50] offers a clear and instructive perspective on how strings fundamentally differ from point particles, highlighting the implications of this distinction for our ability to probe and resolve structures at extremely short distances. Let consider a free string expressed in light-cone coordinates as

$$X^\pm = \frac{X^0 \pm X^1}{\sqrt{2}}, \quad X^+ = 2\alpha' P^+ \tau, \quad (1.49)$$

where τ is the worldsheet time parameter. The transverse coordinates X^i exhibit a mean squared transverse size of

$$\langle (\Delta X^i)^2 \rangle \sim \alpha' \log n_{\max}, \quad (1.50)$$

where $n_{\max}$ is the mode cutoff. This implies that strings cannot probe arbitrarily short distances, introducing a natural minimal length scale of order $\sqrt{\alpha'}$, which plays a crucial role in regularizing ultraviolet divergences and resolving singularities inherent in point-particle theories.

Remarkably, string theory predicts the precise dimensionality of spacetime required for consistency such as for Bosonic string theory one need $D = 26$, Superstring theory $D = 10$ and M-theory $D = 11$ [51, 52]. These extra dimensions are compactified on small manifolds, typically of *Calabi–Yau type* [51], ensuring that low-energy physics appears effectively four-dimensional. However, this gives rise to the *vacuum selection problem*, with more than 10^{100} possible compactifications, leaving the specific realization describing our Universe is unknown. Moreover, supersymmetry predicts superpartners for all known particles, yet none have been experimentally observed to date. Despite these challenges, string theory remains a mathematically elegant and internally consistent framework for unifying quantum gravity and gauge interactions, providing deep insights into the structure of spacetime.

1.4 A Brief Summary

Despite the remarkable success of both quantum theory and general relativity in their respective domains, these two pillars of modern physics remain fundamentally incompatible when describing extreme regimes such as black hole singularities or the very early universe near the Planck scale. In this chapter, we have reviewed various approaches to quantum gravity that aim to resolve these inconsistencies, providing valuable insights into the unification of the two frameworks. Nevertheless, the quest for a complete and consistent theory of quantum gravity remains an open challenge. A key and universal feature emerging from nearly all quantum gravity frameworks is the existence of a *minimal measurable length*. In contrast to classical GR and conventional QFT, where spacetime is treated as a continuous and smooth manifold at all scales, quantum gravity suggests that this continuum picture breaks down near the Planck length $l_P \sim 10^{-35}$ meters. This fundamental limitation leads to the discretization of spacetime and modifies the Heisenberg uncertainty principle into a *generalized uncertainty principle (GUP)*, which implies a finite resolution limit for spatial measurements.

In the next chapter, we will explore the concept of minimal measurable length in detail, examining various *Gedanken experiments* and different quantum gravity approaches that motivate this profound insight.

Chapter 2

HISTORY OF MINIMAL LENGTH

2.1 Historical Origin

From Newtonian mechanics to modern quantum field theory, physical theories have traditionally been built on the assumption of a continuous and infinitely divisible spacetime manifold. This continuum is imagined as an unbroken collection of zero-dimensional points extending without bound. Yet, from antiquity, the question has persisted: *Can space truly be divided into arbitrarily small pieces, or is there an ultimate, indivisible “atom” of space?* This philosophical dilemma echoes back to Zeno’s paradoxes and has haunted natural philosophy for centuries.

In the early 20th century, David Hilbert himself raised doubts about the physical reality of the continuum. He argued that

“a homogeneous continuum admitting infinite divisibility is not encountered in nature, and thus “infinitesimals” remain metaphysical constructs rather than physical entities.”

Such a view has troubling implications for fundamental physics, as Feynman questioned

“Why should it take an infinite amount of logic to figure out what one tiny piece of spacetime is going to do?”

The earliest modern suggestion of a discrete spacetime came from Robert Levi in 1927, who proposed the *chronon*, a hypothetical smallest unit of time. However, the possibility of fundamental limits to the divisibility of space and time remained a far-reaching speculation. In the 1930s, with the advent of quantum field theory (QFT), many physicists began to suspect that the introduction of a fundamental length scale might cure the troublesome divergence in the theory. The ideas of fundamental finite length and minimum frequency was studied in these years including Fint [53], March [54], and Goudsmit [55], among others. They have the common consideration of fundamental length to be in the realm of subatomic physics to be of the order of a femtometer (10^{-15} m).

Among them, one exception is the young Russian physicist, Matvei Bronstein, often recognized as the father of quantum gravity. Already in 1936, Bronstein realized that gravity is fundamentally distinct from electrodynamics as it forbids arbitrarily high concentrations of charge (energy) in small regions of spacetime. If the concentration is too large in a small region, the region collapses into a black hole. Using the weak-field approximation of general relativity, he argued that this imposes an unavoidable limit on the precision with which the gravitational field expressed through the Christoffel symbols can be measured. His insight was an early precursor to the modern idea of a Planck-scale limit. Unfortunately, the history of a promising young physicist ended when he was executed in 1938 at the age of 31.

Earlier, Werner Heisenberg pursued another route to resolving divergences in QFT. He speculated that introducing noncommuting spacetime coordinates could regularize the infinities, particularly the electron self-energy. In correspondence with his student Rudolf Peierls in 1930, Heisenberg suggested deforming the canonical commutators so that position operators no longer commute, i.e. $[\hat{x}^\mu, \hat{x}^\nu] \neq 0$. However, he was unable to give this proposal a consistent mathematical foundation. Although Heisenberg remained hopeful that a fundamental length might shed light on the non-renormalizability of the Fermi theory of β -decay, progress in quantum electrodynamics (QED) soon made divergences manageable through renormalization. Consequently, the discrete approach of space remained unappealing due to the problem of Lorentz invariance.

It was only in 1947 that Hartland Snyder gave Heisenberg's idea a firm mathematical foundation. Snyder introduced modified commutation relations between spacetime and momentum operators, which made spacetime coordinates non-commutative while

maintaining Lorentz invariance. This modification implied a natural limitation on resolution, effectively introducing a minimal length, though Snyder himself did not explicitly highlight this consequence. Importantly, his approach avoided the arbitrary cutoffs in momentum space that many found objectionable. Nevertheless, difficulties with the incorporation of translations limited the acceptance of his model. Meanwhile, Bronstein's gravitational perspective quietly resurfaced. In 1960, Peres and Rosen analyzed the uncertainties in measuring the gravitational field itself, pointing out that attempting to probe regions smaller than the Schwarzschild radius leads to fundamental measurement uncertainties. However, they did not connect this directly with the limitations on measuring distances itself.

The decisive step came in 1964 with Alden Mead, who systematically examined thought experiments combining quantum mechanics and general relativity. He showed that gravity amplifies quantum uncertainties in such a way that no distance shorter than the Planck length can ever be resolved. Since gravity couples universally, this limitation, though negligible in everyday experiments, is inescapable in principle. Mead later recalled that his proposal was met with resistance: the prevailing view of the 1960s was that the Planck length could never play a fundamental role in physics. Many physicists instead favored other candidate fundamental lengths, such as the proton Compton wavelength, despite a lack of evidence.

In the 1970s, the issue reemerged with discovery of black hole thermodynamics by Hawking. His derivation of black hole radiation raised the notorious “trans-Planckian problem.” Because modes approaching a black hole horizon undergo infinite blueshift, contributions from trans-Planckian frequencies seemed necessary, even though physics at such scales was undefined. This motivated proposals to introduce a minimal wavelength or maximal frequency to resolve the paradox. Similar concerns arose in inflationary cosmology as tracing today's low-energy fluctuations backward in time implied that they originated at trans-Planckian scales. William Unruh (1995) and others suggested modified dispersion relations or generalized uncertainty principles (GUPs) as remedies, ensuring the smallest resolvable length.

Parallel algebraic developments appeared as well when Majid and Ruegg introduced the κ -Poincaré algebra, a Hopf algebra deformation of the Poincaré symmetry that naturally

accommodates non-commutative spacetime. Around the same time, Kempf and collaborators formulated a consistent framework of quantum mechanics and quantum field theory incorporating a minimal length, often expressed via a GUP. These approaches demonstrated that modifying the canonical commutators or dispersion relations could yield a smallest measurable scale in a Lorentz-compatible manner.

Although initially controversial, the concept of a minimal length later gained significant support from string theory. Studies of high-energy string scattering revealed that the Planck length functions as a fundamental limit of resolution. Since then, evidence for minimal length has arisen across a wide array of approaches, including quantum groups, noncommutative geometry, loop quantum gravity, Horava–Lifshitz gravity, causal dynamical triangulations, and even mechanisms of supersymmetry breaking.

Today, one does not need to adopt a specific theory of quantum gravity to motivate the existence of a minimal length. The combination of general relativity and quantum mechanics alone already implies its necessity, while effective field theory arguments provide further support. As comprehensive reviews now demonstrate, the minimal length has emerged as a recurring and robust feature across diverse and independent lines of reasoning. Unless otherwise stated, natural units ($\hbar = c = 1$) are adopted throughout this chapter.

2.2 Gedanken experiments

Gedanken experiments, known as “thought experiments,” have played a crucial role in the development of physics throughout history as a poor theoretical physicist’s way to limit of the theory. These limitations may arise from the lack of instruments or equipment, or from situations that are practically impossible to realize. Fortunately, technological advances have sometimes transformed thought experiments into real experiments, as was the case with the Einstein-Podolsky-Rosen (EPR) paradox of 1935. The advantages of the Gedanken experiments are that they allow theorists to explore domains of parameter space that are entirely inaccessible to current or foreseeable experimental capability. By doing so, thought experiments often uncover inconsistencies, paradoxes, or unexpected consequences, thereby highlighting potential weaknesses or gaps in existing theoretical frameworks. Also, a thought experiment connects theory to

reality by requiring careful consideration of what constitutes a measurable entity. In this Thesis, we will discuss some of the thought experiments related to minimal measurable length.

2.2.1 Heisenberg Microscope with Newtonian gravity

The Heisenberg microscope provides an intuitive experimental motivation for the uncertainty principle. Consider a photon of frequency ω used to measure the position of a particle along the x -axis. The scattered photon that enters the lens of the microscope must fall within an angle θ , and from classical optics, the resolution limit of the measurement is given by

$$\Delta x \gtrsim \frac{1}{2\pi\omega \sin \theta}. \quad (2.1)$$

Equation (2.1) represents the diffraction limit of the microscope. Increasing the photon frequency decreases its wavelength and therefore improves the spatial resolution of the measurement.

At the same time, the photon used for the measurement recoils upon scattering and transfers its momentum to the particle. However, the exact direction of the photon after scattering is uncertain within the angle θ , the corresponding uncertainty in the momentum of the particle is given as

$$\Delta p \gtrsim \omega \sin \theta. \quad (2.2)$$

The photon carries momentum $p_\gamma = \omega$ in the adopted natural units. Since the scattered photon can emerge anywhere within the angular aperture θ , only its transverse momentum component is uncertain. Combining these two estimates yields the familiar Heisenberg uncertainty relation,

$$\Delta x \Delta p \gtrsim \frac{1}{2\pi}. \quad (2.3)$$

This uncertainty in simultaneously measuring momentum and position of the particle is not merely a limitation of the measurement method but rather a fundamental property of the quantum nature of matter. Moreover, the photon does not scatter at a sharply localized point but interacts with the particle over a region of size r .

The above argument neglects the gravitational interaction between the probing photon and the measured particle. Since the photon carries energy ω , Einstein mass-energy equivalence implies that it also acts as a gravitational source. Consequently, during the measurement process the particle experiences an additional gravitational acceleration. Including gravitational effects into this scenario, following the treatment of Mead [56], the photon carries energy and therefore exerts a gravitational acceleration on the particle,

$$g \approx \frac{G\omega}{r^2}. \quad (2.4)$$

Assuming the particle is nonrelativistic compared to the photon, it acquires a velocity

$$v \approx gr \approx \frac{G\omega}{r}. \quad (2.5)$$

Over a time interval $\approx r$, the particle travels a distance

$$l = vr \approx G\omega. \quad (2.6)$$

Since the photon direction is uncertain within the angle θ , this leads to an additional position uncertainty of

$$\Delta x \gtrsim G\omega \sin \theta. \quad (2.7)$$

Combining the uncertainties from Eqs. (2.1) and (2.7), one finds a minimal position uncertainty of the order of the Planck length,

$$\Delta x \gtrsim \sqrt{G} = \ell_P. \quad (2.8)$$

The momentum measurement of the photon is also affected. The momentum of photon increases by an amount $\sim Gm\omega/r$, where m is the mass of the particle. Consequently, the uncertainty in the particle's momentum becomes

$$\Delta p \gtrsim \omega \left(1 + \frac{Gm}{r} \right) \sin \theta, \quad (2.9)$$

and the corresponding position uncertainty is estimated as

$$\Delta x \gtrsim \omega \left(\frac{r}{m} + G \right) \sin \theta. \quad (2.10)$$

This bound is larger than the gravitational uncertainty obtained in Eq. (2.7).

Adler and Santiago [57] further argued that the uncertainty of the momentum of particle should be at least of the order of the photon's momentum, i.e. $\Delta p \sim \omega$. This leads to

$$\Delta x \gtrsim G \Delta p. \quad (2.11)$$

If the usual uncertainty (from quantum mechanics) and the additional gravitational uncertainty are added linearly, one arrives at the generalized uncertainty relation

$$\Delta x \gtrsim \frac{1}{\Delta p} + G \Delta p. \quad (2.12)$$

This shows the standard uncertainty relation gets modified when gravity is taken into account with incorporating additional uncertainty due to gravity.

2.2.2 Heisenberg Microscope with General Relativity

The uncertainty relations acquire important refinements when considered in the framework of general relativity. Before going into the details, it is instructive to recall a general relativistic argument regarding black hole formation under conditions of high energy density localized in a region of spacetime. According to the Hoop Conjecture, if an energy ω is confined to a region such that the circumference in every spatial direction satisfies $R \leq 4\pi G\omega$, then the region will eventually collapse to form a black hole.

Let us now consider a particle whose position we wish to measure using a test particle. We assume that the test particle has a rest mass m_0 and a four-momentum $(\omega, \vec{k})$. The velocity v of the test particle is then given by

$$v = \frac{k}{\sqrt{m_0^2 + k^2}}, \quad (2.13)$$

where $k = |\vec{k}|$ and $k^2 = \omega^2 - m_0^2$. We assume that the test particle is moving along the x -axis.

To obtain the spacetime metric generated by the test particle, we first transform into its rest frame by performing a Lorentz boost along the x -direction. In this rest frame (denoted by primed coordinates), the measured particle moves in the $-x'$ direction, and

the spacetime geometry is described by the Schwarzschild metric. The relevant metric components are

$$g'_{00} = 1 + \frac{2Gm_0}{|x'|}, \quad g'_{11} = -\frac{1}{g'_{00}}, \quad g'_{22} = g'_{33} = -1, \quad (2.14)$$

with all other components vanishing. Introducing the Newtonian gravitational potential,

$$\Phi' = -\frac{Gm_0}{|x'|},$$

the metric may be written as $g'_{00} = 1 + 2\Phi'$, where $\Phi' < 0$. Transforming back to the laboratory frame by a Lorentz boost, the non-vanishing metric components become

$$g_{00} = \frac{1 + 2\Phi}{1 + 2\Phi(1 - v^2)} + 2\Phi, \quad g_{11} = -\frac{-1 + 2\Phi v^2}{1 + 2\Phi(1 - v^2)} + 2v^2\Phi, \quad (2.15)$$

$$g_{01} = g_{10} = -\frac{2v\Phi}{1 + \Phi(1 - v^2)} - 2v\Phi, \quad g_{22} = g_{33} = -1, \quad (2.16)$$

where

$$\Phi = \frac{\Phi'}{1 - v^2} = -\frac{G\omega}{r}, \quad (2.17)$$

and $r = vt - x$ denotes the mean separation between the test particle (t.p.) and the measured particle (m.p.). To avoid the formation of an event horizon in the rest frame, the metric component must satisfy $g'_{00} = 1 + 2\Phi' > 0$, which is equivalent to $-2\Phi' < 1$, since $\Phi' < 0$. Using Eq. (2.17), this condition becomes

$$-2\Phi' = \frac{2G\omega}{r}(1 - v^2) < 1. \quad (2.18)$$

Furthermore, the uncertainty in position is bounded by both $\Delta x \gtrsim 1/\omega$ and $\Delta x \gtrsim r$. Consequently,

$$\Delta x^2 \gtrsim \frac{r}{\omega} \gtrsim 2G(1 - v^2). \quad (2.19)$$

In the non-relativistic limit, $v^2 \ll 1$, this reproduces the lower bound on the spatial resolution obtained previously. We now consider the complementary ultrarelativistic limit, $1 - v^2 \ll 1$, which, according to Eq. (2.17), corresponds to the regime $-\Phi \gg 1$.

Motion of the measured particle: To proceed, we must estimate how much the m.p. is displaced by the gravitational attraction of the nearby t.p. The worldline of the m.p.

must remain timelike, which imposes the condition

$$ds^2 = (g_{00} + 2g_{10}v' + g_{11}v'^2)dt^2 \geq 0, \quad (2.20)$$

where v' denotes the velocity of the m.p. in the x -direction. Using the boosted metric Eq. (2.15), and following Mead, we introduce the abbreviation

$$\beta = 1 + 2\Phi(1 - v^2), \quad (2.21)$$

which, from Eq. (2.18), satisfies $0 < \beta < 1$. After some algebra (subtracting -1 and dividing $v' - 1$), and noting that $\Phi \leq 0$ and $v \leq 1$, Eq. (2.20) implies

$$v' \geq \frac{1 + 2\Phi(1 + \beta)}{1 - 2\Phi v^2(1 + \beta)}, \quad (2.22)$$

and equivalently

$$\frac{v'}{1 - v'} \geq -\frac{1}{2}(1 + 2\Phi). \quad (2.23)$$

Interaction time and displacement: Physically, we are interested in the displacement of the m.p. due to the gravitational influence of the t.p. The faster the two particles pass by each other, the shorter the interaction time, and consequently the smaller the displacement. For the shortest possible interaction time we set $v \rightarrow 1$, which corresponds to the minimal influence.

The time τ required for the t.p. to move a distance r away from the m.p. satisfies

$$\tau \gtrsim \frac{r}{1 - v'}, \quad (2.24)$$

and during this interval the m.p. is displaced by

$$l = v'\tau \gtrsim r \frac{v'}{1 - v'} \gtrsim \frac{r}{2}(-1 - 2\Phi). \quad (2.25)$$

In the limit $-\Phi \gg 1$, this reduces to

$$l \gtrsim G\omega. \quad (2.26)$$

Projecting onto the x -axis, we obtain as in the Newtonian gravity case that the uncertainty in the position of the measured particle due to the limited angular resolution of

the photon ϵ is

$$\Delta x \gtrsim G\omega \sin \epsilon. \quad (2.27)$$

Combining this with the usual Heisenberg microscope uncertainty, $\Delta x \gtrsim \frac{1}{2\pi\omega \sin \epsilon}$, one finally arrives at the minimal bound

$$\Delta x \gtrsim l_P, \quad (2.28)$$

where l_P is the Planck length. This adds a additional uncertainty in the measurement in position then the usual uncertainty as we discussed in above section.

2.2.3 Black Hole Horizons

As discussed in the previous subsection on the Heisenberg microscope, the uncertainty in position measurements increases when a high-energy test particle is used, since the particle itself generates a Schwarzschild geometry. A related question arises for black holes: to what precision can the size of a black hole itself be measured?

When the mass of a black hole approaches the Planck mass, the associated horizon radius $r \sim GM$ becomes comparable to its Compton wavelength $\lambda = 1/M$. This implies that quantum fluctuations in the position of the black hole affect the very definition of the horizon.

A more elaborate argument was developed by Maggiore through a thought experiment aimed at measuring the area of a non-rotating, charged black hole horizon. In Boyer-Lindquist coordinates, the event horizon is located at

$$r_H = GM \left[1 + \sqrt{1 - \frac{Q^2}{GM^2}} \right], \quad (2.29)$$

where Q is the charge and M is the mass of the black hole.

To determine the black hole area, one may detect its Hawking radiation and attempt to trace it back to its emission point with the best possible accuracy. For the case of an black hole, where $Q^2 = GM^2$, the Hawking temperature vanishes. In this case, Maggiore perturbed the black hole by sending in a photon from spatial infinity and then waited for it to be re-emitted.

If the microscope detects a photon of frequency ω , the finite wavelength of the photon introduces the standard uncertainty in localizing its origin. In addition, during the emission process the black hole mass changes from $M + \omega$ to M , which in turn shifts the horizon radius that we are attempting to measure. If the photon energy is known only up to an uncertainty Δp , this error propagates into the uncertainty of the black hole radius as

$$\Delta r_H \sim \left| \frac{\partial r_H}{\partial M} \right| \Delta p. \quad (2.30)$$

Using Eq. (2.29), and under the assumption that no naked singularities exist in nature (i.e. $Q^2 \geq GM^2$), one finds

$$\Delta r_H \gtrsim 2G\Delta p. \quad (2.31)$$

Maggiore then proposed that the two uncertainties the standard one, which is inversely proportional to the photon energy, and the additional one obtained in Eq. (2.31) should be added linearly, yielding

$$\Delta r_H \gtrsim \frac{1}{\Delta p} + \beta G\Delta p, \quad (2.32)$$

where β is a dimensionless constant to be fixed by a specific theory.

It is important to note that the uncertainty considered by Maggiore differs in nature from that of Mead, although both share a common origin. Here the uncertainty arises from the impossibility of directly measuring a black hole without altering its horizon via the emission of a particle carrying energy. The shorter the wavelength of the emitted particle, the larger the induced distortion.

2.3 Loop Quantum Gravity

In the previous chapter, we introduced the basic ideas of canonical quantum gravity, or more specifically, loop quantum gravity (LQG). We noted that from the densitized triads one can reconstruct the spatial metric, while from the connection A together with the triads one can recover the extrinsic curvature, thereby providing a full description of spacetime geometry. This formulation not only recasts G.R in the language of gauge theory but also supplies a natural canonical structure. At the classical level, the

fundamental Poisson brackets are given by

$$\{A_a^i(x), \tilde{E}_j^b(y)\} = \beta \delta_a^b \delta_j^i \delta^3(x - y), \quad (2.33)$$

where β is the Barbero–Immirzi parameter. Upon quantization, these classical variables are promoted to operators acting on wave functionals of the connection

$$\hat{A}_a^i \Psi(A) = A_a^i \Psi(A), \quad \hat{\tilde{E}}_i^a \Psi(A) = -i \frac{\delta \Psi(A)}{\delta A_a^i}, \quad (2.34)$$

and the Poisson bracket is then replaced by the standard commutator as

$$[\hat{A}_b^j(y), \hat{\tilde{E}}_i^a(x)] = i \delta_b^a \delta_i^j \delta^3(x - y). \quad (2.35)$$

As in ordinary quantum mechanics, one can choose different representations of the quantum states. In LQG, one particularly natural representation is the *loop representation*, where states are expanded in terms of traces of holonomies around closed loops. However, this loop basis is overcomplete (two different combinations of loops may describe the same state). A more refined and mathematically suitable basis is provided by *spin networks* ψ_s . A spin network is a graph with edges and vertices: the edges are labeled by irreducible representations of $SU(2)$, and the vertices are labeled by $SU(2)$ intertwiners that couple the incident edges. Thus, the kinematical states of LQG are represented by such labeled graphs. The power of this quantization scheme lies in the fact that one can construct geometric operators such as the *area* and *volume* operators and compute their spectra. In particular, the expansion in the spin-network basis makes it possible to explicitly calculate eigenvalues of these operators. In this thesis, however, we restrict ourselves to providing only an overview of the approach, without delving into the detailed calculations.

The Area Operator: Consider a two-dimensional surface Σ parameterized by coordinates (x^1, x^2) , with the third coordinate fixed at $x^3 = 0$. The area of this surface is given as

$$A_\Sigma = \int_\Sigma dx^1 dx^2 \sqrt{\det q^{(2)}}, \quad (2.36)$$

where the induced metric determinant is

$$\det q^{(2)} = q_{11}q_{22} - q_{12}^2.$$

In terms of the densitized triads, this expression can be rewritten as

$$A_\Sigma = \int_\Sigma dx^1 dx^2 \sqrt{\tilde{E}_i^3 \tilde{E}^{3i}}. \quad (2.37)$$

To promote this to a quantum operator, one replaces the classical triads with operators. Since the expression involves the square root of products of operators, one introduces a suitable regularization by averaging over smearing functions and then taking the limit where the smearing functions approach delta functions. Acting on a spin-network state, the regulated area operator yields the eigenvalue equation:

$$\hat{A}_\Sigma \psi_s = 8\pi \ell_P^2 \beta \sum_i \sqrt{j_i(j_i + 1)} \psi_s, \quad (2.38)$$

where the sum is taken over all spin-network edges that intersect the surface Σ , and j_i are the $1/2$ representation labels associated with those edges.

This result implies that the area spectrum in LQG is *discrete*, with the smallest nonzero eigenvalue given by

$$A_{\min} = 4\pi\sqrt{3}\beta\ell_P^2. \quad (2.39)$$

A similar construction can be carried out for the *volume operator*. In this case, the relevant expression involves products of three triads associated with a three-dimensional region. After quantization and regularization, one finds that the volume spectrum is likewise discrete and admits a nonzero smallest eigenvalue of the order of the Planck volume, ℓ_P^3 .

These results lead to a profound interpretation of spin networks. The *edges* of the graph correspond to fundamental quanta of area, each carrying a discrete area contribution proportional to $\ell_P^2 \sqrt{j(j+1)}$. The *vertices* of the graph correspond to quanta of three-dimensional volume. Thus, in LQG, space itself is quantized such that the geometry is not continuous but rather built up from discrete excitations of area and volume carried by the combinatorial structure of spin networks.

2.4 String Theory

In order to understand minimal length corrections within string theory, it is important to study one of its fundamental symmetries, known as *T-duality*.

When a spacelike dimension is compactified on a circle of radius R , closed strings acquire a new property of winding around the compactified dimension. The number of times a string wraps around this compactified dimension is denoted by a integer w , called the *winding number*.

For simplicity, let us consider a closed string in the presence of a single additional compact dimension. Along this compact direction the coordinate satisfies the boundary condition

$$X^4(\tau, \sigma + 2\pi) = X^4(\tau, \sigma) + 2\pi w R, \quad (2.40)$$

where $w \in \mathbb{Z}$ is the winding number. In addition, the momentum in the compact direction is quantized in integer multiples of $1/R$. The mode expansion for $X^4(\tau, \sigma)$ can therefore be written as

$$X^4(\tau, \sigma) = x_0^4 + \frac{n l_s^2}{R} \tau + w R \sigma + i \frac{\alpha'}{2} \sum_{n \neq 0} \left(\frac{\alpha_n^i}{n} e^{in(\tau+\sigma)} + \frac{\tilde{\alpha}_n^i}{n} e^{in(\tau-\sigma)} \right), \quad (2.41)$$

where x_0^4 is the initial position of the string in the compact dimension, $n \in \mathbb{Z}$ labels the quantized momentum modes, l_s is the string length ($l_s^2 = \alpha'$), and $\alpha_n^i, \tilde{\alpha}_n^i$ are the oscillator modes.

The conjugate momentum in the compact direction is given by

$$P^4(\tau, \sigma) = \frac{n}{R} + \frac{i}{\sqrt{2} l_s} \sum_{n \neq 0} \left(\alpha_n^i e^{in(\tau+\sigma)} + \tilde{\alpha}_n^i e^{in(\tau-\sigma)} \right). \quad (2.42)$$

The total energy of the string includes contributions from the momentum modes, winding modes, and oscillatory excitations. After renormalizing the divergent zero-point energy,

the mass spectrum takes the form

$$E^2 = \sum_{\mu \neq 4} P^\mu P_\mu + m^2, \quad (2.43)$$

$$m^2 = \frac{n^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} + \frac{2}{\alpha'} \left(N + \tilde{N} - \frac{D-2}{12} \right), \quad (2.44)$$

where μ runs over the non-compact dimensions, and N and $\tilde{N}$ denote the excitation levels of left- and right-moving modes, respectively. One also notes that the mass spectrum remains invariant under the exchange

$$R \longleftrightarrow \frac{l_s^2}{R}, \quad n \longleftrightarrow w. \quad (2.45)$$

This symmetry is known as *target-space duality*, or more commonly, *T-duality*.

T-duality implies that a compact dimension of radius R is physically indistinguishable from one of radius l_s^2/R . In particular, this symmetry suggests that distances smaller than the string length l_s cannot be probed, since probing a very small radius is equivalent to probing a large radius. Consequently, T-duality naturally leads to the idea of a minimal length scale in string theory.

A simple illustration is, consider the quantized momentum of a string on a spacetime circle of radius R from Eq.(2.42)

$$p = \frac{n}{R}, \quad n \in \mathbb{Z}, \quad (2.46)$$

and the corresponding energy of these momentum excitations (Kaluza-Klein modes) is

$$E_n = \frac{n^2}{R^2}. \quad (2.47)$$

On the other hand, the same string may wind w times around the compact dimension through T-duality. In this case, the energy associated with winding is

$$\tilde{E}_w = \frac{|w|R}{\alpha'}. \quad (2.48)$$

Thus, momentum modes dominate when R is large, while winding modes dominate when R is small. T-duality ensures that these two descriptions are physically equivalent. Since Kaluza-Klein momentum modes obey the usual uncertainty principle, $\Delta x \sim 1/\Delta p$, we

can relate the uncertainty in position to the string energy. From (2.47), one finds

$$\Delta p \approx \frac{\Delta E}{c} \approx \frac{\Delta x}{\alpha'}. \quad (2.49)$$

This leads to a modified uncertainty relation of the form

$$\Delta x \gtrsim \frac{1}{2\Delta p} + \alpha' \Delta p. \quad (2.50)$$

At low energies (small Δp), the first term dominates and one recovers the standard Heisenberg uncertainty principle. However, at high energies (large Δp), the second term dominates, indicating an additional “fuzziness” in spacetime. This implies the existence of a minimal resolvable length, of order $\Delta x_{\min} \sim l_s \equiv \sqrt{\alpha'}$

2.5 Summary

The concept of a minimal measurable length has captivated the attention of physicists since the mid-20th century, originating from early ideas such as the *chronon* and the notion of spacetime quanta. This chapter is devoted to the historical development of the idea of a minimal measurable length, presenting various arguments both in favor and against its necessity. Over time, thought experiments (Gedanken experiments) have played a pivotal role in revealing how gravity and quantum mechanics interplay to suggest a fundamental limit to spatial resolution. We discussed two significant Gedanken experiments in detail: the Heisenberg microscope modified by gravitational effects and the role of black hole horizons in limiting the precision of position measurements, as argued in works by Mead (1964) [56] and Adler et al. (1999)[58]. These investigations demonstrate that attempting to localize a particle within distances comparable to the Planck length inevitably leads to the formation of a micro black hole, preventing further localization.

From a theoretical perspective, different quantum gravity approaches support the idea of a minimal length. In Loop Quantum Gravity (LQG), the quantization of geometric operators such as area and volume leads to discrete spectra, implying a minimal measurable area and volume. In contrast, String Theory naturally incorporates a minimal

length scale due to the finite string tension α' , limiting the resolution of spatial structures and preventing probes beyond the string scale.

All these diverse approaches converge towards a common insight of fundamentally discrete spacetime structure and exhibits a fuzzy structure at the smallest scales. This realization has motivated both theoretical investigations and experimental efforts to detect possible quantum gravity signatures. Notably, modifications in dispersion relations, black hole thermodynamics, and deviations from standard quantum field theory predictions serve as promising windows into Planck-scale physics.

The next chapter is dedicated to the phenomenological implications of quantum gravity, focusing in particular on the Generalized Uncertainty Principle (GUP) as a well-motivated and widely studied framework. This phenomenological approach provides a concrete way to incorporate minimal length effects into quantum theory and explore their observable consequences.

Chapter 3

MODIFYING THE HEISENBERG UNCERTAINTY

Phenomenology is commonly described as “the science of phenomena as distinct from that of the nature of being,” or in physics, “the study of observable consequences of a theoretical model.”

For more than seventy years, the problem of quantum gravity has been one of the challenging problems in theoretical physics. As we discussed in the above chapters that a consistent quantum theory of gravity should be capable of resolving the conceptual and mathematical tensions that arise from the coexistence of general relativity and quantum mechanics. Historically, progress in physics has often been driven by experimental data such that unexpected observations challenge existing frameworks and motivate the development of new phenomenological models designed to address these puzzles. Frequently, models that at first appear incomplete or unsatisfactory turn out to be crucial stepping stones toward the correct theoretical description of new physical phenomena.

However, in the case of quantum gravity, the situation is different. Since we currently do not have the direct experimental guidance near the Planck scale, theorists are often compelled to construct candidate frameworks guided primarily by principles of mathematical consistency, conceptual elegance, or unification. This has led to long-standing debates about the merits of different approaches (such as string theory or loop quantum gravity) without a definitive experimental test to confirm them. The hope of eventually

identifying the correct theory of quantum gravity therefore relies on a process of theoretical exploration combined with indirect or novel experimental strategies, in a spirit of trial and error.

There are two complementary strategies that have emerged in the search for quantum gravity. The first is the so-called *top-to-bottom* approach, which attempts to build a fundamental theory starting from assumptions about the structure of spacetime at scales nearly seventeen orders of magnitude below those accessible to current experiments. Prominent examples include string theory and loop quantum gravity. The second strategy is the *bottom-to-top* approach, often referred to as *quantum gravity phenomenology*. This program attempts to extract possible experimental signatures of quantum gravity by working upward from known physics, guided by tentative insights from top-down theories. The aim is to identify characteristic features of spacetime or matter at the Planck scale that could manifest in observable phenomena, particularly in light of existing or forthcoming experimental sensitivities.

One of the earliest experiment at the interface of quantum and gravitational realms was performed by Colella, Overhauser, and Werner (commonly known as the *COW experiment*). The setup tested the validity of the Schrödinger equation for matter waves in the presence of Earth gravitational potential, described as

$$i\hbar \frac{\partial \Psi(t, \mathbf{r})}{\partial t} = \left[-\frac{\hbar^2}{2m_I} \nabla^2 + m_G \Phi(\mathbf{r}) \right] \Psi(t, \mathbf{r}), \quad (3.1)$$

where $\Psi(t, \mathbf{r})$ is the wavefunction of the particle, $\Phi(\mathbf{r})$ is the Newtonian potential of Earth, and m_I and m_G denote the inertial and gravitational masses, respectively. This framework is sometimes referred to as the Schrödinger-Newton model.

To avoid confounding effects from electric charges and electromagnetic interactions, the experiment was conducted using neutrons. The essential idea was to observe the interference pattern of neutron beams subjected to Earth gravitational field. The COW experiment exploited the fact that Earth gravitational potential arises from the coherent contribution of an enormous number of particles. Thus, despite the weakness of gravity at the single-particle level, the cumulative effect is sufficiently strong to introduce measurable phase shifts in the neutron wavefunction. The observed phase difference provided the first direct evidence that gravity produces observable effects in a quantum system.

Although quantum gravitational effects are expected to dominate only near the Planck scale in both length and energy, the COW experiment raised an important conceptual question about the scales at which the quantum aspects of spacetime become relevant. The experiment shows that, although our current experiments are still orders of magnitude away from probing such high energies, it is reasonable to expect that there will be some low-energy artifacts of the QG effects that show up in current experiments.

3.1 Planck-Scale Symmetries

3.1.1 Lorentz Symmetry

In the classical description, spacetime is modeled as a continuous and smooth manifold. Special relativity (SR), formulated in Minkowski spacetime, is built upon fundamental symmetries such as homogeneity and isotropy and admits a single frame-independent scale (the speed of light c) which governs the transformation laws between inertial observers. These transformations form the Poincaré group, with the Lorentz group as a subgroup. Within this framework, length itself is no longer a Lorentz-invariant quantity. Quantum gravity (QG) effects, however, are expected to become significant at extremely high energies near the Planck scale. The existence of a minimal measurable length, often proportional to the Planck length, challenges the classical notion of Lorentz invariance. In such scenarios, the fundamental symmetries of spacetime may be violated, modified, or generalized.

If spacetime is discretized at the Planck scale, an additional fundamental scale or structure is introduced, thereby affecting its symmetries. This possibility was already suggested by 't Hooft in 1996. In particular, continuous symmetry transformations such as Lorentz transformations appear incompatible with a discrete network of points. Furthermore, Lorentz symmetry is often at odds with frameworks involving spacetime non-commutativity. Such non-commutativity naturally introduces a minimal measurable length, analogous to the role of the uncertainty principle in quantum mechanics. As a consequence, the transformation laws between inertial observers are modified.

In this context, infinitesimal symmetries are no longer described by the standard Poincaré algebra but instead by deformations into Hopf algebras. Explicit studies in non-commutative

spacetimes have shown that the Lie algebra structure of the Poincaré symmetries is either broken into smaller subalgebras or deformed into Hopf-algebraic structures. Remarkably, this allows the rotational generators to remain unaffected, and the momentum operators to remain commutative, while the boost sector undergoes deformation.

A concrete realization of spacetime discretization arises in loop quantum gravity (LQG), one of the most developed approaches to quantum gravity. LQG does not describe spacetime as a rigid discrete lattice of points but instead introduces discreteness in a more sophisticated way, through quantized spectra of area and volume operators. In this framework, spacetime points themselves lose any well-defined identity. Nevertheless, it appears that even this refined form of discretization remains incompatible with the classical notion of Lorentz invariance.

3.1.2 CPT Symmetry

One of the most intriguing implications of quantum gravity (QG) theories is the possible violation of CPT symmetry due to the existence of a minimal measurable length, typically of the order of the Planck length $l_P \sim 10^{-35}$ meters. In standard quantum field theory, the CPT theorem is deeply connected to the principles of locality, unitarity, and Lorentz invariance. The CPT symmetry consists of three fundamental discrete transformations: charge conjugation (C), parity transformation (P), and time reversal (T).

However, in quantum gravity scenarios where spacetime acquires a discrete or “foamy” structure, such as in models of quantum spacetime foam or Loop Quantum Gravity, the standard assumptions underlying the CPT theorem no longer hold. Quantum gravity-induced decoherence provides a mechanism by which pure quantum states evolve into mixed states due to interactions with the quantum gravitational environment. This leads to a fundamentally ill-defined CPT operator and a breakdown of unitarity, distinguishing these effects from simple Lorentz violation models. Furthermore, modified dispersion relations expressed as

$$E^2 c^2 - p^2 = m^2 c^2 + \sum_{n=1}^N \eta^{(n)} \frac{p^n}{M_{\text{Pl}}^{n-2}},$$

where $\eta^{(n)}$ are dimensionless parameters fixing the strength of the Lorentz-violating corrections, and M_{Pl} is the Planck mass that explicitly break Lorentz invariance and induce CPT violation. Such modifications are conceptually similar to those proposed in the framework of Doubly Special Relativity (DSR). Using this modified dispersion relation, one can compute the shift in particle reaction thresholds in ultra-high-energy cosmic rays (UHECRs). In particular, the critical energy scale at which Lorentz violation becomes significant is given by

$$E_{\text{cr}} \sim (m^2 M_{\text{Pl}}^{n-2})^{1/n},$$

where m is the mass of the heaviest particle involved in the interaction. Above this critical energy, Lorentz-violating effects are expected to become manifest, potentially altering the reaction cross-sections of UHECRs.

Although stringent experimental bounds derived from precision measurements in neutral meson systems and high-energy astrophysical observations strongly constrain large CPT violations, more subtle effects remain plausible and are under active theoretical and experimental investigation. In particular, cosmological scenarios involving de Sitter horizons or large extra dimensions predict novel phenomenology, such as modified Einstein-Podolsky-Rosen (EPR) correlations in entangled meson factories or the potential observation of gravitons and microscopic black holes in high-energy collider experiments.

Some quantum gravity models predict an even stronger influence due to modifications of gravity at short distances, particularly in scenarios with large extra spatial dimensions. For instance, String Theory suggests that the fundamental Planck scale could be significantly lowered by the presence of extra dimensions, thus bringing quantum gravitational phenomena into the reach of current or near-future experiments. In such a case, distinct experimental signatures like missing energy due to graviton emission or mini black hole production and subsequent evaporation could serve as a striking confirmation of the quantum gravity paradigm.

3.2 The Generalized Uncertainty Principle

In Chapter 2, we discussed the emergence of a minimal measurable length as a common feature in various approaches to quantizing gravity. The quantization of gravity suggests that probing extremely small scales of spacetime requires high-energy probes, but at such energies, gravitational effects significantly perturb the spacetime geometry itself. As a consequence, the presence of a minimal uncertainty in position motivates a modification of one of the cornerstones of quantum mechanics: the Heisenberg Uncertainty Principle (HUP).

The standard HUP asserts that position and momentum cannot simultaneously be measured with arbitrary precision

$$\Delta x \Delta p \geq \frac{\hbar}{2}.$$

However, this relation does not accommodate a fundamental minimal length scale. To incorporate such effects, the Generalized Uncertainty Principle (GUP) was first proposed by Kempf, Mangano, and Mann (KMM) in 1995 [59], inspired by arguments from string theory and Gedanken experiments involving black holes as

$$[\mathbf{x}, \mathbf{p}] = i\hbar (1 + \mathcal{C}\mathbf{p}^2), \quad (3.2)$$

where $\mathcal{C}$ is a parameter characterizing the scale at which quantum gravitational effects become significant. Different papers and reviews and work on different approaches and work are in Ref. [60–71].

From the Schrödinger–Robertson uncertainty relation, one obtains the GUP form

$$\Delta x \Delta p \geq \frac{\hbar}{2} (1 + \mathcal{C}(\Delta p)^2), \quad (3.3)$$

where $(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2$. The dimension of $\mathcal{C}$ is inverse square momentum, and this relation implies a minimal position uncertainty of

$$\Delta x_{\min} = \hbar\sqrt{\mathcal{C}},$$

which is achieved when

$$\Delta p = \frac{1}{\sqrt{\mathcal{C}}}.$$

A direct consequence of this modified commutation relation is a deformation of the classical phase space structure. In the classical limit ($\hbar \rightarrow 0$), the commutator reduces to a deformed Poisson bracket

$$\frac{1}{i\hbar}[\mathbf{x}, \mathbf{p}] \longleftrightarrow \{x, p\} = 1 + \mathcal{C}p^2, \quad (3.4)$$

where x and p are classical phase space variables, and $\mathcal{C}$ can be expressed as

$$\mathcal{C} = \frac{\mathcal{C}_0}{(m_{Pl})^2} \equiv \frac{G_N \mathcal{C}_0}{\hbar c^3},$$

with $\mathcal{C}_0$ a dimensionless parameter encoding the strength of quantum gravity effects. Recent studies by Casadio and Scardigli have pointed out a potential inconsistency in this approach, that in transition from the commutator to Poisson bracket the minimal correction term involving $\mathcal{C}$ that have $\hbar^{-1}$ in it diverges as $\hbar \rightarrow 0$. A resolution is to assume $\mathcal{C}_0$ is vanishingly small, making the whole correction negligibly small but finite in the classical regime.

