

# **SINGULAR VALUE INEQUALITIES ON COMMUTATORS OF OPERATORS ON A HILBERT SPACE**

Thesis Submitted for the Award of the Degree of

**DOCTOR OF PHILOSOPHY**

**in**

**Mathematics**

**By**

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**LOVELY PROFESSIONAL UNIVERSITY, PUNJAB  
2026**

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# Declaration of Authorship

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I, hereby declared that the presented work in the thesis entitled “**SINGULAR VALUE INEQUALITIES ON COMMUTATORS OF OPERATORS ON A HILBERT SPACE**” in fulfilment of degree of **Doctor of Philosophy (Ph. D.)** is outcome of research work carried out by me under the supervision of **Dr. ISHA GARG**, working as **Associate Professor**, in the **School of Chemical Engineering and Physical Sciences** of Lovely Professional University, Punjab, India. In keeping with general practice of reporting scientific observations, due acknowledgements have been made whenever work described here has been based on findings of other investigator. This work has not been submitted in part or full to any other University or Institute for the award of any degree.



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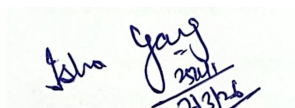
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## Certificate From Supervisor

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This is to certify that the work reported in the Ph. D. thesis entitled “**SINGULAR VALUE INEQUALITIES ON COMMUTATORS OF OPERATORS ON A HILBERT SPACE**” submitted in fulfillment of the requirement for the award of degree of **Doctor of Philosophy (Ph.D.)** in the **School of Chemical Engineering and Physical Sciences**, is a research work carried out by **MANINDERJIT KAUR**, (Registration No. **42000229**), is bonafide record of her original work carried out under my supervision and that no part of thesis has been submitted for any other degree, diploma or equivalent course.

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# Abstract

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The present thesis, titled “**SINGULAR VALUE INEQUALITIES ON COMMUTATORS OF OPERATORS ON A HILBERT SPACE**”, embodies the original research carried out by me under the valuable guidance and supervision of **Dr. ISHA GARG**, Associate Professor, Department of Mathematics, Lovely Professional University, Phagwara, Punjab. This work is being submitted to the Department of Mathematics, School of Chemical Engineering and Physical Sciences, Lovely Professional University, Phagwara-144411, Punjab, India, in partial fulfillment of the requirements for the award of the degree of Doctor of Philosophy in Mathematics.

Singular value inequalities are crucial in analyzing transformations in fields like quantum computing and data science. This thesis focuses on refining these inequalities and expanding their applications in both theoretical and practical contexts.

The journey of singular value inequalities began over a century ago with Erhard Schmidt’s foundational work in 1907, who introduced the concept through integral equations. This early period, spanning 1907 to 1967, saw significant contributions from researchers like Bateman, Picard, Smithies, Chang, Weyl, Horn, Fan, Ault, and Marcus. Their efforts established core results, linking eigenvalues and singular values, and developing bounds for matrix products and traces. However, these early results were often constrained to specific settings, such as finite-dimensional matrices or operators with symmetry, limiting their scope in more complex systems.

Advancements from 1990 to 2025, as explored by Bhatia, Kittaneh, Audeh, Hirzallah, Garg, and others, have significantly expanded the field. Modern research has extended classical results to operator sums, products, commutators, and norm inequalities, often removing restrictive assumptions like positivity. For instance, inequalities for real and imaginary parts of operators, as well as bounds for complex sums and triangle-like expressions, have been developed, offering new insights into operator behavior. These developments have

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been driven by applications in quantum control, numerical methods, and data compression, where singular values play a critical role in simplifying large-scale systems and ensuring computational accuracy. Despite this progress, the literature reveals several gaps: many results rely on positivity assumptions, lack lower bounds for generalized commutators, and fail to address block matrices or operator convexity/concavity, highlighting the need for more general frameworks.

Building on historical developments from Schmidt (1907) to modern works by Bhatia, Kittaneh, and others (1990–2025), the study addresses limitations in existing inequalities by proposing novel bounds for operator products and commutators. Key contributions include a conjecture for operators  $\mathcal{T}_1, \mathcal{T}_2, \mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{L}(\mathcal{H})$  with  $\mathcal{S}_1$  compact, providing bounds for the singular values of  $\mathcal{T}_1\mathcal{S}_1\mathcal{T}_2 + \mathcal{S}_2$ , extended to block matrices, and another for expressions like  $\mathcal{T}_1\mathcal{S}_1\mathcal{T}_2 + \mathcal{T}_2\mathcal{S}_2\mathcal{T}_1 + \mathcal{S}_1 + \mathcal{S}_2$ , incorporating operator convexity and concavity. New upper and lower bounds for generalized commutators and anti-commutators are derived without positivity assumptions, addressing literature gaps. This thesis addresses these challenges by proposing novel singular value inequalities that overcome existing limitations. The practical implications of these contributions are far-reaching. By relaxing restrictive assumptions, the new inequalities enable more accurate analysis of quantum systems, where operators often lack positivity. In data science, they facilitate improved matrix decomposition techniques, crucial for handling large datasets. In engineering, they enhance the reliability of simulations by providing tighter bounds on operator behavior. These advancements not only strengthen the theoretical framework of operator theory but also bridge the gap between mathematical tools and real-world applications.

This thesis focuses on advancing singular value inequalities by exploring generalized commutators, anti-commutators, composite operator forms such as  $\mathcal{T}_1\mathcal{S}_1\mathcal{T}_2 + \mathcal{S}_2$ , and operator-valued functional expressions. The goal is to strengthen existing bounds, extend known results, and resolve open problems that hold relevance across multiple disciplines.

In Chapter 1, we cover the foundational definitions, essential results, and an in-depth literature review. This chapter establishes the theoretical framework by introducing bounded linear operators on Hilbert spaces, their norms, singular values, eigenvalues, and notations for bounded operators and unitarily invariant norms. We trace the evolution of singular value inequalities from the early 20th century through contributions by key researchers, spanning from 1907 to 1967, and extend the review to significant advancements from 1990 to 2025. We critically analyze these developments, identifying gaps such as the lack of lower bounds for commutators, limited exploration of block matrices, and the absence of operator convexity/concavity considerations, setting the stage for the study's objectives and original contributions in the following chapters.

In Chapter 2, we analyze the spectral behavior of singular value inequalities for expressions of the form  $\mathcal{T}_1\mathcal{S}_1 \pm \mathcal{S}_2\mathcal{T}_2$  within bounded linear operators. We extend existing concepts to include  $m$ -commutator and anti-commutator block frameworks, leveraging unitary equivalence

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and constraints on self-adjoint operators. The introduction of a non-negative increasing concave function broadens the applicability to logarithmic and power-based transformations, with examples and determinant analyses clarifying invertibility conditions.

In Chapter 3, we delve deeper into singular value inequalities for  $\mathcal{T}_1\mathcal{S}_1\mathcal{T}_2 + \mathcal{S}_2$  with one compact operator, proving a conjecture that provides a tighter and more general bound. We extend this framework to multi-block operators and contractions, applying functional transformations to adapt the inequality to various unitarily invariant norms. The analysis enriches spectral properties with new insights into operator algebras, validated through examples and remarks that demonstrate the sharpness of the bounds, concluding with potential extensions to non-compact operators.

In Chapter 4, we examine the derivation of novel upper and lower bounds for generalized commutators and anti-commutators, removing restrictive positivity assumptions. We develop a general framework with essential adjustment terms, validated by counterexamples, and extend the results to anti-commutators and operator sums for refined bounds on real and imaginary components.

In Chapter 5, we explore enhancements to singular value inequalities by integrating operator convexity and concavity, proving a conjecture for  $\mathcal{T}_1\mathcal{S}_1\mathcal{T}_2 + \mathcal{T}_2\mathcal{S}_2\mathcal{T}_1 + \mathcal{S}_1 + \mathcal{S}_2$ . We derive refined bounds using functional analysis, with applications to logarithmic and power transformations, and extend the framework for normal operators through case studies in data compression and quantum control. The validation of bound sharpness through numerical examples concludes with proposals for future research into operator-valued functions and higher-dimensional systems.

A list of references is provided at the end of the thesis, which, while not exhaustive, includes all the research papers and books cited throughout the main content.

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# Acknowledgment

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I am profoundly indebted to my parents and in-laws, whose unconditional love and selfless

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sacrifices have been the bedrock of my achievements. My parents nurtured my inquisitive spirit from childhood, their endless support and prayers fueling my academic pursuits. Their collective belief in my potential has been my guiding force, and I dedicate this milestone to them with all my heart.

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**March 2, 2026**

**MANINDERJIT KAUR**

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# List of Symbols and Abbreviations

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The following abbreviations and notations are used throughout this thesis, particularly in the context of operator theory and singular value inequalities.

| Abbreviations                                                                                                                                                                          | Meaning                                                                               |
|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------|
| $\mathbb{C}$                                                                                                                                                                           | The set of complex numbers.                                                           |
| $\mathbb{C}^k$                                                                                                                                                                         | The $k$ -dimensional complex Euclidean space, a finite-dimensional Hilbert space.     |
| $\mathbb{N}$                                                                                                                                                                           | The set of natural numbers, $\{1, 2, 3, \dots\}$ .                                    |
| $\mathbb{R}$                                                                                                                                                                           | The set of real numbers.                                                              |
| $\mathcal{H}$                                                                                                                                                                          | Complex Hilbert space, finite or infinite-dimensional                                 |
| $\mathfrak{L}(\mathcal{H})$                                                                                                                                                            | The set of bounded linear operators on a Hilbert space $\mathcal{H}$ .                |
| $\mathfrak{C}(\mathcal{H})$                                                                                                                                                            | The set of compact operators on a Hilbert space $\mathcal{H}$ .                       |
| $\mathcal{P}(\mathcal{H})$                                                                                                                                                             | The set of positive operators on a Hilbert space $\mathcal{H}$ .                      |
| $\mathcal{N}(\mathcal{H})$                                                                                                                                                             | The set of negative operators on a Hilbert space $\mathcal{H}$ .                      |
| $\mathcal{SP}(\mathcal{H})$                                                                                                                                                            | The set of strictly-positive operators on a Hilbert space $\mathcal{H}$ .             |
| $\mathcal{SN}(\mathcal{H})$                                                                                                                                                            | The set of strictly-negative operators on a Hilbert space $\mathcal{H}$ .             |
| $\mathfrak{M}[k]$                                                                                                                                                                      | The set of $k \times k$ matrices with complex entries                                 |
| $\mathcal{AGMI}$                                                                                                                                                                       | Arithmetic-Geometric Mean Inequality, a key inequality in operator theory.            |
| $\langle \cdot, \cdot \rangle$                                                                                                                                                         | Inner product on a Hilbert space $\mathcal{H}$                                        |
| $\mathcal{SVD}$                                                                                                                                                                        | Singular Value Decomposition                                                          |
| $\mathcal{PD}$                                                                                                                                                                         | Polar Decomposition                                                                   |
| $\mathcal{T}_1, \mathcal{T}_2, \mathcal{S}_1, \mathcal{S}_2, \mathcal{P}, \mathcal{Q}, \mathcal{R},$<br>$\mathcal{L}, \mathcal{C}, \mathcal{D}, \mathcal{M}, \mathcal{N}, \mathcal{M}$ | Interchangeable symbols for bounded linear operators in $\mathfrak{L}(\mathcal{H})$ . |
| $\mathcal{A}, \mathcal{B}$                                                                                                                                                             | Interchangeable symbols for matrices in $\mathfrak{M}[k]$                             |
| $\mathcal{U}_1, \mathcal{U}_2$                                                                                                                                                         | Unitary operators in $\mathfrak{L}(\mathcal{H})$                                      |
| $\mathcal{D}$                                                                                                                                                                          | Diagonal operator in $\mathfrak{L}(\mathcal{H})$                                      |
| $\mathcal{I}$                                                                                                                                                                          | Identity operator in $\mathfrak{L}(\mathcal{H})$                                      |

|                                    |                                                                                                                                                         |
|------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------|
| $\mathcal{O}$                      | Zero operator in $\mathfrak{L}(\mathcal{H})$                                                                                                            |
| $\mathcal{T}_1^*$                  | Adjoint of operator $\mathcal{T}_1$                                                                                                                     |
| $\mathcal{T}_1'$                   | Transpose of operator $\mathcal{T}_1$                                                                                                                   |
| $ \mathcal{T}_1 $                  | Positive square root of $\mathcal{T}_1^* \mathcal{T}_1$ , i.e., $(\mathcal{T}_1^* \mathcal{T}_1)^{1/2}$                                                 |
| $\oplus$                           | Direct sum of operators on Hilbert spaces                                                                                                               |
| $\Re(\mathcal{T}_1)$               | Real part of operator $\mathcal{T}_1$ , defined as $\frac{1}{2}(\mathcal{T}_1 + \mathcal{T}_1^*)$                                                       |
| $\Im(\mathcal{T}_1)$               | Imaginary part of operator $\mathcal{T}_1$ , defined as $\frac{1}{2i}(\mathcal{T}_1 - \mathcal{T}_1^*)$                                                 |
| $[\mathcal{T}_1, \mathcal{T}_2]$   | Commutator of operators $\mathcal{T}_1$ and $\mathcal{T}_2$ , defined as $\mathcal{T}_1 \mathcal{T}_2 - \mathcal{T}_2 \mathcal{T}_1$                    |
| $\{\mathcal{T}_1, \mathcal{T}_2\}$ | Anti-commutator of operators $\mathcal{T}_1$ and $\mathcal{T}_2$ , defined as $\mathcal{T}_1 \mathcal{T}_2 + \mathcal{T}_2 \mathcal{T}_1$               |
| $\ \mathcal{T}_1\ $                | Spectral norm of operator $\mathcal{T}_1$ , equal to the largest singular value                                                                         |
| $\ \mathcal{T}_1\ _{\mathcal{P}}$  | Schatten $\mathcal{P}$ -norm of operator $\mathcal{T}_1$ , defined as $\left(\sum_{i=1}^k \sigma_i^{\mathcal{P}}(\mathcal{T}_1)\right)^{1/\mathcal{P}}$ |
| $\ \mathcal{T}_1\ $                | Unitarily invariant norm of operator $\mathcal{T}_1$                                                                                                    |
| $\sigma_j(\mathcal{T}_1)$          | $j$ -th singular value of operator $\mathcal{T}_1$ , in descending order                                                                                |
| $\lambda_j(\mathcal{T}_1)$         | $j$ -th eigen value of operator $\mathcal{T}_1$                                                                                                         |
| $\iota$                            | Imaginary unit, $\sqrt{-1}$                                                                                                                             |
| $\det(\mathcal{T}_1)$              | Determinant of operator $\mathcal{T}_1$ (in finite dimensions)                                                                                          |
| $\mathcal{T}_1 \geq \mathcal{T}_2$ | Positive, i.e., $\mathcal{T}_1 - \mathcal{T}_2$ is positive                                                                                             |

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# Chapter 1

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## Introduction

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### 1.1 Introduction

Operator theory, a central branch of functional analysis, has profoundly shaped modern mathematics through foundational work by Schmidt [57], and Von Neumann [52]. This field primarily investigates linear transformations on Hilbert spaces  $\mathcal{H}$ , and their finite-dimensional analogues represented as matrices in  $\mathfrak{M}[k]$ , the algebra of  $k \times k$  complex matrices. The study of bounded linear operators, denoted by  $\mathfrak{L}(\mathcal{H})$ , reveals a rich structure characterized by singular values, operator norms, and spectral properties, which are pivotal to both theoretical developments and practical applications.

A key component of this framework is the theory of singular value inequalities, which plays an instrumental role in operator analysis and matrix theory. These inequalities provide critical tools for bounding and comparing operator expressions and are widely applied in areas such as quantum mechanics, numerical computation, and data science. The origins of these ideas trace back to the early 20th century, beginning with Schmidt's work on integral equations [57], and have since evolved through significant contributions in matrix analysis and operator algebras.

This Chapter introduces the essential notations, definitions, motivations, and preliminary concepts that serve as a basis for the investigations carried out in the subsequent chapters. To provide a comprehensive perspective, this chapter also outlines the historical development of singular value inequalities from 1907 to 1967 and highlights important progress made between 1990 and 2025.

## 1.2 Notations

This thesis employs standard notations in operator theory, beginning with matrices and extending to operators. Unless otherwise stated,  $\mathcal{H}$  denotes a finite-dimensional Hilbert space, such as  $\mathbb{C}^k$  for some positive integer  $k$ , equipped with an inner product  $\langle \cdot, \cdot \rangle$  and the induced norm  $\| \cdot \|$ . The notation  $\mathfrak{M}[k]$  refers to the algebra of matrices of size  $k \times k$  with complex entries, where both addition and multiplication are defined. Throughout this chapter,  $\mathfrak{L}(\mathcal{H})$  denotes the algebra of bounded linear operators on  $\mathcal{H}$ . Operators are denoted by symbols such as  $\mathcal{T}_1, \mathcal{T}_2, \mathcal{S}_1, \mathcal{S}_2$ , with  $\mathcal{I}$  as the identity operator and  $\mathcal{O}$  as the zero operator, and the adjoint of  $\mathcal{T}_1 \in \mathfrak{L}(\mathcal{H})$  is  $\mathcal{T}_1^*$ . The usual operator norm

$$\|\mathcal{T}_2\| = \sup_{\|\mathfrak{x}\|=1} \|\mathcal{T}_2\mathfrak{x}\|, \quad \mathcal{T}_2 \in \mathfrak{L}(\mathcal{H}),$$

which coincides with the spectral norm  $\sigma_1(\mathcal{T}_2)$  in finite-dimensional fields.

The  $j$ -th singular value of  $\mathcal{T}_1 \in \mathfrak{L}(\mathcal{H})$ , denoted  $\sigma_j(\mathcal{T}_1)$ , is ordered as  $\sigma_1(\mathcal{T}_1) \geq \sigma_2(\mathcal{T}_1) \geq \dots \geq 0$  and defined as the square root of the  $j$ -th eigenvalue of  $\mathcal{T}_1^* \mathcal{T}_1$  in non-increasing order, equivalently expressed as

$$\sigma_j(\mathcal{T}_1) = \sup_{\dim \mathcal{V}=j} \inf_{\substack{\mathfrak{x} \in \mathcal{V} \\ \|\mathfrak{x}\|=1}} \|\mathcal{T}_1\mathfrak{x}\|,$$

where the supremum is taken over all  $j$ -dimensional subspaces  $\mathcal{V} \subset \mathcal{H}$ . The  $j$ -th eigenvalue of  $\mathcal{T}_1$ , when defined (typically for self-adjoint operators), is  $\lambda_j(\mathcal{T}_1)$ , ordered by magnitude. The real and imaginary components of  $\mathcal{T}_1$  are  $\Re(\mathcal{T}_1)$  and  $\Im(\mathcal{T}_1)$ , respectively. Unitarily invariant norms, denoted  $\|\cdot\|$ , encompass the spectral and Schatten norms, which remain invariant under unitary transformations. Let  $\mathcal{A} \in \mathfrak{M}[k]$ . Its transpose, written as  $\mathcal{A}'$ , is a simple flip of  $\mathcal{A}$  over its main diagonal. Thus, if  $\mathcal{A} = [a_{r\blacktriangleleft s}]$ , then transpose is given by  $\mathcal{A}' = [a_{s\blacktriangleleft r}]$  where  $1 \leq r, s \leq k$ .

Furthermore, when a real-valued function  $f : \mathcal{F} \rightarrow \mathbb{R}$  is utilized on a self-adjoint operator  $\mathcal{T}_1$ , resulting in  $f(\mathcal{T}_1)$  is interpreted through the functional calculus framework. These notations collectively facilitate the formulation and analysis of singular value inequalities presented in this thesis.

## 1.3 Basic Definitions

**Definition 1.3.1** (Inner Product).  $\mathcal{H}$  is equipped with an inner product  $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$ . It takes two elements from  $\mathcal{H}$  and returns a complex number. This function must satisfy a collection of axioms, including linearity, conjugate symmetry, and positivity, for all vectors  $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{H}$  and scalar  $\vartheta \in \mathbb{C}$ :

1. **Conjugate Symmetry:** The relation  $\langle \mathfrak{x}, \mathfrak{y} \rangle = \overline{\langle \mathfrak{y}, \mathfrak{x} \rangle}$  holds true.

2. **Linearity in the First Argument:** The expression  $\langle \vartheta \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \vartheta \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$ .
3. **Conjugate Linearity in the Second Argument:** The equation  $\langle \mathbf{x}, \vartheta \mathbf{y} + \mathbf{z} \rangle = \overline{\vartheta} \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle$  is satisfied.
4. **Positive Definiteness:** The value  $\langle \mathbf{x}, \mathbf{x} \rangle$  is non-negative, and  $\langle \mathbf{x}, \mathbf{x} \rangle = 0$  if and only if  $\mathbf{x} = 0$ .

By convention, the inner product is taken to be linear in its first variable and conjugate linear in the second. The norm induced by this inner product is given by  $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ .

**Example 1.3.1.** Consider the space  $\mathcal{H} = \mathbb{C}^2$  equipped with the canonical basis  $\{e_1, e_2\}$ . The inner product of two vectors  $\mathbf{x} = (x_1, x_2)$  and  $\mathbf{y} = (y_1, y_2)$  is given by

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 \overline{y_1} + x_2 \overline{y_2}.$$

For  $\mathbf{x} = (1, 2i)$  and  $\mathbf{y} = (2, -1)$ ,  $\langle \mathbf{x}, \mathbf{y} \rangle = 1 \cdot \overline{2} + 2i \cdot \overline{-1} = 2 + 2i \cdot (-1) = 2 - 2i$ , and  $\|\mathbf{x}\| = \sqrt{|1|^2 + |2i|^2} = \sqrt{1 + 4} = \sqrt{5}$ .

**Definition 1.3.2** (Linear Operator). A map  $\mathcal{T}_2 : \mathcal{H} \rightarrow \mathcal{H}$  is called linear if it is additive and homogeneous over  $\mathbb{C}$ , i.e.,

1. **Additivity:**  $\mathcal{T}_2(\mathbf{x} + \mathbf{y}) = \mathcal{T}_2\mathbf{x} + \mathcal{T}_2\mathbf{y}$ ,
2. **Homogeneity:**  $\mathcal{T}_2(\vartheta \mathbf{x}) = \vartheta \mathcal{T}_2\mathbf{x}$ ,

or all  $\mathbf{x}, \mathbf{y} \in \mathcal{H}$  and all scalar  $\vartheta \in \mathbb{C}$ .

**Definition 1.3.3** (Bounded Linear Operator). A linear operator  $\mathcal{T}_2 : \mathcal{H} \rightarrow \mathcal{H}$  is bounded if there exists  $M \geq 0$  such that:

$$\|\mathcal{T}_2\mathbf{x}\| \leq M\|\mathbf{x}\| \quad \text{for all } \mathbf{x} \in \mathcal{H}.$$

The operator norm is defined as:

$$\|\mathcal{T}_2\| = \sup_{\|\mathbf{x}\|=1} \|\mathcal{T}_2\mathbf{x}\| = \sup_{\mathbf{x} \neq 0} \frac{\|\mathcal{T}_2\mathbf{x}\|}{\|\mathbf{x}\|}.$$

**Proposition 1.3.1.** [14] The operator norm on  $\mathfrak{L}(\mathcal{H})$  satisfies:

1. **Non-negativity:**  $\|\mathcal{T}_2\| \geq 0$ , with  $\|\mathcal{T}_2\| = 0$  if and only if  $\mathcal{T}_2 = \mathcal{O}$ .
2. **Homogeneity:**  $\|\vartheta \mathcal{T}_2\| = |\vartheta| \|\mathcal{T}_2\|$  for any scalar  $\vartheta$ .
3. **Triangle Inequality:**  $\|\mathcal{T}_1 + \mathcal{T}_2\| \leq \|\mathcal{T}_1\| + \|\mathcal{T}_2\|$ .

- Proof.** 1. For any  $\mathcal{T}_2 \in \mathfrak{L}(\mathcal{H})$ ,  $\|\mathcal{T}_2\| = \sup_{\|\mathbf{x}\|=1} \|\mathcal{T}_2\mathbf{x}\| \geq 0$ , as norms are non-negative. If  $\|\mathcal{T}_2\| = 0$ , then  $\|\mathcal{T}_2\mathbf{x}\| = 0$  for all  $\mathbf{x}$  with  $\|\mathbf{x}\| = 1$ , implying  $\mathcal{T}_2\mathbf{x} = 0$  for all  $\mathbf{x}$  (by linearity), so  $\mathcal{T}_2 = \mathcal{O}$ . Conversely, if  $\mathcal{T}_2 = \mathcal{O}$ , then  $\|\mathcal{T}_2\mathbf{x}\| = 0$ .
2. For any scalar  $\vartheta$ ,  $\|\vartheta\mathcal{T}_2\| = \sup_{\|\mathbf{x}\|=1} \|\vartheta\mathcal{T}_2\mathbf{x}\| = \sup_{\|\mathbf{x}\|=1} |\vartheta| \|\mathcal{T}_2\mathbf{x}\| = |\vartheta| \sup_{\|\mathbf{x}\|=1} \|\mathcal{T}_2\mathbf{x}\| = |\vartheta| \|\mathcal{T}_2\|$ .
3. For any  $\mathcal{T}_1, \mathcal{T}_2 \in \mathfrak{L}(\mathcal{H})$  and using the triangle inequality for the vector norm, consider:

$$\begin{aligned} \|\mathcal{T}_1 + \mathcal{T}_2\| &= \sup_{\|\mathbf{x}\|=1} \|(\mathcal{T}_1 + \mathcal{T}_2)\mathbf{x}\| = \sup_{\|\mathbf{x}\|=1} \|\mathcal{T}_1\mathbf{x} + \mathcal{T}_2\mathbf{x}\| \\ &\leq \sup_{\|\mathbf{x}\|=1} (\|\mathcal{T}_1\mathbf{x}\| + \|\mathcal{T}_2\mathbf{x}\|) \\ &\leq \sup_{\|\mathbf{x}\|=1} \|\mathcal{T}_1\mathbf{x}\| + \sup_{\|\mathbf{x}\|=1} \|\mathcal{T}_2\mathbf{x}\| = \|\mathcal{T}_1\| + \|\mathcal{T}_2\|, \end{aligned}$$

□

**Definition 1.3.4** (Identity and Zero Operators). The identity operator  $\mathcal{I} \in \mathfrak{L}(\mathcal{H})$  satisfies  $\mathcal{I}\mathbf{x} = \mathbf{x}$  for all  $\mathbf{x} \in \mathcal{H}$ , while the zero operator  $\mathcal{O}$  satisfies  $\mathcal{O}\mathbf{x} = 0$  for all  $\mathbf{x} \in \mathcal{H}$ .

**Example 1.3.2.** For  $\mathcal{H} = \mathbb{C}^2$ , the identity operator  $\mathcal{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  preserves any vector  $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$  as  $\mathcal{I}\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$ , whereas the zero operator  $\mathcal{O} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$  transforms any vector  $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$  into  $\mathcal{O}\mathbf{x} = (0, 0)$ .

## 1.4 Operator Representations and Fundamental Operations

In finite-dimensional Hilbert spaces, operators are represented by  $k \times k$  complex matrices, simplifying algebraic manipulations. The determinant of an operator  $\mathcal{T}_1 \in \mathfrak{L}(\mathcal{H})$ , denoted  $\det(\mathcal{T}_1)$ , corresponds to the determinant of its matrix, and  $\mathcal{T}_1$  is invertible if  $\det(\mathcal{T}_1) \neq 0$ , ensuring a one-to-one and onto mapping, see [14, 42].

### 1.4.1 One-to-One Correspondence with Matrices

For a Hilbert space  $\mathcal{H}$  of finite dimension  $k$ , which is isometrically isomorphic to  $\mathbb{C}^k$  every bounded linear operator  $\mathcal{T}_2 \in \mathfrak{L}(\mathcal{H})$  can be uniquely represented by a complex matrix of order  $k$ . Let  $\{e_1, e_2, \dots, e_k\}$  be a fixed orthonormal basis for  $\mathcal{H}$ . The action of  $\mathcal{T}_2$  on the basis vectors is prescribed by

$$\mathcal{T}_2 e_s = \sum_{r=1}^k a_{r\blacktriangleleft s} e_r \quad (s = 1, 2, \dots, k),$$

where the complex coefficients  $a_{r\blacktriangle s} \in \mathbb{C}$  are the entries of an associated  $k \times k$  matrix

$$\mathcal{A} = a_{r\blacktriangle s} \quad \text{for} \quad 1 \leq r, s \leq k.$$

Now consider an arbitrary vector  $\mathbf{x}$  in the span of the basis, expressed in coordinates as

$$\mathbf{x} = \sum_{s=1}^k \mathbf{x}_s e_s,$$

where  $\mathbf{x}_s \in \mathbb{C}$  are the components of  $\mathbf{x}$  with respect to  $\{e_s\}_{s=1}^k$ . We write the coordinate vector compactly as

$$\mathbf{x} := \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_k \end{bmatrix} \in \mathbb{C}^k.$$

By linearity of  $\mathcal{T}_2$ ,

$$\begin{aligned} \mathcal{T}_2 \mathbf{x} &= \mathcal{T}_2 \left( \sum_{s=1}^k \mathbf{x}_s e_s \right) \\ &= \sum_{s=1}^k \mathbf{x}_s \mathcal{T}_2 e_s \\ &= \sum_{s=1}^k \mathbf{x}_s \left( \sum_{r=1}^k a_{r\blacktriangle s} e_r \right) \\ &= \sum_{r=1}^k \left( \sum_{s=1}^k a_{r\blacktriangle s} \mathbf{x}_s \right) e_r. \end{aligned}$$

The coefficient of  $e_r$  in the final expansion is precisely the  $r$ -th entry of the ordinary matrix-vector product  $\mathcal{A} \mathbf{x}$ :

$$\sum_{s=1}^k a_{r\blacktriangle s} \mathbf{x}_s = (\mathcal{A} \mathbf{x})_r.$$

Consequently,

$$\mathcal{T}_2 \mathbf{x} = \sum_{r=1}^k (\mathcal{A} \mathbf{x})_r e_r = \mathcal{A} \mathbf{x},$$

where the right-hand side is interpreted as the vector whose coordinate representation with respect to the basis  $\{e_r\}$  is given by the usual multiplication of the matrix  $\mathcal{A}$  by the column vector  $\mathbf{x}$ .

The mapping is injective: if  $\mathcal{T}_{21}$  and  $\mathcal{T}_{22}$  yield the same matrix, then  $\mathcal{T}_{21}e_s = \mathcal{T}_{22}e_s$  for all  $s$ , so  $\mathcal{T}_{21} = \mathcal{T}_{22}$  by linearity and spanning of the basis. It is surjective: any matrix  $\mathcal{A} \in \mathbb{C}^{k \times k}$  defines an operator  $\mathcal{T}_2$  via  $\mathcal{T}_2e_s = \sum_{r=1}^k a_{r\blacktriangleleft s}e_r$ . Changing bases via a unitary matrix  $\mathcal{U}_1$  transforms the matrix as  $\mathcal{A} \rightarrow \mathcal{U}_1^* \mathcal{A} \mathcal{U}_1$ , preserving operator properties like singular values and norms.

**Example 1.4.1.** Consider  $\mathcal{H} = \mathbb{C}^2$  with the standard basis  $\{e_1 = [1, 0]', e_2 = [0, 1]'\}$ . Define the linear operator  $\mathcal{T}_2$  such that  $\mathcal{T}_2e_1 = 2e_1 + e_2$  and  $\mathcal{T}_2e_2 = e_1$ .  $\mathcal{A}$  is the matrix representation of  $\mathcal{T}_2$  relative to the chosen basis:

$$\mathcal{A} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix},$$

since  $\mathcal{T}_2e_1 = [2, 1]'$  and  $\mathcal{T}_2e_2 = [1, 0]'$ . For  $\mathbf{x} = \mathbf{x}_1e_1 + \mathbf{x}_2e_2$ , compute:

$$\mathcal{T}_2\mathbf{x} = \mathbf{x}_1(2e_1 + e_2) + \mathbf{x}_2e_1 = (2\mathbf{x}_1 + \mathbf{x}_2)e_1 + \mathbf{x}_1e_2,$$

matching  $\mathcal{A}\mathbf{x} = \begin{bmatrix} 2\mathbf{x}_1 + \mathbf{x}_2 \\ \mathbf{x}_1 \end{bmatrix}$ . Every  $2 \times 2$  matrix corresponds to a unique operator, confirming the one-to-one mapping.

## 1.4.2 Foundational Concepts in Operator Theory

In finite-dimensional Hilbert spaces, matrices and bounded linear operators are in one-to-one correspondence. This allows matrix-based concepts to be naturally expressed using operators. As this thesis focuses on singular value inequalities and operator expressions, it is more consistent to use operator-based definitions. Hence, all definitions will pertain to operators unless otherwise specified.

**Definition 1.4.1** (Transpose Operator). Consider a bounded linear operator  $\mathcal{T}_1 \in \mathfrak{L}(\mathcal{H})$  defined on  $\mathcal{H}$ . Once an orthonormal basis is chosen for  $\mathcal{H}$ , the operator  $\mathcal{T}_1$  can be expressed in matrix form by  $\mathcal{A}$ . The transpose operator  $\mathcal{T}_1'$  is defined such that its matrix representation is  $\mathcal{A}'$ , where the entry  $a'_{r\blacktriangleleft s} = a_{s\blacktriangleleft r}$  for all indices  $r, s$  (where  $1 \leq r, s \leq k$ ).

**Definition 1.4.2** (Adjoint Operator). Let  $\mathcal{T}_1 \in \mathfrak{L}(\mathcal{H})$ . The *adjoint* of  $\mathcal{T}_1$ , denoted  $\mathcal{T}_1^*$ , refers to the unique operator whose action satisfies the defining property of an adjoint operator, and when expressed as a matrix  $\mathcal{A}$  with respect to an orthonormal basis, corresponds to the conjugate transpose matrix  $\mathcal{A}^* = (\overline{\mathcal{A}})'$ , where  $\overline{\mathcal{A}}$  is the matrix with each entry  $a_{r\blacktriangleleft s}$  replaced by its complex conjugate  $\overline{a_{r\blacktriangleleft s}}$  and  $\mathcal{A}' = a_{s\blacktriangleleft r}$  reflects the transpose operation.

**Example 1.4.2.** Consider the operator  $\mathcal{T}_1 \in \mathfrak{L}(\mathcal{H})$  represented by the matrix

$$\mathcal{A} = \begin{bmatrix} 1 & 2i \\ 0 & 3 \end{bmatrix}.$$

The transpose of  $\mathcal{A}$  is

$$\mathcal{A}' = \begin{bmatrix} 1 & 0 \\ 2\iota & 3 \end{bmatrix},$$

while the adjoint (conjugate transpose) is

$$\mathcal{A}^* = (\overline{\mathcal{A}})' = \begin{bmatrix} 1 & 0 \\ -2\iota & 3 \end{bmatrix},$$

obtained by first taking the transpose and then applying complex conjugation to each entry.

**Proposition 1.4.1.** [14] *For any  $\mathcal{T}_1 \in \mathfrak{L}(\mathcal{H})$ , the adjoint satisfies:*

1.  $(\mathcal{T}_1^*)^* = \mathcal{T}_1$ .
2.  $\|\mathcal{T}_1^*\| = \|\mathcal{T}_1\|$ .

**Proof.** 1. For any  $\mathbf{x}, \mathbf{y} \in \mathcal{H}$ , we have:

$$\langle (\mathcal{T}_1^*)^* \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathcal{T}_1^* \mathbf{y} \rangle = \overline{\langle \mathcal{T}_1^* \mathbf{y}, \mathbf{x} \rangle} = \overline{\langle \mathbf{y}, \mathcal{T}_1 \mathbf{x} \rangle} = \langle \mathcal{T}_1 \mathbf{x}, \mathbf{y} \rangle.$$

which directly implies,  $(\mathcal{T}_1^*)^* = \mathcal{T}_1$ .

2. Next,

$$\|\mathcal{T}_1^*\| = \sup_{\|\mathbf{x}\|=1} \|\mathcal{T}_1^* \mathbf{x}\| = \sup_{\|\mathbf{x}\|=1} \sup_{\|\mathbf{y}\|=1} |\langle \mathcal{T}_1^* \mathbf{x}, \mathbf{y} \rangle| = \sup_{\|\mathbf{x}\|=1} \sup_{\|\mathbf{y}\|=1} |\langle \mathbf{x}, \mathcal{T}_1 \mathbf{y} \rangle| = \sup_{\|\mathbf{y}\|=1} \|\mathcal{T}_1 \mathbf{y}\| = \|\mathcal{T}_1\|.$$

Therefore, the operator norm of  $\mathcal{T}_1^*$  is equal to that of  $\mathcal{T}_1$ , confirming the norm-preserving property of the adjoint operator.

□

**Definition 1.4.3** (Eigenvalues). For an operator  $\mathcal{T}_1 \in \mathfrak{L}(\mathcal{H})$  with a corresponding square matrix  $\mathcal{A} \in \mathfrak{M}[k]$ . A complex number  $\lambda \in \mathbb{C}$  is called an eigenvalue of a matrix (or operator) when there exists a non-zero vector  $\mathbf{x} \in \mathbb{C}^k$ , known as its corresponding eigenvector, such that:

$$\mathcal{T}_1 \mathbf{x} = \lambda \mathbf{x},$$

or equivalently,  $\mathcal{A} \mathbf{x} = \lambda \mathbf{x}$ . Eigenvalues of a square matrix  $\mathcal{A}$  are the complex numbers  $\lambda$  for which the characteristic equation vanishes:

$$\det(\mathcal{A} - \lambda \mathcal{I}) = 0.$$

**Example 1.4.3.** Let  $\mathcal{T}_1 \in \mathfrak{L}(\mathcal{H})$  be an operator with its corresponding matrix  $\mathcal{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \in$

$\mathfrak{M}[2]$  in the standard orthonormal basis. The characteristic equation is computed as:

$$\det(\mathcal{A} - \lambda \mathcal{J}) = \det \begin{bmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{bmatrix} = (2 - \lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 = 0,$$

with roots  $\lambda = 1$  and  $\lambda = 3$ . Thus, the eigenvalues of  $\mathcal{T}_1$  are  $\lambda_1 = 3$  and  $\lambda_2 = 1$ .

Moreover, specific classes of operators, defined below, exhibit unique structural traits that underscore their importance in both theoretical frameworks and practical applications.

**Definition 1.4.4** (Self-Adjoint Operator). An operator  $\mathcal{T}_1 \in \mathfrak{L}(\mathcal{H})$  is called *self-adjoint* when it coincides with its adjoint, that is,  $\mathcal{T}_1^* = \mathcal{T}_1$ . In finite dimensions, such operators are characterized by having all their eigenvalues real.

**Example 1.4.4.** Consider the matrix  $\mathcal{A} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ . This matrix is self-adjoint, as it equals its conjugate transpose  $\mathcal{A}^* = \overline{\mathcal{A}'} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ , noting that all entries are real, so  $\overline{\mathcal{A}'} = \mathcal{A}$ . To confirm its spectral properties, we compute the eigenvalues by solving the characteristic equation  $\det(\mathcal{A} - \lambda \mathcal{J}) = 0$ , where  $\mathcal{J} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  is the identity matrix. The determinant is:

$$\det \begin{bmatrix} 2 - \lambda & 1 \\ 1 & 3 - \lambda \end{bmatrix} = (2 - \lambda)(3 - \lambda) - (1)(1) = 6 - 2\lambda - 3\lambda + \lambda^2 - 1 = \lambda^2 - 5\lambda + 5.$$

Setting the characteristic polynomial to zero,  $\lambda^2 - 5\lambda + 5 = 0$ , and using quadratic formula:

$$\lambda = \frac{5 \pm \sqrt{25 - 20}}{2} = \frac{5 \pm \sqrt{5}}{2}.$$

Thus, the eigenvalues are  $\lambda_1 = \frac{5 + \sqrt{5}}{2}$  and  $\lambda_2 = \frac{5 - \sqrt{5}}{2}$ , both real, consistent with the self-adjoint property.

**Definition 1.4.5** (Normal Operator). An operator  $\mathcal{T}_1 \in \mathfrak{L}(\mathcal{H})$  is said to be *normal* if it commutes with its adjoint, that is,  $\mathcal{T}_1 \mathcal{T}_1^* = \mathcal{T}_1^* \mathcal{T}_1$ .

**Example 1.4.5.** Consider the matrix  $\mathcal{T}_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ . This matrix is normal because it satisfies the condition  $\mathcal{T}_1 \mathcal{T}_1^* = \mathcal{T}_1^* \mathcal{T}_1$ , where  $\mathcal{T}_1^* = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ .

Then

$$\mathcal{T}_1 \mathcal{T}_1^* = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and,

$$\mathcal{T}_1^* \mathcal{T}_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus  $\mathcal{T}_1 \mathcal{T}_1^* = \mathcal{T}_1^* \mathcal{T}_1$ , confirming that  $\mathcal{T}_1$  is normal.

**Definition 1.4.6** (Unitary Operator). An operator  $\mathcal{U}_1 \in \mathfrak{L}(\mathcal{H})$  is called *unitary* if it satisfies  $\mathcal{U}_1^* \mathcal{U}_1 = \mathcal{U}_1 \mathcal{U}_1^* = \mathcal{I}$ . Such operators leave the inner product between vectors unchanged.