The presence of a minimal length implies significant consequences for the Hilbert space representation of quantum states. Physical states must remain normalizable, possess well-defined expectation values for position and momentum, and respect a nonzero minimal uncertainty in position:

$$(\Delta x)^2 = \langle \psi | (\mathbf{x} - \langle \mathbf{x} \rangle)^2 | \psi \rangle \geq \Delta x_{\min}^2. \quad (3.5)$$

As a result, true position eigenstates $|x\rangle$ no longer exist, because they would have zero uncertainty, contradicting the existence of a minimal length. Furthermore, the position operator is no longer essentially self-adjoint but only symmetric, ensuring real expectation values but no complete set of eigenstates.

The Hilbert space representation is thus modified from the standard position representation to a *generalized Bargmann-Fock representation* [59]. In this framework, one can define a quasi-position representation based on maximally localized states $|\psi_\lambda^{\text{ml}}\rangle$, satisfying

$$\langle \psi_\lambda^{\text{ml}} | \mathbf{x} | \psi_\lambda^{\text{ml}} \rangle = \lambda, \quad (3.6)$$

where λ denotes the quasi-position coordinate.

In this representation, the momentum and position operators act on quasi-position wave-functions $\psi(\lambda)$ as follows:

$$\mathbf{p} \cdot \psi(\lambda) = \frac{\tan(-i\hbar\sqrt{\mathcal{C}} \partial_\lambda)}{\sqrt{\mathcal{C}}} \psi(\lambda), \quad (3.7)$$

$$\mathbf{x} \cdot \psi(\lambda) = \left(\lambda + i\hbar\mathcal{C} \frac{\tan(-i\hbar\sqrt{\mathcal{C}} \partial_\lambda)}{\sqrt{\mathcal{C}}} \right) \psi(\lambda). \quad (3.8)$$

In the limit $\mathcal{C} \rightarrow 0$, the standard Heisenberg representation is smoothly recovered. However, the position operator in Eq. (3.8) is neither diagonalizable nor essentially self-adjoint on any dense domain of $\mathbf{x}^2$, implying that the spectrum of the position operator is continuous but subject to a fundamental minimal uncertainty. In contrast, the momentum spectrum remains continuous and have no measurable minimum and maximum. Also, in the momentum space the momentum eigenfunctions are Dirac delta function.

Building upon the foundational KMM GUP, further generalizations have been proposed to extend the validity of minimal length effects in quantum gravity. In particular, Nouicer introduced a GUP inspired by field theories on non-anticommutative superspaces, which captures a wider range of quantum gravitational phenomena [72]. In this formulation, the deformed commutation relation takes the form

$$[\mathbf{x}, \mathbf{p}] = i\hbar \exp(\mathcal{C}p^2), \quad (3.9)$$

applying the Schrödinger-Robertson uncertainty relation leads to the modified uncertainty relation as

$$\Delta x \Delta p \geq \frac{\hbar}{2} \exp(\mathcal{C}(\Delta p)^2). \quad (3.10)$$

In the momentum space representation, the operators satisfying this algebra

$$\mathbf{p} \cdot \phi(p) = p \phi(p), \quad (3.11)$$

$$\mathbf{x} \cdot \phi(p) = i\hbar \exp(\mathcal{C}p^2) \frac{\partial \phi(p)}{\partial p}. \quad (3.12)$$

To ensure the self-adjointness of the position operator, the scalar product in this representation is modified

$$\langle \psi | \phi \rangle = \int_{-\infty}^{\infty} \exp(-\mathcal{C}p^2) \psi^*(p) \phi(p) dp, \quad (3.13)$$

$$\langle p | p' \rangle = \exp(-\mathcal{C}p^2) \delta(p - p'). \quad (3.14)$$

In Nouicer's GUP the minimal measurable uncertainty in the position is given as

$$(\Delta x)_{\min}^{\text{Nouicer}} = \sqrt{\frac{e}{2}} \hbar \sqrt{\mathcal{C}}.$$

Later, Pedram proposed a further modification of the GUP to incorporate both a minimal position uncertainty and a maximal momentum cutoff [73]. The deformed commutation relation is expressed as

$$[\mathbf{x}, \mathbf{p}] = \frac{i\hbar}{1 - \mathcal{C}p^2}, \quad (3.15)$$

where the denominator introduces a singularity at $p^2 = \frac{1}{\mathcal{C}}$, effectively implying that the momentum of any particle is bounded by

$$|p| < \frac{1}{\sqrt{\mathcal{C}}}.$$

The corresponding uncertainty relation becomes

$$\Delta x \Delta p \geq \frac{\hbar/2}{1 - \mathcal{C}(\Delta p)^2}. \quad (3.16)$$

The momentum space representation of the position and momentum operators in the GUP framework proposed by Pedram is

$$\mathbf{p} \cdot \phi(p) = p \phi(p), \quad (3.17)$$

$$\mathbf{x} \cdot \phi(p) = \frac{i\hbar}{1 - \mathcal{C}p^2} \frac{\partial \phi(p)}{\partial p}. \quad (3.18)$$

TABLE 3.1: Comparison of different GUP models.

GUP Model	Deformed Commutator	Minimal Length Uncertainty	Maximal Observable Momentum
HUP	$[X, P] = i\hbar$	$\times$	$\times$
KMM	$[X, P] = i\hbar(1 + \mathcal{C}P^2)$	$\checkmark$	$\times$
Nouicer	$[X, P] = i\hbar \exp(\mathcal{C}P^2)$	$\checkmark$	$\times$
Pedram	$[X, P] = \frac{i\hbar}{1 - \mathcal{C}P^2}$	$\checkmark$	$\checkmark$

TABLE 3.2: Minimal measurable length and maximal momenta in three GUP models.

GUP Framework	Minimal length $(\Delta X)_{min}$	Maximal momentum P_{max}
KMM	$\hbar\sqrt{\mathcal{C}}$	—
Nouicer	$\hbar\sqrt{\mathcal{C}}\sqrt{\frac{e}{2}}$	—
Pedram	$\frac{3\sqrt{3}}{4}\hbar\sqrt{\mathcal{C}}$	$\frac{1}{\sqrt{\mathcal{C}}}$

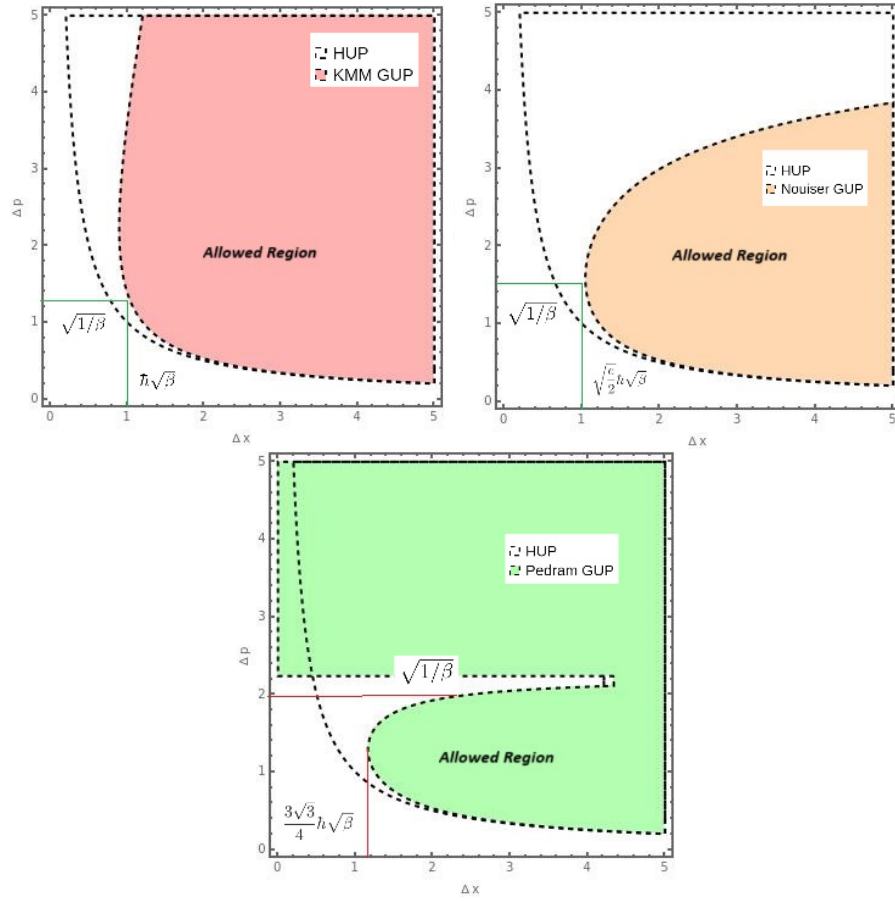


FIGURE 3.1: In this plot, we just replaced the GUP paramter $\mathcal{C}$ by β . Figure (a) on the upper left shows a comparison between HUP and KMM GUP, where the minimal length is given by $\hbar\sqrt{\beta}$. Figure (b) represents the comparison between Nouicer GUP and HUP, which again only includes minimal length but no maximal observable momentum. The final figure compares Pedram GUP with HUP, and we observe that in this GUP, both minimal length and maximal observable momentum are present.

To maintain the symmetricity of the position operator, the scalar product is appropriately modified as

$$\langle \psi | \phi \rangle = \int_{-1/\sqrt{\mathcal{C}}}^{1/\sqrt{\mathcal{C}}} \frac{dp}{1 - \mathcal{C}p^2} \psi^*(p) \phi(p), \quad (3.19)$$

$$\langle p | p' \rangle = \frac{\delta(p - p')}{1 - \mathcal{C}p^2}. \quad (3.20)$$

A concise comparison of the various GUP models discussed above is summarized in **Table 3.2**, which includes the corresponding minimal length scales. Additionally, **Figure 3.1** illustrates the behavior of the uncertainty relations for the KMM, Nouicer, and Pedram models, plotted against the standard Heisenberg Uncertainty Principle (HUP). It is noteworthy that the minimal position uncertainties $(\Delta X)_{\min}^{\text{KMM}} < (\Delta X)_{\min}^{\text{Nouicer}} < (\Delta X)_{\min}^{\text{Pedram}}$ shown in **Table 3.2**. In each panel, the black dashed curve represents the standard HUP, while the coloured curve denotes the corresponding GUP relation. The shaded region represents the physically allowed values of $(\Delta x, \Delta p)$ satisfying the corresponding uncertainty inequality.

Figure 3.1(a) shows the KMM quadratic GUP. In contrast to the HUP, the curve develops a minimal length at $(\Delta x)_{\min} = \hbar\sqrt{\beta}$, indicating that no physical state can probe distances smaller than this value. However, the momentum uncertainty remains unbounded, implying that the KMM model predicts a minimal measurable length but no maximal observable momentum.

Figure 3.1(b) presents the Nouicer exponential GUP. Similar to the KMM model, the uncertainty curve possesses a non-zero minimum position uncertainty, demonstrating the existence of a minimal measurable length. Nevertheless, the momentum uncertainty can increase without bound, and therefore this model also does not predict a maximal observable momentum.

Figure 3.1(c) corresponds to the Pedram GUP. Unlike the previous two models, the uncertainty curve terminates at a finite value of Δp , reflecting the singularity of the deformed commutator at $p = 1/\sqrt{\beta}$. Consequently, this model simultaneously predicts a non-zero minimal measurable length and a finite maximal observable momentum.

Beyond the commonly studied quadratic GUP models, a more generalized form has been proposed by Ali, Das, and Vagenas in 2011 [74]. This formulation is motivated by both

Doubly Special Relativity (DSR), ST, and Gedanken experiments in black hole physics, such that it incorporates a linear dependence on momentum in addition to the quadratic term. The resulting deformed commutation relation is

$$[q_i, p_j] = i\hbar \left\{ \delta_{ij} - \alpha \left(p\delta_{ij} + \frac{p_i p_j}{p} \right) + \alpha^2 [p^2 \delta_{ij} + 3p_i p_j] \right\}, \quad (3.21)$$

where $\alpha = \frac{\alpha_0}{m_P c} = \frac{\alpha_0 l_P}{\hbar}$ is the deformation parameter, with m_P denoting the Planck mass and $l_P \approx 10^{-35}$ m the Planck length. This GUP model implies the existence of both a minimal measurable length and a maximal observable momentum, expressed as

$$\Delta x \geq (\Delta x)_{\min} \approx \alpha_0 l_P, \quad (3.22)$$

$$\Delta p \leq (\Delta p)_{\max} \approx \frac{m_P c}{\alpha_0}. \quad (3.23)$$

To analyze the modified phase space structure, we express the deformed momentum p_j in terms of a low-energy momentum p_{0j} . Since Eq. (3.21) contains terms quadratic in p_i , the most general ansatz consistent with the algebraic structure is

$$p_j = p_{0j} + a p_0 p_{0j} + b p_0^2 p_{0j}, \quad (3.24)$$

where $a \sim \alpha$ and $b \sim \alpha^2$. Using the commutator relation and assuming the canonical structure for p_{0j} , we evaluate

$$\begin{aligned} [x_i, p_j] &= [x_i, p_{0j}] + a ([x_i, p_0] p_{0j} + p_0 [x_i, p_{0j}]) \\ &\quad + b ([x_i, p_0] p_0 p_{0j} + p_0 [x_i, p_0] p_{0j} + p_0^2 [x_i, p_{0j}]). \end{aligned} \quad (3.25)$$

The most general form of the deformed commutator consistent with Eq. (3.21) is thus

$$[x_i, p_j] = i\hbar \left(\delta_{ij} + \alpha_1 p \delta_{ij} + \alpha_2 \frac{p_i p_j}{p} + \beta_1 p^2 \delta_{ij} + \beta_2 p_i p_j \right), \quad (3.26)$$

now evaluating the Jacobi identity

$$[[x_i, x_j], p_k] + [[x_j, p_k], x_i] + [[p_k, x_i], x_j] = 0, \quad (3.27)$$

and matching terms, we obtain

$$[x_i, p_j] = i\hbar\delta_{ij} + i\hbar a \left(p\delta_{ij} + \frac{p_i p_j}{p} \right) + i\hbar(2b - a^2)p_i p_j + i\hbar(b - a^2)p^2\delta_{ij}. \quad (3.28)$$

On comparison with the original form in Eq. (3.21) gives the identification

$$a = -\alpha, \quad b = 2\alpha^2.$$

Therefore, the modified momentum and position operators can be expressed in terms of the low-energy momentum as

$$x_i = x_{0i}, \quad p_i = p_{0i}(1 - \alpha p_0 + 2\alpha^2 p_0^2). \quad (3.29)$$

This hierarchical structure illustrates how progressively stronger modifications of the commutator bracket correspond to increasingly significant deviations from the classical Heisenberg Uncertainty Principle. Such generalized models of GUP offer a richer phenomenological framework for probing quantum gravity effects at the Planck scale, particularly in contexts where both minimal length and maximal momentum are expected to play a role.

3.3 Relativistic Generalized Uncertainty Principle

In phenomenological studies of quantum gravity, GUP is employed to incorporate the existence of a minimal length scale by modifying the standard Heisenberg uncertainty principle. At low energies, this approach provides a valid and useful approximation. However, as we approach high-energy regimes, a fundamental issue emerges that is the violation of Lorentz covariance. This problem arises because a minimal length is not a Lorentz-invariant quantity. The breaking of Lorentz symmetry implies that a special choice of frame of reference which brings back the notion of aether, which stands in contradiction to the fundamental principles of Special Relativity.

Several attempts have been made to implement GUP in a relativistic context and within quantum field theory frameworks. Nevertheless, in these formulations, the minimal measurable length remains frame-dependent and hence not Lorentz invariant. To resolve

this issue, it is possible to formulate a Lorentz-covariant version of GUP that provides a frame-independent minimal length scale and reduces to the familiar non-relativistic GUP in the appropriate low-energy limit.

The GUP-modified commutator relation in D -dimensional space can be expressed as

$$[x_i, p_j] = i\hbar \delta_{ij} (1 + \mathcal{C} \vec{p}^2), \quad (3.30)$$

where $i, j \in \{1, 2, \dots, D\}$ and $\vec{p}^2 = p^k p_k$. Here, x_i and p_j represent *the physical position and momentum operators*, which are not canonically conjugate in the conventional sense, making their relationship less straightforward.

To address this complication, we first introduce canonically conjugate four-vectors x_0^μ and p_0^μ , defined

$$p_0^\mu = -i\hbar \frac{\partial}{\partial x_{0\mu}}, \quad [x_0^\mu, p_0^\nu] = i\hbar \eta^{\mu\nu}, \quad (3.31)$$

where $\eta^{\mu\nu}$ is the Minkowski metric. We then assume that the physical position and momentum operators are general functions of the auxiliary variables

$$x^\mu = x^\mu(x_0, p_0), \quad p^\mu = p^\mu(x_0, p_0). \quad (3.32)$$

Using a Taylor expansion of these physical operators in terms of the auxiliary variables, any general function $f(x_0^\mu, p_0^\mu)$ can be expressed as:

$$T(x_0^0, \dots, x_0^d, p_0^0, \dots, p_0^d) = \sum_{n_0=0}^{\infty} \dots \sum_{n_d=0}^{\infty} \sum_{m_0=0}^{\infty} \dots \sum_{m_d=0}^{\infty} \frac{(x_0^0 - a_0)^{n_0} \dots (x_0^d - a_0)^{n_d} (p_0^0 - b_0)^{m_0} \dots (p_0^d - b_d)^{m_d}}{n_0! \dots n_d! m_0! \dots m_d!} \quad (3.33)$$

$$\left(\frac{\partial^{n_0 + \dots + n_d m_0 + \dots + m_d} f(x_0^0, \dots, x_0^d, p_0^0, \dots, p_0^d)}{\partial x_0^{0n_0} \dots \partial x_0^{dn_d} \partial p_0^{0m_0} \dots \partial p_0^{dm_d}} \right) (a_0, \dots, a_d, b_0, \dots, b_d), \quad (3.34)$$

where a^μ and b^ν are constants. Applying this expansion to the physical position and momentum operators, and computing the commutator between them, we observe an important fact: the commutator between the two momentum operators must vanish, i.e.,

$$[p^\mu, p^\nu] = 0. \quad (3.35)$$

This ensures that momentum eigenvalues exist and can be measured with arbitrary precision. It also implies that the physical momentum operator cannot depend on the auxiliary position x_0^μ , but only on the auxiliary momentum p_0^μ .

The general expression for the commutator between two arbitrary functions of auxiliary variables takes the form:

$$[f(x_0, p_0), g(x_0, p_0)] \approx [p_0, p_0] \frac{i\hbar^3}{(l_p)^2} + [x_0, x_0] \frac{i\hbar^3}{(m_p c)^2} + [x_0, p_0] + \mathcal{O}(x_0^2, p_0^2). \quad (3.36)$$

Therefore, for the physical momentum operators to commute, they must be functions of the auxiliary momentum alone.

To ensure that the minimal uncertainty in position is Lorentz invariant, the physical position and momentum operators must themselves transform as four-vectors under the Poincaré algebra. Consequently, only terms in the Taylor expansion containing exactly one free index are allowed. Under these constraints, the physical position and momentum operators reduce to infinite polynomials of the auxiliary variables:

$$\begin{aligned} x^\mu(x_0, p_0) = & A_1 x_0^\mu + A_2 p_0^\mu + A_3 x_0^\mu x_0^\nu p_{0\nu} + A_4 x_0^\nu x_{0\nu} p_0^\mu \\ & + A_5 x_0^\mu x_{0\nu} p_0^\nu + A_6 x_0^\mu p_0^\nu p_{0\nu} + A_7 x_0^\nu p_0^\mu p_{0\nu} + A_8 p_0^\mu p_0^\nu p_{0\nu} + \dots, \end{aligned} \quad (3.37)$$

$$p^\mu(p_0) = C_1 p_0^\mu + C_2 p_0^\mu p_0^\nu p_{0\nu} + \dots, \quad (3.38)$$

where A_i and C_i are coefficients with appropriate dimensions.

Given the perturbative nature of our study, and to remain within a phenomenological framework, we neglect higher-order terms such as A_8 because they are extremely suppressed in any conceivable experimental setting.

After computing the spatial part of the commutator between physical position and momentum operators, and comparing it with the quadratic GUP form, we obtain:

$$x^\mu = x_0^\mu - \alpha_1 \beta_1 p_0^\rho p_{0\rho} + \alpha_3 \beta_1 p_0^\mu p_0^\rho x_{0\rho} + \alpha_4 \hbar \beta_1 p_0^\mu, \quad (3.39)$$

$$p^\mu = p_0^\mu (1 + \alpha_4 \beta_1 p_0^\rho p_{0\rho}), \quad (3.40)$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are dimensionless parameters, and β_1 has units of inverse squared Planck momentum.

The most general quadratic Lorentz-covariant GUP, which preserves a minimal measurable length, is therefore expressed as:

$$[x^\mu, p^\nu] = i\hbar [(1 + (\alpha_4 - \alpha_1)\beta_1 p^\rho p_\rho) \eta^{\mu\nu} + (\alpha_2 + 2\alpha_4)\beta_1 p^\mu p^\nu]. \quad (3.41)$$

This expression reduces to the standard Heisenberg algebra as $\beta_1 \rightarrow 0$, and to the non-relativistic GUP as $c \rightarrow \infty$.

From this, the non-commutativity of position operators follows naturally:

$$[x^\mu, x^\nu] = i\hbar\beta_1 \frac{-2\alpha_1 + \alpha_2}{1 + (\alpha_4 - \alpha_1)\beta_1 p^\rho p_\rho} (x^\mu p^\nu - x^\nu p^\mu). \quad (3.42)$$

This demonstrates the emergence of a non-commutative (fuzzy) spacetime structure. However, a conceptual challenge remains, this algebra does not close in a simple way, raising important questions about whether spacetime can be fully described by a single algebraic structure. We will carry out further discussion and application of the RGUP in the atomic spectra in Chapter 6.

3.4 Summary

In this chapter, we explored the phenomenological implications of quantum gravity (QG), focusing on the Generalized Uncertainty Principle (GUP) as a compelling approach to incorporate the idea of a minimal measurable length into quantum mechanics. Starting from the well-known Heisenberg Uncertainty Principle, we reviewed its limitations in the presence of gravitational effects, particularly at scales near the Planck length $l_P \sim 10^{-35}$ meters.

We discussed several GUP models proposed in the literature, including the KMM model, Nouicer's exponential GUP, and Pedram's formulation with both minimal length and maximal observable momentum. These models consistently predict that a fundamental limit exists in measuring positions or momenta, leading to significant modifications of the standard quantum mechanical framework. A detailed comparison of the different GUP models was presented, illustrating how the minimal uncertainty in position and the existence of a maximal momentum emerge in each case.

Building on the KMM GUP model, Todorinov [75] proposed a Lorentz-covariant formulation of the GUP that avoids frame-dependent effects and provides a consistent description of minimal length in a relativistic setting. By introducing auxiliary canonical variables and expressing physical position and momentum operators in terms of these, we derived a general quadratic GUP commutator in a manifestly Lorentz-covariant form.

An important outcome of this analysis is the natural emergence of non-commutative geometry in spacetime. The position operators no longer commute, signaling a fuzzy spacetime structure induced by quantum gravitational effects. While the modified algebra preserves consistency with known physics in the low-energy limit, it opens new avenues for exploring fundamental Planck-scale physics.

In the next and upcoming chapter, we will apply the GUP and RGUP framework to different cosmological models and atomic spectral systems, aiming to extract phenomenological predictions and potential test in laboratory experiments.

Chapter 4

LITERATURE REVIEW

The quest to unify quantum mechanics and general relativity has long been one of the most profound pursuits in theoretical physics. At the heart of this unification lies the radical idea that spacetime, contrary to classical intuition, might not be smooth and continuous but instead possesses a discrete structure at the Planck scale. This realization emerges naturally when one attempts to reconcile quantum theory with gravity, giving rise to the concept of a minimal measurable length — a notion elegantly encapsulated in the **Generalized Uncertainty Principle (GUP)**. The GUP modifies Heisenberg’s uncertainty principle to include quantum gravitational effects, suggesting that the act of measuring spacetime itself becomes fundamentally limited at high energies.

This idea found its first mathematically rigorous form in the seminal work of **Kempf, Mangano, and Mann (1995)** [59], who developed a consistent Hilbert space representation of the minimal length uncertainty relation. Their work demonstrated that if nature admits a smallest measurable length, the canonical commutation relations between position and momentum must necessarily be deformed. This deformation introduces nontrivial corrections to quantum mechanics, influencing the structure of the wavefunction, the energy spectrum, and the very geometry of spacetime. Their approach bridged the abstract mathematical formulation of quantum gravity with measurable quantum phenomena, setting the stage for the modern exploration of minimal-length physics.

The early 2000s witnessed a surge of interest in the physical implications of this modified uncertainty principle. **Adler and Santiago (2001)** [60] explored the gravitational origin of the uncertainty principle, showing that when gravitational effects are included, the

minimal length arises naturally, thereby linking quantum fluctuations to spacetime curvature. Building upon this foundation, **Ali, Das, and Vagenas (2009)** [74] proposed a linear form of the GUP that remained compatible with **Doubly Special Relativity (DSR)**, a theory introducing an invariant energy scale alongside the speed of light. This formulation offered a unified framework connecting minimal length effects with Lorentz symmetry deformation, a concept deeply intertwined with Planck-scale physics.

As the framework matured, physicists began to explore how GUP might manifest across different domains of physics. **Scardigli and Casadio (2016)** [61] demonstrated that GUP could introduce subtle yet measurable corrections to classical gravitational systems, such as the precession of planetary orbits. Similarly, **Tawfik and Diab (2015)** [62] provided a comprehensive review of the GUP's applications in cosmology, thermodynamics, and astrophysics, emphasizing its universal role as a phenomenological bridge between quantum mechanics and general relativity.

In parallel, **Pedram (2012)** [63] extended the formalism to higher orders, proposing a generalized model that preserved Lorentz covariance while introducing both minimal length and maximal momentum. Such formulations hinted at the deep algebraic restructuring of spacetime at high energies. These efforts collectively built the conceptual foundation for the application of GUP in early universe cosmology, black hole thermodynamics, and atomic physics.

The inflationary epoch of the universe provided a particularly fertile ground for exploring GUP-inspired corrections. Quantum fluctuations of scalar fields during inflation seed the large-scale structure of the cosmos, and any modification to their uncertainty relations could leave observable imprints on the cosmic microwave background (CMB). **Nozari and Kamali (2012)** [64] examined the impact of minimal length on inflationary parameters and showed that GUP deforms the slow-roll dynamics and alters the spectral index of primordial perturbations. Later studies, including those by **Shababi and Mousavi (2018)** [65], confirmed that these deformations could modify the duration and exit conditions of inflation, suggesting that the early universe might carry a quantum gravitational fingerprint encoded in its expansion history.

Simultaneously, black hole thermodynamics emerged as another natural arena for testing the implications of GUP. In the pioneering works of **Scardigli (1999)** [66], it was shown

that introducing GUP into Hawking’s framework regularizes the black hole temperature, preventing its divergence as the mass approaches the Planck scale. This correction implies the existence of stable black hole remnants — a potential resolution to the information loss paradox. Further refinements by **Cavaglià et al. (2004)** [67] and **Nouicer (2007)** [?] introduced logarithmic and inverse-area corrections to black hole entropy, providing a deeper thermodynamic consistency with quantum gravity predictions.

The reach of GUP, however, extends beyond cosmology and black holes — it reshapes the very symmetries underlying quantum mechanics. **Bosso and Das (2017)** [68] investigated how the GUP-deformed commutation relations affect angular momentum algebra. They discovered that at Planckian scales, the classical $SU(2)$ symmetry structure of angular momentum transforms into a deformed algebra that leads to small but conceptually significant corrections in eigenvalue spectra. These deformations indicate that spacetime symmetries themselves might acquire a quantum group structure, pointing toward the possibility of q -deformed or noncommutative spacetime symmetries. **Quesne and Tkachuk (2004)** [69] provided a Lorentz-covariant form of the deformed algebra, ensuring mathematical consistency and physical interpretability.

A particularly intriguing aspect of GUP phenomenology lies in its effect on atomic systems, which serve as a bridge between fundamental theory and laboratory-scale observables. **Benczik et al. (2002)** [70] and **Das and Vagenas (2008)** [71] showed that GUP induces minute but detectable shifts in the hydrogen atom spectrum, the Lamb shift, and the Landau levels. Such predictions can be extended to the Stark and Zeeman effects, where external electric or magnetic fields amplify the sensitivity of atomic systems to quantum gravitational corrections. These studies open exciting possibilities for indirect experimental tests of Planck-scale physics through precision spectroscopy, especially under strong-field astrophysical conditions like those near magnetars or white dwarfs.

Despite these remarkable advancements, the journey toward understanding the true implications of GUP remains far from complete. Existing theories of quantum gravity — string theory, loop quantum gravity (LQG), and doubly special relativity (DSR) — all predict a minimal length but differ in conceptual foundations and phenomenological predictions. String theory, though elegant, relies on a fixed background and has not yet recovered the Standard Model or provided experimental confirmation. Loop quantum

gravity offers background independence but faces difficulties in achieving a well-defined semiclassical limit and ensuring covariance of its constraint algebra. DSR, on the other hand, introduces energy-dependent modifications to Lorentz transformations, yet its experimental predictions remain ambiguous.

In this landscape of competing yet incomplete frameworks, the Generalized Uncertainty Principle stands out as a unifying phenomenological approach — simple, mathematically robust, and rich in predictive power. It captures the essence of minimal length effects without requiring a full quantum gravity theory. However, several critical questions remain unanswered. How does GUP modify the dynamics of inflationary scalar fields and influence the universe’s early evolution? How does the deformation of angular momentum algebra reshape our understanding of symmetry and conservation laws at the Planck scale? And can atomic spectroscopy truly serve as a window into quantum gravitational phenomena?

This thesis seeks to address these interconnected questions by systematically exploring the implications of the Generalized Uncertainty Principle on the fabric of spacetime and observable physics. Specifically, the objectives of this research are:

- **To investigate Generalized Uncertainty Principle (GUP) modified cosmological dynamics in the inflationary era**, analyzing its influence on scalar field evolution, slow-roll parameters, and the inflationary potential.
- **To analyze Generalized Uncertainty Principle (GUP) deformed algebraic structure of angular momentum**, uncovering how the modified commutation relations alter the quantization and symmetry structure of physical systems.
- **To study the signatures of Generalized Uncertainty Principle (GUP) in the atomic spectra**, including modifications in the fine structure, Zeeman, and Stark effects, with particular attention to strong-field astrophysical regimes.

Through these investigations, the present work aims to bridge the gap between abstract quantum gravity formalism and measurable physical consequences. By embedding GUP into diverse physical contexts — from the earliest moments of the universe to the smallest quantum systems — this study endeavors to reveal how the minimal length hypothesis might reshape our fundamental understanding of the universe.

Chapter 5

COSMOLOGICAL IMPLICATIONS

In the previous chapter, we discussed two of the most successful yet fundamentally incompatible frameworks in modern physics, one that governs the microscopic world, *Quantum Mechanics (QM)*, and the other that accurately describes gravitational phenomena at large scales, *General Relativity (GR)*. The unification of these two pillars of physics represents one of the greatest challenges in theoretical physics and forms the foundation of the ongoing quest for a *quantum theory of gravity*. A key insight emerging from this pursuit is that probing increasingly smaller length scales requires progressively higher energies, as dictated by the Heisenberg Uncertainty Principle. At sufficiently high energies, approaching the Planck scale, gravitational effects become significant, and the classical description of spacetime begins to break down. Such regimes provide ideal laboratories for exploring quantum gravitational effects, particularly in extreme environments such as black hole singularities and the early Universe near the Big Bang.

The Hot Big Bang cosmology has been remarkably successful in describing the large-scale evolution of the Universe. Within this framework, the Universe is assumed to be homogeneous and isotropic on cosmic scales, an assumption encapsulated in the *Cosmological Principle*. The *Standard Model of Cosmology (SMC)*, built upon this principle, employs the *Friedmann–Lemaître–Robertson–Walker (FLRW)* metric to describe the geometry of spacetime. For a spatially flat and spherically symmetric Universe, the metric takes

the form

$$ds^2 = -N^2(t) dt^2 + a^2(t) [dr^2 + r^2 d\Omega^2], \quad (5.1)$$

where $a(t)$ is the scale factor representing the cosmic expansion, and $N(t)$ is the lapse function ensuring time reparametrization invariance. The FLRW metric represents a maximally symmetric spacetime geometry that upholds the Copernican principle, forming the foundation of the SMC. One of its remarkable achievements is the prediction of the Cosmic Microwave Background (CMB) radiation. Despite its successes, however, the SMC encounters several fundamental challenges, notably the horizon problem, which refers to the absence of a causal connection between distant regions of the Universe. Additional issues include the flatness and entropy problems [76–81].

The inflationary paradigm, proposed by A. Guth in 1981, elegantly resolves these shortcomings by introducing a phase of nearly exponential expansion in the very early Universe. In this framework, a scalar field endowed with an appropriate potential, commonly referred to as the inflaton, acts as the driving agent of inflation. Beyond addressing the puzzles of the SMC, inflation provides a natural mechanism for the origin of large-scale structures observed in the Universe. Although the cosmos appears remarkably homogeneous and isotropic on large scales [82], tiny fluctuations of order 10^{-5} detected in the CMB serve as the seeds for cosmic structure formation. These small perturbations justify the application of cosmological perturbation theory, where the zeroth-order background remains the classical FLRW spacetime, while first-order deviations account for inhomogeneities. Physically, such perturbations originate from quantum fluctuations of the inflaton field, which in turn induce fluctuations in the gravitational field. These coupled perturbations lead to variations in energy density, ultimately giving rise to the clustering of matter and the formation of galaxies and cosmic structures. During inflation, these initially microscopic quantum fluctuations are stretched to super-Hubble scales, later re-entering the Hubble horizon as the Universe expands, thus offering a causal explanation for the observed large-scale structure [83–85].

Within the inflationary framework, the background spacetime is generally treated as classical, while first-order perturbations are quantized under the linearized theory of gravity. However, as one approaches the Planck scale [86, 87], this approximation becomes inadequate, since quantum effects of spacetime itself are expected to become significant. The absence of a unified framework capable of reconciling the principles

of GR and QM reflects one of the most profound challenges in modern cosmology. To address this, several approaches have been proposed, among which the *Generalized Uncertainty Principle (GUP)* offers a particularly promising avenue. The GUP provides a phenomenological way to incorporate quantum gravitational effects into low-energy cosmological dynamics. It arises naturally in various candidate theories of quantum gravity, such as String Theory[88–94], Doubly Special Relativity (DSR) [95, 96], and noncommutative geometry, by introducing a minimal measurable length and modifying the canonical commutation relations between position and momentum operators.

The idea that gravity could affect quantum uncertainty was first proposed by Mead in 1964 [56], who argued that at extremely high energies, gravitational effects would impose a lower bound on the precision with which position and momentum can be simultaneously measured. Subsequent developments have shown that these modifications yield important corrections in black hole thermodynamics, leading to a finite black hole temperature and entropy consistent with the Bekenstein–Hawking relation [58, 97–99]. In cosmology, GUP corrections have been shown to influence early-Universe dynamics, inflationary perturbations, and even the generation of primordial magnetic fields [100–108]. Thus, the GUP serves as a theoretical bridge between quantum gravity and observable cosmological phenomena.

Incorporating the effects of the GUP into cosmological models allows for the exploration of how quantum-gravity-induced deformations might alter the dynamical evolution of the Universe. One such avenue involves the study of scalar field models, particularly those involving quintessence and phantom fields, which have been instrumental in explaining cosmic acceleration. The *quintessential inflation model* extends the standard inflationary paradigm by employing a single classical scalar field, known as the *quintessence field*, to describe both early- and late-time cosmic acceleration. The field evolves under a suitable potential that enables a smooth transition from an inflationary phase to the current dark-energy-dominated era. Quintessence models often exhibit attractor behavior, meaning that their cosmological trajectories are largely insensitive to initial conditions, thereby alleviating the fine-tuning problem. The incorporation of quintessence into inflationary theory was pioneered by Peebles and Vilenkin [109], and such models have provided an elegant framework for understanding the evolution of the Universe [110–115].

In contrast, the *phantom field* represents a more exotic form of dark energy characterized by a negative kinetic term in the Lagrangian, leading to an equation of state parameter $\omega = p/\rho < -1$. This configuration results in a super-accelerated expansion that can culminate in a future singularity known as the *Big Rip*, where the scale factor diverges within a finite cosmic time. Despite its pathological outcome, phantom cosmology provides a valuable theoretical framework for testing the boundaries of gravitational theory and examining how quantum effects, such as those introduced by the GUP, might regulate or prevent such singularities [116–123].

In light of these developments, this chapter is devoted to exploring the cosmological dynamics of both phantom and quintessence scalar fields under the influence of GUP-induced corrections. Starting from the Einstein–Hilbert action, we construct the GUP-corrected Hamiltonian formalism and derive the modified Friedmann, Raychaudhuri, and Klein–Gordon equations. The resulting system is then analyzed using dynamical systems techniques to extract qualitative and stability information about the cosmological evolution. This comprehensive investigation, we aim to shed light on how quantum-gravity-inspired modifications may alter the behavior of scalar fields and potentially provide new insights into the early and late-time evolution of the cosmos.

5.1 Case I: Phantom Scalar Field

SMC is based on the “Copernican Principle”, which says that the universe is homogeneous and isotropic on a large scale. This is encoded in the maximally symmetric flat FLRW universe

$$ds^2 = -N^2(t)dt^2 + a^2(t) [dr^2 + r^2 d\Omega^2]. \quad (5.2)$$

Since GR is a field theory, its dynamics can be obtained from the Euler-Lagrange equation by varying the metric and the matter field of the Einstein-Hilbert action:

$$S_{EH} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} R + \mathcal{L}_m, \quad (5.3)$$

to give the Einstein’s equations $G_{\mu\nu} = \kappa T_{\mu\nu}$, Where $G_{\mu\nu}$ is the Einstein Tensor, $T_{\mu\nu}$ the energy momentum tensor, and $\kappa \equiv 8\pi G/c^4$ is set equal to one for the rest of the chapter owing to the usage of the natural units.

The discovery of late time acceleration of the universe has led to two major approaches to address the issue of the acceleration. One is based on modifying the gravity sector and the other is on the matter sector [124–127]. In this chapter, we focus on the second approach, that is, to modify the matter content to address the early epoch of the universe. To this effect, we adopt a phantom scalar field. This has been extensively studied in the context of the late-time era of evolution. The fact that the phantom field produces a phase of accelerated expansion of the universe makes it interesting to investigate its implication in inflationary dynamics as well.

One of our prime focuses is to explore the tail-end dynamics of the universe where QG effects is still important but not necessarily dominant. In the domain of LQC, this has been reported as *transition phase* from quantum to the classical universe in pre-inflationary dynamics [128–130]. However, this chapter will take a radically different approach by directly invoking a GUP in the cosmological Hamiltonian. This is an effective way of modeling the quantum corrected background evolution of the universe. In this section and in what follows, we review the construction of the GUP-deformed background equation of motion for a phantom scalar field.

5.1.1 A Classical View

In this subsection, we consider the Einstein–Hilbert action with a minimally coupled phantom scalar field and a positive cosmological constant,

$$S_{\text{EH}} = \int \sqrt{-g} \left[\frac{1}{2\kappa} (R - 2\Lambda) + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] d^4x, \quad (5.4)$$

on the background of a maximally symmetric spacetime described by Eq.(5.1), where $V(\phi)$ is the potential of the scalar field ϕ . Given the flat FLRW background, our action takes the following form

$$S_{\text{EH}} = V_0 \int dt \left[-\frac{3a\dot{a}^2}{N} - a^3 \left(\frac{\dot{\phi}^2}{2N} + N(\Lambda + V) \right) \right], \quad (5.5)$$

where V_0 is the volume of a fiducial cells, introduced to facilitate our calculations in a non-compact flat FLRW spacetime. Later, we can take the limit $\lim V_0 \rightarrow \infty$, as the final results will be independent of its values. Therefore, for the sake of simplicity, it can

be set to 1 without loss of generality. Recall that in this chapter we choose the natural units so that $\kappa = 1$.

Thus, from Eq.(5.5) we can see that the the Lagrangian density is given by

$$\mathcal{L} = -\frac{3a\dot{a}^2}{N} - a^3 \left(\frac{\dot{\phi}^2}{2N} + N(\Lambda + V) \right), \quad (5.6)$$

from which we can see that $\mathcal{L}$ does not depend on $\dot{N}(t)$. Hence there is no dynamics in the lapse function $N(t)$, as now we have $P_N \equiv \frac{\partial \mathcal{L}}{\partial N} = 0$. Therefore, the dynamics of the system are completely contained in the equations of motion for (a, P_a, ϕ, P_ϕ) governed by the Hamiltonian

$$\mathcal{H} = -N \left(\frac{P_a^2}{12a} + \frac{P_\phi^2}{2a^3} - a^3(\Lambda + V) \right), \quad (5.7)$$

which is obtained from Eq.(5.6) through the Legendre transformation, where $P_a \equiv \frac{\delta \mathcal{L}}{\delta \dot{a}}$ and $P_\phi = \frac{\delta \mathcal{L}}{\delta \dot{\phi}}$ are the conjugate momentum to a and ϕ , respectively, with the symplectic structure,

$$\{a, P_a\} = 1, \quad \{\phi, P_\phi\} = 1. \quad (5.8)$$

Then, the corresponding Friedmann, Raychaudhuri and Klein-Gordon equations (see appendix B) are given respectively by

$$3H^2 = -\frac{\dot{\phi}^2}{2} + \Lambda + V(\phi), \quad (5.9)$$

$$2\frac{\ddot{a}}{a^2} + \left(\frac{\dot{a}}{a}\right)^2 = \frac{\dot{\phi}^2}{2} + \Lambda + V(\phi), \quad (5.10)$$

$$\ddot{\phi} + 3\dot{\phi}\left(\frac{\dot{a}}{a}\right) - \frac{dV(\phi)}{d\phi} = 0. \quad (5.11)$$

.

5.1.2 Dynamics in the Presence of Minimal Length

In this subsection, we incorporate the higher-order correction of the uncertainty principle in the cosmological Hamiltonian without the potential $V(\phi)$. To achieve our goal, we first perform a canonical transformation of the phase space in x and y variables such that our Hamiltonian gets simplified and making it easier to incorporate the effects of

GUP as follows:

$$x = \frac{a^{3/2}}{\mu} \sin(\mu\phi), \quad y = \frac{a^{3/2}}{\mu} \cos(\mu\phi), \quad (5.12)$$

while preserving the dynamics. From Eq.(5.12), with $\mu = \sqrt{3/8}$ we obtain the following

$$\dot{x}^2 \sin^2(\mu\phi) + \dot{y}^2 \cos^2(\mu\phi) + 2\dot{x}\dot{y} \sin(\mu\phi) \cos(\mu\phi) = 6a\dot{a}^2, \quad (5.13)$$

$$\dot{x}^2 \cos^2(\mu\phi) + \dot{y}^2 \sin^2(\mu\phi) - 2\dot{x}\dot{y} \cos(\mu\phi) \sin(\mu\phi) = a^3 \dot{\phi}^2. \quad (5.14)$$

In addition, by considering the sum of the squares of the components in Eq.(5.12), we have

$$x^2 + y^2 = \frac{a^3}{\mu^2} \sin^2(\mu\phi) + \frac{a^3}{\mu^2} \cos^2(\mu\phi) = \frac{8a^3}{3}. \quad (5.15)$$

Now, from Eq.(5.15) we observe that the physical volume of the universe under study can be elegantly expressed as the radius of the circle, with $(0, 0)$ as the center in the plane containing the configuration variable (x, y) . Since the Friedmann equation is nothing but the fractional rate of change of volume, intuitively, one can speculate that the knowledge of the dynamics of the pair (x, y) suffices to predict the evolution of the universe.

Having set the stage in terms of the Cartesian pair (x, y) , we now return to the question of dynamics. We obtain the final form of the Lagrangian in terms of configuration coordinates $(x, y, \dot{x}, \dot{y})$ to be

$$\mathcal{L} = - \left[\frac{\dot{x}^2 + \dot{y}^2}{2N} + \frac{3}{8}(x^2 + y^2)\Lambda N \right]. \quad (5.16)$$

A quick look at the form of the Lagrangian suggests that dynamics is symmetric w.r.t the origin of the circle in the plane of (x, y) .

Given this, it is straightforward to obtain the canonically transformed Hamiltonian using the Legendre transformation. Thus, the final form of the Hamiltonian takes the form

$$\mathcal{H}_0 = N \left[\frac{P_x^2}{2} + \frac{P_y^2}{2} + \frac{\omega^2}{2}(x^2 + y^2) \right], \quad (5.17)$$

where we set $\omega^2 = -\frac{3}{4}\Lambda$, which, formally, represents the frequency of two independent harmonic oscillators. The utility of the canonical transformation of Eq.(5.12) is clear from the elegant expression of Eq.(5.17), and the dynamics of a universe with a phantom

scalar and a positive cosmological constant can be expressed as a system of two decoupled simple harmonic oscillators in the new phase space representation.

However, Eq. (5.17) is still classical, though expressed in a different form. The minimal uncertainty in position at the Planck scale is a feature present in many QG theories [131]. On the other hand, the HUP, one of the pillars of QM, allows for an arbitrary value of uncertainty in position as long as the product of the position and momentum uncertainties is larger than $\frac{\hbar}{2}$ [107, 132, 133]. With the existing HUP, we have an arbitrary choice of uncertainty, and hence precision in position, which motivates modifying the Heisenberg relation. One of the most notable and heuristic approaches is the GUP, first introduced by Kempf, Mangano, and Mann (KMM) in 1995, where the uncertainty relation in one dimension is obtained via the deformed commutator bracket

$$[x, p] = i\hbar(1 + \alpha^2 p^2). \quad (5.18)$$

Later, a more general form of GUP was proposed by Ali, Das, and Vagenas [134] in 2011, which incorporates linear and quadratic dependence on momentum in different QG theories. The presence of the quadratic term is dictated by String/M Theory [135, 136] and black hole physics [137–140], while the linear momentum-dependent term is motivated by Doubly Special Relativity (DSR) [141]. The GUP deformed commutator bracket is expressed as

$$[q_i, p_j] = i\hbar \left\{ \delta_{ij} - \alpha \left(p\delta_{ij} + \frac{p_i p_j}{p} \right) + \alpha^2 [p^2 \delta_{ij} + 3p_i p_j] \right\}, \quad (5.19)$$

where $\alpha = \alpha_0/M_{Pl}c = \alpha_0 l_{Pl}/\hbar$, M_{Pl} = Planck mass, $l_{Pl} \approx 10^{-35}m$ = Planck length, and $M_{Pl}c^2 \approx 10^{19}GeV$ = Planck energy.

To write down the dynamics due to momentum deformation owing to GUP, we introduce the semi-classical canonical variables q_i and P_i and GUP in the WDW equation in our cosmological model as followed by [142, 143]. The modified commutator relation incorporating the linear and quadratic dependence of momentum [131, 134] is

$$[q_i, P_j] = i\hbar \delta_{ij} (1 - 2\beta\gamma P_0 + 4\epsilon\gamma^2 P_0^2). \quad (5.20)$$

Using this commutator relation, we find that the canonical variable is approximately expressed as

$$q_i = q_{0i}, \quad P_i = P_{0i} \left(1 - \beta\gamma P_0 + 2\gamma^2 \frac{\beta^2 + 2\epsilon}{3} P_0^2 \right), \quad (5.21)$$

where $P_0^2 = P_{0j}P_{0j}$, γ is related to the scales where quantum-gravitational effects became relevant, typically defined by a value proportional to the inverse of the Planck momentum as $\gamma \equiv \frac{\gamma_0}{M_{Plc}}$ with $\gamma_0 \sim 1$. The parameters β and ϵ are dimensionless and highlight the terms originating from linear and quadratic contribution to GUP. For $\beta = 0$, we recover the KMM GUP model.

Now we calculate the GUP distorted Hamiltonian up to the order of γ^2 [144], which is

$$\mathcal{H} = \mathcal{H}_0 - \beta\gamma(P_{0x}^2 + p_{0y}^2)^{3/2} + \gamma^2(P_{0x}^2 + p_{0y}^2)^2 \left(\frac{\beta^2}{6} + \frac{2\epsilon}{3} \right) + \mathcal{O}(\gamma^3), \quad (5.22)$$

where $\mathcal{H}_0 \equiv \frac{P_{0x}^2}{2} + \frac{P_{0y}^2}{2} + \frac{\omega^2}{2}(x^2 + y^2)$, is the unperturbed Hamiltonian before introducing GUP. From now on, the subscript 0 will be used to denote the unperturbed version of Eq.(5.17). For example, the unperturbed x, y are represented as (q_{0x}, q_{0y}) while the unperturbed pair (P_x, P_y) as (P_{0x}, P_{0y}) .

In the final step, we re-express the GUP deformed Hamiltonian Eq.(5.22) in terms of the cosmological phase-space variables. This is achieved by applying the inverse transformation to express Eq.(5.22) in the cosmological variables, namely, the expansion factor, the scalar field, and their corresponding conjugate momenta, where

$$P_{0x} \equiv \dot{x} = \frac{3}{2} \frac{a^{1/2} \dot{a}}{\mu} \sin(\mu\phi) + (a^{3/2} \dot{\phi}) \cos(\mu\phi), \quad (5.23)$$

$$P_{0y} \equiv \dot{y} = \frac{3}{2} \frac{a^{1/2} \dot{a}}{\mu} \cos(\mu\phi) - (a^{3/2} \dot{\phi}) \sin(\mu\phi). \quad (5.24)$$

Because there are no dynamics in the lapse function, without loss of generality we choose $N(t) = 1$. Then, we find that $P_a \equiv \frac{\partial \mathcal{L}}{\partial \dot{a}} = -6\dot{a}a$, $P_\phi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = -a^3 \dot{\phi}$. Substituting P_a and P_ϕ into Eq.(5.23) we obtain

$$P_{0x} = -\frac{P_a}{4a^{1/2}\mu} \sin(\mu\phi) - \frac{P_\phi}{a^{3/2}} \cos(\mu\phi), \quad (5.25)$$

$$P_{0y} = -\frac{P_a}{4a^{1/2}\mu} \cos(\mu\phi) + \frac{P_\phi}{a^{3/2}} \sin(\mu\phi). \quad (5.26)$$

Then, applying P_{0x} and P_{0y} in the momentum-deformed Hamiltonian due to GUP correction in Eq.(5.22), we get

$$\mathcal{H}_{\text{GUP}} = -N \left[\frac{\omega^2}{2} a^3 + \frac{P_a^2}{12a} + \frac{P_\phi^2}{2a^3} - \beta\gamma \left(\frac{P_a^2}{6a} + \frac{P_\phi^2}{a^3} \right)^{3/2} + \gamma^2 \left(\frac{P_a^2}{6a} + \frac{P_\phi^2}{a^3} \right)^2 \left(\frac{\beta^2}{6} + \frac{2\epsilon}{3} \right) + \mathcal{O}(\gamma^3) \right]. \quad (5.27)$$

Now, the full dynamics as dictated by the Hamiltonian of Eq.(5.27) involve all the three parameters β , ϵ , and γ . To simplify the analysis and obtain physically relevant dynamics, we focus on the γ -axis of the quantum parameter phase space, although the three quantum parameters β , ϵ , and γ are in principle independent. So, the GUP-modified Hamiltonian is

$$\mathcal{H}_{\text{GUP}} = -N \left[\frac{\omega^2}{2} a^3 + \frac{P_a^2}{12a} + \frac{P_\phi^2}{2a^3} + 2\gamma^2 \left(\frac{P_a^4}{108a^2} + \frac{P_\phi^4}{3a^6} + \frac{P_a^2 P_\phi^2}{9a^4} \right) \right]. \quad (5.28)$$

This is the required GUP distorted Hamiltonian in the cosmological phase space dominated by a phantom scalar field with a positive cosmological constant up to second-order perturbation. We obtain, using Hamilton's equation

$$-\dot{P}_a = \frac{\partial \mathcal{H}_{\text{GUP}}}{\partial a} = \frac{-P_a^2}{12a^2} - \frac{3P_\phi^2}{2a^4} + 3a^2 \frac{\omega^2}{2} - \gamma^2 \left(\frac{P_a^4}{27a^3} + \frac{4P_\phi^4}{a^7} + \frac{8P_a^2 P_\phi^2}{9a^5} \right) \quad (5.29)$$

by substituting from the Lagrangian canonical equation

$$6\dot{a}^2 + 6a\ddot{a} = 3\dot{a}^2 + \frac{3a^2\dot{\phi}^2}{2} - \frac{3a^2\omega^2}{2} + \gamma^2 \left(48a\dot{a}^4 + 4a^5\dot{\phi}^4 - 32a^3\dot{a}^2\dot{\phi}^2 \right).$$

The Raychaudhuri equation reads as,

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 = \frac{\dot{\phi}^2}{2} - \frac{\omega^2}{2} + \gamma^2 \left(16a^3 H^4 + \frac{4a^3\dot{\phi}^4}{3} + \frac{32a^3 H^2 \dot{\phi}^2}{3} \right), \quad (5.30)$$

similarly, the modified Friedmann equation is obtained by Hamilton's equation and the Lagrangian canonical equation,

$$\frac{\partial \mathcal{L}}{\partial \dot{a}} = p_a = -6\dot{a}a \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = p_\phi = -a^3\dot{\phi} \quad (5.31)$$

with the GUP-modified Hamiltonian

$$\mathcal{H}_{GUP} = N \left[\frac{P_a^2}{12a} + \frac{P_\phi^2}{2a^3} + \frac{a^3\omega^2}{2} + 2\gamma^2 \left(\frac{P_a^4}{108a^2} + \frac{P_\phi^4}{3a^6} + \frac{P_a^2 P_\phi^2}{9a^4} \right) \right]. \quad (5.32)$$

Varying action with respect to the *lapse function* N gives the GUP distorted Hamiltonian constraint $\mathcal{H}'_{GUP} = 0$, given in [145].

$$\begin{aligned} \mathcal{H}'_{GUP} = & 3\dot{a}^2 a + \frac{a^3 \dot{\phi}^2}{2} + \frac{a^3 \omega^2}{2} + 2\gamma^2 \left(12a^2 \dot{a}^4 + \frac{a^6 \dot{\phi}^4}{3} + 4a^4 \dot{a}^2 \dot{\phi}^2 \right) \\ & \left(\frac{\dot{a}}{a} \right)^2 + \frac{1}{3} \left[\frac{\dot{\phi}^2}{2} + \frac{\omega^2}{2} + 2\gamma^2 \left(12a^3 \left(\frac{\dot{a}}{a} \right)^4 + \frac{a^3 \dot{\phi}^4}{3} + 4a^3 \dot{\phi}^2 \left(\frac{\dot{a}}{a} \right)^2 \right) \right] = 0 \end{aligned}$$

or,

$$3H^2 = - \left(\frac{\dot{\phi}^2}{2} + \frac{\omega^2}{2} \right) - 2\gamma^2 \left(12H^4 a^3 + \frac{a^3 \dot{\phi}^4}{3} + 4a^3 H^2 \dot{\phi}^2 \right). \quad (5.33)$$

The Klein-Gordon equation is

$$\ddot{\phi} + 3\dot{\phi}H = 0. \quad (5.34)$$

We notice that Eqs.(5.30) and (5.33) directly incorporate quantum corrections, whereas for the Klein-Gordon equation (5.34), there is no explicit dependence on quantum corrections. Any quantum deformation entering Eq.(5.34) arises only implicitly through the Hubble parameter. This situation is similar to the models of LQC [146, 147]. in this chapter, we consider only the quadratic momentum deformed GUP and use the modified Hamiltonian in Eq.(5.28) to study the deformed dynamics of cosmology with a cosmological constant and an arbitrary potential.

5.1.2.1 Discussion on the Friedmann Equation

In this subsection, we simplify the Friedmann Eq.(5.33). Defining a new parameter $\alpha \equiv 2\gamma^2$, we find that Eq.(5.33) can be written as a quadratic equation in terms of $\tilde{H} = H^2$ as

$$3\tilde{H}^2 + \left(\dot{\phi}^2 + \frac{3}{4\alpha a^3} \right) \tilde{H} - C_0 = 0, \quad (5.35)$$

where

$$C \equiv - \left(\frac{\dot{\phi}^2}{2} + \frac{\omega^2}{2} + \frac{\alpha a^3 \dot{\phi}^4}{3} \right), \quad C_0 \equiv \frac{C}{4\alpha a^3}. \quad (5.36)$$

Solving the quadratic equation gives us

$$\tilde{H} = \frac{-\left(\dot{\phi}^2 + \frac{3}{4\alpha a^3}\right) \pm \sqrt{\left(\dot{\phi}^2 + \frac{3}{4\alpha a^3}\right)^2 + 12C_0}}{6}. \quad (5.37)$$

Since $\tilde{H} = H^2$, the right-hand side of the Eq.(5.37) must be greater than zero. This implies

$$\sqrt{\left(\dot{\phi}^2 + \frac{3}{4\alpha a^3}\right)^2 + 12C_0} > \left(\dot{\phi}^2 + \frac{3}{4\alpha a^3}\right), \quad (5.38)$$

which is equivalent to

$$C_0 > 0, \quad (5.39)$$

or

$$\frac{\dot{\phi}^2}{2} + \frac{\alpha a^3 \dot{\phi}^4}{3} < \frac{3\Lambda}{8}. \quad (5.40)$$

Rewriting Eq.(5.37)

$$H^2 = -\frac{\dot{\phi}^2}{6} - \frac{1}{8\alpha a^3} + \left(\frac{\dot{\phi}^2}{6} + \frac{1}{8\alpha a^3}\right) \sqrt{1 + \frac{12C_0}{\left(\dot{\phi}^2 + \frac{3}{4\alpha a^3}\right)^2}}, \quad (5.41)$$

and applying the conditions Eq.(5.38), where $M \equiv \dot{\phi}^2 + \frac{3}{4\alpha a^3}$, we have

$$\sqrt{1 + \frac{12C_0}{M^2}} > 1. \quad (5.42)$$

and

$$\frac{12C_0}{M^2} = \frac{6\dot{\phi}^2 - \frac{9}{2}\Lambda + 4\alpha a^3 \dot{\phi}^4}{6\dot{\phi}^2 + \frac{9}{4\alpha a^3} + 4\alpha a^3 \dot{\phi}^4} \ll 1. \quad (5.43)$$

Comparing the numerator and denominator of the Eq.(5.43), we find that the first term and the last term are the same, while in the second term, the denominator has a α^{-1} -dependence. Since α , which represents the GUP correction, is taken to be very small and during the early epoch the scale factor $a(t)$ is also very small, which makes the denominator very large.

Applying binomial expansion of the Eq.(5.41) we get,

$$H^2 = -\frac{\dot{\phi}^2}{6} - \frac{1}{8\alpha a^3} + \left(\frac{\dot{\phi}^2}{6} + \frac{1}{8\alpha a^3}\right) \left[1 + \frac{1}{2} \left(\frac{12C_0}{M^2}\right) - \frac{1}{2} \left(\frac{1}{2} - 1\right) \left(\frac{1}{2!}\right) \left(\frac{12C_0}{M^2}\right)^2 + \mathcal{O}(3)\right]. \quad (5.44)$$

Considering term up to the first order in $12C_0/M^2$, we find

$$H^2 = \frac{1}{4\alpha a^3 \dot{\phi}^2 + 3} \left(-\frac{\dot{\phi}^2}{2} + \frac{3\Lambda}{8} - \frac{\alpha a^3 \dot{\phi}^4}{3} \right), \quad (5.45)$$

which is the required Friedmann equation with GUP modification for a phantom scalar field with a positive cosmological constant.

5.1.3 Phantom scalar field with an arbitrary potential

Now, we construct the GUP-modified Friedmann equation for arbitrary potential by applying the change in variables shown in the previous section. The Lagrangian can be written in terms of $(x, y, \dot{x}, \dot{y})$ by applying the procedure prescribed in Sec.(5.1.2), so we find

$$\mathcal{L} = - \left[\frac{\dot{x}^2 + \dot{y}^2}{2} + \frac{3}{8}(x^2 + y^2)V(\phi) \right]. \quad (5.46)$$

In addition, the unperturbed Hamiltonian can be obtained by the Legendre transformation of Eq.(5.46)

$$H_0 = - \left[\frac{P_x^2}{2} + \frac{P_y^2}{2} - \frac{3}{8}(x^2 + y^2)V(\phi) \right]. \quad (5.47)$$

Introducing GUP, as given in Sec.(5.1.2) to the unperturbed Hamiltonian Eq.(5.47), we find

$$\mathcal{H}_{GUP} = -N \left[\frac{P_a^2}{12a} + \frac{P_\phi^2}{2a^3} - a^3 V(\phi) + 2\gamma^2 \left(\frac{P_a^4}{108a^2} + \frac{P_\phi^4}{3a^6} + \frac{P_a^2 P_\phi^2}{9a^4} \right) \right]. \quad (5.48)$$

From the above GUP corrected Hamiltonian one can easily obtain the Raychaudhuri equation, given by

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 = \frac{\dot{\phi}^2}{2} + V(\phi) + \gamma^2 \left(16a^3 H^4 + \frac{4a^3 \dot{\phi}^4}{3} + \frac{32a^3 H^2 \dot{\phi}^2}{3} \right). \quad (5.49)$$

Following the same procedure we have discussed earlier and after some algebraic manipulations, the Friedmann equation for an arbitrary potential can be written as

$$H^2 = \frac{1}{4\alpha a^3 \dot{\phi}^2 + 3} \left(-\frac{\dot{\phi}^2}{2} + V(\phi) - \frac{\alpha a^3 \dot{\phi}^4}{3} \right), \quad (5.50)$$

and the KG equation as

$$\ddot{\phi} + 3\dot{\phi}H - \frac{dV(\phi)}{d\phi} = 0. \quad (5.51)$$

With the same reasons as explained previously, now the KG equation also remains the same. We note that despite the inclusion of the potential term there is no explicit quantum correction to the above KG equation. In addition to the reason explained in Sec.(5.1.2) concerning GUP correction only for kinetic part of the phantom scalar field, however, in the present section, the potential is treated classically in the light of the models of LQC [146, 147]. The justification of this treatment is based on the fact that as the potential begins to dominate on the onset inflationary era the quantum effects of spacetime starts diluting. Thus it validates the employment of effective dynamics.

In SMC, the factor $\alpha a^3 \dot{\phi}^4/3$ is absent in the numerator of Eq.(5.50), retaining only the energy density term. The additional terms arise solely from the quantum corrections due to GUP. In addition, Eq.(5.50) reveals the possibility of singularity resolution. The occurrence of non-singular bounce in phantom models has already been studied in [148–152]. The possibility of a non-singular bounce is further enhanced by the presence of a negative quantum corrected term, $-\alpha a^3 \dot{\phi}^4/3$. Though $\dot{a} = 0$ is a necessary condition for the occurrence of a bounce, the solution must meet $\ddot{a} > 0$ at the bounce for the contracting universe to reverse its trajectory and begin expanding.- This condition is satisfied by the Raychaudhuri equation (5.49), as its right-hand side is positive.

5.2 Dynamical System Analysis

A nonlinear system is generally difficult to study analytically. However, the method of dynamical system analysis (DSA) serves as a powerful tool for extracting qualitative information. Einstein's equations, when applied to the flat FLRW spacetime, become a set of coupled second-order ordinary differential equations. However, with a suitable choice of variables, they can be transformed to the first-order autonomous differential equations.

In this method the system's dynamics is cast as a set of first-order autonomous differential equations, and the fixed points are defined as the points where the vector flow of the dynamical variables vanishes. The precise nature of the fixed points is obtained

by examining the behavior of the leading order perturbation around the fixed points. Mathematically, the signs of the eigenvalues of the Jacobian matrix evaluated at the fixed points indicate the nature of the fixed points. For an extensive review of DSA, we refer readers to [153–156].

Phase portraits, on the other hand, are visual representations of the trajectories of a dynamic system. It provides insights into the qualitative behavior of the system pictorially. This is achieved by drawing a tangent at each point given by the flow vectors of the autonomous differential equations. When applied to cosmology, it offers an intuitive understanding of the fate of the Universe even without solving the equations.

In the subsequent subsections, we construct the autonomous equations and henceforth perform DSA for the background with the GUP modification introduced in the above sections.

We know that Einstein's equations are second order in nature. However, to perform a DSA, they must be transformed into a set of first-order differential equations. A widely practiced method is to begin by normalizing the Friedmann equation by the square of the Hubble parameter to make each term dimensionless. In the process, it brings all the components contributing to the Hubble rate on equal footing. The next step is to write the EoM for each independent dimensionless dynamical variable obtained with the help of the Raychaudhuri and Klien-Gordon equations. The final set of the equations, when expressed entirely in terms of newly defined dimensionless variables, constitutes the required autonomous system.

In the following, let us consider some dynamical cases with different potentials $V(\phi)$, separately.

5.2.1 For $V(\phi) = V_0\phi^2$

Following the above-mentioned recipe, we perform DSA by constructing autonomous equations for the chosen potential. This will allow us to perform fixed point analysis and study the behavior of phase portraits. From Eq.(5.50), the expansion normalized Friedmann equation can be written as

$$4\alpha a^3 \dot{\phi}^2 + 3 = -\frac{\dot{\phi}^2}{2H^2} + \frac{V(\phi)}{H^2} - \frac{2\alpha a^3 \dot{\phi}^2}{3} \frac{\dot{\phi}^2}{2H^2}. \quad (5.52)$$

A suitable choice of dimensionless dynamical variables, also called the expansion normalised (EN) variables, are

$$x \equiv \sqrt{\frac{\dot{\phi}^2}{6H^2}}, \quad y \equiv \sqrt{\frac{V_0\phi^2}{3H^2}}, \quad z \equiv \sqrt{\alpha a^3 \dot{\phi}^2}. \quad (5.53)$$

Then, the expansion-normalized Friedmann equation reads

$$x^2 + y^2 - \frac{2}{3}z^2x^2 - \frac{4}{3}z^2 = 1. \quad (5.54)$$

It is observed that the GUP corrections denoted by α is presented only in the variable z in Eq.(5.54) as can be seen from Eq.(5.53). In addition, from Eq.(5.51) we find that the KG equation reads

$$\ddot{\phi} + 3\dot{\phi}H - 2V_0\phi = 0. \quad (5.55)$$

From Eq.(5.49), on the other hand, we obtain

$$\frac{\dot{H}}{H^2} = \frac{3}{2}(x^2 + y^2 - 1) + \frac{(2 + 6x^4 + 8x^2)(-1 - x^2 + y^2)}{(2x^4 + 4x^2)}. \quad (5.56)$$

It should be noted that the EN variables alone fail to close the autonomous system for the power law potential. For example, a new dynamic variable depending on ϕ appears in the form of $\lambda \equiv -\frac{V_{,\phi}}{V}$. The physical phase space for a power law system is always represented by a positive y half-cylinder stretching from $\lambda = 0$ to $+\infty$ due to the symmetry [157–160], which means that the phase space is not compact. To make phase space compact, we choose a new dynamic variable u , which is

$$u = \frac{\lambda}{\lambda + 1}. \quad (5.57)$$

This transformation makes our phase space compact with the range $0 \leq u \leq 1$. finally we can write our system of equations in terms of u . Then, the autonomous set of dynamical equations for quadratic potential with the GUP correction finally read

$$f(x, y) \equiv \frac{dx}{dN} = -3(1-u)x - u \left(\sqrt{6} y^2 \right) - (1-u)x \left\{ \frac{3}{2}(x^2 + y^2 - 1) + \frac{(2 + 6x^4 + 8x^2)(-1 - x^2 + y^2)}{2x^4 + 4x^2} \right\}, \quad (5.58)$$

$$g(x, y) \equiv \frac{dy}{dN} = \sqrt{6} y x u - y(1-u) \left\{ \frac{3}{2}(x^2 + y^2 - 1) + \frac{(2 + 6x^4 + 8x^2)(-1 - x^2 + y^2)}{2x^4 + 4x^2} \right\}, \quad (5.59)$$

$$h(x, y) \equiv \frac{du}{dN} = -\sqrt{6}(\Gamma - 1)(1-u)xz^2, \quad \Gamma = \frac{VV_{,\phi\phi}}{(V_{,\phi})^2}. \quad (5.60)$$

The cosmological dynamics for the universe with a quadratic potential and the GUP corrections are completely contained in the three equations (5.58), (5.59) and (5.60). We also observe that the GUP corrections appear only in the third autonomous equation, for $h(x, y)$.

Now, we perform a thorough analysis of the fixed points of the cosmological system dictated by a phantom scalar field with quadratic potential in a GUP-modified scenario. Later, we compare the results with those without GUP.

Fixed points are obtained by setting $d\mathcal{F}/dN = 0$ simultaneously, where $\mathcal{F} \equiv (x, y, u)$. This physically means that the system becomes stationary at the points. Then, the fixed points along with their behavior are tabulated in Table 5.2. To compare our results with the original dynamics, we turn off the quantum perturbation by setting $\alpha = 0$ in Eqs.(5.58-5.60), and the results are summarized in Table 5.1.

Comparison of Tables 5.1 and 5.2 shows that the introduction of the GUP corrections substantially modifies the phase-space structure of the cosmological system. In the classical case, six equilibrium points are obtained, including two stable points, three saddle points, and one neutral point. After incorporating the GUP corrections, however, only two saddle points survive. This reduction in the number of fixed points should not be interpreted as a loss of physical information; rather, it reflects the modification of the underlying cosmological dynamics by quantum-gravity effects.

The origin of this change can be traced to the GUP-induced modification of the Friedmann constraint and the autonomous system through the additional quantum variable

$z = \sqrt{\alpha a^3 \dot{\phi}^2}$, which encodes the minimal-length corrections. Consequently, the conditions required for the existence of equilibrium solutions are altered, and several classical fixed points no longer satisfy the quantum-corrected dynamical equations. In other words, quantum-gravity effects restrict the class of dynamically admissible cosmological trajectories near the Planck regime.

From a physical perspective, the purpose of introducing the GUP is not to generate a larger number of fixed points but to investigate how the existence of a fundamental minimal length modifies the qualitative evolution of the Universe. The disappearance of the classical stable and neutral points indicates that some cosmological states allowed in the classical theory become inaccessible once quantum-gravity corrections are included. Therefore, the surviving saddle points represent the only equilibrium configurations compatible with the GUP-modified cosmological dynamics, illustrating that Planck-scale effects can significantly alter the global phase-space structure even though they become negligible in the classical limit $\alpha \rightarrow 0$.

The persistence of only saddle points in the GUP-corrected phase space has important physical implications. Since saddle points are stable only along specific directions of the phase space, they do not represent asymptotic late-time attractors but rather transient cosmological states. Consequently, the GUP-corrected Universe evolves through these intermediate configurations before departing toward other regions of the phase space. The disappearance of the classical stable points indicates that the inclusion of minimal-length effects modifies the asymptotic structure of the cosmological evolution, preventing the system from approaching the same late-time equilibrium predicted by the classical theory. This behavior is consistent with the expectation that quantum-gravity effects become significant only in the high-energy regime, where they alter the early-Universe dynamics and reshape the global phase-space structure. As the Universe expands and the energy scale decreases, the quantum corrections become negligible, allowing the cosmological evolution to smoothly approach the classical regime.

5.2.2 Exponential Potential

In this subsection, we consider a potential of the form

$$V(\phi) = V_0 e^{-k\phi}, \quad (5.61)$$

TABLE 5.1: **Fixed points and the stability analysis for the Potential $V(\phi) = V_0\phi^2$ without GUP**

	x	y	u	E_1	E_2	E_3	Stability
A	0	0	c	$-3(1-c)/2$	$3(1-c)/2$	0	saddle point
B	0	0	0	$-3/2$	$3/2$	0	saddle point
C	0	0	1	0	0	0	neutral point
D	c	0	1	$2\sqrt{6}c$	$-\sqrt{6}c$	0	saddle point
E	0	-1	0	-3	-3	0	stable point
F	0	1	0	-3	-3	0	stable point

TABLE 5.2: **Fixed points and the stability analysis for the Potential $V(\phi) = V_0\phi^2$ with GUP**

	x	y	u	E_1	E_2	E_3	Stability
A	$+c(\text{other than } 0)$	0	1	0	$2\sqrt{6}c$	$-\left(\frac{2\sqrt{6}c^3+\sqrt{6}c^5}{c^2(2+c^2)}\right)$	saddle point
B	$-c(\text{other than } 0)$	0	1	0	$-2\sqrt{6}c$	$\left(\frac{2\sqrt{6}c^3+\sqrt{6}c^5}{c^2(2+c^2)}\right)$	saddle point

to study the GUP modified background dynamics, where V_0 and k are constants. Then, the EN variables are

$$x \equiv \sqrt{\frac{\dot{\phi}^2}{6H^2}}, \quad y \equiv \sqrt{\frac{V_0 e^{-k\phi}}{3H^2}}, \quad z \equiv \sqrt{\alpha a^3 \dot{\phi}^2}. \quad (5.62)$$

it can be shown that now the expansion-normalized Friedmann equation in the form of the EN variables reads

$$x^2 + y^2 - \frac{2}{3}z^2x^2 - \frac{4}{3}z^2 = 1, \quad (5.63)$$

and from Eq.(5.51) the Klein Gordon equation reads

$$\ddot{\phi} + 3\dot{\phi}H + V_0 k e^{-k\phi} = 0. \quad (5.64)$$

Furthermore, the Raychaudhuri equation (5.49) in terms of the EN variables is given by

$$\frac{\dot{H}}{H^2} = \frac{3}{2}(x^2 + y^2 - 1) + \frac{(2 + 6x^4 + 8x^2)(-1 - x^2 + y^2)}{(2x^4 + 4x^2)}. \quad (5.65)$$

Because our potential is exponential, we can get $\Gamma = \frac{V V_{,\phi\phi}}{V_{,\phi}^2}$ equal to 1. While constructing the autonomous equation, we obtain a factor of $Q \equiv -\frac{V_{,\phi}}{V} = k$. With these, we can write the autonomous differential equations as

$$\tilde{f}(x, y) = \frac{dx}{dN} = -3x - \sqrt{\frac{3}{2}}Qy^2 - x \left[\frac{3}{2}(x^2 + y^2 - 1) + \frac{(2 + 6x^4 + 8x^2)(-1 - x^2 + y^2)}{(2x^4 + 4x^2)} \right], \quad (5.66)$$

$$\tilde{g}(x, y) = \frac{dy}{dN} = -\sqrt{\frac{3}{2}}Qyx - y \left[\frac{3}{2}(x^2 + y^2 - 1) + \frac{(2 + 6x^4 + 8x^2)(-1 - x^2 + y^2)}{(2x^4 + 4x^2)} \right]. \quad (5.67)$$

The original dynamics without the quantum corrections can be easily obtained as a limiting case by setting $\alpha = 0$.

Fixed Points

The fixed points of the system with the eigenvalues of the Jacobian are summarized in Tables 5.3 and 5.4, respectively, with and without GUP.

TABLE 5.3: Fixed points and corresponding eigenvalues of the Potential $V(\phi) = V_0 e^{-k\phi}$ without GUP

	x	y	E_1	E_2	Stability
A	$-Q/\sqrt{6}$	$\sqrt{(Q^2 + 6)/6}$	$-(6 + Q^2)/2$	$-(3 + Q^2)$	stable point
B	$-Q/\sqrt{6}$	$-\sqrt{(Q^2 + 6)/6}$	$-(6 + Q^2)/2$	$-(3 + Q^2)$	stable point
C	0	0	$-3/2$	$3/2$	saddle point

TABLE 5.4: Fixed points and corresponding eigenvalues of the potential $V(\phi) = V_0 e^{-k\phi}$ with GUP.

	x	y	E_1	E_2	Stability
A	$-\frac{Q}{\sqrt{6}}$	$\sqrt{\frac{Q^2 + 6}{6}}$	$-\frac{72Q^2 + 18Q^4 + Q^6}{2Q^2(12 + Q^2)}$	$-\frac{72 + 84Q^2 + 21Q^4 + Q^6}{Q^2(12 + Q^2)}$	Stable point
B	$-\frac{Q}{\sqrt{6}}$	$-\sqrt{\frac{Q^2 + 6}{6}}$	$-\frac{72Q^2 + 18Q^4 + Q^6}{2Q^2(12 + Q^2)}$	$-\frac{72 + 84Q^2 + 21Q^4 + Q^6}{Q^2(12 + Q^2)}$	Stable point

The rigorous fixed point analysis shows that the introduction of GUP corrections completely alters the fixed points and hence their stability in the case of exponential potential. It is found that the dynamics without GUP have two stable fixed points and one

saddle point, whereas the GUP-modified dynamics for the exponential potential leave us only with two stable fixed points.

Comparison of Tables 5.3 and 5.4 shows that the inclusion of the GUP corrections modifies the phase-space structure without changing the physically relevant late-time attractors. In the classical system, the autonomous equations admit two stable fixed points together with a saddle point located at the origin. After incorporating the GUP corrections, the saddle point disappears, whereas both stable fixed points remain intact with modified eigenvalues. This demonstrates that the quantum-gravity corrections preserve the asymptotic behaviour of the cosmological solutions while modifying the transient evolution of the system.

The disappearance of the saddle point indicates that the corresponding classical equilibrium configuration is no longer compatible with the quantum-corrected Friedmann constraint. Consequently, the associated transient cosmological trajectory is removed from the physical phase space. In contrast, the persistence of the two stable fixed points implies that the late-time evolution remains robust against the inclusion of the GUP corrections. Therefore, although the minimal-length effects modify the early cosmological dynamics and alter the phase-space topology, they do not change the attractor behaviour for the exponential potential.

From the perspective of effective quantum cosmology, this result is physically significant. Since the GUP corrections are expected to become important only near the Planck regime, they primarily modify the high-energy evolution of the Universe. As the energy density decreases during the expansion, the quantum corrections become negligible, allowing the cosmological dynamics to evolve towards the same stable attractor as in the classical theory.

The surviving stable fixed point corresponds to an attractor solution in which the normalized kinetic and potential energy densities of the phantom scalar field approach constant values. Consequently, irrespective of the initial conditions, neighbouring cosmological trajectories converge towards this equilibrium state at late times. The persistence of this attractor after incorporating the GUP corrections demonstrates that the late-time cosmological behaviour predicted by the exponential potential is robust against Planck-scale modifications, while the transient evolution leading to the attractor is modified by the quantum corrections.

Phase Portrait

The phase portraits corresponding to the exponential potential are shown in Fig. 5.1 for the representative choice $Q = 1$. Since

$$y = \sqrt{\frac{V(\phi)}{3H^2}},$$

and the exponential potential satisfies $V(\phi) > 0$, only the upper half of the phase space ($y \geq 0$) is physically admissible. The lower half ($y < 0$) corresponds to the negative branch of the square root and therefore does not represent independent physical solutions. Consequently, only the fixed point $A = (-0.40, 1.08)$ is physically relevant, while point B is shown only for mathematical completeness.

The arrows represent the direction of the cosmological evolution with respect to the e-folding time $N = \ln a$. Trajectories evolve along the integral curves toward the stable fixed points and away from the saddle point. In the absence of GUP corrections (left panel), the phase space contains two stable fixed points, A and B , together with a saddle point $C = (0, 0)$. The saddle point acts as a transient configuration: nearby trajectories may approach it along one direction but eventually depart toward one of the stable attractors.

After incorporating the GUP corrections (right panel), the saddle point disappears, whereas the stable attractors remain unchanged. The disappearance of the saddle point indicates that the corresponding classical equilibrium solution is no longer compatible with the GUP-modified Friedmann constraint. Consequently, the transient cosmological phase associated with this equilibrium is removed from the quantum-corrected phase space.

The persistence of the stable fixed point A demonstrates that the late-time attractor of the exponential potential is robust against the inclusion of Planck-scale corrections. Therefore, the GUP corrections primarily modify the transient evolution and the global topology of the phase space, while leaving the asymptotic cosmological behaviour essentially unchanged. This result is consistent with the expectation that quantum-gravity effects are significant only in the high-energy regime and gradually become negligible as the Universe evolves toward the classical limit.

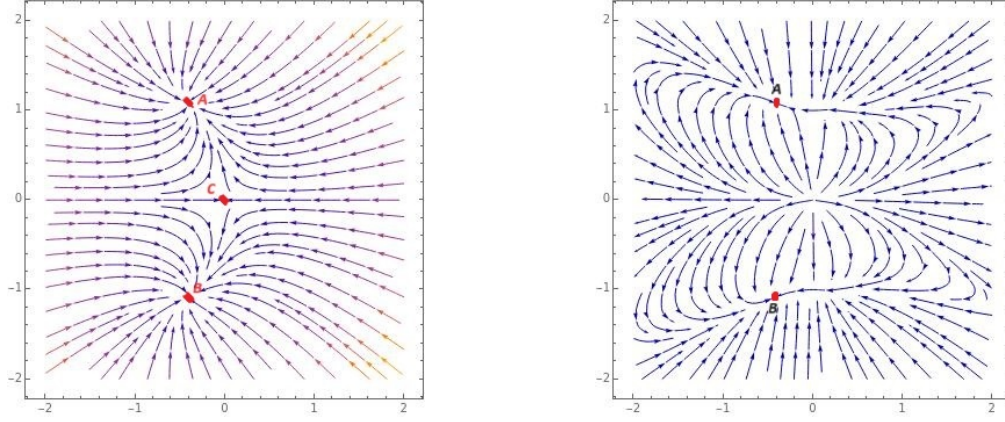


FIGURE 5.1: Phase portrait for potential $V(\phi) = V_0 e^{-k\phi}$ for value of $Q = 1$. *Left* without GUP fluctuation where the stable fixed points are at $\mathbf{A}=(-0.40, 1.08)$, $\mathbf{B}=(-0.40, -1.08)$ and a saddle fixed point at $\mathbf{C}=(0,0)$ and *Right* with GUP where the stable fixed points are $\mathbf{A}=(-0.40, 1.08)$, $\mathbf{B}=(-0.40, -1.08)$

5.3 GUP-modified inflationary scenario

In this section, we present inflationary dynamics under the influence of GUP for quadratic and exponential potentials. To understand the change in the behavior of the background dynamics of our cosmology in the presence of GUP distortions, we need to study the behavior of cosmologically relevant parameters, such as, the expansion factor $a(t)$, the scalar field $\phi(t)$, the EoS (w_{eff}), and the slow-climb parameters ϵ and η with and without GUP deformations [161]. In what follows, we do this case by case for each potential considered in this article. For the numerical integration, we work in natural units, $c = \hbar = 8\pi G = 1$, so that all dynamical variables are expressed in Planck units. Consequently, the cosmic time t appearing in the numerical solutions is a dimensionless time variable measured in units of the Planck time (equivalently, the inverse Planck mass). Therefore, the time axis in all numerical plots should be interpreted as the normalized cosmological time.

5.3.1 Quadratic Potential $V(\phi) = \mu^2 \phi^2$

During cosmic inflation, the behavior of a phantom field differs from that of a normal scalar field. While a normal scalar field undergoes a slow roll along its potential, a phantom field exhibits a slow climb along its potential. This distinction is evident both mathematically and graphically, as demonstrated in [162] for power-law potentials.

After introducing GUP corrections to the Friedmann equations, the slow climb parameters defined as $\epsilon \equiv -\frac{\dot{H}}{H^2}$, $\eta \equiv \frac{V''}{3H^2}$, and $\delta \equiv \eta - \epsilon$ can be expressed as follows

$$\epsilon = \frac{3}{2} \left(\frac{\dot{\phi}^2}{6H^2} + \frac{V_0\phi^2}{3H^2} - 1 \right) + 4\alpha a^3 H^2 + \frac{\alpha a^3 \dot{\phi}^4}{3H^2} + \frac{8}{3}\alpha a^3 \dot{\phi}^2, \quad \eta = \frac{2V_0}{3H^2}, \quad (5.68)$$

for $V(\phi) = V_0\phi^2$.

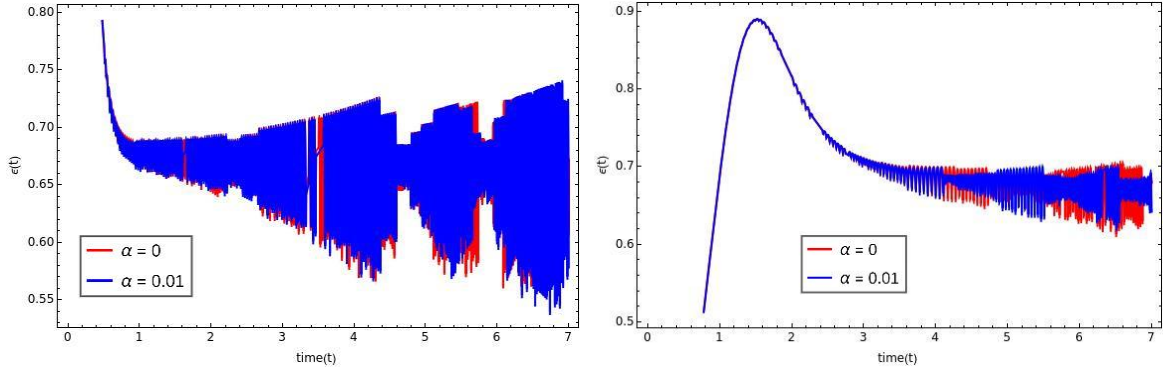


FIGURE 5.2: Evolution of the slow-climb parameter $\epsilon(t)$ for the quadratic potential $V(\phi) = V_0\phi^2$ in Planck units. The numerical solutions are obtained with $V_0 = 1$ and over the dimensionless time interval $0 \leq t \leq 7$. The red curves correspond to the classical case ($\alpha = 0$), while the blue curves represent the GUP-corrected case with $\alpha = 0.01$. *Left panel:* Initial conditions $a(0) = 1$, $\dot{a}(0) = 1$, $\phi(0) = 0.1$, and $\dot{\phi}(0) = 0$. *Right panel:* Initial conditions $a(0) = 4$, $\dot{a}(0) = 4$, $\phi(0) = 0.1$, and $\dot{\phi}(0) = 0$.

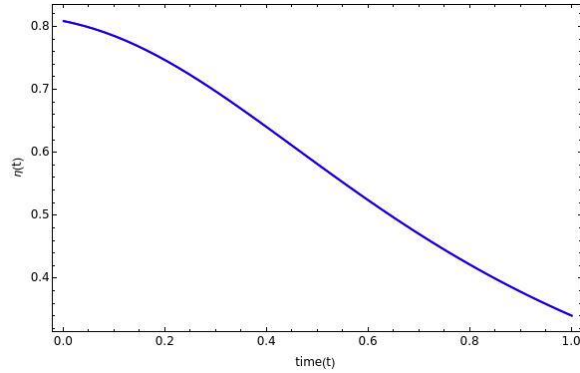


FIGURE 5.3: $\eta(t)$ vs t for the square potential.