**Example 1.4.6.** Consider the matrix  $\mathcal{U}_1 = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ . To verify it is unitary, we compute the conjugate transpose  $\mathcal{U}_1^* = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$  (since entries are real). Then:

$$\mathcal{U}_1^* \mathcal{U}_1 = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathcal{I},$$

and similarly  $\mathcal{U}_1 \mathcal{U}_1^* = \mathcal{I}$ , using  $\cos^2 \theta + \sin^2 \theta = 1$ . Thus,  $\mathcal{U}_1$  is unitary.

**Definition 1.4.7** (Contractive Operator). An operator  $\mathcal{T}_1 \in \mathfrak{L}(\mathcal{H})$  is termed *contractive* when its operator norm satisfies  $\|\mathcal{T}_1\| \leq 1$ . The operator norm, which represents the maximum singular value of  $\mathcal{T}_1$ , will be discussed further in Section 1.7.

**Example 1.4.7.** The matrix  $\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$  is contractive because its operator norm is 0.5, which is less than 1.

**Definition 1.4.8** (Direct Sum). For operators  $\mathcal{T}_{11}, \mathcal{T}_{12}, \dots, \mathcal{T}_{1m} \in \mathfrak{L}(\mathcal{H})$ , the direct sum is defined as:

$$\bigoplus_{i=1}^m \mathcal{T}_{1i} = \begin{bmatrix} \mathcal{T}_{11} & \mathcal{O} & \cdots & \mathcal{O} \\ \mathcal{O} & \mathcal{T}_{12} & \cdots & \mathcal{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{O} & \mathcal{O} & \cdots & \mathcal{T}_{1m} \end{bmatrix},$$

which acts on the Hilbert space  $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$  consisting of  $m$  copies of  $\mathcal{H}$ . Thus,  $\bigoplus_{i=1}^m \mathcal{T}_{1i}$  represents a block-diagonal operator whose diagonal blocks are the operators  $\mathcal{T}_{1i}$  and all off-diagonal entries are the zero operator  $\mathcal{O}$ .

The operator norm associated with this block-diagonal operator, denoted by,  $\|\bigoplus_{i=1}^m \mathcal{T}_{1i}\|$  satisfies

$$\left\| \bigoplus_{i=1}^m \mathcal{T}_{1i} \right\| = \max_{1 \leq i \leq m} \|\mathcal{T}_{1i}\| = [\|\mathcal{T}_{11}\|, \|\mathcal{T}_{12}\|, \dots, \|\mathcal{T}_{1m}\|]_{\infty}.$$

### 1.4.3 Positive and Negative Operators

Positive operators are vital in areas such as optimization, quantum mechanics, and statistical modeling due to their consistent spectral characteristics.

**Definition 1.4.9.** (Positive and Strictly Positive Operators). A self-adjoint operator  $\mathcal{T}_1 \in \mathfrak{L}(\mathcal{H})$ , is considered **positive** if the inner product satisfies:

$$\langle \mathbf{x}, \mathcal{T}_1 \mathbf{x} \rangle \geq 0 \quad \text{for every } \mathbf{x} \in \mathcal{H}.$$

It is **strictly positive** if the inner product is strictly positive for all non-zero vectors,. This means that for every non-zero vector  $\mathbf{x} \in \mathcal{H}$ , the inner product  $\langle \mathbf{x}, \mathcal{T}_1 \mathbf{x} \rangle$  is strictly positive.

If  $\mathcal{T}_1$  is positive (or strictly positive), then  $-\mathcal{T}_1$  is negative (or strictly negative). The relation  $\mathcal{T}_1 \geq \mathcal{T}_2$  holds if  $\mathcal{T}_1 - \mathcal{T}_2$  is a positive operator.

We denote the set of positive operators in  $\mathfrak{L}(\mathcal{H})$  by  $\mathcal{P}(\mathcal{H})$ , the strictly positive operators by  $\mathcal{SP}(\mathcal{H})$ , the negative operators by  $\mathcal{N}(\mathcal{H})$ , and the strictly negative operators by  $\mathcal{SN}(\mathcal{H})$ .

**Example 1.4.8.** Consider the matrices  $\mathcal{T}_1 = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$  and  $\mathcal{T}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ .

First, for  $\mathcal{T}_1$ :  $\mathcal{T}_1 \mathbf{x} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} 4\mathbf{x}_1 \\ 2\mathbf{x}_2 \end{bmatrix}$  where  $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{C}^2$ . The inner product in this case is computed as

$$\langle \mathbf{x}, \mathcal{T}_1 \mathbf{x} \rangle = \mathbf{x}_1 \overline{(4\mathbf{x}_1)} + \mathbf{x}_2 \overline{(2\mathbf{x}_2)} = 4|\mathbf{x}_1|^2 + 2|\mathbf{x}_2|^2.$$

Since  $|\mathbf{x}_1|^2 \geq 0$  and  $|\mathbf{x}_2|^2 \geq 0$ , and at least one is positive for  $\mathbf{x} \neq 0$ ,  $4|\mathbf{x}_1|^2 + 2|\mathbf{x}_2|^2 > 0$  for any non-zero  $\mathbf{x}$ . Thus,  $\mathcal{T}_1$  is strictly positive.

Next, for  $\mathcal{T}_2$ :  $\mathcal{T}_2 \mathbf{x} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1 + \mathbf{x}_2 \\ \mathbf{x}_1 + \mathbf{x}_2 \end{bmatrix}$ . The corresponding inner product is:  $\langle \mathbf{x}, \mathcal{T}_2 \mathbf{x} \rangle = |\mathbf{x}_1 + \mathbf{x}_2|^2$ . Since  $|\mathbf{x}_1 + \mathbf{x}_2|^2 \geq 0$  for all  $\mathbf{x}$ ,  $\mathcal{T}_2$  is positive. However, it equals zero when  $\mathbf{x}_1 + \mathbf{x}_2 = 0$ , e.g., for  $\mathbf{x} = (1, -1) \neq 0$ , where  $\langle \mathbf{x}, \mathcal{T}_2 \mathbf{x} \rangle = |1 - 1|^2 = 0$ . Thus,  $\mathcal{T}_2$  is positive but not strictly positive.

**Remark 1.4.1.** For any bounded operator  $\mathcal{T}_1 \in \mathfrak{L}(\mathcal{H})$ , the operator  $\mathcal{T}_1^* \mathcal{T}_1$  is positive, as for each  $\mathbf{x} \in \mathcal{H}$ , we have

$$\langle \mathbf{x}, \mathcal{T}_1^* \mathcal{T}_1 \mathbf{x} \rangle = \|\mathcal{T}_1 \mathbf{x}\|^2 \geq 0.$$

The positive square root  $|\mathcal{T}_1| = (\mathcal{T}_1^* \mathcal{T}_1)^{1/2}$ , referred to as the modulus of  $\mathcal{T}_1$ , plays a fundamental role in various operator decompositions, which we will discuss in detail in the next section.

## 1.5 Singular Value and Polar Decomposition

Singular values extend the concept of eigenvalues to non-normal operators, a key concept in matrix analysis, see [14]. This section outlines definitions, examples, and key properties, including several propositions.

**Definition 1.5.1** (Singular Value Decomposition ( $\mathcal{SVD}$ )). For every operator  $\mathcal{T}_1 \in \mathfrak{L}(\mathcal{H})$ , let  $\sigma_1(\mathcal{T}_1) \geq \sigma_2(\mathcal{T}_1) \geq \cdots \geq \sigma_k(\mathcal{T}_1)$  be the singular values of  $\mathcal{T}_1$ . The  $\mathcal{SVD}$  provides a representation of an operator  $\mathcal{T}_1$  in the form  $\mathcal{T}_1 = \mathcal{U}_1 \Sigma \mathcal{U}_2^*$ . Here,  $\mathcal{U}_1$  and  $\mathcal{U}_2$  are unitary operators, and  $\Sigma$  is a diagonal matrix containing the singular values of  $\mathcal{T}_1$  along its main diagonal. Explicitly, the matrix  $\Sigma$  is represented as

$$\Sigma = \begin{bmatrix} t_1 & 0 & \cdots & 0 \\ 0 & t_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_k \end{bmatrix},$$

where the set of diagonal entries is  $\{t_j \mid j = 1, 2, \dots, k\}$ , and each  $t_j$  corresponds to the singular value  $\sigma_j(\mathcal{T}_1)$ . The columns of  $\mathcal{U}_1$  are the eigenvectors of  $\mathcal{T}_1 \mathcal{T}_1^*$ , and the columns of  $\mathcal{U}_2$  are the eigenvectors of  $\mathcal{T}_1^* \mathcal{T}_1$ .

**Definition 1.5.2** (Singular Values). For an operator  $\mathcal{T}_1 \in \mathfrak{L}(\mathcal{H})$ , the singular values  $\sigma_1(\mathcal{T}_1) \geq \sigma_2(\mathcal{T}_1) \geq \cdots \geq \sigma_k(\mathcal{T}_1) \geq 0$  are the eigenvalues of  $|\mathcal{T}_1| = (\mathcal{T}_1^* \mathcal{T}_1)^{1/2}$ , listed in descending order. For a self-adjoint operator  $\mathcal{T}_1$ , each singular value  $\sigma_j(\mathcal{T}_1)$  is equal to the absolute value of the corresponding eigenvalue  $\lambda_j(\mathcal{T}_1)$ . In other words, the singular values of  $\mathcal{T}_1$  coincide with the magnitudes of its eigenvalues.

**Example 1.5.1.** Consider the matrix  $\mathcal{T}_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ . The singular values of  $\mathcal{T}_1$  are the eigenvalues of  $|\mathcal{T}_1| = (\mathcal{T}_1^* \mathcal{T}_1)^{1/2}$ . We compute them step by step. Since the entries of  $\mathcal{T}_1$  are real,

$$\mathcal{T}_1^* = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}.$$

Next,

$$\mathcal{T}_1^* \mathcal{T}_1 = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix}.$$

Now, to find the eigenvalues of  $\mathcal{T}_1^* \mathcal{T}_1$ ,

$$\det(\mathcal{T}_1^* \mathcal{T}_1 - \lambda \mathcal{I}) = 0,$$

$$\det \begin{bmatrix} 10 - \lambda & 14 \\ 14 & 20 - \lambda \end{bmatrix} = (10 - \lambda)(20 - \lambda) - 14 \cdot 14 = \lambda^2 - 30\lambda + 4.$$

Using the quadratic formula: we get

$$\lambda = \frac{30 \pm \sqrt{30^2 - 4 \cdot 1 \cdot 4}}{2} = \frac{30 \pm \sqrt{900 - 16}}{2} = \frac{30 \pm \sqrt{884}}{2}.$$

Since  $\sqrt{884} \approx 29.732$ , the eigenvalues are approximately  $\lambda_1 \approx \frac{30 + 29.732}{2} \approx 29.866$  and  $\lambda_2 \approx \frac{30 - 29.732}{2} \approx 0.134$ .

The singular values  $\sigma_j$  are obtained as the positive square roots of the eigenvalues of  $\mathcal{T}_1^* \mathcal{T}_1$ , and they are conventionally arranged in non-increasing (descending) order.

$$\sigma_1 = \sqrt{\lambda_1} \approx \sqrt{29.866} \approx 5.466, \text{ and } \sigma_2 = \sqrt{\lambda_2} \approx \sqrt{0.134} \approx 0.366.$$

Thus, the singular values of  $\mathcal{T}_1$  are  $\sigma_1 \approx 5.466$  and  $\sigma_2 \approx 0.366$ .

**Example 1.5.2.** Now, consider a self-adjoint matrix  $\mathcal{T}_1 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ . Its eigenvalues are the diagonal entries  $\lambda_1 = 2$  and  $\lambda_2 = -1$ , found by solving  $\det(\mathcal{T}_1 - \lambda \mathcal{I}) = 0$ . Since the singular values correspond to the absolute values of the eigenvalues, we obtain  $\sigma_1 = |2| = 2$  and  $\sigma_2 = |-1| = 1$ .

**Definition 1.5.3** (Polar Decomposition ( $\mathcal{PD}$ )). Every operator  $\mathcal{T}_1 \in \mathfrak{L}(\mathcal{H})$  possesses a polar decomposition, expressed as  $\mathcal{T}_1 = \mathcal{U}_1 \mathcal{P}$ , where  $\mathcal{U}_1$  is a unitary operator and  $\mathcal{P} = |\mathcal{T}_1| \in \mathcal{P}(\mathcal{H})$ . The positive part  $\mathcal{P}$  is uniquely determined, while the unitary part  $\mathcal{U}_1$  is uniquely defined whenever  $\mathcal{T}_1$  is invertible.

**Example 1.5.3.** Consider the matrix  $\mathcal{T}_1 = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$ . We compute its polar decomposition

$\mathcal{T}_1 = \mathcal{U}_1 |\mathcal{T}_1|$ , where  $|\mathcal{T}_1| = (\mathcal{T}_1^* \mathcal{T}_1)^{1/2}$  is the modulus and  $\mathcal{U}_1$  is unitary. We have,  $\mathcal{T}_1^* = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$ .

Then,

$$\mathcal{T}_1^* \mathcal{T}_1 = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}.$$

The matrix  $\mathcal{T}_1^* \mathcal{T}_1$  is diagonal with entries 4 and 4, so its eigenvalues are both 4. The positive square root of these eigenvalues is 2, leading to:

$$|\mathcal{T}_1| = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Let's test  $\mathcal{U}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ :

$$\mathcal{U}_1 |\mathcal{T}_1| = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = \mathcal{T}_1.$$

This confirms  $\mathcal{U}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ , an unitary matrix exists.

**Remark 1.5.1.** Singular values are invariant under unitary transformations:  $\sigma_j(\mathcal{U}_1 \mathcal{T}_1) = \sigma_j(\mathcal{T}_1)$  for unitary  $\mathcal{U}_1$ , a property critical for applications like principal component analysis.

**Definition 1.5.4** (Jordan Decomposition). For a self-adjoint operator  $\mathcal{T}_1 \in \mathfrak{L}(\mathcal{H})$ , having eigenvalues  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l \geq \lambda_{l+1} \geq \dots \geq \lambda_k$ , where  $\lambda_l \geq 0 > \lambda_{l+1}$ . Consider  $\mathcal{U}_1$  a unitary operator such that  $\mathcal{T}_1 = \mathcal{U}_1 \mathcal{D} \mathcal{U}_1^*$ ,  $\mathcal{D}$  is a diagonal operator  $\mathcal{D} = \text{diag}(\lambda_1, \dots, \lambda_k)$ . Then

$$\mathcal{T}_1 = \mathcal{T}_1^+ - \mathcal{T}_1^-$$

is the Jordan decomposition of  $\mathcal{T}_1$ , both  $\mathcal{T}_1^+$  and  $\mathcal{T}_1^-$  are positive operators. Here,  $\mathcal{T}_1^+ = \mathcal{U}_1 \mathcal{D}^+ \mathcal{U}_1^*$  and  $\mathcal{T}_1^- = \mathcal{U}_1 \mathcal{D}^- \mathcal{U}_1^*$ , where the diagonal matrices  $\mathcal{D}^+$  and  $\mathcal{D}^-$  are defined as  $\mathcal{D}^+ = (\lambda_1, \dots, \lambda_l, 0, \dots, 0)$  and  $\mathcal{D}^- = (0, \dots, 0, -\lambda_{l+1}, \dots, -\lambda_k)$ , respectively.

**Lemma 1.5.1.** [34] Let  $\mathcal{T}_{11}$  and  $\mathcal{T}_{12} \in \mathcal{P}(\mathcal{H})$ , and let  $\mathcal{T}_1 = \mathcal{T}_{11} - \mathcal{T}_{12}$ . Let  $\mathcal{T}_1 = \mathcal{T}_{11}^+ - \mathcal{T}_{12}^-$  be the Jordan decomposition of self-adjoint operator  $\mathcal{T}_1$ . Then, for each  $j = 1, 2, \dots, k$ ,

$$\sigma_j(\mathcal{T}_1^+) \leq \sigma_j(\mathcal{T}_{11}) \quad \text{and} \quad \sigma_j(\mathcal{T}_1^-) \leq \sigma_j(\mathcal{T}_{12}).$$

Operators can be decomposed into self-adjoint components, analogous to complex numbers, facilitating analysis in quantum mechanics.

**Definition 1.5.5** (Real and Imaginary Parts). The operator  $\mathcal{T}_1$  can be decomposed into its real and imaginary components as follows:

$$\Re(\mathcal{T}_1) = \frac{1}{2}(\mathcal{T}_1 + \mathcal{T}_1^*), \quad \Im(\mathcal{T}_1) = \frac{1}{2\iota}(\mathcal{T}_1 - \mathcal{T}_1^*),$$

where  $\iota = \sqrt{-1}$  denotes the imaginary unit. Both  $\Re(\mathcal{T}_1)$  and  $\Im(\mathcal{T}_1)$  are self-adjoint.

**Example 1.5.4.** For  $\mathcal{T}_1 = \begin{bmatrix} 1 & \iota \\ 0 & 2 \end{bmatrix}$ , compute  $\mathcal{T}_1^* = \begin{bmatrix} 1 & 0 \\ -\iota & 2 \end{bmatrix}$ . Then:

$$\Re(\mathcal{T}_1) = \frac{1}{2} \left( \begin{bmatrix} 1 & \iota \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -\iota & 2 \end{bmatrix} \right) = \begin{bmatrix} 1 & \iota/2 \\ -\iota/2 & 2 \end{bmatrix}, \quad \Im(\mathcal{T}_1) = \frac{1}{2\iota} \left( \begin{bmatrix} 1 & \iota \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ -\iota & 2 \end{bmatrix} \right) = \begin{bmatrix} 0 & 1/2 \\ -1/2 & 0 \end{bmatrix}.$$

**Proposition 1.5.1.** An operator  $\mathcal{T}_1 \in \mathfrak{L}(\mathcal{H})$  is contractive if and only if the block operator

$$\begin{bmatrix} \mathcal{T}_1 & \mathcal{T}_1 \\ \mathcal{T}_1^* & \mathcal{T}_1 \end{bmatrix}$$

is positive.

**Proof.** To begin with, let us examine the simplest scenario where  $\mathcal{H}$  is one-dimensional. Then,

for a complex number  $t$ , the inequality  $|t| \leq 1$  holds if and only if the matrix  $\begin{bmatrix} 1 & t \\ \bar{t} & 1 \end{bmatrix}$  is positive. For spaces of higher dimensions, the analysis can be extended by employing  $\mathcal{SVD}$  as  $\mathcal{T}_1 = \mathcal{U}_1 \Sigma \mathcal{U}_2^*$ , where  $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_k)$ . Then:

$$\begin{bmatrix} \mathcal{J} & \mathcal{T}_1 \\ \mathcal{T}_1^* & \mathcal{J} \end{bmatrix} = \begin{bmatrix} \mathcal{U}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{U}_2^* \end{bmatrix} \begin{bmatrix} \mathcal{J} & \Sigma \\ \Sigma & \mathcal{J} \end{bmatrix} \begin{bmatrix} \mathcal{U}_1^* & \mathcal{O} \\ \mathcal{O} & \mathcal{U}_2 \end{bmatrix}.$$

This shows that the block matrix is unitarily equivalent to

$$\begin{bmatrix} \mathcal{J} & \Sigma \\ \Sigma & \mathcal{J} \end{bmatrix},$$

which can be decomposed as a direct sum:

$$\bigoplus_{j=1}^k \begin{bmatrix} 1 & \sigma_j \\ \sigma_j & 1 \end{bmatrix}.$$

Each  $2 \times 2$  block is positive if and only if  $\sigma_j \leq 1$ . Thus, the original matrix is positive if and only if all singular values  $\sigma_j \leq 1$ , i.e.,  $\|\mathcal{T}_1\| \leq 1$ .  $\square$

**Lemma 1.5.2.** Suppose  $\mathcal{T}_1 \in \mathcal{P}(\mathcal{H})$ . Then the block operator

$$\begin{bmatrix} \mathcal{T}_1 & \mathcal{T}_1 \\ \mathcal{T}_1 & \mathcal{T}_1 \end{bmatrix}$$

is also positive, and the converse is true as well.

**Proof.** Since  $\mathcal{T}_1 \geq 0$ , we can write  $\mathcal{T}_1 = \mathcal{T}_1^{1/2} \mathcal{T}_1^{1/2}$ . Then the block matrix becomes:

$$\begin{bmatrix} \mathcal{T}_1 & \mathcal{T}_1 \\ \mathcal{T}_1 & \mathcal{T}_1 \end{bmatrix} = \begin{bmatrix} \mathcal{T}_1^{1/2} & \mathcal{O} \\ \mathcal{T}_1^{1/2} & \mathcal{O} \end{bmatrix} \begin{bmatrix} \mathcal{T}_1^{1/2} & \mathcal{T}_1^{1/2} \\ \mathcal{O} & \mathcal{O} \end{bmatrix} = \begin{bmatrix} \mathcal{T}_1^{1/2} & \mathcal{O} \\ \mathcal{T}_1^{1/2} & \mathcal{O} \end{bmatrix} \begin{bmatrix} \mathcal{T}_1^{1/2} & \mathcal{O} \\ \mathcal{T}_1^{1/2} & \mathcal{O} \end{bmatrix}^* \geq \mathcal{O}.$$

$\square$

**Corollary 1.5.1.** Let  $\mathcal{T}_1 \in \mathfrak{L}(\mathcal{H})$ . Then the block matrix

$$\begin{bmatrix} |\mathcal{T}_1| & \mathcal{T}_1^* \\ \mathcal{T}_1 & |\mathcal{T}_1^*| \end{bmatrix}$$

is positive.

**Proof.** Consider the  $\mathcal{PD}$  of  $\mathcal{T}_1$  given by  $\mathcal{T}_1 = \mathcal{U}_1 \mathcal{P}$ , where  $\mathcal{U}_1$  is unitary and  $\mathcal{P} = |\mathcal{T}_1| \in \mathcal{P}(\mathcal{H})$ .

Then we can express:

$$\begin{bmatrix} |\mathcal{T}_1| & \mathcal{T}_1^* \\ \mathcal{T}_1 & |\mathcal{T}_1^*| \end{bmatrix} = \begin{bmatrix} \mathcal{P} & \mathcal{P}\mathcal{U}_1^* \\ \mathcal{U}_1\mathcal{P} & \mathcal{U}_1\mathcal{P}\mathcal{U}_1^* \end{bmatrix}.$$

This can be written as:

$$\begin{bmatrix} \mathcal{I} & \mathcal{O} \\ \mathcal{O} & \mathcal{U}_1 \end{bmatrix} \begin{bmatrix} \mathcal{P} & \mathcal{P} \\ \mathcal{P} & \mathcal{P} \end{bmatrix} \begin{bmatrix} \mathcal{I} & \mathcal{O} \\ \mathcal{O} & \mathcal{U}_1^* \end{bmatrix},$$

Since the middle block matrix is positive, and the above representation corresponds to a similarity transformation by a unitary block matrix, it follows from lemma 1.5.2 that the original block matrix is also positive.  $\square$

## 1.6 Commutators and Anti-Commutators

Commutators and anti-commutators encapsulate non-commutative and additive interactions, pivotal in quantum mechanics (e.g., uncertainty principles) and control theory.

**Definition 1.6.1** (Generalized Commutator and Anti-Commutator). For operators  $\mathcal{T}_1, \mathcal{T}_2, \mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{L}(\mathcal{H})$ , the generalized commutator is:

$$\mathcal{T}_1\mathcal{S}_1 - \mathcal{S}_2\mathcal{T}_2,$$

and the anti-commutator is:

$$\mathcal{T}_1\mathcal{S}_1 + \mathcal{S}_2\mathcal{T}_2.$$

These generalize the standard commutator  $[\mathcal{T}_1, \mathcal{T}_2] = \mathcal{T}_1\mathcal{T}_2 - \mathcal{T}_2\mathcal{T}_1$  and anti-commutator  $\{\mathcal{T}_1, \mathcal{T}_2\} = \mathcal{T}_1\mathcal{T}_2 + \mathcal{T}_2\mathcal{T}_1$ . The historical foundation of commutator theory originates from quantum physics. A notable early result was established by Wintner in 1947 [62], who formulated the commutation relation:

$$\mathcal{P}\mathcal{Q} - \mathcal{Q}\mathcal{P} = \frac{-\iota h}{2\pi}\mathcal{I},$$

where  $\mathcal{P}$  and  $\mathcal{Q}$  denote the momentum and position operators in quantum mechanics,  $h$  is Planck's constant,  $\iota$  is the imaginary unit, and  $\mathcal{I}$  is the identity operator on  $\mathcal{H}$ . Wintner observed that  $\mathcal{P}$  and  $\mathcal{Q}$  could not both be bounded operators—an insight foundational to quantum theory.

Following this, Wielandt [61] independently verified Wintner's result using a distinct method, offering a mathematically rigorous perspective. Later, Putnam [54] generalized Wintner's findings. He showed that in finite-dimensional spaces, the result holds without additional assumptions, while in infinite-dimensional settings, at least one of the involved operators must be self-adjoint for the commutation relation to hold.

These foundational results have since inspired extensive research into the structure of commutators and anti-commutators, especially their norm and singular value inequalities, which are essential tools in modern operator theory.

## 1.7 Operator Norms

**Definition 1.7.1** (Spectral Norm). The *spectral norm* of  $\mathcal{T}_1 \in \mathfrak{L}(\mathcal{H})$  is defined as the maximum among all singular values of  $\mathcal{T}_1$ . Formally, the spectral norm is expressed as

$$\|\mathcal{T}_1\|_{\text{spectral}} = \sigma_1(\mathcal{T}_1),$$

where  $\sigma_1(\mathcal{T}_1)$  is the first singular value of  $\mathcal{T}_1$ .

**Definition 1.7.2** (Operator Norm). The operator norm of  $\mathcal{T}_1 \in \mathfrak{L}(\mathcal{H})$  is:

$$\|\mathcal{T}_1\| = \sup_{\|\mathbf{x}\|=1} \|\mathcal{T}_1\mathbf{x}\|,$$

where the supremum is taken over all unit vectors  $\mathbf{x} \in \mathcal{H}$ .

**Theorem 1.7.1.** Given a finite-dimensional Hilbert space  $\mathcal{H}$  and any  $\mathcal{T}_1 \in \mathfrak{L}(\mathcal{H})$ , it holds that

$$\|\mathcal{T}_1\| = \|\mathcal{T}_1\|_{\text{spectral}} = \sigma_1(\mathcal{T}_1),$$

thus confirming the equivalence of the operator norm and the spectral norm.

**Proof.** The operator norm is defined as:

$$\|\mathcal{T}_1\| = \sup_{\|\mathbf{x}\|=1} \|\mathcal{T}_1\mathbf{x}\| = \sup_{\|\mathbf{x}\|=1} \sqrt{\langle \mathcal{T}_1\mathbf{x}, \mathcal{T}_1\mathbf{x} \rangle} = \sup_{\|\mathbf{x}\|=1} \sqrt{\langle \mathbf{x}, \mathcal{T}_1^* \mathcal{T}_1 \mathbf{x} \rangle}.$$

Since  $\mathcal{H}$  is finite-dimensional,  $\mathcal{T}_1^* \mathcal{T}_1$  has a complete set of eigenvalues. Consequently, the maximum value of the quadratic form  $\langle \mathbf{x}, \mathcal{T}_1^* \mathcal{T}_1 \mathbf{x} \rangle$  over all unit vectors  $\|\mathbf{x}\| = 1$  is attained precisely when  $\mathbf{x}$  is a unit eigenvector associated with the largest eigenvalue  $\lambda_1(\mathcal{T}_1^* \mathcal{T}_1)$ , and this eigenvalue satisfies

$$\lambda_1(\mathcal{T}_1^* \mathcal{T}_1) = \sigma_1(\mathcal{T}_1)^2.$$

Thus:

$$\|\mathcal{T}_1\| = \sqrt{\sup_{\|\mathbf{x}\|=1} \langle \mathbf{x}, \mathcal{T}_1^* \mathcal{T}_1 \mathbf{x} \rangle} = \sqrt{\sigma_1(\mathcal{T}_1)^2} = \sigma_1(\mathcal{T}_1) = \|\mathcal{T}_1\|_{\text{spectral}}.$$

□

In operator analysis, one of the fundamental concepts is that the operator norm and the spectral norm are equal. Throughout this thesis, we will use the operator norm notation in place of the spectral norm for simplicity and consistency.

**Definition 1.7.3** (Schatten  $p$ -Norm). For  $1 \leq p \leq \infty$ , the Schatten  $p$ -norm of  $\mathcal{T}_1 \in \mathfrak{L}(\mathcal{H})$  is:

$$\|\mathcal{T}_1\|_p = \left( \sum_{j=1}^k \sigma_j^p(\mathcal{T}_1) \right)^{1/p}.$$

For  $p = \infty$ , it is the operator norm; for  $p = 2$ , the Frobenius norm; for  $p = 1$ , the trace norm.

**Example 1.7.2.** For  $\mathcal{T}_1$  with singular values  $\sigma_1 = 3$ ,  $\sigma_2 = 1$ , the Schatten 2-norm is  $\|\mathcal{T}_1\|_2 = \sqrt{3^2 + 1^2} = \sqrt{10}$ , and the Schatten 1-norm is  $\|\mathcal{T}_1\|_1 = 3 + 1 = 4$ .

**Definition 1.7.4** (Unitarily Invariant Norm). A norm  $\|\cdot\|$  on  $\mathfrak{L}(\mathcal{H})$  is unitarily invariant if:

$$\|\mathcal{U}_1 \mathcal{T}_1 \mathcal{U}_2\| = \|\mathcal{T}_1\| \quad \text{for all unitary } \mathcal{U}_1, \mathcal{U}_2 \in \mathfrak{L}(\mathcal{H}).$$

The operator and Schatten  $p$ -norms are unitarily invariant, ideal for basis-independent analysis.

**Proposition 1.7.1** (Properties of the Operator Norm). *Let  $\mathcal{T}_1, \mathcal{T}_2 \in \mathfrak{L}(\mathcal{H})$  and let the operator norm be defined as  $\|\mathcal{T}_1\| = \sup_{\|\mathbf{x}\|=1} \|\mathcal{T}_1 \mathbf{x}\|$ . The following properties hold:*

1. *The operator norm of the positive operators  $\mathcal{T}_1^* \mathcal{T}_1$  and  $\mathcal{T}_1 \mathcal{T}_1^*$  satisfies*

$$\|\mathcal{T}_1^* \mathcal{T}_1\| = \|\mathcal{T}_1 \mathcal{T}_1^*\| = \|\mathcal{T}_1\|^2.$$

2. *The absolute value of  $\mathcal{T}_1$ , defined as  $|\mathcal{T}_1| = (\mathcal{T}_1^* \mathcal{T}_1)^{1/2}$ , and its adjoint counterpart  $|\mathcal{T}_1^*| = (\mathcal{T}_1 \mathcal{T}_1^*)^{1/2}$ , have equal norms, both coinciding with the norm of  $\mathcal{T}_1$  and  $\mathcal{T}_1^*$ :*

$$\||\mathcal{T}_1|\| = \||\mathcal{T}_1^*|\| = \|\mathcal{T}_1\| = \|\mathcal{T}_1^*\|.$$

3. *For any pair of bounded linear operators  $\mathcal{T}_1$  and  $\mathcal{T}_2$ :*

$$\|\mathcal{T}_1 \mathcal{T}_2\| = \|\mathcal{T}_2 \mathcal{T}_1\|.$$

4. *For every non-negative real exponent  $r \geq 0$ , the norm of the power of the absolute value of  $\mathcal{T}_1$  is the power of the norm of  $\mathcal{T}_1$ :*

$$\||\mathcal{T}_1|^r\| = \|\mathcal{T}_1\|^r.$$

These properties, derived from matrix analysis principles [14, 15], ensure robust norm behavior in theoretical and computational contexts.

## 1.8 Operator Monotone and Convex Functions

Functional calculus allows scalar functions to be applied to operators, enabling the derivation of inequalities critical for optimization.

**Definition 1.8.1** (Operator Monotone Function). A function  $f : \mathcal{J} \rightarrow \mathbb{R}$ , on an interval  $\mathcal{J} \subseteq \mathbb{R}$ , is operator monotone increasing if, for any pair of self-adjoint operators  $\mathcal{T}_1, \mathcal{T}_2 \in \mathfrak{L}(\mathcal{H})$  whose

eigenvalues lie in  $\mathcal{J}$ ,

$$\mathcal{T}_1 \geq \mathcal{T}_2$$

implies

$$f(\mathcal{T}_1) \geq f(\mathcal{T}_2),$$

where the operator  $f(\mathcal{T}_1)$  is defined via the functional calculus by applying  $f$  to each eigenvalue of  $\mathcal{T}_1$ . Correspondingly,  $-f$  is operator monotone increasing, if the function  $f$  is operator monotone decreasing.

**Example 1.8.1.** The function  $f(t) = t$  is operator monotone increasing, as  $\mathcal{T}_1 \geq \mathcal{T}_2$  implies  $f(\mathcal{T}_1) = \mathcal{T}_1 \geq \mathcal{T}_2$ . The function  $f(t) = -t$  is monotone decreasing.

**Definition 1.8.2** (Operator Convex Function). A function  $f : \mathcal{J} \rightarrow \mathbb{R}$ , on an interval  $\mathcal{J} \subseteq \mathbb{R}$ , is said to be operator convex if, for any self-adjoint operators  $\mathcal{T}_1, \mathcal{T}_2 \in \mathfrak{L}(\mathcal{H})$  with spectra contained in  $\mathcal{J}$  and for every scalar  $\vartheta \in [0, 1]$ :

$$f(\vartheta \mathcal{T}_1 + (1 - \vartheta) \mathcal{T}_2) \leq \vartheta f(\mathcal{T}_1) + (1 - \vartheta) f(\mathcal{T}_2).$$

If  $-f$  is convex,  $f$  is concave.

Functions such as  $f(t) = t^r$  for  $0 \leq r \leq 1$  are monotone increasing and concave on  $(0, \infty)$ , forming the basis for many inequalities [14, Chapter 5]. We assume  $f(0) = 0$  for concave functions to handle small eigenvalues.

**Proposition 1.8.1.** Let  $f : (0, \infty) \rightarrow \mathbb{R}$  be a continuous function. Then  $f$  is operator monotone increasing on the positive reals if and only if it is operator concave on the same domain.

**Example 1.8.2.** Consider  $f(t) = -\frac{1}{t}$ , a continuous function on  $(0, \infty)$ . For  $\mathcal{T}_1 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ ,  $\mathcal{T}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathcal{P}(\mathcal{H})$ , and  $t = 0.5$ :

- For operator monotonicity: Since  $\mathcal{T}_1 \geq \mathcal{T}_2$ ,  $\left( \mathcal{T}_1 - \mathcal{T}_2 = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \geq 0 \right)$ , then:

$$f(\mathcal{T}_1) = -\begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}, \quad f(\mathcal{T}_2) = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$f(\mathcal{T}_1) - f(\mathcal{T}_2) = \begin{bmatrix} -\frac{1}{4} + 1 & 0 \\ 0 & -\frac{1}{4} + 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & 0 \\ 0 & \frac{3}{4} \end{bmatrix} \geq 0,$$

so  $f(\mathcal{T}_1) \geq f(\mathcal{T}_2)$ .

- For operator concavity:

$$t\mathcal{T}_1 + (1-t)\mathcal{T}_2 = 0.5 \cdot \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} + 0.5 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2.5 & 0 \\ 0 & 2.5 \end{bmatrix},$$

$$f(t\mathcal{T}_1 + (1-t)\mathcal{T}_2) = -\frac{1}{\begin{bmatrix} 2.5 & 0 \\ 0 & 2.5 \end{bmatrix}} = \begin{bmatrix} -\frac{2}{5} & 0 \\ 0 & -\frac{2}{5} \end{bmatrix},$$

$$tf(\mathcal{T}_1) + (1-t)f(\mathcal{T}_2) = 0.5 \cdot \begin{bmatrix} -\frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \end{bmatrix} + 0.5 \cdot \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{5}{8} & 0 \\ 0 & -\frac{5}{8} \end{bmatrix}.$$

$$f(t\mathcal{T}_1 + (1-t)\mathcal{T}_2) - (tf(\mathcal{T}_1) + (1-t)f(\mathcal{T}_2)) = \begin{bmatrix} -\frac{2}{5} + \frac{5}{8} & 0 \\ 0 & -\frac{2}{5} + \frac{5}{8} \end{bmatrix} = \begin{bmatrix} \frac{9}{40} & 0 \\ 0 & \frac{9}{40} \end{bmatrix} \geq 0,$$

$$\text{so } f(t\mathcal{T}_1 + (1-t)\mathcal{T}_2) \geq tf(\mathcal{T}_1) + (1-t)f(\mathcal{T}_2).$$

Thus,  $f(t) = -\frac{1}{t}$  is both operator monotone and operator concave, verifying the theorem.

## 1.9 Historical Development

### 1.9.1 Origin of Singular Values

In 1907, Erhard Schmidt introduced the concept of singular values within the context of integral equations, initially terming them “eigenvalues” [57]. He studied real integral equations with symmetric and non-symmetric kernels, a vibrant area of research at the time. This overview as discussed in Kaur and Garg [43] offers a cohesive historical perspective and contextual depth to support the technical results discussed in later chapters.

For a symmetric, continuous kernel  $\delta(\mathfrak{x}, \mathfrak{z})$  on  $[a, b] \times [a, b]$ , Schmidt considered the eigenvalue problem:

$$\Delta(\mathfrak{x}) = \lambda \int_a^b \delta(\mathfrak{x}, \mathfrak{z}) \Delta(\mathfrak{z}) d\mathfrak{z}, \quad (1.1)$$

where  $\lambda$  is an eigenvalue and  $\Delta(\mathfrak{x})$  is the corresponding eigenfunction, both real due to the kernel’s symmetry.

**Example 1.9.1.** Consider the symmetric kernel  $\delta(\mathfrak{x}, \mathfrak{z}) = \mathfrak{x}\mathfrak{z}$  on  $[0, 1] \times [0, 1]$ . The eigenvalue problem becomes:

$$\Delta(\mathfrak{x}) = \lambda \int_0^1 \mathfrak{x}\mathfrak{z} \Delta(\mathfrak{z}) d\mathfrak{z} = \lambda \mathfrak{x} \int_0^1 \mathfrak{z} \Delta(\mathfrak{z}) d\mathfrak{z}.$$

Let  $t = \int_0^1 \mathfrak{z} \Delta(\mathfrak{z}) d\mathfrak{z}$ . Then,  $\Delta(\mathfrak{x}) = \lambda t \mathfrak{x}$ . After substitution, the equation becomes

$$t = \lambda t \int_0^1 \mathfrak{z}^2 d\mathfrak{z}.$$

Evaluating the definite integral:

$$\int_0^1 \mathfrak{z}^2 d\mathfrak{z} = \left[ \frac{\mathfrak{z}^3}{3} \right]_0^1 = \frac{1}{3},$$

so

$$c t = \lambda t \cdot \frac{1}{3}.$$

For non-trivial solutions, either  $t = 0$  (implying  $\Delta(\mathfrak{x}) = 0$ ) or  $\lambda = 3$ , with  $\Delta(\mathfrak{x}) = t \mathfrak{x}$ . This illustrates a simple eigenvalue and eigenfunction pair.

For non-symmetric kernels, Schmidt showed there exist non-zero functions  $\Delta(x)$  and  $\Gamma(x)$  satisfying:

$$\Delta(\mathfrak{x}) = \lambda \int_a^b \delta(\mathfrak{x}, \mathfrak{z}) \Gamma(\mathfrak{z}) d\mathfrak{z}, \quad \Gamma(\mathfrak{x}) = \lambda \int_a^b \delta(\mathfrak{x}, \mathfrak{z}) \Delta(\mathfrak{z}) d\mathfrak{z}, \quad (1.2)$$

and introduced the symmetric, positive definite kernel:

$$\delta_{\mathcal{J}}(\mathfrak{x}, \mathfrak{z}) = \int_a^b \delta(\mathfrak{x}, \mathfrak{y}) \delta(\mathfrak{z}, \mathfrak{y}) d\mathfrak{y}, \quad (1.3)$$

with eigenvalue  $\lambda^2$ . This kernel facilitated the generalization of eigenvalues and eigenfunctions to non-symmetric cases, laying the groundwork for singular value theory.

**Example 1.9.2.** For a non-symmetric kernel  $\delta(\mathfrak{x}, \mathfrak{z}) = \mathfrak{x}e^{\mathfrak{y}}$  on  $[0, 1] \times [0, 1]$ , compute:

$$\delta_{\mathcal{J}}(\mathfrak{x}, \mathfrak{z}) = \int_a^b \delta(\mathfrak{x}, \mathfrak{y}) \delta(\mathfrak{z}, \mathfrak{y}) d\mathfrak{y}.$$

Substituting the given form of  $\mathcal{K}$ , we obtain

$$\delta_{\mathcal{J}}(\mathfrak{x}, \mathfrak{z}) = \int_0^1 (\mathfrak{x}e^{\mathfrak{y}}) \cdot (\mathfrak{z}e^{\mathfrak{y}}) d\mathfrak{y} = \mathfrak{x}\mathfrak{z} \int_0^1 e^{2\mathfrak{y}} d\mathfrak{y}.$$

Now evaluate the definite integral:

$$\int_0^1 e^{2\mathfrak{y}} d\mathfrak{y} = \left[ \frac{1}{2} e^{2\mathfrak{y}} \right]_{\mathfrak{y}=0}^{\mathfrak{y}=1} = \frac{1}{2} e^2 - \frac{1}{2} e^0 = \frac{e^2 - 1}{2}.$$

Thus,

$$\delta_{\mathcal{J}}(\mathfrak{x}, \mathfrak{z}) = \mathfrak{x}\mathfrak{z} \cdot \frac{e^2 - 1}{2}.$$

This  $\delta_{\mathcal{J}}(\mathfrak{x}, \mathfrak{z}) = \frac{e^2 - 1}{2} \mathfrak{x}\mathfrak{z}$  is symmetric and positive, resembling the symmetric case but derived

from a non-symmetric kernel.

H. Bateman [13] extended Schmidt's work by defining singular values as the reciprocals of eigenvalues of the kernel, providing an alternative perspective. Emile Picard [53] formalized the term “singular values” for symmetric kernels, distinguishing them from eigenvalues and solidifying their role in integral equation theory.

In 1937, Frank Smithies [58] generalized Schmidt's theory but struggled to directly connect eigenvalues and singular values for non-symmetric kernels. In 1949, Schmidt's student T. S. Chang [26], inspired by Smithies, established an indirect connection. Chang proved that if the series of powers of singular values converges:

$$\sum_j \sigma_j^r < \infty,$$

Consequently, the  $r$ th power series of the absolute values of the eigenvalues also converges:

$$\sum_j |\lambda_j|^r < \infty.$$

In 1949, Weyl [60] proved a foundational inequality that connects the eigenvalues with singular values:

$$|\lambda_1 \lambda_2 \cdots \lambda_n| \leq \sigma_1 \sigma_2 \cdots \sigma_n, \quad n = 1, 2, \dots, k, \quad (1.4)$$

where  $\lambda_j$  are the eigenvalues of  $\mathcal{T}_1$ , ordered by  $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_k|$ , and  $\sigma_j$  are its singular values, ordered by  $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k \geq 0$ . The proof of this inequality relies on the properties of the compound operator, defined as follows.

**Definition 1.9.1** (Compound Operator). The  $n$ -th compound operator of  $\mathcal{T}_1$ , denoted  $\mathcal{T}_1^{(n)}$ , is a matrix of size  $\binom{k}{n} \times \binom{k}{n}$  whose entries are the  $n \times n$  principal minors of  $\mathcal{T}_1$ . Specifically, for any two ordered  $n$ -element subsets  $I, J \subseteq \{1, \dots, k\}$ , the corresponding entry is

$$(\mathcal{T}_1^n)_{I,J} = \det(\mathcal{T}_{1I,J}),$$

where  $\mathcal{T}_{1I,J}$  is the submatrix of  $\mathcal{T}_1$  formed by rows in  $I$  and columns in  $J$ . In the special case  $n = k$ , the operator  $\mathcal{T}_1^{(k)}$  reduces to a  $1 \times 1$  matrix, and its eigenvalue is simply

$$\lambda_1(\mathcal{T}_1^{(k)}) = \det(\mathcal{T}_1) = \prod_{j=1}^k \lambda_j(\mathcal{T}_1).$$

More generally, for  $1 \leq n \leq k$ , the eigenvalues of  $\mathcal{T}_1^{(n)}$  are all possible products

$$\prod_{j \in \alpha} \lambda_j(\mathcal{T}_1),$$

taken over increasing  $n$ -element index sets  $\alpha \subset \{1, \dots, k\}$ .

The compound operator  $\mathcal{T}_1^{(n)}$  is essential in proving Weyl's inequality, as it allows the eigenvalue product  $\lambda_1(\mathcal{T}_1) \cdots \lambda_n(\mathcal{T}_1)$  to be expressed as  $\lambda_1(\mathcal{T}_1^{(n)})$ , which is then compared to the singular value product  $\sigma_1(\mathcal{T}_1^{(n)}) = \sigma_1(\mathcal{T}_1) \cdots \sigma_n(\mathcal{T}_1)$  using the Binet–Cauchy theorem [25] and the operator radius inequality [33, 42].

**Proposition 1.9.1.** *For  $\mathcal{T}_1 \in \mathfrak{L}(\mathcal{H})$ , let  $\lambda_j(\mathcal{T}_1)$  be the eigenvalues ordered by magnitude  $|\lambda_1(\mathcal{T}_1)| \geq \cdots \geq |\lambda_k(\mathcal{T}_1)|$ , and  $\sigma_j(\mathcal{T}_1)$  be the singular values ordered  $\sigma_1(\mathcal{T}_1) \geq \cdots \geq \sigma_k(\mathcal{T}_1) \geq 0$ . Then*

$$\prod_{j=1}^n |\lambda_j(\mathcal{T}_1)| \leq \prod_{j=1}^n \sigma_j(\mathcal{T}_1),$$

with equality holding when  $n = k$ .

## 1.9.2 Origin of Singular Value Inequalities

### 1.9.2.1 Singular Value Inequalities: Horn and Fan

Roger Horn [41] and Kay Fan [29] established foundational singular value inequalities for matrices. For  $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{P}(\mathcal{H})$ , the determinant satisfies:

$$\det(\mathcal{T}_1 \mathcal{T}_2) = \det(\mathcal{T}_1) \det(\mathcal{T}_2). \quad (1.5)$$

However, this does not generally hold for singular values. Horn [41] provided a weaker inequality:

$$\sigma_1(\mathcal{T}_1 \mathcal{T}_2) \sigma_2(\mathcal{T}_1 \mathcal{T}_2) \cdots \sigma_n(\mathcal{T}_1 \mathcal{T}_2) \leq (\sigma_1(\mathcal{T}_1) \sigma_2(\mathcal{T}_1) \cdots \sigma_n(\mathcal{T}_1)) (\sigma_1(\mathcal{T}_2) \sigma_2(\mathcal{T}_2) \cdots \sigma_n(\mathcal{T}_2)). \quad (1.6)$$

for  $n = 1, 2, \dots, k$ . While equality holds when  $n = k$ , i.e., considering the full product of all singular values, it may not hold when  $n < k$ . In particular, if both  $\mathcal{T}_1$  and  $\mathcal{T}_2$  are positive, then

$$\prod_{j=1}^k \sigma_j(\mathcal{T}_1 \mathcal{T}_2) = \prod_{j=1}^k \sigma_j(\mathcal{T}_1) \prod_{j=1}^k \sigma_j(\mathcal{T}_2),$$

but the inequality can be strict when  $n < k$ , as demonstrated by the following example.

**Example 1.9.3.** Let:

$$\mathcal{T}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{T}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

both positive. Then  $\mathcal{T}_1 \mathcal{T}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ , so  $\sigma_1(\mathcal{T}_1 \mathcal{T}_2) = 0$ . Singular values are  $\sigma_1(\mathcal{T}_1) = 1$ ,  $\sigma_1(\mathcal{T}_2) = 1$ , but

$$\sigma_1(\mathcal{T}_1 \mathcal{T}_2) = 0 \neq \sigma_1(\mathcal{T}_1) \sigma_1(\mathcal{T}_2) = 1,$$

showing that the product inequality fails for partial products. Nonetheless, Horn's inequality (1.6)

remains valid:

$$\sigma_1(\mathcal{T}_1\mathcal{T}_2) = 0 \leq \sigma_1(\mathcal{T}_1)\sigma_1(\mathcal{T}_2) = 1.$$

Ky Fan [29] extended these results to completely continuous operators in Hilbert spaces, generalizing von Neumann's theorem [52] and connecting to classical results like Muirhead's theorem [35]. Fan's work focused on maximum properties of singular values, enhancing the theoretical framework.

## 1.10 Current Developments

The exploration of singular value inequalities has evolved notably since the early 20th century. Modern research from 1990 to 2025 has extended historical results, such as Weyl's eigenvalue-singular value inequalities (1.4) and Horn's product inequalities (1.6), to operator sums, products, commutators, and norm inequalities.

Equations and inequalities are the heartbeat of mathematics, shaping everything from basic calculations to cutting-edge technologies. Many inequalities, first discovered for real or complex numbers, reveal simple yet powerful relationships. Over decades, mathematicians have worked to extend these inequalities to matrices, which act like grids of numbers used to model complex systems in science and engineering. However, matrices behave differently, and not all scalar inequalities hold as expected; some work directly, while others fail or require adjustment. To overcome these challenges, researchers have developed weaker versions of these inequalities using singular values, which measure key properties of matrices, much like a ruler measures length. This section explores three classical inequalities for real or complex numbers and their singular value counterparts, drawing on foundational works by Bhatia and Kittaneh [17, 21] and Garg and Aujla [32]. These inequalities are central to the development of new singular value inequalities for operator transformations.

### 1.10.1 Inequalities and Singular Value Extensions

#### 1.10.1.1 Arithmetic-Geometric Mean Type Inequality

The classical arithmetic-geometric mean ( $\mathcal{AGM}$ ) inequality for real numbers  $t_1, t_2$  is given by:

$$2|t_1 t_2| \leq t_1^2 + t_2^2.$$

This inequality has been extended to the operator setting by Bhatia and Kittaneh, as follows.

**Proposition 1.10.1.** [17] For  $\mathcal{T}_1, \mathcal{T}_2 \in \mathfrak{L}(\mathcal{H})$ . Then

$$2\sigma_j(\mathcal{T}_1\mathcal{T}_2^*) \leq \sigma_j(\mathcal{T}_1^*\mathcal{T}_1 + \mathcal{T}_2^*\mathcal{T}_2), \quad j = 1, 2, \dots, k.$$

If both  $\mathcal{T}_1$  and  $\mathcal{T}_2$  are self-adjoint, this further simplifies to:

$$2\sigma_j(\mathcal{T}_1\mathcal{T}_2) \leq \sigma_j(\mathcal{T}_1^2 + \mathcal{T}_2^2), \quad j = 1, 2, \dots, k.$$

**Proof.** For  $\mathcal{T}_1, \mathcal{T}_2 \in \mathfrak{L}(\mathcal{H})$ . Let  $\mathcal{P} = \begin{bmatrix} \mathcal{T}_1 & \mathcal{T}_2 \\ \mathcal{O} & \mathcal{O} \end{bmatrix}$ , be an operator on  $\mathcal{H} \oplus \mathcal{H}$ . Note that

$$\mathcal{P}\mathcal{P}^* = \begin{bmatrix} \mathcal{T}_1\mathcal{T}_1^* + \mathcal{T}_2\mathcal{T}_2^* & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{bmatrix}, \quad \mathcal{P}^*\mathcal{P} = \begin{bmatrix} \mathcal{T}_1^*\mathcal{T}_1 & \mathcal{T}_1^*\mathcal{T}_2 \\ \mathcal{T}_2^*\mathcal{T}_1 & \mathcal{T}_2^*\mathcal{T}_2 \end{bmatrix}.$$

Let  $\mathcal{U}_1 = \begin{bmatrix} \mathcal{I} & \mathcal{O} \\ \mathcal{O} & -\mathcal{I} \end{bmatrix}$  be an unitary operator. We then define the block matrix

$$\mathcal{N} = \begin{bmatrix} \mathcal{O} & 2\mathcal{T}_1^*\mathcal{T}_2 \\ 2\mathcal{T}_2^*\mathcal{T}_1 & \mathcal{O} \end{bmatrix},$$

which can be expressed as

$$\mathcal{N} = \mathcal{P}^*\mathcal{P} - \mathcal{U}_1(\mathcal{P}^*\mathcal{P})\mathcal{U}_1^*.$$

Right hand side of above equation is Jordan decomposition of  $\mathcal{Q}$  as a difference of two positive operators. Hence, by Lemma 1.5.1 and using  $\sigma_j(\mathcal{P}^*\mathcal{P}) = \sigma_j(\mathcal{P}\mathcal{P}^*)$  we get,

$$\sigma_j(\mathcal{Q}^\pm) \leq \sigma_j(\mathcal{T}_1\mathcal{T}_1^* + \mathcal{T}_2\mathcal{T}_2^*),$$

It is clear that  $\sigma_j(\mathcal{Q}^+)$  and  $\sigma_j(\mathcal{Q}^-)$  both are bounded above by  $\sigma_j(\mathcal{T}_1\mathcal{T}_1^* + \mathcal{T}_2\mathcal{T}_2^*)$ . Hence,

$$\sigma_j(\mathcal{Q}) \leq \sigma_j(\mathcal{Z}), \quad \text{for each } j = 1, 2, \dots, 2k,$$

where

$$\mathcal{Z} = \begin{bmatrix} \mathcal{T}_1\mathcal{T}_1^* + \mathcal{T}_2\mathcal{T}_2^* & \mathcal{O} \\ \mathcal{O} & \mathcal{T}_1\mathcal{T}_1^* + \mathcal{T}_2\mathcal{T}_2^* \end{bmatrix}.$$

The singular values of the block operator  $\mathcal{Q}$  consist of the singular values of  $2(\mathcal{T}_1^*\mathcal{T}_2)$ , each repeated with multiplicity two. Similarly, the singular values of  $\mathcal{Z}$  are precisely the singular values of  $2(\mathcal{T}_1\mathcal{T}_1^* + \mathcal{T}_2\mathcal{T}_2^*)$ , also taken with double multiplicity. Hence, we get

$$2\sigma_j(\mathcal{T}_1^*\mathcal{T}_2) \leq \sigma_j(\mathcal{T}_1\mathcal{T}_1^* + \mathcal{T}_2\mathcal{T}_2^*), \quad \text{for } j = 1, 2, \dots, k.$$

□

This inequality plays a critical role in understanding relationships between operator sums and products.

We now recall Bhatia and Davis's result, they extended this analysis to norm bounds involving generalized anti-commutators. They proved in [16, 22]:

**Proposition 1.10.2.** *The following inequalities holds for  $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{P}(\mathcal{H})$  and  $\mathcal{S}_1 \in \mathfrak{L}(\mathcal{H})$ ,*

$$2\|\mathcal{T}_1^{1/2}\mathcal{S}_1\mathcal{T}_2^{1/2}\| \leq \|\mathcal{T}_1^\rho\mathcal{S}_1\mathcal{T}_2^{1-\rho} + \mathcal{T}_1^{1-\rho}\mathcal{S}_1\mathcal{T}_2^\rho\| \leq \|\mathcal{T}_1\mathcal{S}_1 + \mathcal{S}_1\mathcal{T}_2\|, \quad \rho \in [0, 1], \quad (1.7)$$

$$\|\mathcal{T}_1^\rho\mathcal{S}_1\mathcal{T}_2^{1-\rho} - \mathcal{T}_1^{1-\rho}\mathcal{S}_1\mathcal{T}_2^\rho\| \leq |2\rho - 1|\|\mathcal{T}_1\mathcal{S}_1 - \mathcal{S}_1\mathcal{T}_2\|, \quad \rho \in [0, 1], \quad (1.8)$$

where  $\|\cdot\|$  is a unitarily invariant norm.

In 1992, Bhatia and Kittaneh [45] provided an alternative proof of (1.7) using a different approach. Zhan [65, 63] conjectured and Audenaert [11] proved second inequality of (1.7) in terms of singular values as follows

**Proposition 1.10.3.** *For  $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{P}(\mathcal{H})$ , and we fix an integer index  $j$  such that  $1 \leq j \leq k$ . Then*

$$\sigma_j(\mathcal{T}_1^\rho\mathcal{T}_2^{1-\rho} + \mathcal{T}_1^{1-\rho}\mathcal{T}_2^\rho) \leq \sigma_j(\mathcal{T}_1 + \mathcal{T}_2) \quad \rho \in [0, 1],$$

Audeh [7], Hirzallah [37], Bhatia and Kittaneh [20, 18], Albadawi and Hussien [4], Abbas and Ghabries [1], Rahama et al. [55] refined AGMI and proved various related inequalities, developing generalized inequalities.

### 1.10.1.2 Complex Sum Inequality

A classical scalar inequality asserts:

$$|\iota t_1 + t_2| \leq \sqrt{2}|\iota t_1 + t_2|,$$

for real numbers  $t_1, t_2$ . In [21], Bhatia extended this concept to operators:

**Proposition 1.10.4.** *Let  $\mathcal{T}_1, \mathcal{T}_2 \in \mathfrak{L}(\mathcal{H})$  be two operators. Then*

1. *For self-adjoint operators  $\mathcal{T}_1$  and  $\mathcal{T}_2$ ,*

$$\sum_{j=1}^k \sigma_j(\mathcal{T}_1 + \mathcal{T}_2) \leq \sum_{j=1}^k \sqrt{2}\sigma_j(\mathcal{T}_1 + \iota\mathcal{T}_2).$$

2. *If  $\mathcal{T}_1 \in \mathcal{P}(\mathcal{H})$  and  $\mathcal{T}_2$  is self-adjoint,*

$$\prod_{j=1}^k \sigma_j(\mathcal{T}_1 + \mathcal{T}_2) \leq \prod_{j=1}^k \sqrt{2}\sigma_j(\mathcal{T}_1 + \iota\mathcal{T}_2).$$

3. For  $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{P}(\mathcal{H})$ ,

$$\sigma_j(\mathcal{T}_1 + \mathcal{T}_2) \leq \sqrt{2}\sigma_j(\mathcal{T}_1 + \iota\mathcal{T}_2).$$

These inequalities demonstrate how complex combinations provide upper bounds for singular values in operator theory. The following application of theorem (1.10.4) establish inequalities for positive operators under concave and convex mappings .

**Proposition 1.10.5.** [5] *Given  $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{P}(\mathcal{H})$ , and scalars  $\vartheta, \mu \in (0, 1)$  forming a convex partition ( $\vartheta + \mu = 1$ ), let  $f$  be a concave function mapping non-negative reals to non-negative reals. Then, for each index  $j = 1, \dots, k$ :*

$$\sigma_j(\vartheta f(\mathcal{T}_1) + \mu f(\mathcal{T}_2)) \leq \sigma_j(f(2|\vartheta\mathcal{T}_1 + \iota\mu\mathcal{T}_2|)), \quad (1.9)$$

Consider a strictly increasing convex function  $f$  defined on  $[0, \infty)$  with values in  $[0, \infty)$  and satisfying

$$f(0) = 0.$$

Then, for each index  $j$  from 1 to  $k$ :

$$\sigma_j(f(\vartheta\mathcal{T}_1 + \mu\mathcal{T}_2)) \leq 2\sigma_j(\vartheta f(\mathcal{T}_1) + \iota\mu f(\mathcal{T}_2)), \quad (1.10)$$

### 1.10.1.3 Operator Version of Triangle Inequality

A classical triangle-like inequality for complex numbers states:

$$|\mathcal{t}_1 + \mathcal{t}_2|^2 \leq (1 + |\mathcal{t}_1|^2)(1 + |\mathcal{t}_2|^2).$$

Garg and Aujla [32] extended this to operator form:

**Proposition 1.10.6.** *Let  $\mathcal{T}_1, \mathcal{T}_2 \in \mathfrak{L}(\mathcal{H})$  be operators. For  $1 \leq n \leq k$ ,  $1 \leq \rho \leq 2$  and a concave function  $f : [0, \infty) \rightarrow [0, \infty)$ ,*

$$\prod_{j=1}^n \sigma_j(|\mathcal{T}_1 + \mathcal{T}_2|^\rho) \leq \prod_{j=1}^n \sigma_j(\mathcal{T}_k + |\mathcal{T}_1|^\rho) \cdot \prod_{j=1}^n \sigma_j(\mathcal{T}_k + |\mathcal{T}_2|^\rho),$$

$$\prod_{j=1}^n \sigma_j(\mathcal{T}_k + f(|\mathcal{T}_1 + \mathcal{T}_2|)) \leq \prod_{j=1}^n \sigma_j(\mathcal{T}_k + f(|\mathcal{T}_1|)) \cdot \prod_{j=1}^n \sigma_j(\mathcal{T}_k + f(|\mathcal{T}_2|)).$$

First result is the extension of triangle inequality to operators form and second result is the extension of well known Rotfel'd's inequality [56]. Another proof of these results is proved by Zhao [67].

These inequalities extend classical scalar bounds to the operator setting, offering refined control over the behavior of singular values in structured operator expressions.

## 1.11 Overview of the Chapter

This chapter provides a comprehensive overview of the evolution and significance of singular value inequalities within the framework of operator theory. Tracing back to the early 20th century, foundational contributions by Schmidt, Bateman, Picard, Smithies, Chang, Weyl, Horn, Ky Fan, Ault, and Marcus laid the groundwork for understanding integral equations and matrix inequalities. These early developments (1907–1967), supported by illustrative examples and rigorous proofs, introduced key concepts central to the analysis of linear operators.

In more recent decades (1990–2025), the field has seen substantial growth through the work of Bhatia, Kittaneh, Audeh, Hirzallah, Garg, and others, who have extended classical results to cover more general operator settings. These include singular value inequalities for sums, products, and commutators of operators, often relaxing assumptions such as positivity or normality. The incorporation of bounded, compact, and normal operators has allowed these results to be applied more broadly across disciplines like quantum mechanics, numerical computation, and data science.

In parallel, the chapter introduces the essential mathematical framework required for this thesis, focusing on bounded linear operators on Hilbert spaces  $\mathcal{H}$ , and their structural properties such as norms, singular values  $\sigma_j$ , and eigenvalues  $\lambda_j$ . Notations including  $\mathfrak{L}(\mathcal{H})$  for bounded operators and  $||| \cdot |||$  for unitarily invariant norms are also presented. Special attention is given to expressions involving generalized commutators, anti-commutators, forms like  $\mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2 + \mathcal{S}_2$ , and operator-valued functions, which form the central theme of the subsequent chapters.

By integrating both historical development and theoretical groundwork, this chapter sets the stage for the original contributions presented in Chapters 2 to 5, aimed at addressing open problems and expanding the scope of singular value inequalities in operator theory.

# Chapter 2

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## Singular Value inequalities for $\mathcal{T}_1\mathcal{S}_1 \pm \mathcal{S}_2\mathcal{T}_2$

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### 2.1 Introduction

Among the core problems in operator theory are singular value inequalities involving sums of positive operators, with extensive results established for operators in the space  $\mathfrak{L}(\mathcal{H})$ . A significant contribution in this area was made by Kittaneh in 1997 [46], who established a key inequality for the singular values of operator sums. Specifically,  $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{P}(\mathcal{H})$ , Kittaneh proposed:

$$\sigma_j((\mathcal{T}_1 + \mathcal{T}_2) \oplus \mathcal{O}) \leq \sigma_j\left((\mathcal{T}_1 \oplus \mathcal{T}_2) + \left(|\mathcal{T}_2^{1/2}\mathcal{T}_1^{1/2}| \oplus |\mathcal{T}_1^{1/2}\mathcal{T}_2^{1/2}|\right)\right), \quad j = 1, 2, \dots, k. \quad (2.1)$$

The result delivers a singular value estimate for  $\mathcal{T}_1 + \mathcal{T}_2$  via comparison with a direct sum structure that incorporates both the original operators and their square root products.

Building on this foundation, Kittaneh refined the result in 2006 [47] by reorganizing the direct sum terms, yielding a more streamlined inequality:

$$\sigma_j((\mathcal{T}_1 + \mathcal{T}_2) \oplus \mathcal{O}) \leq \sigma_j\left(\left(\mathcal{T}_1 + |\mathcal{T}_2^{1/2}\mathcal{T}_1^{1/2}|\right) \oplus \left(\mathcal{T}_2 + |\mathcal{T}_1^{1/2}\mathcal{T}_2^{1/2}|\right)\right), \quad j = 1, 2, \dots, k. \quad (2.2)$$

This refinement enhances the applicability of the original result by restructuring the right-hand side to involve sums of individual operators with their respective cross-terms. When specialized to the spectral norm, inequality (2.2) implies:

$$\|\mathcal{T}_1 + \mathcal{T}_2\| \leq \left[\|\mathcal{T}_1 + |\mathcal{T}_2^{1/2}\mathcal{T}_1^{1/2}|\|, \|\mathcal{T}_2 + |\mathcal{T}_1^{1/2}\mathcal{T}_2^{1/2}|\|\right]_{\infty}, \quad (2.3)$$

providing a concise bound for the operator norm of the sum  $\mathcal{T}_1 + \mathcal{T}_2$ .

Furthermore, Kittaneh extended the framework of inequality (2.2) to Schatten  $p$ -norms for  $1 \leq p < \infty$  and  $j = 1, 2, \dots, k$ , resulting in:

$$\|\mathcal{T}_1 + \mathcal{T}_2\|_p \leq \left( \left\| \mathcal{T}_1 + |\mathcal{T}_2^{1/2} \mathcal{T}_1^{1/2}| \right\|_p^p + \left\| \mathcal{T}_2 + |\mathcal{T}_1^{1/2} \mathcal{T}_2^{1/2}| \right\|_p^p \right)^{1/p}. \quad (2.4)$$

This generalization broadens the scope of the inequality to a wider class of operator norms, enhancing its utility in functional analysis.

Additionally, when the earlier inequality (2.1) is applied to the spectral norm, it yields:

$$\|\mathcal{T}_1 + \mathcal{T}_2\| \leq [\|\mathcal{T}_1\|, \|\mathcal{T}_2\|]_\infty + \left\| |\mathcal{T}_1^{1/2} \mathcal{T}_2^{1/2}| \right\|, \quad (2.5)$$

offering a bound that complements the refined inequality (2.3).

These developments represent a significant advancement over the well-known Davidson-Power inequality, as noted in [28]. Moreover, Bourin and Mhanna [24] and Audeh [8, 10] provided an equivalent and more generalized inequalities, further enriching the theoretical landscape. These results collectively pave the way for novel generalizations of the inequalities, contributing to a deeper understanding of operator sums in Hilbert spaces.

It should be noted here that Kittaneh's inequality (2.2) is the trivial consequence of inequality presented by Audeh [8] to generalized anti-commutators, proving for  $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{P}(\mathcal{H})$  and  $\mathcal{S}_1 \in \mathfrak{L}(\mathcal{H})$ :

$$\sigma_j((\mathcal{T}_1\mathcal{S}_1 + \mathcal{S}_1\mathcal{T}_2) \oplus \mathcal{O}) \leq \sigma_j(\mathcal{M} \oplus \mathcal{N}), \quad j = 1, 2, \dots, k, \quad (2.6)$$

where:

$$\begin{aligned} \mathcal{M} &= \mathcal{P} + |\mathcal{T}_2^{1/2} \mathcal{S}_1^* \mathcal{T}_1^{1/2}|, \quad \mathcal{N} = \mathcal{Q} + |\mathcal{T}_1^{1/2} \mathcal{S}_1 \mathcal{T}_2^{1/2}|, \\ \mathcal{P} &= \frac{1}{2} \mathcal{T}_1 + \frac{1}{2} \mathcal{T}_1^{1/2} |\mathcal{S}_1^*|^2 \mathcal{T}_1^{1/2}, \quad \mathcal{Q} = \frac{1}{2} \mathcal{T}_2 + \frac{1}{2} \mathcal{T}_2^{1/2} |\mathcal{S}_1|^2 \mathcal{T}_2^{1/2}. \end{aligned}$$

This generalizes (2.2) when  $\mathcal{S}_1 = \mathcal{I}$ , addressing operator expressions in quantum control theory.

Zhao [68] generalized this to  $\mathcal{T}_1\mathcal{S}_1 + \mathcal{S}_2\mathcal{T}_2$  for  $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{P}(\mathcal{H})$  and  $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{L}(\mathcal{H})$ , and index  $j$  from 1 to  $k$ , proving:

$$\sigma_j((\mathcal{T}_1\mathcal{S}_1 + \mathcal{S}_2\mathcal{T}_2) \oplus \mathcal{O}) \leq \sigma_j((\mathcal{M} + \mathcal{P}) \oplus (\mathcal{N} + \mathcal{Q})), \quad (2.7)$$

where:

$$\begin{aligned} \mathcal{M} &= \frac{1}{2} \mathcal{T}_1 + \frac{1}{2} \mathcal{T}_1^{1/2} |\mathcal{S}_1^*|^2 \mathcal{T}_1^{1/2}, \quad \mathcal{N} = \frac{1}{2} \mathcal{T}_2 + \frac{1}{2} \mathcal{T}_2^{1/2} |\mathcal{S}_2|^2 \mathcal{T}_2^{1/2}, \\ \mathcal{P} &= \frac{1}{2} |\mathcal{T}_2^{1/2} (\mathcal{S}_1 + \mathcal{S}_2)^* \mathcal{T}_1^{1/2}|, \quad \mathcal{Q} = \frac{1}{2} |\mathcal{T}_1^{1/2} (\mathcal{S}_1 + \mathcal{S}_2) \mathcal{T}_2^{1/2}|. \end{aligned}$$

Findings of Audeh [8] and Zhao [68] serve as extensions of several earlier findings, including

those by Bhatia and Kittaneh [19], Kittaneh [49, 50, 48] and Wang and Du [59], thereby enriching the theoretical framework surrounding singular value inequalities for commutator-type operator expressions.

In a seminal work by Kittaneh [50], a singular value inequality was established for generalized commutators. This inequality states that for operators  $\mathcal{T}_1, \mathcal{T}_2, \mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{L}(\mathcal{H})$  where  $\mathcal{S}_1$  and  $\mathcal{S}_2 \in \mathfrak{C}(\mathcal{H})$ , the singular values satisfy:

$$\sigma_j(\mathcal{T}_1\mathcal{S}_1 - \mathcal{S}_2\mathcal{T}_2) \leq 2[\|\mathcal{T}_1\|, \|\mathcal{T}_2\|]_\infty \sigma_j(\mathcal{S}_1 \oplus \mathcal{S}_2) \text{ for } j = 1, 2, \dots \quad (2.8)$$

The following inequality by Hirzallah [38] extends inequality (2.8) (to a more general form) as follows for operators  $\mathcal{T}_1, \mathcal{T}_2, \mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{L}(\mathcal{H})$  with  $\mathcal{S}_1$  and  $\mathcal{S}_2 \in \mathfrak{C}(\mathcal{H})$ .

$$\sigma_j(\mathcal{T}_1\mathcal{S}_1 - \mathcal{S}_2\mathcal{T}_2) \leq (\|\mathcal{T}_1\| + \|\mathcal{T}_2\|) \sigma_j(\mathcal{S}_1 \oplus \mathcal{S}_2)$$

for all  $j = 1, 2, \dots$ , while Audeh [6] further advanced this line of research by establishing related inequalities for the  $m$ -th order commutators.

In line with this progression, we have extended inequality (2.7) to address blocks involving  $m$ -commutators. The core findings of this chapter have been published in Kaur and Garg [44].

## 2.2 Singular Value Inequalities for blocks of $m$ -commutators

This section develops singular value inequalities for block operator expressions involving generalized  $m$ -commutators and anti-commutators. These findings extend traditional singular value bounds to more complex operator systems represented via direct sums. The subsequent lemmas, theorems, and corollaries are designed to progressively build toward this objective, providing both theoretical generalizations and practical utilities for managing such block configurations.

**Lemma 2.2.1.** *If  $\mathcal{T}_1, \mathcal{T}_2 \in \mathfrak{L}(\mathcal{H})$ , then:*

$$\sigma_j(\mathcal{T}_1 \oplus \mathcal{T}_2) = \sigma_j(\mathcal{T}_2 \oplus \mathcal{T}_1), \quad \text{holding for } j = 1, 2, \dots, k.$$

**Proof.** We proceed by noting that:

$$\begin{bmatrix} \mathcal{T}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{T}_2 \end{bmatrix} \text{ and } \begin{bmatrix} \mathcal{T}_2 & \mathcal{O} \\ \mathcal{O} & \mathcal{T}_1 \end{bmatrix}$$

are unitarily equivalent via the unitary operator:

$$\mathcal{U}_1 = \begin{bmatrix} \mathcal{O} & -\mathcal{I} \\ -\mathcal{I} & \mathcal{O} \end{bmatrix}.$$

Note that:

$$\mathcal{U}_1^* \begin{bmatrix} \mathcal{T}_2 & \mathcal{O} \\ \mathcal{O} & \mathcal{T}_1 \end{bmatrix} \mathcal{U}_1 = \begin{bmatrix} \mathcal{O} & \mathcal{J} \\ \mathcal{J} & \mathcal{O} \end{bmatrix} \begin{bmatrix} \mathcal{T}_2 & \mathcal{O} \\ \mathcal{O} & \mathcal{T}_1 \end{bmatrix} \begin{bmatrix} \mathcal{O} & -\mathcal{J} \\ -\mathcal{J} & \mathcal{O} \end{bmatrix} = \begin{bmatrix} \mathcal{T}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{T}_2 \end{bmatrix}.$$

Since unitary equivalence preserves singular values,  $\sigma_j(\mathcal{T}_1 \oplus \mathcal{T}_2) = \sigma_j(\mathcal{T}_2 \oplus \mathcal{T}_1)$ .  $\square$

**Lemma 2.2.2.** *If  $\mathcal{T}_1 \in \mathfrak{L}(\mathcal{H})$  is self-adjoint operator. Then  $\pm\mathcal{T}_1 \leq |\mathcal{T}_1|$ .*

**Proof.** Every self-adjoint operator can be factorized as

$$\mathcal{T}_1 = \mathcal{U}_1 \mathcal{D} \mathcal{U}_1^*$$

where  $\mathcal{U}_1$  is any unitary operator and  $\mathcal{D}$  is a diagonal operator whose diagonal entries are precisely the eigenvalues of  $\mathcal{T}_1$ . Note that

$$\mathcal{D} \leq |\mathcal{D}|,$$

pre-multiplying by  $\mathcal{U}_1$  and post-multiplying by  $\mathcal{U}_1^*$  on both sides we have,

$$\mathcal{U}_1 \mathcal{D} \mathcal{U}_1^* \leq \mathcal{U}_1 |\mathcal{D}| \mathcal{U}_1^*,$$

$$\implies \mathcal{T}_1 \leq |\mathcal{T}_1|,$$

In similar steps,

$$-\mathcal{T}_1 \leq |\mathcal{T}_1|.$$

The desired conclusion follows immediately from merging the two established inequalities, i.e.,  $\pm\mathcal{T}_1 \leq |\mathcal{T}_1|$ .  $\square$

Now we will present the following theorem for direct sum of  $m$ - commutators using above two lemmas and Proposition 1.10.1 .

**Theorem 2.2.1.** *Let  $\mathcal{T}_{11}, \mathcal{T}_{12}, \dots, \mathcal{T}_{1m}; \mathcal{T}_{21}, \mathcal{T}_{22}, \dots, \mathcal{T}_{2m} \in \mathcal{P}(\mathcal{H})$  and  $\mathcal{S}_{11}, \mathcal{S}_{12}, \dots, \mathcal{S}_{1m}; \mathcal{S}_{21}, \mathcal{S}_{22}, \dots, \mathcal{S}_{2m} \in \mathfrak{L}(\mathcal{H})$ . Then:*

$$\sigma_j \left( \bigoplus_{i=1}^m (\mathcal{T}_{1i}\mathcal{S}_{1i} - \mathcal{S}_{2i}\mathcal{T}_{2i}) \right) \leq \sigma_j \left( \bigoplus_{i=1}^m \mathcal{M}_i \oplus_{i=1}^m \mathcal{N}_i \right), \quad (2.9)$$

where

$$\begin{aligned}\mathcal{M}_i &= \mathcal{P}_i + \frac{1}{2}|\mathcal{T}_{2i}^{1/2}(\mathcal{S}_{2i} - \mathcal{S}_{1i})^*\mathcal{T}_{1i}^{1/2}|, \\ \mathcal{N}_i &= \mathcal{Q}_i + \frac{1}{2}|\mathcal{T}_{1i}^{1/2}(\mathcal{S}_{2i} - \mathcal{S}_{1i})\mathcal{T}_{2i}^{1/2}|, \\ \mathcal{P}_i &= \frac{1}{2}\mathcal{T}_{1i} + \frac{1}{2}\mathcal{T}_{1i}^{1/2}|\mathcal{S}_{1i}^*|^2\mathcal{T}_{1i}^{1/2}, \\ \mathcal{Q}_i &= \frac{1}{2}\mathcal{T}_{2i} + \frac{1}{2}\mathcal{T}_{2i}^{1/2}|\mathcal{S}_{2i}|^2\mathcal{T}_{2i}^{1/2}.\end{aligned}$$

**Proof.** Consider two operators  $\mathcal{M}_1$  and  $\mathcal{M}_2^*$  given by,

$$\mathcal{M}_1 = \begin{bmatrix} \mathcal{T}_{11}^{1/2} & \mathcal{S}_{21}\mathcal{T}_{21}^{1/2} & \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{T}_{12}^{1/2} & \mathcal{S}_{22}\mathcal{T}_{22}^{1/2} & \dots & \mathcal{O} & \mathcal{O} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \dots & \mathcal{T}_{1m}^{1/2} & \mathcal{S}_{2m}\mathcal{T}_{2m}^{1/2} \end{bmatrix},$$

$$\mathcal{M}_2^* = \begin{bmatrix} \mathcal{T}_{11}^{1/2}\mathcal{S}_{11} & \mathcal{O} & \dots & \mathcal{O} \\ -\mathcal{T}_{21}^{1/2} & \mathcal{O} & \dots & \mathcal{O} \\ \mathcal{O} & \mathcal{T}_{12}^{1/2}\mathcal{S}_{12} & \dots & \mathcal{O} \\ \mathcal{O} & -\mathcal{T}_{22}^{1/2} & \dots & \mathcal{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{O} & \mathcal{O} & \dots & \mathcal{T}_{1m}^{1/2}\mathcal{S}_{1m} \\ \mathcal{O} & \mathcal{O} & \dots & -\mathcal{T}_{2m}^{1/2} \end{bmatrix}.$$

Then, for  $j = 1, 2, \dots$ , and using Proposition 1.10.1, we have

$$\sigma_j \left( \bigoplus_{i=1}^m (\mathcal{T}_{1i}\mathcal{S}_{1i} - \mathcal{S}_{2i}\mathcal{T}_{2i}) \oplus \mathcal{O} \right) = \sigma_j(\mathcal{M}_1\mathcal{M}_2^*) \leq \frac{1}{2}\sigma_j(\mathcal{M}_1^*\mathcal{M}_1 + \mathcal{M}_2^*\mathcal{M}_2).$$

Using matrix calculation, we obtain

$$\begin{aligned}\frac{1}{2}\sigma_j(\mathcal{M}_1^*\mathcal{M}_1 + \mathcal{M}_2^*\mathcal{M}_2) &= \sigma_j \begin{bmatrix} \mathcal{P}_1 & \mathcal{D}_1 & \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} & \mathcal{O} \\ \mathcal{E}_1 & \mathcal{Q}_1 & \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{P}_2 & \mathcal{D}_2 & \dots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{E}_2 & \mathcal{Q}_2 & \dots & \mathcal{O} & \mathcal{O} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \dots & \mathcal{P}_m & \mathcal{D}_m \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \dots & \mathcal{E}_m & \mathcal{Q}_m \end{bmatrix}, \\ &= \sigma_j \left( \begin{bmatrix} \mathcal{P}_1 & \mathcal{O} & \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{Q}_1 & \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{P}_2 & \mathcal{O} & \dots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{Q}_2 & \dots & \mathcal{O} & \mathcal{O} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \dots & \mathcal{P}_m & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} & \mathcal{Q}_m \end{bmatrix} + \begin{bmatrix} \mathcal{O} & \mathcal{D}_1 & \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} & \mathcal{O} \\ \mathcal{E}_1 & \mathcal{O} & \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{D}_2 & \dots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{E}_2 & \mathcal{O} & \dots & \mathcal{O} & \mathcal{O} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} & \mathcal{D}_m \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \dots & \mathcal{E}_m & \mathcal{O} \end{bmatrix} \right),\end{aligned}$$

where  $\mathcal{C}_i, \mathcal{D}_i, \mathcal{P}_i, \mathcal{Q}_i$  for  $i = 1, 2, \dots, m$  are defined as follows

$$\begin{aligned}\mathcal{C}_i &= \frac{1}{2} \left( \mathcal{T}_{2i}^{1/2} (\mathcal{S}_{2i} - \mathcal{S}_{1i})^* \mathcal{T}_{1i}^{1/2} \right), & \mathcal{D}_i &= \frac{1}{2} \left( \mathcal{T}_{1i}^{1/2} (\mathcal{S}_{2i} - \mathcal{S}_{1i}) \mathcal{T}_{2i}^{1/2} \right), \\ \mathcal{P}_i &= \frac{1}{2} \mathcal{T}_{1i} + \frac{1}{2} \mathcal{T}_{1i}^{1/2} |\mathcal{S}_{1i}^*|^2 \mathcal{T}_{1i}^{1/2}, & \mathcal{Q}_i &= \frac{1}{2} \mathcal{T}_{2i} + \frac{1}{2} \mathcal{T}_{2i}^{1/2} |\mathcal{S}_{2i}|^2 \mathcal{T}_{2i}^{1/2}.\end{aligned}$$

Using Lemma 2.2.2 above matrix can be written as,

$$\begin{aligned}& \leq \sigma_j \left( \begin{bmatrix} \mathcal{P}_1 & \mathcal{O} & \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{Q}_1 & \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{P}_2 & \mathcal{O} & \dots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{Q}_2 & \dots & \mathcal{O} & \mathcal{O} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \dots & \mathcal{P}_m & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} & \mathcal{Q}_m \end{bmatrix} + \begin{bmatrix} |\mathcal{C}_1| & \mathcal{O} & \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & |\mathcal{D}_1| & \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & |\mathcal{C}_2| & \mathcal{O} & \dots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & |\mathcal{D}_2| & \dots & \mathcal{O} & \mathcal{O} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \dots & |\mathcal{C}_m| & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} & |\mathcal{D}_m| \end{bmatrix} \right) \\ & = \sigma_j \begin{bmatrix} \mathcal{P}_1 + |\mathcal{C}_1| & \mathcal{O} & \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{Q}_1 + |\mathcal{D}_1| & \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{P}_2 + |\mathcal{C}_2| & \mathcal{O} & \dots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{Q}_2 + |\mathcal{D}_2| & \dots & \mathcal{O} & \mathcal{O} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \dots & \mathcal{P}_m + |\mathcal{C}_m| & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} & \mathcal{Q}_m + |\mathcal{D}_m| \end{bmatrix} \\ & \quad \begin{bmatrix} \mathcal{M}_1 & \mathcal{O} & \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{N}_1 & \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{M}_2 & \mathcal{O} & \dots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{N}_2 & \dots & \mathcal{O} & \mathcal{O} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \dots & \mathcal{M}_m & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} & \mathcal{N}_m \end{bmatrix},\end{aligned}$$

where  $\mathcal{M}_i = \mathcal{P}_i + \frac{1}{2} |\mathcal{T}_{2i}^{1/2} (\mathcal{S}_{2i} - \mathcal{S}_{1i})^* \mathcal{T}_{1i}^{1/2}|$  and  $\mathcal{N}_i = \mathcal{Q}_i + \frac{1}{2} |\mathcal{T}_{1i}^{1/2} (\mathcal{S}_{2i} - \mathcal{S}_{1i}) \mathcal{T}_{2i}^{1/2}|$ .