In the cosmological context, EoS is a useful parameter to understand and classify the acceleration and deceleration phases of our universe. For $w = 0$, it corresponds to non-relativistic matter such as cold dark matter (CDM) or non-relativistic baryonic matter and for $w = 1/3$ it refers to radiation dominated. For $w = -1, -1 < w < -1/3$ and $w < -1$ refer to the cosmological constant, Quintessence, and Phantom eras, respectively.

The Raychaudhuri equation in SMC can be written in terms of EoS as follows

$$\frac{\ddot{a}}{a} = -(1 + 3w)\frac{\rho}{6}, \quad (5.69)$$

while the GUP modified Raychaudhuri Eq.(5.49) reads

$$\frac{\ddot{a}}{a} = - \left[1 + 3 \left(\frac{\frac{-\dot{\phi}^2}{2} - V(\phi) - \frac{2\alpha a^3 \dot{\phi}^4}{3} - \frac{4\alpha a^3 \dot{\phi}^2 V(\phi)}{3}}{-\frac{\dot{\phi}^2}{2} + V(\phi) - \frac{\alpha a^3 \dot{\phi}^4}{3}} \right) \right] \frac{\rho}{6}. \quad (5.70)$$

Hence, we get

$$w = \frac{\frac{-\dot{\phi}^2}{2} - V(\phi) - \frac{2\alpha a^3 \dot{\phi}^4}{3} - \frac{4\alpha a^3 \dot{\phi}^2 V(\phi)}{3}}{-\frac{\dot{\phi}^2}{2} + V(\phi) - \frac{\alpha a^3 \dot{\phi}^4}{3}}. \quad (5.71)$$

Then, for the quadratic potential, we have

$$w = \frac{\frac{-\dot{\phi}^2}{2} - V_0 \phi^2 - \frac{2\alpha a^3 \dot{\phi}^4}{3} - \frac{4\alpha a^3 \dot{\phi}^2 V_0 \phi^2}{3}}{-\frac{\dot{\phi}^2}{2} + V_0 \phi^2 - \frac{\alpha a^3 \dot{\phi}^4}{3}}. \quad (5.72)$$

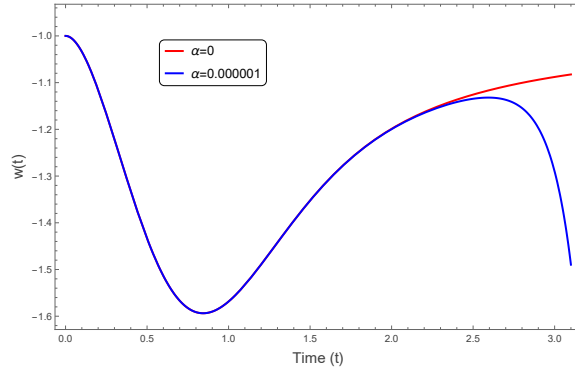


FIGURE 5.4: Comparison of EoS vs time for potential $V = \mu^2 \phi^2$ for different values of α .

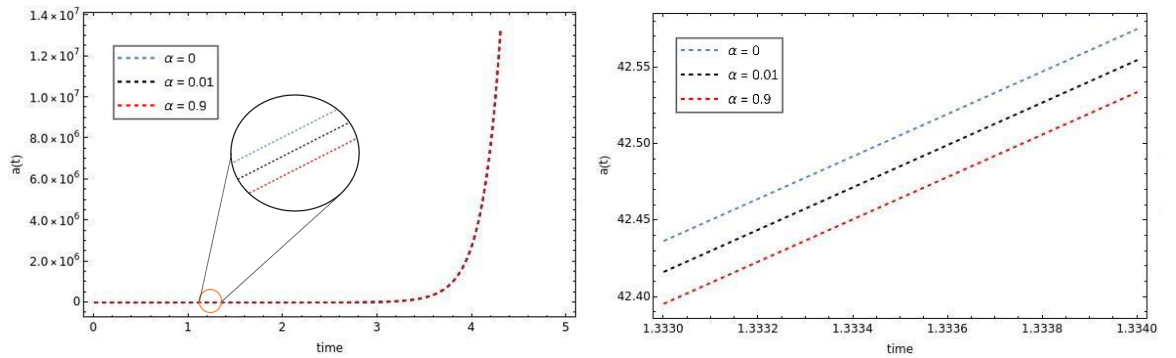


FIGURE 5.5: *Left panel* : Plot of $a(t)$ along the Y-axis vs time(t) along the X-axis for three values of $\alpha = 0, 0.01, 0.9$ respectively in $V = \mu^2 \phi^2$ potential with initial condition as $a(0) = 1$ and $\phi(0) = 1$ and $\dot{\phi}(0) = 0.1$. *Right panel* : Magnified view.

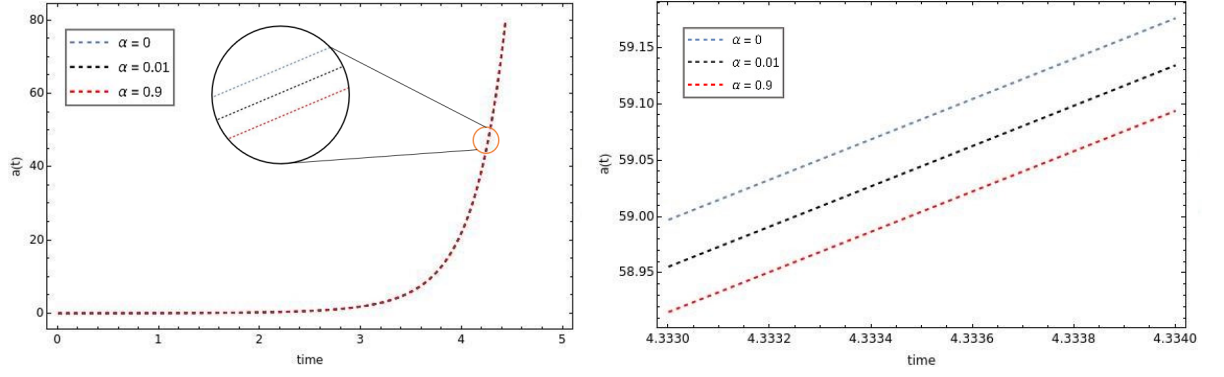


FIGURE 5.6: *Left panel* : Plot of $a(t)$ along the Y-axis vs time(t) along the X-axis for three values of $\alpha = 0, 0.01, 0.9$ respectively in $V = \mu^2 \phi^2$ potential with initial condition as $a(0) = 0.1$ and $\phi(0) = 0.1$. *Right panel* : Magnified view.

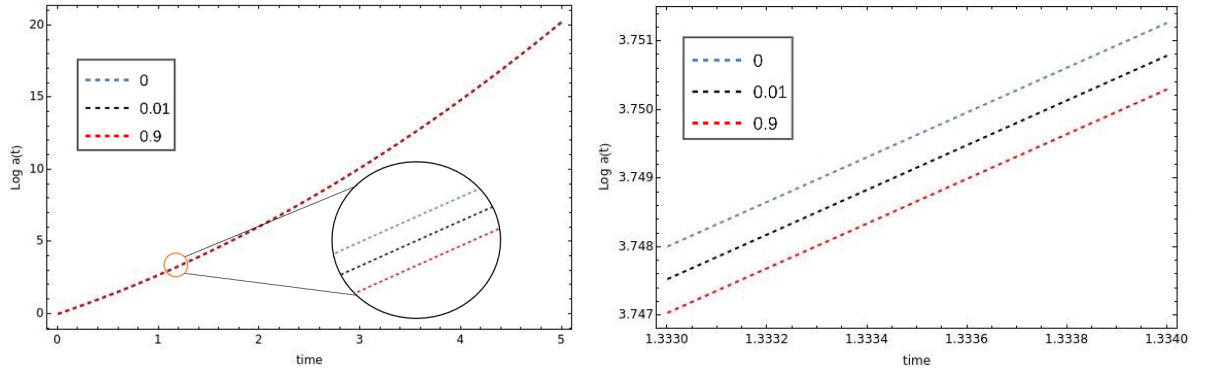


FIGURE 5.7: *Left panel* : Plot of logarithm of $a(t)$ along the Y-axis vs time(t) along the X-axis for three values of $\alpha = 0, 0.01, 0.9$ respectively in $V = \mu^2 \phi^2$ potential with initial condition as $a(0) = 1$ and $\phi(0) = 1$. *Right panel* : Magnified view.

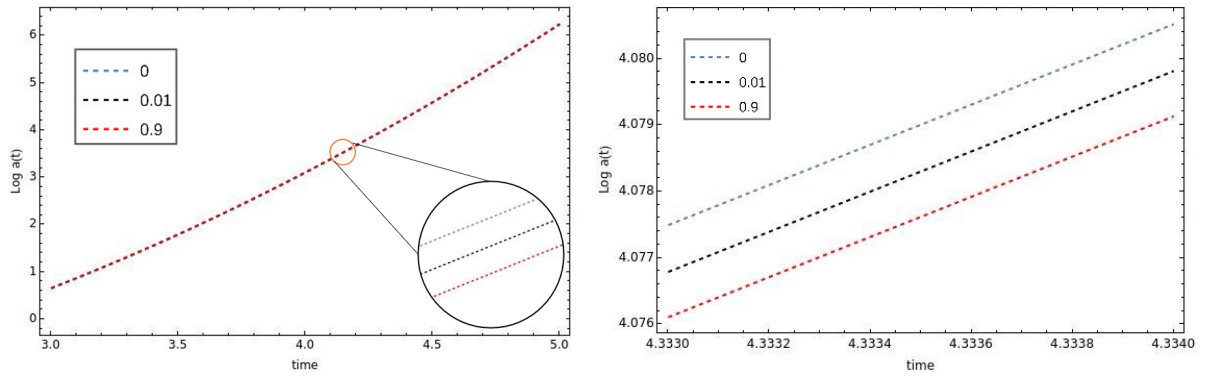


FIGURE 5.8: *Left panel* : Plot of logarithm of $a(t)$ along the Y-axis vs time(t) along the X-axis for three values of $\alpha = 0, 0.01, 0.9$ respectively in $V = \mu^2 \phi^2$ potential with initial condition as $a(0) = 0.1$ and $\phi(0) = 0.1$. *Right panel* : Magnified view.

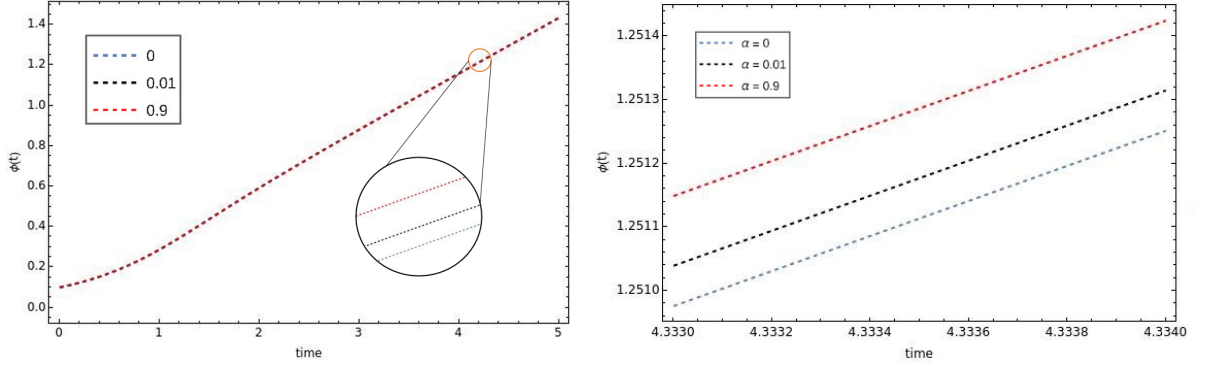


FIGURE 5.9: *Left panel* : Plot of $\phi(t)$ along the Y-axis vs time(t) along the X-axis for three values of $\alpha = 0, 0.01, 0.9$ respectively in $V = \mu^2\phi^2$ potential with initial condition as $a(0) = 1$ and $\phi(0) = 1$. *Right panel* : Magnified view.

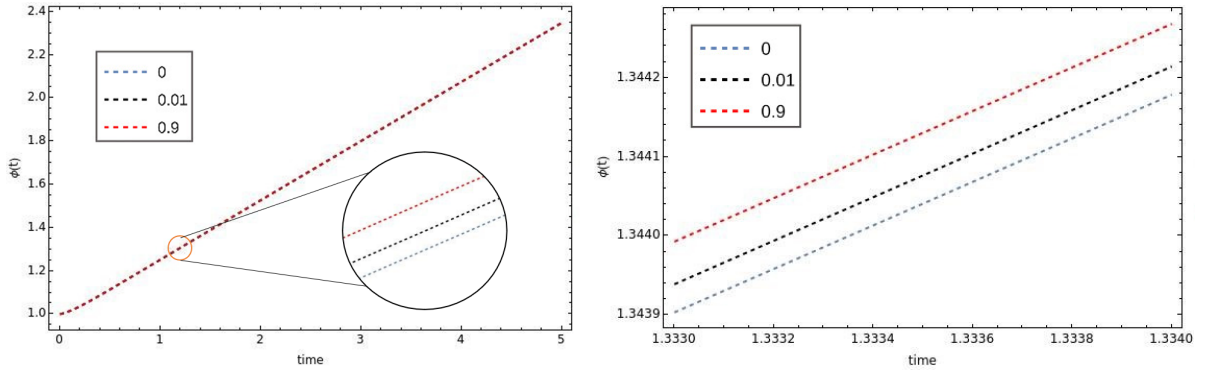


FIGURE 5.10: *Left panel* : Plot of $\phi(t)$ along the Y-axis vs time(t) along the X-axis for three values of $\alpha = 0, 0.01, 0.9$ respectively in $V = \mu^2\phi^2$ potential with initial condition as $a(0) = 0.1$ and $\phi(0) = 0.1$. *Right panel* : Magnified view.

Fig. 5.5 illustrates the evolution of the scale factor for different values of the GUP parameter. In all cases the scale factor grows nearly exponentially, confirming that the inflationary phase is preserved even after the inclusion of quantum-gravity corrections. Although the curves almost overlap in the left panel, the enlarged view on the right reveals that the GUP parameter produces small but systematic deviations from the classical evolution. As the value of α increases, the scale factor departs progressively from the classical solution, demonstrating that the minimal-length corrections modify the expansion history of the Universe.

The physical implication of this distortion is that the GUP corrections change the expansion rate and consequently alter the accumulated number of inflationary e-folds,

$$N = \ln \left(\frac{a_f}{a_i} \right), \quad (5.73)$$

without changing the qualitative inflationary behavior. Since the scale factor remains an increasing function of time, the Universe continues to undergo accelerated expansion. The GUP corrections therefore act as higher-order quantum modifications that quantitatively alter the inflationary dynamics while preserving the existence of the inflationary phase. Figure 5.6 shows that the same qualitative behaviour persists for different choices of the initial conditions, indicating that the GUP-induced modification of the expansion is robust and not an artifact of a particular initial configuration.

The behavior of ϕ w.r.t the cosmic time t for different initial conditions is shown in Figs. 5.9 and 5.10. A magnified view of the effect of the GUP deformation for different values of α is provided on the right-hand side of the figures. On the left-hand sides of the figures, the scalar field starts from a very low value and then increases linearly upward for different initial conditions. As we can observe from the graph of ϕ , the inclusion of a higher value of the GUP strength, α in the dynamics causes the evolution to distort upwards from the original dynamics without GUP.

Fig. 5.4 compares the evolution of the effective equation-of-state parameter for the classical ($\alpha = 0$) and GUP-corrected ($\alpha \neq 0$) cosmological models. In both cases, the equation-of-state parameter remains negative throughout the evolution, indicating that the Universe is dominated by a dark-energy-like component. In the classical case ($\alpha = 0$), the equation of state is determined solely by the competition between the negative kinetic energy of the phantom scalar field and its potential energy, causing the solution to remain close to the phantom divide, $w = -1$. Upon introducing the GUP corrections, the equation-of-state parameter decreases further below -1 , implying a stronger phantom behavior. This originates from the additional quantum-gravity contributions proportional to $\alpha a^3 \dot{\phi}^4$ and $\alpha a^3 \dot{\phi}^2 V(\phi)$ in the effective pressure. These terms contribute negatively to the numerator of Eq. (5.72), making the pressure more negative than in the classical case, while the corresponding correction to the effective energy density is comparatively smaller. Consequently, the ratio $w = p/\rho$ shifts to lower values, driving the system deeper into the phantom regime. As expected, in the limit $\alpha \rightarrow 0$, the GUP contributions vanish continuously and the standard phantom cosmology is recovered.

Figs. 5.2 and 5.3 illustrate the evolution of the slow-climb parameters $|\epsilon|$ and $|\eta|$ obtained by numerically solving the GUP-modified cosmological equations under two different sets of initial conditions. The numerical integration is performed over the time interval

$0 \leq t \leq 7$, and the results are shown for the classical case ($\alpha = 0$) and for the GUP-corrected case ($\alpha = 0.01$). In both cases the slow-climb parameters remain much smaller than unity throughout the evolution, satisfying the conditions $|\epsilon| \ll 1$, $|\eta| \ll 1$, which confirms that the inflationary regime is maintained.

The effect of the GUP correction is relatively small, as expected since α is treated as a perturbative quantum-gravity parameter. Nevertheless, the GUP modifies the detailed time evolution of the slow-climb parameters, leading to small deviations from the classical trajectories, particularly at early times when the quantum corrections are strongest. As the Universe expands, the influence of the GUP gradually decreases and the quantum-corrected solutions approach the classical behaviour. This demonstrates that the minimal-length corrections primarily affect the early stages of the cosmological evolution while preserving the slow-climb inflationary phase.

5.3.2 Exponential Potential $V(\phi) = V_0 e^{-k\phi}$

In this subsection, we study the GUP-modified background dynamics for exponential potential. We determine the slow roll parameters and equation of state and subsequently present them for different initial conditions. Then, the slow climb parameters are

$$\epsilon = \frac{3}{2} \left(\frac{\dot{\phi}^2}{6H^2} + \frac{V_0 e^{-k\phi}}{3H^2} - 1 \right) + 4\alpha a^3 H^2 + \frac{\alpha a^3 \dot{\phi}^4}{3H^2} + \frac{8}{3} \alpha a^3 \dot{\phi}^2, \quad \eta = \frac{V_0 k^2 e^{-k\phi}}{3H^2}. \quad (5.74)$$

The effective equation of state (EoS) for $V(\phi) = V_0 e^{-k\phi}$ is

$$w = \frac{\frac{-\dot{\phi}^2}{2} - V_0 e^{-k\phi} - \frac{2\alpha a^3 \dot{\phi}^4}{3} - \frac{4\alpha a^3 \dot{\phi}^2 e^{-k\phi}}{3}}{-\frac{\dot{\phi}^2}{2} + V_0 e^{-k\phi} - \frac{\alpha a^3 \dot{\phi}^4}{3}}. \quad (5.75)$$

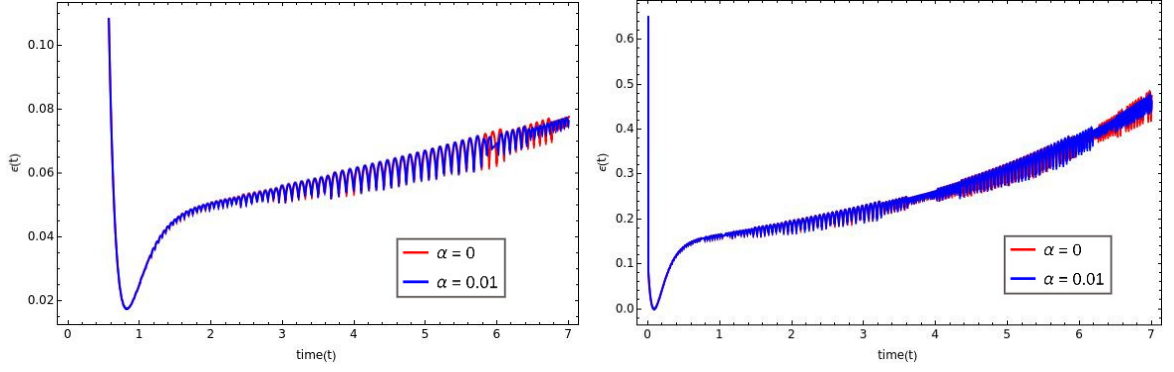


FIGURE 5.11: *Left panel:* For $V(\phi) = V_0 e^{-k\phi}$ with initial condition as $a(0) = 1$, $\dot{a}(0) = 1$ and $\dot{\phi}(0) = 0.1$ plot of $\epsilon(t)$ along Y-axis vs time(t) along X-axis for three different values of α where, $\alpha = 0$ with no GUP fluctuation. *Right panel:* For initial condition as $a(0) = 0.1$, $\dot{a}(0) = 0.1$ and $\dot{\phi}(0) = 0.1$.

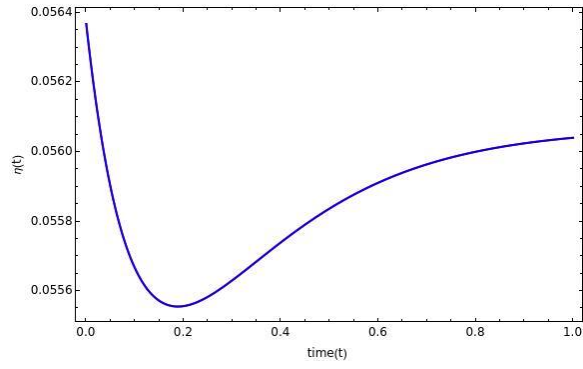


FIGURE 5.12: $\eta(t)$ vs t for exponential potential.

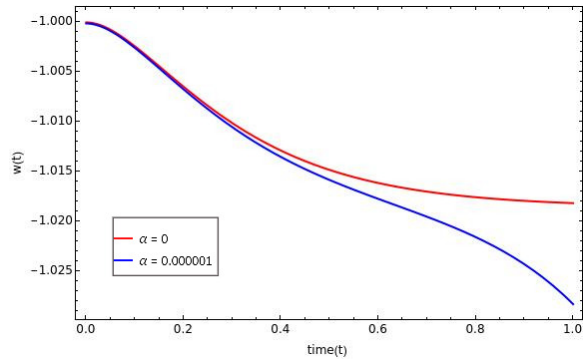


FIGURE 5.13: Comparison of EoS vs time graph for potential $V = e^{-k\phi}$ for different values of α .

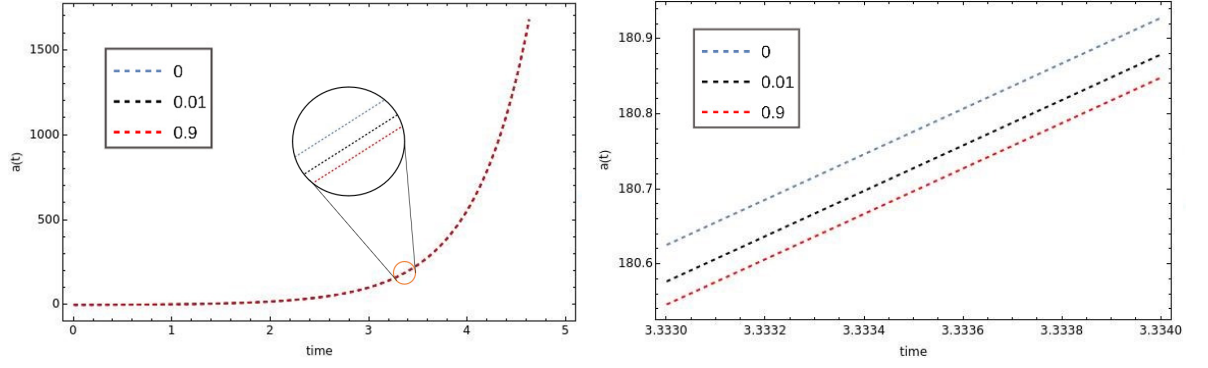


FIGURE 5.14: *Left panel:* Plot of $a(t)$ along Y-axis vs time (t) along X-axis for three values of $\alpha = 0, 0.01, 0.9$ respectively in $V = e^{-k\phi}$ potential with initial condition $a(0) = 1$ and $\phi(0) = 1$. *Right panel:* Magnified view.

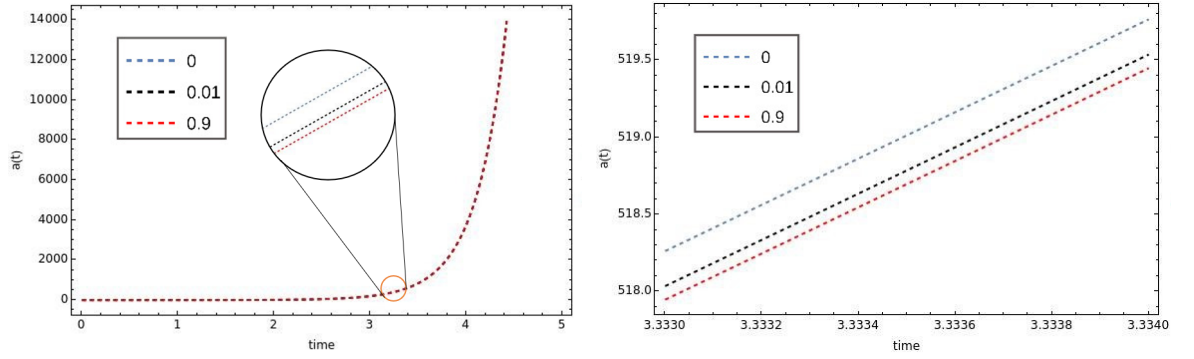


FIGURE 5.15: *Left panel:* Plot of $a(t)$ along Y-axis vs time (t) along X-axis for three values of $\alpha = 0, 0.01, 0.9$ respectively in $V = e^{-k\phi}$ potential with initial condition $a(0) = 0.1$ and $\phi(0) = 0.1$. *Right panel:* Magnified view.

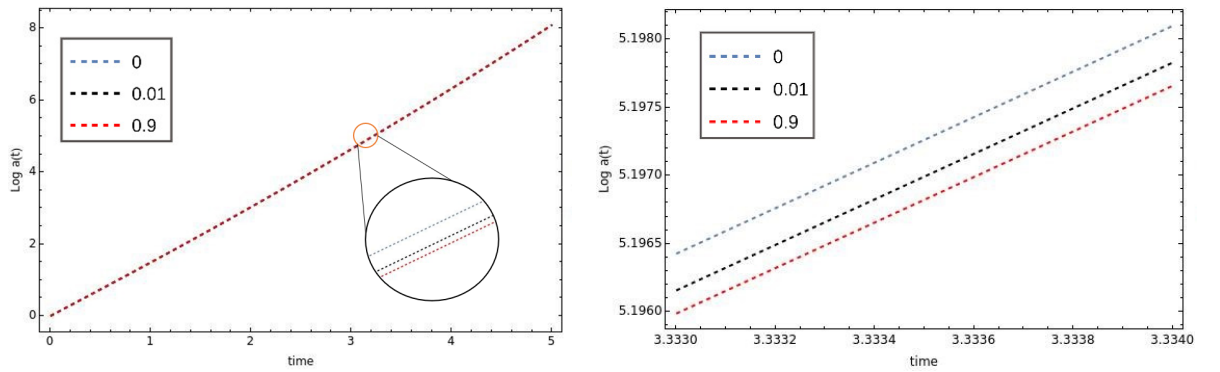


FIGURE 5.16: *Left panel:* Plot of $\log[a(t)]$ along Y-axis vs time (t) along X-axis for three values of $\alpha = 0, 0.01, 0.9$ respectively in $V = e^{-k\phi}$ potential with initial condition $a(0) = 1$ and $\phi(0) = 1$. *Right panel:* Magnified view.

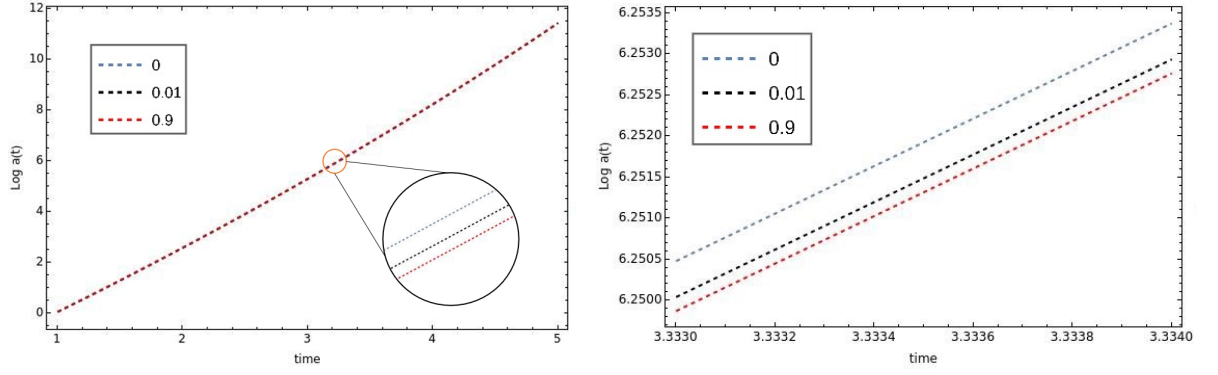


FIGURE 5.17: *Left panel:* Plot of $\log[a(t)]$ along Y-axis vs time (t) along X-axis for three values of $\alpha = 0, 0.01, 0.9$ respectively in $V = e^{-k\phi}$ potential with initial condition $a(0) = 0.1$ and $\phi(0) = 0.1$. *Right panel:* Magnified view.

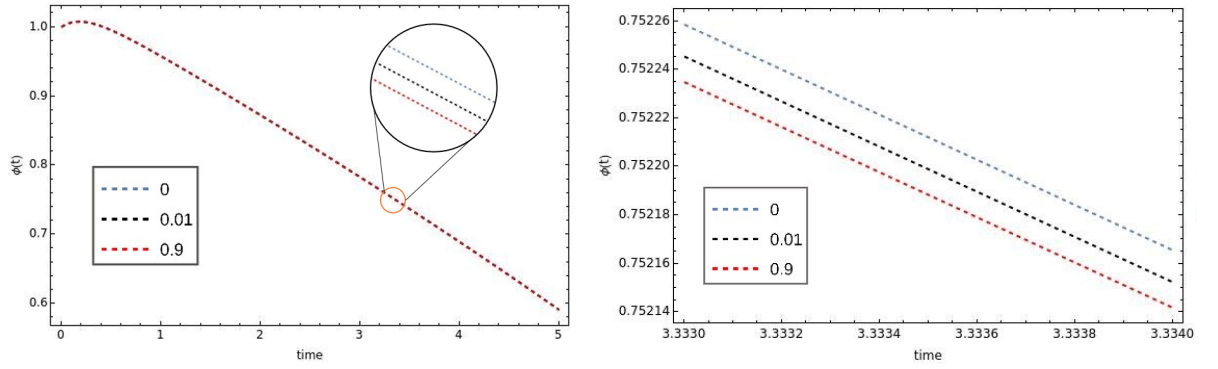


FIGURE 5.18: *Left panel:* Plot of $\phi(t)$ along Y-axis vs time (t) along X-axis for three values of $\alpha = 0, 0.01, 0.9$ respectively in $V = e^{-k\phi}$ potential with initial condition $a(0) = 1$ and $\phi(0) = 1$. *Right panel:* Magnified view.

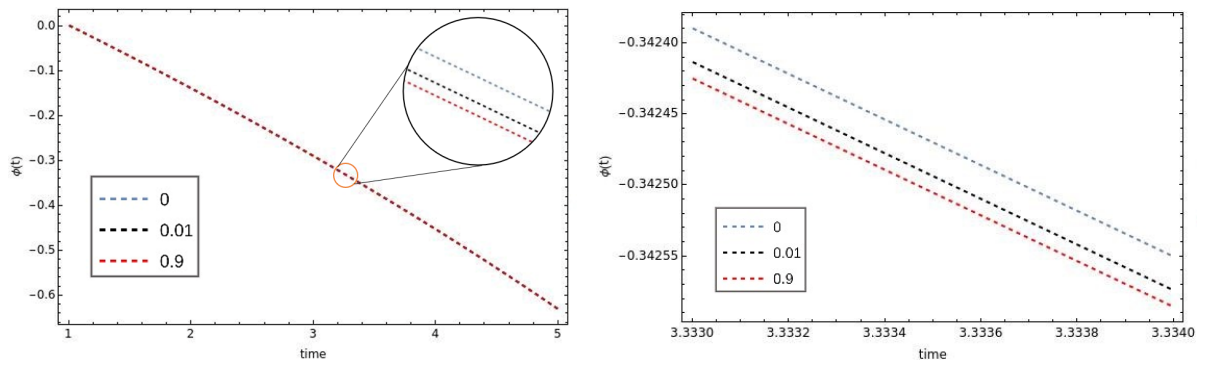


FIGURE 5.19: *Left panel:* Plot of $\phi(t)$ along Y-axis vs time (t) along X-axis for three values of $\alpha = 0, 0.01, 0.9$ respectively in $V = e^{-k\phi}$ potential with initial condition $a(0) = 0.1$ and $\phi(0) = 0.1$. *Right panel:* Magnified view.

We analyze the effect of the GUP modification on the background dynamics for the exponential potential in Figs. 5.14–5.19. Throughout this analysis, the dimensionless

time variable t is expressed in Planck units, while the remaining model parameters and initial conditions are given in the corresponding figure captions. The left panel of Fig. 5.14 shows that the expansion factor exhibits an approximately exponential growth, confirming the inflationary evolution of the universe. Since the deviations introduced by the GUP are relatively small during the early stages of evolution, the three curves almost overlap. Therefore, the highlighted region is enlarged in the right panel to clearly display the effect of the GUP parameter. It is observed that the solution with $\alpha = 0.01$ remains very close to the standard ($\alpha = 0$) evolution, whereas increasing the deformation parameter to $\alpha = 0.9$ produces a noticeable deviation from the classical trajectory. The GUP correction therefore modifies the expansion history by changing the expansion rate and consequently the accumulated number of e-foldings during inflation. Larger values of the deformation parameter lead to stronger departures from the classical solution, demonstrating that quantum-gravity corrections become increasingly significant as the strength of the GUP deformation increases. Similar qualitative behavior is observed in the evolution of the scalar field and its velocity shown in Figs. 5.14–5.19.

The behavior of the EoS parameter in the phantom field with and without the effect of GUP is represented in Fig. 5.13. The EoS starts with a value of -1, indicating proximity to the cosmological constant era, and transitions toward the phantom-dominated era for both with and without GUP dynamics.

The behavior of ϕ for exponential potentials with different initial conditions are shown in Figs. 5.18 and 5.19. In addition, a magnified view of the GUP deformation for different values of α is shown on the right-hand side of Figs. 5.18 and 5.19. On the left-hand side of the scalar field, it starts from the highest value and then decreases linearly downward for different initial conditions. We observe from the graph of ϕ that the inclusion of a comparable level of GUP in the dynamics causes the line to distort downward from the original blue line, which represents the dynamics without GUP. Furthermore, the plots of $|\epsilon|$ and $|\eta|$ in Figs. 5.11 and 5.12 indicate inflation in the case of exponential potential.

5.4 Case II: Quintessence Scalar Field

The Friedmann–Lemaître–Robertson–Walker (FLRW) metric for the spatially flat universe ($k = 0$) which guarantees homogeneity and isotropy is given by

$$ds^2 = -N^2(t)dt^2 + a^2(t) [dr^2 + r^2 d\Omega^2], \quad (5.76)$$

The Einstein–Hilbert (EH) action incorporating the metric and matter field for the scalar field is given:

$$S_{EH} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} R + \mathcal{L}_m, \quad (5.77)$$

which gives $G_{\mu\nu} = T_{\mu\nu}$, where $G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}$ is the energy-momentum tensor. Considering natural units we set $\kappa \equiv \frac{1}{8\pi G}$ equal to one for rest of the chapter.

According to GR, matter defines geometry (Mach principle) and this geometry dictates the flow of matter in the universe. This implies that the matter content of the universe determines the expansion of universe for a given epoch. The second approach is to modify the geometric sector of EH action Eq.(5.77), which gives rise to a plethora of models [163–165].

In this chapter, our focus lies on the former approach, wherein we employ the quintessence scalar field to examine the dynamics of the evolution of the universe. The main objective is to delve into the dynamics of the universe, where quantum gravity effects remain significant but not predominant.

We adopt the approach of modeling the quantum-corrected background evolution of the universe by incorporating the GUP directly into the cosmological Hamiltonian. In this section, we will review the method of constructing the equation of motion for the quintessence scalar field without GUP deformation.

The Lagrangian for quintessence scalar field with arbitrary potential is

$$\mathcal{L} = \frac{1}{2\kappa} \sqrt{|g|} [R + \kappa \{g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - 2V(\phi)\}], \quad (5.78)$$

for FLRW universe with chosen signature as $(-, +, +, +)$, Lagrangian (5.78) reduces to [166],

$$\mathcal{L} = -\frac{3a\dot{a}^2}{N} + a^3 \left(\frac{\dot{\phi}^2}{2N} - NV(\phi) \right). \quad (5.79)$$

One can obtain Hamiltonian through the Legendre transformation, defined as $\mathcal{H} \equiv \dot{a}P_a + \dot{\phi}P_\phi - \mathcal{L}$, to give classical Hamiltonian as

$$\mathcal{H} = N \left[-\frac{P_a^2}{12a} + \frac{P_\phi^2}{2a^3} + a^3 V(\phi) \right]. \quad (5.80)$$

The classical Hamiltonian given by Eq.(5.80) governs the background dynamics of a universe dominated by a quintessence scalar field with the following symplectic structure:

$$\{a, P_a\} = 1, \quad \{\phi, P_\phi\} = 1, \quad (5.81)$$

The conjugate momenta to a and ϕ are denoted as $P_a \equiv \frac{\delta \mathcal{L}}{\delta \dot{a}}$ and $P_\phi = \frac{\mathcal{L}}{\delta \dot{\phi}}$ respectively. Therefore, the complete phase space comprises (a, P_a, ϕ, P_ϕ) . While the symplectic algebra represented by Eq.(5.81) captures the kinematical structure of the theory, the dynamical evolution is determined by the Poisson flow of the phase space variables with respect to the Hamiltonian $\mathcal{H}$.

$$\begin{aligned} \dot{a} &= \{a, \mathcal{H}\}, & \dot{P}_a &= \{P_a, \mathcal{H}\}, \\ \dot{\phi} &= \{\phi, \mathcal{H}\}, & \dot{P}_\phi &= \{P_\phi, \mathcal{H}\}. \end{aligned} \quad (5.82)$$

Using the above Eq.(5.82) the Friedmann, Raychaudhuri and Klien-Gordon equations for the Quintessence scalar field are obtained as follows:

$$3H^2 = \frac{\dot{\phi}^2}{2} + V(\phi), \quad (5.83)$$

$$\ddot{\phi} + 3\dot{\phi}\frac{\dot{a}}{a} + \frac{dV(\phi)}{d\phi} = 0, \quad (5.84)$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = -\frac{\dot{\phi}^2}{2} + V(\phi). \quad (5.85)$$

So far, our analysis has been purely classical. However, formulating the theory in canonical language paves the way for canonical quantization. In the following section, we will

explore how GUP correction deforms the symplectic structure in question. Once this is understood, it is straight forward to obtain the quantum-corrected dynamics.

5.5 Quintessence scalar field in GUP-modified setup

In this section, we incorporate the effects of the GUP into the dynamics to examine how the GUP modifies the equation of motion for the quintessence scalar field. We invoke the higher-order correction of uncertainty principle to the cosmological Hamiltonian. In order to achieve this, we start with choosing the phase space x and y which incorporate scale factor a and scalar field ϕ as:

$$x = \frac{a^{3/2}}{\mu} \sinh(\mu\phi), \quad y = \frac{a^{3/2}}{\mu} \cosh(\mu\phi), \quad (5.86)$$

while dynamics remain the same. We note that the differentiation of Eq.(5.86) gives

$$\dot{x} = \frac{3}{2} \frac{a^{1/2} \dot{a}}{\mu} \sinh(\mu\phi) + (a^{3/2} \dot{\phi}) \cosh(\mu\phi), \quad (5.87)$$

$$\dot{y} = \frac{3}{2} \frac{a^{1/2} \dot{a}}{\mu} \cosh(\mu\phi) + (a^{3/2} \dot{\phi}) \sinh(\mu\phi). \quad (5.88)$$

Now, multiplying both sides of Eq.(5.87) by $\sinh(\mu\phi)$ and Eq.(5.88) by $\cosh(\mu\phi)$ and subtracting them together gives

$$\dot{y} \cosh(\mu\phi) - \dot{x} \sinh(\mu\phi) = \frac{3}{2} \frac{a^{1/2}}{\mu} \dot{a}. \quad (5.89)$$

Next by squaring both sides of Eq.(5.89), followed by dividing the expression by $2N$ and considering $\mu = \sqrt{3/8}$ one obtains,

$$\frac{\dot{y}^2 \cosh^2(\mu\phi) + \dot{x}^2 \sinh^2(\mu\phi) - 2\dot{x}\dot{y} \sinh(\mu\phi) \cosh(\mu\phi)}{2N} = \frac{3a\dot{a}^2}{N}. \quad (5.90)$$

Instead, multiply both sides of Eq.(5.87) by $\cosh(\mu\phi)$ and Eq.(5.88) by $\sinh(\mu\phi)$ and subtracting

$$\dot{x} \cosh(\mu\phi) - \dot{y} \sinh(\mu\phi) = a^{3/2} \dot{\phi}. \quad (5.91)$$

This is followed by squaring Eq.(5.91) and dividing it by $2N$ to obtain

$$\frac{\dot{x}^2 \cosh^2(\mu\phi) + \dot{y}^2 \sinh^2(\mu\phi) - 2\dot{x}\dot{y} \cosh(\mu\phi) \sinh(\mu\phi)}{2N} = \frac{a^3 \dot{\phi}^2}{2N}. \quad (5.92)$$

In subtracting, by considering the squares of the components in Eq.(5.86), we have:

$$\begin{aligned} y^2 - x^2 &= \frac{a^3}{\mu^2} \cosh^2(\mu\phi) - \frac{a^3}{\mu^2} \sinh^2(\mu\phi) \\ \implies y^2 - x^2 &= \frac{a^3}{\mu^2} = \frac{8a^3}{3}. \end{aligned} \quad (5.93)$$

The equation above represents a hyperbola centered at the origin $(0, 0)$ with axes along the x - and y -directions. Specifically, it is an open hyperbola extending along the y -axis, as y^2 has a positive sign. After converting all the terms in Cartesian pair (x, y) , we now return to the question of dynamics. Using Eq.(5.90) Eq.(5.92) and Eq.(5.93) we can determine the form of the Lagrangian in configuration coordinates $(x, y, \dot{x}, \dot{y})$ to be

$$\mathcal{L} = \left[\frac{\dot{x}^2 + \dot{y}^2}{2N} - \frac{3}{8}(y^2 - x^2)V(\phi)N \right]. \quad (5.94)$$

Using the above Lagrangian, the corresponding Hamiltonian is obtained by performing the Legendre transformation. The resulting Hamiltonian is

$$\mathcal{H}_0 = N \left[\frac{P_x^2}{2} - \frac{P_y^2}{2} + (y^2 - x^2)V(\phi) \right]. \quad (5.95)$$

The canonical transformation recasts the minisuperspace dynamics into the form of a *ghost oscillator*. The form of an oscillator-ghost-oscillator system is a structure that frequently appears in canonical quantum cosmology [167]. Unlike an ordinary two-dimensional harmonic oscillator, whose Hamiltonian possesses a positive-definite kinetic sector and is bounded from below, the Hamiltonian in Eq. (5.95) contains an indefinite kinetic structure through the combination $P_x^2/2 - P_y^2/2$. The negative sign should not be interpreted as the existence of a propagating negative-energy particle or a fundamental ghost instability. Rather, it originates from the Lorentzian signature of the minisuperspace metric inherited from the Einstein-Hilbert action after the canonical transformation. The variables x and y are canonically transformed combinations of the gravitational and scalar-field degrees of freedom, and therefore the opposite signs in their kinetic terms reflect the geometry of minisuperspace rather than the physical properties of the scalar field itself.