Hence by lemma 2.2.1,  $\sigma_j \left( \bigoplus_{i=1}^m (\mathcal{T}_{1i}\mathcal{S}_{1i} - \mathcal{S}_{2i}\mathcal{T}_{2i}) \oplus \mathcal{O} \right) \leq \sigma_j \left( \bigoplus_{i=1}^m \mathcal{M}_i \oplus \bigoplus_{i=1}^m \mathcal{N}_i \right)$ .  $\square$

**Corollary 2.2.1.** Let  $\mathcal{T}_{1i}, \mathcal{T}_{2i}, \mathcal{S}_{1i}, \mathcal{S}_{2i} \in \mathfrak{L}(\mathcal{H})$ , with  $\mathcal{T}_{1i}, \mathcal{T}_{2i} \in \mathcal{P}(\mathcal{H})$  for all  $i = 1, \dots, m$ .

Then:

$$\sigma_j \left( \bigoplus_{i=1}^m (\mathcal{T}_{1i} \mathcal{S}_{1i} + \mathcal{S}_{2i} \mathcal{T}_{2i}) \right) \leq \sigma_j \left( \bigoplus_{i=1}^m \mathcal{M}_i \oplus \bigoplus_{i=1}^m \mathcal{N}_i \right), \quad (2.10)$$

where:

$$\begin{aligned} \mathcal{P}_i &= \frac{1}{2} \mathcal{T}_{1i} + \frac{1}{2} \mathcal{T}_{1i}^{1/2} |\mathcal{S}_{1i}^*|^2 \mathcal{T}_{1i}^{1/2}, \\ \mathcal{Q}_i &= \frac{1}{2} \mathcal{T}_{2i} + \frac{1}{2} \mathcal{T}_{2i}^{1/2} |\mathcal{S}_{2i}|^2 \mathcal{T}_{2i}^{1/2}, \\ \mathcal{M}_i &= \mathcal{P}_i + \frac{1}{2} \left| \mathcal{T}_{2i}^{1/2} (\mathcal{S}_{1i} + \mathcal{S}_{2i})^* \mathcal{T}_{1i}^{1/2} \right|, \\ \mathcal{N}_i &= \mathcal{Q}_i + \frac{1}{2} \left| \mathcal{T}_{1i}^{1/2} (\mathcal{S}_{1i} + \mathcal{S}_{2i}) \mathcal{T}_{2i}^{1/2} \right|. \end{aligned}$$

**Proof.** By adapting the proof technique from Theorem 2.2.1, with modification in the block operator  $\mathcal{M}_2^*$ . Specifically, we define the operator  $\mathcal{M}_2^*$  as:

$$\mathcal{M}_2^* = \begin{bmatrix} \mathcal{T}_{11}^{1/2} \mathcal{S}_{11} & \mathcal{O} & \dots & \mathcal{O} \\ \mathcal{T}_{21}^{1/2} & \mathcal{O} & \dots & \mathcal{O} \\ \mathcal{O} & \mathcal{T}_{12}^{1/2} \mathcal{S}_{12} & \dots & \mathcal{O} \\ \mathcal{O} & \mathcal{T}_{22}^{1/2} & \dots & \mathcal{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{O} & \mathcal{O} & \dots & \mathcal{T}_{1m}^{1/2} \mathcal{S}_{1m} \\ \mathcal{O} & \mathcal{O} & \dots & \mathcal{T}_{2m}^{1/2} \end{bmatrix}.$$

The rest of the proof proceeds identically, utilizing the same block matrix decomposition, application of singular value inequalities, and operator norm estimates. Therefore, the inequality follows with the modified  $\mathcal{M}_2^*$ , concluding the proof.  $\square$

Next, to facilitate our extensions, we first introduce two foundational lemmas concerning operator properties and functional transformations. These results provide the necessary tools for our subsequent theorems and corollaries.

**Lemma 2.2.3.** [12] Let  $\mathcal{T}_1 \in \mathfrak{L}(\mathcal{H})$ , and let  $f : (0, \infty) \rightarrow (0, \infty)$  be a strictly increasing function. Then, for  $j = 1, 2, \dots, k$ ,

- (a) For every singular value  $\sigma_j(\mathcal{T}_1)$ , the function  $f$  preserves its correspondence with the operator  $f(|\mathcal{T}_1|)$ . Concretely,

$$f(\sigma_j(\mathcal{T}_1)) = \sigma_j(f(|\mathcal{T}_1|)).$$

- (b) In the special case where  $\mathcal{T}_1$  is self-adjoint, the function  $f$  maps eigenvalues in a consistent

way: applying  $f$  to  $\lambda_j(\mathcal{T}_1)$  is the same as taking the  $j$ -th eigenvalue of  $f(\mathcal{T}_1)$ . Thus,

$$f(\lambda_j(\mathcal{T}_1)) = \lambda_j(f(\mathcal{T}_1)).$$

**Proof.** (a) For any operator  $\mathcal{T}_1$ , the singular values are defined as

$$\sigma_j(\mathcal{T}_1) = \lambda_j^{1/2}(\mathcal{T}_1^* \mathcal{T}_1) = \lambda_j^{1/2}(|\mathcal{T}_1|^2).$$

Since  $f$  is a strictly increasing function on the positive reals. So, the singular values of  $f(|\mathcal{T}_1|)$  are:

$$\sigma_j(f(|\mathcal{T}_1|)) = \lambda_j^{1/2}(f(|\mathcal{T}_1|^2)) = \lambda_j^{1/2}(f(|\mathcal{T}_1|^2)),$$

since  $f(|\mathcal{T}_1|^2) = f(|\mathcal{T}_1|^2)$  under the functional calculus for positive operators. Using the fact that  $\lambda_j^{1/2}(|\mathcal{T}_1|^2)$  coincides with the  $j$ -th singular value of  $\mathcal{T}_1$ , applying the function  $f$  gives

$$\sigma_j(f(|\mathcal{T}_1|)) = f(\lambda_j^{1/2}(|\mathcal{T}_1|^2)) = f(\sigma_j(\mathcal{T}_1)).$$

(b) For the special case where  $\mathcal{T}_1$  is self-adjoint ( $\mathcal{T}_1 = \mathcal{T}_1^*$ ), the proof follows from part (a) with modifications: since  $\mathcal{T}_1^* \mathcal{T}_1 = \mathcal{T}_1^2$ . So,

$$\lambda_j(|\mathcal{T}_1|) = |\lambda_j(\mathcal{T}_1)| = \sigma_j(\mathcal{T}_1).$$

Applying the functional calculus to  $f(|\mathcal{T}_1|)$  yields eigenvalues  $f(\lambda_j(|\mathcal{T}_1|)) = f(\sigma_j(\mathcal{T}_1))$ , aligning with the general result due to the spectral decomposition of self-adjoint operators.  $\square$

**Lemma 2.2.4.** [23] For normal operators  $\mathcal{T}_1, \mathcal{T}_2$  in  $\mathfrak{L}(\mathcal{H})$ , and a concave non-negative function  $f$  defined on  $[0, \infty)$ . Then,

$$\|f(|\mathcal{T}_1 + \mathcal{T}_2|)\| \leq \|f(|\mathcal{T}_1|) + f(|\mathcal{T}_2|)\|.$$

**Theorem 2.2.2.** Let  $\mathcal{T}_{11}, \mathcal{T}_{12}, \dots, \mathcal{T}_{1m}; \mathcal{T}_{21}, \mathcal{T}_{22}, \dots, \mathcal{T}_{2m} \in \mathcal{P}(\mathcal{H})$  and  $\mathcal{S}_{11}, \mathcal{S}_{12}, \dots, \mathcal{S}_{1m}; \mathcal{S}_{21}, \mathcal{S}_{22}, \dots, \mathcal{S}_{2m} \in \mathfrak{L}(\mathcal{H})$ . Let  $f$  be a non-negative increasing concave function defined on  $[0, \infty)$ . Then,

$$\left\| f \left( \bigoplus_{i=1}^m (\mathcal{T}_{1i}\mathcal{S}_{1i} - \mathcal{S}_{2i}\mathcal{T}_{2i}) \oplus \mathcal{O} \right) \right\| \leq \| \mathcal{M} \oplus \mathcal{N} \|, \quad (2.11)$$

where

$$\begin{aligned}\mathcal{M}_i &= f(\mathcal{P}_i) + f\left(\frac{1}{2}\left|\mathcal{T}_{2i}^{1/2}(\mathcal{S}_{2i} - \mathcal{S}_{1i})^* \mathcal{T}_{1i}^{1/2}\right|\right), \\ \mathcal{N}_i &= f(\mathcal{Q}_i) + f\left(\frac{1}{2}\left|\mathcal{T}_{1i}^{1/2}(\mathcal{S}_{2i} - \mathcal{S}_{1i}) \mathcal{T}_{2i}^{1/2}\right|\right), \\ \mathcal{P}_i &= \frac{1}{2}\mathcal{T}_{1i} + \frac{1}{2}\mathcal{T}_{1i}^{1/2}|\mathcal{S}_{1i}^*|^2\mathcal{T}_{1i}^{1/2}, \\ \mathcal{Q}_i &= \frac{1}{2}\mathcal{T}_{2i} + \frac{1}{2}\mathcal{T}_{2i}^{1/2}|\mathcal{S}_{2i}|^2\mathcal{T}_{2i}^{1/2},\end{aligned}$$

$$\text{and } \mathcal{M} = \bigoplus_{i=1}^m \mathcal{M}_i, \mathcal{N} = \bigoplus_{i=1}^m \mathcal{N}_i.$$

**Proof.** Consider the operators  $\mathcal{M}_1$  and  $\mathcal{M}_2^*$  defined as:

$$\begin{aligned}\mathcal{M}_1 &= \begin{pmatrix} \mathcal{T}_{11}^{1/2} & \mathcal{S}_{21}\mathcal{T}_{21}^{1/2} & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{T}_{12}^{1/2} & \mathcal{S}_{22}\mathcal{T}_{22}^{1/2} & \cdots & \mathcal{O} & \mathcal{O} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{T}_{1m}^{1/2} & \mathcal{S}_{2m}\mathcal{T}_{2m}^{1/2} \end{pmatrix}, \\ \mathcal{M}_2^* &= \begin{pmatrix} \mathcal{T}_{11}^{1/2}\mathcal{S}_{11} & \mathcal{O} & \cdots & \mathcal{O} \\ -\mathcal{T}_{21}^{1/2} & \mathcal{O} & \cdots & \mathcal{O} \\ \mathcal{O} & \mathcal{T}_{12}^{1/2}\mathcal{S}_{12} & \cdots & \mathcal{O} \\ \mathcal{O} & -\mathcal{T}_{22}^{1/2} & \cdots & \mathcal{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{O} & \mathcal{O} & \cdots & \mathcal{T}_{1m}^{1/2}\mathcal{S}_{1m} \\ \mathcal{O} & \mathcal{O} & \cdots & -\mathcal{T}_{2m}^{1/2} \end{pmatrix}.\end{aligned}$$

Given that  $f$  is an increasing and concave function, we apply a standard inequality involving singular values for each  $j = 1, 2, \dots$ . Additionally, Proposition 1.10.1 and Lemma 2.2.3 are employed to establish the resulting bound.

$$\sigma_j\left(f\left(\left|\bigoplus_{i=1}^m(\mathcal{T}_{1i}\mathcal{S}_{1i} - \mathcal{S}_{2i}\mathcal{T}_{2i}) \oplus \mathcal{O}\right|\right)\right) = \sigma_j(f(|\mathcal{M}_1\mathcal{M}_2^*|)) = f(\sigma_j(\mathcal{M}_1\mathcal{M}_2^*)) \leq f\left(\frac{1}{2}\sigma_j(\mathcal{M}_1^*\mathcal{M}_1 + \mathcal{M}_2^*\mathcal{M}_2)\right). \quad (2.12)$$

Through matrix computations,  $\mathcal{M}_1^*\mathcal{M}_1 + \mathcal{M}_2^*\mathcal{M}_2$  yields where  $\mathcal{C}_i$ ,  $\mathcal{D}_i$ ,  $\mathcal{P}_i$  and  $\mathcal{Q}_i$  are as

defined above:

$$\begin{aligned} \mathcal{M}_1^*\mathcal{M}_1 + \mathcal{M}_2^*\mathcal{M}_2 &= \begin{pmatrix} \mathcal{P}_1 & \mathcal{D}_1 & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{O} \\ \mathcal{C}_1 & \mathcal{Q}_1 & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{P}_2 & \mathcal{D}_2 & \cdots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{C}_2 & \mathcal{Q}_2 & \cdots & \mathcal{O} & \mathcal{O} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{P}_m & \mathcal{D}_m \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{C}_m & \mathcal{Q}_m \end{pmatrix}, \\ &= \begin{pmatrix} \mathcal{P}_1 & \mathcal{O} & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{Q}_1 & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{P}_2 & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{Q}_2 & \cdots & \mathcal{O} & \mathcal{O} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{P}_m & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{Q}_m \end{pmatrix} + \begin{pmatrix} |\mathcal{C}_1| & \mathcal{O} & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & |\mathcal{D}_1| & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & |\mathcal{C}_2| & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & |\mathcal{D}_2| & \cdots & \mathcal{O} & \mathcal{O} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \cdots & |\mathcal{C}_m| & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} & |\mathcal{D}_m| \end{pmatrix}. \end{aligned}$$

Applying the concavity of  $f$  and Lemma 2.2.2 and 2.2.4, we obtain:

$$\begin{aligned} f\left(\frac{1}{2}\sigma_j(\mathcal{M}_1^*\mathcal{M}_1 + \mathcal{M}_2^*\mathcal{M}_2)\right) &\leq \\ \sigma_j\left(f\left(\left\|\begin{bmatrix} \mathcal{P}_1 & \mathcal{O} & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{Q}_1 & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{P}_2 & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{Q}_2 & \cdots & \mathcal{O} & \mathcal{O} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{P}_m & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{Q}_m \end{bmatrix}\right\| + f\left(\left\|\begin{bmatrix} |\mathcal{C}_1| & \mathcal{O} & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & |\mathcal{D}_1| & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & |\mathcal{C}_2| & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & |\mathcal{D}_2| & \cdots & \mathcal{O} & \mathcal{O} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \cdots & |\mathcal{C}_m| & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} & |\mathcal{D}_m| \end{bmatrix}\right\| \right)\right). \end{aligned}$$

The off-diagonal terms are absorbed into the absolute values, and applying  $f$  on all terms, we get:

$$\sigma_j\left(f\left(\left\|\bigoplus_{i=1}^m (\mathcal{T}_{1i}\mathcal{S}_{1i} - \mathcal{S}_{2i}\mathcal{T}_{2i}) \oplus \mathcal{O}\right\|\right)\right) = \sigma_j\left(\left\|\begin{bmatrix} \mathcal{M}_1 & \mathcal{O} & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{N}_1 & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{M}_2 & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{N}_2 & \cdots & \mathcal{O} & \mathcal{O} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{M}_m & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{N}_m \end{bmatrix}\right\|\right),$$

where

$$\mathcal{M}_i = f(\mathcal{P}_i) + f\left(\frac{1}{2}\left|\mathcal{T}_{2i}^{1/2}(\mathcal{S}_{2i} - \mathcal{S}_{1i})^*\mathcal{T}_{1i}^{1/2}\right|\right), \quad \mathcal{N}_i = f(\mathcal{Q}_i) + f\left(\frac{1}{2}\left|\mathcal{T}_{1i}^{1/2}(\mathcal{S}_{2i} - \mathcal{S}_{1i})\mathcal{T}_{2i}^{1/2}\right|\right).$$

Using the direct sum property from Lemma 2.2.1, the inequality holds as:

$$\left\| f\left(\left|\bigoplus_{i=1}^m (\mathcal{T}_{1i}\mathcal{S}_{1i} - \mathcal{S}_{2i}\mathcal{T}_{2i}) \oplus \mathcal{O}\right|\right) \right\| \leq \|\mathcal{M}_i \oplus \mathcal{N}_i\|.$$

□

**Corollary 2.2.2.** Let  $\mathcal{T}_{11}, \mathcal{T}_{12}, \dots, \mathcal{T}_{1m}; \mathcal{T}_{21}, \mathcal{T}_{22}, \dots, \mathcal{T}_{2m} \in \mathcal{P}(\mathcal{H})$  and  $\mathcal{S}_{11}, \mathcal{S}_{12}, \dots, \mathcal{S}_{1m}; \mathcal{S}_{21}, \mathcal{S}_{22}, \dots, \mathcal{S}_{2m} \in \mathfrak{L}(\mathcal{H})$ . Then, for every unitarily invariant norm,

$$\left\| \log\left(\left|\bigoplus_{i=1}^m (\mathcal{T}_{1i}\mathcal{S}_{1i} - \mathcal{S}_{2i}\mathcal{T}_{2i}) \oplus \mathcal{O}\right| + \mathcal{I}\right) \right\| \leq \|\mathcal{M} \oplus \mathcal{N}\|,$$

where

$$\begin{aligned} \mathcal{M}_i &= \log(\mathcal{P}_i + \mathcal{I}) + \log\left(\frac{1}{2}\left|\mathcal{T}_{2i}^{1/2}(\mathcal{S}_{2i} - \mathcal{S}_{1i})^*\mathcal{T}_{1i}^{1/2}\right| + \mathcal{I}\right), \\ \mathcal{N}_i &= \log(\mathcal{Q}_i + \mathcal{I}) + \log\left(\frac{1}{2}\left|\mathcal{T}_{1i}^{1/2}(\mathcal{S}_{2i} - \mathcal{S}_{1i})\mathcal{T}_{2i}^{1/2}\right| + \mathcal{I}\right), \\ \mathcal{P}_i &= \frac{1}{2}\mathcal{T}_{1i} + \frac{1}{2}\mathcal{T}_{1i}^{1/2}|\mathcal{S}_{1i}^*|^2\mathcal{T}_{1i}^{1/2}, \\ \mathcal{Q}_i &= \frac{1}{2}\mathcal{T}_{2i} + \frac{1}{2}\mathcal{T}_{2i}^{1/2}|\mathcal{S}_{2i}|^2\mathcal{T}_{2i}^{1/2}, \end{aligned}$$

$$\text{and } \mathcal{M} = \bigoplus_{i=1}^m \mathcal{M}_i, \mathcal{N} = \bigoplus_{i=1}^m \mathcal{N}_i.$$

**Proof.** By specializing Theorem 2.11 to the function

$$f(t) = \log(t + 1),$$

which is non-negative, strictly increasing, and concave on  $[0, \infty)$ . Applying this function to the inequality preserves the unitarily invariant norm structure, yielding the stated bound. □

**Corollary 2.2.3.** Let  $\mathcal{T}_{11}, \mathcal{T}_{12}, \dots, \mathcal{T}_{1m}; \mathcal{T}_{21}, \mathcal{T}_{22}, \dots, \mathcal{T}_{2m} \in \mathcal{P}(\mathcal{H})$  and  $\mathcal{S}_{11}, \mathcal{S}_{12}, \dots, \mathcal{S}_{1m}; \mathcal{S}_{21}, \mathcal{S}_{22}, \dots, \mathcal{S}_{2m} \in \mathfrak{L}(\mathcal{H})$ . Then, for every exponent  $\rho \in (0, 1]$  and any unitarily invariant norm  $\|\cdot\|$ ,

$$\left\| \left|\bigoplus_{i=1}^m (\mathcal{T}_{1i}\mathcal{S}_{1i} - \mathcal{S}_{2i}\mathcal{T}_{2i}) \oplus \mathcal{O}\right|^\rho \right\| \leq \|\mathcal{M} \oplus \mathcal{N}\|,$$

where

$$\begin{aligned}\mathcal{M}_i &= \mathcal{P}_i^\rho + \frac{1}{2^r} \left| \mathcal{T}_{2i}^{1/2} (\mathcal{S}_{2i} - \mathcal{S}_{1i})^* \mathcal{T}_{1i}^{1/2} \right|^\rho, \\ \mathcal{N}_i &= \mathcal{Q}_i^\rho + \frac{1}{2^r} \left| \mathcal{T}_{1i}^{1/2} (\mathcal{S}_{2i} - \mathcal{S}_{1i}) \mathcal{T}_{2i}^{1/2} \right|^\rho, \\ \mathcal{P}_i &= \frac{1}{2} \mathcal{T}_{1i} + \frac{1}{2} \mathcal{T}_{1i}^{1/2} |\mathcal{S}_{1i}^*|^2 \mathcal{T}_{1i}^{1/2}, \\ \mathcal{Q}_i &= \frac{1}{2} \mathcal{T}_{2i} + \frac{1}{2} \mathcal{T}_{2i}^{1/2} |\mathcal{S}_{2i}|^2 \mathcal{T}_{2i}^{1/2},\end{aligned}$$

$$\text{and } \mathcal{M} = \bigoplus_{i=1}^m \mathcal{M}_i, \mathcal{N} = \bigoplus_{i=1}^m \mathcal{N}_i.$$

**Proof.** The inequality is a direct outcome of Theorem 2.11 by selecting,

$$f(t) = t^\rho \text{ with } \rho \in (0, 1]$$

where  $f$  remains non-negative, increasing, and concave on  $[0, \infty)$ . This choice ensures the singular value transformation holds under the given norm conditions.  $\square$

## 2.3 Connections to Known Inequalities: Remarks and Illustrative Examples

The following remarks and example are provided to demonstrate how the proposed singular value inequalities for  $m$ -block operator expressions reduce to several well-established results when the block size  $m = 1$ . These reductions not only validate the general results but also highlight their broader applicability by encapsulating classical operator inequalities as special cases.

**Remark 2.3.1.** For  $m = 1$ , setting  $\mathcal{T}_{11} = \mathcal{T}_1$ ,  $\mathcal{T}_{21} = \mathcal{T}_2$  and  $\mathcal{S}_{11} = \mathcal{S}_{21} = \mathcal{S}_1$  in inequality (2.9) reduces the general  $m$ -block inequality to the single-block version established by Audeh [8].

This transition is significant in practical contexts like quantum systems, where each operator block corresponds to a subsystem.

**Remark 2.3.2.** Choosing  $m = 1$ ,  $\mathcal{T}_{11} = \mathcal{T}_1$ ,  $\mathcal{T}_{21} = \mathcal{T}_2$  and  $\mathcal{S}_{11} = \mathcal{S}_{21} = \mathcal{S}$ , the inequality (2.9) specializes to a well-known inequality due to Zhan [64], involving singular values of operator differences.

This highlights how classical inequalities are embedded within the broader  $m$ -block framework.

The subsequent counterexample illustrates that omitting the unitary factor  $\mathcal{U}_1$  would violate the spectral bounds, underscoring its essential role in the inequality.

**Example 2.3.1.** Let  $\mathcal{T}_1$  and  $\mathcal{T}_2$  be positive operators on  $\mathbb{R}^2$ , represented by the matrices:

$$\mathcal{A} = \begin{bmatrix} 10 & 15 \\ 15 & 25 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 5 & 4 \\ 4 & 15 \end{bmatrix}.$$

We consider the matrix form of Zhan's inequality:

$$|(\mathcal{T}_1 - \mathcal{T}_2) \oplus \mathcal{O}| \leq \mathcal{U}_1^*(\mathcal{T}_1 \oplus \mathcal{T}_2)\mathcal{U}_1, \quad (2.13)$$

where  $\mathcal{U}_1$  is unitary. To test this, calculate:

$$\mathcal{P} = |(\mathcal{A} - \mathcal{B}) \oplus \mathcal{O}| - (\mathcal{A} \oplus \mathcal{B}) = \begin{bmatrix} -0.3816 & -7.6865 & 0 & 0 \\ -7.6865 & -12.0573 & 0 & 0 \\ 0 & 0 & -5 & -4 \\ 0 & 0 & -4 & -15 \end{bmatrix}.$$

The eigenvalues of the matrix  $\mathcal{P}$  are given by  $-16.40, -15.87, -3.60, 3.43$ , where the presence of a positive eigenvalue confirms that  $\mathcal{P}$  is not a negative definite matrix. This demonstrates that the inequality in (2.13) does not hold unless a suitable unitary operator  $\mathcal{U}_1$  is applied.

**Remark 2.3.3.** We now compute the determinant of the block-structured matrix  $\mathcal{P}$  as defined in Example 2.3.1. The matrix  $\mathcal{P}$  is constructed as a difference of two block-diagonal operators:

$$\mathcal{P} = |(\mathcal{T}_1 - \mathcal{T}_2) \oplus \mathcal{O}| - (\mathcal{T}_1 \oplus \mathcal{T}_2).$$

To evaluate  $\det(\mathcal{P})$ , we first recognize the block-triangular form induced by this structure. Specifically, after appropriate unitary similarity or direct inspection,  $\mathcal{P}$  can be expressed in a form that separates the contributions from  $\mathcal{T}_1 - \mathcal{T}_2$  and  $\mathcal{T}_2$ . This leads to the factorization:

$$\det(\mathcal{P}) = \det(|\mathcal{T}_1 - \mathcal{T}_2| - \mathcal{T}_1) \cdot \det(-\mathcal{T}_2).$$

The term  $\det(-\mathcal{T}_2)$  arises from the lower block involving  $\mathcal{T}_2$ , and since  $-\mathcal{T}_2 = (-1) \cdot \mathcal{T}_2$ , we have:

$$\det(-\mathcal{T}_2) = (-1)^{\dim \mathcal{H}} \det(\mathcal{T}_2) = (-1)^k \det(\mathcal{T}_2),$$

where  $k = \dim \mathcal{H}$  is the dimension of the underlying Hilbert space. Thus, combining both parts, the full determinant is:

$$\det(\mathcal{P}) = \det(|\mathcal{T}_1 - \mathcal{T}_2| - \mathcal{T}_1) \cdot (-1)^k \det(\mathcal{T}_2).$$

This identity highlights how the absolute value operator  $|\mathcal{T}_1 - \mathcal{T}_2|$  interacts with  $\mathcal{T}_1$  in the upper block, while the sign flip in the lower block contributes the alternating factor  $(-1)^k$ .

This observation reveals that whenever  $\det(\mathcal{T}_2) = 0$ , it follows immediately that  $\det(\mathcal{P}) = 0$ , meaning  $\mathcal{P}$  is singular. In other words, non-invertibility of  $\mathcal{T}_2$  is a sufficient condition for the

singularity of  $\mathcal{P}$ . However, this condition is not necessary that is,  $\mathcal{P}$  can still be singular even when  $\mathcal{T}_2$  is invertible. To illustrate this, consider the following concrete example:

$$\mathcal{A} = \begin{bmatrix} 2 & 4 \\ 4 & 9 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 3 & 5 \\ 5 & 9 \end{bmatrix}.$$

Here,  $\mathcal{T}_2$  (corresponding to  $\mathcal{B}$ ) is invertible, yet  $\mathcal{P}$  remains non-invertible due to eigenvalue overlap. This counterexample clearly demonstrates that invertibility of  $\mathcal{T}_2$  alone does not guarantee invertibility of  $\mathcal{P}$ , additional structural conditions on  $\mathcal{T}_1$  and  $\mathcal{T}_2$  are required.

We now highlight how a proper choice of unitary operator  $\mathcal{U}_1$  is also essential in the matrix version of Kittaneh's inequality.

**Remark 2.3.4.** To derive the refined inequality established by Kittaneh [47] for the case  $m = 1$ , we set  $\mathcal{T}_{11} = \mathcal{T}_1$ ,  $\mathcal{T}_{21} = \mathcal{T}_2$ , and  $\mathcal{S}_{11} = \mathcal{S}_{21} = \mathcal{S}$  in inequality (2.10). This leads to the matrix form:

$$(\mathcal{T}_1 + \mathcal{T}_2) \oplus \mathcal{O} \leq \mathcal{U}_1^* \left( \left( \mathcal{T}_1 + |\mathcal{T}_2^{1/2} \mathcal{T}_1^{1/2}| \right) \oplus \left( \mathcal{T}_2 + |\mathcal{T}_1^{1/2} \mathcal{T}_2^{1/2}| \right) \right) \mathcal{U}_1,$$

where  $\mathcal{U}_1$  is unitary. To illustrate this, let us revisit matrices  $\mathcal{A}$  and  $\mathcal{B}$  from Example 2.3.1. Compute the matrix

$$\begin{aligned} \mathcal{Q} &= |(\mathcal{A} + \mathcal{B}) \oplus \mathcal{O}| - \left( \left( \mathcal{A} + |\mathcal{B}^{1/2} \mathcal{A}^{1/2}| \right) \oplus \left( \mathcal{B} + |\mathcal{A}^{1/2} \mathcal{B}^{1/2}| \right) \right) \\ &= \begin{bmatrix} -2.1545 & -5.4241 & 0 & 0 \\ -5.4241 & -2.7816 & 0 & 0 \\ 0 & 0 & -10.5432 & -12.3122 \\ 0 & 0 & -12.3122 & -34.3930 \end{bmatrix}. \end{aligned}$$

The eigenvalues of matrix  $\mathcal{Q}$  are approximately  $-39.61, -7.90, -5.33, 2.97$ . The presence of a positive eigenvalue implies that  $\mathcal{Q}$  is not negative definite. This highlights the necessity of choosing an appropriate  $\mathcal{U}_1$  is crucial for ensuring the inequality remains valid under the spectral norm.

## 2.4 Conclusion

The exploration of singular value inequalities for expressions of the form  $\mathcal{T}_1\mathcal{S}_1 \pm \mathcal{S}_2\mathcal{T}_2$  in  $\mathfrak{L}(\mathcal{H})$  has uncovered significant new perspectives. Stemming from Kittaneh's pioneering inequalities of 1997 and 2006, which laid the groundwork for bounding operator sums across spectral and Schatten  $p$ -norms, this chapter has pushed these concepts into more complex territories. The introduction of  $m$ -commutator and anti-commutator block frameworks, as detailed in the enhanced Theorem 2.2.1 and Corollary 2.2.1, marks a substantial progression. These findings build on the unitary equivalence principles from Lemma 2.2.1 and the constraints on self-adjoint operators from Lemma 2.2.2, offering robust generalizations that extend the contributions of

Audeh, Zhao, and earlier scholars.

The incorporation of a non-negative increasing concave function  $f$  in Theorem 2.2.2, along with its practical applications in Corollaries 2.2.2 and 2.2.3, broadens the applicability to logarithmic and power-based transformations. The remarks illustrate the flexibility of these results, demonstrating how specific parameter choices reconnect to classical bounds by Zhan and Kitaneh, while Example 2.3.1 underscores the essential role of unitary operators in maintaining these inequalities. The determinant analysis further clarifies the conditions affecting operator invertibility, enriching the understanding of matrix properties in this context.

These developments fortify the theoretical foundation of operator theory and provide valuable tools for applications in quantum control and numerical linear algebra, where multi-block systems are increasingly relevant. The results, documented in Kaur and Garg [44], affirm the chapter's contribution to the evolving landscape of singular value inequalities.

# Chapter 3

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## Singular Value Inequalities for $\mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2 + \mathcal{S}_2$

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### 3.1 Introduction

This chapter focuses on the development and analysis of new singular value inequalities related to operator expressions of the form  $\mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2 + \mathcal{S}_2$ . These investigations build upon foundational results discussed in Chapter 1, particularly the work of Audeh [8] and Zhao [68] on generalized commutators and anti-commutators.

Section 3.2 introduces the essential lemmas and technical tools required for the main results, while the proof of the proposed inequality for  $\mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2 + \mathcal{S}_2$  is detailed using techniques from operator theory. Section 3.2.1 illustrates the applications of the derived inequalities, including their extensions to block matrices and comparisons with known results. The final section, Section 3.2.2, discusses the broader significance of these contributions in the context of operator theory, laying the groundwork for further generalizations in subsequent chapters.

Let  $\mathcal{T}_1, \mathcal{T}_2, \mathcal{S}_1, \mathcal{T}_{1i}, \mathcal{T}_{2i}, \mathcal{S}_{1i} \in \mathfrak{L}(\mathcal{H})$  for  $i = 1, \dots, m$ . Throughout this chapter, we denote

$$\mathcal{P}_i := \mathcal{T}_{1i} \mathcal{S}_{1i} \mathcal{T}_{2i} \quad (i = 1, \dots, m), \quad \text{and} \quad \mathcal{P} := \mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2$$

for the bilinear operator products under consideration. By applying the triangle inequality and the submultiplicative property of unitarily invariant norms, following bound holds:

$$\left\| \sum_{i=1}^m \mathcal{T}_{1i} \mathcal{S}_{1i} \mathcal{T}_{2i} \right\| \leq \left( \sum_{i=1}^m \|\mathcal{T}_{1i}\| \|\mathcal{T}_{2i}\| \right) \|\mathcal{S}_1\|, \quad (3.1)$$

Since unitarily invariant norms are monotone with respect to singular values, a natural question arises: Does a similar inequality hold for individual singular values? That is, if  $\mathcal{T}_{1i}, \mathcal{T}_{2i} \in \mathfrak{L}(\mathcal{H})$  and  $\mathcal{S}_1 \in \mathfrak{C}(\mathcal{H})$ , does

$$\sigma_j \left( \sum_{i=1}^m \mathcal{T}_{1i} \mathcal{S}_1 \mathcal{T}_{2i} \right) \leq \left( \sum_{i=1}^m \|\mathcal{T}_{1i}\| \|\mathcal{T}_{2i}\| \right) \sigma_j(\mathcal{S}_1), \quad j = 1, 2, \dots, k, \quad (3.2)$$

hold? This relation does hold in the special case when  $m = 1$ . More precisely, for  $\mathcal{T}_{11}, \mathcal{T}_{22} \in \mathfrak{L}(\mathcal{H})$  and  $\mathcal{S}_1 \in \mathfrak{C}(\mathcal{H})$ , we have: (see, e.g., [14], p. 75)

$$\sigma_j(\mathcal{P}) \leq \|\mathcal{T}_{11}\| \|\mathcal{T}_{22}\| \sigma_j(\mathcal{S}_1), \quad j = 1, 2, \dots, k.$$

However, when  $m > 1$ , inequality (3.2) does not generally hold for  $j > 1$ , as demonstrated by the following counterexample. Consider matrices:

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad A_2 = B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \mathcal{S}_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

In this setup, we compute:

$$\sigma_2(A_1 \mathcal{S}_1 B_1 + A_2 \mathcal{S}_1 B_2) = 2 > 0 = (\|A_1\| \|B_1\| + \|A_2\| \|B_2\|) \sigma_2(\mathcal{S}_1),$$

which contradicts inequality (3.2), thereby illustrating that it fails in general for higher singular values.

To address this limitation, Hirzallah and Kittaneh [40] proposed a refined inequality for operators  $\mathcal{T}_{1i}, \mathcal{T}_{2i} \in \mathfrak{L}(\mathcal{H})$  and compact operators  $\mathcal{S}_{1i} \in \mathfrak{C}(\mathcal{H})$ :

$$\sigma_j \left( \sum_{i=1}^m \mathcal{P}_i \right) \leq \left( \sum_{i=1}^m \|\mathcal{T}_{1i}\| \|\mathcal{T}_{2i}\| \right) \sigma_j \left( \bigoplus_{i=1}^m \mathcal{S}_{1i} \right), \quad j = 1, 2, \dots, k. \quad (3.3)$$

This result extends his earlier work on singular value inequalities for commutators and operator products [38].

Furthermore, in [39], it was shown that for operators  $\mathcal{T}_1, \mathcal{S}_1, \mathcal{T}_2, \mathcal{S}_2 \in \mathfrak{L}(\mathcal{H})$ , where  $\mathcal{T}_1$  and  $\mathcal{T}_2$  are normal and  $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{C}(\mathcal{H})$ , the following inequality holds:

$$\sigma_j(\mathcal{P} + \mathcal{S}_2) \leq 2^{1-1/\rho} \sigma_j \left( (|\mathcal{T}_1|^\rho + \mathcal{I})^{1/\rho} \mathcal{S}_1 (|\mathcal{T}_2|^\rho + \mathcal{I})^{1/\rho} \oplus (|\mathcal{T}_1|^\rho + \mathcal{I})^{1/\rho} \mathcal{S}_2 (|\mathcal{T}_2|^\rho + \mathcal{I})^{1/\rho} \right), \\ \rho \geq 2, \quad j = 1, 2, \dots, k. \quad (3.4)$$

This result generalizes inequality (3.1) and provides further insight into singular value bounds for operator expressions involving products and sums.

The literature on singular value inequalities reveals substantial progress, yet several unresolved challenges persist, limiting the applicability of existing results in operator theory and matrix analysis. Many results, including those by Audeh [8] and Zhao[68], often assume operator positivity, which restricts their utility for general operators in  $\mathfrak{L}(\mathcal{H})$ . To this end, we propose the following conjecture for operator products involving compact operators.

**Conjecture 3.1.1.** Let  $\mathcal{T}_1, \mathcal{T}_2, \mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{L}(\mathcal{H})$ , with  $\mathcal{S}_1$  being compact operator and  $j = 1, 2, \dots k$ . Then,

$$\sigma_j(\mathcal{P} + \mathcal{S}_2) \leq (\|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{S}_2\|) \sigma_j(\mathcal{S}_1 \oplus \mathcal{J}).$$

The proof of this conjecture is established in this chapter, along with a comparative analysis against earlier results, including those by Hirzallah [39]. We emphasize the central role played by the term  $\sigma_j(\mathcal{S}_1 \oplus \mathcal{J})$  in obtaining effective singular value estimates.

Additionally, we extend our investigation to the multi-block setting and provide new singular value inequalities that generalize earlier contributions by Hirzallah and Kittaneh [40] and also refine inequality (3.1). These contributions help build a more unified theory encompassing direct sum structures, commutators, and anti-commutators under the lens of compact and bounded operator interactions.

This part of the thesis has been published in the journal *Advances in Operator Theory* by Kaur and Garg [44], underscoring its contribution to the field and its potential for further theoretical and practical developments.

## 3.2 Singular Value Inequalities for $\mathcal{T}_1\mathcal{S}_1\mathcal{T}_2 + \mathcal{S}_2$

This section derives a significant inequality for the singular values of the operator sum  $\mathcal{T}_1\mathcal{S}_1\mathcal{T}_2 + \mathcal{S}_2$ , where  $\mathcal{T}_1, \mathcal{T}_2, \mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{L}(\mathcal{H})$ ,  $\mathcal{S}_1$  is compact, and  $\mathcal{H}$  is finite-dimensional (with  $\mathcal{P}$  as defined in Section 3.1). To establish the main result, we first recall the min-max theorem for singular values, which is instrumental in bounding operator expressions. Then we present a lemma that provides a foundational bound for the singular values of  $\mathcal{T}_1\mathcal{S}_1\mathcal{T}_2$ , followed by the main theorem and its proof.

The min-max theorem provides a variational characterization of singular values, essential for our proofs. It generalizes the Courant-Fischer theorem [[14], p.75] for eigenvalues to the context of singular values.

**Proposition 3.2.1** (Min-Max Characterization). *Let  $\mathcal{T}_1 \in \mathfrak{L}(\mathcal{H})$ . Then the  $j$ -th singular value of  $\mathcal{T}_1$  is given by:*

$$\sigma_j(\mathcal{T}_1) = \sup_{\dim \mathcal{V}=j} \inf_{\substack{\mathfrak{z} \in \mathcal{V} \\ \|\mathfrak{z}\|=1}} \|\mathcal{T}_1 \mathfrak{z}\|,$$

where the supremum is taken over all  $j$ -dimensional subspaces  $\mathcal{V} \subset \mathcal{H}$ .

**Proof.** Because the singular values of an operator  $\mathcal{T}_1$  are defined as the non-negative square roots of the eigenvalues of the positive operator  $\mathcal{T}_1^* \mathcal{T}_1$ , we can apply the Courant-Fischer theorem to  $\mathcal{T}_1^* \mathcal{T}_1$ . Specifically,  $\lambda_j(\mathcal{T}_1^* \mathcal{T}_1)$ , is equal to the square of the  $j$ -th singular value of  $\mathcal{T}_1$ :

$$\lambda_j(\mathcal{T}_1^* \mathcal{T}_1) = \sigma_j(\mathcal{T}_1)^2.$$

The Courant–Fischer variational principle provides the following characterization:

$$\lambda_j(\mathcal{T}_1^* \mathcal{T}_1) = \sup_{\dim \mathcal{V}=j} \inf_{\substack{\mathfrak{z} \in \mathcal{V} \\ \|\mathfrak{z}\|=1}} \langle \mathcal{T}_1^* \mathcal{T}_1 \mathfrak{z}, \mathfrak{z} \rangle = \sup_{\dim \mathcal{V}=j} \inf_{\substack{\mathfrak{z} \in \mathcal{V} \\ \|\mathfrak{z}\|=1}} \|\mathcal{T}_1 \mathfrak{z}\|^2.$$

Taking the square root (preserving non-negativity), we deduce

$$\sigma_j(\mathcal{T}_1) = \sqrt{\lambda_j(\mathcal{T}_1^* \mathcal{T}_1)} = \sup_{\dim \mathcal{V}=j} \inf_{\substack{\mathfrak{z} \in \mathcal{V} \\ \|\mathfrak{z}\|=1}} \|\mathcal{T}_1 \mathfrak{z}\|.$$

□

We now present a lemma that bounds the singular values of the operator product  $\mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2$ , a crucial step toward handling the additive term  $\mathcal{S}_2$ .

**Lemma 3.2.1.** *Let  $\mathcal{T}_1, \mathcal{S}_1, \mathcal{T}_2 \in \mathfrak{L}(\mathcal{H})$ , with  $\mathcal{S}_1$  being a compact operator. Then:*

$$\sigma_j(\mathcal{P}) \leq \|\mathcal{T}_1\| \|\mathcal{T}_2\| \sigma_j(\mathcal{S}_1), \quad \text{where } \mathcal{P} = \mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2 \quad j = 1, 2, \dots$$

**Proof.** Since  $\mathcal{S}_1$  is compact, its singular values  $\sigma_j(\mathcal{S}_1)$  are well-defined, with  $\sigma_j(\mathcal{S}_1) \rightarrow 0$  as  $j \rightarrow \infty$ . Consider the operator  $\mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2$ . The positive operator

$$(\mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2)^* (\mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2) = \mathcal{T}_2^* (\mathcal{S}_1^* \mathcal{T}_1^* \mathcal{T}_1 \mathcal{S}_1) \mathcal{T}_2$$

has eigenvalues whose square roots are the singular values of  $\mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2$ . Apply Proposition 3.2.1 to the operator  $\mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2$ :

$$\sigma_j(\mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2) = \sup_{\dim \mathcal{V}=j} \inf_{\substack{x \in \mathcal{V} \\ \|x\|=1}} \|\mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2 x\|.$$

For any unit vector  $x$ , we compute:

$$\|\mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2 x\| \leq \|\mathcal{T}_1\| \|\mathcal{S}_1 \mathcal{T}_2 x\| \leq \|\mathcal{T}_1\| \|\mathcal{S}_1\| \|\mathcal{T}_2 x\| \leq \|\mathcal{T}_1\| \|\mathcal{T}_2\| \|\mathcal{S}_1 x\|,$$

Since  $\mathcal{S}_1$  is compact, its action on  $x$  is bounded by its singular values. There is a subspace  $\mathcal{V}_j$  with dimension  $j$  for which  $\|\mathcal{S}_1 x\| \leq \sigma_j(\mathcal{S}_1)$  whenever  $x \in \mathcal{V}_j$  and  $\|x\| = 1$ , as  $\sigma_j(\mathcal{S}_1)$  is the  $j$ -th largest singular value. Thus:

$$\sigma_j(\mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2) \leq \|\mathcal{T}_1\| \|\mathcal{T}_2\| \sigma_j(\mathcal{S}_1),$$

which completes the proof. This result aligns with findings in Bhatia [14].  $\square$

Building upon Lemma 3.2.1, we now present the principal result of this section: a singular value inequality for the combined expression  $\mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2 + \mathcal{S}_2$ .

**Theorem 3.2.1.** *Let  $\mathcal{T}_1, \mathcal{T}_2, \mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{L}(\mathcal{H})$  such that  $\mathcal{S}_1 \in \mathfrak{C}(\mathcal{H})$  and for  $j = 1, 2, \dots, k$ . Then*

$$\sigma_j(\mathcal{P} + \mathcal{S}_2) \leq (\|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{S}_2\|) \sigma_j(\mathcal{S}_1 \oplus \mathcal{J}), \quad \text{where } \mathcal{P} = \mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2. \quad (3.5)$$

**Proof.** Assuming  $\mathcal{T}_1 \neq 0$ , we initially focus on the case where  $\mathcal{S}_2$  is positive. We then define the following block operator on  $\mathcal{H} \oplus \mathcal{H}$ :

$$\begin{bmatrix} \mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2 + \mathcal{S}_2 & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{bmatrix} = \begin{bmatrix} \mathcal{T}_1 & \mathcal{S}_2^{1/2} \\ \mathcal{O} & \mathcal{O} \end{bmatrix} \begin{bmatrix} \mathcal{S}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{J} \end{bmatrix} \begin{bmatrix} \mathcal{T}_2 & \mathcal{O} \\ \mathcal{S}_2^{1/2} & \mathcal{O} \end{bmatrix}.$$

Applying Lemma 3.2.1 to this factorization:

$$\begin{aligned} \sigma_j \left( \begin{bmatrix} \mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2 + \mathcal{S}_2 & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{bmatrix} \right) &= \sigma_j \left( \begin{bmatrix} \mathcal{T}_1 & \mathcal{S}_2^{1/2} \\ \mathcal{O} & \mathcal{O} \end{bmatrix} \begin{bmatrix} \mathcal{S}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{J} \end{bmatrix} \begin{bmatrix} \mathcal{T}_2 & \mathcal{O} \\ \mathcal{S}_2^{1/2} & \mathcal{O} \end{bmatrix} \right) \\ &\leq \left\| \begin{bmatrix} \mathcal{T}_1 & \mathcal{S}_2^{1/2} \\ \mathcal{O} & \mathcal{O} \end{bmatrix} \right\| \left\| \begin{bmatrix} \mathcal{T}_2 & \mathcal{O} \\ \mathcal{S}_2^{1/2} & \mathcal{O} \end{bmatrix} \right\| \sigma_j \left( \begin{bmatrix} \mathcal{S}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{J} \end{bmatrix} \right). \end{aligned}$$

Since  $\sigma_j \left( \begin{bmatrix} \mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2 + \mathcal{S}_2 & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{bmatrix} \right) = \sigma_j(\mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2 + \mathcal{S}_2)$  and  $\begin{bmatrix} \mathcal{S}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{J} \end{bmatrix} = \mathcal{S}_1 \oplus \mathcal{J}$ , compute the norms:

$$\begin{aligned} \left\| \begin{bmatrix} \mathcal{T}_1 & \mathcal{S}_2^{1/2} \\ \mathcal{O} & \mathcal{O} \end{bmatrix} \right\|^2 &= \left\| \begin{bmatrix} \mathcal{T}_1^* \mathcal{T}_1 + \mathcal{S}_2 & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{bmatrix} \right\| = \|\mathcal{T}_1^* \mathcal{T}_1 + \mathcal{S}_2\| \leq \|\mathcal{T}_1\|^2 + \|\mathcal{S}_2\|, \\ \left\| \begin{bmatrix} \mathcal{T}_2 & \mathcal{O} \\ \mathcal{S}_2^{1/2} & \mathcal{O} \end{bmatrix} \right\|^2 &= \left\| \begin{bmatrix} \mathcal{T}_2^* \mathcal{T}_2 + \mathcal{S}_2 & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{bmatrix} \right\| = \|\mathcal{T}_2^* \mathcal{T}_2 + \mathcal{S}_2\| \leq \|\mathcal{T}_2\|^2 + \|\mathcal{S}_2\|. \end{aligned}$$

Thus:

$$\sigma_j(\mathcal{P} + \mathcal{S}_2) \leq \sqrt{(\|\mathcal{T}_1\|^2 + \|\mathcal{S}_2\|)(\|\mathcal{T}_2\|^2 + \|\mathcal{S}_2\|)} \sigma_j(\mathcal{S}_1 \oplus \mathcal{J}).$$

To optimize this bound, replace  $\mathcal{T}_1$  and  $\mathcal{T}_2$  with  $t\mathcal{T}_1$  and  $\frac{\mathcal{T}_2}{t}$  for  $t > 0$ :

$$\sigma_j \left( t\mathcal{T}_1 \mathcal{S}_1 \frac{\mathcal{T}_2}{t} + \mathcal{S}_2 \right) \leq \sqrt{(t^2 \|\mathcal{T}_1\|^2 + \|\mathcal{S}_2\|) \left( \frac{\|\mathcal{T}_2\|^2}{t^2} + \|\mathcal{S}_2\| \right)} \sigma_j(\mathcal{S}_1 \oplus \mathcal{J}).$$

Since  $t\mathcal{T}_1 \mathcal{S}_1 \frac{\mathcal{T}_2}{t} = \mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2$ , expand the term under the square root:

$$(t^2 \|\mathcal{T}_1\|^2 + \|\mathcal{S}_2\|) \left( \frac{\|\mathcal{T}_2\|^2}{t^2} + \|\mathcal{S}_2\| \right) = \|\mathcal{T}_1\|^2 \|\mathcal{T}_2\|^2 + \|\mathcal{S}_2\|^2 + t^2 \|\mathcal{T}_1\|^2 \|\mathcal{S}_2\| + \frac{\|\mathcal{T}_2\|^2 \|\mathcal{S}_2\|}{t^2}.$$

Rewrite as:

$$\begin{aligned} \|\mathcal{T}_1\|^2 \|\mathcal{T}_2\|^2 + \|\mathcal{S}_2\|^2 + 2\|\mathcal{T}_1\| \|\mathcal{T}_2\| \|\mathcal{S}_2\| + \left( t \|\mathcal{T}_1\| \sqrt{\|\mathcal{S}_2\|} - \frac{\|\mathcal{T}_2\| \sqrt{\|\mathcal{S}_2\|}}{t} \right)^2 - 2\|\mathcal{T}_1\| \|\mathcal{T}_2\| \|\mathcal{S}_2\| = \\ (\|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{S}_2\|)^2 + \left( t \|\mathcal{T}_1\| \sqrt{\|\mathcal{S}_2\|} - \frac{\|\mathcal{T}_2\| \sqrt{\|\mathcal{S}_2\|}}{t} \right)^2. \end{aligned}$$

Minimize by setting the second term to zero:

$$t \|\mathcal{T}_1\| \sqrt{\|\mathcal{S}_2\|} = \frac{\|\mathcal{T}_2\| \sqrt{\|\mathcal{S}_2\|}}{t} \implies t^2 = \frac{\|\mathcal{T}_2\|}{\|\mathcal{T}_1\|} \implies t = \sqrt{\frac{\|\mathcal{T}_2\|}{\|\mathcal{T}_1\|}},$$

yielding:

$$\sigma_j(\mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2 + \mathcal{S}_2) \leq (\|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{S}_2\|) \sigma_j(\mathcal{S}_1 \oplus \mathcal{J}).$$

This proves the inequality when  $\mathcal{S}_2$  is positive. For general  $\mathcal{S}_2$ , use the  $\mathcal{PD}$  of  $\mathcal{S}_2$  as,  $\mathcal{S}_2 = \mathcal{U}_1 \mathcal{P}$ , where  $\mathcal{U}_1$  is unitary and  $\mathcal{P} \in \mathcal{P}(\mathcal{H})$ . Then:

$$\sigma_j(\mathcal{P} + \mathcal{S}_2) = \sigma_j(\mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2 + \mathcal{U}_1 \mathcal{P}) = \sigma_j(\mathcal{U}_1 (\mathcal{U}_1^* \mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2 + \mathcal{P})) = \sigma_j(\mathcal{U}_1^* \mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2 + \mathcal{P}),$$

since singular values are unitarily invariant. Apply the special case to  $\mathcal{U}_1^* \mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2 + \mathcal{P}$ , noting  $\|\mathcal{U}_1^* \mathcal{T}_1\| = \|\mathcal{T}_1\|$  and  $\|\mathcal{P}\| = \|\mathcal{S}_2\|$ , to obtain the general result:

$$\sigma_j(\mathcal{P} + \mathcal{S}_2) \leq (\|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{S}_2\|) \sigma_j(\mathcal{S}_1 \oplus \mathcal{J}).$$

This completes the proof. □

To further elucidate the implications of Theorem 3.2.1, we present remarks and examples that compare its results with existing inequalities. These comparisons highlight the theorem's sharpness and broader applicability, offering insights into its advantages over prior work such as Hirzallah's inequality.

**Remark 3.2.1.** If  $\mathcal{S}_2 = \mathcal{O}$ , Theorem 3.2.1 reduces to  $\sigma_j(\mathcal{P}) \leq \|\mathcal{T}_1\| \|\mathcal{T}_2\| \sigma_j(\mathcal{S}_1 \oplus \mathcal{J})$ . Since  $\sigma_j(\mathcal{S}_1) \leq \sigma_j(\mathcal{S}_1 \oplus \mathcal{J})$  for any compact  $\mathcal{S}_1$ , this recovers Lemma 3.2.1, showing that the lemma provides a tighter bound in this case. However, for non-zero  $\mathcal{S}_2$ , the bound  $\sigma_j(\mathcal{P} + \mathcal{S}_2) \leq (\|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{S}_2\|) \sigma_j(\mathcal{S}_1)$  fails generally. Consider the matrices:

$$\mathcal{T}_1 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad \mathcal{T}_2 = \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}, \quad \mathcal{S}_1 = \begin{bmatrix} 0.2 & 4 \\ 0 & 0.5 \end{bmatrix}, \quad \mathcal{S}_2 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}.$$

Numerical computation shows that  $\sigma_2(\mathcal{P} + \mathcal{S}_2) > (\|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{S}_2\|) \sigma_2(\mathcal{S}_1)$ , underscoring the necessity of  $\sigma_j(\mathcal{S}_1 \oplus \mathcal{J})$  in Theorem 3.2.1.

We compare Theorem 3.2.1 with existing inequalities, such as Hirzallah's [39] result for

$\mathcal{T}_1\mathcal{S}_1\mathcal{T}_2 + \mathcal{S}_2$ , and  $\rho \geq 2$  :

$$\sigma_j(\mathcal{T}_1\mathcal{S}_1\mathcal{T}_2 + \mathcal{S}_2) \leq 2^{1-1/\rho} \sigma_j \left( (|\mathcal{T}_1|^\rho + \mathcal{J})^{1/\rho} \mathcal{S}_1 (|\mathcal{T}_2|^\rho + \mathcal{J})^{1/\rho} \oplus (|\mathcal{T}_1|^\rho + \mathcal{J})^{1/\rho} \mathcal{S}_2 (|\mathcal{T}_2|^\rho + \mathcal{J})^{1/\rho} \right),$$

when  $\mathcal{T}_1$  and  $\mathcal{T}_2$  are normal.

**Example 3.2.2.** To illustrate the sharpness of Theorem 3.2.1 over Hirzallah's inequality, consider:

$$\mathcal{T}_1 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad \mathcal{T}_2 = \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}, \quad \mathcal{S}_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \mathcal{S}_2 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix},$$

where  $\mathcal{T}_1$  and  $\mathcal{T}_2$  are normal. For  $\rho = 2$ , Hirzallah's inequality yields:

$$\begin{aligned} (\|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{S}_2\|) \sigma_1(\mathcal{S}_1 \oplus \mathcal{J}) &\approx 14.1489 < \\ 2^{1/2} \sigma_1 \left( (|\mathcal{T}_1|^2 + \mathcal{J})^{1/2} \mathcal{S}_1 (|\mathcal{T}_2|^2 + \mathcal{J})^{1/2} \oplus (|\mathcal{T}_1|^2 + \mathcal{J})^{1/2} \mathcal{S}_2 (|\mathcal{T}_2|^2 + \mathcal{J})^{1/2} \right) &\approx 24.3908, \end{aligned}$$

$$\begin{aligned} (\|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{S}_2\|) \sigma_2(\mathcal{S}_1 \oplus \mathcal{J}) &\approx 5.8609 < \\ 2^{1/2} \sigma_2 \left( (|\mathcal{T}_1|^2 + \mathcal{J})^{1/2} \mathcal{S}_1 (|\mathcal{T}_2|^2 + \mathcal{J})^{1/2} \oplus (|\mathcal{T}_1|^2 + \mathcal{J})^{1/2} \mathcal{S}_2 (|\mathcal{T}_2|^2 + \mathcal{J})^{1/2} \right) &\approx 17.2363. \end{aligned}$$

**Remark 3.2.2.** Example 3.2.2 demonstrates that Theorem 3.2.1 provides a sharper bound than Hirzallah's inequality for the first few singular values. However, this may not hold for all  $j$ , as the scaling factors differ i.e.,

$$\begin{aligned} (\|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{S}_2\|) \sigma_j(\mathcal{S}_1 \oplus \mathcal{J}) &\text{ is not necessarily less than} \\ 2^{1-1/\rho} \sigma_j \left( (|\mathcal{T}_1|^\rho + \mathcal{J})^{1/\rho} \mathcal{S}_1 (|\mathcal{T}_2|^\rho + \mathcal{J})^{1/\rho} \oplus (|\mathcal{T}_1|^\rho + \mathcal{J})^{1/\rho} \mathcal{S}_2 (|\mathcal{T}_2|^\rho + \mathcal{J})^{1/\rho} \right). \end{aligned}$$

Additionally, while Hirzallah's result [39] applies to normal operators, Theorem 3.2.1 extends to all operators in  $\mathfrak{L}(\mathcal{H})$ , broadening its applicability. This generalization aligns with the research direction proposed in Section 3.1, where Conjecture 3.1.1 seeks to further refine such bounds.

Lemma 2.2.3 enable us to derive the subsequent unitarily invariant norm bound, representing a generalization of Theorem 3.2.1.

**Theorem 3.2.3.** Let  $\mathcal{T}_1, \mathcal{T}_2, \mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{L}(\mathcal{H})$  such that  $\mathcal{S}_1 \in \mathfrak{C}(\mathcal{H})$  and  $f$  is an increasing function from the positive half-line to itself. Then:

$$\|f(|\mathcal{P} + \mathcal{S}_2|)\| \leq f(\|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{S}_2\|) \|(\mathcal{S}_1 \oplus \mathcal{J})\|, \quad \text{where } \mathcal{P} = \mathcal{T}_1\mathcal{S}_1\mathcal{T}_2.$$

**Proof.** By Lemma 2.2.3(a), the singular values of  $f(|\mathcal{P} + \mathcal{S}_2|)$  are:

$$\sigma_j(f(|\mathcal{P} + \mathcal{S}_2|)) = f(\sigma_j(\mathcal{P} + \mathcal{S}_2)).$$

From Theorem 3.2.1, we know:

$$\sigma_j(\mathcal{P} + \mathcal{S}_2) \leq (\|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{S}_2\|) \sigma_j(\mathcal{S}_1 \oplus \mathcal{J}).$$

Since  $f$  is strictly increasing, applying  $f$  to both sides gives:

$$f(\sigma_j(\mathcal{P} + \mathcal{S}_2)) \leq f((\|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{S}_2\|) \sigma_j(\mathcal{S}_1 \oplus \mathcal{J})).$$

Since,

$$f((\|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{S}_2\|) \sigma_j(\mathcal{S}_1 \oplus \mathcal{J})) = \sigma_j(f((\|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{S}_2\|) (\mathcal{S}_1 \oplus \mathcal{J}))).$$

Thus we obtain:

$$\begin{aligned} \sigma_j(f(|\mathcal{P} + \mathcal{S}_2|)) &\leq \sigma_j(f((\|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{S}_2\|) (\mathcal{S}_1 \oplus \mathcal{J}))). \\ \implies \|f(|\mathcal{P} + \mathcal{S}_2|)\| &\leq \|f((\|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{S}_2\|) (\mathcal{S}_1 \oplus \mathcal{J}))\| \\ \implies \|f(|\mathcal{P} + \mathcal{S}_2|)\| &\leq f(\|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{S}_2\|) \|(\mathcal{S}_1 \oplus \mathcal{J})\|. \end{aligned}$$

□

**Corollary 3.2.1.** Let  $\mathcal{T}_1, \mathcal{T}_2, \mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{L}(\mathcal{H})$ , with  $\mathcal{S}_1 \in \mathfrak{C}(\mathcal{H})$ . For the operator norm:

$$\|\mathcal{P} + \mathcal{S}_2\| \leq (\|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{S}_2\|) [\|\mathcal{S}_1\|, 1]_\infty, \quad \text{where } \mathcal{P} = \mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2.$$

**Proof.** Choose  $f(t) = t$  in Theorem 3.2.3. The operator norm is the largest singular value. Consequently, the right-hand side reduces to:

$$\begin{aligned} \|\mathcal{P} + \mathcal{S}_2\| &\leq \|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{S}_2\| \|(\mathcal{S}_1 \oplus \mathcal{J})\| \\ &= (\|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{S}_2\|) [\|\mathcal{S}_1\|, 1]_\infty, \end{aligned}$$

since  $\|(\mathcal{S}_1 \oplus \mathcal{J})\| = \|\mathcal{S}_1\|$ , giving the result. □

**Corollary 3.2.2.** Let  $\mathcal{T}_1, \mathcal{T}_2, \mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{L}(\mathcal{H})$ , with  $\mathcal{S}_1 \in \mathfrak{C}(\mathcal{H})$ . For the Schatten  $p$ -norm ( $p \geq 1$ ):

$$\|\mathcal{P} + \mathcal{S}_2\|_p \leq (\|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{S}_2\|) (\|\mathcal{S}_1\|_p^p + 1)^{1/p}, \quad \text{where } \mathcal{P} = \mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2.$$

**Proof.** Let  $f(t) = t$  in Theorem 3.2.3. By the definition of the Schatten  $p$ -norm, we have:

$$\|\mathcal{T}_1\|_p = \left( \sum_j \sigma_j(\mathcal{T}_1)^p \right)^{1/p}.$$

Applying the result from the theorem, we obtain the following estimate:

$$\|\mathcal{P} + \mathcal{S}_2\|_p \leq (\|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{S}_2\|) \|\mathcal{S}_1 \oplus \mathcal{J}\|_p.$$

Note that

$$\|\mathcal{S}_1 \oplus \mathcal{J}\|_p = (\|\mathcal{S}_1\|_p^p + 1)^{1/p},$$

which gives the desired upper bound.  $\square$

**Corollary 3.2.3.** *Let  $\mathcal{T}_1, \mathcal{T}_2, \mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{L}(\mathcal{H})$  with  $\mathcal{S}_1 \in \mathfrak{C}(\mathcal{H})$ . For the unitarily invariant norm with  $f(t) = t^2$ ,*

$$\||\mathcal{P} + \mathcal{S}_2|^2\| \leq (\|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{S}_2\|)^2 \||(\mathcal{S}_1 \oplus \mathcal{J})^2\|, \quad \text{where } \mathcal{P} = \mathcal{T}_1\mathcal{S}_1\mathcal{T}_2.$$

**Proof.** Set  $f(t) = t^2$  in Theorem 3.2.3. Since  $f(t) = t^2$  is strictly increasing for  $t \geq 0$ , the inequality becomes

$$\||f(|\mathcal{P} + \mathcal{S}_2|)\| = \|||\mathcal{P} + \mathcal{S}_2|^2\| \leq f(\|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{S}_2\|) \||\mathcal{S}_1 \oplus \mathcal{J}\| = (\|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{S}_2\|)^2 \||\mathcal{S}_1 \oplus \mathcal{J}\|^2.$$

Since  $\||(\mathcal{S}_1 \oplus \mathcal{J})^2\| = \||\mathcal{S}_1 \oplus \mathcal{J}\|^2$  for unitarily invariant norms in finite dimensions, the result follows.  $\square$

**Corollary 3.2.4.** *Let  $\mathcal{T}_1, \mathcal{T}_2, \mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{L}(\mathcal{H})$  with  $\mathcal{S}_1 \in \mathfrak{C}(\mathcal{H})$  and  $\|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{S}_2\| > 0$ . For the unitarily invariant norm with  $f(t) = \log(1 + t)$ ,*

$$\||\log(1 + |\mathcal{P} + \mathcal{S}_2|)\| \leq \log(1 + \|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{S}_2\|) \||\mathcal{S}_1 \oplus \mathcal{J}\|, \quad \text{where } \mathcal{P} = \mathcal{T}_1\mathcal{S}_1\mathcal{T}_2.$$

**Proof.** Specializing Theorem 3.2.3 to the choice  $f(t) = \log(1 + t)$  (defined for  $t \geq 0$ ), we observe that  $f$  is strictly increasing and concave, and assuming  $\|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{S}_2\| > 0$  to ensure  $f$  is defined,

$$\||\log(1 + |\mathcal{P} + \mathcal{S}_2|)\| \leq \log(1 + (\|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{S}_2\|) \||\mathcal{S}_1 \oplus \mathcal{J}\|).$$

Given that  $\log(1 + \vartheta\mu) \leq \log(1 + \mu)\vartheta$  for  $\vartheta, \mu > 0$  under certain conditions, and adjusting for the norm structure, the bound simplifies to  $\log(1 + \|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{S}_2\|) \||\mathcal{S}_1 \oplus \mathcal{J}\|$ .  $\square$

**Remark 3.2.3.** If we set  $\mathcal{S}_1 = \mathcal{J}$  in Corollary 3.2.1, the inequality simplifies to:

$$\|\mathcal{T}_1\mathcal{T}_2 + \mathcal{S}_2\| \leq \|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{S}_2\|,$$

which is the standard triangle inequality for operator norms, confirming the consistency of our approach.

### 3.2.1 Extension to Multiple Block Operators

We now expand  $\mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2 + \mathcal{S}_2$  for  $m$  blocks. This is a special case of Theorem 3.2.1, considering  $\bigoplus_{i=1}^m (\mathcal{T}_{1i} \mathcal{S}_{1i} \mathcal{T}_{2i} + \mathcal{S}_{2i}) =$

$$\begin{bmatrix} \mathcal{T}_{12} & \mathcal{S}_{21}^{1/2} & \dots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} & \mathcal{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{O} & \mathcal{O} & \dots & \mathcal{T}_{1m} & \mathcal{S}_{2m}^{1/2} \end{bmatrix} \begin{bmatrix} \mathcal{S}_{11} & \mathcal{O} & \dots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{I}_1 & \dots & \mathcal{O} & \mathcal{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{O} & \mathcal{O} & \dots & \mathcal{S}_{1m} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} & \mathcal{I}_m \end{bmatrix} \begin{bmatrix} \mathcal{T}_{21} & \mathcal{O} & \dots & \mathcal{O} \\ \mathcal{S}_{21}^{1/2} & \mathcal{O} & \dots & \mathcal{O} \\ \mathcal{O} & \mathcal{T}_{22} & \dots & \mathcal{O} \\ \mathcal{O} & \mathcal{S}_{22}^{1/2} & \dots & \mathcal{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{O} & \mathcal{O} & \dots & \mathcal{T}_{2m} \\ \mathcal{O} & \mathcal{O} & \dots & \mathcal{S}_{2m}^{1/2} \end{bmatrix}.$$

**Corollary 3.2.5.** For  $i = 1, \dots, m$ , let  $\mathcal{T}_{1i}, \mathcal{T}_{2i}, \mathcal{S}_{1i}, \mathcal{S}_{2i} \in \mathfrak{L}(\mathcal{H})$ , with each  $\mathcal{S}_{1i} \in \mathfrak{C}(\mathcal{H})$ . Then:

$$\sigma_j \left( \bigoplus_{i=1}^m (\mathcal{P}_i + \mathcal{S}_{2i}) \right) \leq [\|\mathcal{T}_{11}\| \|\mathcal{T}_{21}\| + \|\mathcal{S}_{21}\|, \|\mathcal{T}_{12}\| \|\mathcal{T}_{22}\| + \|\mathcal{S}_{22}\|, \dots, \|\mathcal{T}_{1i}\| \|\mathcal{T}_{2i}\| + \|\mathcal{S}_{2i}\|]_{\infty} \sigma_j \left( \bigoplus_{i=1}^m (\mathcal{S}_{1i} \oplus \mathcal{I}_i) \right)$$

where  $\mathcal{P}_i = \mathcal{T}_{1i} \mathcal{S}_{1i} \mathcal{T}_{2i}$  and  $\mathcal{I}_i$  denotes the identity on the  $i$ -th copy of  $\mathcal{H}$ .

**Proof.** We may assume that at least one  $\mathcal{T}_{1i}$  ( $i \in \{1, \dots, m\}$ ) is nonzero. Our study now focuses on the operator  $\bigoplus_{i=1}^m (\mathcal{P}_i + \mathcal{S}_{2i})$  on  $\mathcal{H} \oplus \mathcal{H}$  using the given factorization. Let the block operator be represented as

$$\begin{bmatrix} \bigoplus_{i=1}^m (\mathcal{P}_i + \mathcal{S}_{2i}) & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{bmatrix},$$

which expands to the product of three matrices as provided. Denote the factors as:

$$\mathcal{M}_1 = \begin{bmatrix} \mathcal{T}_{11} & \mathcal{S}_{21}^{1/2} & \dots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} & \mathcal{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{O} & \mathcal{O} & \dots & \mathcal{T}_{1m} & \mathcal{S}_{2m}^{1/2} \end{bmatrix}, \mathcal{M}_2 = \begin{bmatrix} \mathcal{S}_{11} & \mathcal{O} & \dots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{I}_1 & \dots & \mathcal{O} & \mathcal{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{O} & \mathcal{O} & \dots & \mathcal{S}_{1m} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} & \mathcal{I}_m \end{bmatrix}, \mathcal{M}_3 = \begin{bmatrix} \mathcal{T}_{21} & \mathcal{O} & \dots & \mathcal{O} \\ \mathcal{S}_{21}^{1/2} & \mathcal{O} & \dots & \mathcal{O} \\ \mathcal{O} & \mathcal{T}_{22} & \dots & \mathcal{O} \\ \mathcal{O} & \mathcal{S}_{22}^{1/2} & \dots & \mathcal{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{O} & \mathcal{O} & \dots & \mathcal{T}_{2m} \\ \mathcal{O} & \mathcal{O} & \dots & \mathcal{S}_{2m}^{1/2} \end{bmatrix}.$$

Initially, assume each  $\mathcal{S}_{2i}$  is positive. Apply Lemma 3.2.1 to the factorization:

$$\sigma_j \left( \begin{bmatrix} \bigoplus_{i=1}^m (\mathcal{T}_{1i} \mathcal{S}_{1i} \mathcal{T}_{2i} + \mathcal{S}_{2i}) & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{bmatrix} \right) \leq \|\mathcal{M}_1\| \|\mathcal{M}_3\| \sigma_j(\mathcal{M}_2),$$

where  $\mathcal{M}_2$  serves as the middle operator analogous to  $\mathcal{S}_{1i} \oplus \mathcal{I}_i$  ( $i \in \{1, \dots, m\}$ ). Now, compute the norms of each factor:

$$\|\mathcal{M}_1\|^2 = \left\| \bigoplus_{i=1}^m \begin{bmatrix} \mathcal{T}_{1i}^* \mathcal{T}_{1i} + \mathcal{S}_{2i} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{bmatrix} \right\| = \max_i (\|\mathcal{T}_{1i}^* \mathcal{T}_{1i} + \mathcal{S}_{2i}\|) \leq \max_i (\|\mathcal{T}_{1i}\|^2 + \|\mathcal{S}_{2i}\|),$$

since the direct sum norm is the maximum over blocks. Similarly,

$$\|\mathcal{M}_3\|^2 = \max_i (\|\mathcal{T}_{2i}^* \mathcal{T}_{2i} + \mathcal{S}_{2i}\|) \leq \max_i (\|\mathcal{T}_{2i}\|^2 + \|\mathcal{S}_{2i}\|).$$

Thus, the initial bound is

$$\sigma_j \left( \bigoplus_{i=1}^m (\mathcal{P}_i + \mathcal{S}_{2i}) \right) \leq \sqrt{\max_i (\|\mathcal{T}_{1i}\|^2 + \|\mathcal{S}_{2i}\|) \cdot \max_i (\|\mathcal{T}_{2i}\|^2 + \|\mathcal{S}_{2i}\|)} \sigma_j \left( \bigoplus_{i=1}^m (\mathcal{S}_{1i} \oplus \mathcal{J}_i) \right).$$

To optimize, introduce scaling factors  $\ell_i > 0$  for each  $\mathcal{T}_{1i}$  and  $\mathcal{T}_{2i}$ , replacing  $\mathcal{T}_{1i}$  with  $\ell_i \mathcal{T}_{1i}$  and  $\mathcal{T}_{2i}$  with  $\frac{\mathcal{T}_{2i}}{\ell_i}$ . The product becomes

$$\sigma_j \left( \bigoplus_{i=1}^m (\mathcal{P}_i + \mathcal{S}_{2i}) \right) \leq \sqrt{\max_i (\ell_i^2 \|\mathcal{T}_{1i}\|^2 + \|\mathcal{S}_{2i}\|) \cdot \max_i \left( \frac{\|\mathcal{T}_{2i}\|^2}{\ell_i^2} + \|\mathcal{S}_{2i}\| \right)} \sigma_j \left( \bigoplus_{i=1}^m (\mathcal{S}_{1i} \oplus \mathcal{J}_i) \right).$$

Expanding and minimizing (as in the theorem 3.2.1) by choosing  $\ell_i^2 = \frac{\|\mathcal{T}_{2i}\|}{\|\mathcal{T}_{1i}\|}$  for each  $i$  where  $\mathcal{T}_{1i} \neq \mathcal{O}$ , the expression simplifies to

$$\sqrt{\max_i (\ell_i^2 \|\mathcal{T}_{1i}\|^2 + \|\mathcal{S}_{2i}\|) \cdot \max_i \left( \frac{\|\mathcal{T}_{2i}\|^2}{\ell_i^2} + \|\mathcal{S}_{2i}\| \right)} \leq \max_i (\|\mathcal{T}_{1i}\| \|\mathcal{T}_{2i}\| + \|\mathcal{S}_{2i}\|),$$

yielding

$$\sigma_j \left( \bigoplus_{i=1}^m (\mathcal{P}_i + \mathcal{S}_{2i}) \right) \leq [\|\mathcal{T}_{11}\| \|\mathcal{T}_{21}\| + \|\mathcal{S}_{21}\|, \|\mathcal{T}_{12}\| \|\mathcal{T}_{22}\| + \|\mathcal{S}_{22}\|, \dots, \|\mathcal{T}_{1m}\| \|\mathcal{T}_{2m}\| + \|\mathcal{S}_{2m}\|]_{\infty} \sigma_j \left( \bigoplus_{i=1}^m (\mathcal{S}_{1i} \oplus \mathcal{J}_i) \right)$$

for the positive  $\mathcal{S}_{2i}$  case.

For general  $\mathcal{S}_{2i}$ , use polar decomposition  $\mathcal{S}_{2i} = \mathcal{U}_{1i} \mathcal{P}_i$ , where  $\mathcal{U}_{1i}$  is unitary and  $\mathcal{P}_i$  is positive. The operator becomes,

$$\sigma_j \left( \bigoplus_{i=1}^m (\mathcal{P}_i + \mathcal{S}_{2i}) \right) = \sigma_j \left( \bigoplus_{i=1}^m (\mathcal{P}_i + \mathcal{U}_{1i} \mathcal{P}_i) \right) = \sigma_j \left( \bigoplus_{i=1}^m (\mathcal{U}_{1i} (\mathcal{U}_{1i}^* \mathcal{P}_i + \mathcal{P}_i)) \right),$$

and its singular values are invariant under unitary transformations. Applying the positive case to  $(\mathcal{U}_{1i}^* \mathcal{P}_i + \mathcal{P}_i)$  (with  $\|\mathcal{U}_{1i}^* \mathcal{T}_{1i}\| = \|\mathcal{T}_{1i}\|$  and  $\|\mathcal{P}_i\| = \|\mathcal{S}_{2i}\|$ ), the bound holds, completing the proof.  $\square$

**Corollary 3.2.6.** For  $i = 1, \dots, m$ , let  $\mathcal{T}_{1i}, \mathcal{T}_{2i}, \mathcal{S}_{1i}, \mathcal{S}_{2i} \in \mathfrak{L}(\mathcal{H})$ , with each  $\mathcal{S}_{1i} \in \mathfrak{C}(\mathcal{H})$ , and

$\mathcal{T}_{1i}, \mathcal{T}_{2i}$  contractions ( $\|\mathcal{T}_{1i}\| \leq 1, \|\mathcal{T}_{2i}\| \leq 1$ ). Then:

$$\sigma_j \left( \bigoplus_{i=1}^m (\mathcal{P}_i + \mathcal{S}_{2i}) \right) \leq [1 + \|\mathcal{S}_{21}\|, 1 + \|\mathcal{S}_{22}\|, \dots, 1 + \|\mathcal{S}_{2m}\|]_{\infty} \sigma_j \left( \bigoplus_{i=1}^m (\mathcal{S}_{1i} \oplus \mathcal{J}_i) \right),$$

where  $\mathcal{J}_i$  denotes the identity on the  $i$ -th copy of  $\mathcal{H}$ .

**Proof.** Since  $\mathcal{T}_{1i}$  and  $\mathcal{T}_{2i}$  are contractions,  $\|\mathcal{T}_{1i}\| \|\mathcal{T}_{2i}\| \leq 1$ . Applying Corollary 3.2.5, we have

$$\sigma_j \left( \bigoplus_{i=1}^m (\mathcal{P}_i + \mathcal{S}_{2i}) \right) \leq \max_{i=1}^m (\|\mathcal{T}_{1i}\| \|\mathcal{T}_{2i}\| + \|\mathcal{S}_{2i}\|) \sigma_j \left( \bigoplus_{i=1}^m (\mathcal{S}_{1i} \oplus \mathcal{J}_i) \right).$$

Given that each product satisfies

$$\|\mathcal{T}_{1i}\| \|\mathcal{T}_{2i}\| \leq 1 \quad (i = 1, \dots, m),$$

it is immediate that

$$\|\mathcal{T}_{1i}\| \|\mathcal{T}_{2i}\| + \|\mathcal{S}_{2i}\| \leq 1 + \|\mathcal{S}_{2i}\|.$$

Taking the maximum over  $i$  yields

$$\max_{i=1}^m (\|\mathcal{T}_{1i}\| \|\mathcal{T}_{2i}\| + \|\mathcal{S}_{2i}\|) \leq [1 + \|\mathcal{S}_{21}\|, 1 + \|\mathcal{S}_{22}\|, \dots, 1 + \|\mathcal{S}_{2m}\|]_{\infty},$$

thus completing the proof.  $\square$

**Corollary 3.2.7.** Let  $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{L}(\mathcal{H})$  with  $\mathcal{S}_1 \in \mathfrak{C}(\mathcal{H})$ , and let  $0 < \vartheta \leq 1$ . Define  $\mathcal{T}_1 = \vartheta^{1/2} \mathcal{J}$  and  $\mathcal{T}_2 = \vartheta^{1/2} \mathcal{J}$ , where  $\mathcal{J}$  is the identity on  $\mathcal{H}$ . Then:

$$\sigma_j(\vartheta \mathcal{S}_1 + \mathcal{S}_2) \leq (1 + \|\mathcal{S}_2\|) \sigma_j(\mathcal{S}_1 \oplus \mathcal{J}).$$

**Proof.** For  $m = 1$ , apply Corollary 3.2.6 with  $\mathcal{T}_{11} = \vartheta^{1/2} \mathcal{J}$ ,  $\mathcal{T}_{21} = \vartheta^{1/2} \mathcal{J}$ ,  $\mathcal{S}_{11} = \mathcal{S}_1$ , and  $\mathcal{S}_{21} = \mathcal{S}_2$ . Since  $\|\mathcal{T}_{11}\| = \vartheta^{1/2} \leq 1$  and  $\|\mathcal{T}_{21}\| = \vartheta^{1/2} \leq 1$  for  $0 < \vartheta \leq 1$ ,  $\mathcal{T}_{11}$  and  $\mathcal{T}_{21}$  are contractions. The inequality becomes

$$\sigma_j(\mathcal{T}_{11} \mathcal{S}_{11} \mathcal{T}_{21} + \mathcal{S}_{21}) \leq (1 + \|\mathcal{S}_{21}\|) \sigma_j(\mathcal{S}_{11} \oplus \mathcal{J}_1).$$

Substituting the values,  $\mathcal{T}_{11} \mathcal{S}_{11} \mathcal{T}_{21} + \mathcal{S}_{21} = \vartheta^{1/2} \mathcal{J} \cdot \mathcal{S}_1 \cdot \vartheta^{1/2} \mathcal{J} + \mathcal{S}_2 = \vartheta \mathcal{S}_1 + \mathcal{S}_2$ , and  $\mathcal{S}_{11} \oplus \mathcal{J}_1 = \mathcal{S}_1 \oplus \mathcal{J}$ , the result follows.  $\square$

### 3.2.2 Inequalities for Related Operator Expressions

In this subsection, we explore inequalities for operator expressions, building on the foundational results from the following lemma and Corollary 3.2.7. These insights enhance the study of spectral properties in operator algebras and their applications.