Various methodologies exist for incorporating GUP [168–171]. One approach involves deforming coordinates while momentum remains unchanged and the other is to deform

momentum [172–176]. In this chapter, we adopt the latter approach where we write the Hamiltonian after introducing momentum deformation using generalized uncertainty. To be specific, we introduce semi-classical canonical variables q_i and P_i , and subsequently apply GUP on the Wheeler-DeWitt (WDW) equation of our cosmological model as follows [142, 144]. The modified commutator relation, which includes both linear and quadratic dependencies on momentum, is given by

$$[q_i, P_j] = i\hbar\delta_{ij}(1 - 2\beta\gamma P_0 + 4\epsilon\gamma^2 P_0^2). \quad (5.96)$$

From this relation, the canonical variable can be approximately expressed as follows:

$$q_i = q_{0i}, \quad P_i = P_{0i} \left(1 - \beta\gamma P_0 + 2\gamma^2 \frac{\beta^2 + 2\epsilon}{3} P_0^2 \right), \quad (5.97)$$

where $P_0^2 = P_{0j}P_0^j$, ϵ, β are the dimensionless parameters, and γ is defined by a value which is inverse of the plank momentum as $\gamma \equiv \frac{\gamma_0}{M_{Pl}\epsilon}$, where M_{Pl} is the Planck mass with $\gamma_0 \sim 1$, these are the parameters where quantum-gravitational effect became relevant. Substituting Eq.(5.97) in the original Hamiltonian Eq.(5.95) and taking terms only upto the order of γ^2 gives:

$$\mathcal{H} = \mathcal{H}_0 + \beta\gamma(P_{0y}^4 - P_{0x}^4) + \gamma^2(P_{0x}^4 - P_{0y}^4) \left(\frac{\beta^2}{6} + \frac{2\epsilon}{3} \right) + \mathcal{O}(\gamma^3), \quad (5.98)$$

where $\mathcal{H}_0 = \frac{P_{0x}^2}{2} - \frac{P_{0y}^2}{2} + (y^2 - x^2)V(\phi)$ is the original Hamiltonian without GUP-distortion. Here subscript 0 denotes the unperturbed Hamiltonian equation. And the unperturbed x, y are denoted as (q_{0x}, q_{0y}) while the corresponding unperturbed momentum pairs (P_x, P_y) are indicated as (P_{0x}, P_{0y}) .

Since we require to re-express Eq.(5.98) in terms of cosmological variables. This can be achieved through the inverse transformation of variables in terms of scale factor, scalar field and there conjugate momenta. Using Eq.(5.87) and Eq.(5.88), we get

$$P_{0x} = \dot{x} = \frac{3}{2} \frac{a^{1/2} \dot{a}}{\mu} \sinh(\mu\phi) + (a^{3/2} \dot{\phi}) \cosh(\mu\phi), \quad (5.99)$$

$$P_{0y} = \dot{y} = \frac{3}{2} \frac{a^{1/2} \dot{a}}{\mu} \cosh(\mu\phi) + (a^{3/2} \dot{\phi}) \sinh(\mu\phi). \quad (5.100)$$

From the Lagrangian we can also obtain the canonical momenta P_a and P_ϕ as

$$\frac{\partial \mathcal{L}}{\partial \dot{a}} = P_a = -\frac{6\dot{a}a}{N},$$

and

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = P_\phi = \frac{a^3 \dot{\phi}}{N},$$

by substituting P_a and P_ϕ in Eq.(5.99) the equations becomes

$$P_{0x} = -\frac{P_a}{4a^{1/2}\mu} \sinh(\mu\phi) + \frac{P_\phi}{a^{3/2}} \cosh(\mu\phi), \quad (5.101)$$

$$P_{0y} = -\frac{P_a}{4a^{1/2}\mu} \cosh(\mu\phi) + \frac{P_\phi}{a^{3/2}} \sinh(\mu\phi). \quad (5.102)$$

Substituting P_{0x} and P_{0y} in the momentum-deformed Hamiltonian Eq.(5.98) and on setting the dimensionless parameters $\epsilon = 1$ and $\beta = 0$ (quadratic form of GUP) gives the GUP deformed Hamiltonian as,

$$\begin{aligned} \mathcal{H}_{\text{GUP}} = N \left[-\frac{P_a^2}{12a} + \frac{P_\phi^2}{2a^3} + V(\phi)a^3 + \alpha \left\{ \left[\left(\frac{P_a^2}{12a} + \frac{P_\phi^2}{2a^3} \right) \cosh(2\mu\phi) \right. \right. \right. \\ \left. \left. \left. - \frac{P_a P_\phi}{4\mu a^2} \sinh(2\mu\phi) \left(-\frac{P_a^2}{12a} + \frac{P_\phi^2}{2a^3} \right) \right] \right\} \right]. \quad (5.103) \end{aligned}$$

This is the required GUP distorted Hamiltonian in the cosmological phase space dominated by a quintessence scalar field with arbitrary potential up to second-order perturbation where $\alpha = 8\gamma^2/3$. From the above Hamiltonian, we can obtain the Friedmann, Raychaudhuri and KG equation as follows:

Using Eq.(5.82) we can obtain the final Raychaudhuri equation:

$$\begin{aligned} 2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = & -\frac{\dot{\phi}^2}{2} + V(\phi) + \alpha a^3 \left[\left(-\left(H^2 + \frac{\phi^2}{2}\right) \cosh(2\mu\phi) + 2\sqrt{\frac{2}{3}} H \dot{\phi} \sinh(2\mu\phi) \right) \left(-3H^2 + \frac{\phi^2}{2} \right) \right. \\ & \left. + \left((3H^2 + \frac{\phi^2}{2}) \cosh(2\mu\phi) + \sqrt{6} H \dot{\phi} \sinh(2\mu\phi) \right) \left(H^2 - \frac{\phi^2}{2} \right) \right]. \quad (5.104) \end{aligned}$$

Varying action with respect to N gives the GUP distorted Hamiltonian constraint $\mathcal{H}'_{GUP} = 0$, given in [145] gives the final Friedmann equation as:

$$3H^2 = \frac{\dot{\phi}^2}{2} + V(\phi) + \alpha a^3 \left[\left((3H^2 + \frac{\dot{\phi}^2}{2}) \cosh(2\mu\phi) + \sqrt{6}H \sinh(2\mu\phi) \right) \left(-3H^2 + \frac{\dot{\phi}^2}{2} \right) \right], \quad (5.105)$$

and the K-G equation is

$$\ddot{\phi} + 3\dot{\phi}H + V_{,\phi}(\phi) = \sqrt{\frac{3}{2}}\alpha a^3 \left[\left(3H^2 + \frac{\dot{\phi}^2}{2} \right) \sinh(2\mu\phi) + \sqrt{6}H\dot{\phi} \cosh(2\mu\phi) \right] \left(3H^2 - \frac{\dot{\phi}^2}{2} \right). \quad (5.106)$$

With the effect of GUP-perturbation our original cosmological equations take the form of Eq.(5.104), Eq.(5.105), and Eq.(5.106). In all these equations the effect of GUP appears explicitly and by turning off the GUP term α we recovered the original equations (5.83), (5.84) and (5.85). Compared to the GUP-distortion in the phantom field in [177] we observe that the phantom KG equation remains the same after introducing GUP-distortion and shows no explicit GUP terms while it gets modified for quintessence.

5.6 Dynamical system analysis

In this section, we carry out stability analysis for our GUP-distorted system. Since our system is highly nonlinear and it is difficult to predict its behavior, the method of dynamical system analysis is used to extract qualitative behavior. In this method, we use a set of dimensionless variables that describe the state of the system in a finite phase space, and to understand the flow of these variables, we construct first-order autonomous differential equations. One can find fixed points by solving these coupled differential equations simultaneously, where a fixed point represents the points where overall dynamics vanish. The nature of these fixed points is evaluated through the sign of eigenvalues from the Jacobian matrix at those points [153–156, 178–180].

We begin with the Friedmann equation as our starting point because it acts as a constraint equation. Next, we normalize it by the square of the Hubble parameter such that it brings all components contributing to the Hubble rate on an equal footing. This normalization ensures that each term becomes dimensionless, allowing us to express them in terms of dimensionless variables called Expansion-Normalized (EN) variables.

In the next section, we construct the autonomous differential equation for a given potential and perform a dynamical system analysis for the GUP-modified background cosmology.

5.6.1 Potential $V(\phi) = V_0 \sinh^{-n}(\mu\phi)$

In this subsection, we study qualitative dynamics (DSA) for a universe dominated by a quintessence scalar field with $V(\phi) = V_0 \sinh^{-n}(\mu\phi)$, considering GUP correction. In order to determine the appropriate potential that fits with the evolution of the universe such that it shows asymptotic behavior like inverse power law in early time and exponential in the later phase is the potential $V(\phi) = V_0 \sinh^{-n}(\mu\phi)$.

The normalized Friedmann Eq.(5.105) takes the form as

$$1 = \frac{\dot{\phi}^2}{6H^2} + \frac{V(\phi)}{3H^2} + \frac{\alpha a^3 \dot{\phi}^4 \cosh(2\mu\phi)}{3H^2} - 3\alpha a^3 H^2 \cosh(2\mu\phi) - \sqrt{6}\alpha a^3 H \dot{\phi} \sinh(2\mu\phi) + \frac{\alpha a^3 \dot{\phi}^3 \sinh(2\mu\phi)}{\sqrt{6}H}. \quad (5.107)$$

now defining new EN variables as

$$x \equiv \sqrt{\frac{\dot{\phi}^2}{6H^2}}, \quad y \equiv \sqrt{\frac{V(\phi)}{3H^2}}, \quad z \equiv \sqrt{3\alpha a^3 H^2 \cosh(2\mu\phi)}, \quad v \equiv \sqrt{\sqrt{6}\alpha a^3 \dot{\phi} H \sinh(2\mu\phi)}. \quad (5.108)$$

and writing the Eq.(5.107) in terms of new variables is written as

$$1 = x^2 + y^2 + x^4 z^2 - z^2 - v^2 + x^2 v^2. \quad (5.109)$$

Raychaudhuri's equation in terms of new variables is

$$\frac{\dot{H}}{H^2} = \frac{3}{2}(-x^2 + y^2 - 1) + z^2 \left(\frac{2}{3} + x^2(2 - 3x^2) \right) - \left\{ (1 - \mu)x^2 - \frac{1}{2} \right\} \left\{ \frac{1 - x^2 - y^2 - z^2(x^4 - 1)}{x^2 - 1} \right\}. \quad (5.110)$$

As we have defined Raychaudhuri and Friedmann equation in terms of new variables we are ready to do the dynamical system analysis. To know the dynamics of the system we need the vector flow of each term as the dynamics evolve with time so firstly, we need

to construct first-order differential equations for each new terms. We are going to take slope of each terms with respect to newly defined time $N = \ln a$ where 'a' is the scale factor. However, constructing autonomous differential equation alone with the help of new variables sometimes fails to close the system [157–160]. To close the system we need to define new variable $\lambda \equiv -\frac{V_{,\phi}}{V}$,

$$\lambda = \frac{n\mu}{\sinh(\mu\phi)}. \quad (5.111)$$

With this we construct the autonomous equations:

$$\begin{aligned} f(x, y, z) \equiv \frac{dx}{dN} = & \left\{ -\frac{3}{4}x^3 + \frac{3}{4x} \right\} \left\{ \frac{1-x^2-y^2-z^2(x^4-1)}{x^2-1} \right\} + 3xz^2 + 2\sqrt{6}x^3z^2 - 3x + \sqrt{\frac{3}{2}}\lambda y^2 \\ & - x \left[\frac{3}{2}(-x^2+y^2-1) + z^2 \left(\frac{2}{3} + x^2(2-3x^2) \right) \right. \\ & \left. - \left\{ (1-\mu)x^2 - \frac{1}{2} \right\} \left\{ \frac{1-x^2-y^2-z^2(x^4-1)}{x^2-1} \right\} \right], \end{aligned} \quad (5.112)$$

$$\begin{aligned} g(x, y, z) \equiv \frac{dy}{dN} = & -\sqrt{\frac{3}{2}}\lambda xy - y \left[\frac{3}{2}(-x^2+y^2-1) \right. \\ & \left. + z^2 \left(\frac{2}{3} + x^2(2-3x^2) \right) - \left\{ (1-\mu)x^2 - \frac{1}{2} \right\} \left\{ \frac{1-x^2-y^2-z^2(x^4-1)}{x^2-1} \right\} \right] \end{aligned} \quad (5.113)$$

$$\begin{aligned} h(x, y, z) \equiv \frac{dz}{dN} = & \frac{3}{2}z + \frac{3}{4z} \left\{ \frac{1-x^2-y^2-z^2(x^4-1)}{x^2-1} \right\} + z \left[\frac{3}{2}(-x^2+y^2-1) \right. \\ & \left. + z^2 \left(\frac{2}{3} + x^2(2-3x^2) \right) - \left\{ (1-\mu)x^2 - \frac{1}{2} \right\} \left\{ \frac{1-x^2-y^2-z^2(x^4-1)}{x^2-1} \right\} \right] \end{aligned} \quad (5.114)$$

$$u(x, y, z) \equiv \frac{d\lambda}{dN} = -\frac{(6\lambda x^2 z^2)(x^2-1)}{1-x^2-y^2-z^2(x^4-1)}. \quad (5.115)$$

5.6.2 Fixed point analysis

In this subsection, we perform dynamical system analysis for our chosen potential with the help of fixed points and linear stability theory to know the nature of each fixed point. Subsequently, we compare our GUP-distorted dynamics with the undistorted original dynamics.

To do this we first need to take the slopes of the above equations equal to zero to find the fixed point. These fixed points represent those points where our dynamics get static.

To know the behavior of fixed points we use the linear stability analysis method.

The analysis is shown in Table 5.6. As depicted from Table 5.6 our dynamics get more richer than the original dynamics shown in Table 5.5. Incorporating the Generalized Uncertainty Principle(GUP) introduce the addition of new variables into the original dynamics. Since our dynamics consist of four variables, it necessitates the construction of a 4×4 Jacobian matrix yielding four eigenvalues. In Table 5.6, 5.5 eigenvalues of the Jacobian matrix associated with the autonomous system at each fixed point are represented as E_i . The study of these eigenvalues is important to determine the nature of stability at different fixed points.

In addition to that the fixed points also get shifted from the original dynamics and stable fixed points get vanishes. It means that with the introduction of GUP, our dynamics get modified with a comparable amount which is totally opposite behavior when we discussed GUP in the phantom field in [177] where after the introduction of GUP dynamics get less richer and fixed points remain the same.

TABLE 5.5: Fixed points and the stability analysis for the potential $V(\phi) = V_0 \sinh^{-n} \mu \phi$ without GUP.

	x	y	λ	E_1	E_2	E_3	Stability
A	0	0	c	$-3/2$	$3/2$	0	saddle point
B	-1	0	0	3	3	$3/2$	unstable point
C	1	0	0	3	3	$-3/2$	saddle point
D	0	-1	0	-3	-3	0	stable point
E	0	1	0	-3	-3	0	stable point

TABLE 5.6: GUP-distorted Fixed points and the stability analysis for the potential $V(\phi) = V_0 \sinh^{-n} \mu \phi$.

	x	y	z	λ	E_1	E_2	E_3	E_4	Stability
A	-0.2	-1.01	-4.8	0	483.5	19.7	-7.1	0.2	saddle point
B	-0.2	1.01	-4.8	0	483.5	19.7	-7.1	0.2	saddle point
C	-0.2	-1.01	4.8	0	483.5	19.7	-7.1	0.2	saddle point
D	-0.2	1.01	4.8	0	483.5	19.7	-7.1	0.2	saddle point
E	0.2	-1.01	-4.8	0	483.5	19.7	-7.1	0.2	saddle point
F	0.2	1.01	-4.8	0	483.5	19.7	-7.1	0.2	saddle point
G	0.2	-1.01	4.8	0	483.5	19.7	-7.1	0.2	saddle point
H	0.2	1.01	4.8	0	483.5	19.7	-7.1	0.2	saddle point

The GUP modification introduces additional higher-order nonlinear corrections together with an extra dynamical degree of freedom associated with the deformation parameter. Consequently, a direct one-to-one comparison between the standard and GUP-modified

phase spaces is not straightforward, since the two systems evolve in spaces of different dimensions. Nevertheless, a comparison of Tables 5.5 and 5.6 reveals several important qualitative differences. In the classical system, the fixed points A , B , and C disappear after the GUP corrections are incorporated, while the stable fixed points $D = (0, -1)$ and $E = (0, 1)$ are displaced from their original locations and split into a family of GUP-deformed critical points because of the additional phase-space variable. Furthermore, all of the GUP-modified fixed points possess identical eigenvalues, indicating that the quantum-gravity corrections introduce a degenerate stability structure.

This reduction and modification of the fixed-point structure should not be interpreted as a loss of physical information. Rather, it reflects the fact that the GUP alters the underlying phase-space geometry through higher-order quantum-gravity corrections. The disappearance and displacement of classical critical points demonstrate that Planck-scale effects modify the asymptotic states of the cosmological system, while the change in the stability properties indicates that the evolution of the universe near these equilibrium configurations is no longer governed solely by classical general relativity. Therefore, the physical importance of incorporating the GUP lies not in producing a larger number of fixed points, but in revealing how a fundamental minimal-length scale changes the existence, location, and stability of cosmological attractors. These modifications provide a possible phenomenological signature of quantum-gravity effects and may lead to observable deviations from the classical cosmological evolution at very high energies.

5.7 Conclusion

In this chapter, we have constructed and analyzed the GUP-corrected effective Hamiltonian derived from the classical Einstein–Hilbert action for cosmological models involving scalar fields, specifically the phantom and quintessence fields. The motivation for introducing the Generalized Uncertainty Principle (GUP) arises from the expectation that a minimal length scale, as suggested by several approaches to quantum gravity, would modify the canonical phase space structure, leading to new dynamics in the early and late universe. Our study has focused on how such GUP-induced momentum deformations alter the background cosmological evolution through modifications of the Friedmann, Raychaudhuri, and Klein–Gordon equations.

We first considered a toy model comprising a minimally coupled phantom scalar field with a positive cosmological constant, and subsequently extended the analysis to include an arbitrary potential. Interestingly, we found that while the Klein–Gordon equation remains free from any explicit GUP corrections, the Friedmann and Raychaudhuri equations receive direct quantum modifications. This feature is reminiscent of certain results in loop quantum cosmology (LQC), where quantum corrections also modify the background equations but not the field equation directly. The resulting system of equations is highly nonlinear and hence requires a qualitative approach through dynamical system analysis (DSA) to capture the essential features of the dynamics.

Through linear stability analysis, we observed that GUP corrections significantly influence the local behavior of the system while keeping the overall global dynamics largely similar. In the case of the quadratic potential, certain fixed points that exist in the classical scenario vanish upon the introduction of GUP corrections, whereas for the exponential potential, the inclusion of GUP terms leads to the disappearance of saddle points and the emergence of two stable attractor points. These results suggest that the minimal length deformation can alter the stability landscape of cosmological trajectories, thereby changing the long-term behavior of the universe. Such features were illustrated and confirmed through the phase portraits and stability tables presented in the analysis.

We also examined the cosmological implications of these corrections in the context of inflationary dynamics. Our results show that both quadratic and exponential potentials support nearly exponential expansion even in the presence of GUP corrections, signifying viable inflationary behavior. The slow-climb parameters for both potentials indicate the existence of an inflationary phase, while the GUP-corrected equation of state parameter approaches $w \simeq -1$ more rapidly than in the uncorrected case. This implies that quantum gravitational effects encoded through GUP can drive the universe toward a stronger accelerated phase, modifying the inflationary duration and exit conditions. Moreover, the deviation of the equation of state from its classical trajectory demonstrates how GUP corrections can leave imprints on late-time cosmology.

On the other hand, the phantom field, despite explaining accelerated expansion, suffers from severe instability issues in perturbation analysis. A natural question that arises is whether these instabilities might be mitigated by the noncommutative structure induced

by GUP corrections. This remains an open problem worth detailed investigation. Furthermore, it would be insightful to extend the present study to quintom models, which are free from such instabilities and can smoothly interpolate between quintessence-like and phantom-like behavior. The framework developed here serves as a foundation for such future studies.

We also constructed the GUP-modified Hamiltonian for a quintessence scalar field, introducing deformation directly into the momentum sector. In this case, unlike the phantom scenario, the Klein–Gordon equation explicitly exhibits GUP dependence. The resulting modifications alter the energy density, pressure, and dynamical equations governing the evolution of the scalar field. By analyzing potentials of the form $V(\phi) = V_0 \sinh^{-n}(\mu\phi)$, we demonstrated how GUP distortion changes the nature and location of fixed points and modifies their stability properties. The disappearance and distortion of specific fixed points in the GUP-corrected regime reveal that the minimal length scale enriches the dynamical structure of the universe, introducing new pathways in phase space evolution.

In summary, this work highlights that GUP-induced deformations significantly affect the qualitative dynamics of cosmological scalar field models, influencing both early-time inflation and late-time acceleration. While the global features remain broadly consistent with classical cosmology, local behaviors, stability conditions, and inflationary parameters are altered in nontrivial ways. These findings emphasize the importance of incorporating minimal length effects in cosmological studies to probe the phenomenological implications of quantum gravity.

From an observational perspective, these deviations could, in principle, leave small imprints on inflationary observables, including the scalar spectral index, the tensor-to-scalar ratio, and the primordial gravitational-wave spectrum. Future high-precision observations from Cosmic Microwave Background (CMB) experiments, large-scale structure surveys, and next-generation gravitational-wave detectors may therefore provide indirect constraints on the GUP deformation parameter. Consequently, sufficiently precise cosmological observations could potentially distinguish between the two scenarios and provide an indirect test of Planck-scale physics. Such analyses have recently been carried out in other GUP-inspired inflationary scenarios, where the theoretical predictions were confronted with the latest observational data from ACT DR6, Planck, DESI

BAO, and BICEP/Keck observations [181]. Extending the present phantom GUP cosmological model to include cosmological perturbations and comparing its predictions with current observational data remains an interesting direction for future research.

Chapter 6

HAWKING RADIATION

The discovery of General Relativity (GR), proposed by Albert Einstein in 1915, provided a mathematical framework to construct many cosmological models as well to explain astrophysical phenomena. The profound success of GR can be seen from prediction like perihelion precession of mercury[182], deflection of light when passing through massive bodies[183], and gravitational redshift of light[184]. In addition to this one of the most intriguing predictions of GR is the mysterious supermassive astronomical objects having a very large gravitational attraction that even light cannot escape are Black Holes[185–188].

Extensive studies on black holes have explored their formation, properties, radiation, and even their shadows, all through the lens of GR [189–193]. Beyond these properties, black holes may offer insights into the origins and fundamental structure of the universe itself [194–196]. One of the defining and particularly fascinating characteristics of black holes is their emission of radiation. Black holes, as described by Hawking, behave as black bodies that emit thermal radiation with a temperature T_H inversely proportional to their mass M . This phenomenon, known as Hawking radiation, leads to a gradual reduction in the black hole’s mass, with the temperature T_H diverging as the mass approaches zero. From the perspective of an external observer, the radiative process makes the black hole appear akin to a quantum system. However, Hawking’s insight in 1975 introduced the concept of the “Hawking information paradox”, highlighting a loss of information associated with the emitted radiation, which implies an increase in the von Neumann entropy of the universe.

Various derivations exist for Hawking radiation, many of which involve quantum effects near the black hole's event horizon. One of the most widely discussed explanations is based on particle-antiparticle pair production at the horizon: one particle escapes while the other falls into the black hole, resulting in radiation from the escaping particle. The infalling particle possesses negative energy, thereby reducing the black hole's mass, while the escaping particle contributes to the observed radiation [197]. Although this mechanism provides an intuitive view, it raises questions about the specific nature and conservation principles involved in the process [198–201]. Two widely used methods in the literature for calculating the imaginary part of the action are the Parikh-Wilczek radial null geodesic method and the Hamilton-Jacobi method.

In 2000, Parikh and Wilczek introduced a new approach by combining quantum mechanics with General Relativity [202]. Using radial null geodesics, they calculated tunneling probabilities by analyzing particle trajectories in a dynamic geometry while enforcing conservation laws. This approach, which accounts for the shrinking event horizon due to radiation, leads to non-thermal corrections to the black hole's emission spectrum and provides a partial resolution to the information loss problem.

On the other hand, the Hamilton-Jacobi method relies on the WKB approximation, a technique commonly used in quantum mechanics to study barrier penetration problems. The key idea is to determine the imaginary part of the classical action $\mathcal{S}$ of an emitted particle, which is directly related to the tunneling probability.

Recent advancements in quantum gravity propose solutions to issues related to information loss and non-thermal emissions. Quantum gravity theories are particularly relevant near black hole horizons and singularities, where extreme energies and minimal length scales dominate. Among these approaches, the Generalized Uncertainty Principle (GUP) is a prominent theory. By modifying the Heisenberg Uncertainty Principle to include linear and quadratic momentum terms, the GUP introduces a minimum observable length scale. This modification is supported by other quantum gravity frameworks, including Loop Quantum Gravity (LQG) [24, 130, 203–206], String Theory (ST) [207–213], and Doubly Special Relativity (DSR) [214, 215]. With the success of GUP-based models in explaining various cosmological phenomena [103, 168–177, 216], this chapter extends GUP corrections to the Schwarzschild black hole to investigate their impact on the Hawking temperature using the Parikh-Wilczek and Hamilton-Jacobi (HJ) methods.

The chapter is structured as follows: In Sec. 6.1.1, we present the GUP-modified Schwarzschild metric in Painlevé-Gullstrand coordinates, providing a stationary and non-singular formulation suitable for our analysis. We then compute the tunneling probability using the Parikh-Wilczek method and derive the associated Hawking temperature for massless particles within the GUP framework. Using a semiclassical WKB approximation, we analyze how GUP modifies the Hawking temperature, introducing a minimal length scale and predicting a stable remnant. Additionally, we explore the implications for black hole information loss by examining correlations among emitted particles, demonstrating that the modified emission spectrum is non-thermal, potentially resolving the information paradox.

In Sec. 6.1.2, we derive the GUP-modified Hawking temperature using the Hamilton-Jacobi method and compare the results with the Parikh-Wilczek approach. Finally, in Sec. IV, we summarize our findings and discuss how GUP corrections contribute to a deeper understanding of black hole thermodynamics and quantum gravitational effects that may preserve information.

6.1 The Tunneling mechanism in GUP framework

In the literature, two widely used approaches exist for deriving the Hawking temperature of black holes one is the Parikh-Wilczek radial null geodesic approach and other is the Hamilton-Jacobi method. The Hamilton-Jacobi method is based on the WKB approximation applied to the Klein-Gordon equation in a curved spacetime background, whereas the Parikh-Wilczek method incorporates quantum tunneling by analyzing the trajectory of emitted particles across the event horizon. In the next section, we extend both methods by incorporating the effects of GUP and derive the corresponding corrections to the Hawking temperature. These modifications account for quantum gravitational effects, which become significant near the Planck scale.

6.1.1 GUP corrected Parikh-Wilczek Radial null geodesic method

The general relativity (GR) predicts a singularity in spacetime at the center of a black hole. It is believed that quantum gravitational effect, which can not be neglected at high curvature regions, could regulate the occurrence of singularity. Various approaches have

been proposed to address this issue using quantum gravity theories [217–219]. In this chapter, we take a bottom-up approach by incorporating the GUP, which modifies the HUP to introduce a minimal length, as proposed by various quantum gravity theories. These modifications include additional terms, with both linear and quadratic corrections, as suggested by ST and DSR. The most well-known form is the quadratic correction introduced by Kempf, Mangano, and Mann (KMM) [74, 220–222] as

$$\Delta x \Delta p \geq \frac{\hbar}{2} \left(1 + \alpha L_P^2 \frac{\Delta p^2}{\hbar^2} \right), \quad (6.1)$$

where α is the dimensionless GUP parameter of order unity and L_P is the Planck length. It is straightforward to see that we recover the well-known HUP when $\alpha \rightarrow 0$. With this the GUP implies the following inequality

$$\frac{\hbar}{\alpha L_P^2} \Delta x \left(1 - \sqrt{1 - \frac{\alpha L_P^2}{\Delta x^2}} \right) \leq \Delta p \leq \frac{\hbar}{\alpha L_P^2} \Delta x \left(1 + \sqrt{1 - \frac{\alpha L_P^2}{\Delta x^2}} \right), \quad (6.2)$$

and imposes a minimum bound on the uncertainty in position, given by

$$(\Delta x)_{min} = \sqrt{\alpha} L_P.$$

Now, by assuming that photons escape the Schwarzschild black hole at the radius $r_H = 2GM$ and that the spectrum of these escaping photons is thermal, as suggested by Adler et al. [223], it is possible to derive the Hawking temperature as

$$T_{GUP} = \frac{M}{\pi \alpha} \left(1 - \sqrt{1 - \frac{\alpha M_p^2}{4M^2}} \right), \quad (6.3)$$

where, $M_p^2 = \frac{\hbar c}{G}$ with $c = 1$. In this context, we adhere to the convention of introducing a $\frac{1}{2\pi}$ factor to obtain the Hawking temperature of the Schwarzschild black hole, $T_{Sch} = \frac{M_p^2}{8\pi M}$, or $T_{Sch} = \frac{1}{8\pi GM}$ with $\hbar = 1$ in the limit of small α .

On the other hand, to incorporate the effects of GUP into the black hole metric, we first consider the classical Schwarzschild black hole, where the line element is given by:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \quad (6.4)$$

with $f(r)$ defined as $\left(1 - \frac{2GM}{r}\right)$. The event horizon is located at $r_s = 2GM$. We

anticipate that the effects of the GUP will modify the metric elements by introducing additional correction terms, such that the original Schwarzschild metric is restored when the GUP effects are removed. Using the approach outlined in [224, 225], the GUP-corrected Schwarzschild metric takes the following form:

$$ds^2 = -g(r)dt^2 + \frac{dr^2}{g(r)} + r^2 d\Omega^2, \quad (6.5)$$

where

$$g(r) = \left(1 - \frac{2GM}{r}\right) h(r), \quad (6.6)$$

which is the most general form we get on preserving the areal radius. The modified Hawking temperature became

$$T = \frac{1}{4\pi} g'(r)|_{r=r_H} = \frac{1}{8\pi GM} h(r_H). \quad (6.7)$$

Now, on comparing Eq.(6.3) and Eq.(6.7) one can deduce the expression

$$h(r_H) = \frac{2r_H^2}{G\alpha} \left(1 - \sqrt{1 - \frac{\alpha L_p^2}{r_H^2}}\right), \quad (6.8)$$

or, in a more general form

$$h(r) = \frac{2r^2}{G\alpha} \left(1 - \sqrt{1 - \frac{\alpha L_p^2}{r^2}}\right). \quad (6.9)$$

For a small α limit, the GUP-modified effective metric is reduced to

$$g(r) = \left(1 - \frac{2GM}{r}\right) \left(1 + \frac{\alpha}{4r^2} + \frac{\alpha^2}{8r^4}\right). \quad (6.10)$$

where, we set $\hbar = L_P = 1$ from now on. Since these corrections are obtained from the GUP-modified Hawking temperature [224, 225], we cannot directly apply it to our model as our approach is to get a GUP-modified Hawking temperature through the tunneling process from the modified Schwarzschild metric. In the next section, we first derive these corrections by treating GUP as a perturbation in the background metric $g_{\mu\nu}$ and then follow the procedure.

6.1.1.1 GUP in curved spacetime

We follow the seminal work presented in [226], using the canonical commutator relation between the position operator x^μ and momentum operator p^ν is expressed in Minkowski space-time are $[x^\mu, p^\nu] = i\eta^{\mu\nu}$. In the context of curved space-time this commutator relation can be expressed as

$$[x^\mu, p^\nu] = ig^{\mu\nu}, \quad (6.11)$$

where $g_{\mu\nu}$ is the background metric of a curved spacetime. The GUP, as one of the approaches to quantum gravity, which introduces corrections to the background metric. We can express the perturbed metric as:

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}, \quad (6.12)$$

where $h_{\mu\nu}$ is a small perturbation, with condition $|h_{\mu\nu}| \ll |g_{\mu\nu}|$, representing the correction introduced by GUP. We assume that the perturbation is proportional to the background metric $g_{\mu\nu}$, which gives the following form as [227]:

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} (1 + \epsilon \tilde{h}), \quad (6.13)$$

where ϵ is a small parameter associated with GUP, and $\tilde{h}$ is a dimensionless perturbation quantifying the relative change to the background metric.

If we consider the effects of GUP in the Schwarzschild metric, the metric function $f(r)$ is modified to include higher-order corrections where the leading-order term is $g(r) = 1 - \frac{2GM}{r}$. The GUP-corrected Schwarzschild metric function can be expressed [224, 225] as:

$$g(r) = \left(1 - \frac{2GM}{r}\right) h(r), \quad (6.14)$$

where $h(r)$ includes higher-order corrections, and the leading order term is given by $(1 - \frac{2GM}{r})$. We also expect that $h(r) \rightarrow 1$ for non-GUP case and for asymptotic flat space for $r \rightarrow \infty$.

In paper [228], the corrections due to GUP can be incorporated into the metric as higher-order terms. One such expansion is:

$$g(r) = 1 - \frac{2GM}{r} + \frac{\epsilon}{r^2} + \frac{\epsilon^2}{r^4}, \quad (6.15)$$

where ϵ represents the first-order GUP correction. From Eq.(8.63, 6.14, 6.15), it is evident that the perturbation term $\tilde{h}_{\mu\nu}$ introduces corrections of the form $\mathcal{O}(1/r^{2n+1})$, where the leading term $\frac{1}{r^2}$ gives the lowest-order correction. Thus, the function $h(r)$, representing the GUP corrections using Eq.(8.63, 6.14, 6.15), can be written as:

$$h(r) = 1 + \frac{\epsilon}{r^2} + \frac{\epsilon^2}{r^4}. \quad (6.16)$$

To match the GUP corrections given in Eq.(6.10) up to second order in $\frac{1}{r^2}$, we set $\epsilon = \frac{\alpha}{4}$, where α is a constant derived from the specific GUP framework. This gives the following perturbed metric:

$$g(r) = 1 - \frac{2GM}{r} + \frac{\alpha}{4r^2}, \quad (6.17)$$

Eq.6.17 and Eq.6.10 are similar which gives the freedom to choose this kind of GUP correction in the line element.

One can easily observe that in the non-GUP case, the event horizon of a Schwarzschild black hole is simply located at $r = 2GM$. However, when incorporating the effects of the GUP, the metric coefficient in Eq. (6.5) leads to the formation of two distinct event horizons with an inner horizon and outer horizon given

$$r_{inner} = \frac{1}{2}[2M - \sqrt{4M - \alpha}], \quad (6.18)$$

$$r_{outer} = \frac{1}{2}[2M + \sqrt{4M - \alpha}], \quad (6.19)$$

Thus, the GUP-corrected Schwarzschild metric is given by:

$$ds^2 = - \left(1 - \frac{2M}{r} + \frac{\alpha}{4r^2}\right) dt_s^2 + \left(1 - \frac{2M}{r} + \frac{\alpha}{4r^2}\right)^{-1} dr^2 + r^2 d\Omega^2, \quad (6.20)$$

where t_s is the Schwarzschild time. The above line element resembles that of Reissner-Nordström electrically charged black hole.

Parikh and Wilczek, in 1999, introduced a direct explanation for Hawking radiation as a tunneling process of particles through the black hole's horizon within a “dynamic geometry” framework [202]. They precisely demonstrated that the tunneling barrier is formed by the energy of the particle itself. Additionally, they showed that as particles tunnel, the black hole's radius shrinks, indicating a dynamic geometry. This concept has

since been extended to tackle the information loss paradox by incorporating quantum gravity effects, through non-commutative geometry [229] and [230].

Since tunneling process occurs across the horizon, it is necessary to choose the stationary and non-singular coordinates at the horizon. This is achieved by introducing new time coordinates as:

$$t = t_s + \int \left[\frac{\sqrt{1 - g_{00}}}{g_{00}} \right] dr, \quad (6.21)$$

where t_s is the Schwarchild time. With this choice, the new Schwarzschild line element in Painlevé-Gullstrand coordinates reads

$$ds^2 = - \left(1 - \frac{2M}{r} + \frac{\alpha}{4r^2} \right) dt^2 + 2 \left(\sqrt{\frac{2M}{r} - \frac{\alpha}{4r^2}} \right) dt dr + dr^2 + r^2 d\Omega^2. \quad (6.22)$$

6.1.1.2 Tunneling process for massless particles

From the Eq.(6.22) one can deduce the equation of radial null geodesics as

$$\dot{r} = \pm 1 - \sqrt{\frac{2M}{r} - \frac{\alpha}{4r^2}}, \quad (6.23)$$

where the plus and minus signs correspond to outgoing (ingoing) geodesics as t increases towards the future. We consider that just inside the outer event horizon, pair creation takes place, creating virtual particle-antiparticle pairs. The absorption and emission of these virtual particles fluctuate the hole's mass through energy conservation law. During the tunneling process, emission of virtual particle with energy ω reduces the mass M to $M - \omega$ through conservation law. These fluctuations will also affect the space-time geometry just outside the horizon making the geometry dynamical.

Consequently, Eq.(6.22) and Eq.(6.23) get modified by considering self-gravitating effects as

$$ds^2 = - \left[1 - \frac{2(M - \omega)}{r} + \frac{\alpha}{4r^2} \right] dt^2 + 2 \left[\sqrt{\frac{2(M - \omega)}{r} - \frac{\alpha}{4r^2}} \right] dt dr + dr^2 + r^2 d\Omega^2, \quad (6.24)$$

and

$$\dot{r} = \pm 1 - \sqrt{\frac{2(M - \omega)}{r} - \frac{\alpha}{4r^2}}. \quad (6.25)$$

We consider that the virtual particle is at r_{in} initially, which is ζ (very small quantity with the dimension of length) smaller than the outer horizon radius crosses somewhere at r_{out} , which is ζ greater than the reduced outer horizon

$$r_{in} = \frac{1}{2} \left[2M + \sqrt{4M^2 - \alpha} \right] - \zeta, \quad (6.26)$$

$$r_{out} = \frac{1}{2} \left[2(M - \omega) + \sqrt{4(M - \omega)^2 - \alpha} \right] + \zeta. \quad (6.27)$$

Considering the outgoing wave traced back toward the outer horizon, its wavelength, as measured by the stationary observer, is extremely blue-shifted. Consequently, the radial wavenumber approaches infinity, making the point particle description valid or justifying the WKB (Wentzel-Kramers-Brillouin) approximation.

One can use the semiclassical procedure to find the transmission coefficient of tunneling particle as an exponential function of imaginary part of action as

$$\Gamma \approx e^{-2\text{Im}S}. \quad (6.28)$$

The imaginary part of action for positive energy particle crosses the horizon r_{in} to r_{out} is expressed as

$$\text{Im}S = \text{Im} \int_{r_{in}}^{r_{out}} p_r dr, \quad (6.29)$$

where p_r is the canonical momentum of a virtual particle with Hamiltonian as $H = M - \omega$. And, we can write it as

$$\text{Im}S = \int_0^\omega \int_{r_{in}}^{r_{out}} \frac{dr(-d\omega)}{\dot{r}}. \quad (6.30)$$

Now, by substituting $\dot{r}$ from Eq. (6.25) into Eq. (6.30), one can clearly see that the integrand has two poles, which can be solved by deforming the contour. For simplicity, one can expand g_{00} around the outer horizon r_{out} , making it have just one pole located at r_{out} [230, 231] which gives

$$\text{Im}S = \int_0^\omega \frac{2\pi \{ 2(M - \omega) + \sqrt{4(M - \omega)^2 - \alpha} \}}{4(M - \omega) \{ 2(M - \omega) + \sqrt{4(M - \omega)^2 - \alpha} \} - 2\alpha}, \quad (6.31)$$

finally, expressed as

$$\text{Im}S \simeq 2\pi[2\omega \left(M - \frac{\omega}{2}\right) + \frac{M}{2}\sqrt{4M^2 - \alpha} - \frac{M - \omega}{2}\sqrt{4(M - \omega)^2 - \alpha}]. \quad (6.32)$$

A similar result can be derived for the ingoing radial motion of a virtual particle with negative energy created outside the black hole and tunneling into it. Now, substituting Eq.(6.32) on Eq.(6.28) gives

$$\Gamma \approx \exp \left\{ -4\pi \left[2\omega \left(M - \frac{\omega}{2} \right) + \frac{M}{2} \sqrt{4M^2 - \alpha} - \frac{M - \omega}{2} \sqrt{4(M - \omega)^2 - \alpha} \right] \right\} = e^{+\Delta S_{BH}}, \quad (6.33)$$

where ΔS_{BH} is the change in Bekenstein-Hawking entropy. On comparing the power of exponential term in Eq.(6.33) to the thermal emission $e^{-\beta\omega}$, T_H at Hawking temperature only up to the first order in ω gives

$$T_H^{GUP} = \frac{\sqrt{4M^2 - \alpha}}{2\pi[2M + \sqrt{4M^2 - \alpha}]^2}. \quad (6.34)$$

It is clear that as the GUP parameter $\alpha \rightarrow 0$ the GUP modified Hawking temperature takes its standard form. Also, the Hawking temperature we obtained in Eq.(6.34) resembles the Hawking temperature obtained in the Reissner-Nordström black hole as

$$\alpha \propto Q^2, \quad (6.35)$$

where Q is the charge parameter in the RN black hole. The quantum effects on a Schwarzschild black hole behave similarly to those on an electrically charged black hole. This analogy leads us to conclude that electric charge behaves similarly to quantum effects. Such studies have already been conducted in a non-commutative background [232]. This implies that the existence of electric charge in spacetime leads to quantum fluctuations in the background. As the GUP parameter is related to the electric charge in Eq.(8.25), we have a lower bound on choosing this GUP parameter as Q is quantized as $\alpha \approx \frac{Q^2}{M_{Pl}^2} = 2.56 \times 10^{-38} / 2.1710^{-8}$ units, indicating a situation akin to minimal length, near the Planck length.

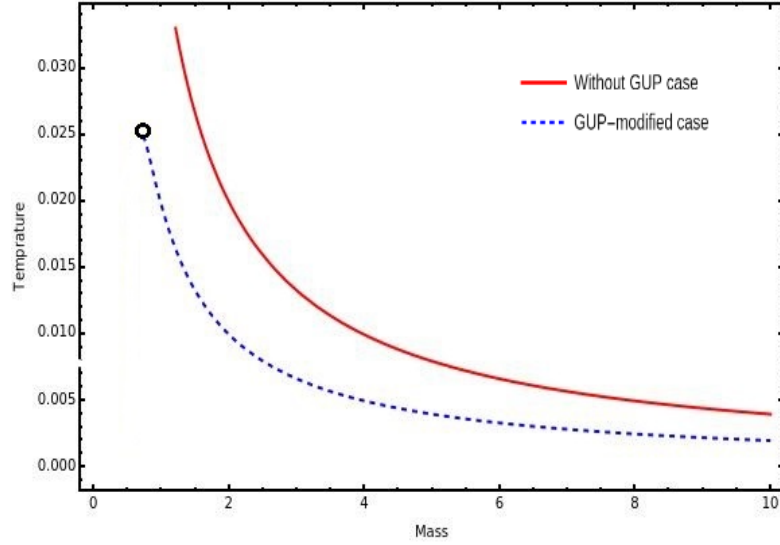


FIGURE 6.1: Graph between temperature and mass of black hole where red line depicts the ordinary case obtained by Parik-Wilczek and the blue dash line represents the GUP-deformed case where $\alpha = 1.5$.

From Fig.(6.1) we depict that for the ordinary case of black hole evaporation, the process terminates with the black hole reaching an infinitely high temperature, ultimately leading to the complete disappearance of the black hole without leaving any remnant behind. For the GUP-deformed scenario, black hole evaporation continues until the black hole reaches a finite mass, leaving behind a remnant with a radius $r = a$. In this scenario, a is the minimal length parameter, the smallest GUP-deformed non-singular Schwarzschild black hole radius. Consequently, this GUP deformation stops the further evaporation of black holes.

We depict from the modified Hawking temperature under GUP predict a non-thermal emission spectrum due to quantum corrections. The finite temperature during the final stages of black hole evaporation suggests that the emitted radiation contains information, potentially resolving the information loss paradox. The GUP corrections leads to address the information loss problem can be shown through the correlation function between the emitted particles as

$$\chi(\omega_1 + \omega_2; \omega_1, \omega_2) \equiv \ln[\Gamma(\omega_1 + \omega_2)] - \ln[\Gamma(\omega_1)\Gamma(\omega_2)], \quad (6.36)$$

where ω_1 and ω_2 are energies of two emitted particles.

$$\begin{aligned} \chi(\omega_1 + \omega_2; \omega_1, \omega_2) = 4\pi & \left[2\omega_1 \left(M - \frac{\omega_1}{2} \right) + 2\omega_2 \left(M - \frac{\omega_2}{2} \right) - 2(\omega_1 + \omega_2) \left(M - \frac{\omega_1 + \omega_2}{2} \right) \right. \\ & - \frac{M - \omega_1}{2} \sqrt{4(M - \omega_1)^2 - \alpha} - \frac{M - \omega_2}{2} \sqrt{4(M - \omega_2)^2 - \alpha} \\ & \left. + \frac{M - (\omega_1 + \omega_2)}{2} \sqrt{4(M - (\omega_1 + \omega_2))^2 - \alpha} + \frac{M}{2} \sqrt{4M^2 - \alpha} \right], \end{aligned} \quad (6.37)$$

since the value of the correlation function is nonzero therefore the radiation is not thermal emission. The tunneling probability of two particles having energy ω_1 and ω_2 is not the same as the tunneling probability of a particle with total energy $(\omega_1 + \omega_2)$. This indicates that there is a correlation between two emitted particles implying that information about the black hole can be encoded in these correlations. In this GUP framework, information is preserved within the remnant, and the GUP enhances these correlations, enabling information to surface from the event horizon, akin to “hidden messengers in Hawking radiation”.

6.1.2 GUP modification in Tunneling through Hamilton-Jacobi method

We start with the modified Heisenberg uncertainty relation of Eq. (6.1), the deformed Heisenberg algebra in four-dimensional Minkowski spacetime with signature $(-+++)$ [233, 234] is given by

$$[X_\mu, P_\nu] = i\hbar [(1 - \alpha(\eta^{\sigma\gamma} P_\sigma P_\gamma))\eta_{\mu\nu} - 2\alpha P_\mu P_\nu], \quad (6.38)$$

from the following commutator relation above one can write the undeformed position operator and the deformed momentum operator as

$$X_\mu = x_\mu, \quad (6.39)$$

$$P_\nu = p_\nu (1 - \alpha\eta^{\sigma\gamma} p_\sigma p_\gamma), \quad (6.40)$$

where x_μ and p_ν are the standard position and momentum operators with the commutator relation $[X_\mu, P_\nu] = i\hbar\eta_{\mu\nu}$. In the framework of the GUP, the Klein-Gordon equation, which governs the dynamics of massless particle described by the wave function Φ , $\eta^{\mu\nu} P_\mu P_\nu = 0$ transforms into a fourth-order partial differential equation [235]

given as

$$\Delta\phi + 2\alpha\hbar^2\Delta(\Delta\phi) + \mathcal{O}(\alpha^2) = 0, \quad (6.41)$$

where $\Delta = \frac{1}{-g}\partial_\mu(g^{\mu\nu}\sqrt{-g}\partial_\nu)$ for any Riemannian spacetime of Lorentzian signature with metric $g_{\mu\nu}$. Since the tunneling process occurs radially across the event horizon only the radial trajectories are relevant. And, for the near-horizon phenomenon, the dynamics are effectively governed by the two-dimensional $(r-t)$ sector of the full spacetime metric, making higher-dimensional contributions negligible. Also, we are considering the background geometry is modified due to the effect of GUP follows the GUP modified Schwarzschild line element in Eq.(6.20) where $g(r)$ is the GUP modified line element. Therefore the KG equation reduces to

$$\begin{aligned} -\frac{1}{g(r)}\partial_t^2\phi + g'(r)\partial_r\phi + g(r)\partial_r^2\phi + 2\beta\hbar^2\left\{\frac{1}{g(r)^2}\partial_t^4\phi + \left(\frac{g'(r)}{g(r)}\right)^2\partial_t^2\phi - \frac{1}{g(r)}\partial_r\partial_t^2\phi + g''(r)\partial_r\phi \right. \\ \left. + 2g'(r)\partial_r^2\phi + g(r)\partial_r^3\phi - \frac{2}{g(r)^2}\partial_t^2\phi - \partial_r^2\partial_t^2\phi \right. \\ \left. + g(r)g'''(r)\partial_r\phi + g(r)g'(r)\partial_r^3\phi + g(r)g''(r)\partial_r^2\phi + g(r)^2\partial_r^4\phi\right\} = 0, \end{aligned} \quad (6.42)$$

by employing the standard ansatz for the semi-classical wave function satisfying the above equation, we assume

$$\phi = \exp\left[-\frac{i}{\hbar}S(r, t)\right]. \quad (6.43)$$

Substituting this expression into Eq. (6.42) and expanding $S(r, t) = S_0(r, t) + \sum_i \hbar^i S_i(r, t)$, we restrict ourselves to the semiclassical value S_0 , which reduce Eq.(6.42) to

$$\frac{(\partial_t S)^2}{g(r)} - g(r)(\partial_r S)^2 + 2\alpha\left\{g(r)^2(\partial_r S)^4 - \frac{(\partial_t S)^4}{g(r)^2} + (\partial_r S)^2(\partial_t S)^2\right\} = 0, \quad (6.44)$$

the above equation, Eq. (6.44), is derived under the semi-classical limit $\hbar \rightarrow 0$, where terms of $\mathcal{O}(\hbar)$ and higher are neglected. We also observe that without the GUP parameter α we recover the usual semi-classical Hamilton-Jacobi equation. Given that the metric in Eq.(6.20) is static, it possesses a timelike Killing vector. Consequently, we seek a solution to Eq. (6.44) which behaves as

$$S_0 = \omega t + \tilde{S}_0(r), \quad (6.45)$$

where ω represents the conserved quantity (represents the energy of particle) associated with the timelike Killing vector. Substituting Eq.(6.45) in Eq.(6.44) gives

$$\tilde{S}_0(r) = \pm \int dr \sqrt{\frac{1}{4\alpha g(r)} - \frac{\omega^2}{2g^2(r)} \pm \frac{1}{4\alpha g^2(r)} \sqrt{g^2(r) - 12\alpha\omega^2 g(r) + 16\alpha^2\omega^4}}, \quad (6.46)$$

now on expanding the inside root and only taking up to the order $\mathcal{O}(\alpha)$ since $\alpha \ll 1$, which gives one of the roots of the equation is

$$\tilde{S}_0 = \pm \int dr \frac{\omega}{g(r)} \quad (6.47)$$

where the integration limits are chosen from the inner r_{in} to the outer horizon radius r_{out} given in Eq.(6.26). The $(\pm)$ sign in front of the integral represents the outgoing and ingoing particles. Now, using Eq.(6.47) and Eq.(6.45)

$$S_0(r, t) = \omega t \pm \omega \int \frac{dr}{g(r)}, \quad (6.48)$$

from Eq.(6.43) we get,

$$P^{(L,R)} = |\phi^{(L,R)}|^2 = \exp -\frac{1}{\hbar} \left(\omega \text{Im}t \pm \omega \text{Im} \int \frac{dr}{g(r)} \right), \quad (6.49)$$

$\phi^{L,R}$ represents the left and right modes of the wavefunction. This gives the ingoing and outgoing probabilities of the particle. Using the classical limit, where the ingoing probability must be unity when there is no reflection and everything is absorbed [236] gives

$$\text{Im}t = -\text{Im} \int \frac{dr}{g(r)}. \quad (6.50)$$

Now using Eq.(6.50), Eq.(6.49) and principle of “*detailed balance*”, we get

$$P^{(R)} = \exp -\frac{4}{\hbar} \omega \text{Im} \int \frac{dr}{f(r)} = \exp -\frac{\omega}{T_H^{GUP}}, \quad (6.51)$$

since $g(r)$ is the GUP-modified Schwarzschild metric element now on doing the contour integral one of the pole lies inside the contour and similar procedure to get Eq.(6.31), we obtain

$$T_H = \frac{\sqrt{4M - \alpha}}{2\pi[2M + \sqrt{4M - \alpha}]^2}. \quad (6.52)$$

One can easily see that as the GUP parameter α vanishes in the above expression we recover the standard results. We also observe that Eqs. (6.52) and (6.34), derived from

two different approaches, lead to the same modification to the Hawking temperature.

6.2 Conclusion

In this chapter, we have investigated the implications of the Generalized Uncertainty Principle (GUP) on black hole thermodynamics, particularly focusing on its effects on Hawking radiation and the evaporation process of the Schwarzschild black hole. The GUP, as a phenomenological approach to quantum gravity, introduces a minimal measurable length scale that modifies the standard Heisenberg uncertainty relation and consequently alters the fundamental structure of spacetime at the Planck scale.

We derived the GUP-modified Hawking temperature using two independent but complementary approaches: the Parikh–Wilczek tunneling method and the Hamilton–Jacobi (HJ) method. In Sec. 6.1.1, we formulated the GUP-deformed Schwarzschild metric and transformed it into the Painlevé–Gullstrand coordinate system to eliminate coordinate singularities and obtain a stationary metric suitable for the tunneling analysis. We then analyzed the quantum tunneling of massless particles across the event horizon using the WKB approximation and obtained the modified Hawking temperature. Our results indicate that the standard Parikh–Wilczek expression for the Hawking temperature acquires additional correction terms proportional to the GUP parameter α , as shown in Eq. (6.34). Importantly, in the limit $\alpha \rightarrow 0$, the conventional Hawking temperature is recovered, ensuring the consistency of the framework.

A crucial consequence of these findings is that, within the GUP framework, black holes do not evaporate completely. Instead, they asymptotically approach a finite remnant mass corresponding to a maximum temperature. This result contrasts sharply with the standard semiclassical picture, in which the black hole temperature diverges as the mass approaches zero. The introduction of a minimal length scale through GUP effectively prevents complete evaporation by establishing a lower bound on the black hole mass. Such remnants are of particular interest in quantum gravity and could potentially serve as stable end-states of black hole evaporation.

Furthermore, we examined the correlations among emitted quanta with different frequencies, ω_1 and ω_2 , to study the non-thermal nature of Hawking radiation in the presence of GUP corrections. Our analysis reveals that GUP introduces nontrivial correlations

between successive emissions, thereby enhancing the information-carrying capacity of the radiation spectrum. This finding suggests that GUP-induced effects may provide a pathway toward resolving the black hole information loss paradox by allowing partial information recovery through correlated emissions and the existence of remnants.

In Sec. 6.1.2, we derived the Hawking temperature independently using the Hamilton–Jacobi method. Starting from the GUP-corrected Klein–Gordon equation and employing the modified momentum operator, we obtained the tunneling probability and corresponding temperature. Remarkably, the resulting expression for the temperature, Eq. (6.52), is in complete agreement with that obtained from the Parikh–Wilczek tunneling framework, thereby confirming the consistency and robustness of the analysis across different formalisms.

In summary, this chapter demonstrates that the incorporation of GUP leads to profound modifications in black hole thermodynamics. The existence of a minimal length scale alters the evaporation process, introduces correlations among emitted particles, and prevents complete evaporation by stabilizing black holes into long-lived remnants. These results provide valuable insights into the interface between quantum mechanics and gravity and contribute to our understanding of the microscopic structure of space-time. Although no direct observational evidence for quantum-gravity-induced black hole remnants currently exists, recent theoretical and phenomenological studies have highlighted several possible astrophysical signatures and observational strategies for probing such remnants through high-energy astrophysical observations and gravitational-wave measurements [237]. Consequently, the present work provides a theoretical framework that may be tested by future astronomical observations capable of constraining the existence of black hole remnants and the associated GUP deformation parameter. Future work may further extend the present analysis to charged and rotating black holes, investigate the thermodynamic stability of the remnants, and explore their cosmological and astrophysical implications.

Chapter 7

ANGULAR MOMENTUM ALGEBRA

Studies of the GUP have revealed changes in several aspects of Quantum Mechanics (QM). For instance, the Hamiltonian describing a minimal coupling with an electromagnetic field is expected to undergo modifications, as demonstrated in [238]. This has been used to show changes in the Landau levels [238]. Additionally, the GUP is found to affect phenomena such as the Lamb shift, potential steps and barriers, which are relevant in quantum configurations like Scanning Tunnelling Microscopes [239, 240], as well as the case of a particle in a box, where the box's length becomes quantized [239–241]. Moreover, the energy levels of a simple harmonic oscillator are also altered [239]. It has been suggested that the GUP could be detected through its effects on quantum optical systems [242–244].

In this Chapter, we discuss GUP corrections to a significant theoretical and experimental area of quantum mechanics mainly in the line of the seminal work in [245]. Specifically, we examine how GUP leads to modifications in the angular momentum algebra. This article is organized as follows: In sec. 7.1, we start with the well-known angular momentum algebra in QM. In Sec. 7.2, we discuss the modifications arising from different models of GUPs to the angular momentum algebra. Finally, in Sec. 7.3, we extend the application of the modified angular momentum algebra to the hydrogen atom and observe the resulting modifications to the energy levels.

7.1 Angular Momentum Algebra in HUP

In this section, we review the standard angular momentum commutator brackets using the HUP and then compare it with the results obtained when applying the GUP. Angular momentum in classical and quantum mechanics plays an important role in understanding many physical phenomenons and their mathematical descriptions like the trajectory of planets and planetary systems (Kepler's law), rotation of rigid bodies, the structure of atoms, and many more. In this section, we start with the classical definition of angular momentum. In classical mechanics, angular momentum measures the “amount of rotation” likewise linear momentum which measures the “amount of motion”. In simple terms, for point particle, it is the cross product of position and momentum. Angular momentum also follows the law of conservation as motion is in the central field. The particle motion is restricted to the plane perpendicular to the angular momentum vector.

The classical description of angular momentum is given as follows: the angular momentum $\vec{L}$ of a particle with mass m and linear momentum $\vec{p}$ located at position $\vec{r}$ is given by the vector cross product:

$$\vec{L} = \vec{r} \times \vec{p}. \quad (7.1)$$

where the magnitude is given by $L = r.p \sin \theta$ and θ is the angle between the vectors $\vec{r}$ and $\vec{p}$. The components in the cartesian coordinate are:

$$L_x = yp_z - zp_y, \quad L_y = zp_x - xp_z, \quad L_z = xp_y - yp_x. \quad (7.2)$$

In quantum mechanics, physical quantities such as angular momentum are represented by operators acting on the wave function of a system, rather than continuous variables. The classical quantities of position and momentum are promoted to operators as we go into the quantum regime. The position operator $\hat{r}$ and the momentum operator $\hat{p}$ in one dimension are defined as:

$$r \equiv \hat{r} \equiv \hat{q}, \quad p \equiv \hat{p} \equiv -i\hbar \frac{d}{dq}. \quad (7.3)$$

Angular momentum is defined through the generator of infinitesimal rotations and its algebraic properties. However, for orbital angular momentum, this definition in Eq.

(7.22) and the generator of rotation are equivalent. This is analogous to linear momentum, which is the generator of translations and, like angular momentum, does not give rise to a minimal uncertainty relation, as it follows the standard Heisenberg algebra. Angular momentum, as the generator of rotation, also does not allow for minimal angular resolution. Following this, the corresponding components of angular momentum in Cartesian coordinates are represented as follows:

$$L_x = -i\hbar \left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right), \quad L_y = -i\hbar \left(z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right), \quad L_z = -i\hbar \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right). \quad (7.4)$$

Using the commutator relation between position and linear momentum ($[q_i, p_j] = i\hbar \delta_{ij}$), one can easily obtain the “fundamental commutation relations of angular momentum” which incorporate all the basic properties of rotation in three dimensions:

$$[L_i, L_j] = i\hbar \epsilon_{ijk} L_k. \quad (7.5)$$

with the Levi-Civita symbols

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } ijk \text{ is an even permutation of } 123 \\ -1 & \text{if } ijk \text{ is an odd permutation of } 123 \\ 0 & \text{otherwise.} \end{cases}$$

To define Eq.(7.25), we used two main concepts: first, that L_k is the generator of rotations about the K^{th} axis, and second, that rotations about different axes do not commute. Additionally, one can verify that the Jacobi identity, an important property of the Lie algebra of angular momentum operators, follows from Eq.(7.25) is given as

$$[L_i, [L_j, L_k]] + [L_j, [L_k, L_i]] + [L_k, [L_i, L_j]] = 0, \quad (7.6)$$

Using Eq.(7.6), it is easy to show that we can define eigenstates of L^2 and L_z simultaneously or in other words L^2 commutes with all of its components as:

$$[L^2, L_j] = [L_i^2, L_j] + [L_j^2, L_j] + [L_k^2, L_j] = 0. \quad (7.7)$$

In fact, consider that

$$[L_i, p_m] = \epsilon_{ijk}[q_j p_k, p_m] = \epsilon_{ijk}[q_j, p_m]p_k = i\hbar\epsilon_{ijk}p_k \quad (7.8)$$

Using Eq.(7.8), we have

$$[L_i, p^2] = [L_i, p_i^2 + p_j^2 + p_k^2] = p_i[L_j, p_i] + [L_j, p_i]p_i + p_j[L_k, p_j] + [L_k, p_j]p_j = 0 \quad (7.9)$$

and from this

$$[L_i, p^2] = 0 = 2p[L_i, p]$$

that means for $p \neq 0$ gives

$$[L_i, p] = 0. \quad (7.10)$$

Similarly, commutator relation of L^2 with p and p^2 is as follows:

$$[L^2, p] = L_i[L_i, p] + [L_i, p]L_i = 0, \quad (7.11)$$

$$[L^2, p^2] = L_i[L_i, p^2] + [L_i, p^2]L_i = 0. \quad (7.12)$$

From the above two Eqs.(7.11,7.12), we observe that the operators L^2 , L_z , p and p^2 commute, allowing us to construct simultaneous eigenstates of these quantities. These eigenstates are also eigenstates of the free particle Hamiltonian, which represents the system's energy in the absence of external forces or potentials. Since the angular momentum operator corresponds to an observable physical quantity, it is Hermitian, meaning it produces real eigenvalues when acting on its eigenstates. In the context of group theory, operators that commute with one another form an abelian group. However, because the components of angular momentum do not commute, the algebra they form is non-abelian.

7.1.1 Ladder operators

Quantum operators like angular momentum are not a convenient way to keep track of eigenvalues and eigenfunctions. For this we introduced Ladder operators as L_+ and L_- on the eigenfunction Y_{lm} where l and m are the quantum numbers. The usual ladder

operators are given as:

$$L_+ = L_x + iL_y, \quad L_- = L_x - iL_y,$$

with commutator relation with the z-component of angular momentum as:

$$[L_z, L_\pm] = [L_z, L_x] \pm i[L_z, L_y] = \hbar(iL_y \pm L_x) = \pm \hbar L_\pm, \quad (7.13)$$

The L^2 commutes with the ladder operators $L_\pm$ as

$$[L^2, L_\pm] = [L^2, L_x] \pm i[L^2, L_y] = 0. \quad (7.14)$$

We also have

$$[L_+, L_-] = [L_x + iL_y, L_x - iL_y] = -2i[L_x, L_y] = 2\hbar L_z. \quad (7.15)$$

The ladder operator also satisfies,

$$L_\pm L_\mp = L^2 - L_z^2 \pm \hbar L_z. \quad (7.16)$$

We also know that ladder operators are not Hermitian since they do not correspond to any physical observable quantities; rather, they act to raise or lower the eigenstates.

The eigenvalues of L^2 and component L_z for the given eigenstate are

$$L^2|nlm\rangle = \hbar^2 l(l+1)|nlm\rangle, \quad L_z|nlm\rangle = \hbar m|nlm\rangle \quad (7.17)$$

where n , l , and m are quantum numbers that take on only integer values. Now using Eq.(7.13), we get

$$L_z L_\pm |nlm\rangle = (m \pm 1)\hbar L_\pm |nlm\rangle \quad (7.18)$$

which means that $L_\pm |nlm\rangle$ act as an eigenstate for L_z with eigenvalue $(m \pm 1)\hbar$. The operators $L_\pm$ are also referred to as raising and lowering operators, as they increase or decrease the eigenvalue of L_z by an amount of $\hbar$. Now using Eq.(8.50) we also obtain

$$L^2 L_\pm |nlm\rangle = L_\pm L^2 |nlm\rangle = l(l+1)\hbar^2 L_\pm |nlm\rangle \quad (7.19)$$

From the above equation, we observe that the ladder operators act on the eigenstates of L^2 without altering its eigenvalue, while they do change the eigenvalue associated with L_z . This point emphasizes that ladder operators affect the quantum number m associated with the L_z component but leave the total angular momentum quantum number l unchanged. We can determine the eigenvalue for the ladder operators in a given state, where the ladder operator shifts the state either one level up or one level down:

$$L_{\pm}|l, m\rangle = \gamma_{\pm}\hbar|l, m \pm 1\rangle \quad (7.20)$$

where, $\gamma_{\pm} = \sqrt{l(l+1) - m(m \pm 1)}$.

In this section, we discussed the standard results of angular momentum algebra in quantum mechanics. All of the aforementioned results are derived under the assumptions of the HUP without considering the implications of a minimal length or minimal angular resolution. In the next section, we will address the concept of minimal length and its effects on angular momentum algebra.

7.2 Modified Angular momentum algebra in the presence of GUP

Angular momentum in classical and quantum mechanics plays an important role in understanding many physical phenomenons and their mathematical descriptions like the trajectory of planets and planetary systems (Kepler's law), rotation of rigid bodies, the structure of atoms, and many more. In this section, we start with the classical definition of angular momentum. In classical mechanics, angular momentum measures the “amount of rotation” likewise linear momentum which measures the “amount of motion”. In simple terms, for point particle, it is the cross product of position and momentum. Angular momentum also follows the law of conservation as motion is in the central field. The particle motion is restricted to the plane perpendicular to the angular momentum vector.

The classical description of the angular momentum $\vec{L}$ of a particle with mass m and linear momentum $\vec{p}$ located at position $\vec{r}$ is given by the vector cross product

$$\vec{L} = \vec{r} \times \vec{p}. \quad (7.21)$$

where the magnitude is given by $L = rp \sin \theta$ and θ is the angle between the vectors $\vec{r}$ and $\vec{p}$. The components in the Cartesian coordinate system are

$$L_x = yp_z - zp_y, \quad L_y = zp_x - xp_z, \quad L_z = xp_y - yp_x. \quad (7.22)$$

In quantum mechanics, physical quantities such as angular momentum are represented by operators acting on the wave function of a system, rather than continuous variables. The classical quantities of position and momentum are promoted to operators as we go into the quantum regime. The standard position operator $\hat{r}$ and the momentum operator $\hat{p}$ in one dimension are

$$r \equiv \hat{r} \equiv \hat{q}, \quad p \equiv \hat{p} \equiv -i\hbar \frac{d}{dq}. \quad (7.23)$$

To incorporate the minimal length effect, the usual position and momentum variables get modified showing a non-commutative geometry. Among the various GUP models that incorporate the minimal length effects predicted by different quantum gravity theories, we adopt in this section a generalized modification through a momentum-dependent function $\mathcal{G}(\beta p)$. The specific form of this function is model-dependent and can represent different versions of GUPs. The different forms of this function are

$$\mathcal{G}(\beta p) = \begin{cases} 1 + \beta p^2, \\ \exp(\beta p^2), \\ \frac{1}{1 - \beta p^2}, \\ \delta_{ij} - \alpha \left(p \delta_{ij} + \frac{p_i p_j}{p} \right) + \alpha^2 (p^2 \delta_{ij} + 3p_i p_j), \end{cases} \quad (7.24)$$

In this section, we study how the effects of the minimal measurable length, where the modified commutator relation between position and momentum alters the standard quantum mechanics of angular momentum and its commutator relations. The modified commutator relation incorporate the minimal measurable length is given by

$$[q_i, p_i] = i\hbar \mathcal{G}(\beta p). \quad (7.25)$$

In the low-energy limit, where quantum gravitational effects can be neglected, we take $\beta \rightarrow 0$, which leads to $\mathcal{G}(\beta p) \approx 1$, thereby recovering standard quantum mechanics. Specifically, we focus on a more generalized form of GUP, incorporating the linear as well as the quadratic corrections. We choose the coordinates to remain invariant, and

only the momentum variable is modified [245–247] as follows:

$$q_i = q_{0,i} \quad p_i = p_{0,i}[1 - k_1 p_0 + (k_2 + k_1^2)p_0^2]. \quad (7.26)$$

The modified commutator relation capturing the linear and quadratic momentum dependence form of GUP is given as

$$[q_i, p_j] = i\hbar \left\{ \delta_{ij} - \gamma k_1 \left(p \delta_{ij} + \frac{p_i p_j}{p} \right) + \gamma^2 [k_2 p^2 \delta_{ij} + (2k_2 + k_1^2) p_i p_j] \right\} \quad (7.27)$$

where γ is related to the scales at which the quantum-gravitational effect became relevant, and is defined as the inverse of the Planck momentum $\gamma = \frac{\gamma_0}{M_{Pl}c}$ with $\gamma_0 \sim 1$, the dimensionless parameters k_1 and k_2 are introduced to explicitly distinguish the contributions arising from the linear and quadratic terms in the GUP framework. In the one dimensional case, setting $k_1 = 0$ and $k_2 = \frac{1}{3}$ reproduces the well-known Kempf-Mangano-Mann (KMM) form of the GUP, as presented in Eq. (5.18). The motivation for choosing this type of GUP, which includes multiple free parameters, lies in the flexibility it provides. By adjusting these parameters, we can explore and analyze various forms of the GUP. The GUP between position and momentum introduces a minimal measurable length. Similarly, if we apply GUP to angular momentum algebra, it leads to the concept of a minimal measurable angle. We expect GUP in the angular variables and their canonical angular momentum. To see this, we use the GUP-modified commutator relation of Eq.(7.25), which changes the commutator relation between the components of angular momentum as

$$\begin{aligned} [L_i, L_j] &= \epsilon_{imn} \epsilon_{jrs} [q_m p_n, q_r p_s] \\ &= \epsilon_{imn} \epsilon_{jrs} \{ q_m [p_n, q_r] p_s + q_r [q_m, p_s] p_n \} \\ &= i\hbar \epsilon_{imn} \epsilon_{jrs} \{ q_r p_n \mathcal{G}(\beta p) - q_m p_s \mathcal{G}(\beta p) \} \\ &= i\hbar (\epsilon_{mni} \epsilon_{mjr} q_r p_n - \epsilon_{nim} \epsilon_{nsj} q_m p_s) \mathcal{G}(\beta p) \\ &= i\hbar \epsilon_{ijk} L_k \mathcal{G}(\beta p), \end{aligned} \quad (7.28)$$

for the form of GUP incorporating the linear as well as quadratic dependence of Eq.(5.19) is written as

$$\begin{aligned}
[L_i, L_j] &= \epsilon_{imn}\epsilon_{jrs}[q_m p_n, q_r p_s] \\
&= \epsilon_{imn}\epsilon_{jrs}\{q_m[p_n, q_r]p_s + q_r[q_m, p_s]p_n\} \\
&= i\hbar\epsilon_{imn}\epsilon_{jrs}\left\{q_r p_n\left[\delta_{ms} - \gamma k_1\left(p\delta_{ms} + \frac{p_m p_s}{p}\right) + \gamma^2[k_2 p^2 \delta_{ij} + (2k_2 + k_1^2)p_m p_s]\right] \right. \\
&\quad \left. - q_m p_s\left[\delta_{nr} - \gamma k_1\left(p\delta_{nr} + \frac{p_n p_r}{p}\right) + \gamma^2[k_2 p^2 \delta_{nr} + (2k_2 + k_1^2)p_n p_r]\right]\right\} \\
&= i\hbar(\epsilon_{mni}\epsilon_{mjr}q_r p_n - \epsilon_{nim}\epsilon_{nsj}q_m p_s)(1 - k_1\gamma p + k_2\gamma^2 p^2) \\
&= i\hbar\epsilon_{ijk}L_k(1 - k_1\gamma p + k_2\gamma^2 p^2),
\end{aligned} \tag{7.29}$$

this results in the usual commutator bracket being modified with additional translational momentum terms. We also observe that angular momentum is now not only the generator of rotation but also of translation. As the GUP parameters $\gamma = 0$, we recover the original commutator algebra. The effects of GUP directly influence the angular momentum term by adding extra terms. We can expand the modified angular momentum in terms of the generators of rotation and translation through various form of GUPs, respectively, as

$$\left\{L_k = L_{0,k}\mathcal{G}(\beta p)\right\} \Rightarrow \begin{cases} \mathcal{G}(\beta p) = 1 + \beta p^2, \\ \mathcal{G}(\beta p) = \exp(\beta p^2), \\ \mathcal{G}(\beta p) = \frac{1}{1 - \beta p^2}, \\ \mathcal{G}(\beta p) = \delta_{ij} - \alpha\left(p\delta_{ij} + \frac{p_i p_j}{p}\right) + \alpha^2[p^2 \delta_{ij} + 3p_i p_j]. \end{cases}$$

Additionally, one can verify that the Jacobi identity, an important property of the Lie algebra of angular momentum operators, using Eq.(7.28) as

$$[L_i, [L_j, L_k]] + [L_j, [L_k, L_i]] + [L_k, [L_i, L_j]] = i\hbar\{\epsilon_{ijk}[L_i, L_i] + \epsilon_{ijk}[L_j, L_j] + \epsilon_{ijk}[L_k, L_k]\}\mathcal{G}(\beta p) = 0, \tag{7.30}$$

it means in the presence of GUP the Jacobi relation remains same. Furthermore, the

eigenstates L^2 and L_z can be simultaneously defined when we consider the GUP modifications. This can be expressed as

$$[L^2, L_j] = [L_i^2, L_j] + [L_j^2, L_j] + [L_k^2, L_j] = 0. \quad (7.31)$$

If we consider the commutator relation between the components of angular momentum and linear momentum, we obtain

$$\begin{aligned} [L_i, p_m] &= \epsilon_{ijk}[q_j p_k, p_m] = \epsilon_{ijk}[q_j, p_m]p_k \\ &= i\hbar\epsilon_{ijk}p_k\mathcal{G}(\beta p) \\ &= i\hbar\epsilon_{imk}p_k\mathcal{G}(\beta p), \end{aligned} \quad (7.32)$$

for the case of linear + quadratic form of GUP, we get

$$\begin{aligned} [L_i, p_m] &= \epsilon_{ijk}[q_j p_k, p_m] = \epsilon_{ijk}[q_j, p_m]p_k \\ &= i\hbar\epsilon_{ijk}p_k \left\{ \delta_{jm} - \gamma k_1 \left(p\delta_{jm} + \frac{p_j p_m}{p} \right) + \gamma^2 [k_2 p^2 \delta_{jm} + (2k_2 + k_1^2)p_j p_m] \right\} \\ &= i\hbar\epsilon_{imk}p_k(1 - \gamma k_1 p + k_2 \gamma^2 p^2), \end{aligned} \quad (7.33)$$

we observe that the relation is modified by an extra term involving $\vec{p}$. The original results remain unchanged if we neglect the quantum gravitational parameters. From the above results, we can easily show that

$$\begin{aligned} [L_i, p^2] &= [L_i, p_i^2 + p_j^2 + p_k^2] = p_i[L_j, p_i] + [L_j, p_i]p_i + p_j[L_k, p_j] + [L_k, p_j]p_j \\ &= i\hbar\mathcal{G}(\beta p)[\epsilon_{jim}p_i p_m + \epsilon_{jim}p_m p_i + \epsilon_{kjn}p_j p_n + \epsilon_{kjn}p_n p_j] \\ &= 0, \end{aligned} \quad (7.34)$$

by the property of $\epsilon_{ijk}p_j p_k + \epsilon_{ijk}p_k p_j = 0$, and utilizing Eq.(7.27). The eigenstate of the component L_z commutes with p and p^2 following the standard QM theory. Consequently, we can define the simultaneous eigenstates for L^2 , L_z , and p . Despite deformation of the commutator algebra induced by the GUP, the fundamental structure of quantum mechanics remains preserved. This indicates that while the algebraic relations are modified, the underlying symmetry structure, such as the form of the angular momentum algebra, remains formally intact when appropriate deformed variables are used. In this framework, it is not the representation that breaks but rather the generators and variables

that get modified, leading to a consistent deformation of the algebra without altering its core Lie structure.

7.2.1 Ladder operators in the presence of Linear + quadratic form of GUP

In this subsection, we calculate the effects of the GUP-deformed algebra on the ladder operators. Using the standard procedure, it will follow the same structure as in the standard theory, but the quantum numbers l and m are now not integers and include correction terms. As mentioned above, we can consider the simultaneous eigenstates of p , L_z , and L^2 as

$$L^2|pl'm\rangle = l'\hbar^2|pl'm\rangle, \quad L_z|pl'm\rangle = m\hbar|pl'm\rangle,$$

where l' denotes the eigenvalue of the deformed angular momentum. Using the Eq.(7.28), the component of angular momentum and ladder operator does not commute and gives

$$[L_z, L_{\pm}] = [L_z, L_x] \pm i[L_z, L_y] = \hbar(iL_y \pm L_x)(1 - \gamma k_1 p + k_2 \gamma^2 p^2) = \pm \hbar L_{\pm}(1 - \zeta), \quad (7.35)$$

where ζ is the GUP corrected factor, $\zeta = k_1 \gamma p - k_2 \gamma^2 p^2$. The ladder operator commutes with the magnitude of angular momentum as

$$[L^2, L_{\pm}] = [L^2, L_x] \pm i[L^2, L_y] = 0, \quad (7.36)$$

this structure is consistent with the standard QM . With the help of GUP-modified commutator relation Eq.(7.28), we also obtain

$$[L_+, L_-] = [L_x + iL_y, L_x - iL_y] = -2i[L_x, L_y] = 2\hbar L_z(1 - \zeta). \quad (7.37)$$

It satisfies,

$$L_{\pm}L_{\mp} = L^2 - L_z^2 \pm \hbar L_z(1 - \zeta). \quad (7.38)$$

Now, using Eq.(7.35), Eq.(7.39) became:

$$L_z L_{\pm} |pl'm\rangle = L_{\pm} [L_z \pm (1 - \zeta)] |pl'm\rangle = \hbar [m \pm (1 - \zeta)] L_{\pm} |pl'm\rangle \quad (7.39)$$

and using Eq.(7.36), we get

$$L^2 L_{\pm} |pl'm\rangle = L_{\pm} L^2 |pl'm\rangle = l' \hbar^2 L_{\pm} |pl'm\rangle, \quad (7.40)$$

and the norm of the state $L_{\pm} |pl'm\rangle$ is obtained as

$$||L_{\pm} |pl'm\rangle||^2 = \langle pl'm | L_{\mp} L_{\pm} |pl'm\rangle = \hbar^2 [l' - m\{m \pm (1 - \gamma k_1 p + k_2 \gamma^2 p^2)\}] \geq 0. \quad (7.41)$$

If we consider L^2 and L_z only, we obtain

$$||L_{\pm} |l'm\rangle||^2 = \langle l'm | L_{\mp} L_{\pm} |l'm\rangle = \hbar^2 [l' - m\{m \pm (1 - \langle \zeta \rangle)\}] \geq 0, \quad (7.42)$$

the GUP modification appears as $\langle \zeta \rangle$, which is the expectation value of the corrections. The term $1 - \langle \zeta \rangle$ is a positive quantity. We also note from the eigenvalues of L^2 and L_z in Eq. (7.38) that $\langle L^2 \rangle \geq \langle L_z^2 \rangle$, which requires $l' \geq m^2$. This implies that there is an upper and lower bound on m , denoted as m_+ and m_- , respectively. Additionally, from Eq. (7.42), we obtain

$$L_{\pm} |pl'm_{\pm}\rangle = 0 \implies l' = m_{\pm}[m_{\pm} \pm (1 - \zeta)] \quad (7.43)$$

we can get $m_{\pm}$ from any starting value of m by multiplying u and v to L_+ and L_- respectively where $u, v \in \mathbb{N}$ as

$$m_+ = m + u(1 - \zeta), \quad m_- = m - v(1 - \zeta) \quad (7.44)$$

and on combining the above two relations, we achieve,

$$m_+ = m_- + (u + v)(1 - \zeta) = m_- + n(1 - \zeta) \quad (7.45)$$

where $n \in \mathbb{N}$. On taking the above equation and Eq.(7.43), one can obtain

$$m_+ = \frac{n}{2}(1 - \zeta) = l(1 - \zeta), \quad m_- = -\frac{n}{2}(1 - \zeta) = -l(1 - \zeta), \quad (7.46)$$

and also we get

$$l' = l(l + 1)(1 - \zeta)^2. \quad (7.47)$$

We noticed that usually, the magnetic quantum number m might not be an integer because the difference between two consecutive eigenvalues of L_z , m_1 and m_2 the difference is $(1 - \zeta)$. Thus, we redefined it as

$$m \rightarrow m(1 - \zeta) \quad (7.48)$$

Now, the new GUP-modified magnetic quantum number m has integer upper and lower bounds as

$$-l \leq m \leq l. \quad (7.49)$$

Next, the eigenvalues of L^2 and L_z for any given state are

$$L^2|plm\rangle = \hbar^2 l(l+1)(1-\zeta)^2|plm\rangle, \quad L_z|plm\rangle = \hbar m(1-\zeta)|plm\rangle. \quad (7.50)$$

To show that the eigenvalues of L^2 and L_z are compatible with the uncertainty relation in Eq. (7.28), we first note that the expectation values of L_x and L_y in an eigenstate of L^2 and L_z are zero:

$$\langle L_x \rangle = \langle L_y \rangle = 0. \quad (7.51)$$

The variances of these components are given by:

$$(\Delta L_x)^2 = \langle L_x^2 \rangle, \quad (\Delta L_y)^2 = \langle L_y^2 \rangle. \quad (7.52)$$

Now, for the right-hand side of Eq. (7.50), the uncertainty relation between L_x and L_y is:

$$\Delta L_x \Delta L_y \geq \frac{|\langle [L_x, L_y] \rangle|}{2} = \frac{\hbar}{2} |\langle L_z \rangle (1 - \langle \zeta \rangle)| = \frac{\hbar^2}{2} |m| (1 - \langle \zeta \rangle)^2, \quad (7.53)$$

where we used the commutator $[L_x, L_y] = i\hbar L_z$ and the GUP correction factor $(1 - \langle \zeta \rangle)$. By the equivalence between L_x and L_y in an eigenstate of L_z , we insert Eq. (7.53) into the left-hand side of Eq. (7.50), yielding:

$$\langle L^2 \rangle = \hbar^2 l(l+1)(1 - \langle \zeta \rangle)^2 = \langle L_x^2 \rangle + \langle L_y^2 \rangle + \langle L_z^2 \rangle \geq \hbar^2 (m^2 + |m|)(1 - \langle \zeta \rangle)^2, \quad (7.54)$$

where the equality holds when $m = \pm l$. We observe that $\langle L_z^2 \rangle$ is bounded by $\langle L^2 \rangle$ due to the uncertainty in the other components, similar to standard quantum mechanics. Now, we will apply the results of the modified angular momentum and ladder operators to a physical system and examine how GUP modifications affect those systems [248–252].

7.3 Application: Modified Hydrogen atom

From the sections above, we now know that by introducing a minimal angular resolution, the angular momentum algebra has changed significantly. To observe these effects, we can apply these corrections to different quantum systems. As we also know, angular momentum algebra plays an important role in understanding atomic systems. In this section, we study the modified energy levels and spectrum of hydrogen atoms based on the GUP-modified angular momentum algebra. We incorporate the effects of GUP directly into the Hamiltonian of the system.