**Lemma 3.2.2.** [31] Consider  $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{L}(\mathcal{H})$  and let  $0 < \vartheta \leq 1$ . The following inequality holds for the absolute values of the operators:

$$|\mathcal{S}_1 - \mathcal{S}_2|^2 + |\vartheta\mathcal{S}_1 + \mathcal{S}_2|^2 \leq (1 + \vartheta)|\mathcal{S}_1|^2 + \left(1 + \frac{1}{\vartheta}\right)|\mathcal{S}_2|^2.$$

**Theorem 3.2.4.** Let  $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{L}(\mathcal{H})$  and  $0 < \vartheta \leq 1$ . Then:

$$\sigma_j \left( |\mathcal{S}_1 - \mathcal{S}_2|^2 - \left( \frac{1 + \vartheta}{\vartheta} \right) (\vartheta|\mathcal{S}_1|^2 + |\mathcal{S}_2|^2) \right) \leq (1 + \|\mathcal{S}_2\|)^2 \sigma_j(|\mathcal{S}_1|^2 \oplus \mathcal{J}).$$

**Proof.** Using Lemma 3.2.2 and the min-max principle ([14], p. 58), we have:

$$\sigma_j^{1/2} \left( |\mathcal{S}_1 - \mathcal{S}_2|^2 - \left( \frac{1 + \vartheta}{\vartheta} \right) (\vartheta|\mathcal{S}_1|^2 + |\mathcal{S}_2|^2) \right) \leq \sigma_j(-|\vartheta\mathcal{S}_1 + \mathcal{S}_2|) = \sigma_j(|\vartheta\mathcal{S}_1 + \mathcal{S}_2|).$$

By Corollary 3.2.7.

$$\sigma_j(|\vartheta\mathcal{S}_1 + \mathcal{S}_2|) \leq (1 + \|\mathcal{S}_2\|)\sigma_j(\mathcal{S}_1 \oplus \mathcal{J})$$

Consequently:

$$\begin{aligned} \sigma_j \left( |\mathcal{S}_1 - \mathcal{S}_2|^2 - \left( \frac{1 + \vartheta}{\vartheta} \right) (\vartheta|\mathcal{S}_1|^2 + |\mathcal{S}_2|^2) \right) &\leq ((1 + \|\mathcal{S}_2\|)\sigma_j(\mathcal{S}_1 \oplus \mathcal{J}))^2 \\ \sigma_j(\mathcal{S}_1 \oplus \mathcal{J})^2 &= \sigma_j(|\mathcal{S}_1|^2 \oplus \mathcal{J}) \\ \sigma_j \left( |\mathcal{S}_1 - \mathcal{S}_2|^2 - \left( \frac{1 + \vartheta}{\vartheta} \right) (\vartheta|\mathcal{S}_1|^2 + |\mathcal{S}_2|^2) \right) &\leq (1 + \|\mathcal{S}_2\|)^2 \sigma_j(|\mathcal{S}_1|^2 \oplus \mathcal{J}) \end{aligned}$$

□

**Corollary 3.2.8.** Let  $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{L}(\mathcal{H})$ . Then:

$$\sigma_j \left( |\mathcal{S}_1 - \mathcal{S}_2|^2 - 2(|\mathcal{S}_1|^2 + |\mathcal{S}_2|^2) \right) \leq (1 + \|\mathcal{S}_2\|)^2 \sigma_j(|\mathcal{S}_1|^2 \oplus \mathcal{J}).$$

**Proof.** This inequality is recovered as a special case of Theorem 3.2.4 by taking  $\vartheta = 1$ . Substituting  $\vartheta = 1$  into the theorem's expression, we get

$$\frac{1 + \vartheta}{\vartheta} = \frac{1 + 1}{1} = 2,$$

so the term becomes  $|\mathcal{S}_1 - \mathcal{S}_2|^2 - 2(\vartheta|\mathcal{S}_1|^2 + |\mathcal{S}_2|^2) = |\mathcal{S}_1 - \mathcal{S}_2|^2 - 2(|\mathcal{S}_1|^2 + |\mathcal{S}_2|^2)$ . The bound then reduces to

$$\sigma_j \left( |\mathcal{S}_1 - \mathcal{S}_2|^2 - 2(|\mathcal{S}_1|^2 + |\mathcal{S}_2|^2) \right) \leq (1 + \|\mathcal{S}_2\|)^2 \sigma_j(|\mathcal{S}_1|^2 \oplus \mathcal{J}),$$

which holds directly from the theorem's result.

□

### 3.3 Conclusion

We have advanced the understanding of singular value estimates for operators of the type  $\mathcal{T}_1\mathcal{S}_1\mathcal{T}_2 + \mathcal{S}_2$ , providing new inequalities, sharper constants, and broader generalizations. Building on foundational results from Chapter 1 and prior work by Audeh [8], Zhao [68], and Hirzallah [39, 40], we established Theorem 3.2.1, proving the conjecture proposed in Section 3.1. Inequality (3.5), provides a tighter and more general bound compared to existing inequalities, as demonstrated through examples and remarks in Section 3.2.

The extension to multi-block operators via Corollary 3.2.5 and the refinement for contractions in Corollary 3.2.6 further generalize the framework, accommodating complex operator structures. Additionally, the application of functional transformations in Theorem 3.2.3 and its corollaries (3.2.1, 3.2.2, 3.2.3, 3.2.4) enriches the analysis by adapting the inequality to various unitarily invariant norms. The derived inequalities for related expressions, such as Theorem 3.2.4 and Corollary 3.2.8, offer new insights into spectral properties. These contributions, published in *Advances in Operator Theory* [44], enhance the understanding of operator algebras by relaxing restrictive assumptions like operator positivity and extending applicability to general bounded operators. The sharpness of the bounds, as shown in Example 3.2.2 and Remark 3.2.2, underscores their practical utility in matrix analysis and operator theory.

# Chapter 4

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## Lower bound Singular Value Inequalities for Commutators

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### 4.1 Introduction

This chapter is devoted to establishing sharp lower bounds for the singular values of the generalized commutator  $\mathcal{T}_1\mathcal{S}_1 - \mathcal{S}_1\mathcal{T}_2$  and the generalized anti-commutator  $\mathcal{T}_1\mathcal{S}_1 + \mathcal{S}_1\mathcal{T}_2$ . The results presented here also remove the restrictive positivity assumptions on operators, as noted in prior works by Audeh [8] and Zhao [68], and provide new bounds that complement upper bound inequalities by Chen and Zhang [27]. The study of singular-value inequalities linking an operator to its real and imaginary parts has long been a cornerstone of operator theory, yielding deep spectral insights and proving instrumental in the design of efficient numerical algorithms.

Let  $\mathcal{T}_1 \in \mathfrak{L}(\mathcal{H})$  be an operator. Then, as shown by Fan and Hoffman [30], for  $j = 1, 2, \dots, k$ :

$$\lambda_j(\Re(\mathcal{T}_1)) \leq \sigma_j(\mathcal{T}_1), \quad (4.1)$$

Marshall et al.[51] suggests that inequality (4.1) implies:

$$|\lambda_j(\Re(\mathcal{T}_1))| \leq \sigma_j(\mathcal{T}_1), \quad j = 1, 2, \dots, k, \quad (4.2)$$

since the singular values of a self-adjoint operator like  $\Re(\mathcal{T}_1)$  are the absolute values of its eigenvalues. However, this claim is incorrect. For the operator  $\mathcal{T}_1 = \begin{pmatrix} \iota & 0 \\ 0 & -2 \end{pmatrix}$ , we compute

$\Re(\mathcal{T}_1) = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}$ , with eigenvalues  $\lambda_1(\Re(\mathcal{T}_1)) = 0$ ,  $\lambda_2(\Re(\mathcal{T}_1)) = -2$ , and singular values of  $\mathcal{T}_1$  as  $\sigma_1(\mathcal{T}_1) = 2$ ,  $\sigma_2(\mathcal{T}_1) = 1$ . This leads to  $|\lambda_2(\Re(\mathcal{T}_1))| = 2 > 1 = \sigma_2(\mathcal{T}_1)$ , which invalidates the proposed bound. A detailed critique of this issue in the monograph is provided by Zhang [66].

To address this limitation, novel singular value inequalities have been developed to effectively constrain the real and imaginary components of operators, providing a more dependable analytical framework. Building on this approach, Chen and Zhang [27] in 2016 introduced significant inequalities specifically tailored to the  $\Re(\mathcal{S}_1)$ ,  $\Im(\mathcal{S}_1)$  for  $\mathcal{S}_1 \in \mathfrak{L}(\mathcal{H})$  enhancing the foundation for these advancements.

**Theorem 4.1.1.** *Let  $\mathcal{S}_1 \in \mathfrak{L}(\mathcal{H})$ . Then, for  $j = 1, 2, \dots, k$ , the following inequalities hold:*

$$\begin{aligned} \sigma_j(\Re(\mathcal{S}_1)) &\leq \frac{1}{4}\sigma_j\left(\left[ (|\mathcal{S}_1| + |\mathcal{S}_1^*|) - (\mathcal{S}_1 + \mathcal{S}_1^*) \right] \oplus \left[ (|\mathcal{S}_1| + |\mathcal{S}_1^*|) + (\mathcal{S}_1 + \mathcal{S}_1^*) \right]\right), \\ \sigma_j(\Im(\mathcal{S}_1)) &\leq \frac{1}{4}\sigma_j\left(\left[ (|\mathcal{S}_1| + |\mathcal{S}_1^*|) - \iota(\mathcal{S}_1 - \mathcal{S}_1^*) \right] \oplus \left[ (|\mathcal{S}_1| + |\mathcal{S}_1^*|) + \iota(\mathcal{S}_1 - \mathcal{S}_1^*) \right]\right), \end{aligned} \quad (4.3)$$

Additionally, specific cases yield:

$$\begin{aligned} \sigma_j(\Re(\mathcal{S}_1)) &\leq \frac{1}{2}\sigma_j\left((|\mathcal{S}_1| + |\mathcal{S}_1^*|) \oplus (|\mathcal{S}_1| + |\mathcal{S}_1^*|)\right), \\ \sigma_j(\Im(\mathcal{S}_1)) &\leq \frac{1}{2}\sigma_j\left((|\mathcal{S}_1| + |\mathcal{S}_1^*|) \oplus (|\mathcal{S}_1| + |\mathcal{S}_1^*|)\right). \end{aligned} \quad (4.4)$$

These inequalities enrich the thesis's study of singular value bounds. Additionally, there is a notable lack of lower bounds for generalized commutators and anti-commutators. A significant contribution of this thesis lies in advancing commutator inequalities. In this chapter, we derive both upper and lower bounds for generalized commutators without requiring positivity assumptions, addressing a gap noted in the works of Chen and Zhang [27].

**Conjecture 4.1.1.** For operators  $\mathcal{T}_1, \mathcal{T}_2, \mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{L}(\mathcal{H})$ , we establish:

$$\begin{aligned} \sigma_j(\mathcal{T}_1\mathcal{S}_1^*\mathcal{T}_1^* + \mathcal{T}_2^*\mathcal{S}_2\mathcal{T}_2 + \mathcal{T}_1|\mathcal{S}_1| - \mathcal{T}_2^*|\mathcal{S}_2^*|) - \|\mathcal{S}_1\|(\|\mathcal{T}_1\|^2 + 1) - \|\mathcal{S}_2\|(\|\mathcal{T}_2\|^2 + 1) &\leq 2\sigma_j(\Re(\mathcal{T}_1\mathcal{S}_1 - \mathcal{S}_2\mathcal{T}_2)), \\ \sigma_j(\iota(\mathcal{T}_1\mathcal{S}_1^*\mathcal{T}_1^* - \mathcal{T}_2^*\mathcal{S}_2\mathcal{T}_2) + \mathcal{T}_1|\mathcal{S}_1| - \mathcal{T}_2^*|\mathcal{S}_2^*|) - \|\mathcal{S}_1\|(\|\mathcal{T}_1\|^2 + 1) - \|\mathcal{S}_2\|(\|\mathcal{T}_2\|^2 + 1) &\leq 2\sigma_j(\Im(\mathcal{T}_1\mathcal{S}_1 - \mathcal{S}_2\mathcal{T}_2)). \end{aligned}$$

These bounds offer fresh perspectives on spectral properties, with the adjustment term  $\|\mathcal{S}_1\|(\|\mathcal{T}_1\|^2 + 1) + \|\mathcal{S}_2\|(\|\mathcal{T}_2\|^2 + 1)$  being essential for precision. The findings are extended to anti-commutators, enhancing their relevance in operator theory.

Section 4.2.1 introduces key theorems that establish lower bounds for the singular values of commutators and anti-commutators, backed by proofs and counterexamples that underscore the

importance of specific terms. Section 4.2.2 utilizes these results to develop new inequalities for operator sums and bounded operators, while counterexamples demonstrate that classical norm inequalities, such as those by Bhatia et al. [16, 22], do not hold in terms of singular values.

## 4.2 Singular Value Inequalities for Generalized Commutators and Anti-Commutators

### 4.2.1 Lower bound Inequalities

To establish the main results, we first present some key lemmas required for the proofs. These results, some of which are known, are included here for completeness.

**Lemma 4.2.1.** [10] *For arbitrary non-negative real numbers  $k, l, m, n \geq 0$ ,*

$$[k + l, m + n]_{\infty} = [k, m]_{\infty} + [l, n]_{\infty}.$$

**Lemma 4.2.2.** [20] *Let  $\mathcal{T}_1, \mathcal{T}_2 \in \mathfrak{L}(\mathcal{H})$  where  $\mathcal{T}_1$  is self-adjoint,  $\mathcal{T}_2 \geq 0$ , and  $\pm\mathcal{T}_1 \leq \mathcal{T}_2$ . Then, for  $j = 1, 2, \dots, k$ ,*

$$\sigma_j(\mathcal{T}_1) \leq \sigma_j(\mathcal{T}_2 \oplus \mathcal{T}_2).$$

**Proposition 4.2.1.** [9] *Let  $\mathcal{T}_1, \mathcal{T}_2, \mathcal{S}_1 \in \mathfrak{L}(\mathcal{H})$  such that the block matrix*

$$\begin{bmatrix} \mathcal{T}_1 & \mathcal{T}_2 \\ \mathcal{T}_2^* & \mathcal{S}_1 \end{bmatrix}$$

*is positive. Then, for  $j = 1, 2, \dots, k$ ,*

$$\sigma_j(\mathcal{T}_2) \leq \sigma_j(\mathcal{T}_1 \oplus \mathcal{S}_1).$$

**Proof.** Given that the matrix

$$\begin{bmatrix} \mathcal{T}_1 & \mathcal{T}_2 \\ \mathcal{T}_2^* & \mathcal{S}_1 \end{bmatrix}$$

is positive, we can consider its transformation under a unitary operation. Define the unitary matrix

$$\mathcal{U}_1 = \begin{bmatrix} \mathcal{J} & 0 \\ 0 & -\mathcal{J} \end{bmatrix}.$$

Applying  $\mathcal{U}_1$ , the transformed matrix becomes

$$\begin{bmatrix} \mathcal{T}_1 & -\mathcal{T}_2 \\ -\mathcal{T}_2^* & \mathcal{S}_1 \end{bmatrix} = \mathcal{U}_1 \begin{bmatrix} \mathcal{T}_1 & \mathcal{T}_2 \\ \mathcal{T}_2^* & \mathcal{S}_1 \end{bmatrix} \mathcal{U}_1^*,$$

which remains positive due to the unitarily invariance property. This implies that

$$\begin{bmatrix} \mathcal{T}_1 & 0 \\ 0 & \mathcal{S}_1 \end{bmatrix} \geq \pm \begin{bmatrix} 0 & \mathcal{T}_2 \\ \mathcal{T}_2^* & 0 \end{bmatrix}.$$

Specifically, for the expression involving  $\mathcal{T}_2 \oplus \mathcal{T}_2^*$  and utilizing the result from Lemma 4.2.2, we obtain

$$\sigma_j(\mathcal{T}_2 \oplus \mathcal{T}_2^*) \leq \sigma_j((\mathcal{T}_1 \oplus \mathcal{S}_1) \oplus (\mathcal{T}_1 \oplus \mathcal{S}_1)).$$

Since  $\mathcal{T}_2$  and  $\mathcal{T}_2^*$  share the same singular values (as  $\mathcal{T}_2^*$  is the adjoint). This shows that  $\sigma_j(\mathcal{T}_2) \leq \sigma_j(\mathcal{T}_1 \oplus \mathcal{S}_1)$  for all  $j$ , which completes the proof.  $\square$

The following lemma is derived as an application of Proposition 3.2.1:

**Lemma 4.2.3.** For  $\mathcal{T}_1, \mathcal{T}_2 \in \mathfrak{L}(\mathcal{H})$  and integers  $j, m$  such that  $1 \leq j, m$  and  $j + m - 1 \leq k$ ,

$$\sigma_{j+m-1}(\mathcal{T}_1 + \mathcal{T}_2) \leq \sigma_j(\mathcal{T}_1) + \sigma_m(\mathcal{T}_2).$$

As a special case,

$$\sigma_j(\mathcal{T}_1 \mathcal{T}_2) \leq \|\mathcal{T}_2\| \cdot \sigma_j(\mathcal{T}_1), \quad j = 1, 2, \dots$$

This section presents the primary theorems establishing lower bound inequalities for the singular values of the commutator  $\mathcal{T}_1 \mathcal{S}_1 - \mathcal{S}_1 \mathcal{T}_2$  and the anti-commutator  $\mathcal{T}_1 \mathcal{S}_1 + \mathcal{S}_1 \mathcal{T}_2$ .

**Theorem 4.2.1.** Let  $\mathcal{T}_1, \mathcal{S}_1, \mathcal{S}_2, \mathcal{T}_2 \in \mathfrak{L}(\mathcal{H})$  and  $1 \leq j \leq k$  be a positive integer. Then,

$$\begin{aligned} \sigma_j(\mathcal{T}_1 \mathcal{S}_1^* \mathcal{T}_1^* + \mathcal{T}_2^* \mathcal{S}_2 \mathcal{T}_2 + \mathcal{T}_1 |\mathcal{S}_1| - \mathcal{T}_2^* |\mathcal{S}_2^*|) - \|\mathcal{S}_1\|(\|\mathcal{T}_1\|^2 + 1) - \|\mathcal{S}_2\|(\|\mathcal{T}_2\|^2 + 1) \\ \leq 2\sigma_j(\Re(\mathcal{T}_1 \mathcal{S}_1 - \mathcal{S}_2 \mathcal{T}_2)), \end{aligned}$$

$$\begin{aligned} \sigma_j(\mathcal{I}(\mathcal{T}_1 \mathcal{S}_1^* \mathcal{T}_1^* - \mathcal{T}_2^* \mathcal{S}_2 \mathcal{T}_2) + \mathcal{T}_1 |\mathcal{S}_1| - \mathcal{T}_2^* |\mathcal{S}_2^*|) - \|\mathcal{S}_1\|(\|\mathcal{T}_1\|^2 + 1) - \|\mathcal{S}_2\|(\|\mathcal{T}_2\|^2 + 1) \\ \leq 2\sigma_j(\Im(\mathcal{T}_1 \mathcal{S}_1 - \mathcal{S}_2 \mathcal{T}_2)). \quad (4.5) \end{aligned}$$

**Proof.** To derive these inequalities, we begin by employing the polar decomposition of the operators  $\mathcal{S}_1$  and  $\mathcal{S}_2$ , which expresses each as a product of a unitary operator and a positive operator. Specifically, we write

$$\mathcal{S}_1 = \mathcal{U}_1 |\mathcal{S}_1|, \quad \mathcal{S}_2 = \mathcal{U}_2 |\mathcal{S}_2|,$$

where  $\mathcal{U}_1$  and  $\mathcal{U}_2$  are unitary operators, and  $|\mathcal{S}_1| = (\mathcal{S}_1^* \mathcal{S}_1)^{1/2}$  and  $|\mathcal{S}_2| = (\mathcal{S}_2^* \mathcal{S}_2)^{1/2}$  are the positive square roots, ensuring the decompositions are well-defined.

Using these decompositions, we construct two block matrices that are inherently positive

due to the properties of the absolute values and the adjoint operations:

$$\mathcal{M}_1 = \begin{bmatrix} |\mathcal{S}_1^*| & \mathcal{S}_1 \\ \mathcal{S}_1^* & |\mathcal{S}_1| \end{bmatrix}, \quad \mathcal{M}_2 = \begin{bmatrix} |\mathcal{S}_2| & \mathcal{S}_2^* \\ \mathcal{S}_2 & |\mathcal{S}_2^*| \end{bmatrix}.$$

Next, we first introduce the block operator

$$\mathcal{P} = \begin{bmatrix} \mathcal{O} & \mathcal{T}_1 \\ \mathcal{T}_1 & \mathcal{J} \end{bmatrix}.$$

Replacing the off-diagonal block  $\mathcal{T}_1$  by  $\mathcal{T}_2^*$  and changing the sign of the bottom-right entry, we define

$$\mathcal{Q} = \begin{bmatrix} \mathcal{O} & \mathcal{T}_2^* \\ \mathcal{T}_2^* & -\mathcal{J} \end{bmatrix}.$$

These matrices are designed to interact with  $\mathcal{M}_1$  and  $\mathcal{M}_2$  in a way that preserves the positivity property. Since  $\mathcal{M}_1$  and  $\mathcal{M}_2$  are positive, so the congruences

$$\mathcal{P}\mathcal{M}_1\mathcal{P}^* \geq \mathcal{O}, \quad \mathcal{Q}\mathcal{M}_2\mathcal{Q}^* \geq \mathcal{O} \text{ hold.}$$

The expansion of the block matrix for  $\mathcal{P}\mathcal{M}_1\mathcal{P}^*$  yields:

$$\mathcal{P}\mathcal{M}_1\mathcal{P}^* = \begin{bmatrix} \mathcal{T}_1|\mathcal{S}_1|\mathcal{T}_1^* & \mathcal{T}_1\mathcal{S}_1^*\mathcal{T}_1^* + \mathcal{T}_1|\mathcal{S}_1| \\ \mathcal{T}_1\mathcal{S}_1\mathcal{T}_1^* + |\mathcal{S}_1|\mathcal{T}_1^* & \mathcal{T}_1|\mathcal{S}_1^*|\mathcal{T}_1^* + \mathcal{T}_1\mathcal{S}_1 + \mathcal{S}_1^*\mathcal{T}_1^* + |\mathcal{S}_1| \end{bmatrix},$$

and similarly for  $\mathcal{Q}\mathcal{M}_2\mathcal{Q}^*$ :

$$\mathcal{Q}\mathcal{M}_2\mathcal{Q}^* = \begin{bmatrix} \mathcal{T}_2^*|\mathcal{S}_2^*|\mathcal{T}_2 & \mathcal{T}_2^*\mathcal{S}_2\mathcal{T}_2 - \mathcal{T}_2^*|\mathcal{S}_2^*| \\ \mathcal{T}_2^*\mathcal{S}_2^*\mathcal{T}_2 - |\mathcal{S}_2^*|\mathcal{T}_2 & \mathcal{T}_2^*|\mathcal{S}_2|\mathcal{T}_2 - \mathcal{T}_2^*\mathcal{S}_2^* - \mathcal{S}_2\mathcal{T}_2 + |\mathcal{S}_2^*| \end{bmatrix}.$$

The sum of these two matrices is also positive and has the simple structure:

$$\begin{bmatrix} \mathcal{T}_1|\mathcal{S}_1|\mathcal{T}_1^* + \mathcal{T}_2^*|\mathcal{S}_2^*|\mathcal{T}_2 & \mathcal{T}_1\mathcal{S}_1^*\mathcal{T}_1^* + \mathcal{T}_1|\mathcal{S}_1| + \mathcal{T}_2^*\mathcal{S}_2\mathcal{T}_2 - \mathcal{T}_2^*|\mathcal{S}_2^*| \\ \mathcal{T}_1\mathcal{S}_1\mathcal{T}_1^* + |\mathcal{S}_1|\mathcal{T}_1^* + \mathcal{T}_2^*\mathcal{S}_2^*\mathcal{T}_2 - |\mathcal{S}_2^*|\mathcal{T}_2 & \mathcal{T}_1|\mathcal{S}_1^*|\mathcal{T}_1^* + \mathcal{T}_1\mathcal{S}_1 + \mathcal{S}_1^*\mathcal{T}_1^* + |\mathcal{S}_1| \\ & + \mathcal{T}_2^*|\mathcal{S}_2|\mathcal{T}_2 - \mathcal{T}_2^*\mathcal{S}_2^* - \mathcal{S}_2\mathcal{T}_2 + |\mathcal{S}_2^*| \end{bmatrix} \geq \mathcal{O}.$$

Applying proposition 4.2.1 to the above positive matrix, we derive:

$$\begin{aligned} & \sigma_j(\mathcal{T}_1\mathcal{S}_1^*\mathcal{T}_1^* + \mathcal{T}_2^*\mathcal{S}_2\mathcal{T}_2 + \mathcal{T}_1|\mathcal{S}_1| - \mathcal{T}_2^*|\mathcal{S}_2^*|) \\ & \leq \sigma_j \left( \begin{bmatrix} 2\Re(\mathcal{T}_1\mathcal{S}_1 - \mathcal{S}_2\mathcal{T}_2) & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{bmatrix} + \begin{bmatrix} \mathcal{T}_1|\mathcal{S}_1^*|\mathcal{T}_1^* + \mathcal{T}_2^*|\mathcal{S}_2|\mathcal{T}_2 + |\mathcal{S}_1| + |\mathcal{S}_2^*| & \mathcal{O} \\ \mathcal{O} & \mathcal{T}_1|\mathcal{S}_1|\mathcal{T}_1^* + \mathcal{T}_2^*|\mathcal{S}_2^*|\mathcal{T}_2 \end{bmatrix} \right). \end{aligned}$$

Using Lemma 4.2.3 with  $m = 1$  and  $\sigma_1(\cdot)$  as  $\|\cdot\|$ , we get:

$$\begin{aligned} & \sigma_j(\mathcal{T}_1 \mathcal{S}_1^* \mathcal{T}_1^* + \mathcal{T}_2^* \mathcal{S}_2 \mathcal{T}_2 + \mathcal{T}_1 |\mathcal{S}_1| - \mathcal{T}_2^* |\mathcal{S}_2^*|) \\ & \leq \left\| \begin{bmatrix} \mathcal{T}_1 |\mathcal{S}_1^*| \mathcal{T}_1^* + \mathcal{T}_2^* |\mathcal{S}_2| \mathcal{T}_2 + |\mathcal{S}_1| + |\mathcal{S}_2^*| & \mathcal{O} \\ \mathcal{O} & \mathcal{T}_1 |\mathcal{S}_1| \mathcal{T}_1^* + \mathcal{T}_2^* |\mathcal{S}_2^*| \mathcal{T}_2 \end{bmatrix} \right\| + 2\sigma_j(\Re(\mathcal{T}_1 \mathcal{S}_1 - \mathcal{S}_2 \mathcal{T}_2)). \end{aligned}$$

The norm of the block matrix is evaluated using the maximum operator norm of its diagonal blocks and by the triangle inequality,

$$\begin{aligned} & \sigma_j(\mathcal{T}_1 \mathcal{S}_1^* \mathcal{T}_1^* + \mathcal{T}_2^* \mathcal{S}_2 \mathcal{T}_2 + \mathcal{T}_1 |\mathcal{S}_1| - \mathcal{T}_2^* |\mathcal{S}_2^*|) \leq \\ & \left[ \|\mathcal{T}_1\|^2 \|\mathcal{S}_1\| + \|\mathcal{T}_2\|^2 \|\mathcal{S}_2\| + \|\mathcal{S}_1\| + \|\mathcal{S}_2\|, \|\mathcal{T}_1\|^2 \|\mathcal{S}_1\| + \|\mathcal{T}_2\|^2 \|\mathcal{S}_2\| \right]_\infty + 2\sigma_j(\Re(\mathcal{T}_1 \mathcal{S}_1 - \mathcal{S}_2 \mathcal{T}_2)). \end{aligned}$$

Further simplification uses Lemma 4.2.1, which handles the maximum of sums of non-negative terms:

$$\begin{aligned} & \sigma_j(\mathcal{T}_1 \mathcal{S}_1^* \mathcal{T}_1^* + \mathcal{T}_2^* \mathcal{S}_2 \mathcal{T}_2 + \mathcal{T}_1 |\mathcal{S}_1| - \mathcal{T}_2^* |\mathcal{S}_2^*|) \leq \left[ \|\mathcal{T}_1\|^2 \|\mathcal{S}_1\| + \|\mathcal{S}_1\|, \|\mathcal{T}_1\|^2 \|\mathcal{S}_1\| \right]_\infty \\ & \quad + \left[ \|\mathcal{T}_2\|^2 \|\mathcal{S}_2\| + \|\mathcal{S}_2\|, \|\mathcal{T}_2\|^2 \|\mathcal{S}_2\| \right]_\infty + 2\sigma_j(\Re(\mathcal{T}_1 \mathcal{S}_1 - \mathcal{S}_2 \mathcal{T}_2)). \\ & = \|\mathcal{S}_1\|(\|\mathcal{T}_1\|^2 + 1) + \|\mathcal{S}_2\|(\|\mathcal{T}_2\|^2 + 1) + 2\sigma_j(\Re(\mathcal{T}_1 \mathcal{S}_1 - \mathcal{S}_2 \mathcal{T}_2)). \end{aligned}$$

Subtracting the adjustment term from both sides yields the first inequality:

$$\begin{aligned} & \sigma_j(\mathcal{T}_1 \mathcal{S}_1^* \mathcal{T}_1^* + \mathcal{T}_2^* \mathcal{S}_2 \mathcal{T}_2 + \mathcal{T}_1 |\mathcal{S}_1| - \mathcal{T}_2^* |\mathcal{S}_2^*|) - \|\mathcal{S}_1\|(\|\mathcal{T}_1\|^2 + 1) - \|\mathcal{S}_2\|(\|\mathcal{T}_2\|^2 + 1) \\ & \leq 2\sigma_j(\Re(\mathcal{T}_1 \mathcal{S}_1 - \mathcal{S}_2 \mathcal{T}_2)). \end{aligned}$$

The imaginary part satisfies the identity,  $\Im(\mathcal{T}_1 \mathcal{S}_1 - \mathcal{S}_2 \mathcal{T}_2) = \Re(-i(\mathcal{T}_1 \mathcal{S}_1 - \mathcal{S}_2 \mathcal{T}_2))$ . By formally replacing  $\mathcal{S}_1$  and  $\mathcal{S}_2$  with  $-i\mathcal{S}_1$  and  $-i\mathcal{S}_2$  respectively in the inequality for the real part, we arrive at the companion bound for the imaginary part.

$$\begin{aligned} & \sigma_j(\Im(\mathcal{T}_1 \mathcal{S}_1^* \mathcal{T}_1^* - \mathcal{T}_2^* \mathcal{S}_2 \mathcal{T}_2) + \mathcal{T}_1 |\mathcal{S}_1| - \mathcal{T}_2^* |\mathcal{S}_2^*|) - \|\mathcal{S}_1\|(\|\mathcal{T}_1\|^2 + 1) - \|\mathcal{S}_2\|(\|\mathcal{T}_2\|^2 + 1) \\ & \leq 2\sigma_j(\Im(\mathcal{T}_1 \mathcal{S}_1 - \mathcal{S}_2 \mathcal{T}_2)). \end{aligned}$$

This completes the proof, establishing the validity of the bounds through the use of polar decomposition and operator inequalities.  $\square$

To demonstrate the necessity of the term  $\|\mathcal{S}_1\|(\|\mathcal{T}_1\|^2 + 1) + \|\mathcal{S}_2\|(\|\mathcal{T}_2\|^2 + 1)$ , we present the following illustrative counterexample.

**Example 4.2.2.** Consider the following simplified inequalities derived from the original Theorem 4.2.1

$$\sigma_j(\mathcal{T}_1 \mathcal{S}_1^* \mathcal{T}_1^* + \mathcal{T}_2^* \mathcal{S}_2 \mathcal{T}_2 + \mathcal{T}_1 |\mathcal{S}_1| - \mathcal{T}_2^* |\mathcal{S}_2^*|) \leq 2\sigma_j(\Re(\mathcal{T}_1 \mathcal{S}_1 - \mathcal{S}_2 \mathcal{T}_2)), \quad (4.6)$$

$$\sigma_j(\iota(\mathcal{T}_1\mathcal{S}_1^*\mathcal{T}_1^* - \mathcal{T}_2^*\mathcal{S}_2\mathcal{T}_2) + \mathcal{T}_1|\mathcal{S}_1| - \mathcal{T}_2^*|\mathcal{S}_2^*|) \leq 2\sigma_j(\Im(\mathcal{T}_1\mathcal{S}_1 - \mathcal{S}_2\mathcal{T}_2)). \quad (4.7)$$

Consider the special instance where  $\mathcal{T}_1 = \mathcal{T}_2$  and  $\mathcal{S}_1 = \mathcal{S}_2 = \mathcal{J}$ . Inserting these operators into inequality (4.6) shows that the right-hand side equals  $\sigma_j(\Re(\mathcal{T}_1\mathcal{J} - \mathcal{J}\mathcal{T}_2)) = \sigma_j(\Re(\mathcal{T}_1 - \mathcal{T}_2)) = \sigma_j(0) = 0$ . However, the left-hand side becomes  $\sigma_j(\mathcal{T}_1\mathcal{T}_1^* + \mathcal{T}_2\mathcal{T}_2^* + \mathcal{T}_1\mathcal{J} - \mathcal{T}_2^*\mathcal{J}) = \sigma_j(2\mathcal{T}_1\mathcal{T}_1^*)$ , which yields non-zero singular values (e.g.,  $\sigma_j(2\mathcal{T}_1\mathcal{T}_1^*) = 2\|\mathcal{T}_1\|^2$  for  $j = 1$ ), leading to a clear contradiction.

For the second inequality (4.7), set  $\mathcal{T}_1 = 2\mathcal{T}_2$  and  $\mathcal{S}_1 = \mathcal{S}_2 = \mathcal{J}$ . The right-hand side is  $\sigma_j(\Im(2\mathcal{T}_2\mathcal{J} - \mathcal{J}\mathcal{T}_2)) = \sigma_j(\Im(\mathcal{T}_2)) = 0$  if  $\mathcal{T}_2$  is self-adjoint, but the left-hand side  $\sigma_j(\iota(2\mathcal{T}_2\mathcal{J}^*2\mathcal{T}_2^* - \mathcal{T}_2^*\mathcal{J}\mathcal{T}_2) + 2\mathcal{T}_2|\mathcal{J}| - \mathcal{T}_2^*|\mathcal{J}^*|)$  simplifies to a non-zero value, again violating the inequality.

These examples highlight that the absence of the term  $\|\mathcal{S}_1\|(\|\mathcal{T}_1\|^2 + 1) + \|\mathcal{S}_2\|(\|\mathcal{T}_2\|^2 + 1)$  undermines the inequalities' correctness, making its inclusion imperative to ensure their validity.  $\square$

Building on the insights from the preceding counterexample, we now extend our analysis to the anti-commutator  $\mathcal{T}_1\mathcal{S}_1 + \mathcal{S}_2\mathcal{T}_2$ . The following theorem establishes lower bound inequalities for this expression, incorporating the norm term identified earlier to ensure consistency across both real and imaginary components.

**Theorem 4.2.3.** *Let  $\mathcal{T}_1, \mathcal{S}_1, \mathcal{S}_2, \mathcal{T}_2 \in \mathfrak{L}(\mathcal{H})$  and  $j$  is an integer in the range  $\{1, 2, \dots, k\}$ . Then*

$$\sigma_j(\mathcal{T}_1\mathcal{S}_1^*\mathcal{T}_1^* + \mathcal{T}_2^*\mathcal{S}_2\mathcal{T}_2 + \mathcal{T}_1|\mathcal{S}_1| + \mathcal{T}_2^*|\mathcal{S}_2^*|) - \|\mathcal{S}_1\|(\|\mathcal{T}_1\|^2 + 1) - \|\mathcal{S}_2\|(\|\mathcal{T}_2\|^2 + 1) \leq 2\sigma_j(\Re(\mathcal{T}_1\mathcal{S}_1 + \mathcal{S}_2\mathcal{T}_2)).$$

Similarly, for the imaginary part, we obtain

$$\sigma_j(\iota(\mathcal{T}_1\mathcal{S}_1^*\mathcal{T}_1^* - \mathcal{T}_2^*\mathcal{S}_2\mathcal{T}_2) + \mathcal{T}_1|\mathcal{S}_1| + \mathcal{T}_2^*|\mathcal{S}_2^*|) - \|\mathcal{S}_1\|(\|\mathcal{T}_1\|^2 + 1) - \|\mathcal{S}_2\|(\|\mathcal{T}_2\|^2 + 1) \leq 2\sigma_j(\Im(\mathcal{T}_1\mathcal{S}_1 + \mathcal{S}_2\mathcal{T}_2)). \quad (4.8)$$

**Proof.** To derive these bounds for the anti-commutator  $\mathcal{T}_1\mathcal{S}_1 + \mathcal{S}_2\mathcal{T}_2$ , we adopt a congruence transformation approach akin to Theorem 4.2.1. Begin with the polar decompositions  $\mathcal{S}_1 = \mathcal{U}_1|\mathcal{S}_1|$  and  $\mathcal{S}_2 = \mathcal{U}_2|\mathcal{S}_2|$ , where  $\mathcal{U}_1, \mathcal{U}_2$  are unitary and  $|\mathcal{S}_1| = (\mathcal{S}_1^*\mathcal{S}_1)^{1/2}$ ,  $|\mathcal{S}_2| = (\mathcal{S}_2^*\mathcal{S}_2)^{1/2}$ .

Construct the positive block matrices:

$$\mathcal{M}_1 = \begin{bmatrix} |\mathcal{S}_1^*| & \mathcal{S}_1 \\ \mathcal{S}_1^* & |\mathcal{S}_1| \end{bmatrix}, \quad \mathcal{M}_2 = \begin{bmatrix} |\mathcal{S}_2| & \mathcal{S}_2^* \\ \mathcal{S}_2 & |\mathcal{S}_2^*| \end{bmatrix}.$$

Introduce the transformation matrix  $\mathcal{Q} = \begin{bmatrix} \mathcal{O} & \mathcal{T}_2^* \\ \mathcal{T}_2^* & \mathcal{J} \end{bmatrix}$  and  $\mathcal{P}$  same as designed in Theorem 4.2.1 to align with the anti-commutator structure. Since  $\mathcal{M}_1$  and  $\mathcal{M}_2$  are positive, the congruences

$\mathcal{P}\mathcal{M}_1\mathcal{P}^* \geq \mathcal{O}$  and a corresponding transformation for  $\mathcal{M}_2$  hold. Expanding  $\mathcal{Q}\mathcal{M}_2\mathcal{Q}^*$  gives:

$$\mathcal{Q}\mathcal{M}_2\mathcal{Q}^* = \begin{bmatrix} \mathcal{T}_2^*|\mathcal{S}_2^*|\mathcal{T}_2 & \mathcal{T}_2^*\mathcal{S}_2\mathcal{T}_2 - \mathcal{T}_2^*|\mathcal{S}_2^*| \\ \mathcal{T}_2^*\mathcal{S}_2^*\mathcal{T}_2 - |\mathcal{S}_2^*|\mathcal{T}_2 & \mathcal{T}_2^*|\mathcal{S}_2|\mathcal{T}_2 - \mathcal{T}_2^*\mathcal{S}_2^* - \mathcal{S}_2\mathcal{T}_2 + |\mathcal{S}_2^*| \end{bmatrix}.$$

Summing these transformed matrices, the relevant block containing  $\mathcal{T}_1\mathcal{S}_1^*\mathcal{T}_1^* + \mathcal{T}_2^*\mathcal{S}_2\mathcal{T}_2 + \mathcal{T}_1|\mathcal{S}_1| + \mathcal{T}_2^*|\mathcal{S}_2^*|$  is isolated. Applying Proposition 4.2.1, we bound:

$$\begin{aligned} \sigma_j(\mathcal{T}_1\mathcal{S}_1^*\mathcal{T}_1^* + \mathcal{T}_2^*\mathcal{S}_2\mathcal{T}_2 + \mathcal{T}_1|\mathcal{S}_1| + \mathcal{T}_2^*|\mathcal{S}_2^*|) &\leq \left\| \begin{bmatrix} \mathcal{T}_1|\mathcal{S}_1^*|\mathcal{T}_1^* + \mathcal{T}_2^*|\mathcal{S}_2|\mathcal{T}_2 + |\mathcal{S}_1| + |\mathcal{S}_2^*| & \mathcal{O} \\ \mathcal{O} & \mathcal{T}_1|\mathcal{S}_1|\mathcal{T}_1^* + \mathcal{T}_2^*|\mathcal{S}_2^*|\mathcal{T}_2 \end{bmatrix} \right\| \\ &\quad + 2\sigma_j(\Re(\mathcal{T}_1\mathcal{S}_1 + \mathcal{S}_2\mathcal{T}_2)). \end{aligned}$$

Using Lemma 4.2.3 and evaluating the adjustment, it simplifies to  $\|\mathcal{S}_1\|(\|\mathcal{T}_1\|^2 + 1) + \|\mathcal{S}_2\|(\|\mathcal{T}_2\|^2 + 1)$ , leading to:

$$\sigma_j(\mathcal{T}_1\mathcal{S}_1^*\mathcal{T}_1^* + \mathcal{T}_2^*\mathcal{S}_2\mathcal{T}_2 + \mathcal{T}_1|\mathcal{S}_1| + \mathcal{T}_2^*|\mathcal{S}_2^*|) - \|\mathcal{S}_1\|(\|\mathcal{T}_1\|^2 + 1) - \|\mathcal{S}_2\|(\|\mathcal{T}_2\|^2 + 1) \leq 2\sigma_j(\Re(\mathcal{T}_1\mathcal{S}_1 + \mathcal{S}_2\mathcal{T}_2)).$$

For the imaginary part, substitute  $\mathcal{S}_1$  with  $-\iota\mathcal{S}_1$  and  $\mathcal{S}_2$  with  $-\iota\mathcal{S}_2$ , adjusting the framework to yield:

$$\sigma_j(\iota(\mathcal{T}_1\mathcal{S}_1^*\mathcal{T}_1^* - \mathcal{T}_2^*\mathcal{S}_2\mathcal{T}_2) + \mathcal{T}_1|\mathcal{S}_1| + \mathcal{T}_2^*|\mathcal{S}_2^*|) - \|\mathcal{S}_1\|(\|\mathcal{T}_1\|^2 + 1) - \|\mathcal{S}_2\|(\|\mathcal{T}_2\|^2 + 1) \leq 2\sigma_j(\Im(\mathcal{T}_1\mathcal{S}_1 + \mathcal{S}_2\mathcal{T}_2)).$$

This completes the proof, confirming the inequalities via congruence and operator bounds.  $\square$

Now, Theorem 4.2.3 applies to derive lower bounds for singular values of operator sums, complementing the upper bounds by Chen and Zhang [27]. We first establish a key singular value inequality for operator sums.

**Corollary 4.2.1.** *Let  $\mathcal{P}, \mathcal{Q} \in \mathfrak{L}(\mathcal{H})$ . For any  $0 \leq \ell \leq 1$  and  $j = 1, 2, \dots, k$ , the following lower bound inequalities are satisfied:*

$$\sigma_j(|\mathcal{Q}|^\ell \mathcal{Q}^* + |\mathcal{P}|^\ell \mathcal{P} + |\mathcal{Q}| + |\mathcal{P}|^\ell |\mathcal{P}^*|^{1-\ell}) - \sum_{\mathcal{S}_1 \in \{\mathcal{P}, \mathcal{Q}\}} (\|\mathcal{S}_1\|^{1-\ell} + \|\mathcal{S}_1\|^{1+\ell}) \leq 2\sigma_j(\Re(\mathcal{P} + \mathcal{Q})),$$

and

$$\sigma_j(\iota(|\mathcal{Q}|^\ell \mathcal{Q}^* - |\mathcal{P}|^\ell \mathcal{P}) + |\mathcal{Q}| + |\mathcal{P}|^\ell |\mathcal{P}^*|^{1-\ell}) - \sum_{\mathcal{S}_1 \in \{\mathcal{P}, \mathcal{Q}\}} (\|\mathcal{S}_1\|^{1-\ell} + \|\mathcal{S}_1\|^{1+\ell}) \leq 2\sigma_j(\Im(\mathcal{P} + \mathcal{Q})).$$

**Proof.** We derive these bounds by adapting Theorem 4.2.3. Set  $\mathcal{T}_1 = |\mathcal{Q}|^\ell$ ,  $\mathcal{T}_2 = |\mathcal{P}|^\ell$ ,

$\mathcal{S}_1^* = \mathcal{U}_2 |\mathcal{Q}|^{1-\ell}$  and  $\mathcal{S}_2 = \mathcal{U}_1 |\mathcal{P}|^{1-\ell}$  (where  $\mathcal{U}_1, \mathcal{U}_2$  are unitary), the left-hand side transforms as follows:

$$\begin{aligned} \sigma_j(|\mathcal{Q}|^\ell \mathcal{U}_2 |\mathcal{Q}|^{1-\ell} |\mathcal{Q}|^\ell + |\mathcal{P}|^\ell \mathcal{U}_1 |\mathcal{P}|^{1-\ell} |\mathcal{P}|^\ell + |\mathcal{Q}|^\ell (\mathcal{U}_2 |\mathcal{Q}|^{1-\ell} |\mathcal{Q}|^{1-\ell} \mathcal{U}_2^*)^{1/2} + |\mathcal{P}|^\ell (\mathcal{U}_1 |\mathcal{P}|^{1-\ell} |\mathcal{P}|^{1-\ell} \mathcal{U}_1^*)^{1/2}) \\ - \|\mathcal{Q}\|^{1-\ell} (1 + \|\mathcal{Q}\|^{2\ell}) - \|\mathcal{P}\|^{1-\ell} (1 + \|\mathcal{P}\|^{2\ell}). \end{aligned}$$

Simplifying the expression with polar decompositions of  $\mathcal{P} = \mathcal{U}_1 |\mathcal{P}|$  and  $\mathcal{Q} = |\mathcal{Q}| \mathcal{U}_2^*$ , we note  $|\mathcal{Q}|^\ell \mathcal{U}_2 |\mathcal{Q}|^{1-\ell} |\mathcal{Q}|^\ell = |\mathcal{Q}|^\ell \mathcal{Q}^*$  and  $|\mathcal{P}|^\ell \mathcal{U}_1 |\mathcal{P}|^{1-\ell} |\mathcal{P}|^\ell = |\mathcal{P}|^\ell \mathcal{P}$ , while the positive terms  $|\mathcal{Q}|^\ell (\mathcal{U}_2 |\mathcal{Q}|^{1-\ell} |\mathcal{Q}|^{1-\ell} \mathcal{U}_2^*)^{1/2} = |\mathcal{Q}|$  and  $|\mathcal{P}|^\ell (\mathcal{U}_1 |\mathcal{P}|^{1-\ell} |\mathcal{P}|^{1-\ell} \mathcal{U}_1^*)^{1/2} = |\mathcal{P}|^\ell |\mathcal{P}^*|^{1-\ell}$ . We can express the norm contributions in the inequality as

$$\sigma_j(|\mathcal{Q}|^\ell \mathcal{Q}^* + |\mathcal{P}|^\ell \mathcal{P} + |\mathcal{Q}| + |\mathcal{P}|^\ell |\mathcal{P}^*|^{1-\ell}) - \|\mathcal{Q}\|^{1-\ell} (1 + \|\mathcal{Q}\|^{2\ell}) - \|\mathcal{P}\|^{1-\ell} (1 + \|\mathcal{P}\|^{2\ell}).$$

Observe that each of the expressions

$$\|\mathcal{Q}\|^{1-\ell} (1 + \|\mathcal{Q}\|^{2\ell}) \quad \text{and} \quad \|\mathcal{P}\|^{1-\ell} (1 + \|\mathcal{P}\|^{2\ell})$$

splits naturally into two terms:

$$\|\mathcal{Q}\|^{1-\ell} + \|\mathcal{Q}\|^{1+\ell}, \quad \|\mathcal{P}\|^{1-\ell} + \|\mathcal{P}\|^{1+\ell}.$$

Substituting these decompositions back into the previous expression leads to the refined form

$$\begin{aligned} \sigma_j(|\mathcal{Q}|^\ell \mathcal{Q}^* + |\mathcal{P}|^\ell \mathcal{P} + |\mathcal{Q}| + |\mathcal{P}|^\ell |\mathcal{P}^*|^{1-\ell}) - \|\mathcal{Q}\|^{1-\ell} - \|\mathcal{Q}\|^{1+\ell} - \|\mathcal{P}\|^{1-\ell} - \|\mathcal{P}\|^{1+\ell} \\ \leq 2 \sigma_j(\Re(\mathcal{P} + \mathcal{Q})). \end{aligned}$$

For the imaginary part, apply the same substitution, resulting in:

$$\sigma_j(i(|\mathcal{Q}|^\ell \mathcal{Q}^* - |\mathcal{P}|^\ell \mathcal{P}) + |\mathcal{Q}| + |\mathcal{P}|^\ell |\mathcal{P}^*|^{1-\ell}) - \|\mathcal{Q}\|^{1-\ell} - \|\mathcal{Q}\|^{1+\ell} - \|\mathcal{P}\|^{1-\ell} - \|\mathcal{P}\|^{1+\ell} \leq 2 \sigma_j(\Im(\mathcal{P} + \mathcal{Q})).$$

□

As a natural extension of Corollary 4.2.1, we derive an inequality for any bounded operator.

**Corollary 4.2.2.** *For any  $\mathcal{S}_1 \in \mathfrak{L}(\mathcal{H})$  and  $1 \leq j \leq k$ ,*

$$\frac{1}{4} [\sigma_j(|\mathcal{S}_1| \mathcal{S}_1^* + |\mathcal{S}_1| \mathcal{S}_1 + 2|\mathcal{S}_1|) - 2 - 2\|\mathcal{S}_1\|^2] \leq \sigma_j(\Re(\mathcal{S}_1)),$$

and

$$\frac{1}{4} [\sigma_j(i(|\mathcal{S}_1| \mathcal{S}_1^* - |\mathcal{S}_1| \mathcal{S}_1) + 2|\mathcal{S}_1|) - 2 - 2\|\mathcal{S}_1\|^2] \leq \sigma_j(\Im(\mathcal{S}_1)).$$

**Proof.** To derive this corollary, we start by substituting  $\mathcal{P} = \mathcal{Q} = \mathcal{S}_1$  and  $t = 1$  into the real part inequality of Corollary 4.2.1, we get:

$$\sigma_j(|\mathcal{S}_1|^1 \mathcal{S}_1^* + |\mathcal{S}_1|^1 \mathcal{S}_1 + |\mathcal{S}_1| + |\mathcal{S}_1|^1 |\mathcal{S}_1^*|^{1-1}) - \sum_{\mathcal{S}_1 \in \{\mathcal{S}_1, \mathcal{S}_1\}} (\|\mathcal{S}_1\|^{1-1} + \|\mathcal{S}_1\|^{1+1}).$$

Since  $|\mathcal{S}_1|^1 = |\mathcal{S}_1|$  and  $|\mathcal{S}_1^*|^{1-1} = |\mathcal{S}_1^*|^0 = \mathcal{I}$ , the term  $|\mathcal{S}_1|^1 |\mathcal{S}_1^*|^{1-1}$  simplifies to  $|\mathcal{S}_1| \cdot \mathcal{I} = |\mathcal{S}_1|$ . The sum of norms becomes:

$$\sum_{\mathcal{S}_1 \in \{\mathcal{S}_1, \mathcal{S}_1\}} (\|\mathcal{S}_1\|^{1-1} + \|\mathcal{S}_1\|^{1+1}) = 2(\|\mathcal{S}_1\|^0 + \|\mathcal{S}_1\|^2) = 2(1 + \|\mathcal{S}_1\|^2).$$

Thus, the inequality transforms to:

$$\sigma_j(|\mathcal{S}_1| \mathcal{S}_1^* + |\mathcal{S}_1| \mathcal{S}_1 + |\mathcal{S}_1| + |\mathcal{S}_1|) - 2(1 + \|\mathcal{S}_1\|^2) \leq 2\sigma_j(\Re(\mathcal{S}_1 + \mathcal{S}_1)).$$

Noting that

$$\Re(\mathcal{S}_1 + \mathcal{S}_1) = \Re(2\mathcal{S}_1) = 2\Re(\mathcal{S}_1),$$

the right-hand side becomes twice the singular value of  $2\Re(\mathcal{S}_1)$ . Dividing the entire inequality by 4 yields the normalized form

$$\frac{1}{4} [\sigma_j(|\mathcal{S}_1| \mathcal{S}_1^* + |\mathcal{S}_1| \mathcal{S}_1 + 2|\mathcal{S}_1|) - 2(1 + \|\mathcal{S}_1\|^2)] \leq \sigma_j(\Re(\mathcal{S}_1)).$$

Similarly for the imaginary part, we get,

$$\frac{1}{4} [\sigma_j(\iota(|\mathcal{S}_1| \mathcal{S}_1^* - |\mathcal{S}_1| \mathcal{S}_1) + 2|\mathcal{S}_1|) - 2 - 2\|\mathcal{S}_1\|^2] \leq \sigma_j(\Im(\mathcal{S}_1)).$$

□

**Remark 4.2.1.** When  $\mathcal{T}_1, \mathcal{T}_2 \in \mathfrak{L}(\mathcal{H})$  are positive operators in Corollary 4.2.1 with  $t \in [0, 1]$ , the following refined bounds apply for  $1 \leq j \leq k$ :

1. For  $\mathcal{T}_1 \neq \mathcal{T}_2$ ,

$$\sigma_j(\mathcal{T}_2^{1+t} + \mathcal{T}_1^{1+t} + \mathcal{T}_2 + \mathcal{T}_1) - \sum_{\mathcal{S}_1 \in \{\mathcal{T}_1, \mathcal{T}_2\}} (\|\mathcal{S}_1\|^{1-t} + \|\mathcal{S}_1\|^{1+t}) \leq 2\sigma_j(\mathcal{T}_1 + \mathcal{T}_2).$$

2. For  $\mathcal{T}_1 = \mathcal{T}_2$ ,

$$\frac{1}{2} [\sigma_j(\mathcal{T}_1^{1+t} + \mathcal{T}_1) - \|\mathcal{T}_1\|^{1-t} - \|\mathcal{T}_1\|^{1+t}] \leq \sigma_j(\mathcal{T}_1).$$

**Lemma 4.2.4.** [39] For  $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{L}(\mathcal{H})$ ,

$$\sigma_j\left(\frac{\mathcal{S}_1 + \mathcal{S}_2}{2}\right) \leq \sigma_j(\mathcal{S}_1 \oplus \mathcal{S}_2), \quad j = 1, 2, \dots, k.$$

Based on this, we establish a lower bound singular value inequality.

**Theorem 4.2.4.** *Let  $\mathcal{T}_1 \in \mathfrak{L}(\mathcal{H})$ . Then, for each index  $1 \leq j \leq k$ :*

$$\frac{1}{2} [\sigma_j(|\mathcal{T}_1^*|\mathcal{T}_1^* + \mathcal{T}_1) - 1 - \|\mathcal{T}_1\|^2] \leq \sigma_j((|\mathcal{T}_1| + \mathcal{T}_1) \oplus (|\mathcal{T}_1| - \mathcal{T}_1)).$$

**Proof.** By the polar decomposition, any operator  $\mathcal{T}_1 \in \mathfrak{L}(\mathcal{H})$  can be written as

$$\mathcal{T}_1 = \mathcal{U}_1^* |\mathcal{T}_1|,$$

where  $\mathcal{U}_1$  is a unitary operator and  $|\mathcal{T}_1| = (\mathcal{T}_1^* \mathcal{T}_1)^{1/2}$ .

Substituting  $\mathcal{T}_1 = \mathcal{U}_1^* |\mathcal{T}_1|$ ,  $\mathcal{S}_1 = \mathcal{U}_1^*$ , and  $\mathcal{T}_2 = \mathcal{S}_2 = \mathcal{O}$  into Theorem 4.2.1, we obtain

$$\frac{1}{2} [\sigma_j(\mathcal{U}_1^* |\mathcal{T}_1| \mathcal{U}_1 |\mathcal{T}_1| \mathcal{U}_1 + \mathcal{U}_1^* |\mathcal{T}_1|) - 1 - \|\mathcal{T}_1\|^2] \leq \sigma_j(\Re(\mathcal{U}_1^* |\mathcal{T}_1| \mathcal{U}_1)).$$

Noting that  $\Re(\mathcal{U}_1^* |\mathcal{T}_1| \mathcal{U}_1) = \mathcal{U}_1^* |\mathcal{T}_1| \mathcal{U}_1$  and using  $\sigma_j(\mathcal{U}_1^* |\mathcal{T}_1| \mathcal{U}_1) = \sigma_j(|\mathcal{T}_1|)$  this reduces to

$$\frac{1}{2} [\sigma_j(|\mathcal{T}_1^*|\mathcal{T}_1^* + \mathcal{T}_1) - 1 - \|\mathcal{T}_1\|^2] \leq \sigma_j(|\mathcal{T}_1|).$$

Now observe that

$$|\mathcal{T}_1| = \frac{(|\mathcal{T}_1| + \mathcal{T}_1) + (|\mathcal{T}_1| - \mathcal{T}_1)}{2}.$$

By invoking Lemma 4.2.4, we obtain an estimate for the singular values of  $|\mathcal{T}_1|$ . In particular, the lemma ensures that the  $j$ -th singular value of  $|\mathcal{T}_1|$  does not exceed the corresponding singular value of the block operator formed from  $|\mathcal{T}_1| + \mathcal{T}_1$  and  $|\mathcal{T}_1| - \mathcal{T}_1$ . Hence,

$$\sigma_j(|\mathcal{T}_1|) \leq \sigma_j((|\mathcal{T}_1| + \mathcal{T}_1) \oplus (|\mathcal{T}_1| - \mathcal{T}_1)).$$

Combining these inequalities, we arrive at the desired result:

$$\frac{1}{2} [\sigma_j(|\mathcal{T}_1^*|\mathcal{T}_1^* + \mathcal{T}_1) - 1 - \|\mathcal{T}_1\|^2] \leq \sigma_j((|\mathcal{T}_1| + \mathcal{T}_1) \oplus (|\mathcal{T}_1| - \mathcal{T}_1)).$$

This completes the proof. □

## 4.2.2 Upper Bound Inequalities

This subsection delves into the limitations of existing results, particularly Audeh's and Zhao's inequalities, which traditionally imposes restrictive conditions such as the positivity of operators  $\mathcal{T}_1$  and  $\mathcal{T}_2$ . To highlight these constraints, we first present a counterexample that demonstrates the failure of Audeh's and Zhao's inequality under general conditions. Subsequently, we introduce a novel theorem that overcomes these limitations by establishing upper bound singular value inequalities for generalized commutators and anti-commutators, free from the positivity

assumption, thereby broadening the applicability of these results.

**Example 4.2.5.** Consider Zhao's inequality (2.7), which asserts an upper bound on the singular values of the expression  $\mathcal{T}_1\mathcal{S}_1 + \mathcal{S}_2\mathcal{T}_2$  under the assumption that  $\mathcal{T}_1$  and  $\mathcal{T}_2$  are positive operators in  $\mathfrak{L}(\mathcal{H})$ . To test its validity without this restriction, let

$$\mathcal{T}_1 = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}, \quad \mathcal{T}_2 = \begin{bmatrix} -2 & -6 \\ -2 & -4 \end{bmatrix}, \quad \mathcal{S}_1 = \mathcal{I}, \quad \mathcal{S}_2 = -\mathcal{I},$$

where  $\mathcal{H} = \mathbb{C}^2$ , and neither  $\mathcal{T}_1$  nor  $\mathcal{T}_2$  is positive. The expression  $\mathcal{T}_1\mathcal{S}_1 + \mathcal{S}_2\mathcal{T}_2$  simplifies to  $\mathcal{T}_1 - \mathcal{T}_2$ :

$$\mathcal{T}_1 - \mathcal{T}_2 = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} -2 & -6 \\ -2 & -4 \end{bmatrix} = \begin{bmatrix} 1 - (-2) & 3 - (-6) \\ 1 - (-2) & 2 - (-4) \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 3 & 6 \end{bmatrix}.$$

The singular values of  $\mathcal{T}_1 - \mathcal{T}_2 = \begin{bmatrix} 3 & 9 \\ 3 & 6 \end{bmatrix}$  are approximately 11.593 and 0.781. But the singular values of R.H.S. are 3.864 and 0.259. Thus, the inequality fails to hold, underscoring the need for a revised approach.  $\square$

Following this counterexample, we present a theorem that addresses the shortcomings of Audeh's and Zhao's result by providing a new upper bound singular value inequality applicable to generalized commutators and anti-commutators without requiring positivity of  $\mathcal{T}_1$  and  $\mathcal{T}_2$ .

**Theorem 4.2.6.** Let  $\mathcal{T}_1, \mathcal{S}_1, \mathcal{S}_2, \mathcal{T}_2 \in \mathfrak{L}(\mathcal{H})$ . For  $j = 1, 2, \dots, k$ :

$$\sigma_j(\mathcal{T}_1\mathcal{S}_1 - \mathcal{S}_2\mathcal{T}_2) \leq \sigma_j((\mathcal{T}_1\mathcal{T}_1^* + \mathcal{S}_2\mathcal{S}_2^*) \oplus (\mathcal{T}_2^*\mathcal{T}_2 + \mathcal{S}_1^*\mathcal{S}_1)). \quad (4.9)$$

**Proof.** To establish this inequality, we construct a block matrix that leverages the properties of positive semi-definiteness and singular values. Consider the matrix product

$$\begin{bmatrix} \mathcal{T}_1 & -\mathcal{S}_2 \\ \mathcal{S}_1^* & \mathcal{T}_2^* \end{bmatrix} \begin{bmatrix} \mathcal{T}_1^* & \mathcal{S}_1 \\ -\mathcal{S}_2^* & \mathcal{T}_2 \end{bmatrix}.$$

Expanding this product yields:

$$\begin{bmatrix} \mathcal{T}_1\mathcal{T}_1^* + \mathcal{S}_2\mathcal{S}_2^* & \mathcal{T}_1\mathcal{S}_1 - \mathcal{S}_2\mathcal{T}_2 \\ \mathcal{S}_1^*\mathcal{T}_1^* + \mathcal{T}_2^*(-\mathcal{S}_2^*) & \mathcal{S}_1^*\mathcal{S}_1 + \mathcal{T}_2^*\mathcal{T}_2 \end{bmatrix}.$$

Since

$$\mathcal{S}_1^*\mathcal{T}_1^* + \mathcal{T}_2^*(-\mathcal{S}_2^*) = \mathcal{S}_1^*\mathcal{T}_1^* - \mathcal{T}_2^*\mathcal{S}_2^*,$$

we may rewrite the expression appearing inside the singular value as an operator of the form

$\mathcal{T}_1\mathcal{S}_1 - \mathcal{S}_2\mathcal{T}_2$ . Applying Proposition 4.2.1 to this operator yields

$$\sigma_j(\mathcal{T}_1\mathcal{S}_1 - \mathcal{S}_2\mathcal{T}_2) \leq \sigma_j((\mathcal{T}_1\mathcal{T}_1^* + \mathcal{S}_2\mathcal{S}_2^*) \oplus (\mathcal{T}_2^*\mathcal{T}_2 + \mathcal{S}_1^*\mathcal{S}_1)).$$

This estimate gives the required upper bound for the singular values and completes the argument.  $\square$

**Corollary 4.2.3.** *An analogous upper bound applies to the generalized anti-commutator. Consider the block matrix*

$$\begin{bmatrix} \mathcal{T}_1 & \mathcal{S}_2 \\ \mathcal{S}_1^* & \mathcal{T}_2^* \end{bmatrix} \begin{bmatrix} \mathcal{T}_1^* & \mathcal{S}_1 \\ \mathcal{S}_2^* & \mathcal{T}_2 \end{bmatrix}.$$

*This product expands to*

$$\begin{bmatrix} \mathcal{T}_1\mathcal{T}_1^* + \mathcal{S}_2\mathcal{S}_2^* & \mathcal{T}_1\mathcal{S}_1 + \mathcal{S}_2\mathcal{T}_2 \\ \mathcal{S}_1^*\mathcal{T}_1^* + \mathcal{T}_2^*\mathcal{S}_2^* & \mathcal{S}_1^*\mathcal{S}_1 + \mathcal{T}_2^*\mathcal{T}_2 \end{bmatrix} \geq \mathcal{O}.$$

Consequently, for  $j = 1, 2, \dots, k$ ,

$$\sigma_j(\mathcal{T}_1\mathcal{S}_1 + \mathcal{S}_2\mathcal{T}_2) \leq \sigma_j((\mathcal{T}_1\mathcal{T}_1^* + \mathcal{S}_2\mathcal{S}_2^*) \oplus (\mathcal{T}_2^*\mathcal{T}_2 + \mathcal{S}_1^*\mathcal{S}_1)).$$

**Remark 4.2.2.** When  $\mathcal{S}_2 = -\mathcal{S}_1$ , Zhao's inequality reduces to

$$\sigma_j((\mathcal{T}_1\mathcal{S}_1 - \mathcal{S}_1\mathcal{T}_2) \oplus \mathcal{O}) \leq \sigma_j\left(\left(\frac{1}{2}\mathcal{T}_1 + \frac{1}{2}\mathcal{T}_1^{1/2}|\mathcal{S}_1^*|^2\mathcal{T}_1^{1/2}\right) \oplus \left(\frac{1}{2}\mathcal{T}_2 + \frac{1}{2}\mathcal{T}_2^{1/2}|\mathcal{S}_1|^2\mathcal{T}_2^{1/2}\right)\right). \quad (4.10)$$

Under the same condition, inequality (4.9) becomes

$$\sigma_j(\mathcal{T}_1\mathcal{S}_1 - \mathcal{S}_1\mathcal{T}_2) \leq \sigma_j((\mathcal{T}_1\mathcal{T}_1^* + \mathcal{S}_1\mathcal{S}_1^*) \oplus (\mathcal{T}_2^*\mathcal{T}_2 + \mathcal{S}_1^*\mathcal{S}_1)). \quad (4.11)$$

This shows that inequality (4.11) can often provide sharper estimates than inequality (4.10).

However, the comparison is not always straightforward, since

$$\sigma_j((\mathcal{T}_1\mathcal{T}_1^* + \mathcal{S}_1\mathcal{S}_1^*) \oplus (\mathcal{T}_2^*\mathcal{T}_2 + \mathcal{S}_1^*\mathcal{S}_1)) \not\leq \sigma_j\left(\left(\frac{1}{2}\mathcal{T}_1 + \frac{1}{2}\mathcal{T}_1^{1/2}|\mathcal{S}_1^*|^2\mathcal{T}_1^{1/2}\right) \oplus \left(\frac{1}{2}\mathcal{T}_2 + \frac{1}{2}\mathcal{T}_2^{1/2}|\mathcal{S}_1|^2\mathcal{T}_2^{1/2}\right)\right)$$

in general. To illustrate this, let

$$\mathcal{T}_1 = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}, \quad \mathcal{T}_2 = \begin{bmatrix} 1 & 2 \\ 5 & 11 \end{bmatrix}, \quad \mathcal{S}_1 = \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix},$$

where  $\mathcal{T}_1$  and  $\mathcal{T}_2$  are positive semi-definite. For these choices, the computations yield

$$\begin{aligned} \sigma_1((\mathcal{T}_1\mathcal{T}_1^* + \mathcal{S}_1\mathcal{S}_1^*) \oplus (\mathcal{T}_2^*\mathcal{T}_2 + \mathcal{S}_1^*\mathcal{S}_1)) &= 168.6088 \\ &\leq \sigma_1\left(\left(\frac{1}{2}\mathcal{T}_1 + \frac{1}{2}\mathcal{T}_1^{1/2}|\mathcal{S}_1^*|^2\mathcal{T}_1^{1/2}\right) \oplus \left(\frac{1}{2}\mathcal{T}_2 + \frac{1}{2}\mathcal{T}_2^{1/2}|\mathcal{S}_1|^2\mathcal{T}_2^{1/2}\right)\right) = 177.7723, \end{aligned}$$

and

$$\begin{aligned} \sigma_1((\mathcal{T}_1\mathcal{T}_1^* + \mathcal{S}_1\mathcal{S}_1^*) \oplus (\mathcal{T}_2^*\mathcal{T}_2 + \mathcal{S}_1^*\mathcal{S}_1)) &= 95.7134 \\ &\leq \sigma_1\left(\left(\frac{1}{2}\mathcal{T}_1 + \frac{1}{2}\mathcal{T}_1^{1/2}|\mathcal{S}_1^*|^2\mathcal{T}_1^{1/2}\right) \oplus \left(\frac{1}{2}\mathcal{T}_2 + \frac{1}{2}\mathcal{T}_2^{1/2}|\mathcal{S}_1|^2\mathcal{T}_2^{1/2}\right)\right) = 160.6817. \end{aligned}$$

This confirms that inequality (4.11) may yield more practical bounds in certain settings, but it does not universally dominate inequality (4.10).  $\square$

**Corollary 4.2.4.** *Let  $\mathcal{T}_1, \mathcal{S}_1, \mathcal{S}_2, \mathcal{T}_2 \in \mathfrak{L}(\mathcal{H})$  with  $\mathcal{S}_1, \mathcal{S}_2 \in \mathcal{P}(\mathcal{H})$  and  $j = 1, 2, \dots, k$ . Then,*

$$\sigma_j(\mathcal{T}_1\mathcal{S}_1 - \mathcal{S}_2\mathcal{T}_2) \leq \sigma_j((\mathcal{T}_1\mathcal{S}_1\mathcal{T}_1^* + \mathcal{S}_2) \oplus (\mathcal{T}_2^*\mathcal{S}_2\mathcal{T}_2 + \mathcal{S}_1)). \quad (4.12)$$

**Proof.** To derive this inequality, we utilize the properties of positive operators. Begin by considering the block matrix formed by the product

$$\begin{bmatrix} \mathcal{T}_1\mathcal{S}_1^{1/2} & -\mathcal{S}_2^{1/2} \\ \mathcal{S}_1^{1/2} & \mathcal{T}_2^*\mathcal{S}_2^{1/2} \end{bmatrix} \begin{bmatrix} \mathcal{S}_1^{1/2}\mathcal{T}_1^* & \mathcal{S}_1^{1/2} \\ -\mathcal{S}_2^{1/2} & \mathcal{S}_2^{1/2}\mathcal{T}_2 \end{bmatrix} \geq \mathcal{O}.$$

Expanding this product, we obtain:

$$\begin{bmatrix} \mathcal{T}_1\mathcal{S}_1^{1/2} \cdot \mathcal{S}_1^{1/2}\mathcal{T}_1^* + (-\mathcal{S}_2^{1/2}) \cdot (-\mathcal{S}_2^{1/2}) & \mathcal{T}_1\mathcal{S}_1^{1/2} \cdot \mathcal{S}_1^{1/2} + (-\mathcal{S}_2^{1/2}) \cdot \mathcal{S}_2^{1/2}\mathcal{T}_2 \\ \mathcal{S}_1^{1/2} \cdot \mathcal{S}_1^{1/2}\mathcal{T}_1^* + \mathcal{T}_2^*\mathcal{S}_2^{1/2} \cdot (-\mathcal{S}_2^{1/2}) & \mathcal{S}_1^{1/2} \cdot \mathcal{S}_1^{1/2} + \mathcal{T}_2^*\mathcal{S}_2^{1/2} \cdot \mathcal{S}_2^{1/2}\mathcal{T}_2 \end{bmatrix} \geq \mathcal{O}.$$

According to Proposition 4.2.1, which states that the singular values of an off-diagonal block are bounded by the singular values of the direct sum of its diagonal blocks, we apply this to the off-diagonal block  $\mathcal{T}_1\mathcal{S}_1 - \mathcal{S}_2\mathcal{T}_2$ . The diagonal blocks are  $\mathcal{T}_1\mathcal{S}_1\mathcal{T}_1^* + \mathcal{S}_2$  and  $\mathcal{S}_1 + \mathcal{T}_2^*\mathcal{S}_2\mathcal{T}_2$ , and their direct sum provides the upper bound. Thus, the inequality  $\sigma_j(\mathcal{T}_1\mathcal{S}_1 - \mathcal{S}_2\mathcal{T}_2) \leq \sigma_j((\mathcal{T}_1\mathcal{S}_1\mathcal{T}_1^* + \mathcal{S}_2) \oplus (\mathcal{T}_2^*\mathcal{S}_2\mathcal{T}_2 + \mathcal{S}_1))$  follows directly.  $\square$

### 4.2.3 Counterexamples for Bhatia and Davis's inequalities

The following examples demonstrate that the singular value inequalities proposed by Bhatia and Davis are not valid for singular values.

**Example 4.2.7.** These inequalities are given by

$$2\sigma_j(\mathcal{T}_1^{1/2}\mathcal{S}_1\mathcal{T}_2^{1/2}) \leq \sigma_j(\mathcal{T}_1^\rho\mathcal{S}_1\mathcal{T}_2^{1-\rho} + \mathcal{T}_1^{1-\rho}\mathcal{S}_1\mathcal{T}_2^\rho) \leq \sigma_j(\mathcal{T}_1\mathcal{S}_1 + \mathcal{S}_1\mathcal{T}_2), \quad \rho \in [0, 1], \quad (4.13)$$

and

$$\sigma_j(\mathcal{T}_1^\rho\mathcal{S}_1\mathcal{T}_2^{1-\rho} - \mathcal{T}_1^{1-\rho}\mathcal{S}_1\mathcal{T}_2^\rho) \leq |2\rho - 1|\sigma_j(\mathcal{T}_1\mathcal{S}_1 - \mathcal{S}_1\mathcal{T}_2), \quad \rho \in [0, 1]. \quad (4.14)$$

These bounds assume certain operator properties, but their validity for singular values can break down when tested with specific operators.

To investigate the first inequality (4.13), consider the operators

$$\mathcal{T}_1 = \mathcal{T}_2 = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \quad \mathcal{S}_1 = \begin{bmatrix} 0.375 & 0.375 \\ -0.125 & 0.875 \end{bmatrix}.$$

Both  $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{P}(\mathcal{H})$  with eigenvalues 4 and 2, and  $\mathcal{T}_1^{1/2} = \mathcal{T}_2^{1/2}$  can be computed as the  $\mathcal{SVD}$ . For the particular choice  $\rho = 1$ , the left-hand side evaluates  $2\sigma_2(\mathcal{T}_1^{1/2}\mathcal{S}_1\mathcal{T}_2^{1/2})$ . First, calculate  $\mathcal{T}_1^{1/2}$ :

$$\mathcal{T}_1^{1/2} = \begin{bmatrix} \sqrt{4} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \text{ (via diagonalization, approximately) } \approx \begin{bmatrix} 2 & 0 \\ 0 & 1.414 \end{bmatrix}.$$

However, numerically,  $\mathcal{T}_1^{1/2}\mathcal{S}_1\mathcal{T}_2^{1/2}$  is computed, and its second singular value  $\sigma_2$  is approximately 0.8966, so  $2\sigma_2 \approx 1.7932$ . Now, compute the right-hand side:  $\mathcal{T}_1\mathcal{S}_1 + \mathcal{S}_1\mathcal{T}_2$  with  $\mathcal{T}_1 = \mathcal{T}_2$ :

$$\begin{aligned} \mathcal{T}_1\mathcal{S}_1 &= \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0.375 & 0.375 \\ -0.125 & 0.875 \end{bmatrix} = \begin{bmatrix} 0.875 & 2.125 \\ 0.125 & 2.875 \end{bmatrix}, \\ \mathcal{S}_1\mathcal{T}_2 &= \begin{bmatrix} 0.875 & 2.125 \\ 0.125 & 2.875 \end{bmatrix}, \\ \mathcal{T}_1\mathcal{S}_1 + \mathcal{S}_1\mathcal{T}_2 &= 2 \begin{bmatrix} 0.875 & 2.125 \\ 0.125 & 2.875 \end{bmatrix}, \end{aligned}$$

whose second singular value  $\sigma_2 \approx 1.7720$ . Since  $1.7932 > 1.7720$ , so inequality  $2\sigma_2(\mathcal{T}_1^{1/2}\mathcal{S}_1\mathcal{T}_2^{1/2}) \leq \sigma_2(\mathcal{T}_1\mathcal{S}_1 + \mathcal{S}_1\mathcal{T}_2)$  fails, indicating the bound does not hold for this case.

For the second inequality (4.14), consider

$$\mathcal{T}_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathcal{T}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad \mathcal{S}_1 = \begin{bmatrix} 0.375 & 0.375 \\ -0.125 & 0.875 \end{bmatrix},$$

with  $\rho = \frac{1}{4}$ . Here,  $|2\rho - 1| = |0.5 - 1| = 0.5$ . Similarly, the inequality does not hold for these singular values, confirming the limitations of the bound for general cases.  $\square$

### 4.3 Conclusion

This chapter has made significant strides in advancing the theory of singular value inequalities, with a particular focus on deriving lower bound inequalities for the commutator  $\mathcal{T}_1\mathcal{S}_1 - \mathcal{S}_1\mathcal{T}_2$  and the anti-commutator  $\mathcal{T}_1\mathcal{S}_1 + \mathcal{S}_1\mathcal{T}_2$ . By removing the restrictive positivity assumptions

imposed by prior works such as those of Audeh [8] and Zhao [68], we have developed a more general framework that complements the upper bound inequalities established by Chen and Zhang [27]. The introduction of novel lower bound inequalities, as presented in Theorem 4.2.1 and Theorem 4.2.3, provides a robust analytical tool for understanding the spectral properties of operators, free from the limitations of earlier constraints.

Key contributions include the establishment of lower bound inequalities that incorporate adjustment terms, such as  $\|\mathcal{S}_1\|(\|\mathcal{T}_1\|^2 + 1) + \|\mathcal{S}_2\|(\|\mathcal{T}_2\|^2 + 1)$ , which were shown to be essential for the validity of the bounds through counterexamples. The illustrative example demonstrated that omitting these terms leads to contradictions, underscoring their necessity in ensuring the inequalities hold. Furthermore, the extension of these results to anti-commutators and operator sums, as seen in Corollary 4.2.1 and Corollary 4.2.2, broadens the applicability of the findings, offering refined bounds for both real and imaginary components.

The critique of classical inequalities, such as those by Fan and Hoffman [30] and the erroneous extension by Marshall et al. [51], highlighted the need for careful validation, a task addressed through counterexamples and the development of more reliable bounds by Chen and Zhang [27]. The upper bound inequalities in Section 4.2.2, exemplified by Theorem 4.2.6 and Corollary 4.2.4, further illustrate the chapter's comprehensive approach by overcoming the positivity restrictions of Audeh's and Zhao's results. The counterexample involving Bhatia and Davis's inequalities further reinforces the limitations of existing frameworks and the necessity of the new inequalities proposed here.

# Chapter 5

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## Singular Value Inequalities Involving Operator Functions

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### 5.1 Introduction

Considerable progress has been achieved in the theory of singular value bounds for expressions of the type  $\mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2 + \mathcal{S}_2$ , thereby deepening our insight into the spectral characteristics of such composite operators. A classical inequality due to Hirzallah and Kittaneh [39] asserts that, for arbitrary bounded operators  $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{L}(\mathcal{H})$ ,

$$\sigma_j(\mathcal{S}_1 + \mathcal{S}_2) \leq 2\sigma_j(\mathcal{S}_1 \oplus \mathcal{S}_2) \quad \text{for all } j \geq 1. \quad (5.1)$$

This foundational inequality provides a benchmark for bounding singular values of operator sums. Building on this, Al-Natoor and Kittaneh [3] refined the result by introducing a bounded operator  $\mathcal{C} \in \mathfrak{L}(\mathcal{H})$ , leading to the inequality

$$\sigma_j(\mathcal{S}_1 \mathcal{C} + \mathcal{C} \mathcal{S}_2) \leq 2\|\mathcal{C}\| \cdot \sigma_j(\mathcal{S}_1 \oplus \mathcal{S}_2). \quad (5.2)$$

These advancements lay a critical foundation for extending singular value inequalities to more complex operator interactions.

More recently, Kaur and Garg [44] proved an inequality involving operators  $\mathcal{T}_1, \mathcal{T}_2, \mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{L}(\mathcal{H})$ , where  $\mathcal{S}_1$  is a compact operator, as follows:

$$\sigma_j(\mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2 + \mathcal{S}_2) \leq (\|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{S}_2\|) \sigma_j(\mathcal{S}_1 \oplus \mathcal{S}).$$

Inspired by these results, we propose an extended inequality that accounts for additional operator terms:

$$\sigma_j(\mathcal{P} + \mathcal{R}) \leq \left( \|\mathcal{T}_1\| \|\mathcal{T}_2\| + [\|\mathcal{S}_1\|, \|\mathcal{S}_2\|]_\infty \right) \sigma_j(\mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \mathcal{J}) + \frac{1}{2} \|\mathcal{P} + \mathcal{R}\|, \\ \text{where } \mathcal{P} = \mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2 + \mathcal{S}_2 \quad \text{and} \quad \mathcal{R} = \mathcal{T}_2 \mathcal{S}_2 \mathcal{T}_1 + \mathcal{S}_1. \quad (5.3)$$

A detailed proof of inequality (5.3), along with further generalizations and sharp estimates, is provided in Section 5.2 that build on the work of Kaur and Garg [44]. The additional norm term  $\frac{1}{2} \|\mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2 + \mathcal{T}_2 \mathcal{S}_2 \mathcal{T}_1 + \mathcal{S}_1 + \mathcal{S}_2\|$  plays a crucial role in maintaining the inequality's validity, a point explored in depth following the proof. Section 5.3 discusses enhancements to existing inequalities, such as (5.1) and (5.2), by incorporating operator convexity and concavity to refine the bounds. This work aims to advance the theory of singular value inequalities, enriching the interplay between spectral properties and functional inequalities within a broader framework.