The usual Hamiltonian for hydrogen atom for a central force with the potential energy $U(r) = \frac{e^2}{r}$ is given as

$$\mathcal{H}\psi(\vec{r}) = \left[\frac{p_r^2}{2m} + \frac{L^2}{2mr^2} - \frac{e^2}{r} \right] \psi(\vec{r}), \quad (7.55)$$

we have the radial momentum term p_r and the square of angular momentum which consists of both radial and angular parts. However, we include the GUP effects in terms of expected values *i.e.* $\langle \zeta \rangle = k_1 \gamma \langle p \rangle - k_2 \gamma \langle p^2 \rangle$ as

$$L^2 |lm\rangle = \hbar^2 l(l+1)(1 - \langle \zeta \rangle)^2 |lm\rangle, \quad L_z |lm\rangle = \hbar m(1 - \langle \zeta \rangle) |lm\rangle, \quad \vec{p} |lm\rangle = \vec{p}_0(1 - \langle \zeta \rangle) |lm\rangle. \quad (7.56)$$

In this context, $\langle \zeta \rangle$ represents the average value of the GUP corrections. It is crucial to assume expectation values for observable physical quantities when considering Planck-scale corrections. From Eq. (7.56), we observe that $\vec{L}$ is proportional to $\vec{L}_0$, and both L_i and L^2 commute with the Hamiltonian $\mathcal{H}$. Using this result, the GUP-modified radial part of the Schrödinger equation becomes

$$\left[\frac{p_{0,r}^2}{2m} (1 - \langle \zeta \rangle)^2 + \frac{\hbar^2 l(l+1)}{2mr^2} (1 - \langle \zeta \rangle)^2 - \frac{e^2}{r} \right] y_l(r) = E y_l(r). \quad (7.57)$$

where E is the energy eigenvalue. The above Eq. (7.57) consists of two parts one is the unperturbed part without the GUP modification, expressed in terms of the unmodified momentum and angular momentum, and other is the perturbative part that includes the corrections. It can be written as:

$$\left[\frac{p_{0,r}^2}{2m} + \frac{\hbar^2 l(l+1)}{2mr^2} - \frac{e^2}{r} \right] y_l(r) + \left[\frac{p_{0,r}^2}{2m} + \frac{\hbar^2 l(l+1)}{2mr^2} \right] \{ \langle \zeta \rangle^2 - 2\langle \zeta \rangle \} y_l(r) = E y_l(r). \quad (7.58)$$

Here, the first term represents the low-energy part of the Hamiltonian without GUP corrections. Now on choosing the variables like,

$$\chi = \sqrt{\frac{-2mE}{\hbar^2}}, \quad a = \frac{\hbar^2}{me^2}, \quad \nu = \frac{1}{a\chi}, \quad z = \frac{2\chi r}{1 - \langle \zeta \rangle}, \quad y_l(r) = z^{(l+1)} e^{-\frac{z}{2}} v(z) \quad (7.59)$$

and substituting these variables in Eq.(7.57) gives

$$\left[z \frac{d^2}{dz^2} + (2l + 2 - z) \frac{d}{dz} + \left(\frac{\nu}{1 - \langle \zeta \rangle} - l - 1 \right) \right] = 0. \quad (7.60)$$

On taking $\gamma = 0$ the above equation changes to the well-known Laguerre equation. With the given condition as

$$n' = \frac{\nu}{1 - \langle \zeta \rangle} - l - 1 \in \mathbb{N}. \quad (7.61)$$

On solving the Eq.(7.60) we obtain the associated Laguerre polynomial

$$L_{n'}^{(2l+1)}(z) = \sum_{i=0}^{n'} \frac{(-1)^i (n' + 2l + 1)! z^i}{(n' - i)! (2l + 1 + i)! i!}. \quad (7.62)$$

Then, the solution for the Schrodinger equation (7.57) for the radial part is

$$y_l(r) = z^{(l+1)} e^{-\frac{z}{2}} \sum_{i=0}^{n'} \frac{(-1)^i (n' + 2l + 1)! z^i}{(n' - i)! (2l + 1 + i)! i!}, \quad (7.63)$$

and the modified principal quantum number is

$$n = \nu = \frac{e^2}{\hbar} \sqrt{\frac{m}{-2E}} = (n' + l + 1)(1 - \langle \zeta \rangle) = n_0(1 - \langle \zeta \rangle), \quad (7.64)$$

where n_0 is the principal quantum number. The GUP-deformed expression of the energy level of the hydrogen atom is

$$\begin{aligned} E_n &= -\frac{e^4}{2\hbar^2} \frac{m}{[(n' + l + 1)(1 - \langle \zeta \rangle)]^2} \\ &= -\frac{e^4}{2\hbar^2 c^2} \frac{mc^2}{n^2} \\ &\approx E_n^{(0)} [1 + 2k_1 \gamma \langle p_0 \rangle + \gamma^2 (3k_1^2 \langle p_0 \rangle^2 - 2k_2 \langle p_0^2 \rangle)], \end{aligned} \quad (7.65)$$

$E_n^{(0)}$ represents the energy for the corresponding energy level in the standard theory.

With GUP corrections, the energy for the hydrogen atom now includes additional terms involving the expectation values of p_0 and p_0^2 in a $|n_0 l m\rangle$ eigenstate. If we consider no

GUP effects by setting $\gamma = 0$, we recover the original energy level expression. In standard QM, the energy emitted or absorbed during a transition between two energy levels is given by the energy difference between the two states. Thus, it is straightforward to determine the frequency and wavelength of the emitted or absorbed lines through the wavelength and difference in energy level relation. It is also well known that the energy of the hydrogen atom is inversely proportional to the square of the principal quantum number. From this, we can also calculate the wavelength of the emitted photon when the atom transitions from energy level E_i to E_f as:

$$\frac{1}{\lambda} = \frac{|E_i - E_f|}{hc} = R_\infty \left| \frac{1}{n_{0,f}^2(1 - \langle \zeta_f \rangle)^2} - \frac{1}{n_{0,i}^2(1 - \langle \zeta_i \rangle)^2} \right|, \quad (7.66)$$

where R_∞ is the Rydberg constant. Equation (7.66) shows that the GUP-modified spectrum not only depends on the principal quantum number but also includes corrections involving the expectation values of the electron's momentum, as well as its angular momentum quantum numbers.

7.4 Conclusion

Quantum mechanics, as is well known, is based on the Heisenberg uncertainty relation. Many quantum gravity theories predict the need for modifications that incorporate the concept of a minimal length. With the development of various quantum gravity theories, identifying a distinct signature of quantum gravity remains a significant challenge, making experimental testing crucial for measuring Planck scale effects in low-energy quantum systems.

In this chapter, we consider a momentum-dependent functional term, $\mathcal{G}(\beta p)$, which captures both minimal length corrections and higher-order modifications to the uncertainty principle. The function $\mathcal{G}(\beta p)$ is model-dependent and thus accommodates various formulations of the Generalized Uncertainty Principle (GUP). We focus specifically on the deformed commutator bracket containing linear and quadratic momentum terms and explore its implications for the angular momentum algebra. Our analysis reveals that the GUP-modified position-momentum commutation relation leads to corrections in the angular momentum algebra, introducing an additional term that reflects the GUP-induced

deformation. A comparison between the standard angular momentum algebra and its GUP-deformed counterpart is presented in Table 7.1.

TABLE 7.1: Comparison of standard Angular Momentum Algebra and GUP-Modified Algebra

Aspect	Standard Angular Momentum	GUP-Modified Algebra
Commutator Relations	$[L_i, L_j] = i\hbar\epsilon_{ijk}L_k$	$[L_i, L_j] = i\hbar\epsilon_{ijk}L_k(1 - k_1\gamma p + k_2\gamma^2 p^2)$
Jacobi Identity	$[L_i, [L_j, L_k]] + [L_j, [L_k, L_i]] + [L_k, [L_i, L_j]] = 0$	same as standard
Commutators with L^2	$[L^2, L_j] = 0$	$[L^2, L_j] = 0$
Commutators with Momentum	$[L_i, p_j] = i\hbar\epsilon_{ijk}p_k$	$[L_i, p_j] = i\hbar\epsilon_{ijk}p_k(1 - \gamma k_1 p + k_2\gamma^2 p^2)$
Commutators with p^2	$[L_i, p^2] = 0$	$[L_i, p^2] = 0$
Eigenstates of L^2 and L_z	Can be simultaneously defined	Can still be defined simultaneously
Commutator with L_z	$[L_z, L_\pm] = \pm\hbar L_\pm$	$[L_z, L_\pm] = \pm\hbar L_\pm(1 - \zeta)$
Norm of the State	$\ L_\pm l, m\rangle\ ^2 \geq 0$	$\ L_\pm p, l', m\rangle\ ^2 \geq 0$ with GUP corrections
Quantum Numbers	Integer values for l and m	Integer values but with bounds on m due to GUP

In Sec. 7.3, we carry out the application of these modifications and examine how they affect the energy levels and energy spectrum of the hydrogen atom. We observe that the principal quantum number changes due to the modification, with additional terms representing the average values of p and p^2 , which affect the energy levels of the hydrogen atom. We also note that the energy spectrum now includes corrections that account for the expected values of the electron's momentum and angular momentum quantum number.

Since higher-order GUPs will also follow the same algebra with additional higher-order terms, we expect the algebra to remain consistent for these higher-order GUPs also. With these modifications to the angular momentum algebra, new avenues for phenomenological testing and theories are opened, which can potentially be tested in the future in contexts such as the Zeeman effect, Stark effect, simple harmonic oscillator (SHO), and others. This modification to the angular momentum algebra provides an opportunity to tighten the upper bound of the GUP parameter as studied in [253] by considering systems with angular momentum, like the Hydrogen atom in our case. We leave this for our future endeavours.

Chapter 8

THE ATOMIC SPECTRA

A key insight common to different QG theories approaches is the emergence of a fundamental minimal length scale, typically on the order of the Planck length $l_p \approx 10^{-35}$ m, beyond which the conventional notion of spacetime continuity breaks down. The presence of a minimal measurable length challenges one of the cornerstones of standard quantum mechanics that is the Heisenberg Uncertainty Principle (HUP) which allows, in principle, arbitrarily precise measurements of position at the cost of increased momentum uncertainty. The introduction of a lower bound on the measurable length implies that the HUP must be modified to accommodate this quantum gravitational limit, leading to the formulation of the Generalized Uncertainty Principle (GUP). The GUP extends the standard uncertainty relation by incorporating a momentum-dependent correction term, expressed as

$$\Delta X \Delta P \geq \frac{\hbar}{2} \left[1 + \beta_1 \left(\frac{l_p}{\hbar} \right)^2 (\Delta P)^2 \right], \quad (8.1)$$

where β_1 is a positive dimensionless parameter, and $(\Delta X)_{\min} = \sqrt{\beta_1} l_p$ represents the minimal measurable length [131, 254, 255]. This modification finds theoretical support from studies of string scattering amplitudes, black hole thermodynamics, and spacetime discreteness [58, 97–99].

The idea of a minimal length is not only significant for quantum gravity phenomenology but also has deep implications in cosmology. During the early universe, where energy scales approach the Planck regime, quantum gravitational effects may have influenced the inflationary dynamics driven by scalar fields such as the inflaton or quintessence field.

These scalar fields provide an elegant mechanism to explain both the rapid exponential expansion in the inflationary epoch and the late-time cosmic acceleration. The introduction of GUP into such cosmological contexts modifies the phase space structure, potentially affecting the inflaton potential, field dynamics, and perturbation spectra [256–260]. Thus, the study of minimal length physics serves as a unifying framework connecting the microscopic quantum world to macroscopic cosmological evolution.

Despite its profound theoretical relevance, direct experimental verification of GUP remains challenging due to the Planck-scale suppression of its effects. Nonetheless, high-precision experiments in atomic, molecular, and optical (AMO) physics offer a promising route to probe such corrections indirectly. Atomic systems, governed by well-established quantum mechanics, provide ideal laboratories to test small deviations induced by generalized uncertainty relations. In particular, the fine and hyperfine structure of atomic spectra, and the shifts induced by external electromagnetic fields, are sensitive to minute perturbations in the canonical commutation relations. Hence, phenomena such as the Zeeman effect (interaction of atomic levels with magnetic fields) and the Stark effect (interaction with electric fields) serve as natural candidates to investigate possible imprints of quantum gravity through RGUP-induced corrections.

In this framework, the Relativistic Generalized Uncertainty Principle (RGUP) extends the GUP to a Lorentz-invariant form, enabling its consistent application to relativistic systems and quantum field theory. The RGUP introduces modified commutation relations between position and momentum operators, incorporating quadratic momentum corrections that recover the standard Heisenberg algebra in the low-energy limit. Such a formulation not only preserves the covariance of physical laws but also allows the exploration of minimal length effects in relativistic quantum systems. Within this setting, atomic transitions and spectral line shifts become sensitive probes of the deformed algebraic structure underlying RGUP.

This chapter aims to examine the influence of the Relativistic Generalized Uncertainty Principle on atomic spectra, focusing on the Zeeman and Stark effects as representative phenomena. By analyzing how the presence of a minimal length modifies the canonical operators and the resulting energy level shifts, we explore whether quantum gravitational effects could, in principle, be detected through precision atomic spectroscopy. Furthermore, understanding RGUP-induced corrections in atomic systems establishes

an essential link between quantum gravity phenomenology, high-energy cosmology, and low-energy experimental physics. Through this approach, atomic spectra emerge not merely as laboratory observables but as potential windows into the fundamental structure of spacetime at the Planck scale.

8.1 Theoretical Framework for RGUP and Minimal Length Corrections

The non-relativistic commutator relations incorporating the minimal observable length for D -dimensional space were first proposed by Kempf, Mangano, and Mann in 1995 [261–263] given by

$$[x^i, p^j] = i\hbar [(1 + \alpha_1 p^2)\delta^{ij} + \alpha_2 p^i p^j], \quad (8.2)$$

$$[x^i, x^j] = i\hbar \frac{(2\alpha_1 - \alpha_2) + (2\alpha_1 + \alpha_2)\alpha_1 p^2}{1 + \alpha_1 p^2} (p^i x^j - p^j x^i), \quad (8.3)$$

$$[p^i, p^j] = 0, \quad (8.4)$$

where, α_1 and α_2 are positive deformation parameters that govern the algebraic structure within the deformed space, where the indices $i, j = 0, 1, 2, \dots, D$. This quadratic momentum deformation in the commutator relation is motivated through String theory [264] and Black hole physics [137, 265]. From the Schrodinger-Robertson uncertainty relation [266–268]

$$(\Delta x)^2 (\Delta p)^2 \geq \frac{1}{4} |\langle [x, p] \rangle|^2, \quad (8.5)$$

one can ascertain the minimum uncertainty in position, expressed as

$$(\Delta x^i)_{\text{minimal}} = \hbar \sqrt{(D\alpha_1 + \alpha_2)}. \quad (8.6)$$

In [269], Stetsko and Tkachuk derived approximate expressions for the deformed position and momentum operators, neglecting higher-order corrections or, equivalently, assuming the deformation parameter to be small. To first-order accuracy in the deformation

parameters, they obtained:

$$x^i = x_0^i + \frac{2\alpha_1 - \alpha_2}{4} (p_0^2 x_0^i + x_0^i p_0^2), \quad (8.7)$$

$$p^i = p_0^i \left(1 + \frac{\alpha_2}{2} p_0^2\right), \quad (8.8)$$

where the subscript “0” represents the standard position and momentum operator. For the special case $\alpha_2 = 2\alpha_1$, the position operators commute, leading to the following modified commutation relations and the modified operators as

$$\left\{ \begin{array}{l} [x^i, p^j] = i\hbar [(1 + 2\alpha_1 p^2)\delta^{ij} + \alpha_1 p^i p^j], \\ [x^i, x^j] = 0, \\ [p^i, p^j] = 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} x^i = x_0^i, \\ p^i = p_0^i (1 + \alpha_1 p_0^2). \end{array} \right. \quad (8.9)$$

In Minkowskian spacetime with metric signature $(-+++)$, the quadratic GUP in Eq.(8.2) takes the form as

$$[x^\mu, p^\nu] = i\hbar(1 + (\epsilon - \alpha)\gamma^2 p^\rho p_\rho)\eta^{\mu\nu} + i\hbar(\alpha' + 2\epsilon)\gamma^2 p^\mu p^\nu, \quad (8.10)$$

where $\mu, \nu \in \{0, 1, 2, 3\}$, and γ is a dimensionless parameter with the inverse dimension of momentum, while α , ϵ , and α' are also dimensionless parameters. The parameter γ is defined as $\gamma = \frac{1}{M_{\text{Pl}}c}$, where M_{Pl} denotes the Planck mass. These parameters become significant in regimes where quantum-gravitational effects play a crucial role, particularly near the Planck scale. At this scale, the dimensionless parameters are generally assumed to be of order unity. For a comprehensive discussion on these parameters, see [261, 270].

We observe that Eq. (8.10) simplifies to Eq. (8.2) in the non-relativistic limit ($c \rightarrow \infty$) and reduces to the standard Heisenberg uncertainty relation as $\gamma \rightarrow 0$. Moreover, the variables x^μ and p^ν correspond to the physical position and momentum, which are not canonically conjugate. To address this, we introduce new four-vectors, x_0^μ and p_0^ν , satisfying the relations

$$p_0^\nu = -i\hbar \frac{\partial}{\partial x_{0\nu}}, \quad [x_0^\mu, p_0^\nu] = i\hbar \eta^{\mu\nu}. \quad (8.11)$$

Following [271], the deformed position and momentum variables, expanded up to second order in γ , are given by

$$x^\mu = x_0^\mu - \alpha\gamma^2 p_0^\rho p_{0\rho} x_0^\mu + \alpha'\gamma^2 p_0^\mu p_0^\rho x_{0\rho} + \xi\hbar\gamma^2 p_0^\mu, \quad (8.12)$$

$$p^\mu = p_0^\mu(1 + \epsilon\gamma^2 p_0^\rho p_{0\rho}), \quad (8.13)$$

where ξ is another dimensionless parameter. The position operator satisfies the commutation relation

$$[x^\mu, x^\nu] = i\hbar\gamma^2 \frac{-2\alpha + \alpha'}{1 + (\epsilon - \alpha)\gamma^2 p^\rho p_\rho} (x^\mu p^\nu - x^\nu p^\mu). \quad (8.14)$$

The last two terms in Eq. (8.12) contain a preferred direction due to the p_0^μ term, thus introducing anisotropy and violating the principle of relativity. To maintain relativistic invariance, we set $\alpha' = 0$ and $\xi = 0$. This formulation of relativistic GUP recovers the KMM algebra in the limit $c \rightarrow \infty$. For the application to the Stark effect, we employ the approximate representation introduced by Stetsko and Tkachuk (or Brau-type) [269], ensuring consistency with the KMM algebra in the first order by setting $\alpha = 0$. Under the linear approximation, the modified commutation relations take the form

$$[x^\mu, p^\nu] = i\hbar(1 + \epsilon\gamma^2 p^\rho p_\rho)\eta^{\mu\nu} + 2i\hbar\epsilon\gamma^2 p^\mu p^\nu, \quad (8.15)$$

$$[x^\mu, x^\nu] = 0. \quad (8.16)$$

These expressions generally hold true when the GUP-corrected parameters are taken to be very small. In this regime, we can interpret x as a quasi-position operator. However, if the deformation parameters are not negligible, the neglected higher-order contributions may become significant, potentially leading to inaccuracies or deviations from the true dynamics of the system. Therefore, in situations where high precision is required or where the deformation parameters are not vanishingly small, it becomes necessary to include second-order or even higher-order terms to ensure a more accurate and reliable description. These relations hold for the modified position and momentum operators

$$x^\mu = x_0^\mu, \quad (8.17)$$

$$p^\mu = p_0^\mu(1 + \epsilon\gamma^2 p_0^\rho p_{0\rho}). \quad (8.18)$$

To derive the corresponding dispersion relation, we consider the squared physical momentum, which satisfies

$$p^\rho p_\rho = -(mc)^2, \quad (8.19)$$

now expressing this in terms of p_0^μ , we obtain

$$p_0^\rho p_{0\rho} (1 + 2\epsilon\gamma^2 p_0^\sigma p_{0\sigma}) = -(mc)^2, \quad (8.20)$$

here, m denotes the particle's mass. Due to the presence of mixed derivatives, solving Eq. (8.20) analytically becomes challenging, even in spherically symmetric scenarios. Instead, we solve for $p_0^\rho p_{0\rho}$, selecting solutions that correctly reduce to $(mc)^2$ as $\gamma \rightarrow 0$. Applying the quadratic formula, we obtain the second-order dispersion relation

$$\begin{aligned} p_0^\rho p_{0\rho} &= -\frac{1}{4\epsilon\gamma^2} + \sqrt{\frac{1}{(4\epsilon\gamma^2)^2} - \frac{(mc)^2}{2\epsilon\gamma^2}}, \\ &\simeq -(mc)^2 - 2\epsilon\gamma^2 (mc)^4 - \mathcal{O}(\gamma^4). \end{aligned} \quad (8.21)$$

We observe that in the non-relativistic limit ($c \rightarrow \infty$), the four-momentum scalar product satisfies

$$p_0^\rho p_{0\rho} = -\left(\frac{E}{c}\right)^2 + p_0^i p_{i0}, \quad (8.22)$$

In the non-relativistic quantum-mechanical representation, one has

$$p_0^\rho p_{0\rho} = -\hbar^2 \nabla^2. \quad (8.23)$$

For simplicity and reduction of many parameters from this point onward, we define $\epsilon\gamma^2 = \beta$ and choose $\hbar = 1$. Throughout this chapter, the relativistic generalized uncertainty principle (RGUP) is treated perturbatively. Since the deformation parameter γ is expected to be of the order of the inverse Planck momentum, we assume $\gamma \ll 1$, so that only the leading corrections up to $\mathcal{O}(\gamma^2)$ are retained, while higher-order terms $\mathcal{O}(\gamma^4)$ and beyond are consistently neglected. This approximation is employed throughout the derivation of the RGUP corrections to the Zeeman and Stark effects.

8.2 Zeeman Effect in presence of RGUP

The Zeeman effect describes the splitting of atomic energy levels when an atom is subjected to the external magnetic field. Depending upon the nature of the magnetic field two cases can occur namely the strong-field (normal Zeeman effect) and the weak-field (anomalous Zeeman effect). In the first case, when the magnetic field is strong, the Russell-Saunders coupling breaks down, leading to the independent precession of the orbital and spin angular momenta. This decoupling characterizes the normal Zeeman effect, where the orbital and spin angular momenta precess individually under the influence of the external magnetic field. In the second case, a weak magnetic field is unable to break the Russell-Saunders coupling which results in preserving the total angular momentum the vector sum of the orbital and spin angular momenta. In this scenario, Larmor precession occurs, where both the spin and orbital angular momenta precess around the external magnetic field.

In this section, we study the effect of relativistic corrections due to GUP in the Zeeman effect, starting by analyzing the influence of the external magnetic field. We extend the procedure proposed by Khosropour in [272] to derive a more general form of the GUP correction with relativistic effects, such that it recovers the non-relativistic GUP and the standard expressions without GUP in specific limits. To start with, we take the relativistic Lagrangian field density with current density as

$$\mathcal{L} = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} - A_\mu \mathcal{J}^\mu \quad (8.24)$$

where $\mu, \nu = 0, 1, 2, 3$ and $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$ and $\mathbf{A}(x) = (A^0(x), A^1(x), A^2(x), A^3(x))$ are electromagnetic field tensor and vector potential respectively. Now using the deformed position and derivative operator in Eq.(8.18) and using the Eq.(8.21) as ..

$$x^\mu \rightarrow X^\mu \equiv x_0^\mu, \quad \partial^\mu \rightarrow D^\mu := \partial_0^\mu (1 - \beta(mc)^2 - \mathcal{O}(\gamma^4)), \quad (8.25)$$

where D^μ is the ordinary derivative for Minkoskian space-time. From this point onward, we will use **boldface** letters to denote four-vectors, which include both spatial and

temporal components. Using Eq.(8.25) the modified electromagnetic field tensor become

$$\begin{aligned}\mathcal{F}_{\mu\nu}(\mathbf{X}) &= D_\mu A_\nu(\mathbf{X}) - D_\nu A_\mu(\mathbf{X}), \\ \mathcal{F}_{\mu\nu}(\mathbf{x}) &= F_{\mu\nu}(\mathbf{x}) - \beta(mc)^2 F_{\mu\nu}(\mathbf{x}) - \mathcal{O}(\gamma^4).\end{aligned}\quad (8.26)$$

We define the magnetic induction vector and electric field induction $\mathbf{B}(\mathbf{x})$, $\mathbf{E}(\mathbf{x})$ as

$$F_{ij} = -\epsilon_{ijk}B^k, \quad F^{ij} = \epsilon^{ijk}B_k, \quad F_{i0} = E^i \quad (8.27)$$

since we are studying the effect of magnetic field in the atomic spectra we consider the electric field vector to be zero. The magnetic induction vector in three-dimensional space with $i, j = 1, 2, 3$ is

$$B^i = \{B_x, B_y, B_z\}. \quad (8.28)$$

Using Eq.(8.27), Eq.(8.28) and Eq.(8.26), one can obtain the GUP-modified magneto-static field as

$$\mathbf{b}_{RGUP}(\mathbf{x}) = (1 - \beta(mc)^2 - \mathcal{O}(\gamma^4))\mathbf{B}(\mathbf{x}), \quad (8.29)$$

The above expression includes the effect of relativistic GUP and as we choose $c \rightarrow \infty$ the effects became non-relativistic and the expression up to the order of γ^2 became

$$\mathbf{b}_{GUP}(\mathbf{x}) = (1 - \beta\hbar^2\nabla^2)\mathbf{B}(\mathbf{x}) \quad (8.30)$$

$$= (1 + \beta p^2)\mathbf{B}(\mathbf{x}) \quad (8.31)$$

and as quantum gravity parameter $\gamma \rightarrow 0$ we recover our original magnetic field. We also note that in the relativistic limit ($c \rightarrow \infty$) we recover the results presented in [272]. Now, we use the corrections obtained from RGUP in the magnetic field to obtain the RGUP-corrections in the Zeeman effect. We first choose the Hamiltonian H_0 for hydrogen-like atoms by taking the relativistic correction only in the kinetic energy term as

$$H_0 = \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} + U^{(0)}, \quad (8.32)$$

where $U^{(0)}$ are the internal interactions of the zeroth order in v/c . We derive the magnetostatic field $\mathbf{B}$ from the vector potential as

$$\mathbf{A} = \frac{1}{2}(\mathbf{B} \times \mathbf{X}). \quad (8.33)$$

To incorporate the effect of an external magnetic field on the charged particle, we use the minimal coupling prescription. In the presence of an electromagnetic field, the canonical momentum $\mathbf{p}$ is replaced by

$$\mathbf{p} \rightarrow \mathbf{p} - \frac{e\mathbf{A}}{c}. \quad (8.34)$$

Since the RGUP modifies the magnetostatic field in Eq.(8.29), it also affects the process of obtaining the Hamiltonian in the presence of relativistic GUP. To address this, we first determine the modified vector potential as

$$\mathbf{A}_{RGUP} = \frac{1}{2}(\mathbf{b}_{RGUP} \times \mathbf{x}) = \frac{1}{2}(1 - \beta(mc)^2)(\mathbf{B} \times \mathbf{x}), \quad (8.35)$$

where $\mathbf{B}$ represents the usual magnetostatic field without RGUP. On choosing $\mathbf{B}$ in positive z -direction, we have

$$\begin{aligned} \mathbf{A}_{RGUP} &= -\frac{1}{2}(1 - \beta(mc)^2)(By\hat{x} - Bx\hat{y}), \\ &= (1 - \beta(mc)^2)\mathbf{A}, \end{aligned} \quad (8.36)$$

where $\mathbf{A} = -\frac{1}{2}(By\hat{x} - Bx\hat{y})$. And, Eq.(8.34) modifies using Eq.(8.36) and Eq.(8.18) as

$$\mathbf{p} \rightarrow \mathbf{P} - \frac{e\mathbf{A}_{RGUP}}{c} = (1 - \beta(mc)^2) \left(\mathbf{p} - \frac{e\mathbf{A}}{c} \right). \quad (8.37)$$

Now, the overall Hamiltonian is written as

$$H_{RGUP} = \frac{1}{2m} \left\{ (1 - \beta(mc)^2) \left(\mathbf{p} - \frac{e\mathbf{A}}{c} \right) \right\}^2 - \frac{1}{8m^3c^2} \left\{ (1 - \beta(mc)^2) \left(\mathbf{p} - \frac{e\mathbf{A}}{c} \right) \right\}^4 + U^{(0)} \quad (8.38)$$

and, upto the order γ^2 , we write

$$\begin{aligned}
 H_{RGUP} = & \frac{(1 - 2\beta(mc)^2)}{2m} \left\{ \mathbf{p}^2 + \frac{e^2 \mathbf{A}^2}{c^2} - \frac{e}{c} (\mathbf{p} \cdot \mathbf{A} + \mathbf{A} \cdot \mathbf{p}) \right\} \\
 & - \frac{(1 + 4\beta(mc)^2)}{8m^3 c^2} \left\{ \mathbf{p}^4 + \frac{e^4 \mathbf{A}^4}{c^4} + \frac{2\mathbf{p}^2 e^2 \mathbf{A}^2}{c^2} + \frac{e^2}{c^2} (\mathbf{p} \cdot \mathbf{A} + \mathbf{A} \cdot \mathbf{p})^2 \right. \\
 & \left. - \left(\frac{2e\mathbf{p}^2}{c} + \frac{2e^3 \mathbf{A}^2}{c^3} \right) (\mathbf{p} \cdot \mathbf{A} + \mathbf{A} \cdot \mathbf{p}) \right\} + U^{(0)}. \quad (8.39)
 \end{aligned}$$

We know that one can replace $\mathbf{p} \cdot \mathbf{A}$ by $\mathbf{A} \cdot \mathbf{p}$ with the help of Coulomb gauge ($\nabla \cdot \mathbf{A} = 0$), such that

$$\mathbf{A} \cdot \mathbf{p} = \frac{B}{2} (-yp_x + xp_y) = \frac{1}{2} BL_z, \quad (8.40)$$

$$\mathbf{A}^2 = \frac{1}{4} B^2 (x^2 + y^2). \quad (8.41)$$

Since the relativistic GUP-modified Hamiltonian contains terms with $\mathbf{A}^2$, these can be neglected because of the quadratic term $B^2(x^2 + y^2)$ that corresponds to higher-order magnetic corrections. In a one-electron system like hydrogen, these quadratic terms are much smaller than the linear terms, as the magnitude of the magnetic interaction for a single electron is relatively weak, and the quadratic terms contribute a much smaller energy shift. The Hamiltonian consists of both an unperturbed and a perturbed part. The perturbation arises from two contributions: H_B , due to the external magnetic field, and H_{LS} , due to the spin-orbit ($\mathbf{L} \cdot \mathbf{S}$) coupling. In the first case, we consider only the contribution from a strong magnetic field, where the external magnetic field is strong enough to dominate over the internal spin-orbit coupling. As a result, the interaction between the spin magnetic moment and the external magnetic field becomes significant. The final effective Hamiltonian in the first case will take the form as

$$H_{RGUP} = \frac{p^2}{2m} - \frac{\beta(mc)^2 p^2}{m} - (1 - 2\beta(mc)^2) \frac{e}{2mc} BL_z - \frac{(1 + 4\beta(mc)^2)}{8m^3 c^2} p^4 + \frac{e}{4m^3 c^3} BL_z p^2 + U^{(0)}. \quad (8.42)$$

We also add the spin magnetic-moment interaction in the presence of a relativistic effect [273, 274] with RGUP as

$$-\mu \cdot \mathbf{b}_{RGUP} = -\frac{e}{mc} \left(1 + \frac{\alpha'}{2\pi} \right) \mathbf{s} \cdot (1 - \beta(mc)^2 - \mathcal{O}(\gamma^4)) \mathbf{B} \simeq -\frac{e}{mc} \left(1 + \frac{\alpha'}{2\pi} \right) (1 - \beta(mc)^2) B s_z, \quad (8.43)$$

where α' is the fine-structure constant and this correction accounts for the relativistic. This gives the total modified Hamiltonian as

$$\begin{aligned} H_0 &= \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} + U^{(0)} \\ H_B^{\text{RGUP}} &= -\frac{\beta(mc)^2 p^2}{m} - (1 - \beta(mc)^2) \frac{eB}{2mc} (L_z + 2s_z) + \frac{eB}{2mc} \beta(mc)^2 L_z \\ &\quad + \frac{eB}{4m^3c^3} L_z p^2 - \frac{\alpha'}{2\pi} (1 - \beta(mc)^2) \frac{e}{mc} B s_z - \frac{\beta(mc)^2}{2m^3c^2} p^4. \end{aligned} \quad (8.44)$$

The Relativistic GUP correction to the Hamiltonian acts as a small perturbation. To determine the effects of the RGUP corrections, we use the J^2 , J_z eigenkets as our basis kets. We will assume that the eigenvalues are not affected by the GUP corrections, allowing us to study the leading-order corrections. The energy shift in the first-order in the deformed space is

$$\begin{aligned} & -\frac{eB}{2mc} \langle J_z + S_z \rangle_{j=L\pm\frac{1}{2}, m'} - \frac{\alpha' eB}{2\pi mc} \langle S_z \rangle_{j=L\pm\frac{1}{2}, m'} + \frac{eB}{4m^3c^3} \langle p^2 (J_z - S_z) \rangle_{j=L\pm\frac{1}{2}, m'} \\ & + \beta(mc)^2 \left[\frac{eB}{2mc} \langle J_z + S_z \rangle_{j=L\pm\frac{1}{2}, m'} + \frac{eB}{2mc} \langle J_z - S_z \rangle_{j=L\pm\frac{1}{2}, m'} + \frac{eB\alpha'}{2\pi mc} \langle S_z \rangle_{j=L\pm\frac{1}{2}, m'} \right. \\ & \quad \left. - \frac{\langle p^2 \rangle}{m} + \frac{\langle p^4 \rangle}{2m^3c^2} \right]. \end{aligned} \quad (8.45)$$

in Eq.(8.45), we note that if we disregard relativistic effects, the second and third terms will vanish, and the factor $(mc)^2$ will change to $-p^2$, retaining the expression for the standard GUP correction as discussed in [272]. The expectation value of J_z is $m'\hbar$ where m' is the magnetic quantum number, and from the standard quantum mechanical results, we get

$$\langle S_z \rangle_{j=L\pm\frac{1}{2}, m'} = \pm \frac{m'\hbar}{2l+1}. \quad (8.46)$$

We also know that for $\langle p^2 \rangle$ and $\langle p^4 \rangle$, we use

$$\nabla^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{\mathbf{L}^2}{\hbar^2 r^2} \right). \quad (8.47)$$

Using the above expressions we obtain Lande's formulae for energy shift $\Delta\mathcal{E}_B$ in the presence of RGUP up to the order of $\mathcal{O}(\gamma^2)$ is obtained as

$$\begin{aligned}
(\Delta\mathcal{E}_B)_{\text{RGUP}} = & -\frac{eB\hbar}{2mc}m' \left[1 \pm \frac{1}{2l+1} \right] \mp \frac{\alpha'eB}{2\pi mc} \frac{m'\hbar}{2l+1} + \frac{eB}{4m^3c^3} \frac{m'\hbar^3}{r_0^2} l(l+1) \left(1 \mp \frac{1}{2l+1} \right) \\
& + \beta(mc)^2 \left[\frac{eB}{2mc} m'\hbar \left(1 \pm \frac{1}{2l+1} \right) + \frac{eB}{2mc} m'\hbar \left(1 \mp \frac{1}{2l+1} \right) \mp \frac{eB\alpha'}{2mc\pi} \frac{m'\hbar}{2l+1} \right. \\
& \left. - \frac{\hbar^2 l(l+1)}{mr_0^2} + \frac{\hbar^4}{2m^3c^2r_0^4} (l(l+1))^2 \right]. \tag{8.48}
\end{aligned}$$

where r_0 is the Bohr radius and energy shifts are expressed in SI units (Joules). The second line in Eq. (8.48) shows the correction to the energy shift due to the RGUP. As the effects of relativity and GUP vanish, we recover the original result for the energy shift, which is proportional to m' in [272] as

$$(\Delta\mathcal{E}_B)_{\text{GUP}} = -\frac{eB\hbar}{2mc}m' \left[1 \pm \frac{1}{2l+1} \right] + \frac{\beta}{m} \left[\frac{\hbar^4}{r_0^4} (l(l+1))^2 - \frac{eBm'}{c} \frac{\hbar^2}{r_0^2} l(l+1) \left(1 \pm \frac{1}{2l+1} \right) \right]. \tag{8.49}$$

With the presence of RGUP, the energy shift is no longer proportional to the magnetic quantum number m' due to the inclusion of the last terms $\langle p^2 \rangle$ and $\langle p^4 \rangle$ in Eq. (8.48). This Eq. (8.48) represents the more general form of the relativistic correction with GUP modification to the Lamb energy shift, extending the energy shift equation (8.49) proposed by Khosropour.

Now, if we consider the effect of $\mathbf{L} \cdot \mathbf{S}$ coupling, we first derive the RGUP-modified Hamiltonian associated with it. For a central (spin-independent) potential $U^{(0)}$, the electron's magnetic moment interacts with the effective magnetic field generated by the moving charge in the electric field. Additionally, accounting for the extra contribution to the energy from Thomas precession, in conjunction with relativity, reduces the factor of spin-orbit coupling by 1/2 (known as the Thomas factor) [275, 276], yielding a modified interaction.

$$H_{LS}^{\text{RGUP}} = \left(\frac{e\mathbf{S}}{mc} \right) \cdot \left[\frac{1}{2mc} \frac{1}{r} (\mathbf{P} \times \mathbf{x}) \frac{dU^{(0)}}{dr} \right], \tag{8.50}$$

Within the present RGUP framework, we assume that the intrinsic spin operator is not modified and continues to satisfy the standard SU(2) algebra. Therefore, the leading quantum-gravity corrections enter only through the deformation of the momentum

operator, while the spin degrees of freedom remain unchanged. Substituting the RGUP-modified momentum operator given in Eq.(8.18) into Eq.(8.50), we obtain

$$\begin{aligned} H_{LS}^{\text{RGUP}} &= \left(\frac{e\mathbf{S}}{mc} \right) \cdot \left[\frac{(1 - \beta(mc)^2)}{2mc} \frac{(\mathbf{p} \times \mathbf{x})}{r} \frac{dU^{(0)}}{dr} \right] \\ &= \frac{(1 - \beta(mc)^2)}{2m^2c^2} \frac{1}{r} \frac{dU^{(0)}}{dr} (\mathbf{L} \cdot \mathbf{S}). \end{aligned} \quad (8.51)$$

Since the external magnetic field is treated as a weak perturbation, the uncoupled basis

$$|l, s = \frac{1}{2}, m_l, m_s\rangle$$

provides a convenient basis for evaluating the leading-order expectation values of the spin-orbit Hamiltonian. The expectation value of $\mathbf{L} \cdot \mathbf{S}$ with respect to $|m_l, m_s\rangle$ is

$$\begin{aligned} \langle \mathbf{L} \cdot \mathbf{S} \rangle &= \langle L_z S_z + \frac{1}{2}(L_+ S_- + L_- S_+) \rangle_{m_l m_s} \\ &= \hbar^2 m_l m_s. \end{aligned} \quad (8.52)$$

The expectation values of the ladder operators vanish because they connect different magnetic sublevels that are orthogonal to the chosen basis state,

$$\langle L_{\pm} \rangle_{m_l} = 0, \quad \langle S_{\pm} \rangle_{m_s} = 0, \quad (8.53)$$

so the expectation value of the Hamiltonian approximate to the leading-order correction became

$$\langle H_{LS}^{\text{RGUP}} \rangle = \frac{(1 - \beta(mc)^2) \hbar^2 m_l m_s}{2m^2c^2} \left\langle \frac{1}{r} \frac{dU^{(0)}}{dr} \right\rangle. \quad (8.54)$$

For non-relativistic ($c \rightarrow \infty$) and without GUP ($\gamma \rightarrow 0$), Eq.(8.54) became the usual $\langle H_{LS} \rangle$ with value equal to $\frac{\hbar^2 m_l m_s}{2m^2c^2} \left\langle \frac{1}{r} \frac{dU^{(0)}}{dr} \right\rangle$. In this case, the perturbed Hamiltonian contains two terms one from the external magnetic field and one from the $\mathbf{L} \cdot \mathbf{S}$ coupling. We observe that the presence of RGUP does not alter the number of Zeeman lines, but it modifies the energy shift spectrum. Furthermore, when comparing the energy shift relations for GUP and RGUP, as shown in Eq. (8.49) and Eq. (8.48), respectively, we find that the modified terms in the GUP case are proportional to the orbital angular momentum quantum number l . However, in the case of RGUP, they depend on a combination of m' and l . This implies that for the s -orbital singlet state ($l = 0$), where m'

is also zero, there is no correction to the energy shift for either GUP or RGUP. For orbitals with $l \neq 0$, both GUP and RGUP introduce additional energy shifts. Specifically, the shifts depend on l , m' , and higher-order relativistic corrections, leading to distinct shifts for different levels. These corrections are particularly visible in higher angular momentum states, where they contribute significantly to Lande's energy shift.

8.3 RGUP-Modified Stark Effect

The Stark effect refers to the splitting of energy levels in a hydrogen-like atom when placed in an external electric field. This phenomenon arises due to the interaction between the applied field and the atomic dipole moment, leading to energy level perturbations that depend on the field strength and quantum numbers of the system. We consider a hydrogen atom with relativistic corrections applied only to the kinetic term, described by the Hamiltonian

$$\mathcal{H} = \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} + U, \quad (8.55)$$

when subjected to a uniform external electric field along the positive z -direction, provides the interaction potential as

$$V = -e|\vec{E}|z. \quad (8.56)$$

On considering the minimal length corrections in the presence of RGUP, the modification in the Hamiltonian arises from two sources, firstly, the correction in the interaction potential due to the modification of the electric field, and secondly the correction in the kinetic term due to the modification of the momentum.

We start with the Lagrangian density for an electrostatic field [277], which is given as

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\epsilon_0 E(x)^2 - \rho(x)\phi(x), \\ &= \frac{1}{2}\epsilon_0 (\partial^i \phi(x))(\partial^i \phi(x)) - \rho(x)\phi(x). \end{aligned} \quad (8.57)$$

where $\phi(x)$ is the electrostatic potential and $\rho(x)$ is the electrostatic charge density. We choose deformation only in the momentum term while the position remains fixed. The modified position and momentum operators from Eq.(8.18) in the form of derivatives

are expressed as

$$\begin{aligned} x^\mu &\rightarrow X^\mu \equiv x_0^\mu, \\ \partial^\mu &\rightarrow D^\mu \equiv \partial_0^\mu (1 - \beta(mc)^2 + \mathcal{O}(\gamma^4)). \end{aligned} \quad (8.58)$$

Now incorporating the effects of RGUP through the modified operators Eq.(8.58) into the Lagrangian density Eq.(8.57), we get

$$\mathcal{L}_{RGUP} = \frac{1}{2}\epsilon_0(1 - 2\beta(mc)^2)E(x)^2 - \rho(x)\phi(x), \quad (8.59)$$

on comparing Eq.(8.59) and Eq.(8.57), we obtain the modified electric field upto the $\mathcal{O}(\beta)$ which is defined as

$$E(x)_{RGUP} = (1 - \beta(mc)^2)E(x), \quad (8.60)$$

one can easily transition to the non-relativistic regime by taking the limit $c \rightarrow \infty$, thereby recovering the minimal length corrections as $-\beta\hbar^2\nabla^2\vec{E}(x)$. The Eq.(8.60) represents the modified electric field, which in turn modifies the Hamiltonian of the hydrogen atom.

On incorporating the RGUP corrections into the Hamiltonian, we noticed that we have to include both modified momentum term and the modified electric field in Eq.(8.55). Therefore, the effective Hamiltonian became

$$\mathcal{H}_{RGUP} = \{1 - 2\beta(mc)^2\} \left\{ \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} \right\} - e(1 - \beta(mc)^2)Ez + U, \quad (8.61)$$

this provide the modified Stark potential after neglecting the higher order terms $\mathcal{O}(\beta^2)$ as

$$V_{RGUP} = -2\beta(mc)^2 \left(\frac{p^2}{2m} - \frac{p^4}{8m^3c^2} \right) - eEz(1 - \beta(mc)^2). \quad (8.62)$$

We use the perturbation theory to obtain the shift in the energy. On assuming the eigenstate and the unperturbed Hamiltonian are known, we can divide our analysis in two parts one of the non-degenerate case, taking only ($n = 1$), and for degenerate case ($n \neq 1$).