## 5.2 Main Results

This section presents groundbreaking contributions to singular value inequalities within operator theory, building on the foundational framework from prior chapters. We introduce original theorems that address identified literature gaps, refining bounds for bounded linear operators on Hilbert spaces. These results incorporate advanced techniques, such as operator convexity and concavity, to provide enhanced spectral understanding with wide-ranging applicability. The exploration begins with foundational lemma from established research, laying the groundwork for innovative proofs.

**Lemma 5.2.1.** [2] *Let  $\mathcal{T}_1, \mathcal{T}_2 \in \mathfrak{L}(\mathcal{H})$ . Then, for each index  $j$ ,*

$$\sigma_j(\mathcal{T}_1 + \mathcal{T}_2) \leq \sigma_j(\mathcal{T}_1 \oplus \mathcal{T}_2) + \frac{1}{2} \|\mathcal{T}_1 + \mathcal{T}_2\|.$$

With these tools in hand, we present our first major result.

**Theorem 5.2.1.** *Let  $\mathcal{T}_1, \mathcal{T}_2, \mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{L}(\mathcal{H})$ , where  $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{C}(\mathcal{H})$ . Let  $f : [0, \infty) \rightarrow [0, \infty)$  be a non-negative and monotonically increasing function. This immediately gives rise to the following inequalities:*

(a) *If  $f$  is concave and satisfies  $f(0) = 0$ , then for each  $j$ ,*

$$\sigma_j(f(\|\mathcal{P} + \mathcal{R}\|)) \leq f(\|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{J}\|) \sigma_j(\mathcal{J} \oplus \mathcal{J}) + f\left(\frac{1}{2} \|\mathcal{P} + \mathcal{R}\|\right), \quad (5.4)$$

(b) If  $f$  is convex, then for each  $j$ ,

$$\sigma_j(f(|\mathcal{P} + \mathcal{R}|)) \leq \frac{1}{2}f(2(\|\mathcal{T}_1\|\|\mathcal{T}_2\| + \|\mathcal{Z}\|)\sigma_j(\mathcal{Z} \oplus \mathcal{J})) + \frac{1}{2}f(\|\mathcal{P} + \mathcal{R}\|), \quad (5.5)$$

$$\text{where } \mathcal{P} = \mathcal{T}_1\mathcal{S}_1\mathcal{T}_2 + \mathcal{S}_2, \mathcal{R} = \mathcal{T}_2\mathcal{S}_2\mathcal{T}_1 + \mathcal{S}_1 \quad \text{and} \quad \mathcal{Z} = \begin{bmatrix} \mathcal{S}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{S}_2^* \end{bmatrix}.$$

**Proof.** By Lemma 2.2.3, we have

$$\sigma_j(f(|\mathcal{P} + \mathcal{R}|)) = f(\sigma_j(\mathcal{P} + \mathcal{R})).$$

Applying Lemma 5.2.1, we derive:

$$f(\sigma_j(\mathcal{P} + \mathcal{R})) \leq f\left(\sigma_j\left((\mathcal{P} \oplus \mathcal{R}) + \frac{1}{2}\|\mathcal{P} + \mathcal{R}\|\right)\right). \quad (5.6)$$

For part (a), since  $f$  is concave with  $f(0) = 0$ , we get:

$$\begin{aligned} f(\sigma_j(\mathcal{P} + \mathcal{R})) &\leq f(\sigma_j(\mathcal{P} \oplus \mathcal{R})) + f\left(\frac{1}{2}\|\mathcal{P} + \mathcal{R}\|\right) \\ &= f(\sigma_j(\mathcal{P} \oplus \mathcal{R}^*)) + f\left(\frac{1}{2}\|\mathcal{P} + \mathcal{R}\|\right). \end{aligned}$$

In block matrix form, above can be written as

$$f(\sigma_j(\mathcal{P} + \mathcal{R})) \leq f\left(\sigma_j\left(\begin{bmatrix} \mathcal{T}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{T}_1^* \end{bmatrix} \begin{bmatrix} \mathcal{S}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{S}_2^* \end{bmatrix} \begin{bmatrix} \mathcal{T}_2 & \mathcal{O} \\ \mathcal{O} & \mathcal{T}_2^* \end{bmatrix} + \begin{bmatrix} \mathcal{S}_2 & \mathcal{O} \\ \mathcal{O} & \mathcal{S}_1^* \end{bmatrix}\right)\right) + f\left(\frac{1}{2}\|\mathcal{P} + \mathcal{R}\|\right).$$

Using Theorem 3.2.1 and the definition of  $\mathcal{Z}$ :

$$f(\sigma_j(\mathcal{P} + \mathcal{R})) \leq f((\|\mathcal{T}_1\|\|\mathcal{T}_2\| + \|\mathcal{Z}\|)\sigma_j(\mathcal{Z} \oplus \mathcal{J})) + f\left(\frac{1}{2}\|\mathcal{P} + \mathcal{R}\|\right),$$

proving part (a).

For part (b), revisiting (5.6) with  $f$  convex and applying Jensen's inequality [36]:

$$\sigma_j(f(|\mathcal{P} + \mathcal{R}|)) \leq \frac{1}{2}f(2\sigma_j(\mathcal{P} + \mathcal{R}^*)) + \frac{1}{2}f(\|\mathcal{P} + \mathcal{R}\|).$$

Following the same steps as in part (a):

$$\sigma_j(f(|\mathcal{P} + \mathcal{R}|)) \leq \frac{1}{2}f(2(\|\mathcal{T}_1\|\|\mathcal{T}_2\| + \|\mathcal{Z}\|)\sigma_j(\mathcal{Z} \oplus \mathcal{J})) + \frac{1}{2}f(\|\mathcal{P} + \mathcal{R}\|),$$

completing the proof.  $\square$

Corollaries emerge by applying Theorem 5.2.1 with specific functions  $f$ .

**Corollary 5.2.1.** *Let  $\mathcal{T}_1, \mathcal{T}_2, \mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{L}(\mathcal{H})$ , where  $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{C}(\mathcal{H})$ . Then, for each  $j = 1, 2, \dots, k$ ,*

$$\sigma_j(\log(|\mathcal{P} + \mathcal{R}| + \mathcal{J})) + \log 2 \leq \log[(\|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{Z}\|) \sigma_j(\mathcal{Z} \oplus \mathcal{J}) + 1] + \log(\|\mathcal{P} + \mathcal{R}\| + 2),$$

where

$$\mathcal{P} = \mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2 + \mathcal{S}_2, \mathcal{R} = \mathcal{T}_2 \mathcal{S}_2 \mathcal{T}_1 + \mathcal{S}_1 \quad \text{and} \quad \mathcal{Z} = \begin{bmatrix} \mathcal{S}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{S}_2^* \end{bmatrix}.$$

**Proof.** We apply Theorem 5.2.1 with the function  $f(t) = \log(t + 1)$ . Since  $f$  is concave (as  $f''(t) = -1/(t + 1)^2 < 0$  for  $t > -1$ ) and satisfies  $f(0) = 0$ , it preserves the inequality structure. This yields

$$\sigma_j(\log(|\mathcal{P} + \mathcal{R}| + \mathcal{J})) \leq \log[(\|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{Z}\|) \sigma_j(\mathcal{Z} \oplus \mathcal{J}) + 1] + \log\left(\frac{1}{2}\|\mathcal{P} + \mathcal{R}\| + 1\right).$$

Rearranging the second logarithmic term gives

$$\log\left(\frac{1}{2}\|\mathcal{P} + \mathcal{R}\| + 1\right) = \log(\|\mathcal{P} + \mathcal{R}\| + 2) - \log 2.$$

Substituting back, we obtain

$$\sigma_j(\log(|\mathcal{P} + \mathcal{R}| + \mathcal{J})) + \log 2 \leq \log[(\|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{Z}\|) \sigma_j(\mathcal{Z} \oplus \mathcal{J}) + 1] + \log(\|\mathcal{P} + \mathcal{R}\| + 2),$$

which completes the proof.  $\square$

**Corollary 5.2.2.** *Let  $\mathcal{T}_1, \mathcal{T}_2, \mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{L}(\mathcal{H})$ , where  $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{C}(\mathcal{H})$ . Then, for every  $0 < \rho \leq 1$  and  $j = 1, 2, \dots, k$ ,*

$$\sigma_j(|\mathcal{P} + \mathcal{R}|^\rho) \leq [(\|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{Z}\|)^\rho \cdot \sigma_j^\rho(\mathcal{Z} \oplus \mathcal{J})] + 2^{-\rho} \|\mathcal{P} + \mathcal{R}\|^\rho,$$

where

$$\mathcal{P} = \mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2 + \mathcal{S}_2, \mathcal{R} = \mathcal{T}_2 \mathcal{S}_2 \mathcal{T}_1 + \mathcal{S}_1 \quad \text{and} \quad \mathcal{Z} = \begin{bmatrix} \mathcal{S}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{S}_2^* \end{bmatrix}.$$

**Proof.** We apply Theorem 5.2.1 with the function  $f(t) = t^\rho$ , where  $0 < \rho \leq 1$ . The function  $f$  is concave (since  $f''(t) = \rho(\rho - 1)t^{\rho-2} \leq 0$  for  $t > 0$ ) and satisfies  $f(0) = 0$ .

Using Theorem 5.2.1 with this choice of  $f$ , we obtain

$$\begin{aligned}\sigma_j(|\mathcal{P} + \mathcal{R}|^\rho) &\leq \left[ (\|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{Z}\|) \sigma_j(\mathcal{Z} \oplus \mathcal{J}) \right]^\rho + \left( \frac{1}{2} \|\mathcal{P} + \mathcal{R}\| \right)^\rho \\ &= \left[ (\|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{Z}\|) \sigma_j(\mathcal{Z} \oplus \mathcal{J}) \right]^\rho + 2^{-\rho} \|\mathcal{P} + \mathcal{R}\|^\rho.\end{aligned}$$

This establishes the desired result.  $\square$

**Corollary 5.2.3.** Let  $\mathcal{T}_1, \mathcal{T}_2, \mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{L}(\mathcal{H})$ , where  $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{C}(\mathcal{H})$ . Then, for every  $\rho \geq 1$  and  $j = 1, 2, \dots, k$ ,

$$\sigma_j(|\mathcal{P} + \mathcal{R}|^\rho) \leq 2^{\rho-1} \left[ (\|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{Z}\|)^\rho \cdot \sigma_j^\rho(\mathcal{Z} \oplus \mathcal{J}) \right] + \frac{1}{2} \|\mathcal{P} + \mathcal{R}\|^\rho,$$

where

$$\mathcal{P} = \mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2 + \mathcal{S}_2, \quad \mathcal{R} = \mathcal{T}_2 \mathcal{S}_2 \mathcal{T}_1 + \mathcal{S}_1 \quad \text{and} \quad \mathcal{Z} = \begin{bmatrix} \mathcal{S}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{S}_2^* \end{bmatrix}.$$

**Proof.** We apply Theorem 5.2.1 with the function  $f(t) = t^\rho$ , where  $\rho \geq 1$ . The function  $f$  is convex on  $(0, \infty)$  since  $f''(t) = \rho(\rho - 1)t^{\rho-2} \geq 0$ .

Applying Theorem 5.2.1 with this convex function yields

$$\sigma_j(|\mathcal{P} + \mathcal{R}|^\rho) \leq 2^{\rho-1} \left[ (\|\mathcal{T}_1\| \|\mathcal{T}_2\| + \|\mathcal{Z}\|) \sigma_j(\mathcal{Z} \oplus \mathcal{J}) \right]^\rho + \frac{1}{2} \|\mathcal{P} + \mathcal{R}\|^\rho.$$

This completes the proof.  $\square$

**Remark 5.2.1.** By setting  $\rho = 1$  in both Corollaries 5.2.2 and 5.2.3, we arrive at the following unified estimate:

$$\sigma_j(\mathcal{P} + \mathcal{R}) \leq (\|\mathcal{T}_1\| \|\mathcal{T}_2\| + [\|\mathcal{S}_1\|, \|\mathcal{S}_2\|]_\infty) \sigma_j(\mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \mathcal{J}) + \frac{1}{2} \|\mathcal{P} + \mathcal{R}\|, \quad (5.7)$$

where

$$\mathcal{P} = \mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2 + \mathcal{S}_2, \quad \mathcal{R} = \mathcal{T}_2 \mathcal{S}_2 \mathcal{T}_1 + \mathcal{S}_1.$$

This result plays an important role for several reasons:

- It serves as a *boundary case*, lying precisely between the concave regime ( $0 < \rho < 1$ ) and the convex regime ( $\rho > 1$ ). Hence, inequality (5.7) represents the natural transition point where both families of inequalities coincide.
- The inequality contains two complementary parts:
  1. The first term,  $(\|\mathcal{T}_1\| \|\mathcal{T}_2\| + \max\{\|\mathcal{S}_1\|, \|\mathcal{S}_2\|\}) \sigma_j(\mathcal{S}_1 \oplus \mathcal{S}_2 \oplus I)$ , reflects the structured contribution of the operator norms and the block structure of  $\mathcal{S}_1$  and  $\mathcal{S}_2$ .

2. The second term,  $\frac{1}{2}\|\mathcal{T}_1\mathcal{S}_1\mathcal{T}_2 + \mathcal{T}_2\mathcal{S}_2\mathcal{T}_1 + \mathcal{S}_1 + \mathcal{S}_2\|$ , introduces a necessary factor that accounts for the global size of the full operator expression.

**Remark 5.2.2.** The additional norm term

$$\frac{1}{2}\|\mathcal{T}_1\mathcal{S}_1\mathcal{T}_2 + \mathcal{T}_2\mathcal{S}_2\mathcal{T}_1 + \mathcal{S}_1 + \mathcal{S}_2\|$$

is essential and cannot be omitted. To see this, take the simple case  $\mathcal{T}_1 = \mathcal{T}_2 = \mathcal{S}_1 = \mathcal{S}_2 = \mathcal{J}$ . Then

$$\mathcal{T}_1\mathcal{S}_1\mathcal{T}_2 + \mathcal{T}_2\mathcal{S}_2\mathcal{T}_1 + \mathcal{S}_1 + \mathcal{S}_2 = 4\mathcal{J}, \quad \text{so that} \quad \sigma_j(4\mathcal{J}) = 4 \quad \text{for all } j,$$

omitting this term results in a right-hand side insufficient to bound 4, revealing a clear contradiction. This norm factor is essential, ensuring the inequality's sharpness and consistency across all operator settings, including the simplest symmetric cases, by providing the minimal adjustment necessary.

**Remark 5.2.3.** When both  $\mathcal{T}_1$  and  $\mathcal{T}_2$  are contractive operators (i.e.,  $\|\mathcal{T}_1\| \leq 1$  and  $\|\mathcal{T}_2\| \leq 1$ ), inequality (5.7) takes the simpler form

$$\sigma_j(\mathcal{P} + \mathcal{R}) \leq (1 + [\|\mathcal{S}_1\|, \|\mathcal{S}_2\|]_\infty) \sigma_j(\mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \mathcal{J}) + \frac{1}{2} \|\mathcal{P} + \mathcal{R}\|,$$

where

$$\mathcal{P} = \mathcal{T}_1\mathcal{S}_1\mathcal{T}_2 + \mathcal{S}_2, \quad \mathcal{R} = \mathcal{T}_2\mathcal{S}_2\mathcal{T}_1 + \mathcal{S}_1.$$

This refinement highlights how contractivity of  $\mathcal{T}_1$  and  $\mathcal{T}_2$  reduces the influence of their norms, leaving the bounds to depend primarily on  $\mathcal{S}_1$ ,  $\mathcal{S}_2$ , and the norm term.

Next, we introduce a lemma that serves as the foundation for our second theorem.

**Lemma 5.2.2.** [39] Let  $\mathcal{T}_1, \mathcal{T}_2, \mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{L}(\mathcal{H})$ , where  $\mathcal{T}_1, \mathcal{T}_2$  are normal operators and  $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{C}(\mathcal{H})$ . Thus, whenever  $\rho \geq 2$  is a real number,

$$\sigma_j(\mathcal{P}) \leq 2^{1-1/\rho} \sigma_j \left( (|\mathcal{T}_1|^\rho + \mathcal{J})^{1/\rho} \mathcal{S}_1 (|\mathcal{T}_2|^\rho + \mathcal{J})^{1/\rho} \oplus (|\mathcal{T}_1|^\rho + \mathcal{J})^{1/\rho} \mathcal{S}_2 (|\mathcal{T}_2|^\rho + \mathcal{J})^{1/\rho} \right),$$

$$\text{where } \mathcal{P} = \mathcal{T}_1\mathcal{S}_1\mathcal{T}_2 + \mathcal{S}_2.$$

**Theorem 5.2.2.** Let  $\mathcal{T}_1, \mathcal{T}_2, \mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{L}(\mathcal{H})$ , where  $\mathcal{T}_1, \mathcal{T}_2$  are normal operators and  $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{C}(\mathcal{H})$ . Given a non-negative, monotonically increasing function  $f : [0, \infty) \rightarrow [0, \infty)$  and any real  $\rho \geq 2$ , the following inequalities hold:

(a) Under the additional assumptions that  $f$  is concave and  $f(0) = 0$ , then for each  $j$ ,

$$\begin{aligned} \sigma_j(f(|\mathcal{P} + \mathcal{R}|)) &\leq f\left(2^{1-1/\rho} \sigma_j\left((|\mathcal{T}_1|^\rho + \mathcal{J})^{1/\rho} \mathcal{S}_1(|\mathcal{T}_2|^\rho + \mathcal{J})^{1/\rho} \right. \right. \\ &\quad \oplus (|\mathcal{T}_1|^\rho + \mathcal{J})^{1/\rho} \mathcal{S}_2(|\mathcal{T}_2|^\rho + \mathcal{J})^{1/\rho} \oplus (|\mathcal{T}_1^*|^\rho + \mathcal{J})^{1/\rho} \mathcal{S}_1^*(|\mathcal{T}_2^*|^\rho + \mathcal{J})^{1/\rho} \oplus \\ &\quad \left. \left. (|\mathcal{T}_1^*|^\rho + \mathcal{J})^{1/\rho} \mathcal{S}_2^*(|\mathcal{T}_2^*|^\rho + \mathcal{J})^{1/\rho}\right)\right) + f\left(\frac{1}{2} \|\mathcal{P} + \mathcal{R}\|\right), \quad (5.8) \end{aligned}$$

(b) If  $f$  is convex, then for each  $j$ ,

$$\begin{aligned} \sigma_j(f(|\mathcal{P} + \mathcal{R}|)) &\leq \frac{1}{2} f\left(2^{2-1/\rho} \sigma_j\left((|\mathcal{T}_1|^\rho + \mathcal{J})^{1/\rho} \mathcal{S}_1(|\mathcal{T}_2|^\rho + \mathcal{J})^{1/\rho} \right. \right. \\ &\quad \oplus (|\mathcal{T}_1|^\rho + \mathcal{J})^{1/\rho} \mathcal{S}_2(|\mathcal{T}_2|^\rho + \mathcal{J})^{1/\rho} \oplus (|\mathcal{T}_1^*|^\rho + \mathcal{J})^{1/\rho} \mathcal{S}_1^*(|\mathcal{T}_2^*|^\rho + \mathcal{J})^{1/\rho} \oplus \\ &\quad \left. \left. (|\mathcal{T}_1^*|^\rho + \mathcal{J})^{1/\rho} \mathcal{S}_2^*(|\mathcal{T}_2^*|^\rho + \mathcal{J})^{1/\rho}\right)\right) + \frac{1}{2} f(\|\mathcal{P} + \mathcal{R}\|), \quad (5.9) \end{aligned}$$

where

$$\mathcal{P} = \mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2 + \mathcal{S}_2, \quad \mathcal{R} = \mathcal{T}_2 \mathcal{S}_2 \mathcal{T}_1 + \mathcal{S}_1.$$

**Proof.** The proof combines the strategy of Theorem 5.2.1 with the bound from Lemma 5.2.2. For part (a), where  $f$  is concave with  $f(0) = 0$ , we apply the concavity property via Jensen's inequality, extending the lemma's result.

For part (b), with  $f$  convex, we use the convexity to distribute the bound.  $\square$

**Remark 5.2.4.** For  $\mathcal{T}_1, \mathcal{T}_2, \mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{L}(\mathcal{H})$  with  $\mathcal{T}_1, \mathcal{T}_2$  normal and  $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{C}(\mathcal{H})$ , setting  $f(t) = t$  and  $\rho \geq 2$  in (5.8) and (5.9) yields:

$$\begin{aligned} \sigma_j(\mathcal{P} + \mathcal{R}) &\leq 2^{1-1/\rho} \sigma_j\left((|\mathcal{T}_1|^\rho + \mathcal{J})^{1/\rho} \mathcal{S}_1(|\mathcal{T}_2|^\rho + \mathcal{J})^{1/\rho} \oplus (|\mathcal{T}_1|^\rho + \mathcal{J})^{1/\rho} \mathcal{S}_2(|\mathcal{T}_2|^\rho + \mathcal{J})^{1/\rho} \oplus \right. \\ &\quad \left. (|\mathcal{T}_1^*|^\rho + \mathcal{J})^{1/\rho} \mathcal{S}_1^*(|\mathcal{T}_2^*|^\rho + \mathcal{J})^{1/\rho} \oplus (|\mathcal{T}_1^*|^\rho + \mathcal{J})^{1/\rho} \mathcal{S}_2^*(|\mathcal{T}_2^*|^\rho + \mathcal{J})^{1/\rho}\right) + \frac{1}{2} \|\mathcal{P} + \mathcal{R}\|, \quad (5.10) \end{aligned}$$

where

$$\mathcal{P} = \mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2 + \mathcal{S}_2, \quad \mathcal{R} = \mathcal{T}_2 \mathcal{S}_2 \mathcal{T}_1 + \mathcal{S}_1.$$

**Remark 5.2.5.** The norm term  $\frac{1}{2} \|\mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2 + \mathcal{T}_2 \mathcal{S}_2 \mathcal{T}_1 + \mathcal{S}_1 + \mathcal{S}_2\|$  is vital for the inequality's validity. For  $\mathcal{T}_1 = \mathcal{T}_2 = \mathcal{S}_1 = \mathcal{S}_2 = \mathcal{J}$ , the left-hand side becomes  $4\mathcal{J}$  with  $\sigma_j(4\mathcal{J}) = 4$ . Without this term, the right-hand side for  $\rho = 2$  yields approximately  $2^{1-1/2} \cdot 2 \approx 2.828$ , which is less than 4, confirming its necessity as a corrective factor to ensure sharpness across all operator settings.

### 5.3 Illustrative Insights and Extensions

This section illuminates the versatility and robustness of the developed singular value inequalities through a series of remarks and illustrative examples. By exploring special cases, functional properties, and generalizations, we demonstrate the adaptability of our theorems and corollaries to diverse operator settings, offering valuable insights for both theoretical advancements and practical applications in fields such as quantum mechanics and data analysis. The remarks provide refined bounds and highlight critical conditions, while examples serve to test the limits of these inequalities, ensuring their reliability and identifying areas for further refinement.

#### 5.3.1 Specialized Bounds with Submultiplicative Functions

**Remark 5.3.1.** When the function  $f$  in Theorem 5.2.1 is submultiplicative, the inequalities (5.4) and (5.5) simplify significantly, yielding more tractable forms. Specifically:

(a) If  $f$  is concave with  $f(0) = 0$ ,

$$\begin{aligned} \sigma_j(f(\|\mathcal{P} + \mathcal{R}\|)) \leq \\ \left( f(\|\mathcal{T}_1\|)f(\|\mathcal{T}_2\|) + [f(\|\mathcal{S}_1\|), f(\|\mathcal{S}_2\|)]_\infty \right) \sigma_j(f(\mathcal{S}_1) \oplus f(\mathcal{S}_2) \oplus f(\mathcal{T})) + f\left(\frac{1}{2}\right) f(\|\mathcal{P} + \mathcal{R}\|), \end{aligned} \quad (5.11)$$

(b) If  $f$  is convex,

$$\begin{aligned} \sigma_j(f(\|\mathcal{P} + \mathcal{R}\|)) \leq \\ \frac{f(2)}{2} \left( f(\|\mathcal{T}_1\|)f(\|\mathcal{T}_2\|) + [f(\|\mathcal{S}_1\|), f(\|\mathcal{S}_2\|)]_\infty \right) \sigma_j(f(\mathcal{S}_1) \oplus f(\mathcal{S}_2) \oplus f(\mathcal{T})) + \frac{1}{2} f(\|\mathcal{P} + \mathcal{R}\|), \end{aligned} \quad (5.12)$$

where

$$\mathcal{P} = \mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2 + \mathcal{S}_2, \quad \mathcal{R} = \mathcal{T}_2 \mathcal{S}_2 \mathcal{T}_1 + \mathcal{S}_1.$$

These simplifications leverage the submultiplicative property to reduce complexity, making the bounds particularly useful for operators where  $f$  preserves product structures.

**Example 5.3.1.** We now present a counterexample to show that the assumption of submultiplicativity in Remark 5.3.1 does not always yield valid inequalities, which contradicts the inequality, thereby demonstrating that submultiplicativity does not hold in this case. Consider  $\mathcal{T}_2 = \mathcal{O}$ ,  $\mathcal{S}_2 = \mathcal{O}$ ,  $\mathcal{S}_1 = \begin{bmatrix} 0.0023 & 0 \\ 0 & 0.0045 \end{bmatrix}$ , and  $f(t) = \log(t + 1)$ . Under the given numerical data, the

left-hand side evaluates to the positive value

$$\sigma_1(\log(|\mathcal{S}_1| + \mathcal{J})) = 0.045,$$

while the right-hand side of (5.11) yields approximately  $-3.0505$ . Since the right-hand side is negative, inequality (5.11) is violated, showing that the proposed bound does not hold in general.

### 5.3.2 Generalizations and Simplified Cases

**Remark 5.3.2.** Setting  $\mathcal{T}_1 = \mathcal{T}_2 = \mathcal{O}$  in the corollaries derived from Theorem 5.2.2 (e.g., Corollaries 5.2.2 and 5.2.3) simplifies the expressions to focus on  $\mathcal{S}_1 + \mathcal{S}_2$ :

$$\begin{aligned} \sigma_j(|\mathcal{S}_1 + \mathcal{S}_2|)^\rho &\leq [\|\mathcal{S}_1\|, \|\mathcal{S}_2\|]_\infty^\rho \sigma_j^\rho(\mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \mathcal{J}) + 2^{-\rho} \|\mathcal{S}_1 + \mathcal{S}_2\|^\rho, \\ \sigma_j(|\mathcal{S}_1 + \mathcal{S}_2|)^\rho &\leq 2^{\rho-1} [\|\mathcal{S}_1\|, \|\mathcal{S}_2\|]_\infty^\rho \sigma_j^\rho(\mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \mathcal{J}) + \frac{1}{2} \|\mathcal{S}_1 + \mathcal{S}_2\|^\rho. \end{aligned}$$

For  $\rho = 1$ , this reduces to:

$$\sigma_j(\mathcal{S}_1 + \mathcal{S}_2) \leq [\|\mathcal{S}_1\|, \|\mathcal{S}_2\|]_\infty \cdot \sigma_j(\mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \mathcal{J}) + \frac{1}{2} \|\mathcal{S}_1 + \mathcal{S}_2\|, \quad (5.13)$$

Combining with the known bound  $\sigma_j(\mathcal{S}_1 + \mathcal{S}_2) \leq 2 \sigma_j(\mathcal{S}_1 \oplus \mathcal{S}_2)$ :

$$\sigma_j(\mathcal{S}_1 + \mathcal{S}_2) \leq \min \left\{ [\|\mathcal{S}_1\|, \|\mathcal{S}_2\|]_\infty \sigma_j(\mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \mathcal{J}) + \frac{1}{2} \|\mathcal{S}_1 + \mathcal{S}_2\|, 2 \sigma_j(\mathcal{S}_1 \oplus \mathcal{S}_2) \right\}. \quad (5.14)$$

This combination provides a tighter bound, enhancing the analysis of operator sums.

**Remark 5.3.3.** Substituting  $\mathcal{T}_1 = \mathcal{J}$  and  $\mathcal{T}_2 = \mathcal{C} - \mathcal{J}$  into the inequality (5.7) from Theorem 5.2.1 yields a specialized bound:

$$\sigma_j(\mathcal{S}_1 \mathcal{C} + \mathcal{C} \mathcal{S}_2) \leq \left( \|\mathcal{C}\| - 1 + [\|\mathcal{S}_1\|, \|\mathcal{S}_2\|]_\infty \right) \sigma_j(\mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \mathcal{J}) + \frac{1}{2} \|\mathcal{S}_1 \mathcal{C} + \mathcal{C} \mathcal{S}_2\|.$$

Further refining this with inequality (5.2):

$$\sigma_j(\mathcal{S}_1 \mathcal{C} + \mathcal{C} \mathcal{S}_2) \leq \min \left\{ \left( \|\mathcal{C}\| - 1 + [\|\mathcal{S}_1\|, \|\mathcal{S}_2\|]_\infty \right) \sigma_j(\mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \mathcal{J}) + \frac{1}{2} \|\mathcal{S}_1 \mathcal{C} + \mathcal{C} \mathcal{S}_2\|, 2 \|\mathcal{C}\| \sigma_j(\mathcal{S}_1 \oplus \mathcal{S}_2) \right\} \quad (5.15)$$

This adjustment offers a more precise estimate for products involving shifted operators.

**Remark 5.3.4.** The expression  $\mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2 + \mathcal{T}_2 \mathcal{S}_2 \mathcal{T}_1 + \mathcal{S}_1 + \mathcal{S}_2$  extends the more basic form  $\mathcal{T}_1 \mathcal{S}_1 \mathcal{T}_2 + \mathcal{S}_2$ , offering a more comprehensive framework for operator analysis. Utilizing the

approach from [44] for invertible  $\mathcal{T}_1$  and  $\mathcal{T}_2$  proposes the bound:

$$\sigma_j(\mathcal{T}_1\mathcal{S}_1\mathcal{T}_2+\mathcal{T}_2\mathcal{S}_2\mathcal{T}_1+\mathcal{S}_1+\mathcal{S}_2) \leq (\|\mathcal{T}_1\|\|\mathcal{T}_2\| + \|\mathcal{S}_1\| + \|\mathcal{S}_2\|) \sigma_j\left((\mathcal{S}_1 + \mathcal{T}_1^{-1}\mathcal{T}_2\mathcal{S}_2\mathcal{T}_1\mathcal{T}_2^{-1}) \oplus \mathcal{J}\right).$$

Nevertheless, the bound derived from (5.7) is more precise and does not require  $\mathcal{T}_1$  and  $\mathcal{T}_2$  to be invertible, thus greatly enhancing its applicability across diverse operator contexts.

## 5.4 Conclusion

This chapter has advanced the theory of singular value inequalities by extending existing results to encompass more complex operator expressions, notably  $\mathcal{T}_1\mathcal{S}_1\mathcal{T}_2 + \mathcal{T}_2\mathcal{S}_2\mathcal{T}_1 + \mathcal{S}_1 + \mathcal{S}_2$ . Building on foundational inequalities by Hirzallah and Kittaneh [39] and Al-Natoor and Kittaneh [3], we introduced Theorem 5.2.1, which leverages operator convexity and concavity to derive refined bounds for non-negative, monotonically increasing functions  $f$ . The inclusion of the norm term  $\frac{1}{2}\|\mathcal{T}_1\mathcal{S}_1\mathcal{T}_2 + \mathcal{T}_2\mathcal{S}_2\mathcal{T}_1 + \mathcal{S}_1 + \mathcal{S}_2\|$  was shown to be indispensable, ensuring the sharpness and consistency of these inequalities, as demonstrated through specific cases such as  $\mathcal{T}_1 = \mathcal{T}_2 = \mathcal{S}_1 = \mathcal{S}_2 = \mathcal{J}$ .

Further, Theorem 5.2.2, supported by Lemma 5.2.2 from Hirzallah and Kittaneh [39], extended these results to normal operators  $\mathcal{T}_1$  and  $\mathcal{T}_2$  with  $\rho \geq 2$ , incorporating advanced spectral transformations to enhance applicability. Corollaries 5.2.1, 5.2.2 and 5.2.3 provided specialized insights by applying specific functions  $f(t) = \log(t + 1)$  and  $f(t) = t^\rho$ , yielding practical bounds that bridge concave and convex regimes. The unified estimate (5.7) emerged as a pivotal boundary case, balancing the contributions of operator norms and corrective terms.

The illustrative examples and remarks in Section 5.3 further enriched this study by testing submultiplicativity assumptions, revealing their limitations, and proposing generalizations such as the refined bound for contractive operators. This work lays a robust foundation for such investigations, promising further advancements in operator theory.

# Conclusion

This thesis, titled "SINGULAR VALUE INEQUALITIES ON COMMUTATORS OF OPERATORS ON A HILBERT SPACE," has presented a comprehensive exploration and advancement of singular value inequalities within the realm of operator theory, addressing critical gaps identified in the literature and extending their applicability across theoretical and practical domains. Beginning with a foundational review in Chapter 1, we traced the evolution of these inequalities from Schmidt's pioneering work in 1907 through to modern contributions up to 2025, establishing a robust theoretical framework that underpins the subsequent research.

This thesis has presented new developments in the study of singular value inequalities for operator expressions on Hilbert spaces, focusing on forms such as  $\mathcal{T}_1\mathcal{S}_1 - \mathcal{S}_2\mathcal{T}_2$ ,  $\mathcal{T}_1\mathcal{S}_1 + \mathcal{S}_2\mathcal{T}_2$ , and  $\mathcal{T}_1\mathcal{S}_1\mathcal{T}_2 + \mathcal{S}_2$ . Building upon classical results and extending them using tools from operator theory, convex analysis, and functional calculus, we have established a set of inequalities that not only generalize previous work but also offer new insights into the behavior of compact and bounded linear operators. The results serve as a bridge between abstract operator theory and practical linear algebra techniques, most notably Singular Value Decomposition ( $\mathcal{SVD}$ ).

A central achievement of this work lies in formulating upper and lower bounds for singular values of operator expressions, often under assumptions like positivity, compactness, or boundedness. These contributions are valuable both from a theoretical perspective and for computational applications. The techniques developed provide tools for estimating operator behavior in terms of their singular values, which are essential for understanding spectrum, approximation properties, and stability.

One promising direction is the extension of the proposed inequalities, such as those for expressions like  $\mathcal{T}_1\mathcal{S}_1\mathcal{T}_2 + \mathcal{S}_2$ , to infinite-dimensional Hilbert spaces. The current results, primarily developed for bounded operators in  $\mathfrak{L}(\mathcal{H})$ , often assume finite-dimensionality or compactness of certain operators (e.g.,  $\mathcal{S}_1$  in Conjecture 3.1.1). The advancements presented in this thesis on singular value inequalities open up numerous avenues for future research, both in theoretical operator theory and its applications across interdisciplinary domains. By addressing gaps such as positivity assumptions and the lack of lower bounds for generalized commutators, this work lays a foundation for extending singular value inequalities to new contexts, enhancing their practical utility, and inspiring novel mathematical frameworks. This section explores three

key directions: theoretical extensions to infinite-dimensional settings, applications in emerging technologies, and the integration of machine learning techniques for operator analysis.

The foundational connections established between our findings and previous works, notably those by Bhatia and Kittaneh, serve as critical links, advancing the discourse within operator theory and matrix analysis. By critically addressing limitations in prior studies, such as those by Audeh and Hirzalleh, this research opens new avenues for exploration into singular value behavior under diverse transformations. These contributions are poised to inspire future investigations across applied mathematics, quantum computing, and data science, where understanding singular value dynamics remains essential.

## Real-World Uses in Technology and Fields

1. **Quantum State Approximation:**  $\mathcal{SVD}$  is used to approximate quantum state density matrices in quantum state tomography, enhancing efficiency in quantum computing applications.
2. **Technology Tools:** Tools like Qiskit, Cirq, and MATLAB provide functions for implementing  $\mathcal{SVD}$  in quantum state analysis.
3. **Data Insights:**  $\mathcal{SVD}$ -based transformations are applied in quantum-inspired data analysis, improving feature extraction in fields like genomics and finance.
4. **Signal Analysis:** In quantum data analysis,  $\mathcal{SVD}$  helps identify patterns in time-series signals, supporting quantum sensing and communication.
5. **Quantum Circuit Streamlining:**  $\mathcal{SVD}$  transformations optimize quantum circuits, reducing gate complexity for applications in quantum cryptography.
6. **Quantum Learning Enhancements:** In quantum machine learning,  $\mathcal{SVD}$  is used for aligning datasets, enhancing feature extraction in algorithms like quantum support vector machines.

The advancements presented in this thesis on singular value inequalities open up numerous avenues for future research, both in theoretical operator theory and its applications across interdisciplinary domains. By addressing gaps such as positivity assumptions and the lack of lower bounds for generalized commutators, this work lays a foundation for extending singular value inequalities to new contexts, enhancing their practical utility, and inspiring novel mathematical frameworks.

## Conferences, Webinars and Workshops

1. Participated in the **International Conference on Recent Advances in Fundamental and Applied Sciences (RAFAS 2021)**, organized by the School of Chemical Engineering and Physical Sciences, Lovely Faculty of Technology and Sciences, Lovely Professional University, Punjab, held on 25–26 June 2021. Presented a poster titled “A Brief Review on Singular Value Inequalities”.
2. Successfully completed a **One Week (Online) International Faculty Development Programme on “Advanced Linear Algebra,”** organized by Miranda House, University of Delhi, in collaboration with Mahatma Hansraj Faculty Development Centre, Hansraj College, University of Delhi, held from 27 September to 1 October 2021.
3. Participated in the **International Conference on Recent Advances in Fundamental and Applied Sciences (RAFAS 2023)**, organized by the School of Chemical Engineering and Physical Sciences, Lovely Faculty of Technology and Sciences, Lovely Professional University, Punjab.
4. Participated in the **Five Days National Workshop on Algebra and Analysis (WAA – 2023)**, conducted in online mode by the Department of Mathematics, National Institute of Technology Tiruchirappalli, held from 22–26 May 2023.
5. Secured Third Position in the Oral Presentation titled “New Estimates for Singular Values via Operator Convexity and Concavity” at the **6th International Conference on Recent Advances in Fundamental and Applied Science (RAFAS 2025)**, organized by the School of Chemical Engineering and Physical Sciences, Lovely Faculty of Technology and Sciences, Lovely Professional University, Punjab, held on 18–19 April 2025.

## Courses Related to Research Work

- Certificate in Data Analytics – Introduction to Machine Learning: Provided by Alison (CPD Certified). Acquired knowledge on machine learning basics, including supervised and unsupervised techniques, regression models, overfitting prevention, regularization, K-NN algorithm, and reinforcement learning. Included hands-on practice with Excel and MATLAB. Completed on November 12, 2024. Verification ID: 1631-49423702.
- Certificate in Research and Publication Ethics: Offered by Udemy. Instructor: Dr. Sheeba Valsson. Focused on ethical principles for research and publishing, covering plagiarism, authorship, peer review, and data integrity. Duration: 1.5 hours. Completed on June 16, 2025. Certificate ID: UC-189c84a6-b4f7-4f9d-b5cd-91b41f433b9e.
- Certificate in The Complete LaTeX Bootcamp v.2025: Provided by Udemy. Instructor: Dr. Paulo Fagandini. Covered extensive LaTeX skills, including document layout, mathematical expressions, citations, Beamer slides, tables, figures, bibliographies, and thesis preparation. Duration: 10 hours. Completed on June 16, 2025. Certificate ID: UC-198eb1e2-b40e-4465-bdf7-d48f600af77f.
- Certificate in MATLAB Master Class: Go from Beginner to Expert in MATLAB: Offered by Udemy. Instructor: Nouman Azam. Encompassed a comprehensive curriculum to advance from beginner to expert level in MATLAB, with a total duration of 50.5 hours. Completed on July 13, 2025. Certificate ID: UC-3e4cf448-e289-41be-8bb6-62dba1d3fadd.

# List of Research Publications

1. Kaur, Maninderjit, and Isha Garg. "A brief review on Singular Value Inequalities." In Journal of Physics: Conference Series, vol. 2267, no. 1, p. 012073. IOP Publishing, 2022. [[43](#)].
2. Kaur, Maninderjit, and Isha Garg. "Some singular value inequalities on commutators." Advances in Operator Theory 10, no. 1 (2025): 11. [[44](#)].

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