8.3.1 Energy Shifts for the Ground State under RGUP Corrections ($n = 1$)

For the non-degenerate case, the energy shift is calculated using the RGUP-perturbed Hamiltonian (Eq. 8.61), where the RGUP terms are treated as perturbative corrections. The energy shift for the ground state of the hydrogen atom is then expressed as

$$E_{\text{shift}} = -2\beta(mc)^2 \left(\frac{\langle 1, 0, 0 | \nabla^2 | 1, 0, 0 \rangle}{2m} - \frac{\langle 1, 0, 0 | \nabla^4 | 1, 0, 0 \rangle}{8m^3c^2} \right) - (1 - \beta(mc)^2)e|E|\langle 1, 0, 0 | z | 1, 0, 0 \rangle$$

$$+ \sum_{j \neq k} \frac{e^2|E|^2 \langle k | \left[-2\beta((mc)^2/e|E|) \left(\frac{\nabla^2}{2m} - \frac{\nabla^4}{8m^3c^2} \right) - (1 - \beta(mc)^2)z \right]^2 | j \rangle}{E_k^0 - E_j^0} + \dots \quad (8.63)$$

We know that for a non-degenerate state, and following the standard results,

$$\langle z \rangle = 0, \quad \langle \nabla^2 \rangle = \frac{\hbar^2}{a_0^2}, \quad \langle \nabla^4 \rangle = \frac{5\hbar^4}{a_0^4}, \quad (8.64)$$

using the above relations, we evaluate the linear correction in the energy shift as

$$E_{\text{shift}} = -2\beta(mc)^2 \left(\frac{\hbar^2}{2ma_0^2} - \frac{5\hbar^4}{8m^3c^2a_0^4} \right), \quad (8.65)$$

where a_0 represents the Bohr radius. One can recover the standard result that linear corrections in the energy shift is equal to zero when $\beta \rightarrow 0$ and $c \rightarrow \infty$ in Eq.(8.63). The above linear correction does not involve $\vec{E}$ in the energy shift expression. In the Stark effect, the modified energy shift involving E arises from the quadratic terms, assuming that cubic and higher-order terms have vanished.

The energy shift with the polarizability α_p of an atom is defined as

$$\Delta_{\text{shift}} = -\frac{1}{2}\alpha_p|\vec{E}|^2, \quad (8.66)$$

for our case

$$\alpha_p = -2e^2 \sum_{n \neq 1} \sum_{l, m} \frac{|\langle n, l, m | V'_{\text{RGUP}} | 1, 0, 0 \rangle|^2}{E_1 - E_n}. \quad (8.67)$$

We have

$$\begin{aligned} \sum_{n \neq 1} \sum_{l,m} |\langle n, l, m | V'_{RGUP} | 1, 0, 0 \rangle|^2 &= \sum_{n,l,m} |\langle n, l, m | V'_{RGUP} | 1, 0, 0 \rangle|^2 \\ &= \langle 1, 0, 0 | (V_{RGUP})^2 | 1, 0, 0 \rangle, \end{aligned} \quad (8.68)$$

which is evaluated as

$$\langle 1, 0, 0 | (V'_{RGUP})^2 | 1, 0, 0 \rangle = \langle 1, 0, 0 | \left[-2(\beta(mc)^2/e|E|) \left(\frac{\nabla^2}{2m} - \frac{\nabla^4}{8m^3c^2} \right) - (1 - \beta(mc)^2)z \right]^2 | 1, 0, 0 \rangle, \quad (8.69)$$

and by neglecting the higher order quantum gravitational terms $\mathcal{O}(\beta^2)$ we get

$$\begin{aligned} (1 - 2\beta(mc)^2) \langle 1, 0, 0 | z^2 | 1, 0, 0 \rangle &+ 4(\beta(mc)^2/eE) \langle 1, 0, 0 | z | 1, 0, 0 \rangle \\ &+ 4(\beta(mc)^2/e|E|) \langle 1, 0, 0 | \left(\frac{\nabla^2 z}{2m} - \frac{\nabla^4 z}{8m^3c^2} \right) | 1, 0, 0 \rangle, \end{aligned} \quad (8.70)$$

since, we know that $\langle \nabla^2 z \rangle = \langle \nabla^4 z \rangle = 0$ and

$$\langle z^2 \rangle = a_0^2, \quad (8.71)$$

$$\langle z \rangle = 0, \quad (8.72)$$

this gives the RGUP-modified Stark potential as

$$|\langle V'_{RGUP} \rangle|^2 = (1 - 2\beta(mc)^2) a_0^2. \quad (8.73)$$

We also know that the energy for nth orbit is given as

$$E_n = -\frac{e^2}{2a_0 n^2}, \quad (8.74)$$

and we can write

$$-E_1 + E_n = \frac{e^2}{2a_0} \left(1 - \frac{1}{n^2} \right) \geq -E_1 + E_2 = \frac{3}{8} \frac{e^2}{a_0}. \quad (8.75)$$

From Eqns.(8.67,8.73,8.75), we find the upper limit for modified polarizability parameter as

$$\alpha_p < (1 - 2\beta(mc)^2) \frac{16a_0^3}{3}, \quad (8.76)$$

and the lower limit for the shift

$$\Delta_{shift} > -(1 - 2\beta(mc)^2) \frac{8a_0^3}{3} |\vec{E}|^2. \quad (8.77)$$

It is easily observe from the above relation that we can recover the standard energy shift expression $-\frac{8a_0^3}{3} |\vec{E}|^2$ by taking the lim $\beta = 0$ [278, 279]. The quantum gravitational effects introduced by the RGUP modification raise the lower limit of the energy shift. From the modified polarizability of the hydrogen atom, we calculate the upper bound on the RGUP parameter. Using Eq. (8.76), we obtained:

$$\beta < \frac{1}{(mc)^2} \left(1 - \frac{3\alpha_p}{16a_0^3} \right), \quad (8.78)$$

where the experimental value of α_p [280] is $0.667 \times 10^{-30} \text{ m}^3$, $m = 9.1 \times 10^{-31} \text{ kg}$, the Bohr radius is $a_0 = 5.3 \times 10^{-11} \text{ m}$, and $c = 3 \times 10^8 \text{ m/s}$. Substituting these values, we obtain the upper bound for β as:

$$\beta < 10^{42}. \quad (8.79)$$

While the upper bound obtained here is not the tightest, it is comparable to other bounds in the literature, such as those derived from the Lamb shift $\beta < 10^{36}$ [281], the scanning tunneling microscope $\beta < 10^{21}$ [281], and harmonic oscillators $\beta < 10^6$ [282]. However, it is tighter than the bounds derived from Landau levels $\beta < 10^{50}$ [281] and other classical phenomena [283–286]. It is important to note that the bounds previously derived are under the framework of GUP (Non-relativistic) corrections, while the bound here is calculated in the presence of RGUP (relativistic) corrections.

TABLE 8.1: Upper bounds on the GUP parameter β from various experiments and our RGUP result.

Experiment/Phenomenon	Upper Bound on β
Lamb shift	10^{36} [281]
Scanning Tunneling Microscope (STM)	10^{21} [281]
Harmonic Oscillators	10^6 [282]
Landau Levels	10^{50} [281]
Gravitational Waves (GW), Light Deflection	$10^{60}, 10^{78}$ [283, 284]
This work (RGUP, relativistic)	10^{42}

8.3.2 Degenerate State Energy Splitting with RGUP Modifications ($n \neq 1$)

We know that the modified Stark potential is given by Eq. (8.62). For the excited states of the hydrogen atom, specifically for the degenerate $n = 2$ level, there exist four degenerate states: $|2, 0, 0\rangle$, $|2, 1, -1\rangle$, $|2, 1, 0\rangle$, and $|2, 1, 1\rangle$. To calculate the energy shift caused by external electric field we must evaluate the matrix elements for the combination of different basis. Since the angular momentum operator L_z commutes with ∇^2 and z , the matrix elements between states with different magnetic quantum numbers ($m \neq m'$) vanish. Thus, the perturbation due to the external electric field only couples specific states, leading to a block-diagonal structure in the Hamiltonian.

The corresponding matrix elements in the basis $\{|2, 0, 0\rangle, |2, 1, -1\rangle, |2, 1, 0\rangle, |2, 1, 1\rangle\}$ can be computed using the expectation values of the Stark potential, incorporating the RGUP modifications. These corrections lead to a shift in energy levels, altering the standard Stark splitting in the presence of quantum gravitational effects.

The corresponding perturbed matrix is given as

$$\langle V_{\text{RGUP}} \rangle = \begin{bmatrix} \langle 2, 0, 0 | V_{\text{RGUP}} | 2, 0, 0 \rangle & 0 & \langle 2, 0, 0 | V_{\text{RGUP}} | 2, 1, 0 \rangle & 0 \\ 0 & \langle 2, 1, -1 | V_{\text{RGUP}} | 2, 1, -1 \rangle & 0 & 0 \\ \langle 2, 1, 0 | V_{\text{RGUP}} | 2, 0, 0 \rangle & 0 & \langle 2, 1, 0 | V_{\text{RGUP}} | 2, 1, 0 \rangle & 0 \\ 0 & 0 & 0 & \langle 2, 1, 1 | V_{\text{RGUP}} | 2, 1, 1 \rangle \end{bmatrix}$$

Now calculating the individual elements of the matrix using Eqns.(8.64,8.71) and the expectation value of p^4 for different l , we get

$$\langle 2, 0, 0 | V_{\text{RGUP}} | 2, 0, 0 \rangle = -2\beta(mc)^2 \left[\frac{\hbar^2}{4ma_0^2} - \frac{13}{16a_0^4} \frac{\hbar^4}{8m^3c^2} \right] \equiv -2\beta(mc)^2 M, \quad (8.80)$$

$$\langle 2, 0, 0 | V_{\text{RGUP}} | 2, 1, 0 \rangle = 3a_0(1 - \beta(mc)^2)e|\vec{E}|, \quad (8.81)$$

$$\langle 2, 1, -1 | V_{\text{RGUP}} | 2, 1, -1 \rangle = -2\beta(mc)^2 \left[\frac{\hbar^2}{4ma_0^2} - \frac{7}{48a_0^4} \frac{\hbar^4}{8m^3c^2} \right] \equiv -2\beta(mc)^2 P, \quad (8.82)$$

$$\langle 2, 1, 0 | V_{\text{RGUP}} | 2, 0, 0 \rangle = 3a_0(1 - \beta(mc)^2)e|\vec{E}|, \quad (8.83)$$

$$\langle 2, 1, 0 | V_{\text{RGUP}} | 2, 1, 0 \rangle = -2\beta(mc)^2 \left[\frac{\hbar^2}{4ma_0^2} - \frac{7}{48a_0^4} \frac{\hbar^4}{8m^3c^2} \right] \equiv -2\beta(mc)^2 P, \quad (8.84)$$

$$\langle 2, 1, 1 | V_{\text{RGUP}} | 2, 1, 1 \rangle = -2\beta(mc)^2 \left[\frac{\hbar^2}{4ma_0^2} - \frac{7}{48a_0^4} \frac{\hbar^4}{8m^3c^2} \right] \equiv -2\beta(mc)^2 P, \quad (8.85)$$

this gives the RGUP-modified matrix as follows

$$\langle V_{\text{RGUP}} \rangle = \begin{bmatrix} -2\beta(mc)^2 M & 0 & 3a_0(1 - \beta(mc)^2)e|\vec{E}| & 0 \\ 0 & -2\beta(mc)^2 P & 0 & 0 \\ 3a_0(1 - \beta(mc)^2)e|\vec{E}| & 0 & -2\beta(mc)^2 P & 0 \\ 0 & 0 & 0 & -2\beta(mc)^2 P \end{bmatrix}$$

the determinant of the above matrix gives the RGUP corrections to the energy shift for degenerate case. The characteristic equation for the given matrix on neglecting the $\mathcal{O}(\beta^2)$ is obtained as

$$\lambda(\lambda + 4\beta(mc)^2 P) \left(\lambda^2 + 2\lambda\beta(mc)^2(M + P) - 9a^2 e^2 |\vec{E}|^2 (1 - 2\beta(mc)^2) \right) + \mathcal{O}(\beta^2) = 0, \quad (8.86)$$

neglecting the higher-order corrections $\mathcal{O}(\beta^2)$, we obtain the shifts in Joules as

$$\Delta'_{\text{shift}} = 0, \quad (8.87)$$

$$\Delta''_{\text{shift}} = -4\beta(mc)^2 P, \quad (8.88)$$

$$\Delta'''_{\text{shift}} = \frac{-2\beta(mc)^2(M + P) \pm 6e|\vec{E}|a_0\sqrt{(1 - 2c^2m^2\beta)}}{2}, \quad (8.89)$$

where M and P are defined in Eq.(8.80). We observe from Eq. (8.89) that the shift contains a linear term in the electric field $\vec{E}$, which characterizes the linear Stark effect. With the RGUP modification, additional terms involving the GUP parameter β appear in the shift expression. These extra terms vanish when the RGUP correction is not considered ($\beta = 0$), leaving only the standard linear dependence on $\vec{E}$.

8.4 Conclusion

In extreme environments of high energies where gravity becomes significant, quantum gravitational corrections become relevant. Such environments, like the intense magnetic fields surrounding neutron stars and magnetars [287–292], are where both relativistic and quantum gravity effects play a crucial role. Investigating atomic spectra in the vicinity of these compact objects presents a valuable opportunity to explore how quantum gravity may influence spectral lines, offering a promising observational avenue to test various quantum gravity theories. In this study, we have extended the framework of the

Relativistic Generalized Uncertainty Principle (RGUP) to investigate the Stark effect in a hydrogen atom, incorporating quantum gravitational corrections within Minkowskian spacetime.

In this chapter, we have explored the implications of the Relativistic Generalized Uncertainty Principle (RGUP) on two important quantum phenomena—the Zeeman and Stark effects in hydrogen-like atomic systems—by introducing relativistic corrections to the kinetic term of the Hamiltonian. Using the Stetsko and Tkachuk formalism [269], we established a consistent relativistic GUP algebra and derived a valid four-momentum operator that smoothly reduces to the non-relativistic squared momentum in the limit $c \rightarrow \infty$, confirming the internal consistency of the framework.

For the Zeeman effect, the RGUP-modified Lande energy shift was derived, revealing that the shift is no longer directly proportional to the magnetic quantum number m' , unlike in the standard formulation. When relativistic corrections are neglected, the expression reduces to the GUP-modified Zeeman effect previously obtained in [272], and by setting the GUP parameter $\gamma \rightarrow 0$, the standard Zeeman effect is recovered. The inclusion of RGUP introduces a more intricate dependence of the energy shift on both the deformation and magnetic parameters, leading to a broader range of possible energy variations. This indicates a subtle interplay between relativistic quantum gravity effects and magnetic interactions, enriching our understanding of atomic spectra under extreme physical conditions.

The analysis was then extended to the Stark effect within the same RGUP framework. For the non-degenerate ground state ($n = 1$), we obtained a linear energy shift

$$E_{\text{shift}} = -2\beta(mc)^2 \left(\frac{\hbar^2}{2ma_0^2} - \frac{5\hbar^4}{8mc^2a_0^4} \right),$$

where $\beta = \epsilon\gamma^2$. This correction is finite and disappears as $\beta \rightarrow 0$, thus recovering the standard zero linear shift result. Furthermore, the quadratic Stark effect was shown to possess an increased lower bound for the energy shift, given by

$$\Delta_{\text{shift}} > - \left(1 - 2\beta(mc)^2 \right) \frac{8a_0^3}{3} |\vec{E}|^2,$$

implying that RGUP raises the energy floor compared to the conventional Stark effect. For the degenerate case ($n = 2$), additional β -dependent terms appear, modifying the

linear dependence on the external electric field. All these corrections vanish in the limit $\beta \rightarrow 0$, confirming that the RGUP framework consistently reproduces standard results in the appropriate limits. From these results, we also established an upper bound on the RGUP parameter of the order 10^{42} , which, though not the tightest among current bounds, surpasses several derived from non-relativistic and classical analyses.

The RGUP modification provides small shifts in the Zeeman and Stark spectra. Although these corrections are expected to be extremely small at presently accessible energies, modern precision atomic spectroscopy, such as hydrogen spectroscopy, alkali-atom spectroscopy, trapped-ion optical clocks, and high-resolution Zeeman spectroscopy, provides an ideal testing ground. Future measurements with increasing precision could therefore be used either to detect these corrections or to place upper bounds on the RGUP parameter.

Chapter 9

Conclusion

In this thesis, we explored the implications of minimal length which is a feature common to several approaches to quantum gravity and examined how such effects modify physical systems across cosmology, black hole physics, quantum mechanics, and atomic spectra. We Begin with a review of the motivations for quantum gravity, we highlighted the emergence of a fundamental length scale from Gedanken experiments, semiclassical arguments, and frameworks such as String Theory and Loop Quantum Gravity. We then developed a consistent formulation of the Generalized Uncertainty Principle (GUP) and its relativistic extension (RGUP), emphasizing the phenomenological role these deformations play in capturing Planck-scale physics. This framework was subsequently applied to a range of physical phenomena.

Our cosmological analysis focused on both phantom and quintessence scalar-field models. We showed that GUP corrections introduce significant modifications to the Friedmann and Klein–Gordon equations, thereby influencing the expansion dynamics in the early and late universe. Through dynamical system analysis for multiple potentials, we demonstrated how minimal length can shift critical points, alter stability properties, and modify attractor solutions. Notably, GUP-induced corrections affect inflationary behavior by altering the slow-roll parameters and possibly modifying the duration of inflation.

In the context of black hole physics, we extended two prominent semiclassical tunneling methods using the Parikh–Wilczek radial-null-geodesic method and the Hamilton–Jacobi method to include GUP corrections. We obtained modified tunneling probabilities and

Hawking temperatures in the presence of minimal measurable length with both the procedure. These results imply that minimal length effects render Hawking radiation slightly non-thermal and support the possibility of black hole remnants as end states of evaporation.

We also investigated the angular momentum algebra under linear and quadratic GUP. On computing the modified commutation relations and constructing the corresponding ladder operators, we showed that minimal length deforms the angular momentum spectrum. As an application, we analyzed the hydrogen atom and derived GUP-corrected energy levels. This motivated a broader study of RGUP effects on atomic spectroscopy, including the Zeeman and Stark effects. We computed both non-degenerate and degenerate state corrections and found distinctive shifts and splittings generated by the minimal-length scale.

Across all these investigations, a unified picture emerges: minimal length acts as a universal regulator that softens ultraviolet divergences and induces Planck-scale corrections across a wide range of physical systems. The results presented in this thesis demonstrate that the GUP and RGUP provide powerful and coherent tools for probing quantum gravity phenomenology, offering insights that span cosmology, gravitation, and quantum mechanics.

Chapter 10

Future Plan

The results presented in this thesis open several promising avenues for further exploration of minimal-length physics. A natural extension of this work is to refine the cosmological implications of GUP and RGUP by studying inflationary perturbations, tensor modes, reheating dynamics, and large-scale structure formation. These investigations may uncover observational signatures of minimal length in the cosmic microwave background or galaxy surveys. In black hole physics, applying GUP-corrected tunneling methods to rotating, charged, and AdS black holes could yield a more complete understanding of evaporation, non-thermal spectra, and the possibility of stable remnants.

On the quantum mechanical side, the deformation of angular momentum algebra suggests deeper algebraic structures worth exploring, including potential connections to quantum groups, noncommutative phase space, and symmetry deformations in curved backgrounds. Extending RGUP corrections to relativistic atomic systems and high-field environments, such as magnetars, may also lead to testable signatures in precision spectroscopy. Finally, constructing a fully covariant GUP-modified quantum field theory and developing numerical simulations of RGUP-corrected cosmological and black hole dynamics represent compelling long-term directions that could further strengthen the theoretical foundations and bring minimal-length phenomenology closer to observational verification.

Appendix A

Derivations of GUP Equations

In this appendix, we apply canonical quantization to the GUP-modified Hamiltonian to derive the GUP-modified Wheeler-DeWitt (WDW) equation [144]. We then explore the semi-classical effects of GUP deformation, ultimately obtaining the classical Hamiltonian, which is used for dynamical system analysis.

On substituting the deformed canonical variables in Eq.(5.95) we obtain Eq.(5.98) which is the GUP deformed Hamiltonian. And, using the canonical quantization, we obtain the corresponding WDW-equation as, $\hat{\mathcal{H}}\psi = 0$. Observing that the Hamiltonian is in the form of a harmonic oscillator, we can proceed with quantization using ladder operators. We consider the perturbation in the Hamiltonian to be of the order of γ^2 , so the usual ladder operators $\tilde{a}$, $\tilde{a}^\dagger$, and the number operator N are applicable only to the unperturbed case. For the perturbed case, new ladder and number operators are required [293], and impose the same conditions as in the unperturbed case,

$$\tilde{a}|\phi_n\rangle = \sqrt{n}|\phi_{n-1}\rangle, \quad \tilde{N}|\phi_n\rangle = n|\phi_n\rangle, \quad \tilde{a}^\dagger|\phi_n\rangle = \sqrt{n+1}|\phi_{n+1}\rangle, \quad \tilde{N} = \tilde{a}^\dagger\tilde{a}, \quad [\tilde{a}, \tilde{a}^\dagger] = 1 \quad (\text{A.1})$$

where $|\phi_n\rangle = |\psi_n^{(0)}\rangle + \gamma|\psi_n^{(1)}\rangle + \dots$, and $|\psi_n^{(0)}\rangle$ is the eigenfunction of the unperturbed Hamiltonian, and $|\psi_n^{(m)}\rangle$ is the m-th order correction to the eigenstate. Since the correction γ is very small, we can express the operators as

$$\tilde{a} = a + \sum_{n=1}^{\infty} \gamma^n \alpha_n, \quad \tilde{a}^\dagger = a^\dagger + \sum_{n=1}^{\infty} \gamma^n \alpha_n^\dagger, \quad \tilde{N} = N + \sum_{n=1}^{\infty} \gamma^n \nu_n, \quad (\text{A.2})$$

we apply the above operator Eq.(A.2) to the wave function

$$\tilde{a}|\phi_n\rangle = \left(a + \sum_{m=1}^{\infty} \gamma^m \alpha_m\right) \left(|\psi_n^{(0)}\rangle + \sum_{m=1}^{\infty} \gamma^m |\psi_n^{(m)}\rangle\right), \quad (\text{A.3})$$

$$\tilde{a}^\dagger|\phi_n\rangle = \left(a^\dagger + \sum_{m=1}^{\infty} \gamma^m \alpha_m\right) \left(|\psi_n^{(0)}\rangle + \sum_{m=1}^{\infty} \gamma^m |\psi_n^{(m)}\rangle\right). \quad (\text{A.4})$$

and using the procedure given in [293] we can expand the operators up to the order of γ we get,

$$\tilde{a} = a - \frac{\beta\gamma}{2} \left(\frac{3|\Lambda|}{2}\right)^{1/4} (a^{\dagger 3} - 6Na^\dagger + 2a^3), \quad \tilde{a}^\dagger = a^\dagger + \frac{\beta\gamma}{4} \left(\frac{3|\Lambda|}{2}\right)^{3/4} (a^{\dagger 3} - 6Na^\dagger + 2a^3). \quad (\text{A.5})$$

Substituting these into Eq.(5.21), after incorporating them into the expressions for momentum and coordinates, we obtain

$$\begin{aligned} p_{0k} &= i \left(\frac{3|\Lambda|}{32}\right)^{1/4} (a_k^\dagger - a_k), \quad \tilde{q}_k = q_{0k}, \quad \tilde{k} = \left(\frac{2}{3|\Lambda|}\right)^{1/4} (a^\dagger + a), \\ p_k &= p_{0k} + \frac{i}{8} \left(\frac{3|\Lambda|}{2}\right)^{3/4} \epsilon \gamma^2 (a^{\dagger 3} - 6Na^\dagger + 2a^3). \end{aligned} \quad (\text{A.6})$$

the expectation values are $\langle \tilde{q}_k \rangle = 0$ and $\langle \tilde{p}_k \rangle = 0$. Finally, using the inverse transformation, the volume of the universe is given by the cube of the expansion factor as $V = a^3(t) = \frac{3|\Lambda|}{8}(x^2 + y^2)$. Thus, we calculate the spectrum for the volume, at the order of γ^2 , as the expected value of this expression,

$$\begin{aligned} V(n) &= \sqrt{\frac{3}{2|\Lambda|}} (2n+1) + \frac{9\beta\gamma}{8} \left(\frac{3}{8|\Lambda|}\right)^{1/4} (4\sqrt{3} - \sqrt{2|\Lambda|}) (1+2n+2n^2) \\ &\quad + \frac{3\beta^2\gamma^2}{64} (8 - 4\sqrt{6|\Lambda|} + 3|\Lambda|) (6+13n+3n^2+2n^3). \end{aligned} \quad (\text{A.7})$$

To derive the GUP modified Wheeler-DeWitt (WDW) equation, we apply canonical quantization to the Hamiltonian described by Eq.(5.98), resulting in

$$\hat{H}_{\text{GUP}}\psi = \hat{H}_0\psi + i\beta\gamma\hat{H}_1\psi + \gamma^2 \left(\frac{\beta^2}{6} + \frac{2\epsilon}{3}\right) \hat{H}_2\psi + 2i\beta\gamma^3 \left(\frac{\beta^2 + 2\epsilon}{3}\right) \hat{H}_3\psi - \gamma^4 \left(\frac{\beta^2 + 2\epsilon}{3}\right) \hat{H}_4\psi = 0, \quad (\text{A.8})$$

where

$$\begin{aligned}\hat{H}_0 &= -\frac{1}{2} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right), \quad \hat{H}_1 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^{3/2}, \quad \hat{H}_2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2, \\ \hat{H}_3 &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^{5/2}, \quad \hat{H}_4 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^3.\end{aligned}\tag{A.9}$$

To simplify our analysis, we will limit our considerations to the case where $\beta = 0$ and focus on terms up to the order of γ^2 . To explore the semiclassical effects of the GUP deformation, we assume the wavefunction is of the form $\psi = e^{i(S(x)+G(y))}$. We then apply the operators $\hat{H}_0$ and $\hat{H}_2$ as defined in Equation (A.9), and finally, perform a WKB-approximation, we obtain

$$\hat{H}_0\psi \approx \frac{1}{2} \left[(S')^2 + (G')^2 \right] \psi + \frac{1}{2} \omega^2 [x_0^2 + y_0^2] \psi, \tag{A.10}$$

$$\hat{H}_2\psi \approx \left[(G')^4 + (S')^4 \right] \psi + 2 \left[(G')^2 (S')^2 \right] \psi, \tag{A.11}$$

where $S' = \frac{dS}{dx}$ and $G' = \frac{dG}{dy}$. Now, using the usual definitions $\frac{\partial S}{\partial x} = P_x$ and $\frac{\partial G}{\partial y} = P_y$, we get the classical Hamiltonian

$$H = \frac{1}{2} (P_x^2 + P_y^2) - \frac{3\Lambda}{8} (x^2 + y^2) + \frac{2}{3} \gamma^2 \epsilon (P_x^4 + 2P_x^2 P_y^2 + P_y^4). \tag{A.12}$$

This equation is same as Eq.(5.98) for $\beta = 0$ and taking only the second order of γ , which is later used to determine the GUP-deformed Friedmann, Raychaudhuri, and Klein-Gordon Equations.

A.1 Proof of the Friedmann, Raychaudhuri and Klein-Gordon Equation

In this appendix, we show the derivation of the Friedmann, Raychaudhuri and Klein-Gordon equation starting with the phantom Lagrangian. Using Eq.(5.6) and Eq.(5.7), and applying the Hamilton equations, we can derive the equations of motion for the system as

$$P_a = \frac{\partial \mathcal{L}}{\partial \dot{a}} = -6a\dot{a}, \quad P_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = -a^3 \dot{\phi}, \quad \dot{P}_a = -\frac{\partial \mathcal{H}}{\partial a}, \quad \dot{P}_\phi = -\frac{\partial \mathcal{H}}{\partial \phi}$$

and by substituting the form of P_a and P_ϕ so obtained from the Lagrangian equation in Eq.(5.7).

$$6\dot{a}^2 + 6a\ddot{a} = 3\dot{a}^2 + \frac{3a^2\dot{\phi}^2}{2} + 3a^2V(\phi) + 3a^2\Lambda,$$

the Raychaudhuri equation is

$$2\frac{\ddot{a}}{a^2} + \left(\frac{\dot{a}}{a}\right)^2 = \frac{\dot{\phi}^2}{2} + V(\phi) + \Lambda. \quad (\text{A.13})$$

The Lagrangian in Eq.(5.6) does not depend on $\dot{N}(t)$, which implies that there is no dynamics associated with the lapse function $N(t)$. Consequently, we have $P_N \equiv \frac{\partial \mathcal{L}}{\partial \dot{N}} = 0$. Now, using Eq.(5.7), we obtain

$$3a\dot{a}^2 + \frac{\dot{\phi}^2 a^3}{2} - a^3(\Lambda + V) = 0,$$

this gives the Friedmann equation as

$$3H^2 = -\frac{\dot{\phi}^2}{2} + \Lambda + V(\phi). \quad (\text{A.14})$$

Similarly, the Klein-Gordon equation is

$$3a^2\dot{a}\dot{\phi} + a^3\ddot{\phi} = a^3V'(\phi),$$

or,

$$\ddot{\phi} + 3\dot{\phi}\frac{\dot{a}}{a} - \frac{dV(\phi)}{d\phi} = 0. \quad (\text{A.15})$$

Appendix B

Calculation of Eigenvalues

To obtain the nature of the fixed point for GUP-deformed dynamics, we require to calculate the eigenvalue of the Jacobian matrix for each fixed point. We construct the Jacobian matrix from the autonomous equations (5.112), (5.113), (5.114) and (5.115) as

$$\mathcal{J} = \begin{bmatrix} f_{,x} & f_{,y} & f_{,z} & f_{,\lambda} \\ g_{,x} & g_{,y} & g_{,z} & g_{,\lambda} \\ h_{,x} & h_{,y} & h_{,z} & 0 \\ u_{,x} & u_{,y} & u_{,z} & u_{,\lambda} \end{bmatrix}_{x,y,z,\lambda}$$

where the subscripts represent differentiation with respect to the variables. To obtain the eigenvalues of the Jacobian, we know

$$\det(\mathcal{J} - E_i I) = 0, \quad (\text{B.1})$$

which gives

$$\begin{aligned} & E_i^4 + E_i^3(-f_{,x} - g_{,y} - h_{,z} - u_{,\lambda}) \\ & + E_i^2(f_{,x}g_{,y} + f_{,x}h_{,z} + f_{,x}u_{,\lambda} - f_{,y}g_{,x} - f_{,z}h_{,x} - f_{,\lambda}u_{,x} + g_{,y}h_{,z} + g_{,y}u_{,\lambda} - g_{,z}h_{,y} - g_{,\lambda}u_{,y} + h_{,z}u_{,\lambda}) \\ & + E_i(-f_{,x}g_{,y}h_{,z} - f_{,x}g_{,y}u_{,\lambda} + f_{,x}g_{,z}h_{,y} + f_{,x}g_{,\lambda}u_{,y} - f_{,x}h_{,z}u_{,\lambda} + f_{,y}g_{,x}h_{,z} + f_{,y}g_{,x}u_{,\lambda} - f_{,y}g_{,z}h_{,x} \\ & - f_{,y}g_{,\lambda}u_{,x} - f_{,z}g_{,x}h_{,y} + f_{,z}g_{,y}h_{,x} + f_{,z}h_{,x}u_{,\lambda} - f_{,\lambda}g_{,x}u_{,y} + f_{,\lambda}g_{,y}u_{,x} - f_{,\lambda}h_{,x}u_{,z} + f_{,\lambda}h_{,z}u_{,x} - g_{,y}h_{,z}u_{,\lambda} \\ & + g_{,z}h_{,y}u_{,\lambda} - g_{,\lambda}h_{,y}u_{,z} + g_{,\lambda}h_{,z}u_{,y}) \\ & + (f_{,x}g_{,y}h_{,z}u_{,\lambda} - f_{,x}g_{,z}h_{,y}u_{,\lambda} + f_{,x}g_{,\lambda}h_{,y}u_{,z} - f_{,x}g_{,\lambda}h_{,z}u_{,y} - f_{,y}g_{,x}h_{,z}u_{,\lambda} + f_{,y}g_{,z}h_{,x}u_{,\lambda} - f_{,y}g_{,\lambda}h_{,x}u_{,z} \\ & + f_{,y}g_{,\lambda}h_{,z}u_{,x} + f_{,z}g_{,x}h_{,y}u_{,\lambda} - f_{,z}g_{,y}h_{,x}u_{,\lambda} + f_{,z}g_{,\lambda}h_{,x}u_{,y} - f_{,z}g_{,\lambda}h_{,y}u_{,x} - f_{,\lambda}g_{,x}h_{,y}u_{,z} + f_{,\lambda}g_{,x}h_{,z}u_{,y} \\ & + f_{,\lambda}g_{,y}h_{,x}u_{,z} - f_{,\lambda}g_{,y}h_{,z}u_{,x} - f_{,\lambda}g_{,z}h_{,x}u_{,y} + f_{,\lambda}g_{,z}h_{,y}u_{,x}) = 0 \end{aligned}$$

And, for $\lambda = 0$ and $\mu = \sqrt{3/8}$, the elements of the Jacobian matrix are given by

$$\begin{aligned} f_{,x} = \frac{1}{12x^2(-1+x^2)^2} & \left[9 - 9y^2 + 9z^2 + 15(11 + \sqrt{6})x^{10}z^2 + x^2(-30 + 3y^2 + 7z^2) \right. \\ & + x^6(-6(17 + 3\sqrt{6}) + 3(-7 + \sqrt{6})y^2 + (316 - 147\sqrt{6})z^2) \\ & + x^8(9(5 + \sqrt{6}) + (-393 + 51\sqrt{6})z^2) \\ & \left. + x^4(78 + 9\sqrt{6} + (39 - 9\sqrt{6})y^2 + (-104 + 81\sqrt{6})z^2) \right], \end{aligned} \quad (\text{B.2})$$

$$f_{,y} = \frac{(-3 + 8x^2 + (-7 + \sqrt{6})x^4)y}{2x(-1 + x^2)}, \quad (\text{B.3})$$

$$f_{,z} = \frac{-9 + 25x^2 + 3(-7 + 9\sqrt{6})x^4 + 3(11 + \sqrt{6})x^6}{6x}, \quad (\text{B.4})$$

$$f_{,\lambda} = \sqrt{\frac{3}{2}}y^2, \quad (\text{B.5})$$

$$\begin{aligned} g_{,x} = \frac{1}{2(-1+x^2)^2}xy & \left(2 + \sqrt{6} - (-2 + \sqrt{6})y^2 - 10z^2 + \sqrt{6}z^2 + 2(8 + \sqrt{6})x^6z^2 \right. \\ & \left. - 2x^2(2 + \sqrt{6} - 18z^2) + x^4(2 + \sqrt{6} - 3(14 + \sqrt{6})z^2) \right), \end{aligned} \quad (\text{B.6})$$

$$\begin{aligned} g_{,y} = \frac{1}{12(-1+x^2)} & \left(3(8 + \sqrt{6})x^6z^2 + 3x^4(2 + \sqrt{6} - 18z^2) + 2(-12 + 36y^2 + z^2) \right. \\ & \left. + x^2(-3(-6 + \sqrt{6}) + 9(-10 + \sqrt{6})y^2 + (28 - 3\sqrt{6})z^2) \right), \end{aligned} \quad (\text{B.7})$$

$$g_{,z} = \frac{1}{6} \left(-2 + 3(-10 + \sqrt{6})x^2 + 3(8 + \sqrt{6})x^4 \right) yz, \quad (\text{B.8})$$

$$g_{,\lambda} = -\sqrt{\frac{3}{2}}xy, \quad (\text{B.9})$$

$$\begin{aligned} h_{,x} = \frac{xy^2(3 + (-2 + \sqrt{6})z^2)}{2(-1+x^2)^2z} & - \frac{1}{2}xz \left(5 + \sqrt{6} + (-10 + \sqrt{6} + 2(8 + \sqrt{6})x^2)z^2 \right), \\ & (\text{B.10}) \end{aligned}$$

$$h_{,y} = -\frac{y(3 + (8 + (-10 + \sqrt{6})x^2)z^2)}{2(-1+x^2)z}, \quad (\text{B.11})$$

$$\begin{aligned} h_{,z} = \frac{1}{4(-1+x^2)z^2} & \left(-3 + 5z^2 - 2z^4 - 3(8 + \sqrt{6})x^6z^4 + y^2(3 - 8z^2) \right. \\ & \left. + x^4z^2(-5 - \sqrt{6} + 54z^2) + x^2(3 + (\sqrt{6} - (-10 + \sqrt{6})y^2)z^2 + (-28 + 3\sqrt{6})z^4) \right), \\ & (\text{B.12}) \end{aligned}$$

$$h_{,\lambda} = 0, \quad (\text{B.13})$$

$$u_{,x} = 0, \quad (\text{B.14})$$

$$u_{,y} = 0, \quad (\text{B.15})$$

$$u_{,z} = 0, \quad (\text{B.16})$$

$$u_{,\lambda} = -\frac{6x^2(-1+x^2)z^2}{1 - x^2 - y^2 - (-1+x^4)z^2}. \quad (\text{B.17})$$

for each value of fixed points (A, B, C, D, E, F, G, H) we obtain the same characteristic equation given as

$$E_i^4 - 496.3E_i^3 + 6048.61E_i^2 + 67214.3E_i - 13680.8 = 0 \quad (\text{B.18})$$

and this gives the eigenvalues as

$$E_1 = 483.5, \quad E_2 = 19.7, \quad E_3 = -7.1, \quad E_4 = 0.2. \quad (\text{B.19})$$

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List of Publications

Research Papers included in this Thesis

1. Gaurav Bhandari, S. D. Pathak, Manabendra Sharma and Anzhong Wang. *GUP deformed background dynamics of phantom field. General Relativity and Gravitation*, Volume 56, Issue 11, **2024**, 139. doi:[10.1007/s10714-024-03323-4](https://doi.org/10.1007/s10714-024-03323-4)
Journal Details: General Relativity and Gravitation, Springer, Impact factor: 2.8, Q1.
2. Gaurav Bhandari, S. D. Pathak, Manabendra Sharma and Anzhong Wang. *Distortion of quintessence dynamics by the generalized uncertainty principle. Annals of Physics*, Volume 473, **February 2025**, 169895. doi:[10.1016/j.aop.2024.169895](https://doi.org/10.1016/j.aop.2024.169895)
Journal Details: Annals of Physics, Elsevier, Impact factor: 3.0, Q1.
3. Gaurav Bhandari, S. D. Pathak, Manabendra Sharma. *Generalized uncertainty principle and the Zeeman effect: Relativistic corrections unveiled. Nuclear Physics B*, Volume 1012, **March 2025**, 116817. doi:[10.1016/j.nuclphysb.2025.116817](https://doi.org/10.1016/j.nuclphysb.2025.116817)
Journal Details: Nuclear Physics B, Elsevier, Impact factor: 2.8, Q1.
4. Gaurav Bhandari, S. D. Pathak, Manabendra Sharma, Maxim Yu Khlopov. *Quantum Gravity Corrections to Hawking Radiation via GUP. The European Physical Journal Plus*, Volume 140, Issue 7, **2025**, 663. doi:[10.1140/epjp/s13360-025-06608-2](https://doi.org/10.1140/epjp/s13360-025-06608-2)
Journal Details: The European Physical Journal Plus, Springer, Impact factor: 2.8, Q2.
5. G. Bhandari, S. D. Pathak. *Stark Energy Shifts due to Quantum Gravity in RGUP Algebra. Nuclear Physics B*, Volume 1018, Issue 7, **September 2025**, 117019. doi:[10.1016/j.nuclphysb.2025.117019](https://doi.org/10.1016/j.nuclphysb.2025.117019)
Journal Details: Nuclear Physics B, Elsevier, Impact factor: 2.8, Q1.
6. G. Bhandari, S. D. Pathak, M. Sharma, A. Wang. *Deformed algebraic structure of angular momenta: GUP perspective., Conference Proceedings, Journal of Holography Applications in Physics*, arXiv preprint, arXiv:2411.18901, **November 2024**.
Status: Preprint available on arXiv.

Research Papers not included in this Thesis

1. G. Bhandari, S. D. Pathak, H. Ghotra. *Unified Framework for Geodesic Dynamics with Conservative, Dissipative, and GUP Effects*. arXiv preprint, arXiv:2509.14663, **September 2025**. *Status*: Preprint available on arXiv.
2. Gaurav Bhandari, S. D. Pathak, Manabendra Sharma. *Quantum Corrected Brans-Dicke Cosmology*. *International Journal of Geometric Methods in Modern Physics*, Volume 22, Issue 14, **2025**, 2550137. doi:[10.1142/S0219887825501373](https://doi.org/10.1142/S0219887825501373)
Journal Details: International Journal of Geometric Methods in Modern Physics, World Scientific, Impact factor: 2.2, Q1.
3. G. Bhandari, S. D. Pathak, V. K. Ojha. *RGUP corrections to scalar and fermionic fields*. *The European Physical Journal Plus*, Volume 140, Issue 7, **July 2025**, 630. doi:[10.1140/epjp/s13360-025-06630-4](https://doi.org/10.1140/epjp/s13360-025-06630-4)
Journal Details: The European Physical Journal Plus, Springer, Impact factor: 2.8, Q2.
4. Hemanshi Bundeliya, G. Bhandari, S. D. Pathak, and V. K. Sharma. *Inflation Driven by Scalar-Neutrino Coupling in a Mass-Varying Neutrino Framework*, submitted to Chinese Physics C.
5. K. Solanki, G. Bhandari, V.K. Ojha, and S.D. Pathak. *Tentative title: Phantom angular momentum at Planck scale*, Under preparation.

Presentations in Conferences/Seminars

- Oral presentation on the topic “Minimal Length corrections in the dynamics of Phantom field” in the International Conference on Astroparticle Physics, Gravitation and Cosmology (APGC-2025) in online mode at GLA University, Mathura, India from June26-28, 2025.
- Oral presentation on “Relativistic GUP and Minimal Length Effects on the Zeeman Shift” in the International Conference on Current Advances in Applied Physics (ICCAAP-2025) held in Maulana Azad National Urdu University, Hyderabad during 20th-21st February 2025. Visits: Visited Inter-University Center for Astronomy and Astrophysics (IUCAA), Pune, Aug 20th - 20th Sep 2025.

Workshops/colloquiums attended

- Participation in one-week workshop of GIAN course on A course on Gravitational wave astronomy 2023, Jamia Millia Islamia, New Delhi, delivered by Prof. Anzhong Wang.
- Participation in School of V International School on Gravity, Cosmology and Astrophysics, ISGCA-2024, Bauman MSTU, Russia.
- Participation in The third Indian Association for General Relativity and Gravitation (IAGRG) school on Gravitation and Cosmology through online mode from 14 October 2024 to 25 October 2024.
- Attended the Workshop On Numerical and Analytical Relativity (NAR-2025), Prayagraj, Uttarpradesh, India from March 07-09, 2025. Conferences:

Research Visits

- Successfully completed a 31 days Visited Inter-University Center for Astronomy and Astrophysics (IUCAA), Pune, Aug 20th - 20th Sep 2025., following selection for the program.